

Multiscale Homogenization with Bounded Ratios and Anomalous Slow Diffusion

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Abstract

We show that the effective diffusivity matrix $D(V^n)$ for the heat operator $\partial_t - (\Delta/2 - \nabla V^n \nabla)$ in a periodic potential $V^n = \sum_{k=0}^n U_k(x/R_k)$ obtained as a superposition of Hölder-continuous periodic potentials U_k (of period $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$, $d \in \mathbb{N}^*$, $U_k(0) = 0$) decays exponentially fast with the number of scales when the scale ratios R_{k+1}/R_k are bounded above and below. From this we deduce the anomalous slow behavior for a Brownian motion in a potential obtained as a superposition of an infinite number of scales, $dy_t = d\omega_t - \nabla V^\infty(y_t)dt$.
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1 Introduction

Homogenization in the presence of a large number of spatial scales is both very important for applications and far from understood from a mathematical standpoint. In the asymptotic regime where the spatial scales separate, i.e., when the ratio between successive scales tends to infinity, multiscale homogenization is now well understood; see, for instance, [1, 3, 6, 11, 27, 30].

Nevertheless, the case of multiscale homogenization when spatial scales are not clearly separated, i.e., when the ratios between scales stay bounded, has been recognized as difficult and important. For instance, Avellaneda [4, p. 267] emphasizes that “the assumption of scale separation invoked in homogenization is not adequate for treating the most general problems of transport and diffusion in self-similar random media.”

The potential use of multiscale homogenization estimates for applications are numerous (see, for instance, [39] for applications to geology, or [14, 16, 32] for applications to differential effective medium theories). The main application of this line of ideas is perhaps to proving superdiffusivity for turbulent diffusion: see, for instance, [4, 5, 12, 13, 18, 19, 20, 21, 22, 23, 24, 26, 29, 42, 43].

We are interested here in subdiffusivity problems. Consider the Brownian motion in a periodic potential, i.e., the diffusion process

$$(1.1) \quad dy_t = d\omega_t - \nabla V(y_t)dt$$

where V is periodic and smooth. It is a basic and simple fact of homogenization theory that y_t behaves in large times like a Brownian motion slower than the Brownian motion ω_t driving the equation; i.e., $y^\epsilon(t) = \epsilon y_{t/\epsilon^2}$ converges in law to a Brownian motion with diffusivity matrix $D(V) < I_d$.

We first treat here the case where V is a periodic n -scale potential with ratios (between successive scales) bounded uniformly on n . We introduce a new approach that enables us to show exponential decay of the effective diffusivity matrix when the number of spatial scales grows to infinity.

From this exponential decay we deduce the anomalous slow behavior of Brownian motions in potential V , when V is a superposition of an infinite number of scales.

We have studied this question with a particular application in mind, i.e., to prove that one of the basic mechanisms of anomalous slow diffusion in complex media is the existence of a large number of spatial scales, without a clear separation between them. This phenomenon has been attested for very regular self-similar fractals (see Barlow and Bass [9] and Osada [34] for the Sierpinski carpet; see also [25]). Our goal is to implement rigorously the idea that the key for the subdiffusivity is a never-ending or perpetual homogenization phenomenon over an infinite number of scales, the point being that our model will not have any self-similarity or local symmetry hypotheses.

Our approach naturally gives much more detailed information in dimension one, and this is the subject of [37].

This approach will be shown in forthcoming works to also give a proof of superdiffusive behavior for diffusion in some multiscale divergence-free fields (see [10] for the simple case of shear flow and [36] for a general situation).

The second section contains the description of our model; the third one, the statement of our results; and the fourth one, the proofs.

2 The Multiscale Medium

For $U \in L^\infty(\mathbb{T}_R^d)$ (we note $\mathbb{T}_R^d := R\mathbb{T}^d$), let m_U be the probability measure on \mathbb{T}^d defined by

$$(2.1) \quad m_U(dx) = \frac{e^{-2U(x)} dx}{\int_{\mathbb{T}_R^d} e^{-2U(x)} dx}.$$

The effective diffusivity $D(U)$ is the symmetric positive definite matrix given by

$$(2.2) \quad |D(U)l| = \inf_{f \in C^\infty(\mathbb{T}_R^d)} \int_{\mathbb{T}_R^d} |l - \nabla f(x)|^2 m_U(dx)$$

for l in \mathbb{S}^{d-1} (the unit sphere of \mathbb{R}^d). Our purpose in this work is to obtain quantitative estimates for the effective diffusivity matrix of multiscale potentials V_0^n given

by a sum of periodic functions with (geometrically) increasing periods:

$$(2.3) \quad V_0^n = \sum_{k=0}^n U_k \left(\frac{x}{R_k} \right).$$

In this formula we have two important ingredients: the potentials U_k and the scale parameters R_k . We will now describe the hypothesis we make on these two items of our model.

2.1 Hypotheses on the Potentials U_k

We will assume that

$$(2.4) \quad U_k \in C^\alpha(\mathbb{T}^d),$$

$$(2.5) \quad U_k(0) = 0.$$

Here $C^\alpha(\mathbb{T}^d)$ denotes the space of α -Hölder-continuous functions on the torus \mathbb{T}^d , with $0 < \alpha \leq 1$. We will also assume that the C^α -norm of the U_k are uniformly bounded, i.e.,

$$(2.6) \quad K_\alpha := \sup_{k \in \mathbb{N}} \sup_{x \neq y} \frac{|U_k(x) - U_k(y)|}{|x - y|^\alpha} < \infty.$$

We will need the notation

$$(2.7) \quad K_0 := \sup_{k \in \mathbb{N}} \text{Osc}(U_k)$$

where the oscillation of U_k is given by $\text{Osc}(U) := \sup U - \inf U$.

We also assume that the effective diffusivity matrices of the U_k 's are uniformly bounded. Let $\lambda_{\min}(D(U_k))$ and $\lambda_{\max}(D(U_k))$ be the smallest and largest eigenvalues of the effective diffusivity matrix $D(U_k)$. We will assume that

$$(2.8) \quad \lambda_{\min} := \inf_{\substack{k \in \mathbb{N} \\ l \in \mathcal{S}^{d-1}}} {}^t l D(U_k) l > 0,$$

$$(2.9) \quad \lambda_{\max} := \sup_{\substack{k \in \mathbb{N} \\ l \in \mathcal{S}^{d-1}}} {}^t l D(U_k) l < 1.$$

2.2 Hypotheses on the Scale Parameters R_k

R_k is a spatial scale parameter growing exponentially fast with k ; more precisely, we will assume that $R_0 = r_0 = 1$ and that the ratios between scales defined by

$$(2.10) \quad r_k = \frac{R_k}{R_{k-1}} \in \mathbb{N}^*,$$

for $k \geq 1$, are integers uniformly bounded away from 1 and ∞ : We will define

$$(2.11) \quad \rho_{\min} := \inf_{k \in \mathbb{N}^*} r_k \quad \text{and} \quad \rho_{\max} := \sup_{k \in \mathbb{N}^*} r_k$$

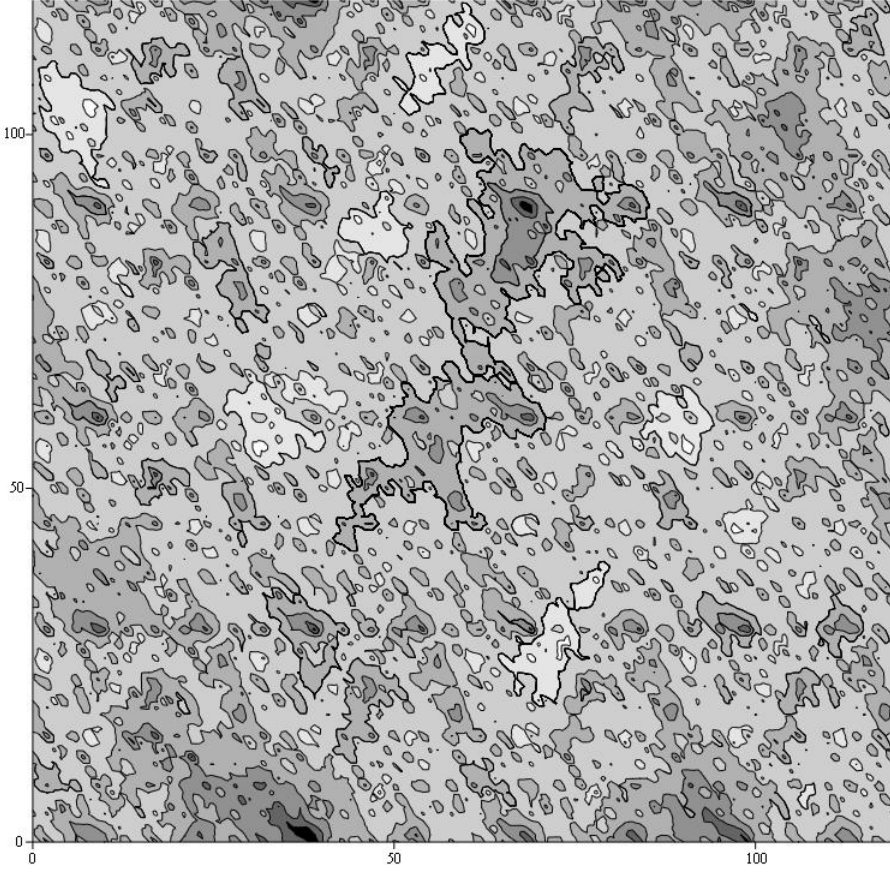


FIGURE 2.1. A simple example of the multiscale potential.

and assume that

$$(2.12) \quad \rho_{\min} \geq 2 \quad \text{and} \quad \rho_{\max} < \infty.$$

As an example, we have illustrated in Figure 2.1 the contour lines of

$$V_0^2(x, y) = \sum_{k=0}^2 U\left(\frac{x}{\rho^k}, \frac{y}{\rho^k}\right)$$

with $\rho = 4$ and

$$U(x, y) = \cos(x + \pi \sin(y) + 1)^2 \sin(\pi \cos(x) - 2y + 2) \cos(\pi \sin(x) + y).$$

3 Main Results

3.1 Quantitative Estimates of the Multiscale Effective Diffusivity

3.1.1 The Central Estimate

Our first objective is to control the minimal and maximal eigenvalues of $D(V_0^n)$. More precisely, letting I_d be the $d \times d$ identity matrix, we will prove the following:

THEOREM 3.1 *Under hypotheses (2.6) and (2.10) and $\rho_{\min}^\alpha \geq K_\alpha$, there exists a constant C depending only on d, α, K_α , and K_0 such that for all $n \geq 1$*

$$(3.1) \quad I_d e^{-n\epsilon} \prod_{k=0}^n \lambda_{\min}(D(U_k)) \leq D(V_0^n) \leq I_d e^{n\epsilon} \prod_{k=0}^n \lambda_{\max}(D(U_k))$$

where

$$(3.2) \quad \epsilon = C \rho_{\min}^{-\alpha/2}.$$

In particular, ϵ tends to 0 when $\rho_{\min} \rightarrow \infty$.

