Discrete time Markov Chains

Let $E$ be a discrete space, finite or at most countable

- $E = \{ \text{red, green, orange}\}$
- $E = \{0, 1\}$
- $E = \{\text{rain, sunshine, snow}\}$
- $E = \{1, 2, \ldots, M\}$
- $E = \mathbb{N}$ \quad ($E = \{0, 1, 2, 3, \ldots\}$ )
- $E = \mathbb{Z}$ \quad ($E = \{-\infty, -4, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ )

**Def.** We say that the sequence of random variables $X = (X_n, n \geq 0)$ with values in $E$ is a Markov Chain if it possesses the Markov property:

For all $n \in \mathbb{N}, y, x_0, \ldots, x_n \in E$ such that

\[ P( X_n = x_n, \ldots, X_0 = x_0 ) > 0 \quad \text{we have} \]

\[ P( X_{n+1} = y \mid X_n = x_n, \ldots, X_0 = x_0 ) = P( X_{n+1} = y \mid X_n = x_n ) \]

Observe that if $P( X_n = x_n ) > 0$ then

\[ \sum_{y \in E} P( X_{n+1} = y \mid X_n = x_n ) = 1 \]
Def A matrix $P = (P(x,y), x, y \in E)$ is said to be a stochastic matrix if its coefficients are positive and the sum of the coefficients over each row is equal to 1:

$$\sum_{y \in E} P(x,y) = 1$$

Def Let $(Q_n, n \geq 1)$ be a sequence of stochastic matrices. We say that the matrices $(Q_n, n \geq 1)$ are transition matrices of the Markov Chain $X$ if for all $n \geq 1$ and $x, y \in E$ such that $1P(x_{n-1} = x) > 0$, we have $y \in E$

$$Q_n(x,y) = 1P(x_n = y | x_{n-1} = x)$$

Let $P$ be a stochastic matrix. We say that the Markov Chain $X$ is homogeneous of transition matrix $P$ if for all $n \geq 0$ and $x, y \in E$ such that $1P(x_n = x) > 0$, we have $y \in E$

$$1P(x_{n+1} = y | x_n = x) = P(x_n, y)$$

By convention, we write

$$1P(x_{n+1} = y | x_n = x_n, \ldots, x_0 = x_0) = P(x_n, y)$$

if $1P(x_n = x_n, \ldots, x_0 = x_0) = 0$
Ex Refined model for the Grøthenburg weather

\( E = \{ \text{rain, sunshine, snow} \} \)

Weather evolves according to \( P_{\text{summer}} \) in May - Sept. \( P_{\text{winter}} \) in Oct. - Apr.

\[
P_{\text{summer}} = \begin{bmatrix} 0.75 & 0.25 & 0 \\ 0.25 & 0.75 & 0 \\ 0.5 & 0.5 & 0 \end{bmatrix}
\]

Draw the transition graph

\[
P_{\text{winter}} = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.15 & 0.7 & 0.15 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}
\]

Ex

\( X_n \): a stock of spare parts at time \( n \) (\# of spare parts at time \( n \))

\( D_{n+1} \): the demand (\( \text{rand. var on } \mathbb{N} \))

\( q \): the supply (constant, quantity of spare parts built per time)

is \( X_n \) a MC?

\[
X_{n+1} = (X_n + q - D_{n+1})^+
\]

Assume \( D_n \) are i.i.d., \( p_k = \Pr(D = k) \), is \( X_n \) an hom MC?

\[
\Pr(x, y) = p_k \quad \text{if } y = x + q - k
\]

\[
\Pr(x, 0) = \Pr(D \geq x + q) = \sum_{k \geq x+q} p_k
\]
The simple symmetric random walk on $\mathbb{Z}$ is defined by

$$S_n = S_0 + \sum_{k=1}^{n} Z_k$$

where $Z = (Z_n, n \geq 1)$ is a sequence of iid random variables.

