Averaging vs Chaos in Turbulent Transport?

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The Model

PDE in $\mathbb{R}^d$

$$\boxed{\partial_t T + \; v \; .\nabla T = \kappa \Delta T}$$

$\kappa > 0$. $v \in (C(\mathbb{R}^d))^d$, $\mathrm{div}(v) = 0$.

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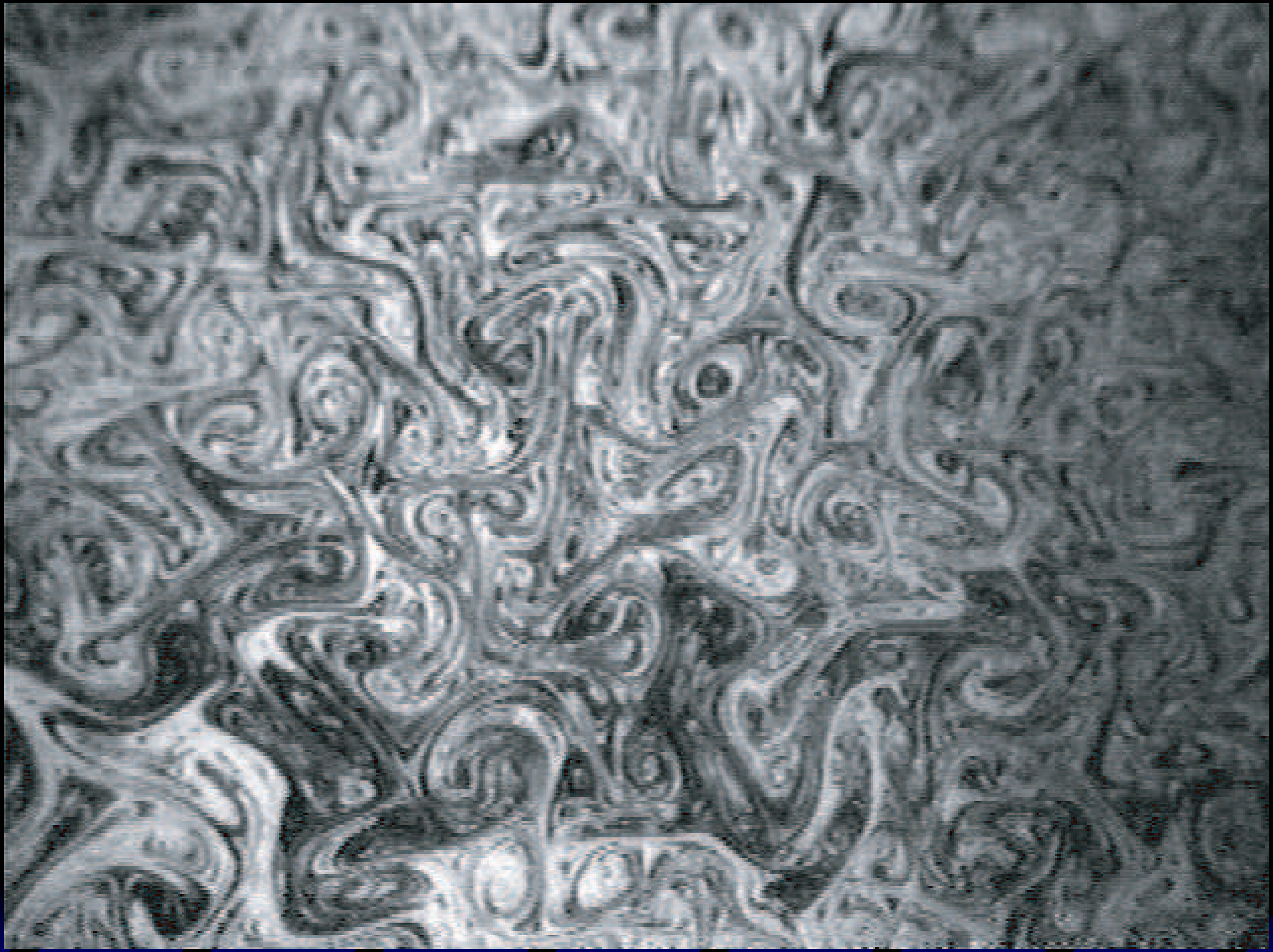
The stream Matrix

$$\boxed{v = \mathrm{div}(\Gamma)}$$

$\Gamma \in (C^1(R^d))^{d\times d}$, $\Gamma_{i,j} = -\Gamma_{j,i}$. $v_i = \sum_{j=1}^d \partial_j \Gamma_{i,j}$.

Motivations: turbulent flows



M. Rutgers

The multi scale decomposition

$$\Gamma := \sum_{n=0}^{\infty} \gamma^n \quad E_n \left(\frac{x}{\rho^n}\right).$$

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Spatial scale

$\rho \in \mathbb{R}^{+,*};$

$$2 \leq \rho < \infty$$

The multi scale decomposition

$$\Gamma := \sum_{n=0}^{\infty} \gamma^n \, E_n\left(\frac{x}{\rho^n}\right).$$

Eddies geometrical parameters

$\mathbb{T}^d :=$ Torus of Dimension d and side 1.

$\forall n,\ \ E_n \in (C^1(\mathbb{T}^d))^{d\times d}$; $E_{n;i,j} = -E_{n;j,i}$; $E_n(0) = 0$.

$$\sup_{n\in\mathbb{N}} \sup_{m,i,j\in\{1,\ldots,d\}} \|\partial_m E_{n;i,j}\|_\infty \leq 1$$

The multi scale decomposition

$$\Gamma := \sum_{n=0}^{\infty} \gamma^n \; E_n \left(\frac{x}{\rho^n}\right).$$

Circulation rates

$\gamma \in \mathbb{R}^{+,*}$;

$$1 < \gamma < \rho$$

The Spectrum is not Kolmogorov

- $v(l)$ velocity of the eddies of size l
- $\mathcal{E}(k)$ The kinetic energy distribution in the Fourier modes
- Kolmogorov

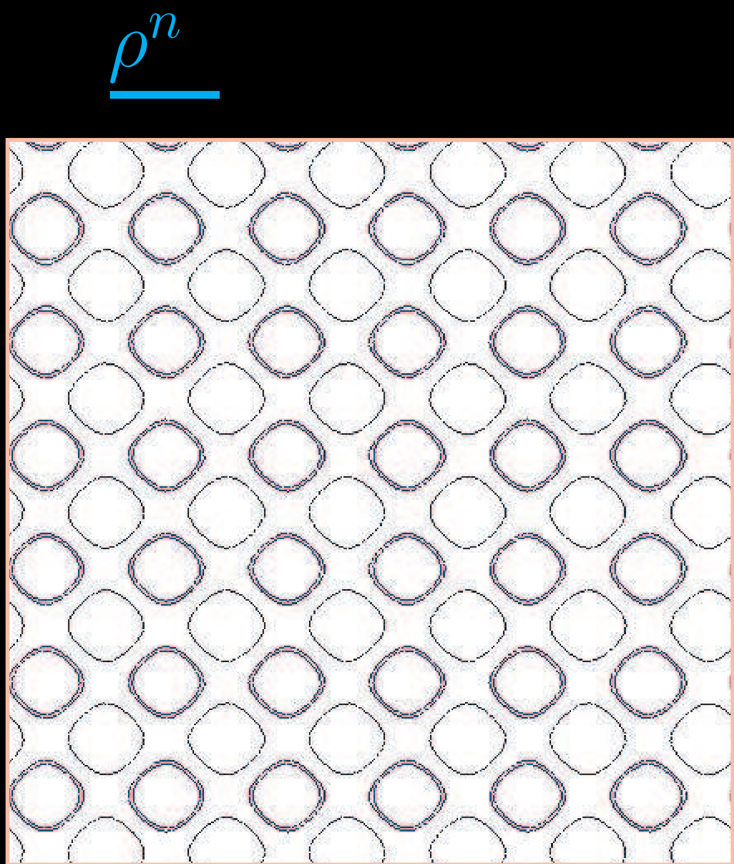
$$v(l) \sim l^{\frac{1}{3}} \qquad \mathcal{E}(k) \sim k^{-\frac{5}{3}}$$

- Our Model

$$v(l) \sim l^{\frac{\ln \gamma}{\ln \rho}-1} \qquad \mathcal{E}(k) \sim k^{1-2\frac{\ln \gamma}{\ln \rho}}$$

- Kolmogorov $\rightarrow \gamma = \rho^{\frac{4}{3}}$ Our model $\gamma < \rho$

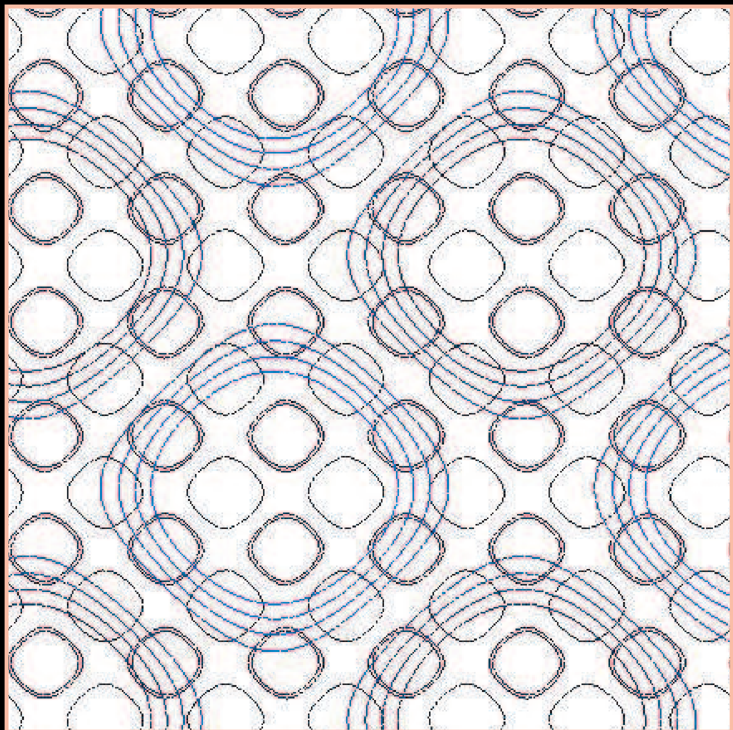
One scale.



Stream lines of the flow $\gamma^n E_n(\frac{x}{\rho^n})$

ρ^{n+1} Two scales.

ρ^{n}



Stream lines of $\gamma^n E_n(\frac{x}{\rho^n})$ $\boxed{and}$ $\gamma^{n+1} E_{n+1}(\frac{x}{\rho^{n+1}})$.

ρ^{n+1}

Two scales.

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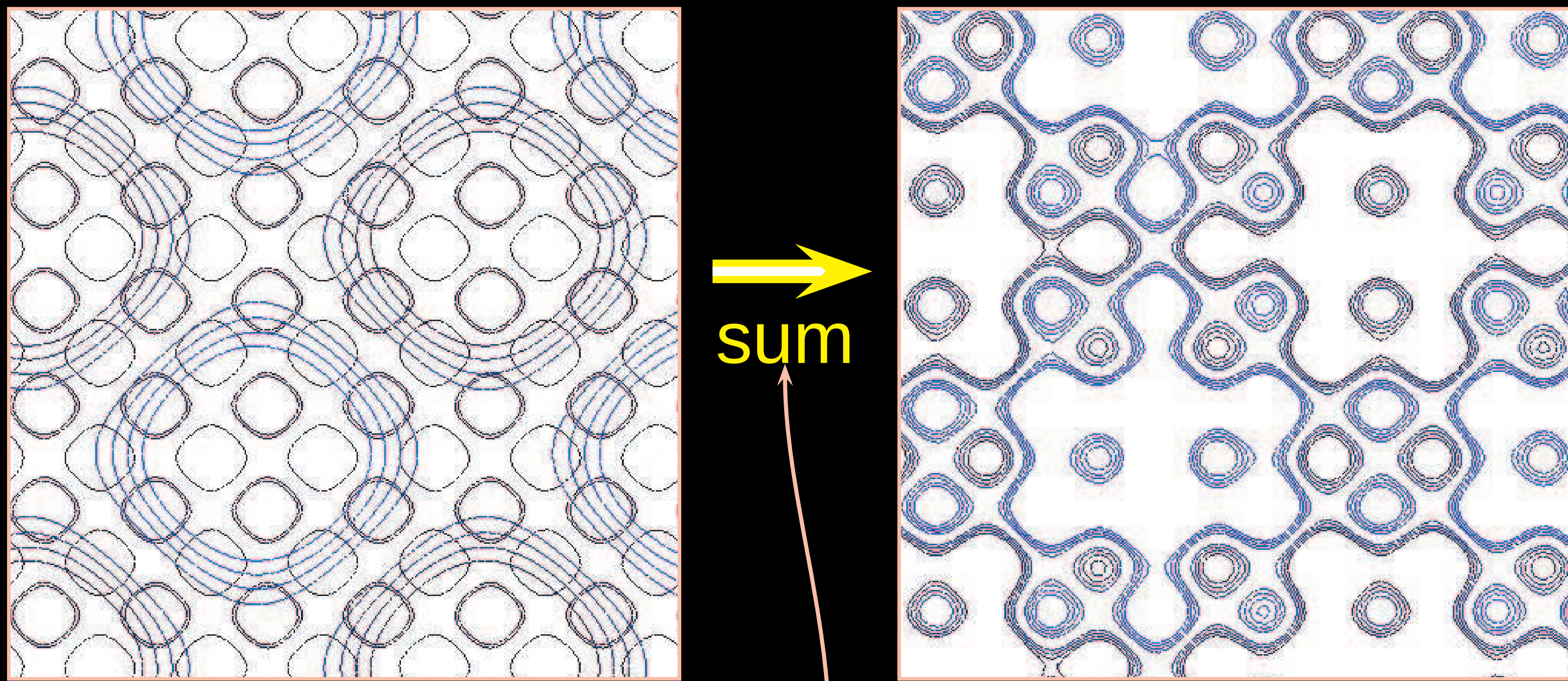


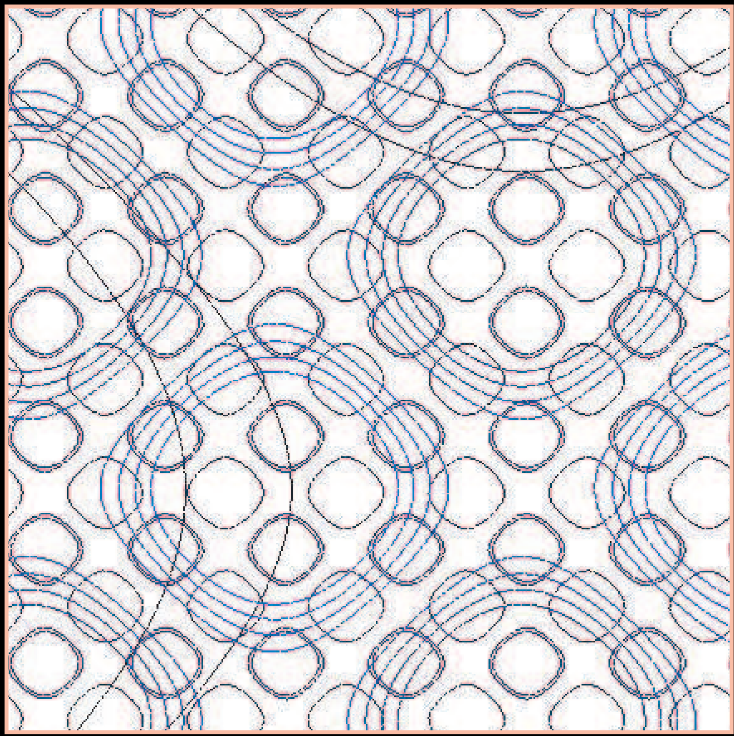
Stream lines of $\gamma^n E_n(\frac{x}{\rho^n})$ $\boxed{+}$ $\gamma^{n+1} E_{n+1}(\frac{x}{\rho^{n+1}})$.

ρ^{n+2}

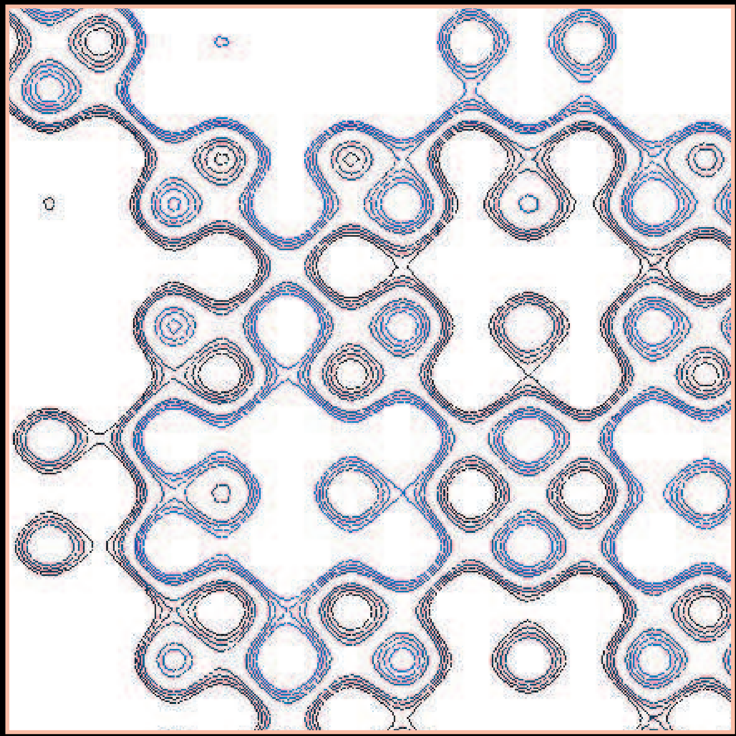
ρ^{n+1}

Three scales.

ρ^{n}



sum

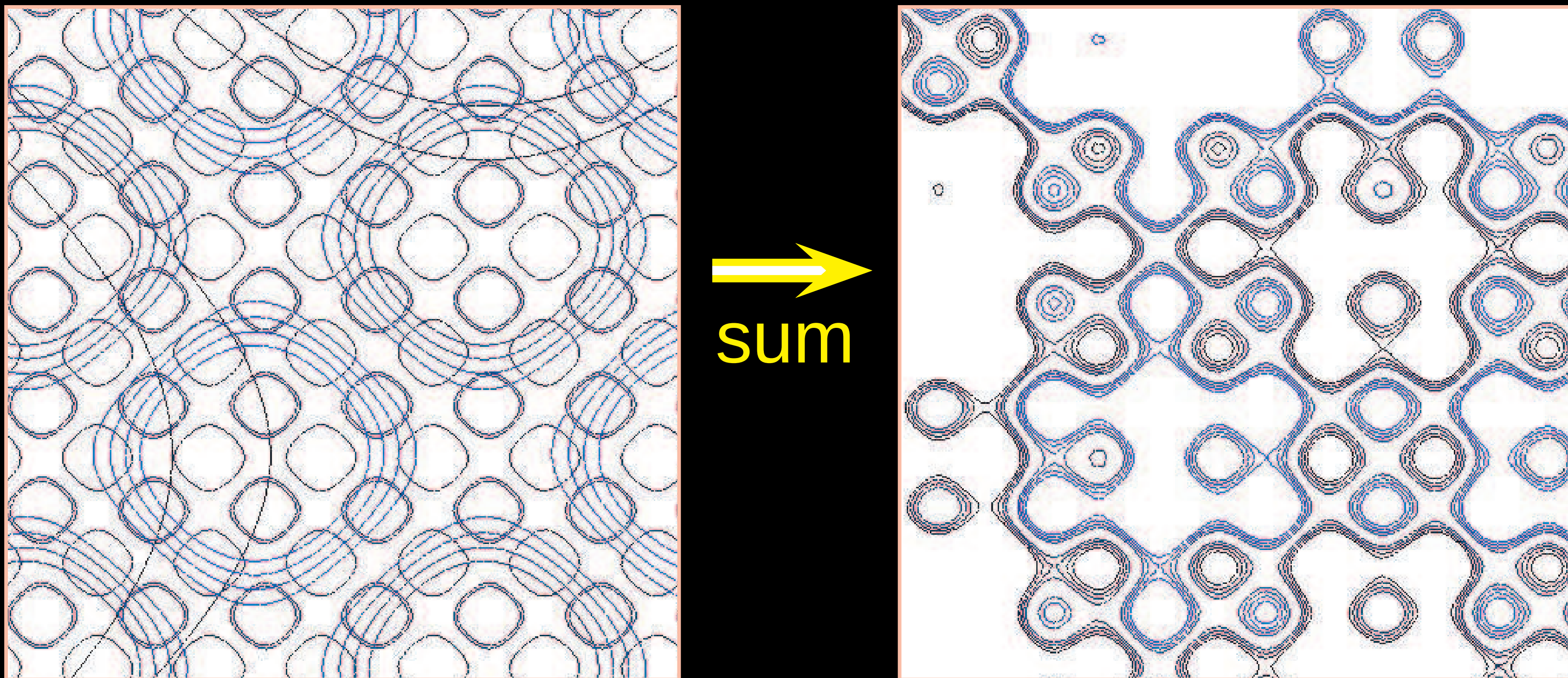


$$\gamma^n E_n\left(\frac{x}{\rho^n}\right) + \gamma^{n+1} E_{n+1}\left(\frac{x}{\rho^{n+1}}\right) + \gamma^{n+2} E_{n+2}\left(\frac{x}{\rho^{n+2}}\right)$$

ρ^{n+2}

ρ^{n+1}

ρ^{n}



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Modification of the ratio ρ

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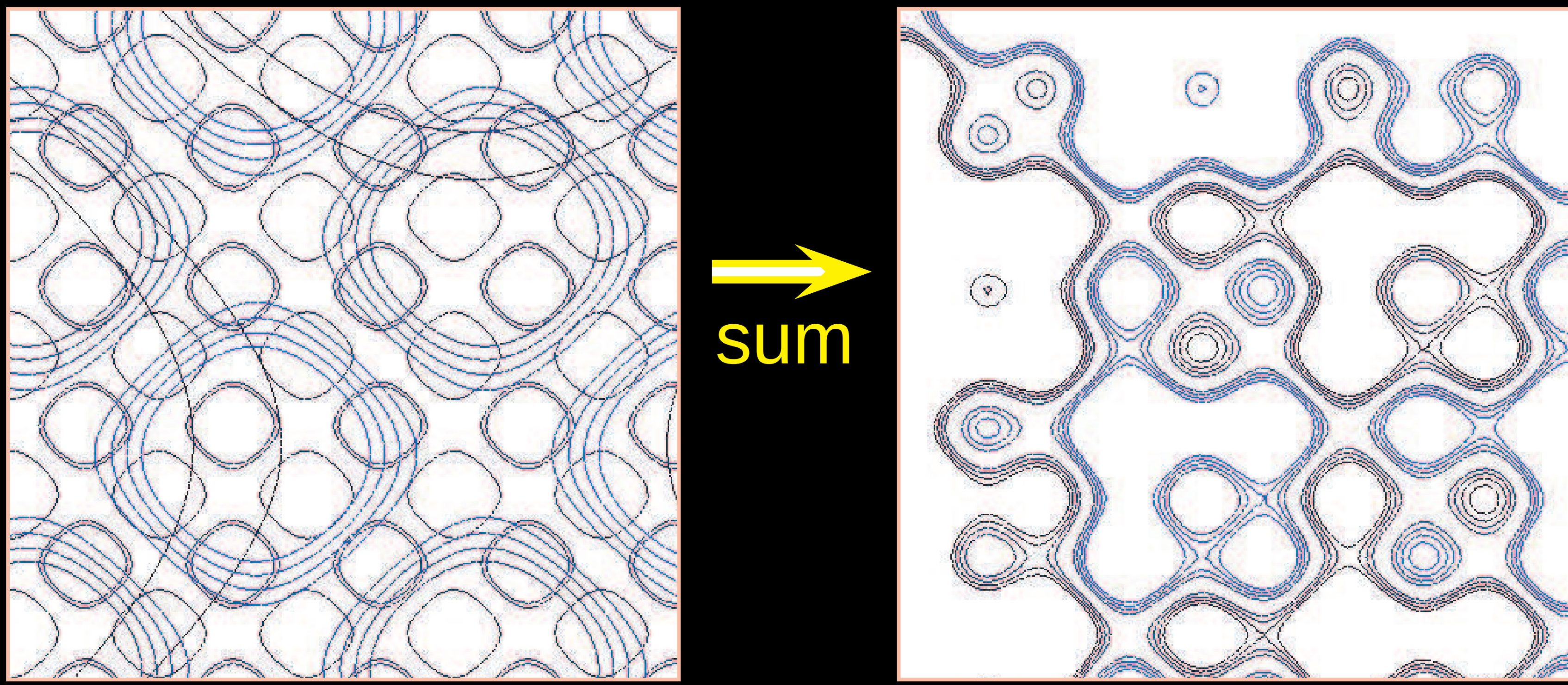
ρ^{n+1}