Remark 3.2. One can interpret this theorem as follows: $D(V_0^n)$ is bounded from above (respectively, from below) by the product of maximal (respectively, minimal) eigenvalues, which are the bounds given by reiterated homogenization under the assumption of complete separation of scales (i.e., $\rho \rightarrow \infty$) times an error term $e^{n\epsilon}$ created by the *interaction* or *overlap* between the different scales.

Remark 3.3. Originally the problem of estimating $D(V_0^n)$ arose in connection with applied sciences, and heuristic theories such as differential effective medium theory (DEM theory) have been developed for that purpose. This theory models a two-phase composite by incrementally adding inclusions of one phase to a background matrix of the other and then recomputing the new effective background material at each increment [14, 16, 32]. Bruggeman first proposed computing the conductivity of a two-component composite structure formed by successive substitutions [15, 17], and this has been generalized by Norris [33] to materials with more than two phases.

More recently Avellaneda [3] gave a rigorous interpretation of the equations obtained by DEM theories, showing this has that they are *homogeneous limit equations* with two very important features: complete separation of scales and “dilution of phases.” That is to say, each “phase” U_k is present at an infinite number of scales in a homogeneous way. Yet two different phases never interact because they always appear at scales whose ratio is ∞ . Moreover, the macroscopic influence of each phase is totally (but nonuniformly) diluted in the infinite number of scales at which it appears. In our context, complete separation of scales would mean that R_{k+1}/R_k grows sufficiently fast to ∞ , and “dilution of phases” would mean that $V_0^n = \sum_{k=0}^n U_k^n(x/R_k)$ with $U_k^n \rightarrow 0$ as $n \rightarrow \infty$. The rigorous tool used by Avellaneda to obtain this interpretation is reiterated homogenization [11].

A very recent work on this topic is the article by Jikov and Kozlov [27], who worked under the assumption of “dilution of phases” and fast separation between scales or, more precisely, under the condition that $\sum_{k=1}^{\infty} k(R_k/R_{k+1})^2 < \infty$. Jikov and Kozlov used the classical toolbox of asymptotic expansion, plugging well-chosen test functions into the cell problem. This method of asymptotic expansion is simply not available in our context.

Theorem 3.1 will be proven by induction on the number of scales. The basic step in this induction is estimate (3.4) on the effective diffusivity for a two-scale periodic medium.

Let $U, T \in C^\alpha(\mathbb{T}^d)$. Let us define for $R \in \mathbb{N}^*$, $S_R U \in C^\alpha(\mathbb{T}^d)$ by $S_R U(x) = U(Rx)$. We will need to estimate $D(S_R U + T)$, the effective diffusivity for a two-scale medium when R is a large integer. Let us define $D(U, T)$, the symmetric definite positive matrix given by

$$(3.3) \quad {}^t l D(U, T) l = \inf_{f \in C^\infty(\mathbb{T}_1^d)} \int_{\mathbb{T}_1^d} {}^t (l - \nabla f(x)) D(U) (l - \nabla f(x)) m_T(dx) \quad \text{for } l \in \mathbb{R}^d.$$

THEOREM 3.4 *Let $R \in \mathbb{N}^*$ and $U, T \in C^\alpha(\mathbb{T}^d)$. If $R^\alpha \geq \|T\|_\alpha$, then there exists a constant C depending only on d , $\text{Osc}(U)$, $\|U\|_\alpha$, and α such that*

$$(3.4) \quad e^{-\epsilon} D(U, T) \leq D(S_R U + T) \leq D(U, T) e^\epsilon$$

with $\epsilon = CR^{-\alpha/2}$.

Remark 3.5. Theorem 3.4 obviously implies that

$$(3.5) \quad D(U, T) = \lim_{R \rightarrow \infty} D(S_R U + T),$$

so that $D(U, T)$ should be interpreted as the effective diffusivity of the two-scale medium for a complete separation of scales. Naturally, $D(U, T)$ is also computable from an explicit cell problem; see (4.12).

Remark 3.6. The estimate given in Theorem 3.4 is stronger than needed for Theorem 3.1. It gives a control of $D(S_R U + T)$ in terms of $D(U, T)$ and not only of the minimal and maximal eigenvalues of $D(U)$ and $D(T)$. In fact, we will only use its Corollary 3.7 given below, which is deduced using the variational formulation (3.3).

COROLLARY 3.7 *Let $R \in \mathbb{N}^*$ and $U, T \in C^\alpha(\mathbb{T}^d)$. If $R^\alpha \geq \|T\|_\alpha$, then there exists a constant C depending only on d , $\text{Osc}(U)$, $\|U\|_\alpha$, and α such that*

$$(3.6) \quad \lambda_{\min}(D(U))D(T)e^{-\epsilon} \leq D(S_R U + T) \leq \lambda_{\max}(D(U))D(T)e^\epsilon$$

with $\epsilon = CR^{-\alpha/2}$.

Remark 3.8. We mentioned that Theorem 3.1 is proven by induction. This induction differs from the one used in reiterated homogenization or DEM theories by the fact that we homogenize on the larger scales first and add at each step a smaller scale.

Let us introduce the following upper and lower exponential rates.

DEFINITION 3.9

$$(3.7) \quad \lambda^+ = \limsup_{n \rightarrow \infty} \frac{1}{n} \ln \lambda_{\max}(D(V_0^n)),$$

$$(3.8) \quad \lambda^- = \liminf_{n \rightarrow \infty} \frac{1}{n} \ln \lambda_{\min}(D(V_0^n)).$$

Theorem 3.1 implies the exponential decay of $D(V_0^n)$ as follows:

COROLLARY 3.10 *Under the hypotheses (2.6), (2.10), and $\rho_{\min}^\alpha \geq K_\alpha$, one has (with ϵ given by (3.2)) for $n \geq 1$*

$$(3.9) \quad I_d e^{-n\epsilon} \lambda_{\min}^{n+1} \leq D(V_0^n) \leq I_d e^{n\epsilon} \lambda_{\max}^{n+1}$$

and

$$(3.10) \quad \lambda^+ \leq \ln \lambda_{\max} + \epsilon,$$

$$(3.11) \quad \lambda^- \geq \ln \lambda_{\min} - \epsilon.$$

In particular, if $\lambda_{\max} < 1$, then there exists a constant

$$\rho_0 = \left(1 + \frac{C_{d, K_0, K_\alpha, \alpha}}{-\ln \lambda_{\max}} \right)^{2/\alpha}$$

such that, for $\rho_{\min} \geq \rho_0$,

$$(3.12) \quad \lambda^+ < 0.$$

Thus one obtains the exponential decay of $D(V_0^n)$ only for a minimal separation between scales, i.e., ρ_{\min} greater than a constant ρ_0 characterized by the medium. It is natural to wonder whether this condition is necessary and what happens below this constant ρ_0 . We will partially answer that question for the simple case when the medium V is self-similar. We will see that it is possible to find models such that, for a certain value C of the separation parameter $\rho_{\min} = C$, $D(V_0^n)$ decays exponentially and for $\rho_{\min} = C + 1$, $D(V_0^n)$ stays bounded away from zero. This will be done using a link with large-deviation theory.

3.1.2 The Self-Similar Case

DEFINITION 3.11 The medium V is called *self-similar* if and only if $\forall n, U_n = U$ and $R_n = \rho^n$ with $\rho \in \mathbb{N}, \rho \geq 2$.

DEFINITION 3.12 For $U \in C^\alpha(\mathbb{T}^d)$ and $\rho \in \mathbb{N}/\{0, 1\}$, we denote by $p_\rho(U)$ the pressure associated to the shift $s_\rho(x) = \rho x$ on \mathbb{T}^d , i.e.,

$$(3.13) \quad p_\rho(U) = \sup_{\mu} \left(\int_{\mathbb{T}^d} U(x) d\mu(x) + h_\rho(\mu) \right)$$

where h_ρ is the Komogorov-Sinai entropy related to the shift s_ρ . We denote

$$(3.14) \quad \mathcal{P}_\rho(U) = p_\rho(U) - p_\rho(0) = p_\rho(U) - d \ln \rho.$$

We refer to [28, 38] for a reminder on the pressure. Let us observe that $\mathcal{P}_\rho(0)$ differs from the standard definition of the topological pressure by the constant $d \ln \rho$ so that $\mathcal{P}_\rho(0) = 0$.

We will relate in the self-similar case the exponential rates λ^+ and λ^- to pressures for the shift s_ρ and to large deviation at level 3 for i.i.d. random variables.