$$P(Z_n = 1) = P(Z_n = -1) = \frac{1}{2}$$

$S_n$ is a homogeneous Markov chain on $\mathbb{Z}$

$$P(x,y) = 0 \quad \text{if} \quad |x-y| \neq 1$$

$$P(x, y) = \frac{1}{2} \quad \text{if} \quad |x-y| = 1$$

$P$: deterministic function

$$\begin{align*}
\text{Ex} & \quad \text{F} \quad \longrightarrow \quad E \\
(\text{Un}, n \geq 1) & \quad \text{a sequence of iid random variables on } F \quad \text{(indep of } X_0) \\
X_{n+1} & = P(X_n, U_n) \\
X_n: & \\text{H.M.C.} \quad P(x, y) = 1 \quad \text{if} \quad P(f(x, U_1) = y) \quad \text{for } x, y \in E
\end{align*}$$
Computer simulations of MC

$U_0, U_1, U_2, \ldots$, iid r.a.v. uniformly distributed in $[0,1]$

$\Psi$, how to simulate a. M.C. $(X_0, X_1, \ldots)$ with state space $E = \{1, 2, \ldots, k\}$

initial distribution $\mu^0$

transition matrix $P$

$\Psi$: initiation function

$\Psi: [0,1] \rightarrow E$ piecewise constant

$\Psi_0(x) = \begin{cases} 1 & \text{for } x \in [0, \mu^0_i] \\ 2 & \text{for } x \in (\mu^0_i, \mu^0_i + \mu^0_j] \\ 3 & \text{for } x \in (\sum_{i=1}^{k-1} \mu^0_i, \sum_{i=1}^{k} \mu^0_i] \end{cases}$

$\phi$: update function

$\phi: [0,1] \times E \rightarrow E$

$\phi(i, x) = \begin{cases} 1 & \text{for } x \in [0, P_{i,1}] \\ 2 & \text{for } x \in (P_{i,1}, P_{i,1} + P_{i,2}] \\ 3 & \text{for } x \in (\sum_{i=1}^{k-1} P_{i,e}, \sum_{i=1}^{k} P_{i,e}] \end{cases}$
Simultaneous:

\[ X_0 = \psi(U_0) \]
\[ X_1 = \phi(X_0, U_1) \]
\[ \vdots \]
\[ X_3 = \phi(X_2, U_3) \]

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Let \( X \) be a H.M.C. compute \( \Pr(X_2 = y \mid X_0 = x) \)

\[
\Pr(X_2 = y \mid X_0 = x) = \frac{\Pr(x_2 = y, x_0 = x)}{\Pr(x_0 = x)}
\]

\[
= \sum_{z \in \mathcal{E}} \frac{\Pr(x_2 = y, x_1 = z, x_0 = x)}{\Pr(x_0 = x)}
\]

\[
= \sum_{z \in \mathcal{E}} \frac{\Pr(x_1 = z, x_0 = x)}{\Pr(x_0 = x)} \cdot \frac{\Pr(x_2 = y, x_1 = z, x_0 = x)}{\Pr(x_2 = y, x_1 = z, x_0 = x)}
\]

\[
= \sum_{z \in \mathcal{E}} \frac{\Pr(x_1 = z \mid x_0 = x) \cdot \Pr(x_2 = y \mid x_1 = z, x_0 = x)}{\Pr(x_0 = x)}
\]

\[
= \sum_{z \in \mathcal{E}} \frac{\Pr(x_1 = z \mid x_0 = x) \cdot \Pr(x_2 = y \mid x_1 = z, x_0 = x)}{\Pr(x_0 = x)}
\]

\[
= \sum_{z \in \mathcal{E}} \Pr(x, z) \cdot P(z, y) = P^2(x, y)
Similarly if \( P^k \) is the \( k \)-th power of \( P \) then
\[
P(X_k = y \mid X_0 = x) = P^k(x, y)
\]
\[
P(X_{k+n} = y \mid X_n = x) = P^k(x, y)
\]

**Proposition** Let \( m \geq 1 \), \( A \subseteq E^m \) and
\[
I_n = \{ (x_1, \ldots, x_{n+m}) \in A \} \quad \text{for} \quad n \geq 0
\]
We consider for \( n \geq 1 \), \( \overline{S}_n = \{ (x_0, \ldots, x_{n-1}) \in B \} \), where \( B \subseteq E^n \). If \( P(X_n = x_n, \overline{S}_n) > 0 \) then we have
\[
P(I_n \mid X_n = x_n, \overline{S}_n) = P(I_n \mid X_n = x_n)
\]
\[
= P(I_0 \mid X_0 = x_0)
\]
\[
= 1
\]
(\* H. M. C. \*)

