Def 7.13 A stochastic process is a collection \( \{X_t, t \in T\} \) of random variables on a prob. space \((\Omega, \mathcal{U}, \mathbb{P})\).

Def 7.14 \(\mathcal{G}(X)\): \(\sigma\)-algebra generated by the events 
\[(X_{t_1}, \ldots, X_{t_k}) \in A \quad A \in \mathcal{B}(\mathbb{R}^k)\]

Theorem 7.15 The (probability) law of \(X\) is uniquely determined by its finite dimensional distribution,
\[\mu_{t_1, \ldots, t_k}(A) = \mathbb{P}\left[(X_{t_1}, \ldots, X_{t_k}) \in A\right] \quad A \in \mathcal{B}(\mathbb{R}^k)\]

Def 7.16 Let \((X_t)_{t \in T}\) be a Gaussian Process.
The covariance function of \(X\) is the function 
\[\Gamma : T \times T \rightarrow \mathbb{R} \text{ defined by } \Gamma(s, t) = \text{Cov}(X_s, X_t) = \mathbb{E}[X_s X_t]\]
**Thm 7.17**

The law of \( X \) is uniquely determined by the function \( \Pi \).

**Def 7.18**

We say that a function \( \Pi \) on \( T \times T \)

is symmetric and of positive type if

\[
\Pi(s,t) = \Pi(t,s)
\]

and

\[
\text{if } c \text{ is a function with finite support on } T
\]

\[
\sum_{T \times T} c(s) c(t) \Pi(s,t) \geq 0
\]

\[
(\text{IE} \left[ \sum_{T} c(s) X(s) \right]^2 \text{ if } \Pi \text{ is the cov. function of } X)
\]
**Thm 7.19**

Let $\Gamma$ be a symmetric function of positive type on $T \times T$, then there exists a Gaussian process $(X_t)_{t \in T}$ whose covariance function is $\Gamma$.

Prof. Kolmogorov existence theorem

**Rk** $T = \mathbb{R}^+$, \[ (X_t)_{t \in \mathbb{R}^+} \text{ is a Brownian motion} \]

$\Gamma(s, t) = sat$

$T = [0, 1]^2$ \[ \text{Brownian sheet} \]

$T = \mathbb{R}^d$ \[ \text{Gaussian field} \]

$T = L^2(E, E, \mu)$

$\Gamma(f, g) = \langle f, g \rangle_{L^2}$ \[ \text{Gaussian measure} \]

$= \int_E f(x) g(x) \mu(dx)$
Definition 7.20

For $T \in \mathbb{R}^+$ and $\mathbb{P}(s, t) = s + t$ (min(s, t))

The process $(X_t, t \in \mathbb{R}^+)$ noted $(B_t, t \in \mathbb{R}^+)$

is called a Brownian Motion.

Proposition 7.21

A process $(X_t, t \in \mathbb{R}^+)$ is a Brownian Motion if

(i) $X_0 = 0$ a.s.

(ii) $\forall 0 \leq s \leq t$ the random variable $X_t - X_s$

is independent from $\mathcal{G}(X_r, r \leq s)$ and has a $N(0, t-s)$ prob. dist.
Corollary 7.22

The process \( X_t (X_t, t \in \mathbb{R}^+) \) is a B.M. (Brownian Motion)

If \( X_0 = 0 \) a.s. and

\[
\forall 0 = t_0 < t_1 < \ldots < t_n \text{ the random variables } X_{t_i} - X_{t_{i-1}} \text{ are independent}
\]

\( X_{t_i} - X_{t_{i-1}} \) is \( \mathcal{N}(0, t_i - t_{i-1}) \)

In particular the density of \( (X_{t_1}, \ldots, X_{t_n}) \) is

\[
\rho_{(Y_1, \ldots, Y_n)} = \frac{1}{(2\pi)^{n/2} \sqrt{(t_1 (t_2 - t_1) \ldots (t_n - t_{n-1})} \exp \left[ - \sum_{i=1}^n \frac{(Y_i - Y_{i-1})^2}{2(t_i - t_{i-1})} \right]
\]
Prop 23

Let $B$ be a BM, then

(i) $B$ is a BM.