ρ^{n}



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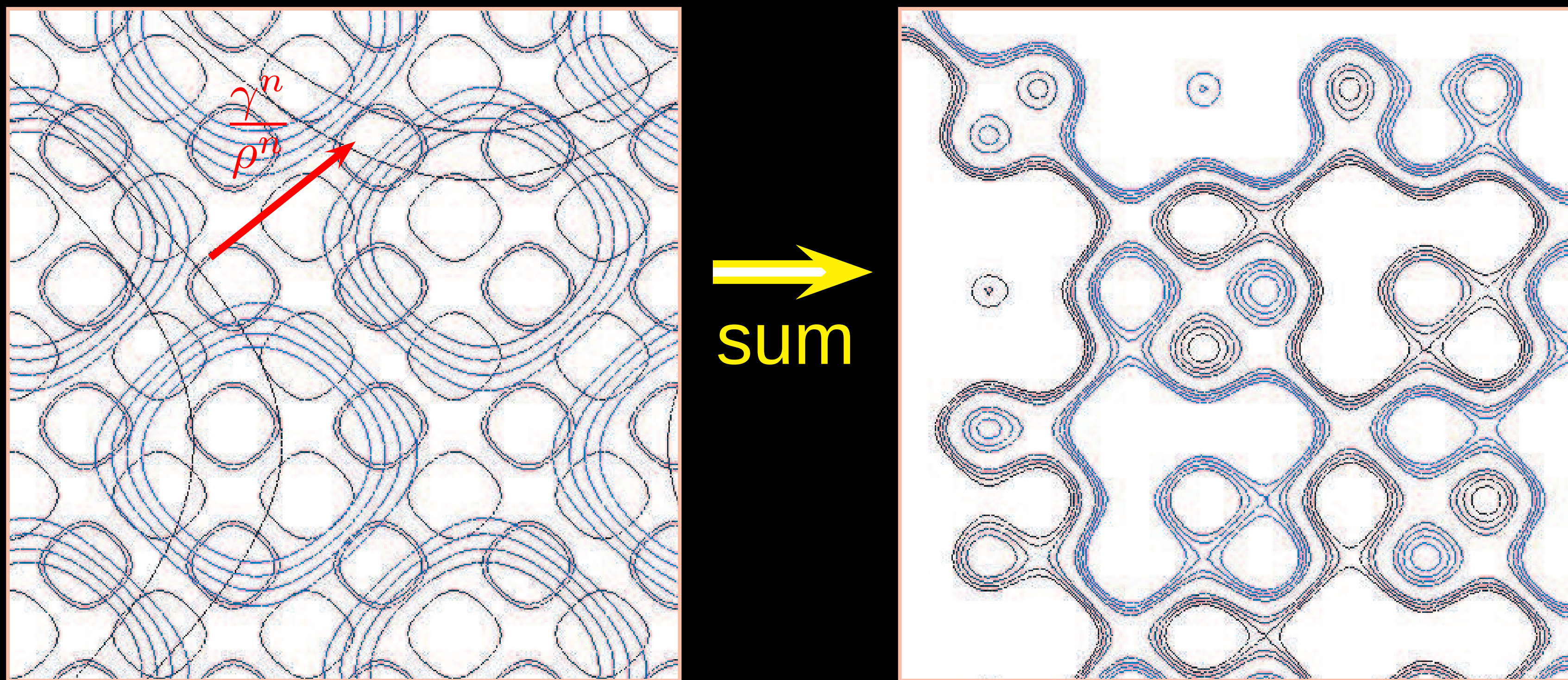
Modification of the ratio ρ

The circulation rates γ^k



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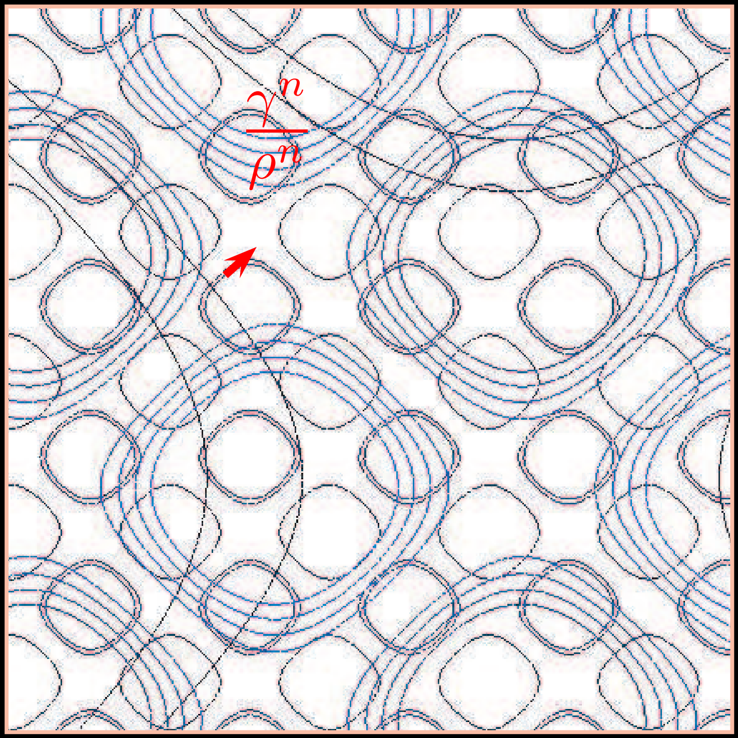

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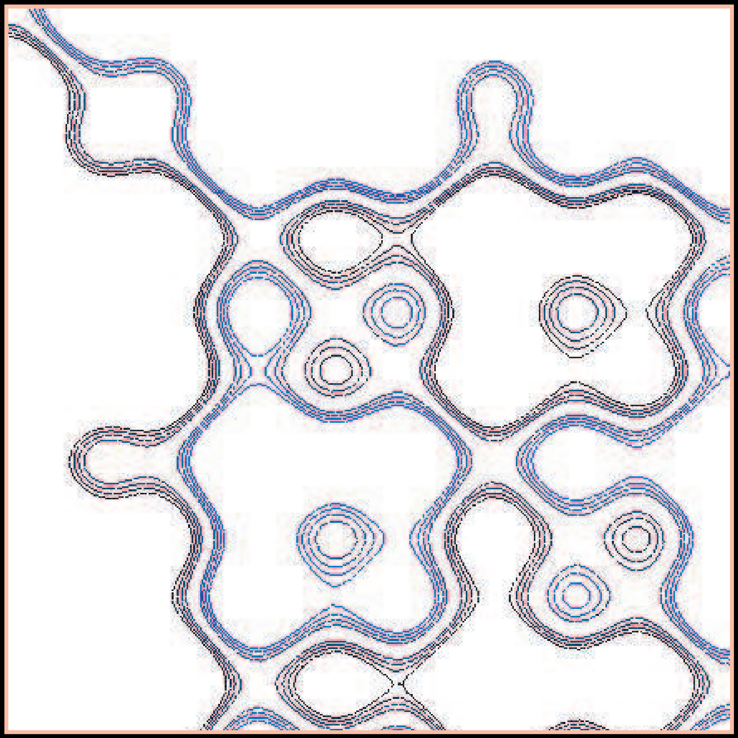
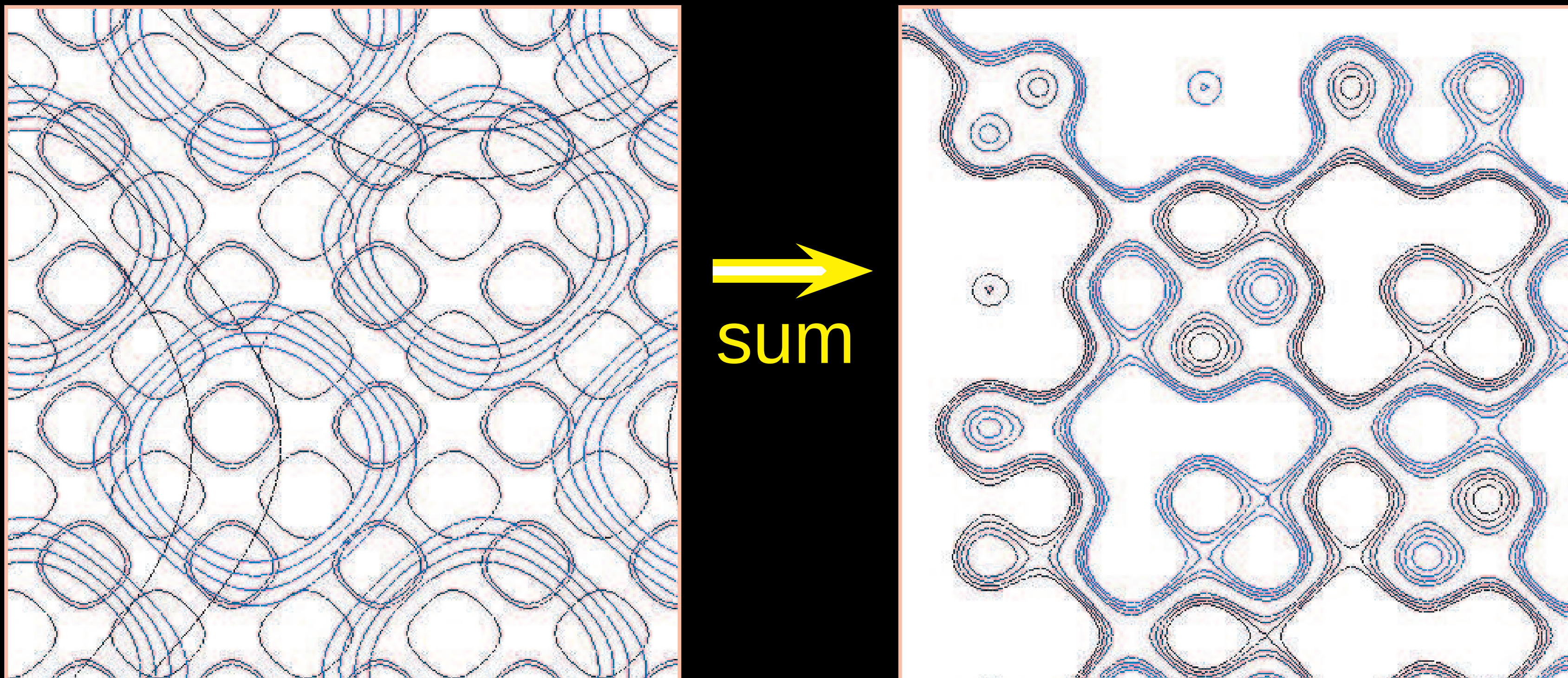


$$\gamma^n E_n\left(\frac{x}{\rho^n}\right) + \gamma^{n+1} E_{n+1}\left(\frac{x}{\rho^{n+1}}\right) + \gamma^{n+2} E_{n+2}\left(\frac{x}{\rho^{n+2}}\right)$$

E_k: Geometry of the eddies



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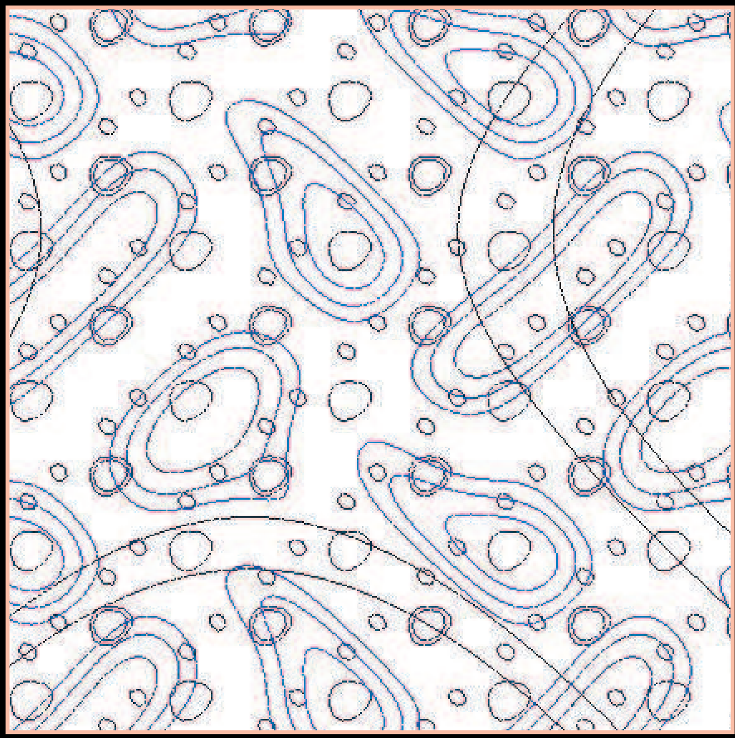
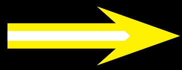
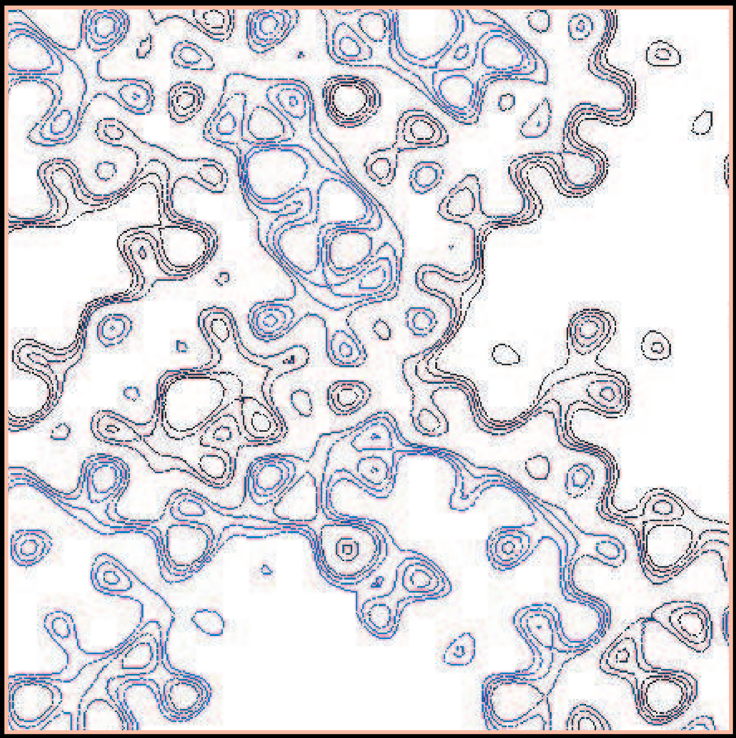






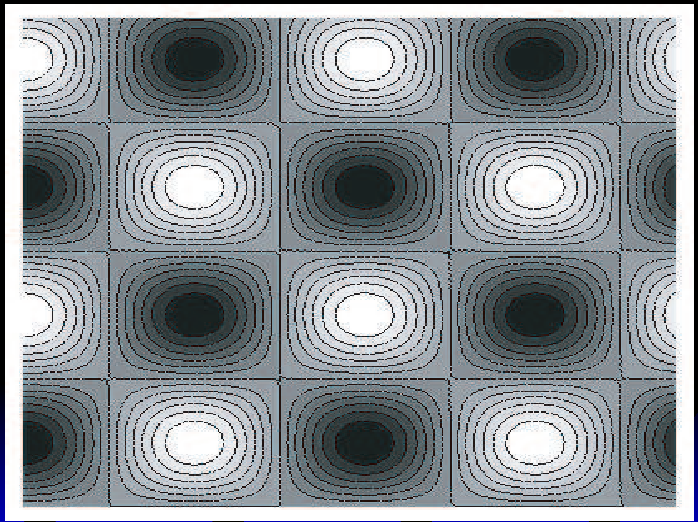
$$\gamma^n E_n\left(\frac{x}{\rho^n}\right) \boxed{+} \gamma^{n+1} E_{n+1}\left(\frac{x}{\rho^{n+1}}\right) \boxed{+} \gamma^{n+2} E_{n+2}\left(\frac{x}{\rho^{n+2}}\right)$$

A simple example of the multi-scale flow

$d = 2$, for all n,

$$E_n(x_1, x_2) = \begin{pmatrix} 0 & h(x_1, x_2) \\ -h(x_1, x_2) & 0 \end{pmatrix}$$

with $h(x_1, x_2) := \cos(2\pi x_1)\sin(2\pi x_2)$

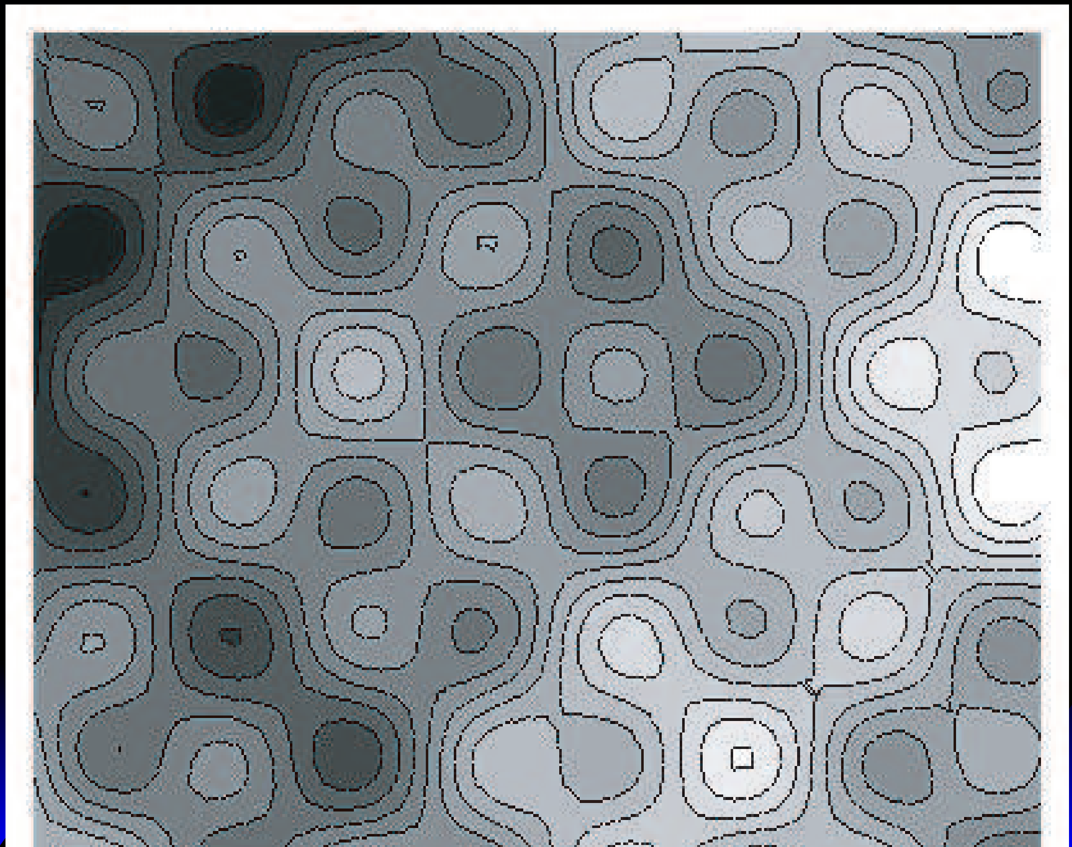


$$h_0^2(x,y) = \sum_{k=0}^{2} \gamma^k h\left(\frac{x}{\rho^k}, \frac{y}{\rho^k}\right)$$

with

$\rho = 2.9$,

$\gamma = 1.25$



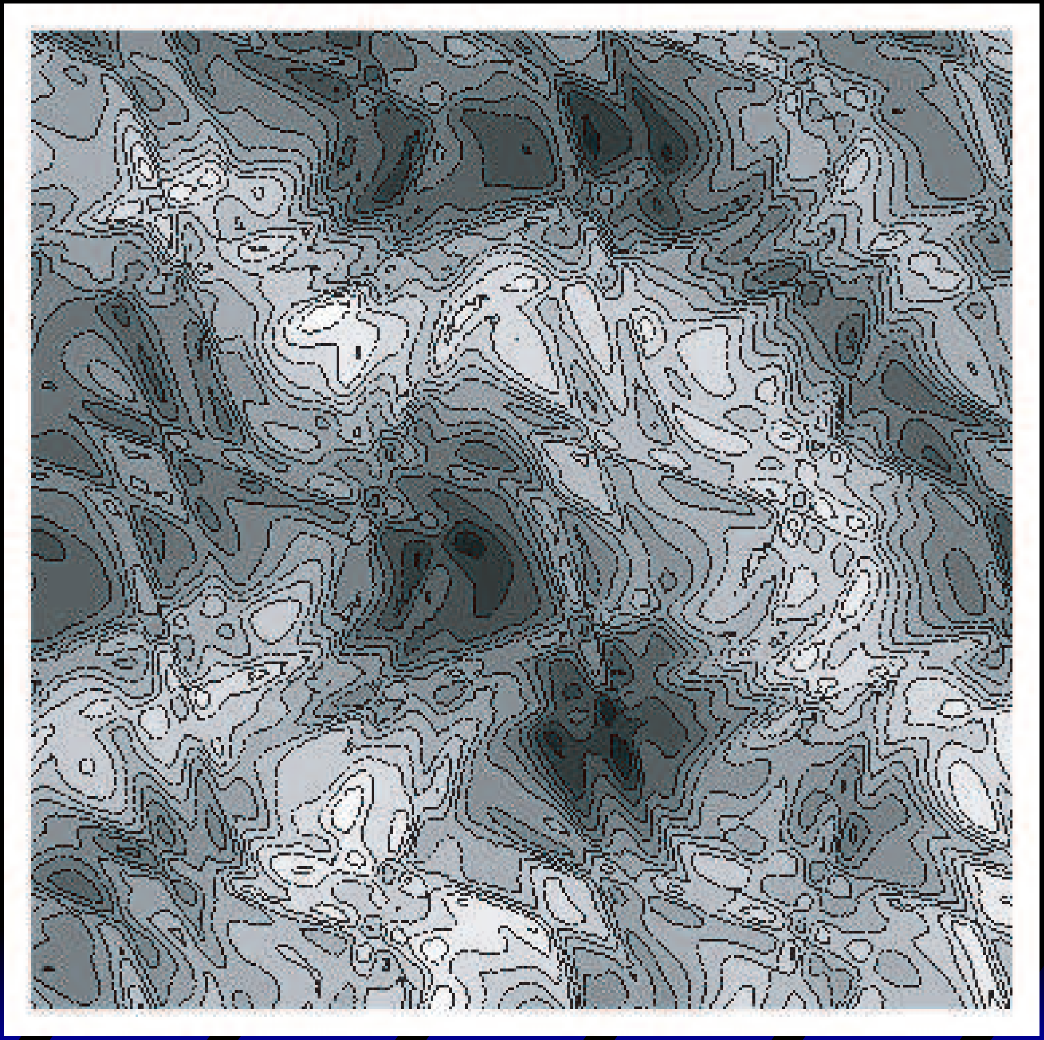
Another example

$$h_0^2(x,y) = \sum_{k=0}^{2} \gamma^k h(\frac{x}{\rho^k}, \frac{y}{\rho^k})$$

with $\rho = 3$, $\gamma = 1.1$ and

$$\begin{aligned} h(x,y) =& 2\sin(2\pi x + 3\cos(2\pi y - 3\sin(2\pi x + 1))) \\ & \sin(2\pi y + 3\cos(2\pi x - 3\sin(2\pi y + 1))) \end{aligned}$$

