Geometric Modeling

Goals

- Model geometry of real world objects
  - car bodies
  - cells
  - turbulent flow
- Model geometry of abstract objects
  - approximation theory
  - smoothness spaces

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Geometric Modeling

Model building

- Interactive modeling
  - build shapes
- Data matching
  - build shapes from measurement
  - interpolation/approximation

Compute with it

- Finite element solvers
- Evolving geometry

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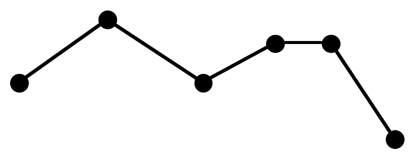
Examples

Curves and Surfaces

- 1D and 2D parameter domains
- higher-D as well, but more unwieldy

Tasks

- interpolate data
- piecewise linear
  - polyline
  - polygon



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Examples

Piecewise linear not good enough

- Higher order polynomials?
  - ill-conditioned
  - keep order bounded
- How to get smoothness?
  - piecewise polynomial

Splines

- Stitch together piecewise polynomial pieces

Subdivision

- Limit of simple steps

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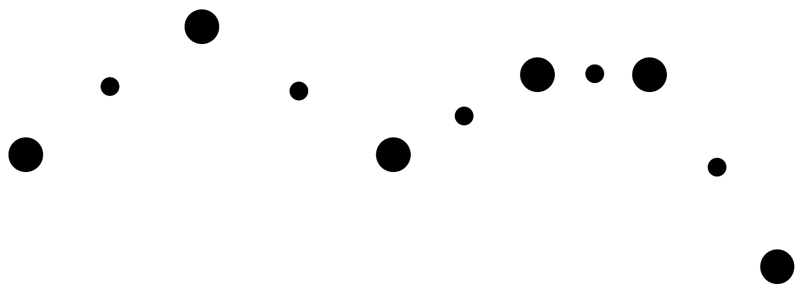
Linear Interpolation

Averaging

- Insert new point as average of neighbors



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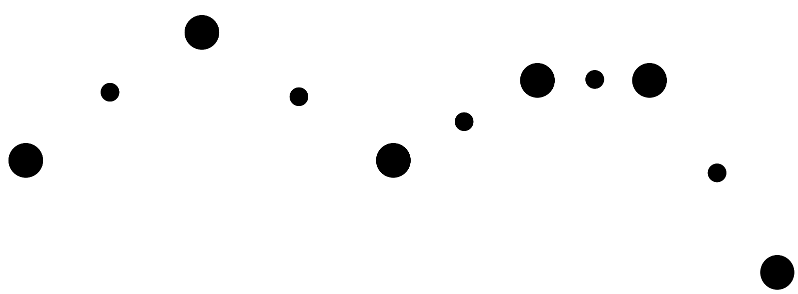
Averaging Again

Chaikin's Algorithm



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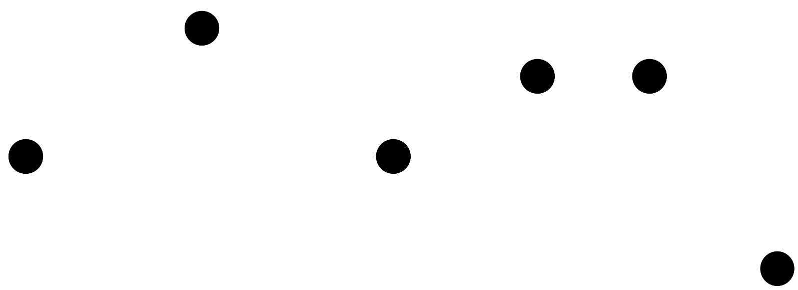
Analysis

In the limit?

- **Simple averaging: piecewise linear**
- **Double averaging: piecewise quadratic**
  - **why?**



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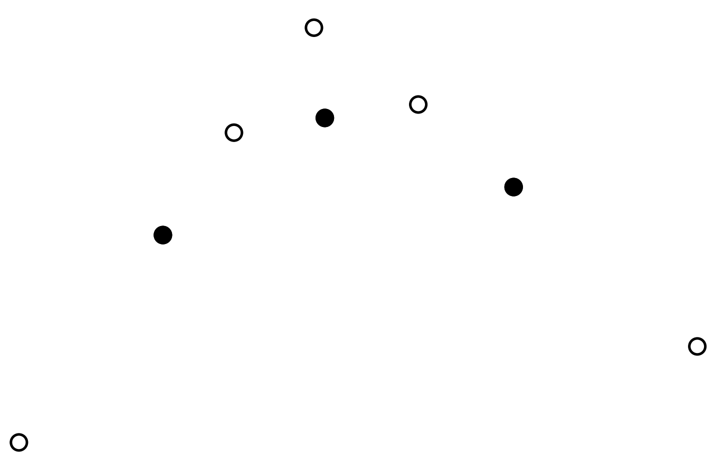
Double Averaging