(ii) $\forall \lambda > 0$, the process

$$B^\lambda_t = \frac{1}{\lambda} B_{\lambda t}$$

is a so a BM.

(iii) $\forall s \geq 0$ the process $B^{(s)}_t = B_{t+s} - B_s$

is a BM. indep. from $\zeta(B_r, r \leq s)$
Ex:

Let \((X_i)_{i \in \mathbb{N}}\) be an infinite sequence of iid random variables such that \(\mathbb{E}[X_i] = 0\) and \(\mathbb{E}[X_i^2] = 1\).

Let \(t \in \mathbb{R}^+\), write

\[ S_n := \sum_{i=1}^{n} X_i \]

a) Is \(\frac{S_n}{n}\) converging as \(n \to \infty\)?

If yes specify the limit and the type of convergence (a.s.? in prob.? in law?)

b) For \(t \in T := [0,1]\) write \([nt]\) the integer part of \(nt\) and

\[ X_n(t) := \frac{S_{[nt]}}{\sqrt{n}} \]

Is \(X_n(t)\) converging as \(n \to \infty\)? If yes explain how, why and specify the limit.

c) Let \(0 = t_0 < t_1 < \ldots < t_m = 1\)

Is \((X_n(t_0), \ldots, X_n(t_m))\) converging in law as \(n \to \infty\)?

If yes explain why and specify the limit law.
If it is known (admit this point) that the stochastic process \( (X_n(t))_{t \in [0,1]} \) converges in law towards a stochastic process \( (Z(t))_{t \in [0,1]} \) iff

\[
\forall \epsilon > 0, \forall \eta > 0 \exists \delta > 0 / \left( \limsup_{n \to \infty} P \left[ \omega \left( X_n, \delta \right) > \eta \right] \leq \epsilon \right)
\]

where \( \omega \left( X_n, \delta \right) \) is the modulus of continuity of \( X_n \).

And for all \( m \geq 0 \) and \( 0 = t_0 < t_1 < \ldots < t_m = 1 \)

the law of \( (X_n(t_0), \ldots, X_n(t_m)) \) converges towards the law of \( (Z(t_0), \ldots, Z(t_m)) \).

Does \( (X_n(t))_{t \in [0,1]} \xrightarrow{n \to \infty} (Z(t))_{t \in [0,1]} \) ?

If you what is \( Z \)?
7.3 Gaussian Measure

Motivation: What is \( \int B_3 \cdot dB_t \) \( B_t \) a Brownian motion indexed by \( \mathbb{R}, \mathbb{R}^2, \ldots, \mathbb{R}^n \) ?

Let \( (E, \mathcal{E}) \) be a measurable space and \( \mu \) a \( \sigma \)-finite measure on \( (E, \mathcal{E}) \)

\[
( \exists (A_i)_{i \in I} \quad / \quad E = \bigcup_{i \in I} A_i \quad \& \quad \forall i \mu(A_i) < \infty )
\]

Def 7.24

A Gaussian measure of intensity \( \mu \) is an isometry \( G \) (linear mapping preserving the inner product)

from \( L^2(E, \mathcal{E}, \mu) \) to a Gaussian space.

\[
G : L^2(E, \mathcal{E}, \mu) \rightarrow H
\]

\( \forall \lambda \in \mathbb{R}, f, g \in L^2(E, \mathcal{E}, \mu) \)

\[
G(\lambda f + g) = \lambda G(f) + G(g)
\]

\[
\mathbb{E}[G(f)G(g)] = \langle f, g \rangle_{L^2(E, \mathcal{E}, \mu)}
\]
\[ If \quad f \in L^2(\mathbb{R}, \mathbb{F}, \mu) \]

\[ G(f) \text{ is a centered Gaussian random variable of variance} \]

\[ \mathbb{E} \left[ G(f)^2 \right] = \| G(f) \|_{L^2(\mathbb{F})}^2 = \| f \|_{L^2(\mathbb{F}, \mu)}^2 \]

In particular, if \( A \in \mathbb{E} \) with \( \mu(A) < \infty \), \( f = \mathbb{1}_A \)

then \( G(\mathbb{1}_A) = G(A) \) is \( \mathcal{N}(0, \mu(A)) \).