In the self-similar case we will write $\lambda^-(-U)$, the exponential rates associated to $D(-V_0^n)$. We write

$$(3.15) \quad Z(U) = -(\mathcal{P}_\rho(2U) + \mathcal{P}_\rho(-2U)).$$

THEOREM 3.13 *If the medium V is self-similar, then we have the following:*

(i) *If $d = 1$, then*

$$(3.16) \quad \lambda^+(U) = \lambda^-(U) = Z(U).$$

(ii) *If $d = 2$, then*

$$(3.17) \quad \lambda^+(U) + \lambda^-(-U) = Z(U).$$

Moreover, if there exists an isometry A of \mathbb{R}^d such that $U(Ax) = -U(x)$ and a reflection B such that $U(Bx) = U(x)$, then $\lambda^-(-U) = \lambda^-(U) = \lambda^+(U)$ so that

$$(3.18) \quad \lambda^+(U) = \lambda^-(U) = \frac{Z(U)}{2}.$$

(iii) *For any d ,*

$$(3.19) \quad Z(U) \leq \lambda^-(U) \leq \lambda^+(U) \leq 0.$$

Remark 3.14. Statement (3.16) is obtained from the explicit formula for $D(V_0^n)$ in $d = 1$; see [37].

Remark 3.15. To be able to use this theorem, it is obviously important to know when $Z(U)$ is strictly negative. A well-known and useful criterion can be stated as $Z(U) < 0$ if and only if U does not belong to the closure of the vector space spanned by cocycles, which can be shown to be equivalent to saying that

$$(3.20) \quad Z(U) < 0 \Leftrightarrow \limsup_{n \rightarrow \infty} \frac{1}{n} \left\| \sum_{k=0}^{n-1} \left(U(\rho^k x) - \int_{\mathbb{T}^d} U(x) dx \right) \right\|_{\infty} > 0.$$

We refer to [37] for the proof of the last statement.

EXAMPLE 3.16 Let $U(x) = \sin(x) - \sin(81x)$ in dimension one. In fact, (3.16) and (3.20) show that $\lambda^+(U) < 0$ as soon as $\rho \geq 82$. For $\rho \leq 81$, the situation is a bit surprising: $\lambda^+(U) < 0$ for $\rho \neq 3, 9, 81$. For these exceptional values $\lambda^+(U) = 0$; in fact, $D(V_0^n)$ remains bounded from below by a strictly positive constant.

This example shows that for a given potential U , even though the multiscale effective diffusivity $D(V_0^n)$ decays exponentially for ρ large enough, one can find isolated values of the scale parameter for which $D(V_0^n)$ remains bounded from below.

Remark 3.17. The symmetry hypotheses given in Theorem 3.13(ii) are only used to prove that for all n , $D(V_0^n) = D(-V_0^n)$ and $\lambda_{\max}(D(V_0^n)) = \lambda_{\min}(D(V_0^n))$ (see Proposition 4.9).

3.2 Subdiffusive Behavior from Homogenization on Infinitely Many Scales

Here we consider the diffusion process given by the Brownian motion in the potential

$$(3.21) \quad V = V_0^\infty = \sum_{k=0}^{\infty} U_k \left(\frac{x}{R_k} \right).$$

We assume in this section that the hypotheses (2.4), (2.5), (2.6), (2.8), (2.9), (2.10), and (2.12) hold. To start with, we will assume that

$$(3.22) \quad \alpha = 1 \quad \text{and that the potentials } U_k \text{ are uniformly } C^1.$$

In particular, V is well-defined and belongs to $C^1(\mathbb{R}^d)$ and $\|\nabla V\|_\infty < \infty$.

The diffusion process associated to the potential V is well-defined by the stochastic differential equation

$$(3.23) \quad dy_t = d\omega_t - \nabla V(y_t)dt.$$

We will show that the multiscale structure of V can lead to an anomalous slow behavior for the process y_t . To describe this subdiffusive phenomenon, we choose to compute the mean exit time from large balls; i.e., let

$$(3.24) \quad \tau(r) = \inf \{t > 0 : |y_t| \geq r\}.$$

We would like to show that $\mathbb{E}_x[\tau(r)]$ grows faster than quadratic in r when $r \rightarrow \infty$ uniformly in x . We cannot obtain such pointwise results in dimension $d > 1$. (See Section 3.2.1 for a discussion; the case $d = 1$ is treated in [37].) But we will start with averaged results on those mean exit times.

The fact that the homogenization results of Section 3.1 can be of some help in estimating the mean exit times is shown by the following lemma:

LEMMA 3.18 For $U \in C^\infty(\mathbb{T}_R^d)$ ($R > 0$) and letting \mathbb{E}^U represent the exit times associated to the diffusion generated by $L_U = \Delta/2 - \nabla U \nabla$, one has

$$(3.25) \quad \begin{aligned} \mathbb{E}_x^U[\tau(x, r)] &\leq C_2 \frac{r^2}{\lambda_{\max}(D(U))} + C_d e^{(9d+15)\text{Osc}(U)} R^2 \\ &\geq C_1 \frac{r^2}{\lambda_{\max}(D(U))} - C_d e^{(9d+15)\text{Osc}(U)} R^2. \end{aligned}$$

Let $m_{V,r}$ be the probability measure on the ball $B(0, r)$ given by

$$(3.26) \quad m_{V,r}(dx) = \frac{e^{-2V(x)} dx}{\int_{B(0,r)} e^{-2V(x)} dx}.$$

We will consider the mean exit time for the process started with initial distribution $m_{V,r}$, i.e.,

$$(3.27) \quad \mathbb{E}_{m_{V,r}}[\tau(r)] = \int_{B(0,r)} \mathbb{E}_x[\tau(r)] m_{V,r}(dx).$$

THEOREM 3.19 Under the hypothesis $\lambda_{\max} < 1$ there exists C_2 depending on $d, \lambda_{\max}, K_0, K_\alpha$, and α such that if $\rho_{\min} > C_2$, then

$$(3.28) \quad \liminf_{r \rightarrow \infty} \frac{\ln \mathbb{E}_{m_{V,r}}[\tau(r)]}{\ln r} > 2.$$

More precisely, there exists $C_3 > 0, C_4 > 0$, and $C_5 > 0$ such that for $r > C_3$,

$$(3.29) \quad \mathbb{E}_{m_{V,r}}[\tau(r)] = r^{2+\nu(r)}$$

with

$$(3.30) \quad \begin{aligned} 0 < C_4 &< \frac{\ln \frac{1}{\lambda_{\max}}}{\ln \rho_{\max}} \left(1 - \frac{C_5}{\ln \rho_{\min}} \right) - \frac{1}{\ln r} C_5 \\ &\leq \nu(r) \leq \frac{\ln \frac{1}{\lambda_{\min}}}{\ln \rho_{\min}} \left(1 + \frac{C_5}{\ln \rho_{\min}} \right) + \frac{1}{\ln r} C_5, \end{aligned}$$

where C_3 and C_5 depend on $(d, K_0, K_\alpha, \alpha)$ and C_4 on $(\lambda_{\max}, \rho_{\max})$.

The proof of this result relies heavily on Theorem 3.1. The idea is that $\mathbb{E}_{m_{V,r}}[\tau(r)]$ is close, when r is large, to $r^2/\lambda_{\max}(D(V_0^n))$ where n is roughly $\sup\{m \in \mathbb{N} : R_m \leq r\}$ so that the exponential decay of $D(V_0^n)$ gives the superquadratic behavior of $\mathbb{E}_{m_{V,r}}[\tau(r)]$, i.e., subdiffusivity.

Remark 3.20. The differentiability hypothesis (3.22), though convenient for defining the process y_t as a solution of the SDE (3.23), is, in fact, useless. The theorem is also meaningful and true with $0 < \alpha < 1$. See Section 4.2 for an explanation.

3.2.1 Pointwise Estimates on the Anomaly

Theorem 3.19 gives the anomalous behavior of the exit times with respect to the invariant measure of the diffusion, and it is desirable to seek pointwise estimates of this anomaly. The additional difficulty is in obtaining quantitative estimates on the stability of divergence form elliptic operators under a perturbation of their principal parts (see Conjecture 3.26). By *stability* we mean here the validity of Condition 3.21 below.

For $U \in C^1(\overline{B(z, r)})$, let \mathbb{E}^U represent the expectation associated to the diffusions generated by $L_U = \frac{1}{2}\Delta - \nabla U \nabla$.

CONDITION 3.21 (Stability Condition) There exists $\mu > 0$ such that for all $n \in \mathbb{N}$, all $z \in \mathbb{R}^d$, and all $r > 0$,

$$(3.31) \quad \frac{1}{\mu} e^{-\mu \text{Osc}_{B(z, r)}(V_{n+1}^\infty)} \inf_{x \in B(z, \frac{r}{2})} E_x^{V_0^n} [\tau(B(z, r))] \leq E_z^V [\tau(B(z, r))],$$

$$(3.32) \quad E_z^V [\tau(B(z, r))] \leq \mu e^{\mu \text{Osc}_{B(z, r)}(V_{n+1}^\infty)} \sup_{x \in B(z, r)} E_x^{V_0^n} [\tau(B(z, r))],$$

where $\text{Osc}_{B(z, r)}(U)$ stands for $\sup_{B(z, r)} U - \inf_{B(z, r)} U$.

Under Condition 3.21, we can obtain sharp pointwise estimates on the mean exit times.