Another example



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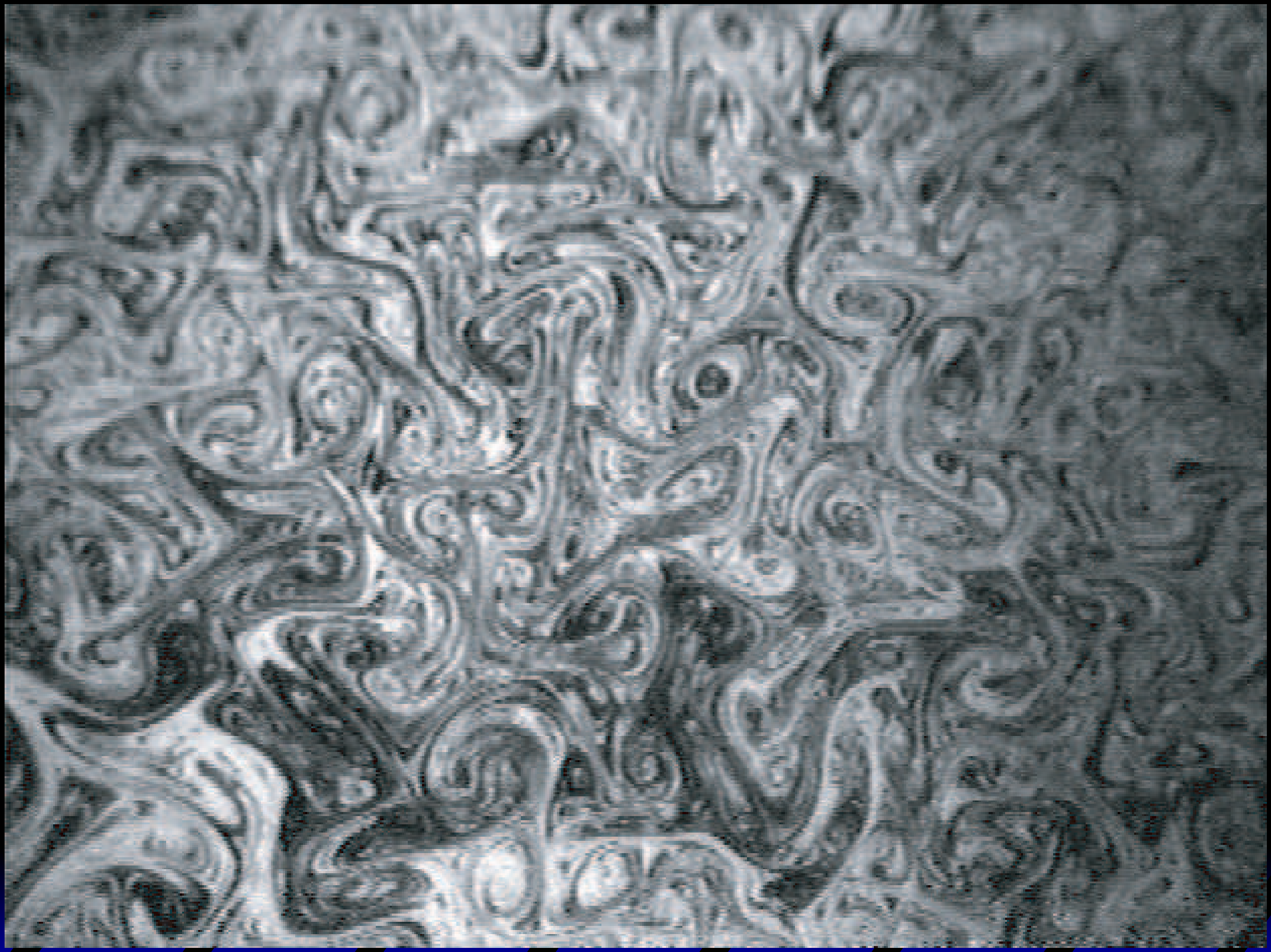
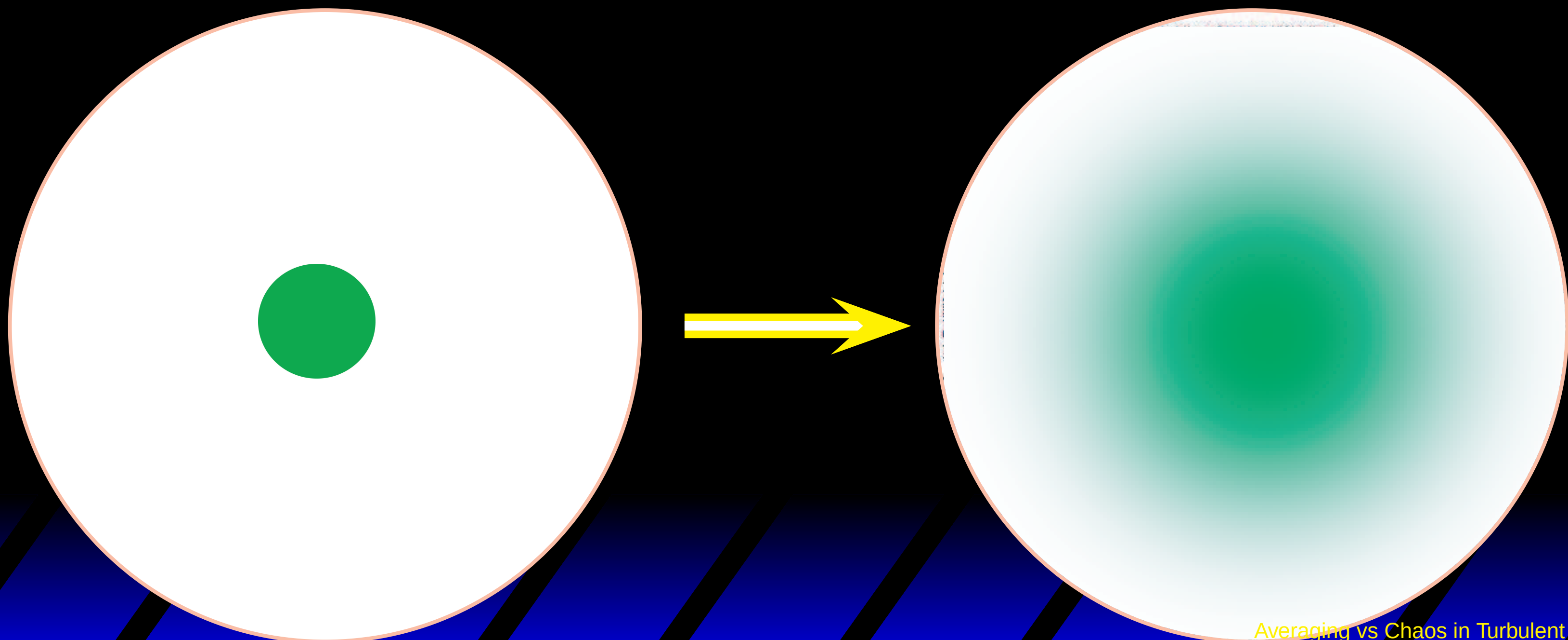


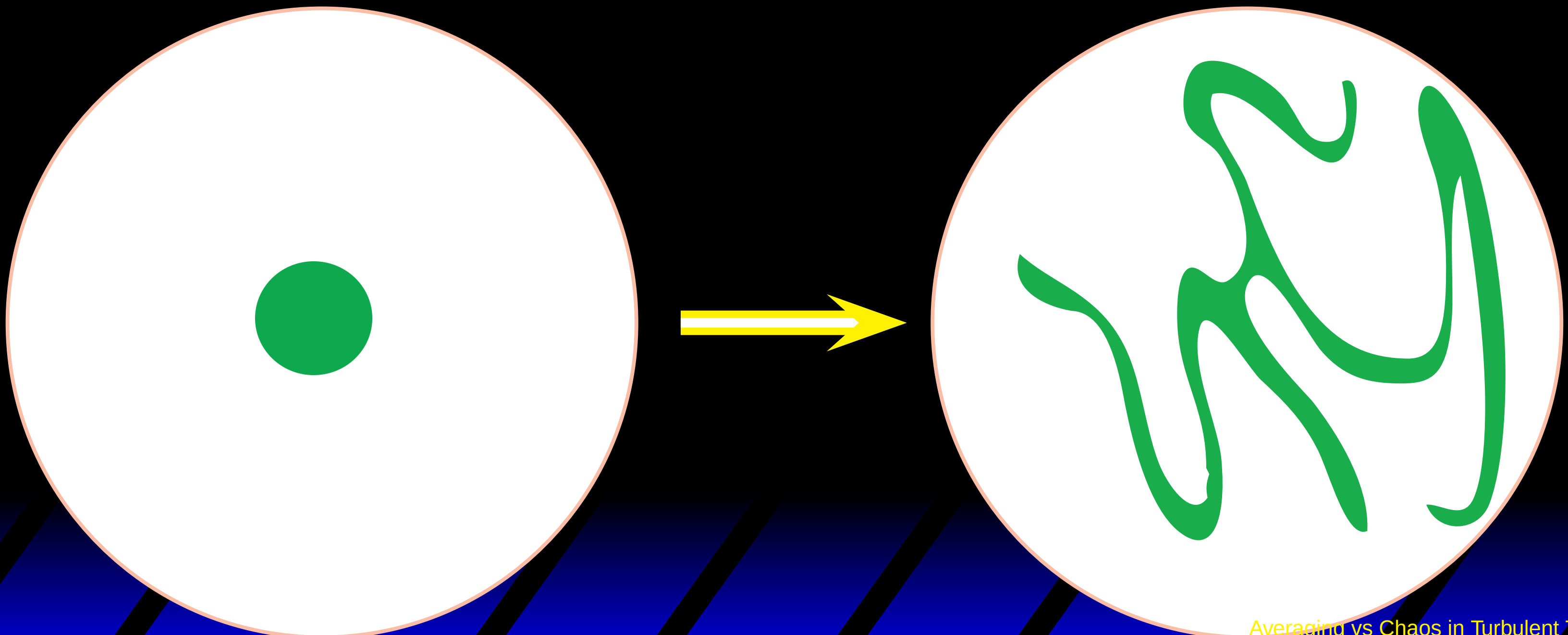
Two opposed descriptions?

- The transport is Superdiffusive, Highly mixing, self-averaging (Kolmogorov 41, Richardson 26, Obukhov 41)



Two opposed descriptions?

- High density gradients, coherent patterns, sensitive to the geometry of the flow, (Poincaré 08, Landau-Lifshitz 42-85, Ruelle-Takens 71)



Questions for our model

- The transport is
- Superdiffusive or not?
- Sensitive to the particular geometry of the flow or not?

Our results: outline

- We can define a parameter $\lambda^- \in \mathbb{R}^+$ from the characteristics $\left(\gamma, (E_n)_{n\in\mathbb{N}}\right)$ of our model, $\leftrightarrow$ inverse of a local Peclet (Reynolds) number
- if $\lambda^- > 0 \rightarrow$ superdiffusive + highly mixing + self-averaging

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- We can define a parameter $\lambda^- \in \mathbb{R}^+$ from the characteristics $\left(\gamma, (E_n)_{n\in\mathbb{N}}\right)$ of our model, $\leftrightarrow$ inverse of a local Peclet (Reynolds) number
- if $\lambda^- > 0 \rightarrow$ superdiffusive + highly mixing + self-averaging
- if $\lambda^- = 0 \rightarrow$ self-averaging collapses, highly sensitive, high gradients

Results

SDE

$$\begin{cases} dy_t = \sqrt{2\kappa}\,d\omega_t + v(y_t)dt \\ y_0 = x \end{cases}$$

ω standard BM in $\mathbb{R}^d$.

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Exit Time

$$\tau(r) := \inf\{t > 0 \;:\; |y_t| \geq r\}$$

Initial Distribution

$$m_r(dx) := \frac{dx}{\int_{B(0,r)} dx} 1_{B(0,r)}$$

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Mean Exit Time

$$\boxed{\mathbb{E}_{m_r}\left[\tau(r)\right] = \frac{1}{\mathrm{Vol}\big(B(0,r)\big)} \int_{B(0,r)} \mathbb{E}_x\left[\tau(r)\right] dx}$$

One Point Fast Motion

Theorem If $\lambda^- > 0$ then $\exists C(d, 1/\lambda^-, 1/\ln\gamma) < \infty$ such that for $\rho > C\gamma$ one has

$$\boxed{\limsup_{r\to\infty} \frac{1}{\ln r} \ln\left(\mathbb{E}_{m_r}\left[\tau(r)\right]\right) < 2}$$

One Point Fast Motion

Theorem For $\lambda^{-} > 0$, $\rho > C\gamma$ and $r > \rho$

$$\mathbb{E}_{m_r}\left[\tau(r)\right] = r^{2-\nu(r)}$$

$$\nu(r) = \frac{\ln \gamma}{\ln \rho}\big(1 + \epsilon(r)\big)$$

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$$|\epsilon(r)| \leq 0.5C\frac{\gamma}{\rho} < 0.5$$

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$$0 < \frac{\ln \gamma}{\ln \rho}$$

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Shear flow models: Avellaneda-Majda (91), Glimm-Zhang (92), Gaudron (00), Komorowski-Fannjiang (01), Ben Arous-Owhadi (01).

Non shear flow, Kraichnan, Gaussian, annealed, one particle:

Piterbarg 97, Komorowski-Olla 02, Fannjiang 02

Fast Transport as an almost sure event

Fast transport event

$$\boxed{H(r) := \{\tau(r) \leq r^{2-\delta}\}}$$

with

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Observe that $\delta > 0$

Fast Transport as an almost sure event

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Theorem If $\lambda^- > 0$ then for $\rho > C\gamma$ one has

$$\boxed{\lim_{r \to \infty} \mathbb{P}_{m_r}[H(r)] = 1}$$

Super-Diffusive two-points motion

z_t second passive tracer

$$dz_t = \sqrt{2\kappa}\, d\bar{\omega}_t + v(z_t)\, dt.$$

$\bar{\omega}_t$ standard BM independent of ω_t.

$$B(0,r,l) := \{(y,z) \in \mathbb{R}^d \times \mathbb{R}^d \,:\, |y-z| < r \text{ and } y^2 + z^2 < l^2\}$$

$$\tau(r,l) := \inf\{t > 0 \,:\, (y_t, z_t) \notin B(0,r,l)\}$$

Super-Diffusive two-points motion

$$m_{r,l}(dy\,dz) := \frac{dy\,dz}{\int_{(y,z)\in B(0,r,l)} dy\,dz} 1_{B(0,r,l)}.$$

Theorem

If $\lambda^- > 0$ then for $\rho > C\gamma$ one has

$$\limsup_{r\to\infty} \lim_{l\to\infty} \frac{1}{\ln r} \ln\left(\mathbb{E}_{m_{r,l}}\left[\tau(r,l)\right]\right) < 2$$

$$\lim_{l\to\infty} \mathbb{E}_{m_{r,l}}\left[\tau(r,l)\right] = r^{2-\nu(r)}$$

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If $\lambda^{-} > 0$

- The particular geometry of the eddies E_n has no influence on the transport
- The transport depends only on the power law $\frac{\ln \gamma}{\ln \rho}$ of the velocity field.

What is λ^- ?

- Renormalization procedure.

A reminder on homogenization

- $\mathcal{M} := \{$positive definite symmetric constant $d \times d$ matrices$\}$

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Let $(a, E) \in \mathcal{M} \times \mathcal{S}(\mathbb{T}^d)$

A reminder on homogenization

Operator

$$\begin{cases} \operatorname{div}\left(a + E\left(\frac{x}{\epsilon}\right)\right)\nabla u_\epsilon(x) = f(x) \quad \text{in } \Omega \\ u_\epsilon = 0 \quad \text{in } \partial\Omega \end{cases}$$

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then $\exists\, \sigma(a, E) \in \mathcal{M}$ such that as $\epsilon \downarrow 0$,

$$\boxed{u_\epsilon \longrightarrow u_0}$$

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As $\epsilon \downarrow 0$,

$$\boxed{\epsilon x_{t/\epsilon^2} \longrightarrow B_t}$$

B_t Brownian Motion with covariance matrix $2\sigma(a, E)$.

A reminder on homogenization

Literature.

- $2\sigma(a, E)$: Effective diffusivity
- $\sigma(a, E)$: Effective conductivity - Eddy viscosity. Dispersion matrix.

A reminder on homogenization

Effective conductivity as a mapping

$$\boxed{\begin{aligned} \sigma \ : \ \mathcal{M} \times \mathcal{S}(\mathbb{T}^d) &\to \mathcal{M} \\ (a, E) &\to \sigma(a, E) \end{aligned}}$$

A reminder on homogenization

Effective conductivity as a mapping

$$\boxed{\begin{aligned}\sigma\ :\ \mathcal{M} \times \mathcal{S}(\mathbb{T}^d) &\to \mathcal{M} \\ (a, E) &\to \sigma(a, E)\end{aligned}}$$

for all $(a, E) \in \mathcal{M} \times \mathcal{S}(\mathbb{T}^d)$

$$\boxed{a \leq \sigma(a, E) \leq a + \int_{\mathbb{T}^d} {}^t E(x) a^{-1} E(x)\, dx}$$

What is λ^- ?

Renormalization sequence $(A_n)_{n\in\mathbb{N}}$
For all $n \in \mathbb{N}$, $A_n \in \mathcal{M}$

$$A_0 = \frac{\kappa}{\gamma} I_d \quad \text{and} \quad A_{n+1} = \frac{1}{\gamma}\sigma(A_n, E_n)$$

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$(A_n)_{n\in\mathbb{N}}$ does not depend on ρ

$$\lambda^- := \liminf_{n\to\infty} \lambda_{\min}(A_n)$$

Interpretation of A_n

$$\boxed{\Gamma^{n-1}(x) := \sum_{k=0}^{n-1} \gamma^k E_k(\frac{x}{\rho^k})}$$

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Assume $\rho \in \mathbb{N}$. Then Γ^{n-1} periodic.

$$\boxed{\operatorname{div}(\kappa I_d + \Gamma^{n-1})\nabla \xrightarrow{Homogen} \operatorname{div} \sigma(\kappa I_d, \Gamma^{n-1})\nabla}$$

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$$\mathrm{div}(\kappa I_d + \Gamma^{n-1})\nabla \xrightarrow{Homogen} \mathrm{div}\, \sigma(\kappa I_d, \Gamma^{n-1})\nabla$$

$$\lim_{\rho \to \infty} \sigma(\kappa I_d, \Gamma^{n-1}) = \gamma^n A_n$$

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Magnitude of the velocity vector field at scale n:

$$V_n \sim \frac{\gamma^n}{\rho^n}$$

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Local Peclet tensor

$$\mathbf{Pe}^n := V_n \rho^n \Big(\lim_{\rho \to \infty} \sigma(\kappa I_d, \Gamma^{n-1}) \Big)^{-1}$$

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Magnitude of the velocity vector field at scale n:

$$V_n \sim \frac{\gamma^n}{\rho^n}$$

Local Reynolds tensor

$$\mathbf{Re}^n := V_n \rho^n \Big(\lim_{\rho\to\infty} \sigma\big(\kappa I_d, \Gamma^{n-1}\big)\Big)^{-1}$$

Interpretation of A_n

Magnitude of the velocity vector field at scale n:

$$V_n \sim \frac{\gamma^n}{\rho^n}$$

Local Reynolds tensor

$$\mathbf{Re}^n := V_n \rho^n \Big(\lim_{\rho \to \infty} \sigma(\kappa I_d, \Gamma^{n-1}) \Big)^{-1}$$

$$A_n = (\mathbf{Pe}^n)^{-1} = (\mathbf{Re}^n)^{-1}$$

When is $\lambda^- > 0$?

$$\boxed{\lambda^+ := \limsup_{n\to\infty} \lambda_{\max}(A_n)}$$

$$\boxed{\mu := \limsup_{n\to\infty} \frac{\lambda_{\max}(A_n)}{\lambda_{\min}(A_n)}}$$

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Theorem

$$\lambda^+ \leq \frac{C_d}{\lambda^-} \quad \text{and} \quad \mu \leq \frac{C_d}{(\lambda^-)^2}$$

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Theorem

$$\lambda^+ \leq \frac{C_d}{\lambda^-} \quad \text{and} \quad \mu \leq \frac{C_d}{(\lambda^-)^2}$$

$$\lambda^- > 0 \Rightarrow \lambda^+ < \infty \quad \text{and} \quad \mu < \infty$$

When is $\lambda^- > 0$?

for $\zeta > 0$

$$\boxed{V(\zeta) := \liminf_{n\to\infty} \frac{\lambda_{\min}\big(\sigma(\zeta I_d, E_n)\big)}{\zeta}}$$

When is $\lambda^- > 0$?

for $\zeta > 0$

$$\boxed{V(\zeta) := \liminf_{n\to\infty} \frac{\lambda_{\min}\big(\sigma(\zeta I_d, E_n)\big)}{\zeta}}$$

V is decreasing and $V \geq 1$ thus

$$\boxed{V(0) := \lim_{\zeta\downarrow 0} V(\zeta)}$$

is well defined.

When is $\lambda^- > 0$?