If \( A_1, \ldots, A_n \in \mathbb{E} \) with \( \mu(A_i) < \infty \), \( A_i \cap A_j = \emptyset \) \( i \neq j \)

then the vector \((G(A_1), \ldots, G(A_n))\) is a Gaussian vector in \( \mathbb{R}^n \) with diagonal covariance matrix

\[ \mathbb{E} \left[ G(A_i) G(A_j) \right] = \langle A_i, A_j \rangle_{L^2(\mu)} \]

\[ G(A_i) \text{ are i.i.d}. \]

\[ A = \bigcup_{i=1}^{\infty} A_i \quad A_i \cap A_j = \emptyset \quad i \neq j \quad \mu(A) < \infty \]

\[ \sum_{i=1}^{\infty} G(A_i) \xrightarrow{n \to \infty} G(A) \]

\[ i \text{ isometry} \]

The properties of the mapping \( A \to G(A) \) look like the properties of measure.

In general, for a fixed \( A \to G(A)(w) \) doesn't define a measure.
Thm 7.25

If $(E, \mathcal{E})$ is a measurable space, $\mu$ a finite measure on $(E, \mathcal{E})$ then there exists a Gaussian measure of intensity $\mu$ on $(E, \mathcal{E})$.

Proof

If $L^2(E, \mathcal{E}, \mu)$ is separable (contains a dense countable subset)

Assume

(ex. $E = \mathbb{R}^+$, $\mu = dx$)

There exists

$\{\varphi_n\}$ an orthonormal basis of $L^2(E, \mathcal{E}, \mu)$

$\varphi_n \in L^2(E, \mathcal{E}, \mu)$, $\left< \varphi_n, \varphi_n \right>_{L^2} = \delta_{nn}$

$\forall f \in L^2(E, \mathcal{E}, \mu)$

$f \sim \sum_{n=1}^{\infty} \left< f, \varphi_n \right> \varphi_n$

$\xi_n$ : iid $\mathcal{N}(0,1)$

$G(f) := \sum_{n=1}^{\infty} \left< f, \varphi_n \right> \xi_n$
$G(\beta)$ is Gaussian

\[ \forall \beta, \gamma \in L^2(E, \varepsilon, \mu) \]

\[ \mathbb{E}[G(\beta)G(\gamma)] = \sum_{m, n=1}^{\infty} \langle \beta, \xi_m \rangle \langle \gamma, \xi_n \rangle \delta_{mn} \]

\[ = \langle \beta, \gamma \rangle \quad L^2(E, \varepsilon, \mu) \]

From now on, we will consider

\[ (E, \varepsilon, \mu) = (\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+), dx) \]

\[ \{ \xi_n \}_{n \in \mathbb{N}} \quad \text{orthonormal basis of} \quad L^2(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+), dx) \]

(Fourier, Haar, Wavelet)

\[ G: L^2(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+), dx) \rightarrow H \]

\[ \sum_{n} \langle \beta, \xi_n \rangle \xi_n \quad \sum_{n} \langle \gamma, \xi_n \rangle \xi_n \]

by isometry

convergence in $L^2(\mathbb{R})$
Thm 7.26

Let $G$ be a Gaussian measure on $L^2(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+), dx)$ of intensity the Lebesgue measure.

The process $(B_t, t \in \mathbb{R}^+)$ defined by

$$B_t = G[\cdot [0,t]]$$

is a Brownian Motion (started from 0, real, 1d)

Proof:

$B_t$ is a Gaussian process since

$$\sum \lambda_i B_t^i = G[\sum \lambda_i \cdot [0,t]]$$

is Gaussian.