THEOREM 3.22 *If V satisfies Condition 3.21 on stability, then there exists a constant C_6 depending on $(d, K_0, K_\alpha, \alpha, \mu, \lambda_{\max})$ such that for $\rho_{\min} > C_6$, one has for all $x \in \mathbb{R}^d$*

$$(3.33) \quad \liminf_{r \rightarrow \infty} \frac{\ln \mathbb{E}_x[\tau(B(x, r))]}{\ln r} > 2.$$

More precisely, there exists a function $\sigma(r)$ such that for $r > C_7$ one has

$$(3.34) \quad C_8 r^{2+\sigma(r)(1-\gamma)} \leq \mathbb{E}_x[\tau(B(x, r))] \leq C_9 r^{2+\sigma(r)(1+\gamma)}$$

with

$$(3.35) \quad \frac{\ln \frac{1}{\lambda_{\max}}}{\ln \rho_{\max}} \left(1 + \frac{C_3}{\ln \rho_{\min}}\right)^{-1} \leq \sigma(r) \leq \frac{\ln \frac{1}{\lambda_{\min}}}{\ln \rho_{\min}} \left(1 + \frac{C_4}{\ln \rho_{\min}}\right)$$

and $\gamma = C_5 K_0 / (\ln \rho_{\min}) < 0.5$, where the constants C_3, C_4, C_7, C_8 , and C_9 depend on $(d, K_0, K_\alpha, \alpha, \mu, \lambda_{\max})$ and C_5 on d .

Here $\tau(B(x, r))$ denotes the exit time from the ball $B(x, r)$.

Remark 3.23. In fact, $\sigma(r)$ can be described rather precisely. Let

$$(3.36) \quad \sigma(r, n) = -\frac{\ln \lambda_{\max} D(V_0^n)}{\ln r}.$$

Define $n_{\text{ef}}(r, C_1, C_2) = \sup\{n \geq 0 : e^{(n+1)C_1 K_0} R_n^2 \leq C_2 r^2\}$. Then there exists C_1 and C_2 depending only on d such that $\sigma(r)$ in Theorem 3.22 is $\sigma(r, n_{\text{ef}}(r, C_1, C_2))$.

Using the precise information of Theorem 3.22, we can estimate the tails of probability transitions for the process y_t (or the tail of the heat kernel for the operator L_V). We get a non-Gaussian upper bound similar to the (more precise) ones proven for fractal diffusions; see [9, 25].

THEOREM 3.24 *If V satisfies Condition 3.21 on stability, then for $\rho_{\min} > C_6$, $r > 0$, and*

$$(3.37) \quad C_{10}r \leq t \leq C_{11}r^{2+\sigma(r)(1-3\gamma)},$$

one has

$$(3.38) \quad \ln \mathbb{P}_x[|y_t - x| \geq r] \leq -C_{13} \frac{r^2}{t} \left(\frac{t}{r}\right)^{\nu(t/h)}$$

with $(C_{17} < 0.5 \ln \rho_{\min})$

$$(3.39) \quad 0 < \frac{\ln \frac{1}{\lambda_{\max}}}{\ln \rho_{\max}} \left(1 - \frac{C_{14}}{\ln \rho_{\min}}\right) \leq \nu(y) \leq \frac{\ln \frac{1}{\lambda_{\min}}}{\ln \rho_{\min}} \left(1 - \frac{C_{15}}{\ln \rho_{\min}}\right),$$

where C_6 , C_{13} , C_{14} , and C_{15} depend on $(d, K_0, K_\alpha, \alpha, \mu, \lambda_{\max})$; C_{17} on d, K_0 ; and C_{10} and C_{11} on $(d, K_0, K_\alpha, \alpha, \mu, \lambda_{\max}, \rho_{\min}, \rho_{\max})$.

Remark 3.25. The non-Gaussian structure of (3.38) is similar to the one obtained for diffusion processes in fractals. Indeed,

$$(3.40) \quad -C \frac{h^2}{t} \left(\frac{t}{h}\right)^\nu = -C \left(\frac{|x - y|^{d_w}}{t}\right)^{\frac{1}{d_w-1}}$$

with $d_w \sim 2 + \nu$.

Next, it has been shown in [37] that for $U \in L^\infty(\mathbb{T}_R^d)$, $\ln p^U(x, y, t)$ is roughly $-t(y - x)D^{-1}(U)(y - x)/t$ for $t > R|x - y|$ (homogenized behavior) where p^U is the heat kernel associated to L_U . Next, writing $n_{\text{ef}}(t/h) = \sup_n \{R_n \leq t/h\}$, the number of scales that can be considered as homogenized in the estimation of the heat kernel tail is obtained from a heuristic computation (which can be made rigorous in dimension one, see [37]) that for $C_{10}h \leq t \leq C_{11}h^{2+\mu}$,

$$(3.41) \quad \ln \mathbb{P}(y_t \geq h) \leq -C \frac{h^2}{t \lambda^{n_{\text{ef}}(t/h)}} - C \frac{h^2}{t} \left(\frac{t}{h}\right)^{-\frac{\ln \lambda}{\ln \rho}} \sim -C \left(\frac{|x - y|^{d_w}}{t}\right)^{\frac{1}{d_w-1}}$$

with $d_w \sim 2 - \frac{\ln \lambda}{\ln \rho}$. Equation (3.41) suggests that the origin of the anomalous shape of the heat kernel for the reflected Brownian motion on the Sierpinski carpet can be explained by a *perpetual homogenization* phenomenon and the formula linking the number of effective scales and the ratio t/h .

The condition $C_{10}h \leq t$ can be translated into ‘‘homogenization has started on at least the first scale’’ ($n_{\text{ef}} \geq 1$), and the second one $t \leq C_{11}h^{2+\mu}$ into ‘‘the heat kernel associated to L_V is far from its diagonal regime.’’ (One can have $h^2/t \ll 1$ before reaching that regime; this is explained by the slowdown of the diffusion.)

Naturally, the weak point of Theorems 3.22 and 3.24 is that checking condition 3.21 seems difficult. But we believe that in fact this condition is always true (we refer to [35, chap. 13]), since this condition is a consequence of the following conjecture; see [37, prop. 2.3].

Conjecture 3.26. There exists a constant C_d depending only on the dimension of the space such that for $\lambda \in C^\infty(\overline{B(0, 1)})$ such that $\lambda > 0$ on $\overline{B(0, 1)}$ and $\phi, \psi \in C^2(\overline{B(0, 1)})$ null on $\partial B(0, 1)$ and both subharmonic with respect to the operator $-\nabla(\lambda \nabla)$, one has

$$(3.42) \quad \int_{B(0,1)} \lambda(x) |\nabla \phi(x) \cdot \nabla \psi(x)| dx \leq C_d \int_{B(0,1)} \lambda(x) \nabla \phi(x) \cdot \nabla \psi(x) dx .$$

It is simple to see [37] that this conjecture is true in dimension one with $C_d = 3$. So proving Conjecture 3.26 would give the pointwise estimates of Theorem 3.22 and the tail estimate for the heat kernel in Theorem 3.24.

Remark 3.27. Here we have assumed that the U_k are uniformly C_1 , but let us observe that since Theorems 3.22 and 3.24 are robust in their dependence on α and K_α (one can choose $\alpha < 1$), one can build a process, with the assumption that the U_k are α -Hölder-continuous, whose mean exit times and heat kernel tail satisfy the estimates given in Theorems 3.22 and 3.24.

4 Proofs

4.1 Multiscale Homogenization with Bounded Ratios

4.1.1 Global Estimates of the Multiscale Effective Diffusivity: Theorem 3.1

The proof of Theorem 3.1 will follow from Corollary 3.7 by a simple induction. Let $n \in \mathbb{N} \setminus \{0, 1\}$, $p \in \mathbb{N}$, $1 \leq p \leq n$, and assume that

$$(4.1) \quad \begin{aligned} I_d e^{-(n-p)\epsilon(\rho_{\min})} \prod_{k=p}^n \lambda_{\min}(D(U_k)) &\leq D(V_p^n) \\ &\leq I_d e^{(n-p)\epsilon(\rho_{\min})} \prod_{k=p}^n \lambda_{\max}(D(U_k)) . \end{aligned}$$

We pass from the quantitative control on $D(V_p^n)$ to a control on $D(V_{p-1}^n)$ by choosing $U(x) = U_{p-1}(x)$, $T(x) = V_p^n(R_n x)$, and $R = R_n/R_{p-1}$ in Theorem 3.4 and observing that $\|T\|_\alpha/R^\alpha \leq (2^\alpha - 1)^{-1} K_\alpha/\rho_{\min}^\alpha$. This proves the induction and henceforth the theorem.

4.1.2 Quantitative Multiscale-Homogenization: Upper Bound in Theorem 3.4

4.1.2.1 We will use the notation introduced in Theorem 3.4. By the variational formula (2.2), $D(U)$ is continuous with respect to U in the L^∞ -norm; thus it is sufficient to prove Theorem 3.4 assuming that U and T are smooth.

First, let us prove that when homogenization takes place on two scales separated by a ratio R , the influence of a translation of the first one with respect to the second one on the global effective diffusivity can easily be controlled. In other words, for $y \in T_1^d$ and letting Θ_y represent the translation operator $T(x) \rightarrow \Theta_y T(x) = T(x + y)$, we obtain

LEMMA 4.1

$$(4.2) \quad e^{-4 \frac{\|T\|_\alpha}{R^\alpha}} D(S_R U + T) \leq D(S_R \Theta_y U + T) \leq e^{4 \frac{\|T\|_\alpha}{R^\alpha}} D(S_R U + T).$$

PROOF: The proof follows by observing that $S_R U + \Theta_y T = \Theta_{[Ry]/R}(S_R U + T) + \Theta_y T - \Theta_{[Ry]/R} T$ where $[Ry]$ stands for the vector with the integral parts of $(yR)_i$ as coordinates. Thus by the variational definition of the effective diffusivity,

$$(4.3) \quad D(S_R U + \Theta_y T) \leq e^{4\|\Theta_y T - \Theta_{[Ry]/R} T\|_\infty} D(\Theta_{[Ry]/R}(S_R U + T)),$$

and (4.2) follows by observing that the effective diffusivity is invariant under a translation of the medium: $D(\Theta_{[Ry]/R}(S_R U + T)) = D(S_R U + T)$. \square

Next we will obtain a quantitative control on $\int_{y \in \mathbb{T}^d} D(S_R U + \Theta_y T) dy$.