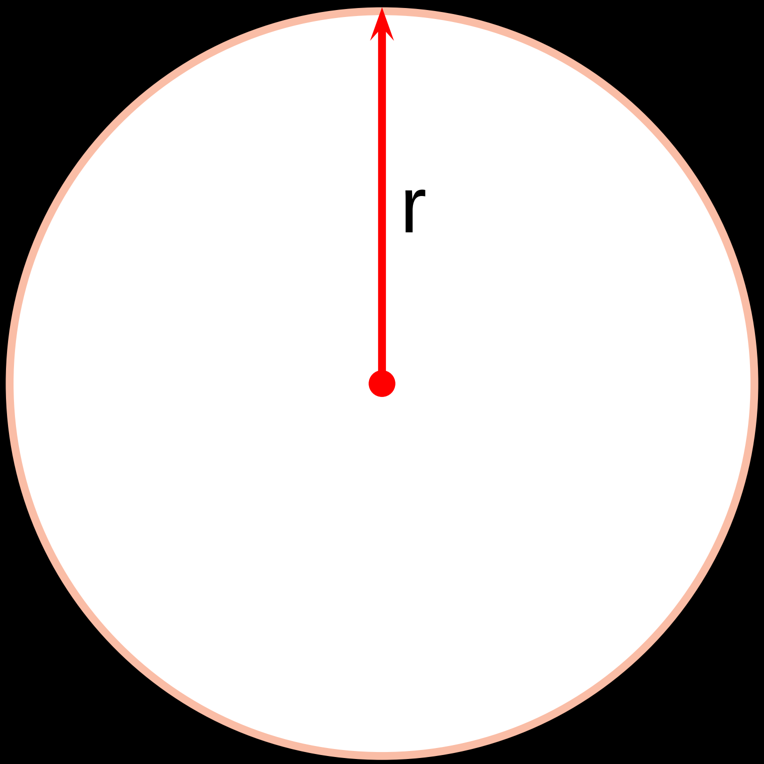
for $\zeta > 0$

$$\boxed{V(\zeta) := \liminf_{n\to\infty} \frac{\lambda_{\min}\big(\sigma(\zeta I_d, E_n)\big)}{\zeta}}$$

Theorem If $\mu < \infty$ and $\gamma < V(0)$ then $\lambda^- > 0$ and

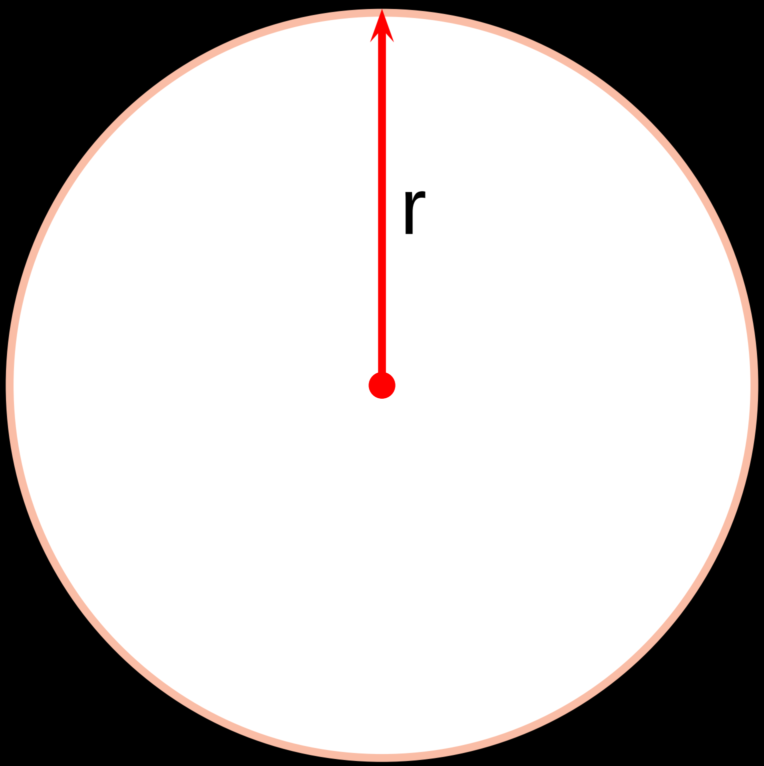
$$\boxed{C_1 \leq \lambda^- \lambda^+ \leq C_2.}$$

Idea of the Proof.



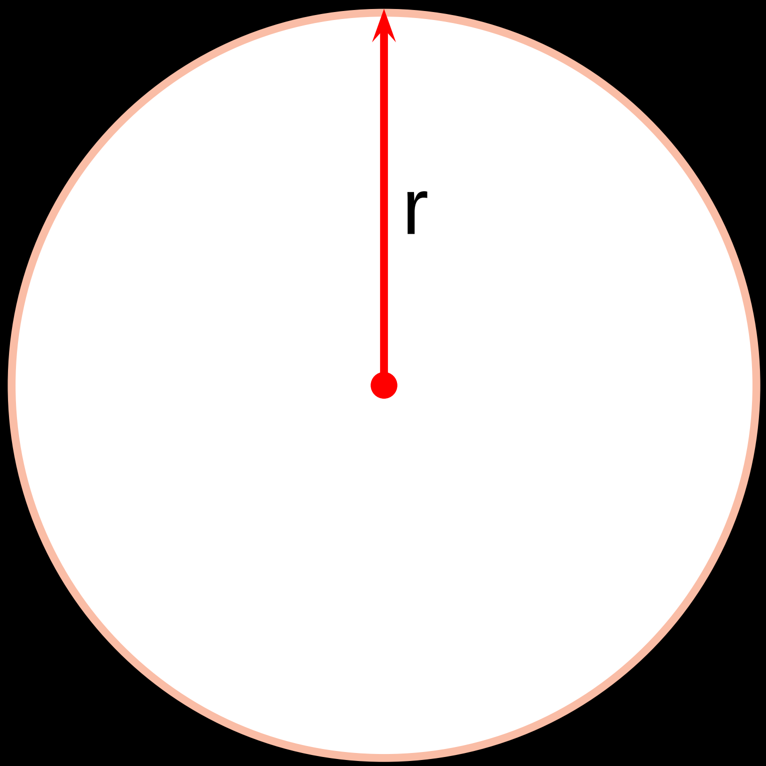
- Ball $B(0, r)$

Idea of the Proof.



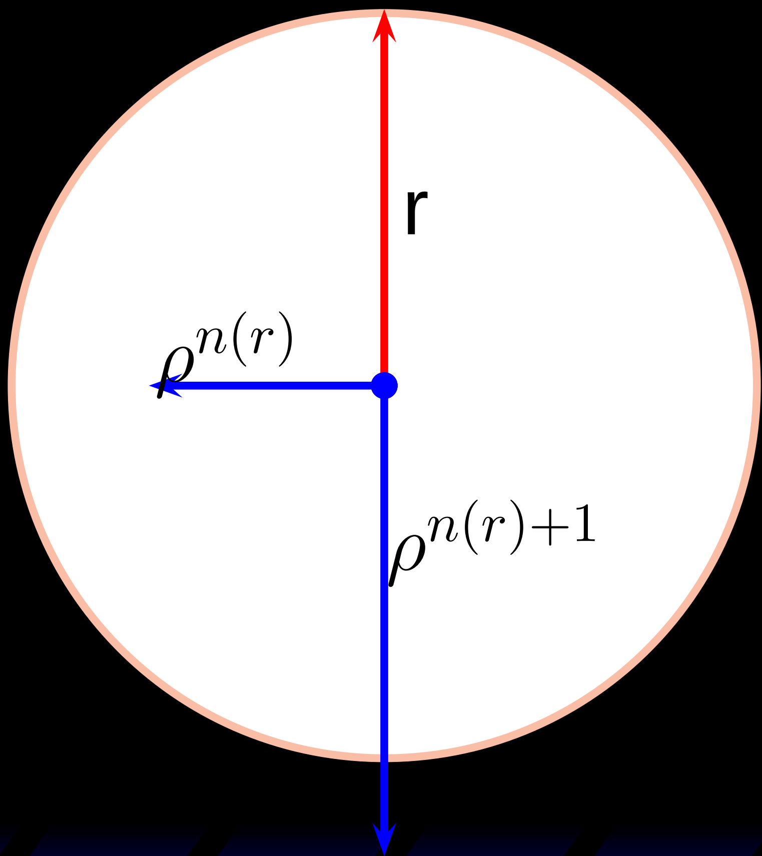
- Ball $B(0, r)$
- We want to compute $\mathbb{E}[\tau(0, r)]$

Idea of the Proof.



- Ball $B(0, r)$
- We want to compute $\mathbb{E}[\tau(0, r)]$
- Main feature of the flow: Infinite number of scales $0, 1, \ldots, \infty$

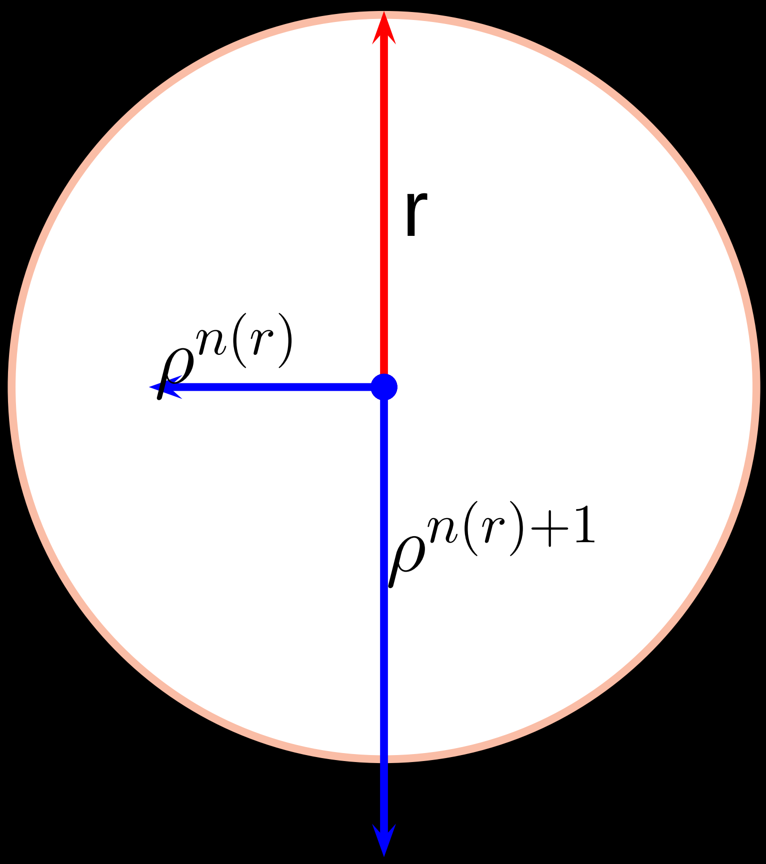
Separation between scales.



- Scale $n(r) = [\ln r / \ln \rho]$

$$\rho^{n(r)} \leq r < \rho^{n(r)+1}$$

Separation between scales.

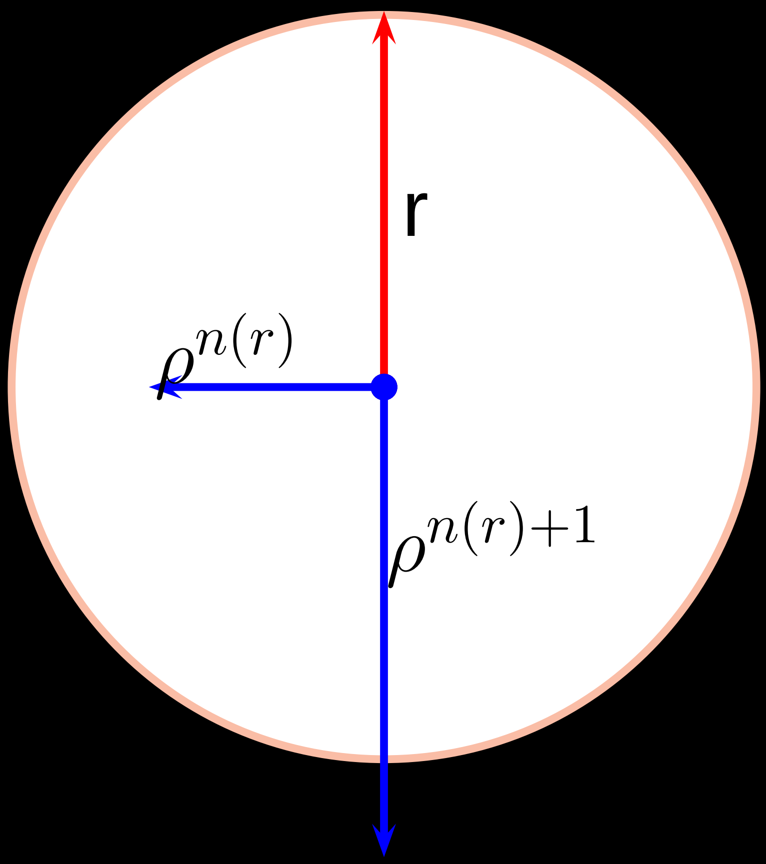


- Scale $n(r) = [\ln r / \ln \rho]$

$$\rho^{n(r)} \leq r < \rho^{n(r)+1}$$

- Small scales $0, \ldots, n(r) - 1$

Separation between scales.

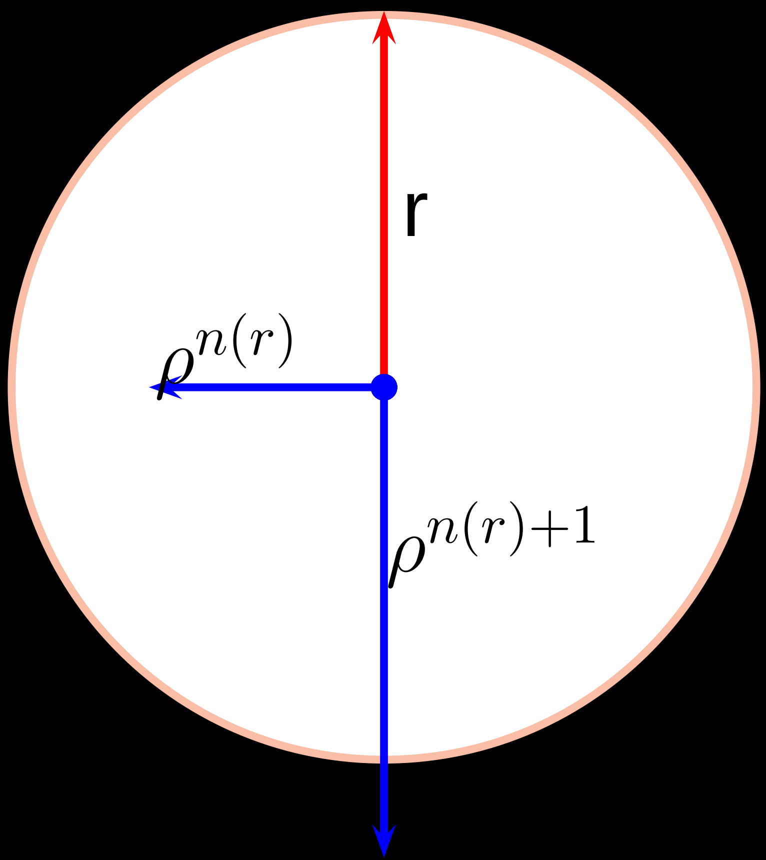


- Scale $n(r) = [\ln r / \ln \rho]$

$$\rho^{n(r)} \leq r < \rho^{n(r)+1}$$

- Small scales $0, \ldots, n(r) - 1$
- Intermediate scales $n(r), n(r) + 1$

Separation between scales.

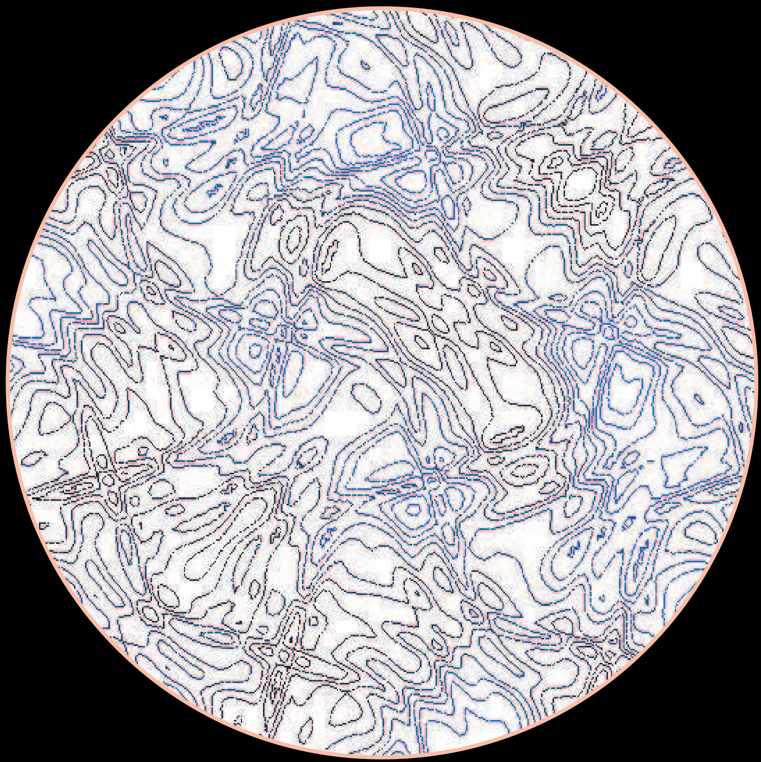


- Scale $n(r) = [\ln r / \ln \rho]$

$$\rho^{n(r)} \leq r < \rho^{n(r)+1}$$

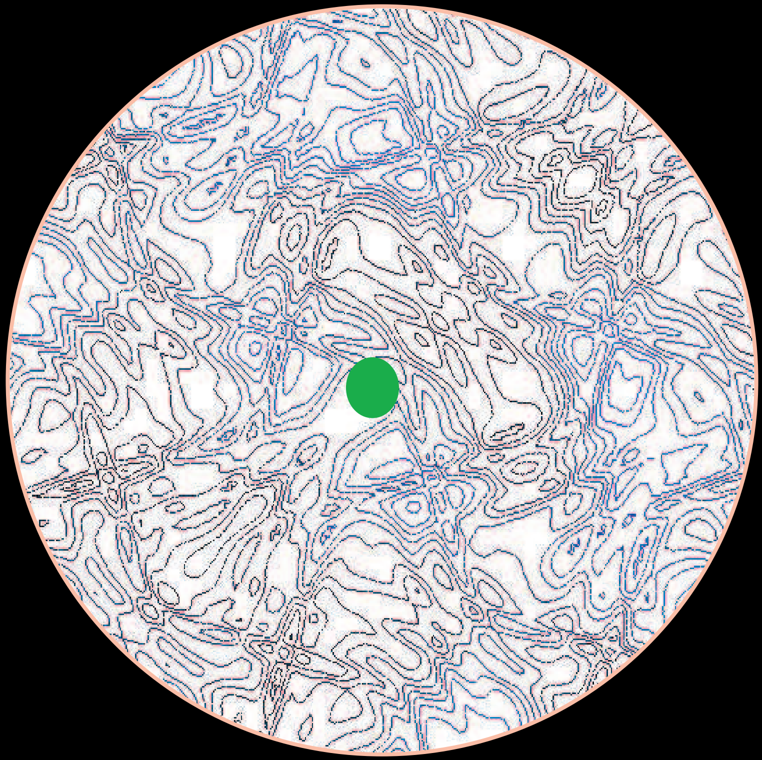
- Small scales
  $0, \ldots, n(r) - 1$
- Intermediate scales
  $n(r), n(r) + 1$
- Large scales
  $n(r) + 2, \ldots, \infty$

All Scales.



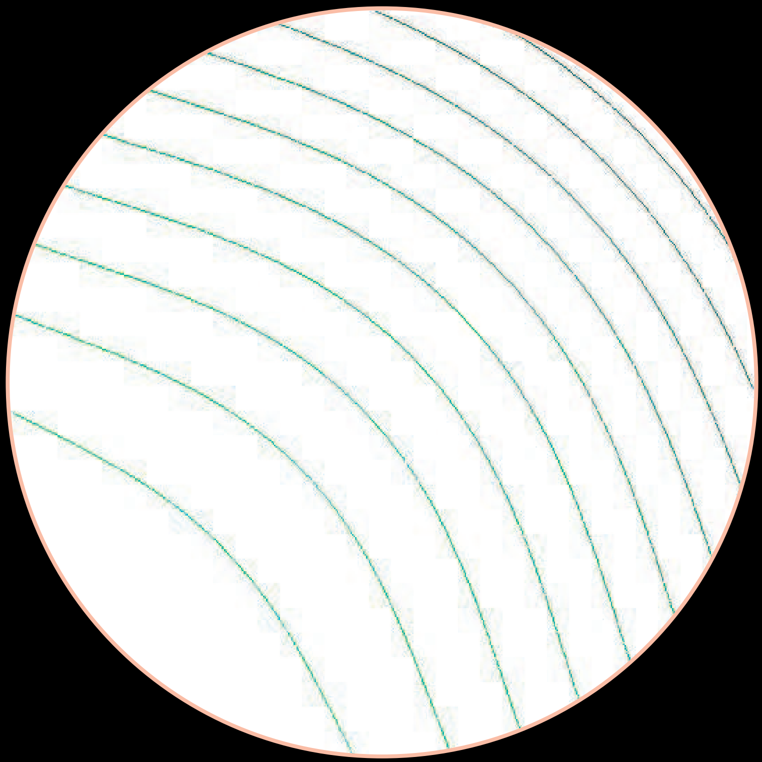
- Medium with infinite number of scales $0, 1, \ldots, \infty$.

All Scales.



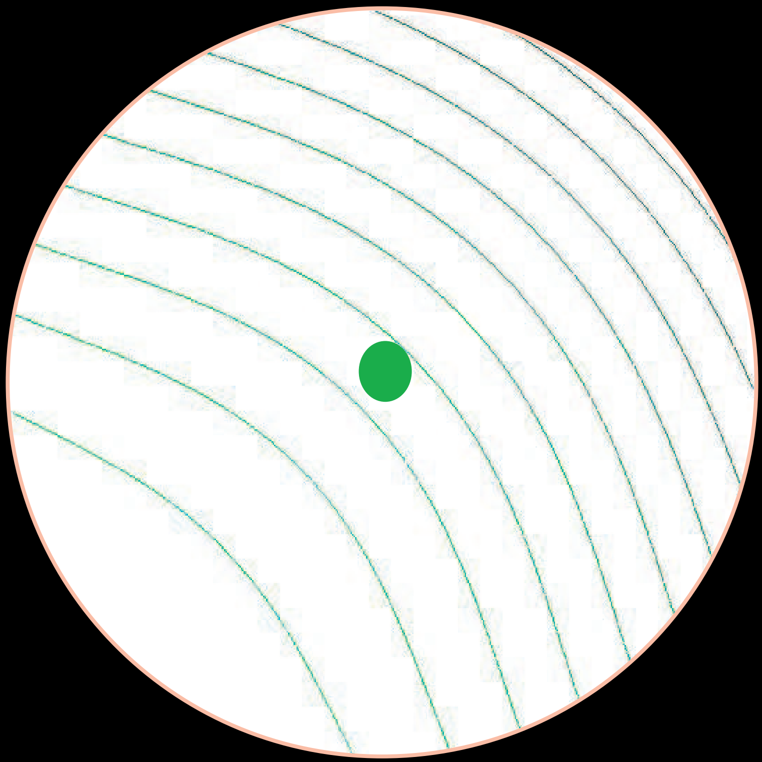
- Medium with infinite number of scales $0, 1, \ldots, \infty$.
- Transport of a drop of dye ?

