Student-t distribution with \( n \) degrees of freedom

\[ X \in (-\infty, \infty) \]

\[
\phi(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi} \Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}}
\]

where \( n \) is the number of d.o.f. and \( \Gamma \) the Gamma function

\[
E[X] = 0
\]

\[
V[X] = \begin{cases} \frac{n}{n-2} & n > 2 \\ \text{undefined} & n \leq 2 \end{cases}
\]

\[
E[e^{iX}] = \frac{K_{\frac{1}{2}}\left(\sqrt{n\,|t|}\right)\left(\sqrt{n\,|t|}\right)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right) 2^{\frac{n}{2}-1}}
\]

\( K_{\frac{1}{2}}(x) \): Bessel function

If \( X = \frac{Z}{\sqrt{V/n}} \)

where \( Z \sim N^2(0,1) \)

\( V \sim \chi^2(n) \)

\( Z \) and \( V \) are indep

then \( X \) is student-t \((n)\)
Cauchy Random Variable

\[ X \in (-\infty, \infty) \]

\[ f(x) = \frac{\alpha / \pi}{x^2 + \alpha^2}, \quad -\infty < x < \infty, \quad \alpha > 0 \]

Mean and variance do not exist

\[ E[e^{itX}] = e^{-\alpha|t|} \]

Ex:
Let \( X_1, \ldots, X_n \) be iid Cauchy

\[ \bar{X} = \frac{X_1 + \ldots + X_n}{n} \]

What is the distribution of \( \bar{X} \)?

so \[ E[e^{it\bar{X}}] = e^{-\alpha|t|} \] \( \Rightarrow \) Cauchy

\( \Rightarrow \) Fat tails in finance

\( \Rightarrow \) Stable law: \( X_1, X_2 \) indep and Cauchy

\( \Rightarrow \) \( aX_1 + bX_2 \) indep Cauchy
1.3.4 Stochastic Processes

Def

(i) A collection \( \{ X(t), t \geq 0 \} \) of random variables is called a stochastic process.

(ii) For each \( w \in \Omega \), the mapping \( t \rightarrow X(t, w) \) is the corresponding sample path.

\[ X(t, w_1) \]

\[ X(t, w_2) \]
1.4 Independence

Motivation

Let $(\Omega, \mathcal{U}, \mathbb{P})$ be a prob. space
Let $A, B \in \mathcal{U}$ be two events

We want a reasonable def of

$$\mathbb{P}(A|B): \text{ the prob of } A \text{ given } B$$

If $w \in \Omega$ is selected at random, we are told $w \in B$

What then is the prob that $w \in A$ also?

In the new prob. space

$$\tilde{\Omega} := \mathcal{U}$$
$$\tilde{\mathcal{U}} := \{C \cap B | C \in \mathcal{U}\}$$

$$\tilde{\mathbb{P}} = \frac{\mathbb{P}}{\mathbb{P}(B)}$$

the prob that $w$ lies in $A$ is

$$\tilde{\mathbb{P}}(A|\tilde{\mathcal{U}}^B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$
Def 1.4.1
We write \( P(A \mid B) := \frac{P(A \cap B)}{P(B)} \) if \( P(B) > 0 \).

"A and B are independent" \( \iff \) any information that the event B has occurred is irrelevant in determining the prob. that A has occurred

\( \iff P(A) = P(A \mid B) = \frac{P(A \cap B)}{P(B)} \)

\( \iff P(A \cap B) = P(A) P(B) \)

Def 1.4.2
Two events A and B are called independent if

\( P(A \cap B) = P(A) P(B) \)

Rk A \& B indep \( \iff \) \( A^c \& B \) indep

\( \iff \) \( A^c \& B^c \) indep
Bayes' Formula

\[ P(A|B) = \frac{P(A) \cdot P(B|A)}{P(B)} \]

\[ = \frac{P(A)}{P(B|A) \cdot P(A) + P(B|A^c) \cdot P(A^c)} \]

**Proof:** \[ P(A|B) = \frac{P(A \cap B)}{P(B)} \]

**Ex:** You have a jar containing 999 fair pennies and one two-headed penny.

Suppose you pick one coin from the jar and get all heads.