**Def** We say that an event \( I \) is a.s. if \( P(I \mid X_0 = x) = 1 \)
\( \forall x \in E \)
Let \( \mathcal{L}_0 \) be the law of \( X_0 \):

\[
\mathcal{L}_0 (x) = 1 \mathbb{P} (X_0 = x) \quad \forall x \in E
\]

\[
1 \mathbb{P} (X_1 = y) = \sum_{x \in E} 1 \mathbb{P} (X_1 = y \mid X_0 = x) \mathbb{P} (X_0 = x) \\
= \sum_{x \in E} \mathcal{L}_0 (x) \mathbb{P} (x, y)
\]

We use the notation

\[
\mathcal{L}_0 \mathbb{P} (y) = \sum_{x \in E} \mathcal{L}_0 (x) \mathbb{P} (x, y)
\]

By induction we check that the law of \( X_n \) is \( \mathcal{L}_0 \mathbb{P}^n \).

Let \( f : E \rightarrow \mathbb{R} \), positive or bounded

\[
\mathbb{E} \left[ f (X_n) \mid X_0 = x \right] = \sum_{y \in E} f (y) \mathbb{P} (X_1 = y \mid X_0 = x) \\
= \sum_{y \in E} \mathbb{P}^n (x, y) f (y) \\
= (\mathbb{P}^n f) (x)
\]

\( \mathbb{P}^n f \): product of the matrix \( \mathbb{P}^n \) by the column vector \( f \)

\[
\mathbb{E} \left[ f (X_n) \right] = \sum_{x \in E} \mathbb{P} (X_n = x) f (x) = \mathcal{L}_0 \mathbb{P}^n f
\]
Ex Suppose $X_n$ is a H.M.C. $(\mathbb{Z}, P)$

Define $Y_n = X_{kn}$

is $Y_n$ a H.M.C.? $(\mathbb{Z}, P^k)$

Ex Virus mutation

Suppose a virus can exist in $N$ different strains, and in each generation either stays the same, or with prob. $\alpha$ mutates to another strain which is chosen at random (uniformly).

What is the prob. that the strain in the $n$th gen is the same as that in the $0$th gen?

**Answer**

$N$-state M.C. $P_{ij} = 1 - \alpha$ for $i \neq j$

We need $P^n_{11}$. Can be simplified as a 2-state M.C.

![Diagram](attachment:image.png)
\[ P = \begin{pmatrix} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{pmatrix} \]

\[ \beta = \frac{\alpha}{N - 1} \]

\[ P^n ? \]

\[ P_{11}^{n+1} = P_{12}^{(n)} P + P_{11}^{(n')} (1 - \alpha) \]

\[ P_{11}^{(n)} + P_{12}^{(n')} = 1 \]

\[ P_{11}^{n+1} = \begin{cases} (1 - \alpha - \beta) P_{11}^n + \beta & P_{11}^n \neq 1 \\ \alpha & \text{if } \alpha + \beta > 0 \end{cases} \]

\[ P_{11}^n = \begin{cases} \frac{\beta}{\alpha + \beta} + \frac{\alpha}{\alpha + \beta} (1 - \alpha - (\beta)^n) & \text{if } \alpha + \beta > 0 \\ 1 & \text{if } \alpha + \beta < 0 \end{cases} \]

answer \[ \frac{1}{N} + (1 - \frac{1}{N}) \left(1 - \frac{\alpha N}{N-1}\right)^h \]
Transmission of a message

A message coded in a binary sequence is transmitted through a network. Each bit is transmitted with a prob of error.

\[ P(0 \rightarrow 0) = 1 - a \]
\[ P(0 \rightarrow 1) = a \]
\[ P(1 \rightarrow 0) = b \]
\[ P(1 \rightarrow 1) = 1 - b \]

\[ \vdots \]

\[ X_n: \text{The result of the transmission at the router} \]
\[ \text{null bits: } X_n \]

Routers behave indep. from each other and errors on bits are indep.

Compute the critical size of the network beyond which the prob to receive an error message is \( > \varepsilon \).