Transport by Large Scales



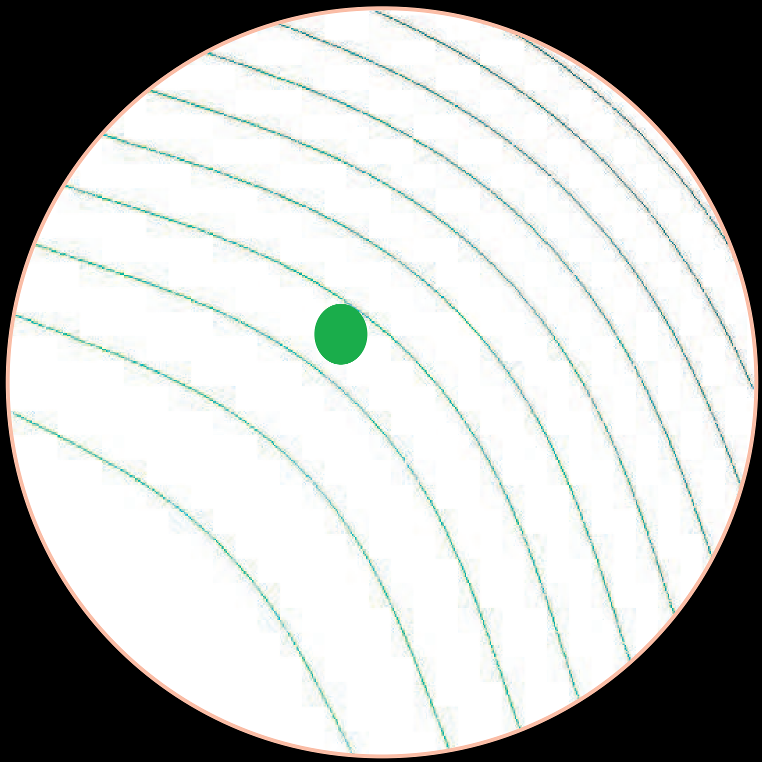
- Large scales

Transport by Large Scales



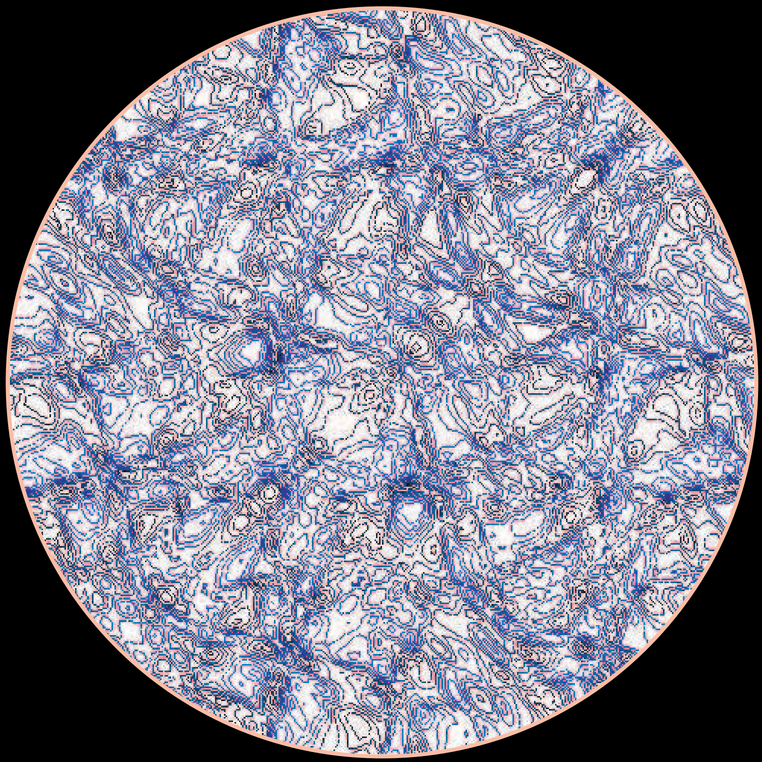
- Large scales
- Their influence on the transport of the drop of dye is

Transport by Large Scales



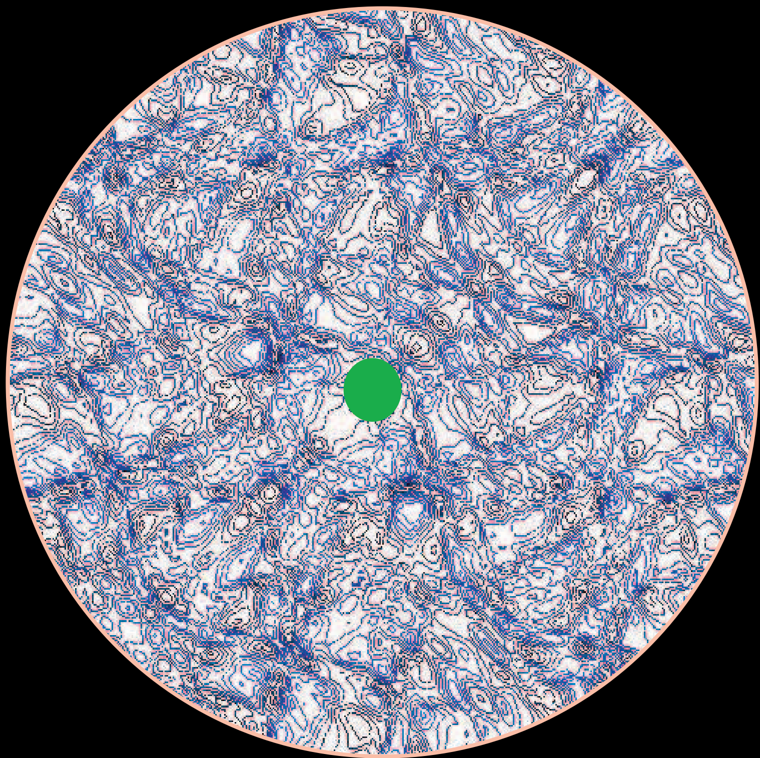
- Large scales
- Their influence on the transport of the drop of dye is negligible

Transport by Small Scales



- Small Scales
  $\longrightarrow$ homogenized.

Transport by Small Scales



- Transport = Diffusion with Effective diffusivity

$$\sigma(\kappa I_d, \Gamma^{n(r)}) \sim \gamma^{n(r)} A_{n(r)}$$

- Exit Time:

$$\tau_D(0, r) \sim \frac{r^2}{\gamma^{n(r)} \lambda(A_{n(r)})}$$

Transport by Small Scales

- Transport by mixing, density gradients smoothed.

Transport by Intermediate Scales

- Intermediate Scales: not homogenized, not negligible.

Transport by Intermediate Scales

- Transport by convection through particular geometry.
- Exit time

$$\tau_C(0,r) \sim \frac{r}{V_{n(r)}} \sim \frac{r^2}{\gamma^{n(r)}}$$

Transport by Intermediate Scales

- Transport by advection, density gradients increased.

If $\lambda^- > 0$

Local Peclet number

$$\mathbf{Pe}(r) := \frac{\tau_C(r)}{\tau_D(r)} \sim \big(\lambda(A_{n(r)})\big)^{-1}$$

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$\lambda^- > 0 \Rightarrow \mathbf{Pe}(r) < \infty \Rightarrow$ At every scale r, advection (irregularities, high gradients) is compensated by averaging (smoothing, dissipating).

If $\lambda^- > 0$

Local Peclet number

$$\mathbf{Pe}(r) := \frac{\tau_C(r)}{\tau_D(r)} \sim \left(\lambda(A_{n(r)})\right)^{-1}$$

$\lambda^- > 0 \Rightarrow \mathbf{Pe}(r) < \infty \Rightarrow$ influence of the intermediate scales on the transport comparable to the influence of the small scales.

$$\boxed{\tau(0,r) \sim \tau_D(0,r) \sim \frac{r^2}{\gamma^{n(r)}}}$$

If $\lambda^- > 0$

Local Peclet number

$$\mathbf{Pe}(r) := \frac{\tau_C(r)}{\tau_D(r)} \sim \left(\lambda(A_{n(r)})\right)^{-1}$$

$$n(r) \sim \frac{\ln r}{\ln \rho}.$$

$$\boxed{\tau(0,r) \sim \frac{r^2}{\gamma^{n(r)}} \sim r^{2-\nu}}$$

with $\nu = \frac{\ln \gamma}{\ln \rho} > 0$.

If $\lambda^{-} = 0$

If $\lambda^- = 0$

As $r \to \infty$

$$\mathbf{Pe}(r) \to \infty$$

If $\lambda^- = 0$

As $r \to \infty$

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- At every scale transport dominated by advection.

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As $r \to \infty$

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- At every scale transport dominated by advection.
- Collapse of the self-averaging property of the flow towards chaos.

If $\lambda^{-} = 0$

As $r \to \infty$

$$\mathbf{Pe}(r) \to \infty$$

- At every scale transport dominated by advection.
- Collapse of the self-averaging property of the flow towards chaos.
- The particular geometry of the eddies can not be neglected even if ρ is large.

Self-Similar Case

Definition

A_n is self-similar and isotropic iff

- $\forall n \in \mathbb{N}$, $E_n = E$; .

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If A_n is self-similar then it is a low order dynamical system.

$$\boxed{A_{n+1} = \frac{1}{\gamma}\sigma(A_n, E)}$$

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$$V(0) := \lim_{\zeta \downarrow 0} \frac{\lambda_{\min}\big(\sigma(\zeta I_d, E)\big)}{\zeta}$$

Self-Similar Case

Theorem If A_n is self-similar and isotropic then

- If $\boxed{\gamma < V(0)}$ then $\lambda^- > 0$ and $\lim_{n\to\infty} A_n = \zeta_0 I_d$ where ζ_0 is the unique solution of $V(\zeta_0) = \gamma$.

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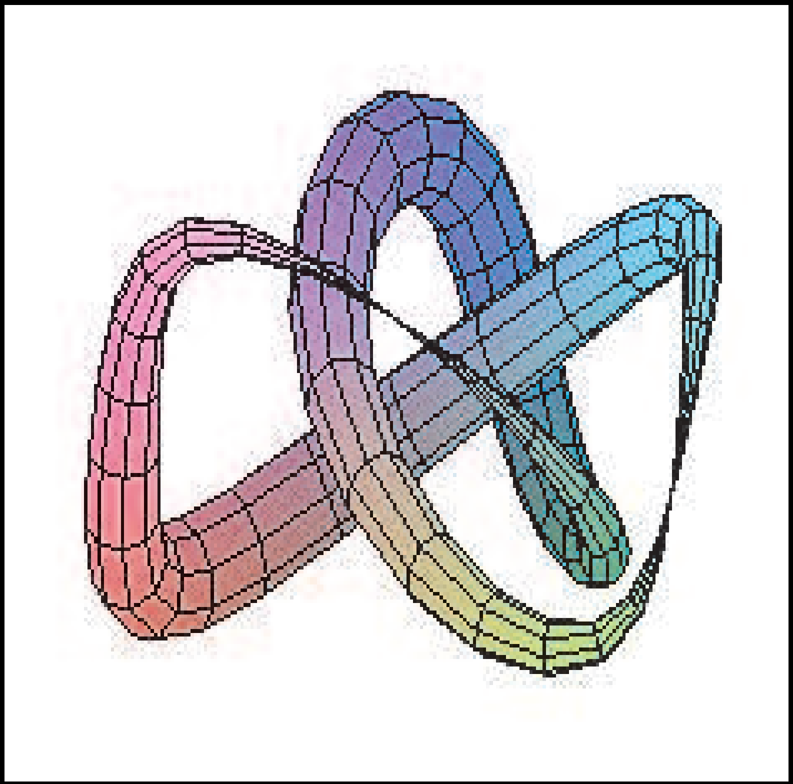
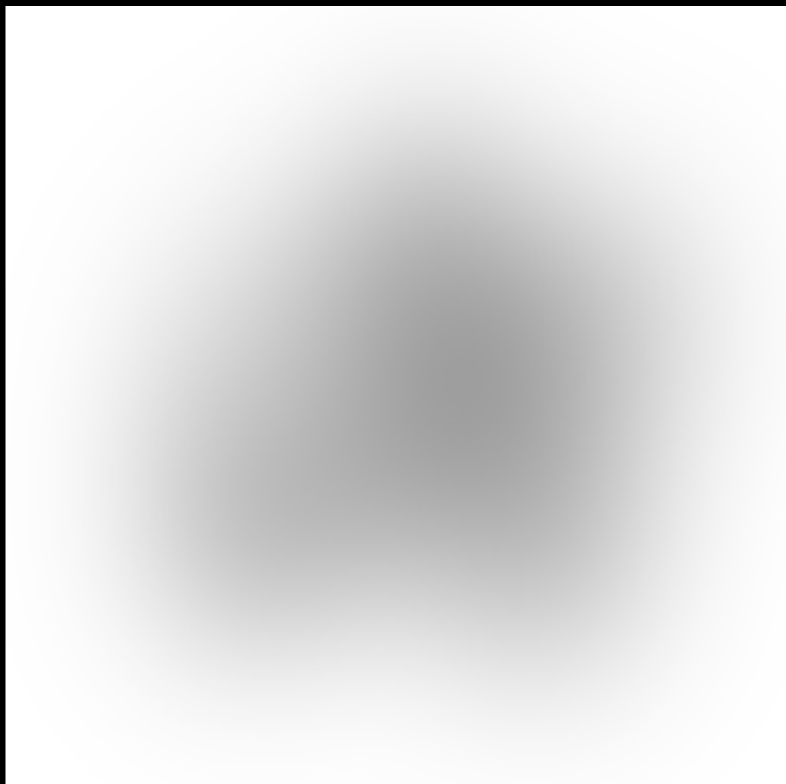
- If $\boxed{\gamma > V(0)}$ then $\lambda^- = 0$ and $\lim_{n\to\infty} \frac{1}{n} \ln \lambda(A_n) = \ln\left(\frac{V(0)}{\gamma}\right)$

Bifurcation



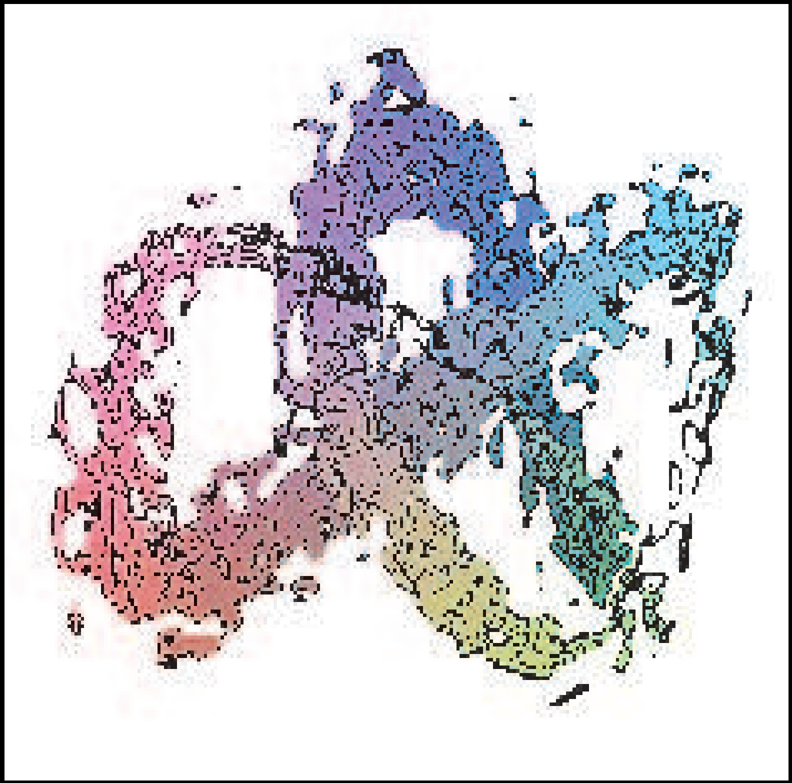
- Flow self similar and isotropic. for all n, $E_n = E$.
- Shape of the eddy E over a period.
- $1 < V(0) < \infty$.

Bifurcation



- $\gamma < V(0)$
- $\lambda^{-} > 0$
- The flow is self averaging

Bifurcation



- $\gamma \geq V(0)$
- $\lambda^{-} = 0$
- The self averaging property collapses.

$\lambda^- = 0$, proof of the collapse

Assume A_n to be self-similar and isotropic.
Spatial scales $\rho^n \to R_n$

$$\Gamma(x) = \sum_{n=0}^{\infty} \gamma^n E(\frac{x}{R_n})$$

$$\rho_{\min} := \inf_{n \in \mathbb{N}} \frac{R_{n+1}}{R_n}$$

$$2 \leq \rho_{\min}$$

$\lambda^- = 0$, proof of the collapse

For $y \in [0,1]^d$

$$\sigma(n,y) := \sigma(\kappa I_d, \Gamma^{n-1} + \gamma^n E(y + \frac{x}{R_n}))$$

$\lambda^- = 0$, proof of the collapse

$$\sigma(n,y) := \sigma(\kappa I_d, \Gamma^{n-1} + \gamma^n E(y + \frac{x}{R_n}))$$

- Averaging paradigm $\Rightarrow$ Relative translation by y has little influence on $\sigma(n,y)$
- for all y,

$$\lim_{\rho_{\min}\to\infty} \sigma(n,y) = \lim_{\rho_{\min}\to\infty} \sigma(n,0)$$

$\lambda^- = 0$, proof of the collapse

$$\sigma(n,y) := \sigma(\kappa I_d, \Gamma^{n-1} + \gamma^n E(y + \frac{x}{R_n}))$$

$$\sigma(n,y,\rho) := \lim_{\frac{R_1}{R_0},\ldots,\frac{R_{n-1}}{R_{n-2}}\rightarrow\infty\,;\,\frac{R_n}{R_{n-1}}=\rho} \sigma(n,y)$$

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$$\sigma(n,y,\rho) = \gamma^{n-1}\sigma(A_{n-1}, E(\rho x) + \gamma E(x+y))$$

$\lambda^- = 0$, proof of the collapse

$$\sigma(n,y) := \sigma(\kappa I_d, \Gamma^{n-1} + \gamma^n E(y + \frac{x}{R_n}))$$

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$$\sigma(n,y,\rho) = \gamma^{n-1}\sigma(A_{n-1}, E(\rho x) + \gamma E(x+y))$$

If $\lambda^- > 0$ then $l \in (R^d)^*$ and $y \in [0,1]^d$,

$$\limsup_{n\to\infty} \frac{{}^t l\sigma(n,y,\rho)l}{{}^t l\sigma(n,0,\rho)l} < 1 + C_d(\rho\lambda^-)^{-\frac{1}{2}}.$$

$\lambda^- = 0$, proof of the collapse

$$\sigma(n,y) := \sigma(\kappa I_d, \Gamma^{n-1} + \gamma^n E(y + \frac{x}{R_n}))$$

$$\sigma(n,y,\rho) := \lim_{\frac{R_1}{R_0},\dots,\frac{R_{n-1}}{R_{n-2}} \to \infty \,;\, \frac{R_n}{R_{n-1}} = \rho} \sigma(n,y)$$

$$\sigma(n,y,\rho) = \gamma^{n-1}\sigma(A_{n-1}, E(\rho x) + \gamma E(x+y))$$

If $\lambda^- = 0$ then for any $\rho > 1$ there exists E and $y \in [0,1]^d$ such that for any $l \in (R^d)^*$,

$$\lim_{n\to\infty} \frac{{}^t l \sigma(n,y,\rho) l}{{}^t l \sigma(n,0,\rho) l} = \infty.$$