LEMMA 4.2 For $R > \|T\|_\alpha$

$$(4.4) \quad \int_{y \in \mathbb{T}^d} D(S_R U + \Theta_y T) dy \leq e^{22 \frac{\|T\|_\alpha}{R^\alpha}} \left(1 + C_d e^{C_d \text{Osc}(U)} \left(\frac{\|T\|_\alpha}{R^\alpha} \right)^{1/2} \right) D(U, T).$$

Let us observe that the combination of Lemma 4.2 with Lemma 4.1 gives the upper bound (3.4) in Theorem 3.4.

Let χ_l^U designate the solution of the cell problem associated to U . We note that for $l \in \mathbb{R}^d$, $L_U = 1/2\Delta - \nabla U \nabla$, $L_U \chi_l = -l \nabla U$, $\chi_l^U(0) = 0$, and

$$(4.5) \quad {}^t l D(U) l = \int_{\mathbb{T}^d} |l - \nabla \chi_l|^2 m_U(dx) = \int_{\mathbb{T}^d} {}^t (l - \nabla \chi_l) \cdot l m_U(dx).$$

Let $\chi^{D(U), T}$ be the \mathbb{T}^d -periodic solution of the following cell problem (which corresponds to a complete homogenization on the smaller scale): For $l \in \mathbb{S}^{d-1}$,

$$(4.6) \quad \nabla(e^{-2T} D(U)(l - \nabla \chi_l^{D(U), T})) = 0.$$

Write for $y \in \mathbb{T}^d$, $x \rightarrow \chi(x, y)$ the \mathbb{T}^d -periodic solution of the cell problem

$$(e^{2(U(Rx+y)+T(x))} \nabla e^{-2(U(Rx+y)+T(x))})(l - \nabla \chi(x, y)) = 0.$$

Let $l \in \mathbb{S}^{d-1}$. Using the formula associating the effective diffusivity to the solution of the cell problem and using that $l - \nabla_x \chi_l(x, y)$ is harmonic with respect to $L_{S_R U + \Theta_y T}$, one obtains

$$(4.7) \quad \int_{y \in \mathbb{T}^d} {}^t l D(S_R \Theta_y U + T) l dy = \int_{\mathbb{T}^d \times \mathbb{T}^d} (l - \nabla_x \chi_l(x, y)) \cdot l dx dy.$$

Writing the decomposition

$$(4.8) \quad \begin{aligned} l = & (I_d - \nabla \chi_{\cdot}^U(Rx + y))(l - \nabla \chi_l^{D(U),T}(x)) \\ & + \nabla \chi_{\cdot}^U(Rx + y)(\nabla \chi_l^{D(U),T}(x) - l) m_{U(Rx+y)+T(\cdot)}(dx) dy, \end{aligned}$$

we get that

$$(4.9) \quad \int_{y \in \mathbb{T}^d} {}^t l D(S_R \Theta_y U + T) l dy = I_1 - I_2,$$

with

$$(4.10) \quad \begin{aligned} I_1 = & \int_{\mathbb{T}^d \times \mathbb{T}^d} (l - \nabla_x \chi_l(x, y))(I_d - \nabla \chi_{\cdot}^U(Rx + y)) \\ & (l - \nabla \chi_l^{D(U),T}(x)) m_{U(Rx+y)+T(\cdot)}(dx) dy \end{aligned}$$

and

$$(4.11) \quad \begin{aligned} I_2 = & \int_{\mathbb{T}^d \times \mathbb{T}^d} (l - \nabla_x \chi_l(x, y)) \nabla \chi_{\cdot}^U(Rx + y) \\ & (\nabla \chi_l^{D(U),T}(x) - l) m_{U(Rx+y)+T(\cdot)}(dx) dy. \end{aligned}$$

It is easy to see that $\chi_{\cdot}^{D(U),T}$ is a minimizer in the variational formula (3.3) associated to $D(U, T)$, which is the effective diffusivity corresponding to two-scale homogenization on U and T with complete separation between the scales; i.e.,

$$(4.12) \quad D(U, T) = \int_{x \in \mathbb{T}^d} {}^t (I_d - \nabla \chi_{\cdot}^{D(U),T}(x)) D(U) (I_d - \nabla \chi_{\cdot}^{D(U),T}(x)) m_T(dx).$$

A simple use of the Cauchy-Schwarz inequality gives an upper bound on I_1 ,

$$(4.13) \quad \begin{aligned} I_1 \leq & \left(\int_{(x,y) \in (\mathbb{T}^d)^2} |l - \nabla_x \chi_l(x, y)|^2 m^{U(Rx+y)+T(x)}(dx) dy \right)^{1/2} \\ & \times \left(\int_{(x,y) \in (\mathbb{T}^d)^2} |(I_d - \nabla \chi_{\cdot}^U(Rx+y))(l - \nabla \chi_l^{D(U),T}(x))|^2 m^{U(Rx+y)+T(x)}(dx) dy \right)^{1/2}. \end{aligned}$$

Integrating first in y in the second term and using the formulae linking effective diffusivities and solutions of the cell problem, we obtain

$$(4.14) \quad I_1 \leq \left(\int_{y \in \mathbb{T}^d} {}^t l D(S_R \Theta_y U + T) l dy \right)^{1/2} \times ({}^t l D(U, T) l)^{1/2} e^{\|T\|_{\alpha}/R^{\alpha}}.$$

We now estimate I_2 . The following lemma together with (4.9) and (4.14) gives Lemma 4.2.

LEMMA 4.3

$$(4.15) \quad |I_2| \leq \left(\int_{y \in \mathbb{T}^d} {}^t l D(S_R \Theta_y U + T) l \, dy \right)^{1/2} \times ({}^t l D(U, T) l)^{1/2} \\ C_d e^{C_d \text{Osc}(U)} e^{4\|T\|_\alpha / R^\alpha} (e^{8\|T\|_\alpha / R^\alpha} - 1)^{1/2}.$$

The proof of this lemma relies heavily on the following elliptic-type estimate.

LEMMA 4.4

$$(4.16) \quad \|\chi_l^U\|_\infty \leq C_d \exp((3d + 2) \text{Osc}(U)) |l|.$$

This lemma is a consequence of [40, theorem 5.4, chap. 5] on elliptic equations with discontinuous coefficients (see also [41]). We give the proof of Lemma 4.4 for the sake of completeness in paragraph 4.1.2.2.

We will now prove Lemma 4.3. First we will estimate the distance between $\chi_l(x, y)$ and $\chi_l(x + y/R, 0)$ for $y \in [0, 1]^d$ in the H^1 -norm.

By the orthogonality property of the solution of the cell problem for $y \in [0, 1]^d$, we have the following:

$$(4.17) \quad \int_{x \in \mathbb{T}^d} \left| \nabla_x \chi_l \left(x + \frac{y}{R}, 0 \right) - \nabla_x \chi_l(x, y) \right|^2 \frac{e^{-2(U(Rx+y)+T(x))}}{\int_{\mathbb{T}^d} e^{-2(U(Rz+y)+T(z))} \, dz} \, dx \, dy \\ = \int_{x \in \mathbb{T}^d} \left| l - \nabla_x \chi_l \left(x + \frac{y}{R}, 0 \right) \right|^2 \frac{e^{-2(U(Rx+y)+T(x))}}{\int_{\mathbb{T}^d} e^{-2(U(Rz+y)+T(z))} \, dz} \, dx \, dy \\ - {}^t l D(S_R \Theta_y U + T) l \\ \leq {}^t l D(S_R U + T) l e^{4\|T\|_\alpha / R^\alpha} - {}^t l D(S_R \Theta_y U + T) l.$$

Thus by Lemma 4.1, for $y \in [0, 1]^d$,

$$(4.18) \quad \int_{x \in \mathbb{T}^d} \left| \nabla_x \chi_l \left(x + \frac{y}{R}, 0 \right) - \nabla_x \chi_l(x, y) \right|^2 m_{U(R \cdot + y) + T(\cdot)}(dx) \, dy \leq \\ {}^t l D(S_R \Theta_y U + T) l (e^{8\|T\|_\alpha / R^\alpha} - 1).$$

Let us introduce

$$(4.19) \quad I_3 = \int_{(x,y) \in \mathbb{T}^d \times [0,1]^d} \left(l - \nabla_x \chi_l \left(x + \frac{y}{R}, 0 \right) \right) \nabla \chi_l^U(Rx + y) (\nabla \chi_l^{D(U),T}(x) - l) \frac{e^{-2(U(Rx+y)+T(x+\frac{y}{R}))}}{\int_{\mathbb{T}^d} e^{-2U(z)} dz \int_{\mathbb{T}^d} e^{-2T(z)} dz} dx dy ;$$

one then has

$$(4.20) \quad |I_2 - I_3| \leq \left(\int_{y \in \mathbb{T}^d} {}^t l D(S_R \Theta_y U + T) l dy \right)^{1/2} \times ({}^t l D(U, T) l)^{1/2} 6e^{4\|T\|_\alpha/R^\alpha + \text{Osc}(U)} (e^{8\|T\|_\alpha/R^\alpha} - 1)^{1/2} .$$

This can be seen by using (4.18), (4.11), the Cauchy-Schwarz inequality (the computation is similar to the one in (4.13)), and the Voigt-Reiss inequality ($D(U) \geq e^{-2\text{Osc}(U)}$), and noticing that for $y \in [0, 1]^d$, $|T(x + y/R) - T(x)| \leq \|T\|_\alpha/R^\alpha$.