What is the probability that the coin you choose is the two-headed one?

**Sol:**

\[ A = \{ \text{the coin is 2-headed} \} \]
\[ B = \{ \text{10 times head} \} \]

\[ P(A|B) = \frac{10^{-3}}{10^{-3} + 2^{-10} \cdot (1-10^{-3})} \]
Monty Hall Dilemma

Suppose you're on a game show, and you're given the choice of 3 doors.

Behind one door is a car, behind the others goats.

You pick a door, say no.1, and the host, who knows what's behind the doors, opens another door

say number 3, which has a goat.

He says to you, "Do you want to pick door no 2?"

Is it to your advantage to switch your choice of doors.
Ex: Some have 3 children, each of which is equally likely to be a boy or a girl, independent of the others.

Define the events:

A = \{ all the children are of the same sex \}
B = \{ There is at most one boy \}
C = \{ The family includes a boy and a girl \}

a) Show that A is independent of B & B is independent of C.

b) Is A independent of C?

Sol: \[ P(A) = \frac{3}{8}, \quad P(B) = \frac{5}{8}, \quad P(C) = \frac{3}{4} \]

\[ P(A \cap B) = \frac{1}{4}, \quad P(B \cap C) = \frac{3}{8} \]
Def 1.4.3

Let $A_1, \ldots, A_n$ be events.

These events are independent if for all choices $1 \leq k_1 < k_2 < \cdots < k_m$

\[ P(A_{k_1} \cap A_{k_2} \cap \cdots \cap A_{k_m}) = P(A_{k_1}) P(A_{k_2}) \cdots P(A_{k_m}) \]

Def 1.4.4

Let $U_i \subseteq \mathcal{U}$ be σ-algebras, for $i = 1, \ldots$

We say that $\{U_i\}_{i=1}^\infty$ are independent if for all choices of $1 \leq k_1 < k_2 < \cdots < k_m$ and of events $A_{k_i} \in U_{k_i}$ we have

\[ P(A_{k_1} \cap A_{k_2} \cap \cdots \cap A_{k_m}) = P(A_{k_1}) P(A_{k_2}) \cdots P(A_{k_m}) \]
**Def 1.4.5**

Let $X_i: \Omega \to \mathbb{R}^n$ be independent random variables $(i=1, \ldots)$

We say that the random variables $X_i, \ldots$ are independent if the $\sigma$-algebras $\{ \mathcal{U}(X_i) \}_{i=1}^\infty$ are independent.

Then $\forall k \geq 2$, $\forall$ Borel sets $B_1, \ldots, B_k \subseteq \mathbb{R}^n$

$$
\mathbb{P} \left[ X_1 \in B_1, X_2 \in B_2, \ldots, X_k \in B_k \right] = \mathbb{P} \left[ X_1 \in B_1 \cap X_2 \in B_2 \right] \cdots \mathbb{P} \left[ X_k \in B_k \right]
$$

**Lemma 1.4.6** Let $X_1, \ldots, X_{m+n}$ be independent $\mathbb{R}^k$-valued random variables. Suppose $f: (\mathbb{R}^k)^n \to \mathbb{R}$ and $g: (\mathbb{R}^k)^m \to \mathbb{R}$

Then $Y = f(X_1, \ldots, X_n)$ and $Z = g(X_{n+1}, \ldots, X_{n+m})$

are independent.
Thm 1.4.6

The random variables \( X_1, \ldots, X_m : \Omega \rightarrow \mathbb{R}^n \) are independent if:

\[
F_{X_1, \ldots, X_m}(x_1, \ldots, x_m) = F_{X_1}(x_1) \cdots F_{X_m}(x_m)
\]

\( \forall x_i \in \mathbb{R}^n, \ i = 1, \ldots, m \)

If the random variables have densities

\[
f_{X_1, \ldots, X_m}(x_1, \ldots, x_m) = f_{X_1}(x_1) \cdots f_{X_m}(x_m)
\]

\( \forall x_i \in \mathbb{R}^n, \ i = 1, \ldots, m \)

where the functions \( f \) are the appropriate densities.