Moreover

$$E[B_s B_t] = \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \mathbb{1}_{[0,s]} \mathbb{1}_{[0,t]} dx \ dx = \sqrt{t} \sqrt{s}$$

$G$ is an isometry.
Construction of the B.M. on $[0,1]$

(Parkinson 8) orthonormal basis of $L^2([0,1], \mathcal{S}_0([0,1]), d\lambda)$

$$B_t \approx \sum_{n=0}^{\infty} \int_{0}^{t} \phi_n(s) \, ds \, \xi_n \quad \xi_n \sim \text{iid } \mathcal{N}(0,1)$$

$$\sum_{n=1}^{M} \int_{0}^{t} \phi_n(s) \, ds \, \xi_n \xrightarrow{\mathcal{L}^2(0)} B_t$$

$$<A_{(0,1)}, \phi_{n}>_{L^2([0,1])}$$

Examples of orthonormal bases $\phi_n$:

- Trigonometric (Fourier series)
  $$\left\{ 1, \sqrt{2}\frac{1}{\sqrt{\pi}} \sin(2\pi n t), \sqrt{2}\frac{1}{\sqrt{\pi}} \cos(2\pi n t), \right\}$$    
  \ \text{for } n=1, 2, ...

Haar functions

$$\left\{ \phi_0, \phi_1, \ldots, \phi_{2^n-1} \right\}$$

$$\phi_0(t) \equiv 1 \quad \text{and with } k = 2^j - 1$$

$$\phi_0^{(k)} = \begin{cases} 2^{n-1} & \frac{k-1}{2^n} < t < \frac{k}{2^n} \\ -2^{n-1} & \frac{k}{2^n} < t < \frac{k+1}{2^n} \\ 0 & \text{otherwise} \end{cases}$$
**Def. 7.27**

If \( B \) is a B.M. and \( G \) the associated measure. We write for \( f \in L^2(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+), d\lambda) \)

\[
\int_0^\infty f(s) dB_s := G(f)
\]

\[
\int_0^t f(s) dB_s := G\left( f[\mathbb{R}^+, t] \right)
\]

The mapping \( f \mapsto \int_0^\infty f(s) dB_s \) is called Wiener integral with respect to the B.M. \( B \)

**Justification of this notation:**

If \( u < v \)

\[
\int_u^v dB_s = G([u, v)) = B_v - B_u
\]

**Def.** A function \( f \in L^2(0, T) \) is called a step function if there exists a partition

\( P = \{0 = t_0 < t_1 < \ldots < t_m = T\} \) such that

\[
f(t) = f_k \text{ for } t_k \leq t < t_{k+1} \quad (k = 0, \ldots, m-1)
\]
Proposition 7.28
If \( f \in L^2(0,T) \) is a step function as above

\[
\int_0^T f(s) \, d\beta_s = \sum_{k=0}^{m-1} f_k \left( \beta_{t_{k+1}} - \beta_{t_k} \right)
\]

Lemma 7.29
If \( f \in L^2(0,T) \), then there exists a sequence of step functions \( f_n \in L^2(0,T) \) such that

\[
\int_0^T |f - f_n|^2 \, ds \xrightarrow{n \to \infty} 0
\]

Moreover, by isometry

\[
\int_0^T f_n \, d\beta_s \xrightarrow{n \to \infty} \int_0^T f(s) \, d\beta_s
\]
**Lemma 7.30**

\[ \forall \alpha, \beta \in \mathbb{R}, f, g \in L^2(0, T) \]

\[ \int_0^T (\alpha f + \beta g) \, d\beta s = \alpha \int_0^T f \, d\beta s + \beta \int_0^T g \, d\beta s \]

\[ \mathbb{E} \left[ \int_0^T f(s) \, d\beta s \right] = 0 \]

\[ \mathbb{E} \left[ \int_0^T g(s) \, d\beta s \right] = \int_0^T \mathbb{E}[g(s)] \, ds \]

\[ \int_0^T f(s) \, d\beta s \text{ is } N(0, \int_0^T f(s)^2 \, ds) \]
2.4 Stochastic Integral