We now want to estimate I_3 . Noting that

$$\begin{aligned} \nabla_y \left(e^{-2(U(Rx+y)+T(x+\frac{y}{R}))} \left(l - \nabla_x \chi_l \left(x + \frac{y}{R}, 0 \right) \right) \right) &= 0, \\ \nabla \chi_l^U(Rx + y) &= \nabla_y \chi_l^U(Rx + y), \end{aligned}$$

and integrating by parts in y , one obtains

$$\begin{aligned} I_3 &= \sum_{i=1}^d \int_{\substack{x \in \mathbb{T}^d \\ y^i \in \partial^i([0,1]^d)}} \left(e^{-2T(x+\frac{y^i+e_i}{R})} \left(l - \nabla_x \chi_l \left(x + \frac{y^i+e_i}{R}, 0 \right) \right) \right. \\ &\quad \left. - e^{-2T(x+\frac{y^i}{R})} \left(l - \nabla_x \chi_l \left(x + \frac{y^i}{R}, 0 \right) \right) \right) \cdot e_i \chi_l^U(Rx + y^i) \\ &\quad (\nabla \chi_l^{D(U),T}(x) - l) \frac{e^{-2U(Rx+y^i)}}{\int_{\mathbb{T}^d} e^{-2U(z)} dz \int_{\mathbb{T}^d} e^{-2T(z)} dz} dx dy^i, \end{aligned}$$

where we have used the notation

$$\partial^i([0, 1]^d) = \{x \in [0, 1]^d : x_i = 0\} .$$

Let us introduce

$$(4.21) \quad I_4 = \sum_{i=1}^d \int_{\substack{x \in \mathbb{T}^d \\ y^i \in \partial^i([0,1]^d)}} \left(-\nabla_x \chi_l \left(x + \frac{y^i + e_i}{R}, 0 \right) + \nabla_x \chi_l \left(x + \frac{y^i}{R}, 0 \right) \right) \cdot e_i \\ \chi_l^U(Rx + y^i) (\nabla \chi_l^{D(U), T}(x) - l) e^{-2T(x+y^i/R)} \frac{e^{-2U(Rx+y^i)}}{\int_{\mathbb{T}^d} e^{-2U(z)} dz \int_{\mathbb{T}^d} e^{-2T(z)} dz} dx dy^i.$$

It is easy to obtain

$$(4.22) \quad |I_4 - I_3| \leq d e^{3 \text{Osc}(U)} e^{2\|T\|_\alpha/R^\alpha} (e^{2\|T\|_\alpha/R^\alpha} - 1) ({}^t l D(S_R U + T) l)^{1/2} \\ \|\chi_l^U\|_\infty ({}^t l D(U, T) l)^{1/2}.$$

We will now establish the fact that although $\chi_l(x, 0)$ is not periodic on $R^{-1}\mathbb{T}^d$, the distance (with respect to the natural H^1 -norm) between the solution of the cell problem $\chi_l(x, 0)$ and its translation $\chi_l(x + e_k/R, 0)$ along the axis of the torus $R^{-1}\mathbb{T}^d$ is small. This is due to the presence of a fast period $R^{-1}\mathbb{T}^d$ in the decomposition $V = S_R U + T$.

Using the standard property of the solution of the cell problem, one obtains

$$(4.23) \quad \int_{\mathbb{T}^d} \left| \nabla \chi_l \left(x + \frac{e_k}{R}, 0 \right) - \nabla \chi_l(x, 0) \right|^2 m_{S_R U + T}(dx) \\ = \int_{\mathbb{T}^d} \left| l - \nabla \chi_l(x, 0) - \nabla \chi_l \left(x + \frac{e_k}{R}, 0 \right) + \nabla \chi_l(x, 0) \right|^2 \\ m_{S_R U + T}(dx) - {}^t l D(S_R U + T) l \\ \leq e^{4\frac{\|T\|_\alpha}{R^\alpha}} \int_{\mathbb{T}^d} \left| l - \nabla \chi_l \left(x + \frac{e_k}{R}, 0 \right) \right|^2 \\ m_{\ominus \frac{e_k}{R}(S_R U + T)}(dx) - {}^t l D(S_R U + T) l,$$

which leads to

$$(4.24) \quad \int_{\mathbb{T}^d} \left| \nabla \chi_l \left(x + \frac{e_k}{R}, 0 \right) - \nabla \chi_l(x, 0) \right|^2 m^{S_R U + T}(dx) \leq \\ {}^t l D(S_R U + T) l (e^{4\|T\|_\alpha/R^\alpha} - 1).$$

By combining this inequality with definition (4.21) of I_4 , we can use the Cauchy-Schwarz inequality to prove that

$$(4.25) \quad |I_4| \leq \sum_{i=1}^d \int_{y^i \in \partial^i(\{0,1\}^d)} \left(\int_{x \in \mathbb{T}^d} \left(\left(-\nabla_x \chi_l \left(x + \frac{y^i + e_i}{R}, 0 \right) + \nabla_x \chi_l \left(x + \frac{y^i}{R}, 0 \right) \right) \cdot e_i \right)^2 \frac{e^{-2U(Rx+y^i)-2T(x+y^i/R)}}{\int_{\mathbb{T}^d} e^{-2U(z)} dz \int_{\mathbb{T}^d} e^{-2T(z)} dz} dx \right)^{\frac{1}{2}} \left(\int_{x \in \mathbb{T}^d} (\chi^U(Rx+y^i)(\nabla \chi_l^{D(U),T}(x)-l))^2 \frac{e^{-2U(Rx+y^i)-2T(x+y^i/R)}}{\int_{\mathbb{T}^d} e^{-2U(z)} dz \int_{\mathbb{T}^d} e^{-2T(z)} dz} dx \right)^{\frac{1}{2}} dy^i .$$

Combining this with (4.24), one obtains

$$(4.26) \quad |I_4| \leq (e^{8\|T\|_\alpha/R^\alpha} - 1)^{1/2} ({}^t l D(S_R U + T) l)^{1/2} d \|\chi^U\|_\infty e^{3 \text{Osc}(U)} ({}^t l D(U, T) l)^{1/2} e^{2\|T\|_\alpha/R^\alpha} .$$

Using Lemma 4.16 to estimate $\|\chi^U\|_\infty$ in (4.26) and combining (4.22) and (4.20), one obtains (4.15) and Lemma 4.3, which proves the upper bound of Theorem 3.4.

4.1.2.2 The purpose of this section is to prove estimate (4.16). First we will recall a theorem concerning elliptic equations with discontinuous coefficients by G. Stampacchia. Its proof in a more general form can be found in [41, theorem 5.4, chap. 5] (see also [40]).

Let us consider the operator (in the weak sense) $L = \nabla(A\nabla)$ defined on some open set $\Omega \subset \mathbb{R}^d$ (for $d \geq 3$) with smooth boundary $\partial\Omega$. A is a $d \times d$ matrix with bounded coefficients in $L^\infty(\Omega)$ such that, for all $\xi \in \mathbb{R}^d$, $\lambda|\xi|^2 \leq {}^t \xi A \xi$, and for all i, j , $|A_{ij}| \leq M$ for some positive constant $0 < \lambda, M < \infty$.

Let $p > d \geq 3$. For $1 \leq i \leq d$, let $f_i \in L^p(\Omega)$. If $\chi \in H_{\text{loc}}^1(\Omega)$ is a local (weak) solution of the equation

$$(4.27) \quad \nabla(A\nabla\chi) = - \sum_{i=1}^d \partial_i f_i ,$$

then χ is in $L^\infty(\Omega)$. Moreover, if $B(x_0, R) \subset \Omega$ we have the following quantitative control:

THEOREM 4.5 *The solution of (4.27) satisfies the following inequality (in the essential supremum sense with $\Omega(x_0, R) = \Omega \cap B(x_0, R)$):*

$$(4.28) \quad \max_{\Omega(x_0, \frac{R}{2})} |\chi| \leq K \left[\left\{ \frac{1}{R^d} \int_{\Omega(x_0, R)} \|\chi\|^2 \right\}^{1/2} + \sum_{i=1}^d \|f_i\|_{L^p(\Omega(x_0, R))} \frac{R^{1-d/p}}{\lambda} \right]$$

with $K = C_d(M/\lambda)^{3d/2}$.

The explicit dependence of the constants in M and λ have been obtained by following the proof of G. Stampacchia [41]. We will now prove (4.16) for $d \geq 3$. (For $d = 1$, this estimate is trivial; for $d = 2$, it is sufficient to consider $U(x_1, x_2)$ as a function on T_1^3 to obtain the result.) χ_l satisfies

$$\nabla(\exp(-2U)\nabla\chi_l) = l \cdot \nabla \exp(-2U);$$

then by Theorem 4.5, for $x_0 \in [0, 1]^d$,

$$\max_{B(x_0, \frac{1}{2})} |\chi_l| \leq C_d \exp(3 \operatorname{Osc}(U)d) \left[\left(\int_{B(x_0, 1)} |\chi_l|^2 \right)^{1/2} + |l| \exp(2 \operatorname{Osc}(U)) \right].$$

Now by periodicity

$$\int_{B(x_0, 1)} |\chi_l|^2 dx \leq \int_{\mathbb{T}^d} |\chi_l|^2 dx,$$

and by the Poincaré inequality (we assume $\int_{\mathbb{T}^d} \chi_l(x) dx = 0$)

$$\int_{\mathbb{T}^d} |\chi_l|^2 dx \leq C_d \int_{\mathbb{T}^d} |\nabla \chi_l|^2 dx.$$

Thus

$$\int_{B(x_0, 1)} |\chi_l|^2 dx \leq C_d \exp(2 \operatorname{Osc}(U)) \int_{\mathbb{T}^d} |\nabla \chi_l|^2 m_U(dx),$$

and since

$$\int_{\mathbb{T}^d} |l - \nabla \chi_l|^2 m_U(dx) = l^2 - \int_{\mathbb{T}^d} |\nabla \chi_l|^2 m_U(dx),$$

one has

$$\int_{\mathbb{T}^d} |\nabla \chi_l|^2 m_U(dx) \leq l^2,$$

and the bound on $\|\chi_l\|_\infty$ is proven.