\( p \): size of the message

Assume \( p = 1 \), \( X_n \) has transition matrix on \( \{0,1\} \)

\[ P = \begin{pmatrix} 1 - a & a \\ b & 1 - b \end{pmatrix} \]
\[ gn = P(X_n = 0) \]
\[ g_{n+1} = (1 - a) \cdot gn + b \cdot (1 - gn) \]
\[ g = \frac{a}{a+b} \quad \text{fix point} \]
\[ gn - g = (1 - a - b)^n (g_0 - g) \]

The probability that the message is erroneous is thus:
\[ \tau_n(0) = \frac{b}{a+b} \quad (1 - a - b)^n \quad \text{if } X_0 = 0 \]
\[ \tau_n(1) = \frac{a}{a+b} + \frac{b}{a+b} \quad (1 - a - b)^n \quad \text{if } X_0 = 1 \]

Message of size \( p \):
\[ X_n = (X_1, \ldots, X_n) \]

\[ \tau_n : \text{prob that the message in } X_n \text{ is not on.} \]
\[ \tau_n = \prod_{i=1}^{n} \tau_n(X_{n-i}) \leq \left[ \alpha \cdot (1 - \alpha) (1 - a - b)^n \right]^p \]
\[ \alpha = \inf \left( \frac{a}{a+b}, \frac{b}{a+b} \right) \]

\( \Rightarrow \) The size \( n \) of the network has to be chosen so that
\[ h < n_c = \frac{1}{\log(1 - \alpha)} - \log \left( \left( \frac{1 - \epsilon}{\epsilon} \right)^{-\alpha} \right) \]
\[ = \frac{\log(1 - \alpha) - \log (1 - a - b)}{- \log (1 - a - b)} \]
Gambler's ruin

Two players A and B play "head or tail."

$\Pr(H) = p \in (0,1)$

successive outcomes iid.

$X_n =$ fortune of A at time $n$

at each step $A \& B$ bet 1 

$X_{n+1} = X_n + Z_n$

$Z_n = 1$ if $\Pr(p)$

$Z_n = -1$ if $1-p = q$

initial fortune of A: $a$

B: $b$

Game ends when a player is ruined

What is the prob. that A wins?

$c = a + b$

$T =$ first time $n$ at which $X_n = 0$ or $c$

$u(0) = \Pr(X_T = c | X_0 = a)$

$u(i) = \Pr(X_T = c | X_0 = i)$

$u(i) = p \cdot u(i+1) + q \cdot u(i-1)$

boundary condition $u(0) = 0$, $u(c) =$

$u(i) = \frac{1 - (\frac{q}{p})^i}{1 - (\frac{q}{p})^c}$

$q/p = \frac{1}{2}$

$u(i) = \frac{i}{c}$
Irreducible and Aperiodic M.C.

Def. We say that a chain is irreducible if the probability, starting from an arbitrary point \( x \in E \) to reach an arbitrary point \( y \in E \) in \( n_{x,y} \) steps, is strictly positive, i.e.: \( \forall x, y \in E, \exists n_{x,y} \geq 1 \) (depending, a priori on \( x, y \)) / \( P^n(x,y) > 0 \)

\[ R_h \ P^n(x,y) > 0 \iff \exists x_0 = x, x_1, \ldots, x_n = y / \prod_{k=1}^{n} P(x_{k-1}, x_k) > 0 \]

Def. We say that state \( i \) leads to state \( j \) \( i \rightarrow j \) if \( \exists n > 0 / P(X_{n+1} = j | X_n = i) > 0 \) (prop. indep. of \( m \) since M.C. is born)

Def. If \( i \rightarrow j \) and \( j \rightarrow i \), we say that the states \( i \) and \( j \) intercommunicate and write \( i \leftrightarrow j \)
Thm: A MC with state space \{1, 2, \ldots\} is said to be irreducible iff \( K_{ij} \neq 0 \) we have \( i \leftrightarrow j \).

(Otherwise, the MC is said to be reducible.)

Ex: \[ P = \begin{bmatrix}
0.5 & 0.5 & 0 & 0 \\
0.3 & 0.7 & 0 & 0 \\
0 & 0 & 0.2 & 0.2 \\
0 & 0 & 0.8 & 0.2
\end{bmatrix} \]

\[
\begin{array}{c}
\overset{0.5}{\text{1}} \xrightarrow{0.5} \overset{0.7}{\text{2}} \\
\overset{0.3}{\text{1}} \xleftarrow{0.3} \overset{0.2}{\text{2}}
\end{array}
\]

\[
\begin{array}{c}
\overset{0.2}{\text{3}} \xrightarrow{0.9} \overset{0.2}{\text{4}} \\
\overset{0.9}{\text{3}} \xleftarrow{0.9} \overset{0.2}{\text{4}}
\end{array}
\]
Aperiodicity