How to define \( \int_0^T X(s, w) \, dB_s(w) \)?

\[
\text{stochastic process}
\]

Def 2.31

A real valued stochastic process \( X \) is called progressively measurable with respect to \( \mathcal{F}_t = \sigma(B_s, s \leq t) \) if for each time \( t \), \( X(t, w) \) is \( \mathcal{F}_t \)-measurable and jointly measurable in the variables \( t \) and \( w \) together.

Def 2.32

We denote by \( L^2(0, T) \) the space of all real valued progressively measurable stochastic processes such that

\[
E \left[ \int_0^T X^2(s, w) \, ds \right] < \infty
\]
Def 7.33

A process \( X \in L^2(0,T) \) is called a step process if there exist a partition \( P = \{0 = t_0 < t_1 < \cdots < t_m = T\} \) such that

\[
X(t, \omega) = X_k(\omega) \quad \text{for} \ t_k \leq t < t_{k+1} \quad (k = 0, \ldots, m-1)
\]

where each \( X_k \) is \( \mathcal{F}_{t_k} \)-measurable.

Def 7.34

Let \( X \in L^2(0,T) \) be a step process as above. Then

\[
\int_0^T X(t)dB_t = \sum_{k=0}^{m-1} X_k (B(t_{k+1}) - B(t_k))
\]

is the Ito stochastic integral of \( X \) on the interval \([0,T]\)
**Lemma 7.35**

\[ a, b \in \mathbb{R}, \quad X, Y \in L^2(0, T) \]

\[ a \int_0^T X_s \, dB_s = a \int_0^T X_s \, dB_s + b \int_0^T Y_s \, dB_s \]

\[ \mathbb{E} \left[ \int_0^T X_s \, dB_s \right] = 0 \]

\[ \mathbb{E} \left[ (\int_0^T X_s \, dB_s)^2 \right] = \mathbb{E} \left[ \int_0^T X_s^2 \, ds \right] \]

**Lemma 7.36**

If \( X \in L^2(0, T) \), there exists a sequence of bounded step processes \( X^n \in L^2(0, T) \) such that

\[ \mathbb{E} \left[ \int_0^T |X_s - X^n_s|^2 \, ds \right] \xrightarrow{n \to \infty} 0 \]
Def 7.37

If $X \in L^2(0, T)$ take a step process $X^n$ as above.

Then

$$E \left[ \left( \int_0^T (X^n_s - X^m_s) \, c(B_s) \right)^2 \right] = E \left[ \int_0^T (X^n_s - X^m_s)^2 \, ds \right]$$

$$\downarrow_{n,m \to \infty} 0$$

So the limit

$$\int_0^T X^n_s \, c(B_s) := \lim_{n \to \infty} \int_0^T X^n_s \, c(B_s)$$

exists in $L^2(\Omega)$ (Cauchy sequence, and the space is complete), and the definition does not depend upon the particular approximating sequence $X^n_s$.
Thm 2.38

\( a, b \in \mathbb{R}, \quad X, Y \in L^2(0,T) \)

\[ \int_0^T (a X_s + b Y_s) \, dB_s = a \int_0^T X_s \, dB_s + b \int_0^T Y_s \, dB_s \]

\[ \mathbb{E} \left[ \int_0^T X_s \, dB_s \right] = 0 \]

\[ \mathbb{E} \left[ (\int_0^T X_s \, dB_s)^2 \right] = \mathbb{E} \left[ \int_0^T X_s^2 \, ds \right] \]

\[ \mathbb{E} \left[ \int_0^T X_s \, dB_s \right] \mathbb{E} \left[ \int_0^T Y_s \, dB_s \right] = \mathbb{E} \left[ \int_0^T X_s Y_s \, ds \right] \]