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Bezier Polynomials

Definition

$$\mathbf{B_i^n(x) = \binom{n}{i} x^i (1-x)^{n-i} \quad x \in [0,1]}$$

Evaluation

$$\mathbf{B_i^n(x) = xB_{i-1}^{n-1}(x) + (1-x)B_i^{n-1}(x)}$$

Use

$$\mathbf{f(x) = \sum_{i=0}^{n} B_i^n(x) b_i}$$

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Bezier Polynomials

Properties

- Partition of unity

$$\mathbf{\sum_{i=0}^{n} B_i^n(x) = \sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i} = (x+1-x)^n}$$

- Stable evaluation (convex combinations only)
- Endpoint interpolation
- Convex hull property
- Affinely invariant

Control Polygon

- Geometrically significant

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Affine Spaces

Affine Combination

$$\mathbf{u_k} \in \mathbf{R^n} \qquad \sum \alpha_k \mathbf{u_k} \qquad \sum \alpha_k = 1$$

Affine Map

$$\mathbf{f}: \mathbf{R^n} \rightarrow \mathbf{R^m} \qquad \mathbf{f}\left(\sum \alpha_k \mathbf{v_k}\right) = \sum \alpha_k \mathbf{f}(\mathbf{u_k})$$

- exactly the linear maps and translation

Affine Space

- subset of vector space closed under affine combinations
- exactly the linear subspace and translation

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Polar Forms

Blossom

- For every polynomial F(u) of degree n there exists a unique symmetric multiaffine map $f(u_1,...,u_n)$ for which F(u)=f(u,...,u). This map is called the polar form or blossom of F.

$$F(u) = \sum_{i=0}^{n} a_i u^i \qquad f(u_1, \ldots, u_n) = \sum_{i=0}^{n} a_i \binom{n}{i}^{-1} \sum_{\substack{S \subseteq \{1,\ldots,n\} \\ |S| = i}} \prod_{j \in S} u_j$$

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Polar Form

$$F(u) = \sum_{i=0}^{n} a_i u^i \qquad f(u_1, \ldots, u_n) = \sum_{i=0}^{n} a_i \binom{n}{i}^{-1} \sum_{\substack{S \subseteq \{1,\ldots,n\} \\ |S| = i}} \prod_{j \in S} u_j$$

Differentiation

$$a_q = \frac{F^{(q)}(0)}{q!} = \binom{n}{q} f(\underbrace{0,\ldots,0}_{n-q}, \underbrace{\hat{1},\ldots,\hat{1}}_{q})$$

$$F^{(q)}(u) = \frac{n!}{(n-q)!} f(u,\ldots,u,\hat{1},\ldots,\hat{1})$$

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Example

$$F(u) = a_0 + a_1 u + a_2 u^2 + a_3 u^3$$

$$\begin{aligned} f(u_1, u_2, u_3) = a_0 & \\ & + \frac{a_1}{3}(u_1 + u_2 + u_3) \\ & + \frac{a_2}{3}(u_1 u_2 + u_2 u_3 + u_3 u_1) \\ & + a_3 u_1 u_2 u_3 \end{aligned}$$

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Continuity

Two Polynomials

- Let F and G be two polynomials of degree n and let u be a point on the real line. Then F and G are C^q continuous at u iff

$$f\left(u^{n-q}u_1,\ldots,u_q\right) = g\left(u^{n-q}u_1,\ldots,u_q\right)$$

$$u_1,\ldots,u_q \in R$$

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Bezier Points

Finding Control Points

- The Bezier control points of F w.r.t [s,t] are given by

$$b_i = f\left(s^{n-i}t^i\right) \qquad i = 0,\ldots,n$$

Proof

$$u = \frac{t-u}{t-s}s + \frac{u-s}{t-s}t$$

$$F(u) = f\left(u^n\right) = \frac{t-u}{t-s}f\left(su^{n-1}\right) + \frac{u-s}{t-s}f\left(u^{n-1}t\right)$$

$$= \ldots = \sum_{i=0}^{n} B_i^{\Delta,n}(u) f\left(s^{n-i}t^i\right)$$

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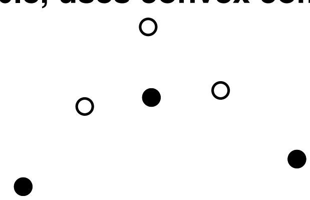
de Casteljau Algorithm

Evaluate Bezier form

$$b_i^0(u) := b_i \qquad i = 0,\ldots,n$$

$$b_i^l(u) := \frac{t-u}{t-s} b_i^{l-1}(u) + \frac{u-s}{t-s} b_{i+1}^{l-1}(u) \qquad l = 1,\ldots,n \quad i = 0,\ldots,n-l$$

- Very stable, uses convex combinations only



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de Casteljau Algorithm

Observations

- Evaluation and subdivision
- Differentiation

$$F'(r) = \frac{n}{s-r}\left(f\left(r^{n-1}s\right) - f\left(r^n\right)\right)$$

- Affine invariance
- C^q conditions

$$c_i = f\left(s^{n-i}t^i\right) = b_{n-i}^i(t)$$

$$F(u) = \sum_{i=0}^{n} B_i^{\Delta,n}(u) b_i \quad [r,s]$$

$$G(u) = \sum_{i=0}^{n} B_i^{\Delta,n}(u) c_i \quad [s,t]$$

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