Thm 1.4.7 If \( X_1, \ldots, X_m \) are independent, real-valued random variables with \( \mathbb{E}[|X_i|] < \infty \) \( i = 1, \ldots, m \) then \( \mathbb{E}[|X_1 \cdots X_m|] < \infty \) and

\[
\mathbb{E}[X_1 \cdots X_m] = \mathbb{E}[X_1] \cdots \mathbb{E}[X_m]
\]
Thm 1.4.8

If \( X_1, \ldots, X_m \) are independent, random variables, with \( V(X_i) < \infty \) \((i = 1, \ldots, m)\), then

\[
V(X_1 + \ldots + X_m) = V(X_1) + \ldots + V(X_m)
\]
2. Limit theorems

2.1 Borel-Cantelli lemma

**Def 2.1.1**

Let $A_1, \ldots, A_n, \ldots$ be events in a prob. space.

Then the event

$$\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m = \{ \omega \in \Omega \mid \omega \text{ belongs to infinitely many of the } A_n \}$$

is called "An infinitely often", abbreviated "An i.o."

---

**Lemma 2.1.2 (Borel-Cantelli)**

If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(An \text{ i.o.}) = 0$

**Prof.** By definition

$$A_n \text{ i.o.} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$$

for each $n$

$$P[An \text{ i.o.}] \leq IP[\bigcup_{m=n}^{\infty} A_m] \leq \sum_{m=n}^{\infty} IP[A_m] \to 0 \text{ as } n \to \infty$$
Application

Def 2.1.3

A sequence of random variables \( \{X_k\}_{k=1}^{\infty} \)
defined on some prob. space converges
in probability to a random variable \( X \)
provided
\[
\lim_{k \to \infty} P[|X_k - X| > \varepsilon] = 0
\]
for each \( \varepsilon > 0 \).

Thm 2.1.4

If \( X_k \to X \) in probability, then there exists a
subsequence \( \{X_{k_j}\}_{j=1}^{\infty} \subset \{X_k\}_{k=1}^{\infty} \)
such that
\[
X_{k_j}(w) \to X(w) \quad \text{for almost every } w
\]
proof For each positive integer $j$ we select $k_j$ so large that

$$\Pr[|X_{k_j} - X| > \frac{1}{j}] \leq \frac{1}{j^2}$$

and also ... $k_{j-1} < k_j < ... , k_j \to \infty$

Let $A_j = \{ |X_{k_j} - X| > \frac{1}{j} \}$

Since $\sum \frac{1}{j^2} < \infty$, the Borel-Cantelli lemma implies

$$\Pr[\bigcap_{j=1}^{\infty} A_j] = 0$$

$\Rightarrow$ with prob. 1 on $\omega$, $\exists J(\omega) < \infty / A_{J(\omega)}$.

$$|X_{k_j(\omega)} - X(\omega)| \leq \frac{1}{j}$$
2.2 Strong Law of Large Numbers

Def. 2.2.1 Let $S_1, \ldots, S_n, \ldots$ be a sequence of random variables defined on the same probability space. Let $S$ be a random variable. We say that $S_n$ converges almost surely to $S$ (noted $S_n \xrightarrow{a.s.} S$) if

$$\lim_{n \to \infty} \mathbb{P}[S_n = S] = 1$$

Ex. If $S_n \xrightarrow{\text{prob.}} S$ is it true that $S_n \xrightarrow{a.s.} S$?

Sol: $S_n \in \{0, 1\}$.

$$\begin{array}{c}
1 & 0 & 0 & 1 & 0 \\
1 & 2 & 3 & 4 & 5 & 6
\end{array}$$

Ex. If $S_n \xrightarrow{a.s.} S$ is it true that $S_n \xrightarrow{\text{prob.}} S$?

Sol: By dominated convergence

$$\lim_{n \to \infty} \mathbb{E} \left[ \mathbb{I} \{ |S_n - S| > \varepsilon \} \right] = \mathbb{E} \left[ \lim_{n \to \infty} \mathbb{I} \{ |S_n - S| > \varepsilon \} \right] = 0$$
Thm 2.2.2 (Strong Law of Large Numbers)

Let \((X_n, n \in \mathbb{N}^*)\) be a sequence of real or vectorial valued i.i.d (independent, identically distributed) integrable random variables defined on the same probability space and such that \(\mathbb{E}[|X_1|] < \infty\).