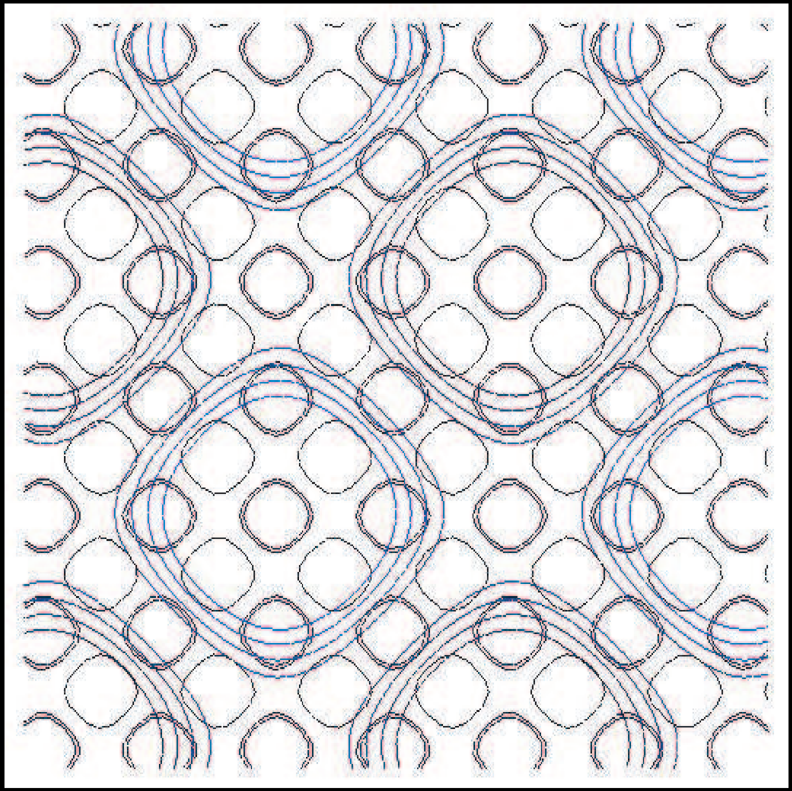
Two scale flows

$$S_\rho E(x) = E(\rho x)$$
$$\Theta_y E(x) = E(x - y)$$

As $\zeta \downarrow 0$

$$\sigma(\zeta I_d, S_\rho E + E) \sim C_1 \zeta I_d.$$

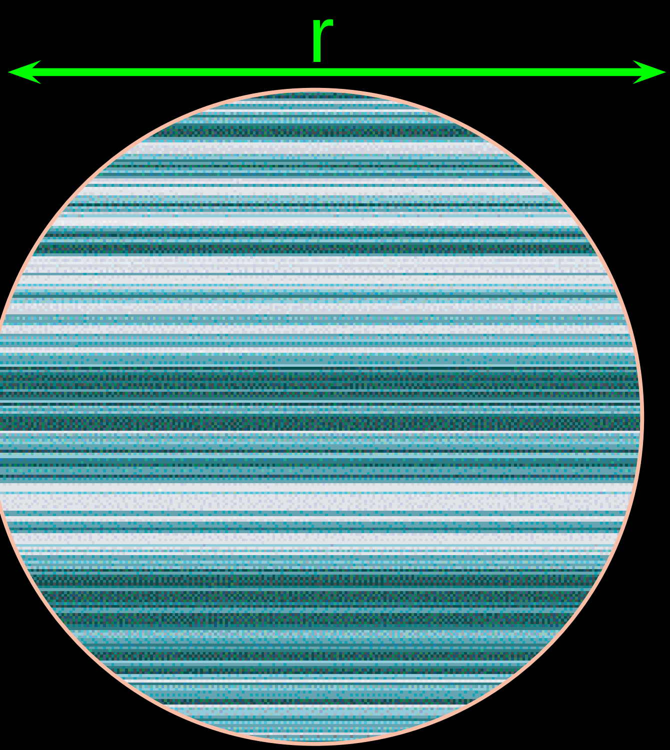
Two scale flows



$$S_\rho E(x) = E(\rho x)$$
$$\Theta_y E(x) = E(x - y)$$
As $\zeta \downarrow 0$

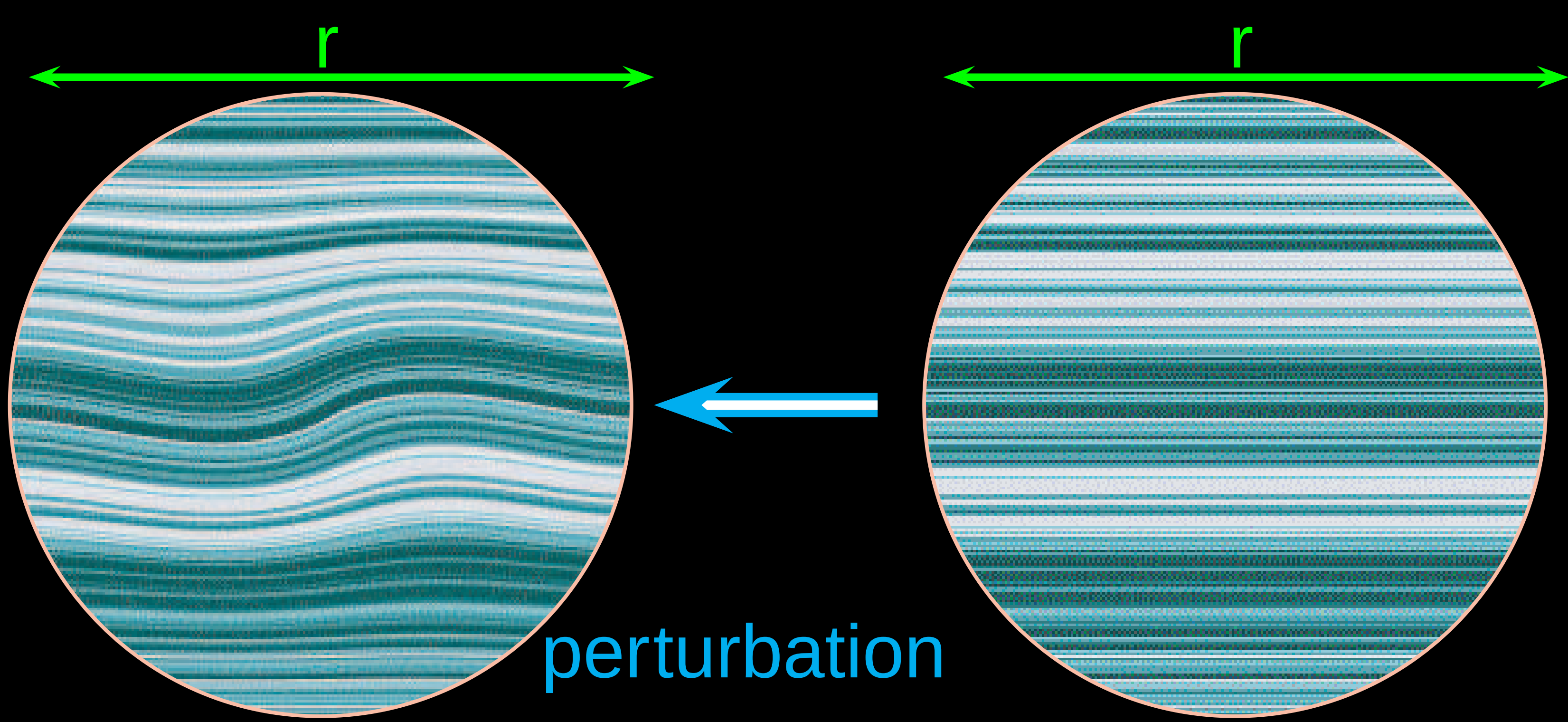
$$\sigma(\zeta I_d, S_\rho E + \Theta_y E) \sim C_2 \zeta^{\frac{1}{2}} I_d$$

On the nature of Turbulence.



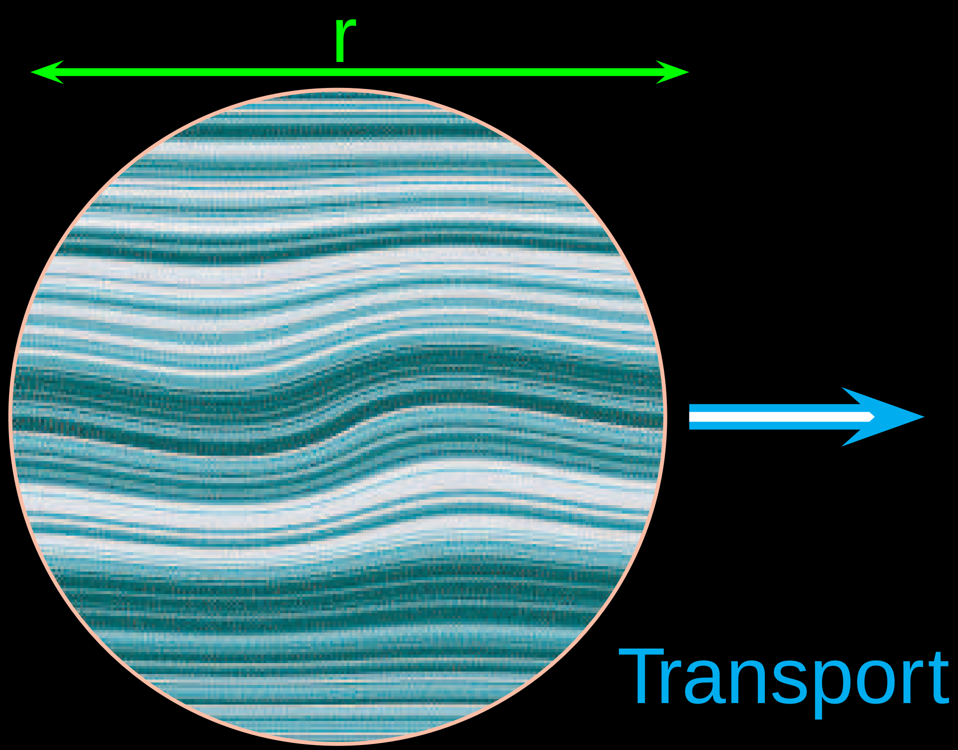
The flow at scale r is laminar.
Viscosity κ, velocity $V(r) = V_0$.

On the nature of Turbulence.



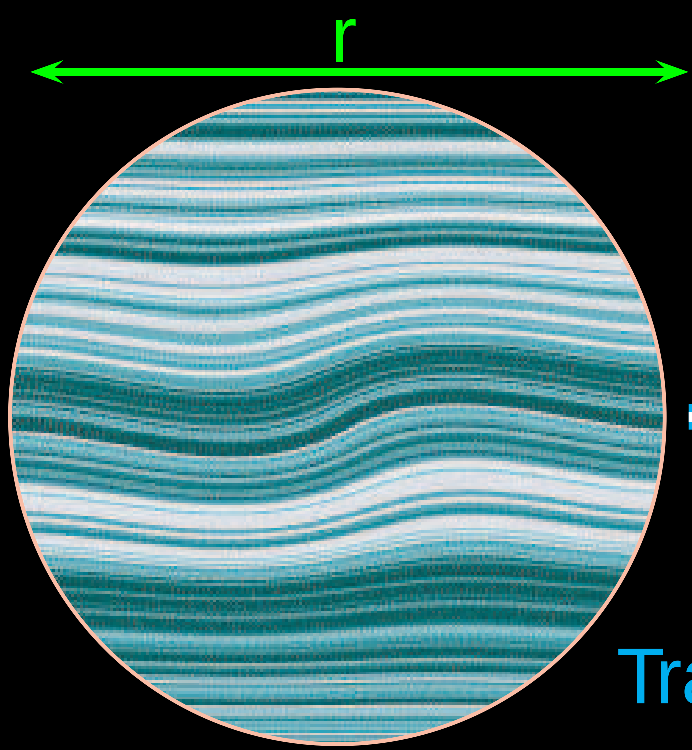
A small perturbation is introduced.

On the nature of Turbulence.



$\tau_D(r)$: exit time of the perturbation by diffusion
$\tau_C(r)$: exit time of the perturbation by convection

On the nature of Turbulence.



Transport

Diffusion

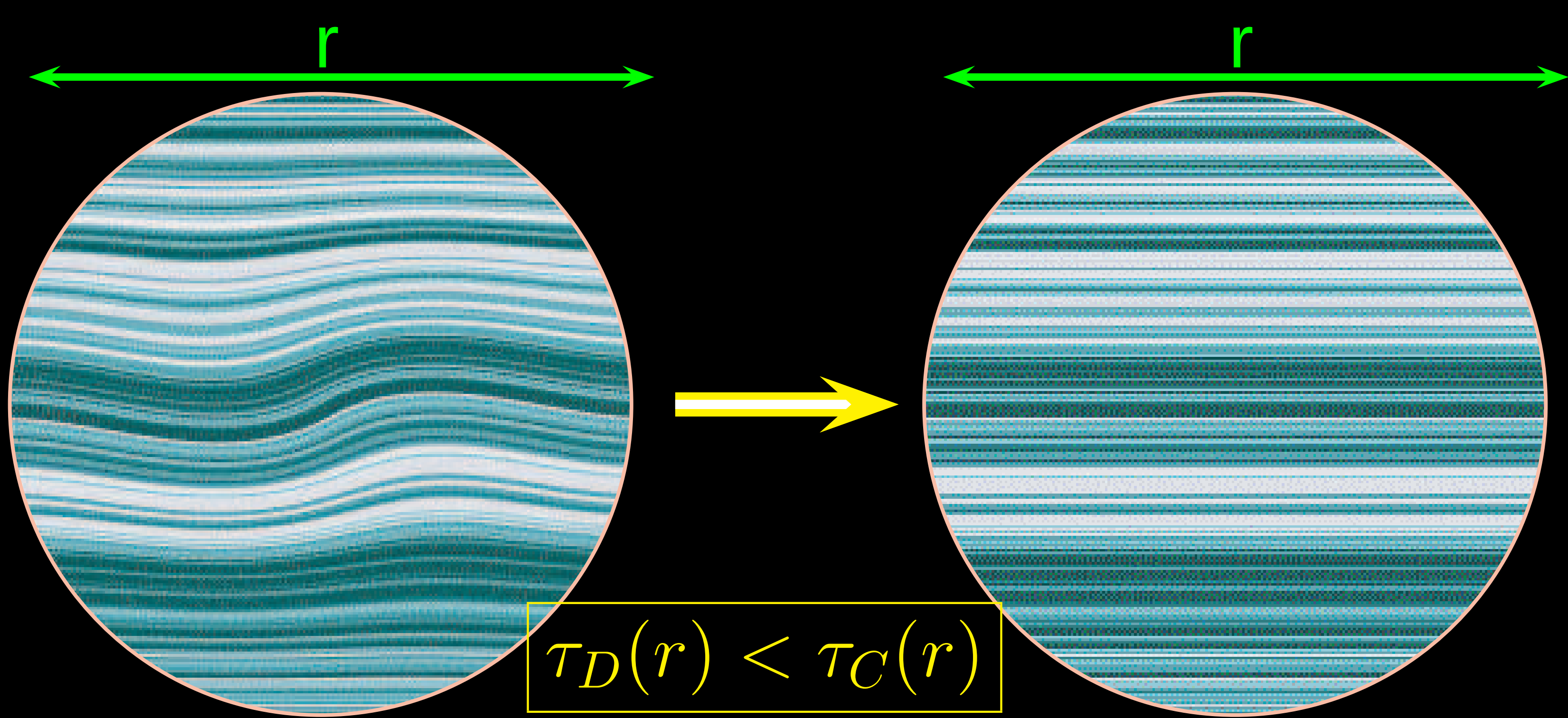
+

Convection

$$\tau_D(r) \sim \frac{r^2}{\kappa}$$

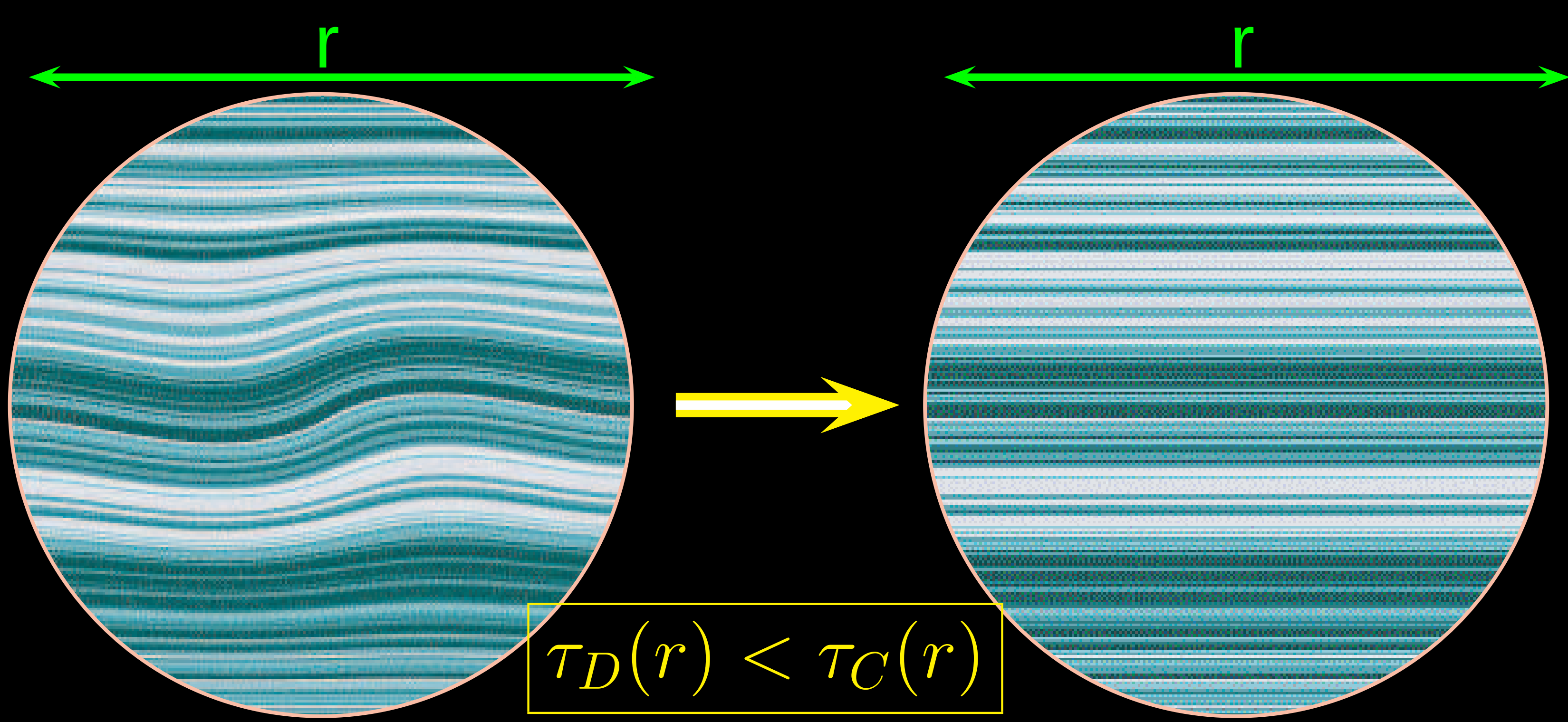
$$\tau_C(r) \sim \frac{r}{V_0}$$

On the nature of Turbulence.



If $\tau_D(r) < \tau_C(r)$ the perturbation exits by diffusion and is smoothed before going out of $B(0, r)$.

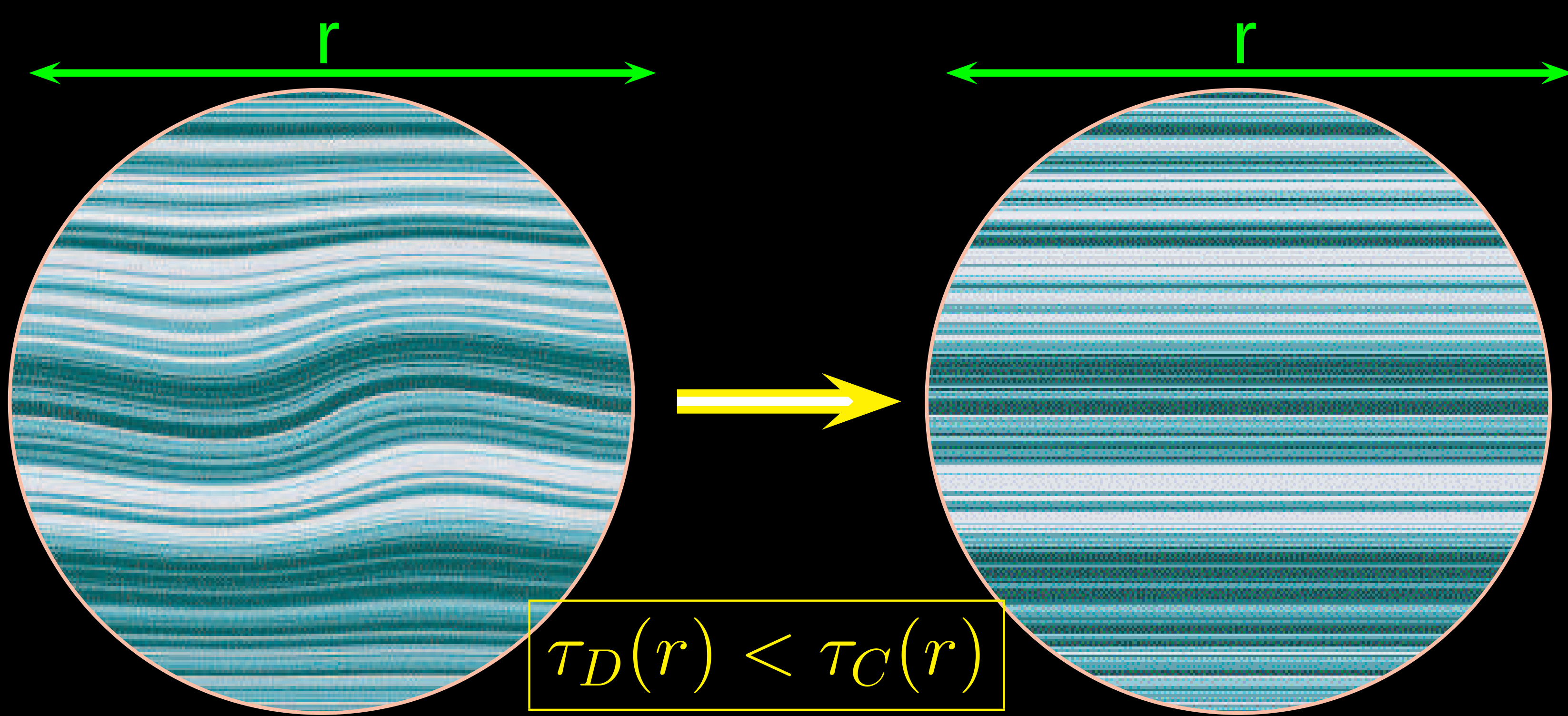
On the nature of Turbulence.



$$\tau_D(r) < \tau_C(r)$$

$\Rightarrow$ the laminar flow is stable at the scale r

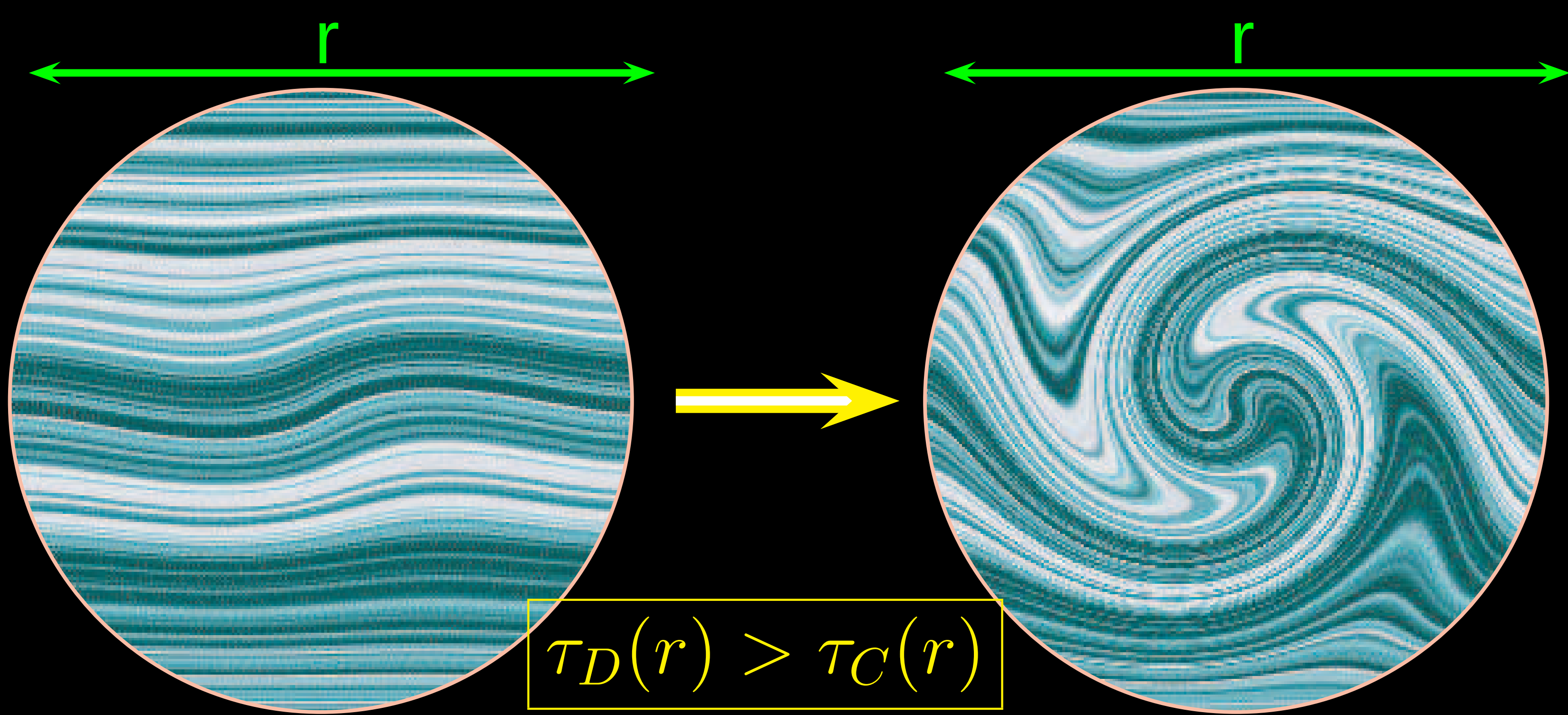
On the nature of Turbulence.