4.1.3 Quantitative Multiscale Homogenization: Lower Bound in Theorem 3.4

4.1.3.1 As for the upper bound, it is sufficient to prove Theorem 3.4 assuming that U and T are smooth, and we will use the notation introduced in paragraph 4.1.2.1. We will prove the following lemma:

LEMMA 4.6 *If $R \geq \|T\|_\alpha$, then for $\xi \in \mathbb{S}^{d-1}$,*

$$(4.29) \quad \int_{\mathbb{T}^d} {}^t \xi D(S_R U + \Theta_y T)^{-1} \xi dy \leq (1 + C_{d, \operatorname{Osc}(U), \|U\|_\alpha, \alpha, \|T\|_\alpha} R^{-\alpha/2}) {}^t \xi D(U, T)^{-1} \xi.$$

This lemma with Lemma 4.1 gives the lower bound in Theorem 3.4. We now prove Lemma 4.6. Let us introduce

$$(4.30) \quad P(x, y) = I_d - \frac{\exp(-2(S_R \Theta_y U + T))}{\int_{\mathbb{T}^d} \exp(-2(S_R \Theta_y U + T)(x)) dx} \\ (I_d - \nabla \chi(x, y)) D(S_R \Theta_y U + T)^{-1},$$

$$(4.31) \quad P^U(x) = I_d - \frac{\exp(-2U(x))}{\int_{\mathbb{T}^d} \exp(-2U(x)) dx} (I_d - \nabla \chi^U(x)) D(U)^{-1},$$

and

$$(4.32) \quad P^{D(U), T}(x) = I_d - \frac{e^{-2T(x)}}{\int_{\mathbb{T}^d} e^{-2T(x)} dx} D(U) (I_d - \nabla \chi^{D(U), T}(x)) D(U, T)^{-1}.$$

We note that $P(x, y)$ minimizes the well-known variational formula associated to $D(S_R \Theta_y U + T)^{-1}$, which is why it will play the same role for the lower bound in Theorem 3.4, the role played by the gradient of the solution of the cell problem $\nabla \chi(x, y)$ for the upper bound. More precisely, for $\xi \in \mathbb{S}^{d-1}$, one obtains as in the proof of the upper bound (by decomposing ξ here)

$$(4.33) \quad \int_{y \in \mathbb{T}^d} {}^t \xi D(S_R \Theta_y U + T)^{-1} \xi dy \\ = \int_{(x, y) \in (\mathbb{T}^d)^2} \left(\int_{\mathbb{T}^d} e^{-2(S_R \Theta_y U + T)(z)} dz \right) e^{2(S_R \Theta_y U + T)(x) {}^t \xi} \\ {}^t (I_d - P(x, y)) \xi dx dy \\ \leq e^{2\|T\|_\alpha / R^\alpha} (I_1 + I_2)$$

with

$$(4.34) \quad I_1 = \int_{\mathbb{T}^d} e^{-2U(z)} dz \int_{\mathbb{T}^d} e^{-2T(z)} dz \int_{(x, y) \in (\mathbb{T}^d)^2} e^{2(S_R \Theta_y U + T)(x) {}^t \xi} {}^t (I_d - P(x, y)) \\ (I_d - P^U(Rx + y)) (I_d - P^{D(U), T}(x)) \xi dx dy$$

and

$$(4.35) \quad I_2 = \int_{\mathbb{T}^d} e^{-2U(z)} dz \int_{\mathbb{T}^d} e^{-2T(z)} dz \int_{(x, y) \in (\mathbb{T}^d)^2} e^{2(S_R \Theta_y U + T)(x) {}^t \xi} {}^t (I_d - P(x, y)) \\ P^U(Rx + y) (I_d - P^{D(U), T}(x)) \xi dx dy.$$

As for the upper bound, using the Cauchy-Schwarz inequality for the integration in x and y , and using

$$(4.36) \quad {}^t\xi D(U, T)^{-1}\xi = \int_{(x,y) \in (\mathbb{T}^d)^2} e^{2(S_R \Theta_y U + T)(x)} ((I_d - P^U(Rx + y))(I_d - P^{D(U), T}(x))\xi)^2 dx dy,$$

one obtains that

$$(4.37) \quad |I_1| \leq e^{\|T\|_\alpha/R^\alpha} \left(\int_{y \in \mathbb{T}^d} {}^t\xi D(S_R \Theta_y U + T)^{-1}\xi dy \right)^{1/2} ({}^t\xi D(U, T)^{-1}\xi)^{1/2}.$$

Thus I_2 will be an error term, and we will prove the following:

LEMMA 4.7

$$(4.38) \quad |I_2| \leq \left(\int_{y \in \mathbb{T}^d} {}^t\xi D(S_R \Theta_y U + T)^{-1}\xi dy \right)^{1/2} ({}^t\xi D(U, T)^{-1}\xi)^{1/2} C_{d, \text{Osc}(U), \|U\|_\alpha, \alpha} e^{4\|T\|_\alpha/R^\alpha} (e^{8\|T\|_\alpha/R^\alpha} - 1)^{1/2}.$$

Let us observe that combining the estimate (4.38) of Lemma 4.7 with (4.37) and (4.33) proves Lemma 4.6.

We will now prove Lemma 4.7. As done in the proof of the upper bound, it is easy to show that, with

$$(4.39) \quad I_3 = \int_{\mathbb{T}^d} e^{-2U(z)} dz \int_{\mathbb{T}^d} e^{-2T(z)} dz \int_{(x,y) \in \mathbb{T}^d \times [0,1]^d} e^{2(U(Rx+y)+T(x+y/R))} {}^t\xi^t \left(I_d - P \left(x + \frac{y}{R}, 0 \right) \right) P^U(Rx + y) (I_d - P^{D(U), T}(x))\xi dx dy,$$

one has

$$(4.40) \quad |I_3 - I_2| \leq 6e^{\text{Osc}(U)} e^{4\|T\|_\alpha/R^\alpha} (e^{8\|T\|_\alpha/R^\alpha} - 1)^{1/2} ({}^t\xi D(S_R U + T)\xi)^{1/2} ({}^t\xi D(U, T)\xi)^{1/2}.$$

The following lemma will be proven in 4.1.3.2:

LEMMA 4.8 $\exists d \times d \times d$ tensors H_{ijm}^U such that $H_{ijm}^U = -H_{jim}^U \in C^\infty(\mathbb{T}^d)$,

$$(4.41) \quad P_{im}^U = \sum_{j=1}^d \partial_j H_{ijm}^U \quad \text{and} \quad \|H_{ijm}^U\|_\infty \leq C_{d, \text{Osc}(U), \|U\|_\alpha, \alpha}.$$

Combining (4.41) with the explicit formula (4.30) for P , one obtains

$$(4.42) \quad I_3 = \frac{\int_{\mathbb{T}^d} e^{-2U(z)} dz \int_{\mathbb{T}^d} e^{-2T(z)} dz}{\int_{\mathbb{T}^d} \exp(-2(S_R U + T)(z)) dz} \int_{(x,y) \in \mathbb{T}^d \times [0,1]^d} \sum_{i,j,k=1}^d ({}^t \xi^t (I_d - P^{D(U),T}(x)))_i \partial_k H_{i,k,j}^U(Rx + y) \left(\left(I_d - \nabla \chi \left(x + \frac{y}{R} \right) \right) D(S_R U + T)^{-1} \xi \right)_j dx dy$$

Thus, using the same notation as in (4.21) and integrating by parts in y , one obtains

$$(4.43) \quad I_3 = \frac{\int_{\mathbb{T}^d} e^{-2U(z)} dz \int_{\mathbb{T}^d} e^{-2T(z)} dz}{\int_{\mathbb{T}^d} \exp(-2(S_R U + T)(z)) dz} \sum_{i,j,k=1}^d \int_{(x,y^k) \in \mathbb{T}^d \times \partial^k([0,1]^d)} ({}^t \xi^t (I_d - P^{D(U),T}(x)))_i H_{i,k,j}^U(Rx + y^k) \left(\left(\nabla \chi \left(x + \frac{y^k}{R} \right) - \nabla \chi \left(x + \frac{y^k + e_k}{R} \right) \right) D(S_R U + T)^{-1} \xi \right)_j dx dy^k,$$

which, using the Cauchy-Schwarz inequality, leads to

$$(4.44) \quad |I_3| \leq C_d e^{2\text{Osc}(U)} \sup_{ijk} \|H_{i,k,j}^U\|_\infty \sum_{k=1}^d \int_{y^k \in \partial^k([0,1]^d)} \left(\int_{x \in \mathbb{T}^d} ((I_d - P^{D(U),T}(x)) \xi)^2 e^{2(U(Rx+y^k)+T(x+y^k/R))} dx \right)^{1/2} \left(\int_{x \in \mathbb{T}^d} \left(\left(\nabla \chi \left(x + \frac{y^k}{R} \right) - \nabla \chi \left(x + \frac{y^k + e_k}{R} \right) \right) D(S_R U + T)^{-1} \xi \right)^2 e^{-2(U(Rx+y^k)+T(x+y^k/R))} dx \right)^{1/2} dy^k.$$

Using the bounds (4.41) and the equation (4.24) to control the natural H^1 -distance between the solution of the cell problem $\chi(x + y^k/R)$ and its translation by e_k/R , one obtains

$$(4.45) \quad |I_3| \leq (\xi D(S_R U + T)^{-1} \xi)^{1/2} ({}^t \xi D(U, T)^{-1} \xi)^{1/2} C_{d,\text{Osc}(U), \|U\|_\alpha, \alpha} e^{4\|T\|_\alpha/R^\alpha} (e^{4\|T\|_\alpha/R^\alpha} - 1)^{1/2}.$$

Combining (4.45) and (4.40), one obtains (4.38), which proves Lemma 4.38.