Then
\[
\frac{X_1 + \ldots + X_n}{n} \xrightarrow{n \to \infty} \mathbb{E}[X_1]
\]

and
\[
\mathbb{E}\left[\left|\frac{X_1 + \ldots + X_n}{n} - \mathbb{E}[X_1]\right|\right] \xrightarrow{n \to \infty} 0
\]

Proof: We will suppose for simplicity
\(X_i(\omega) \in \mathbb{R}\)
\(\mathbb{E}[X_i^4] < \infty\)
\(\mathbb{E}[X_i] = 0\)
(replace \(X_i\) by \(X_i - \mathbb{E}[X_i]\))

Then
\[
\mathbb{E}\left[\left(\sum_{i=1}^{n} X_i\right)^4\right] = \sum_{i=1}^{n} \mathbb{E}[X_i^4] + 3 \sum_{i \neq j} \mathbb{E}[X_i^2 X_j^2]
\]
\[
= n \mathbb{E}[X_i^4] + 3 (n^2 - n) (\mathbb{E}[X_i^2])^2
\]
\[
\leq n^2C \quad \text{for some constant } C
\]
Fix $\varepsilon > 0$.

By Chebychev inequality

$$\Pr \left[ \left| \frac{1}{n} \sum_{i=1}^{n} X_i \right| > \varepsilon \right] \leq \frac{\mathbb{E} \left[ \left( \sum_{i=1}^{n} X_i \right)^4 \right]}{(\varepsilon n)^4}$$

$$\leq \frac{C}{\varepsilon^4} \frac{1}{n^2}$$

By the Borel–Cantelli lemma

$$\Pr \left[ \left| \frac{1}{n} \sum_{i=1}^{n} X_i \right| \geq \frac{1}{n^{\nu_Q}} \right] \leq \frac{C}{n^{3/2}}$$

$$\downarrow$$

$$\Pr \left[ \left| \frac{1}{n} \sum_{i=1}^{n} X_i \right| \geq \frac{1}{n^{\nu_Q}} \text{ l.o.s.} \right] = 0$$

Hence with prob. 1, $\exists n_0(w) / \forall n > n_0(w)$

$$\frac{1}{n} \sum_{i=1}^{n} X_i \leq \frac{1}{n^{\nu_Q}}$$

$$\Rightarrow \frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{a.s.} 0 \quad \text{as } n \to \infty$$
Ex: \( g: \mathbb{R}^d \to \mathbb{R} \)

c/very large (200), \( g \) continuous

How to compute \( \int_{[0,1]^d} g(x) \, dx \)?

Sol: We have at our disposition (rand. number generator)

\((U_n(w), n \in \mathbb{N}^*)\) a sequence of samples of iid rand. var.

unif. distributed on \([0,1]^d\)

Then \( \frac{1}{n} \sum_{k=1}^{n} g(U_k(w)) \xrightarrow[n \to \infty]{} \mathbb{E}[g(U)] = \int_{[0,1]^d} g(x) \, dx \)

Monte-Carlo Method
Two players A & B play a marble game. Each player has both a red and blue marble. They present one marble to each other.

If both show red, A wins $3.
If both show blue, A wins $1.
If the colors do not match, B wins $2.
Is it better to be A, B or does it matter?

**Sol:**

\[ P_A: \text{prob A plays red} \]
\[ P_B: \text{B} \]
\[ G_A: \text{Gains of A} \]

\[
\mathbb{E}[G_A] = 3 P_A P_B + (1-P_A) (1-P_B) + 2 P_A (1-P_B) + 2 P_B (1-P_A)
\]
\[
= 1 \cdot 3 P_A P_B + 3 P_A (1-P_B) + 2 P_B (1-P_A)
\]
\[
= 1 - 2 P_A + P_B 8 (P_A - \frac{3}{8})
\]
\[
= 1 - 2 P_B + P_A 8 (P_B - \frac{3}{8})
\]
\[
\therefore P_B = \frac{7}{8}
\]