For a finite or infinite set \( \{a_1, a_2, \ldots \} \) of positive integers we write \( \gcd(a_1, a_2, \ldots) \) the greatest common divisor of \( a_1, a_2, \ldots \)

The period \( d(i) \) of a state \( i \in E \) is defined as

\[
d(i) = \gcd \{ n \geq 1, (P^n)_{ii} > 0 \}
\]

If \( d(i) = 1 \) then we say that the state \( i \) is aperiodic.

Def. A Markov Chain is said to be aperiodic if all its states are aperiodic. Otherwise, the chain is said to be periodic.

Ex. A Markov Chain is shown below. The states 1 and 3 are aperiodic, but state 2 is periodic with period 2.
Def (Thm) an irreducible

We say that a MC is periodic of period \( d \geq 1 \) if we can decompose the state space \( E \) into a partition of \( d \) subsets \( C_1, \ldots, C_d = C_0 \) such that

\[
P(X_t \in C_k \mid X_0 \in C_{k-1}) = 1 \quad \forall k \in \{1, \ldots, d\}
\]

We say that a chain is aperiodic if its greatest period is 1.

Thm Suppose that we have an aperiodic M.C. \((X_n)_{n \geq 0}\) with state space \( E^d \) and transition matrix \( P \).

Then there exists \( N < \infty \) such that \((P^n)_{i,i} > 0 \) for all \( n > N \) and all \( i \in \{1, \ldots, k\} \).
proof. Lemma from number theory

Lemma. Let \( A = \{a_1, a_2, \ldots \} \) be a set of positive integers which is

(i) non-lattice, meaning that \( \gcd(a_1, a_2, \ldots) \neq 1 \)

and

(ii) closed under additions, meaning that if

\( a \in A \) and \( a' \in A \) then \( aa' \in A \)

Then there exists an integer \( N < \infty \) s.t. \( n \in A \)

\( \forall n \geq N \)

for \( i \in E, \ A_i = \{ n > 1 : (P^n)_i > 0 \} \)

MC aperiodic \( \Rightarrow \) i aperiodic

\( A_i \) non-lattice

\( A_i \) closed under additions

\( a, a' \in A_i \) \( \Rightarrow \)

\( P(X_{a} = 1 | X_0 = i) > 0 \)

\( P(X_{a'} = 1 | X_0 = i) > 0 \)

\( \Rightarrow \)

\( P(X_{a}a' = 1 | X_0 = i) > P(X_a = 1, X_{a'a'} = 1 | X_0 = i) \)

\( \Rightarrow \)

\( P(X_a = 1 | X_0 = i) P(X_{a'a'} = 1 | X_0 = i) > 0 \)
A subsequence and (1) and (2) of the lemma

\[ \exists N_i < \infty \quad (P^n)_{ii} > 0 \quad \forall n \geq N_i \]

\[ N = \max (N_1, \ldots, N_k) \]

**Corollary**

Let \((X_0, X_1, \ldots)\) be an irreducible and aperiodic M.C. with finite state space \(E = \{1, \ldots, k\}\) and transition matrix \(P\).

Then there exists an \(M < \infty\) s.t. \((P^n)_{ij} > 0\) \(\forall i, j \in \{1, \ldots, k\}\) and all \(n \geq M\)

**Proof**

\[ \exists N < \infty \quad (P^n)_{ii} > 0 \quad \forall i \in E, \quad n \geq N \]

Let \((i, j) \in E \times E\)

Irreducibility \(\Rightarrow \exists n_{ij} / (P^{n_{ij}})_{ij} > 0\)

\[ M_{ij} = \min_{m} m + n_{ij} \quad \text{for} \quad m \geq M_{ij} \]

\[ P(X_{m+1} = j | X_m = i) \geq P(X_{m+1} = j, X_{m} = i | X_m = i) \]

\[ P(X_{m-n_{ij}} = i | X_m = i) \quad P(X_{m} = j | X_{m-n_{ij}} = i) \]

\[ \Rightarrow (P^m)_{ij} > 0 \quad \forall m \geq n_{ij} \]

\[ N = \min_{i,j} M_{ij} \]