$$\tau_D(r) < \tau_C(r)$$

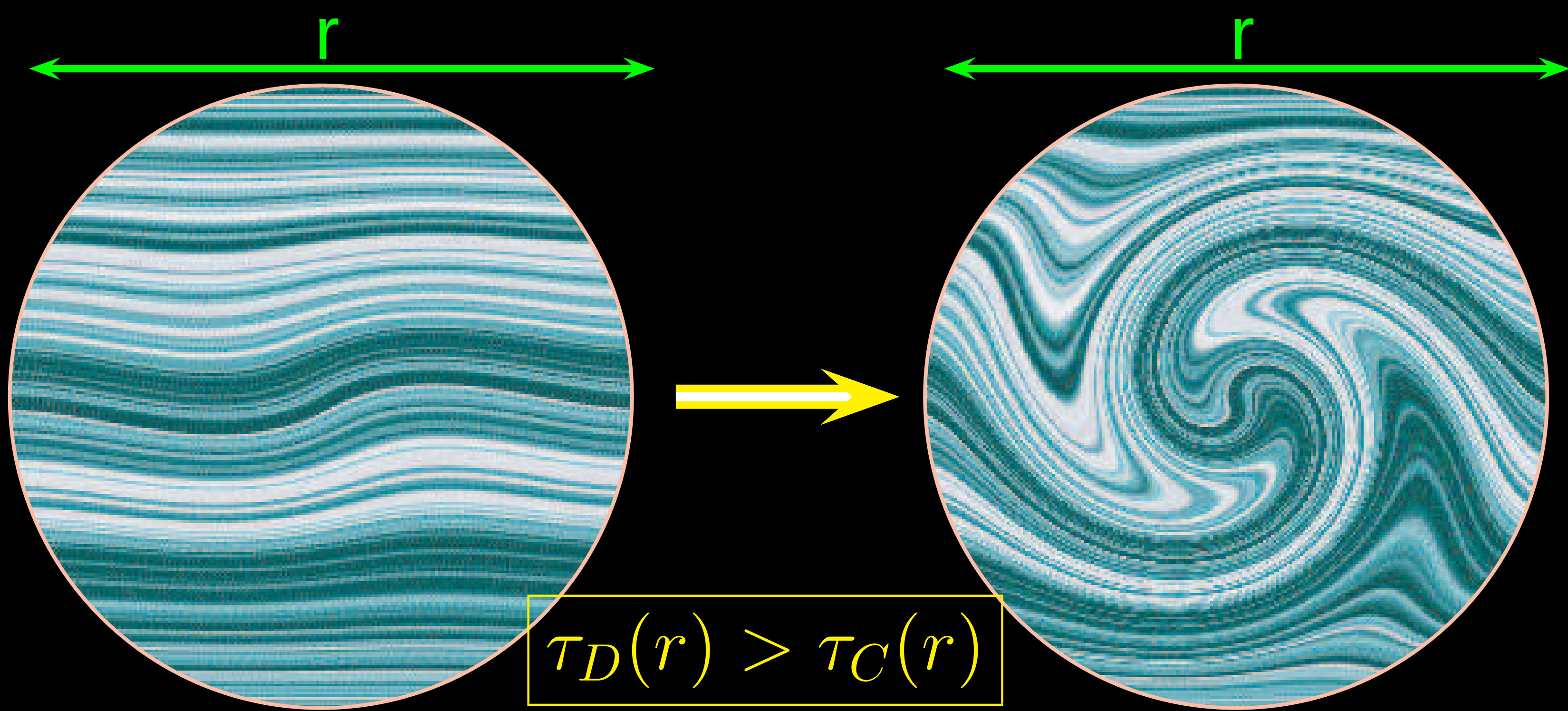
$$\tau_D(r) < \tau_C(r) \Leftrightarrow \frac{r^2}{\kappa} < \frac{r}{V_0} \Leftrightarrow \mathbf{Re} = \frac{rV_0}{\kappa} < 1$$

On the nature of Turbulence.



If $\tau_D(r) > \tau_C(r)$ the perturbation exits by convection and propagates.

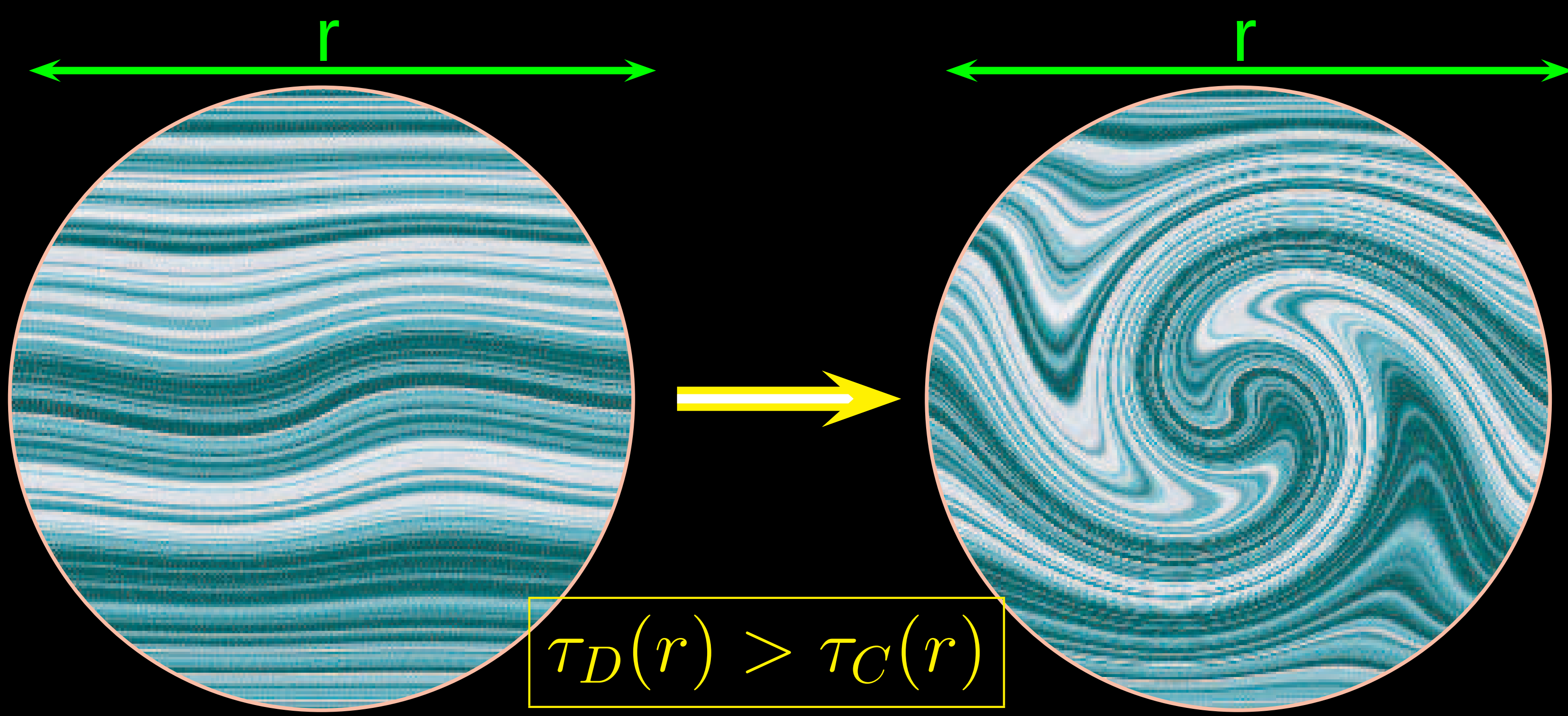
On the nature of Turbulence.



$$\tau_D(r) > \tau_C(r)$$

$\Rightarrow$ The flow is unstable at the scale r and fluctuates and this scale.

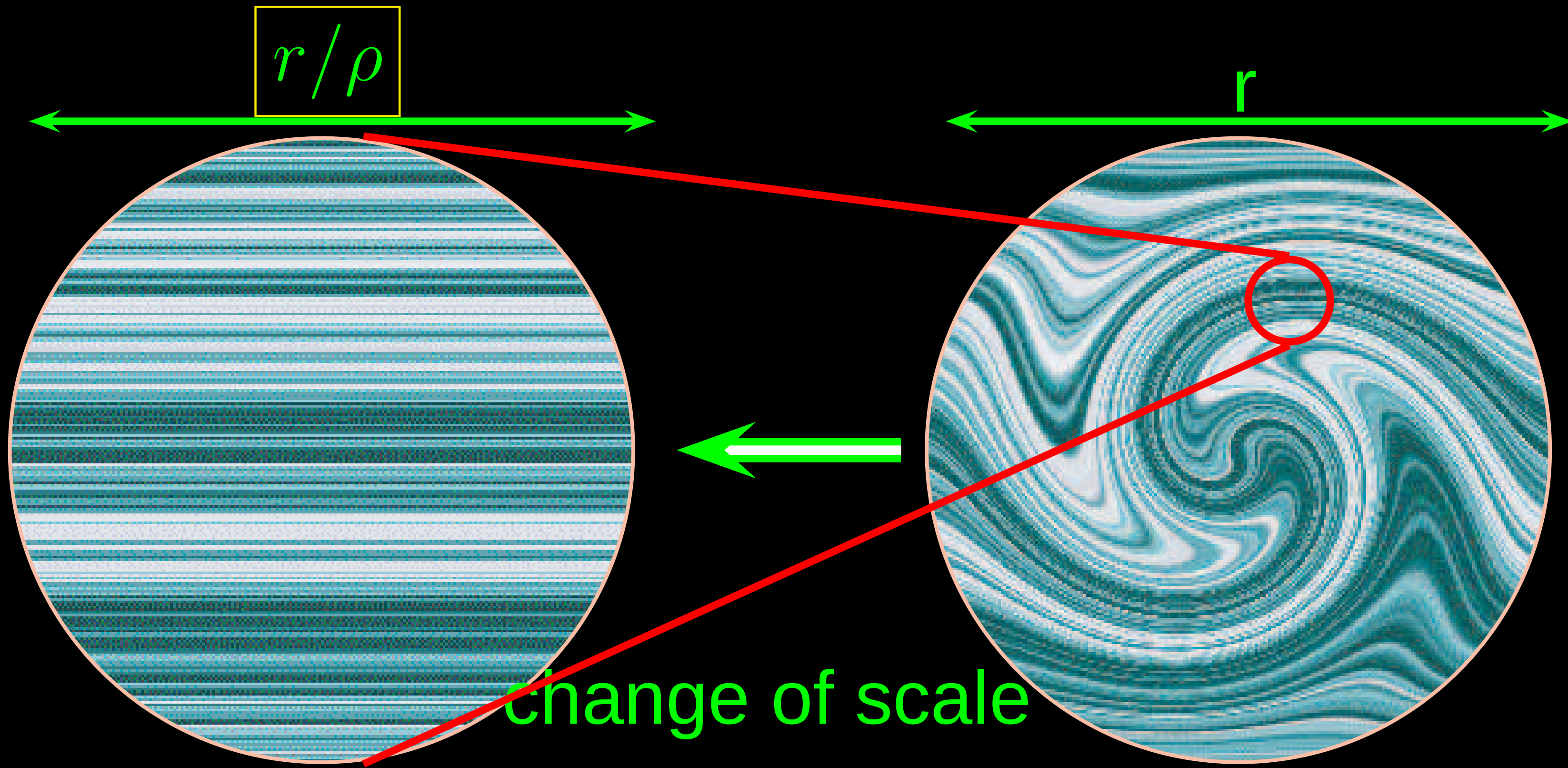
On the nature of Turbulence.



$$\tau_D(r) > \tau_C(r)$$

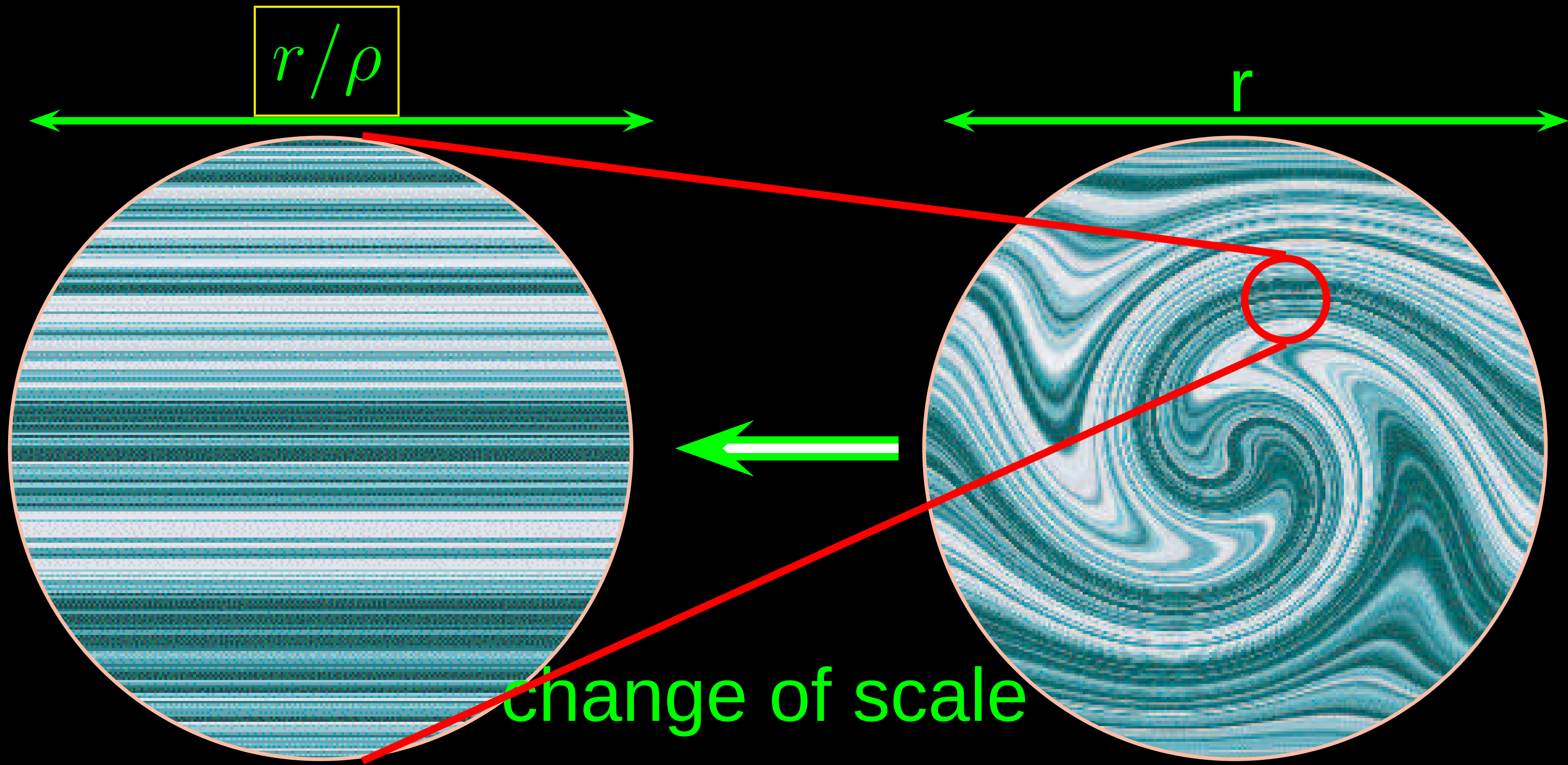
$$\tau_D(r) > \tau_C(r) \Leftrightarrow \frac{r^2}{\kappa} > \frac{r}{V_0} \Leftrightarrow \mathbf{Re} = \frac{rV_0}{\kappa} > 1$$

On the nature of Turbulence.



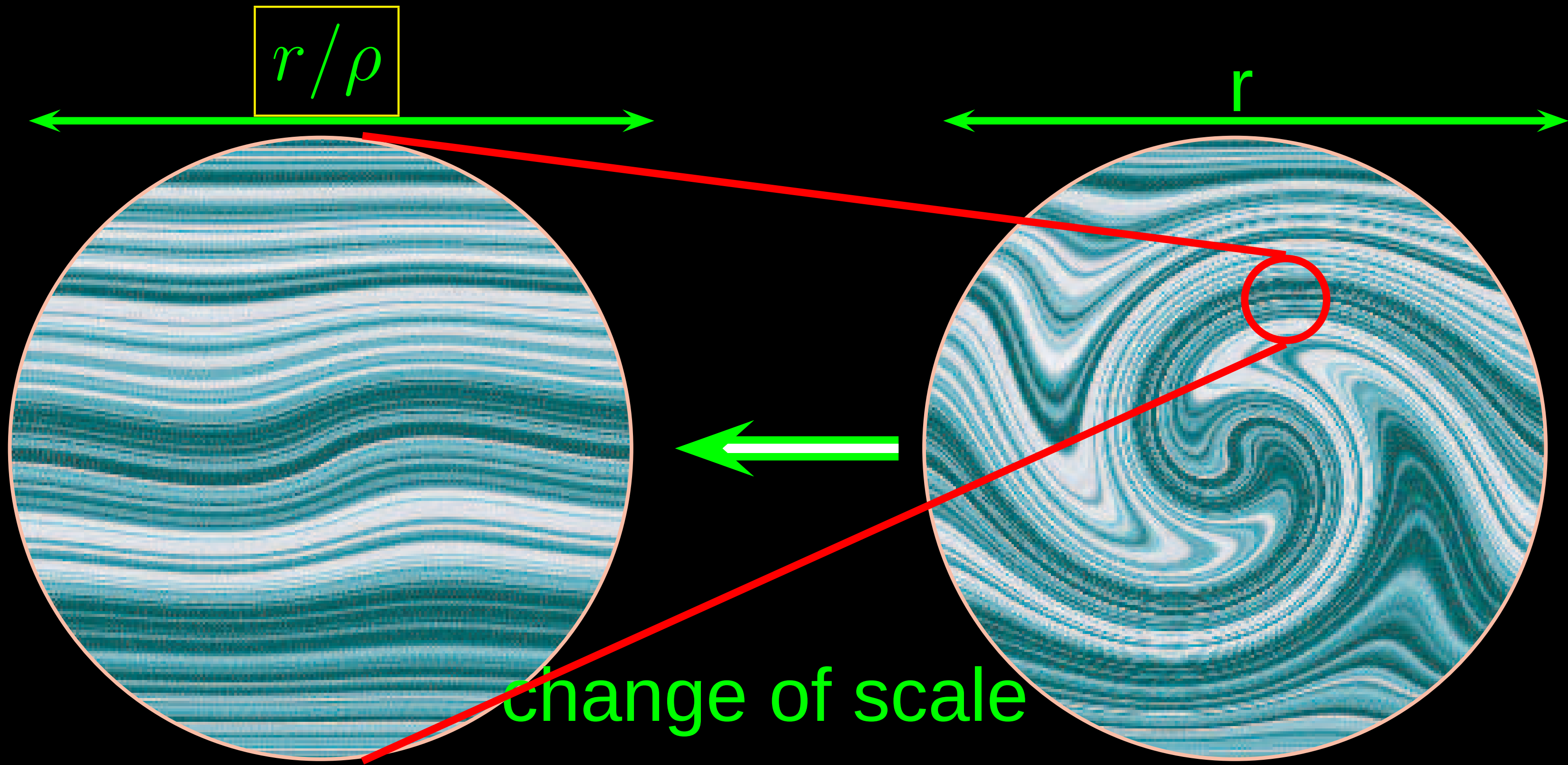
Let's look at the flow at the scale r/ρ

On the nature of Turbulence.



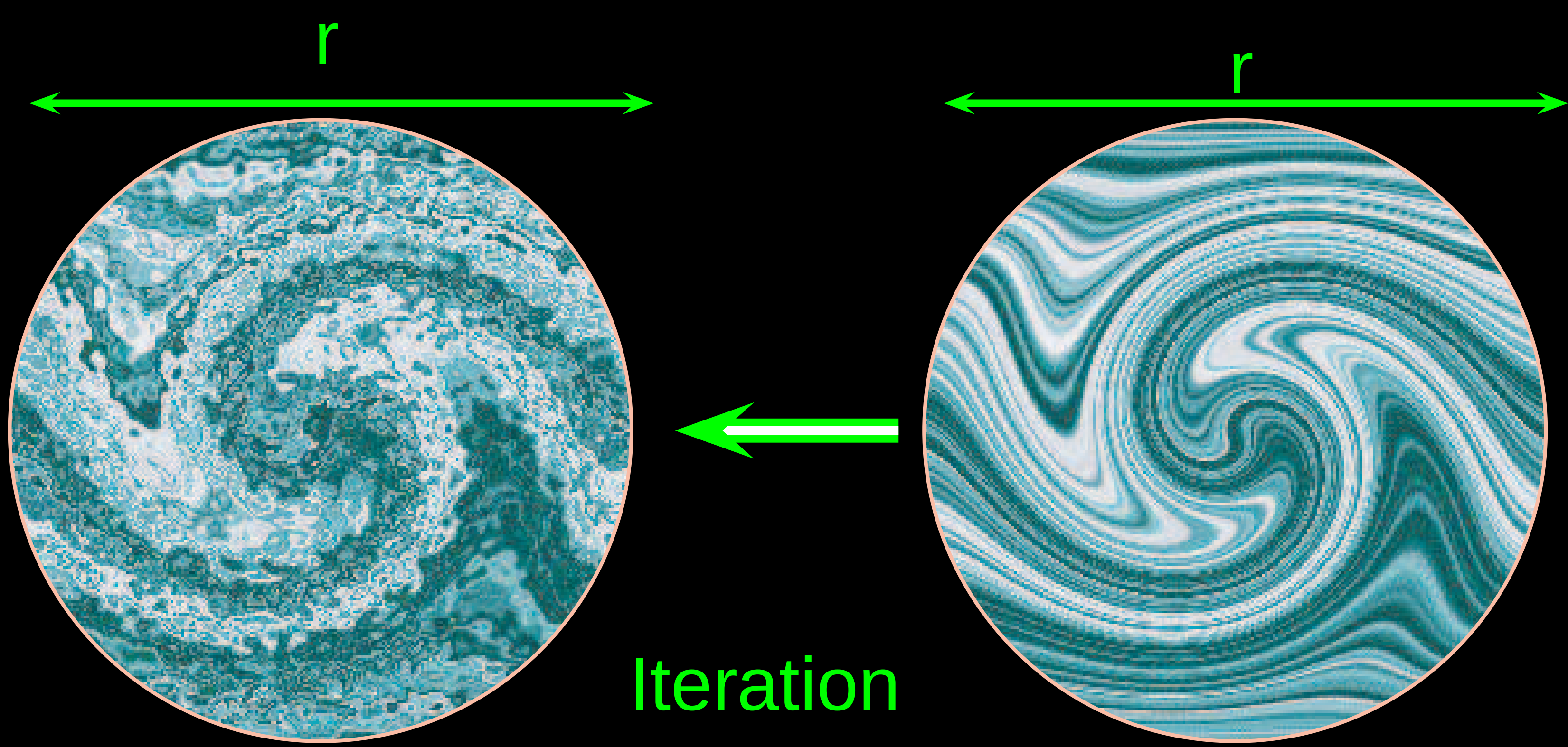
The flow is laminar at this scale r/ρ with velocity $V(r/\rho) = V_0/\gamma$

On the nature of Turbulence.



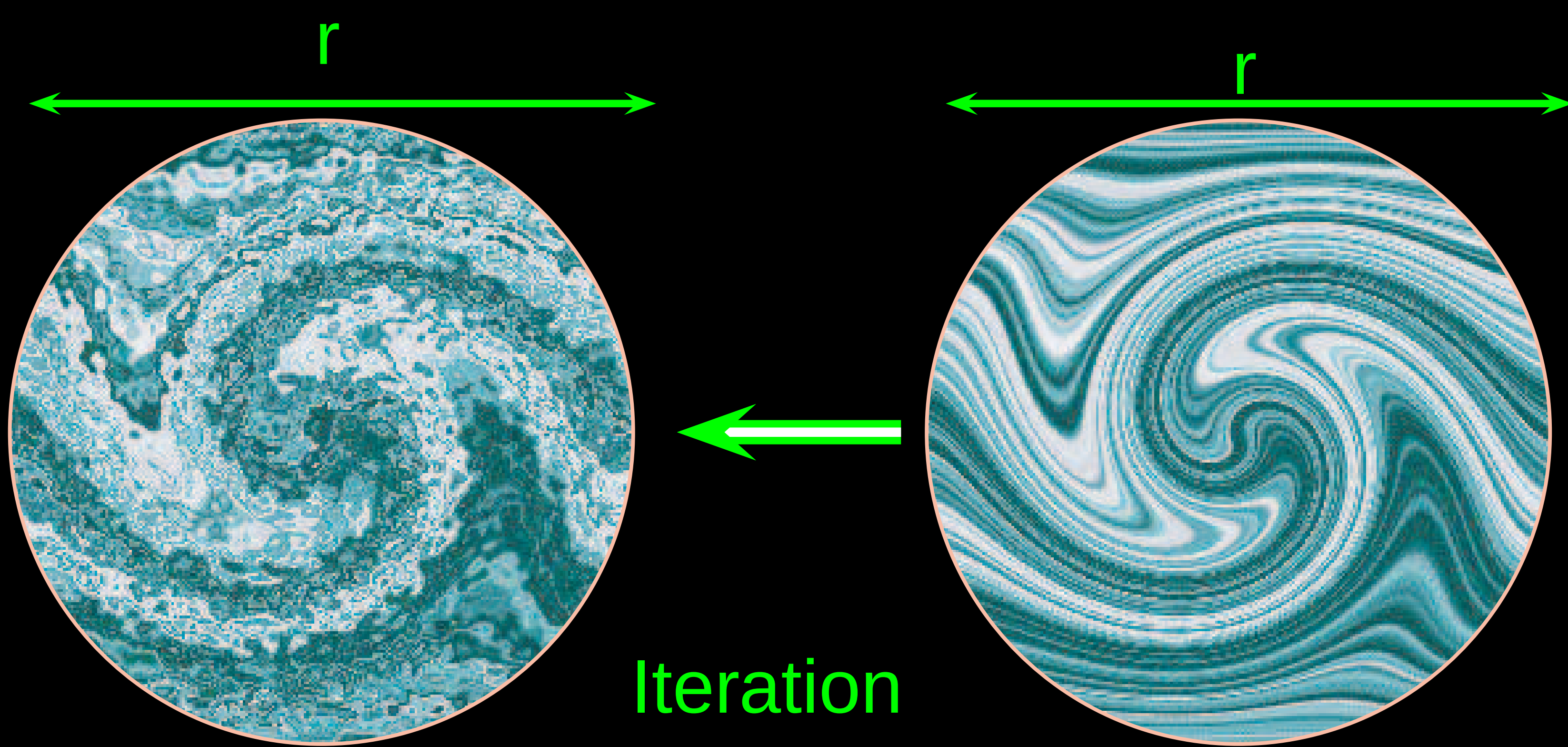
A small perturbation is introduced at the scale r/ρ

On the nature of Turbulence.



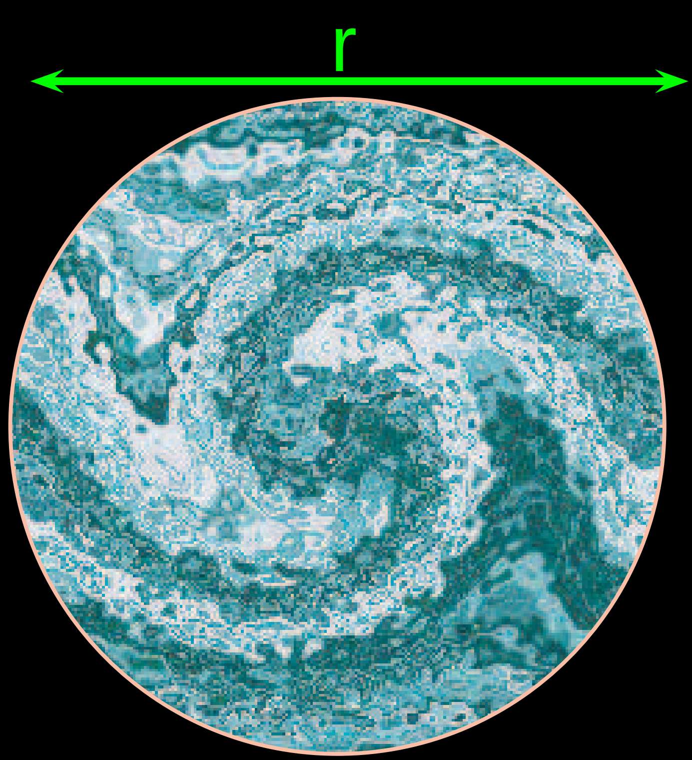
This self similar process is iterated, until the dissipation scale l is reached.

On the nature of Turbulence.



At the dissipation scale l , $\tau_C(l) \sim \tau_D(l)$

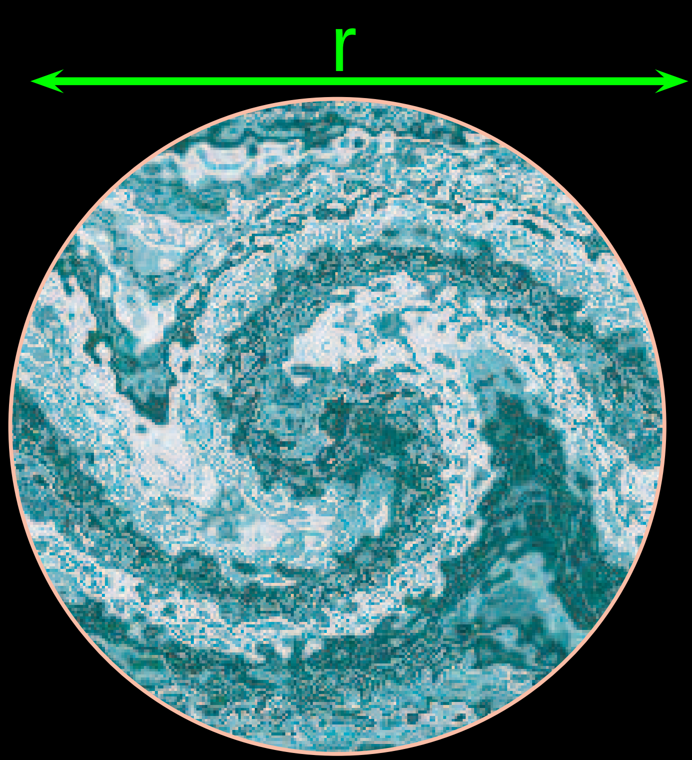
On the nature of Turbulence.



$$\sigma(r) \sim rV(r)$$

The multi-scale structure creates an effective viscosity $\sigma(r) \sim rV(r)$

On the nature of Turbulence.



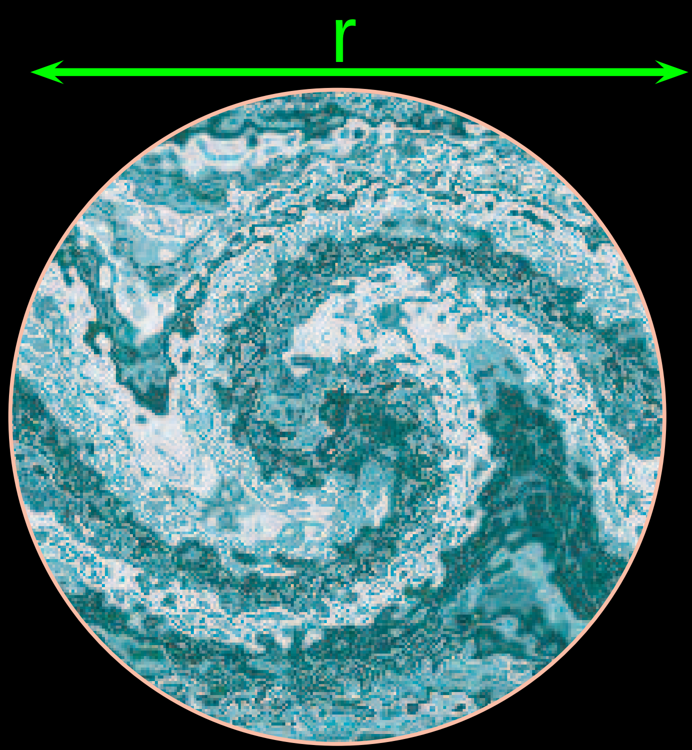
$$\sigma(r) \sim rV(r)$$

$$\tau_C(r) \sim r/V(r)$$

$$\tau_D(r) \sim r^2/\sigma(r)$$

The multi-scale structure stabilizes the flow since $\sigma(r) \sim rV(r) \Rightarrow \tau_D(r) \sim \tau_C(r)$

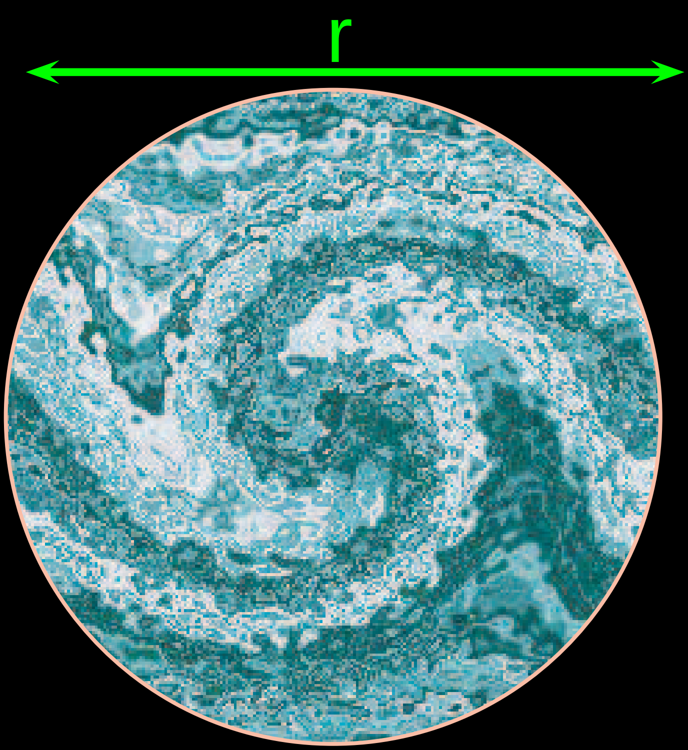
On the nature of Turbulence.



$$V(x) = V_0(x/r)^\alpha$$

We write $\gamma = \rho^{1+\alpha}$

On the nature of Turbulence.



$$V(x) = V_0(x/r)^\alpha$$

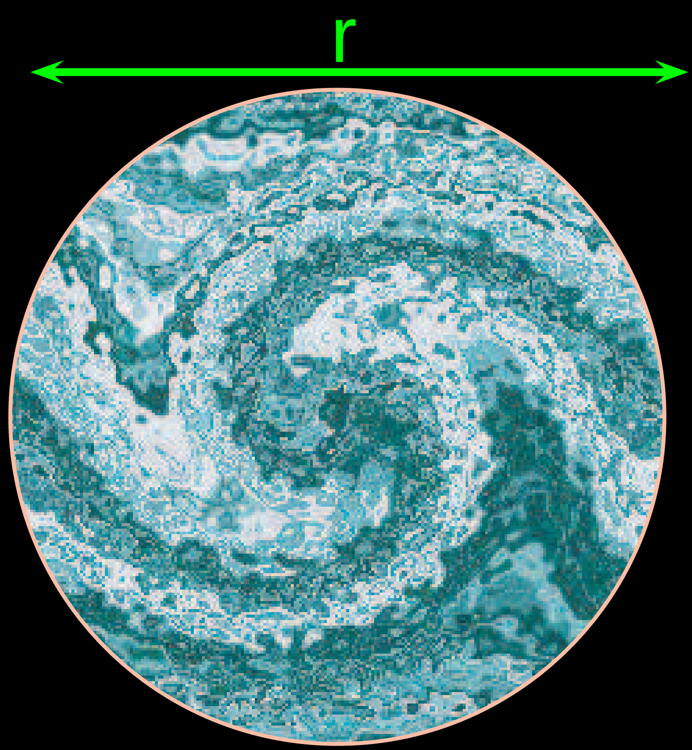
$$\epsilon(x) \sim \frac{(V(x))^3}{x}$$

Kolmogorov $\alpha = 1/3$

The energy dissipation $\epsilon(x)$ at scale x is

$$\sigma(x)(\frac{V(x)}{x})^2 \sim \frac{(V(x))^3}{x}$$

On the nature of Turbulence.



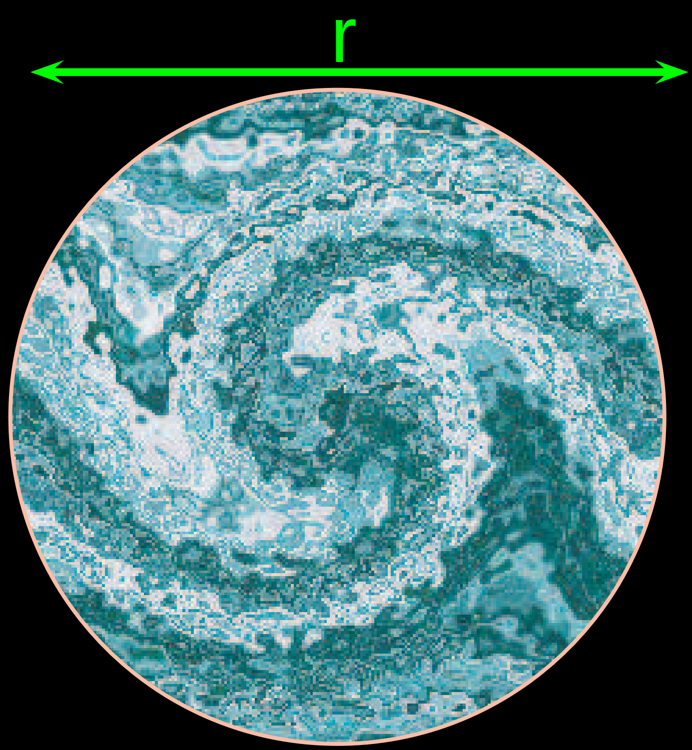
$$V(x) = V_0(x/r)^{\alpha}$$

$$\epsilon(x) \sim \frac{(V(x))^3}{x}$$

Kolmogorov $\alpha = 1/3$

If $\alpha > 1/3$ the larger eddies are dissipated before the smaller ones

On the nature of Turbulence.



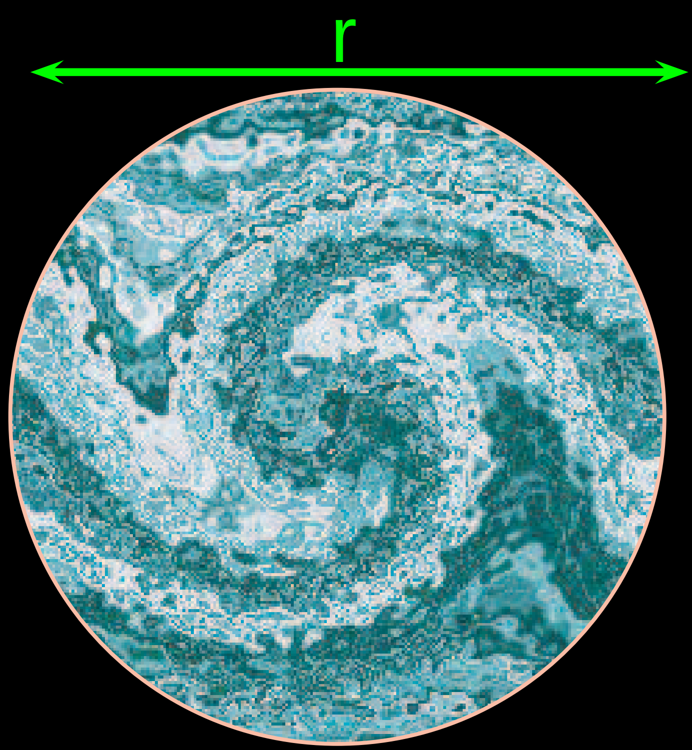
$$V(x) = V_0(x/r)^\alpha$$

$$\epsilon(x) \sim \frac{(V(x))^3}{x}$$

Kolmogorov $\alpha = 1/3$

If $\alpha < 1/3$ the smaller eddies are dissipated before the larger ones

On the nature of Turbulence.



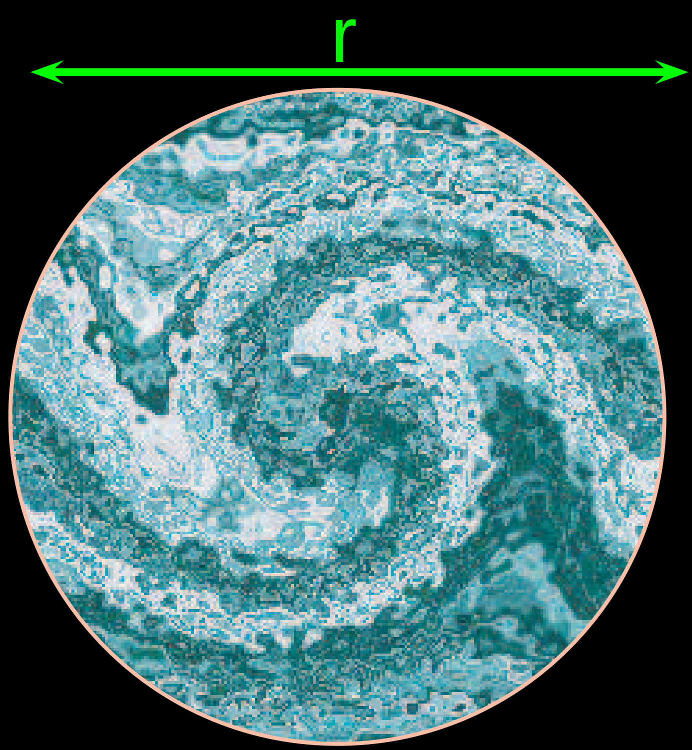
$$V(x) = V_0(x/r)^\alpha$$

$$\epsilon(x) \sim \frac{(V(x))^3}{x}$$

Kolmogorov $\alpha = 1/3$

The relation $\sigma(x) \sim xV(x)$ is at the core of the Kolmogorov law

On the nature of Turbulence.



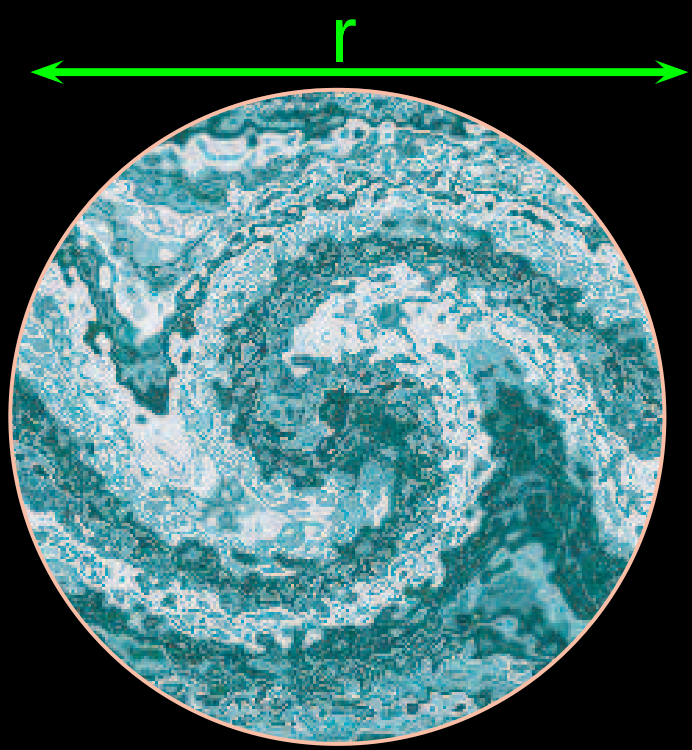
$$V(x) = V_0(x/r)^\alpha$$

$$\epsilon(x) \sim \frac{(V(x))^3}{x}$$

Kolmogorov $\alpha = 1/3$

In the anisotropic case, the relation $\lambda_{\max}(\sigma(x))\lambda_{\min}(\sigma(x)) \sim x^2(V(x))^2$ restores the isotropy of the flow.

On the nature of Turbulence.



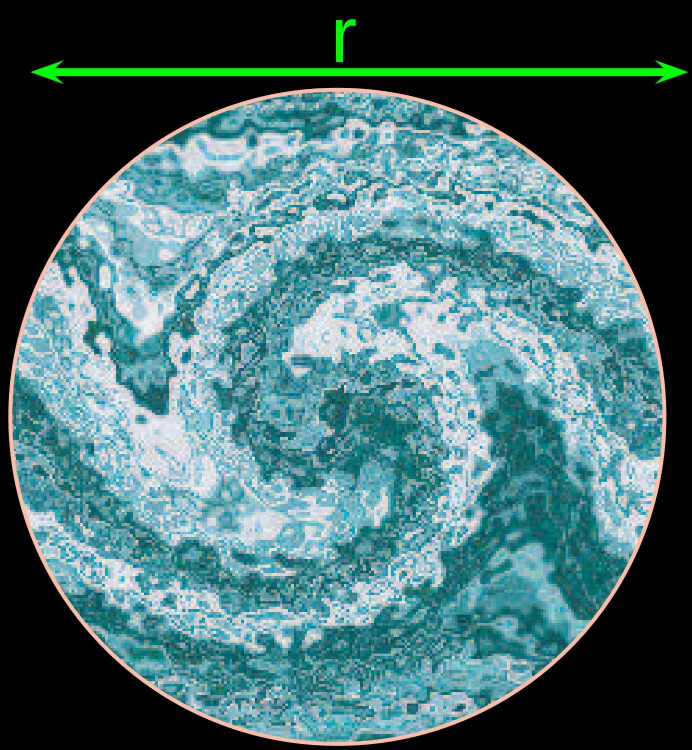
$$V(x) = V_0(x/r)^\alpha$$

$$\epsilon(x) \sim \frac{(V(x))^3}{x}$$

Kolmogorov $\alpha = 1/3$

At the dissipation scale, $\tau_C(l) \sim \tau_D(l) \Leftrightarrow$
$l/V(l) \sim l^2/\kappa$
$\Leftrightarrow r/l \sim (\frac{V_0 r}{\kappa})^{\frac{3}{4}}$

On the nature of Turbulence.



$$V(x) = V_0(x/r)^\alpha$$

$$\epsilon(x) \sim \frac{(V(x))^3}{x}$$

Kolmogorov $\alpha = 1/3$

intermittency at the smaller scales