4.1.3.2 In this paragraph we prove Lemma 4.8. Since $P_{..m}^U$ is a divergence-free vector with mean 0 with respect to Lebesgue measure for each $m \in \{1, 2, \dots, d\}$, by [27, prop. 4.1] there exist skew-symmetric \mathbb{T}^d -periodic smooth matrices $H_{ij1}^U, H_{ij2}^U, \dots, H_{ijd}^U$ ($H_{ijm}^U = -H_{jim}^U$) such that for all m

$$(4.46) \quad P_{im}^U = \sum_{j=1}^d \partial_j H_{ijm}^U.$$

Moreover, writing

$$(4.47) \quad P_{.m}^U = \sum_{k \neq 0} P_{.m}^k e^{2i\pi(k.x)},$$

the Fourier series expansion of P^U , one has (see [27, prop. 4.1])

$$(4.48) \quad H_{njm}^U = \frac{1}{2i\pi} \sum_{k \neq 0} \frac{P_{nm}^k k_j - P_{jm}^k k_n}{k^2} e^{2i\pi(k.x)}.$$

Let us observe that

$$(4.49) \quad H_{njm}^U = B_{nm}^j - B_{jm}^n$$

where B_{nm}^j are the smooth \mathbb{T}^d -periodic solutions of $\Delta B_{nm}^j = \partial_j P_{nm}^U$. By Theorem 4.5 [40, theorem 5.4, chap. 5], if B_{nm} is chosen so that $\int_{\mathbb{T}^d} B_{nm}(x) dx = 0$, then $\|B_{nm}^j\|_\infty \leq C_d \|P_{nm}^U\|_\infty$. Now using theorem 1.1 of [31], it is easy to obtain that $\|\nabla \chi_l^U\|_\infty \leq C_{d, \text{Osc}(U), \|U\|_{\alpha, \alpha}}$; combining this with (4.31), one obtains

$$(4.50) \quad \|B_{nm}^j\|_\infty \leq C_{d, \text{Osc}(U), \|U\|_{\alpha, \alpha}},$$

which leads to (4.41) by equation (4.49).

4.1.4 Explicit Formulae of Effective Diffusivities from Level-3 Large Deviations; Proof of Theorem 3.13

Equation (3.19) follows from the Voigt-Reiss inequality: For $U \in L^\infty(\mathbb{T}^d)$,

$$(4.51) \quad D(U) \geq I_d \left(\int_{\mathbb{T}^d} e^{2U(x)} \int_{\mathbb{T}^d} e^{-2U(x)} \right)^{-1},$$

and the fact that, if $U \in C^\alpha(\mathbb{T}^d)$, then by Varadhan's lemma and the level-3 large deviation associated to the shift s_ρ ,

$$(4.52) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left(\int_{\mathbb{T}^d} e^{\sum_{k=0}^{n-1} U(s_\rho^k x)} dx \right) = \mathcal{P}_\rho(U).$$

We refer to [37] for a more detailed proof of this statement.

In higher dimensions, when the medium is self-similar, one can use the criterion (3.20) associated with the equation (3.19) to characterize ratios for which $D(V_0^n)$

does not converge to 0 at an exponential rate. Equation (3.17), i.e., the extension of the result (3.16) to dimension 2, is done by observing the following proposition:

PROPOSITION 4.9 *For $d = 2$ one has*

$$(4.53) \quad \lambda_{\max}(D(U))\lambda_{\min}(D(-U)) = \lambda_{\min}(D(U))\lambda_{\max}(D(-U)) \\ = \frac{1}{\int_{T_1^d} \exp(2U)dx \int_{T_1^d} \exp(-2U)dx},$$

from which one deduces that if $D(U) = D(-U)$, then

$$(4.54) \quad \lambda_{\min}(D(U))\lambda_{\max}(D(U)) = \left(\int_{T_1^d} \exp(2U)dx \int_{T_1^d} \exp(-2U)dx \right)^{-1}.$$

Let us observe that the assumption $D(U) = D(-U)$ is satisfied if, for instance, $-U_n(x) = U_n(-x)$ or $-U_n(x) = U_n(Ax)$ where A is an isometry of \mathbb{R}^d . The existence of a reflection B such that $U(Bx) = U(x)$ ensures that $\lambda_{\min}(D(U)) = \lambda_{\max}(D(U))$. Thus these symmetry hypotheses combined with (3.17) ensure the validity of (3.18).

It would be interesting to extend (3.17) of Theorem 3.13 to more general cases and higher dimensions. Indeed, Proposition 4.9 is deduced from Proposition 4.10 below, which establishes a strong geometrical link between cohomology and homogenization.

Let $\mathcal{F}_{\text{sol}} = \{p \in (C^\infty(T_1^d))^d : \text{div}(p) = 0 \text{ and } \int_{T_1^d} p \, dx = 0\}$ and $Q(U)$ be the positive definite symmetric matrix associated to the following variational problem. For $l \in \mathbb{S}^d$,

$$(4.55) \quad {}^t l Q(U) l = \inf_{p \in \mathcal{F}_{\text{sol}}} \frac{\int_{T_1^d} |l - p|^2 \exp(2U)dx}{\int_{T_1^d} \exp(2U)dx}.$$

Write in increasing order $\lambda(D(U))_i$ and decreasing order $\lambda(Q(U))_i$, the eigenvalues of $D(U)$ and $Q(U)$, respectively.

PROPOSITION 4.10 *For all $i \in \{1, 2, \dots, d\}$*

$$(4.56) \quad \lambda(D(U))_i \lambda(Q(U))_i = \frac{1}{\int_{T_1^d} \exp(2U)dx \int_{T_1^d} \exp(-2U)dx}.$$

Now we will introduce a geometric interpretation of homogenization that will allow us to prove Proposition 4.9 and equation (3.17) of Theorem 3.13. Let $U \in C^\infty(\mathbb{T}^d)$. It is easy to obtain the following orthogonal decomposition:

$$(4.57) \quad H = (L^2(m_U))^d = H_{\text{pot}} \oplus H_{\text{sol}},$$

where H_{pot} and H_{sol} are the closures (with respect to the intrinsic norm $\|\cdot\|_H$) of the sets of C_{pot} and C_{sol} , the sets of smooth, \mathbb{T}^d -periodic, potential and solenoidal

vector fields, i.e., with $\mathcal{C} = (C^\infty(\mathbb{T}^d))^d$,

$$(4.58) \quad \mathcal{C}_{\text{pot}} = \{ \xi \in \mathcal{C} : \exists f \in C^\infty(\mathbb{T}^d) \text{ with } \xi = \nabla f \},$$

$$(4.59) \quad \mathcal{C}_{\text{sol}} = \{ \xi \in \mathcal{C} : \exists p \in \mathcal{C} \text{ with } \operatorname{div}(p) = 0 \\ \text{and } \xi = p \exp(2U) \int_{\mathbb{T}^d} e^{-2U(x)} dx \}.$$

Thus H is a real Hilbert space equipped with the scalar product

$$(\xi, \nu)_H = \int_{\mathbb{T}^d} \xi(x) \cdot \nu(x) m_U(dx),$$

and by the variational formulation (2.2), for $l \in \mathbb{R}^d$, ${}^t l D(U) l$ is the norm in H of the orthogonal projection of l on H_{sol} , and $l = \nabla \chi_l + \exp(2U) p_l$ is the orthogonal decomposition of l . Writing $\operatorname{dist}(l, H_{\text{pot}})$, the distance of l from H_{pot} in the intrinsic norm $\| \cdot \|_H$, let us observe that by the variational formulation (2.2) we have

$$(4.60) \quad \sqrt{{}^t l D(U) l} = \operatorname{dist}(l, H_{\text{pot}}).$$

Now by duality for all $\xi \in H$,

$$(4.61) \quad \operatorname{dist}(\xi, H_{\text{pot}}) = \sup_{\delta \in \mathcal{C}_{\text{sol}}} \frac{(\delta, \xi)_H}{\|\delta\|_H},$$

from which we deduce the following variational formula for the effective diffusivity by choosing $\xi = l \in \mathbb{R}^d$:

$$(4.62) \quad {}^t l D(U) l = \sup_{\substack{p \in \mathcal{C} \\ \operatorname{div}(p)=0}} \frac{(\int_{\mathbb{T}^d} l \cdot p dx)^2}{\int_{\mathbb{T}^d} p^2 \exp(2U) dx \int_{\mathbb{T}^d} \exp(-2U) dx}.$$

Note that (4.62) gives back Voigt-Reiss's inequality by choosing $p = l$.

Let $Q(U)$ be the positive definite, symmetric matrix given by the variational formula (4.55). Then the following proposition is a direct consequence of equation (4.62).

PROPOSITION 4.11 *For all $l \in \mathbb{S}^{d-1}$,*

$$(4.63) \quad {}^t l D(U) l = \frac{1}{\int_{\mathbb{T}^d} \exp(2U) dx \int_{\mathbb{T}^d} \exp(-2U) dx} \sup_{\xi \in \mathbb{S}^{d-1}} \frac{(l \cdot \xi)^2}{{}^t \xi Q(U) \xi}.$$

Choosing an orthonormal basis diagonalizing $Q(U)$, it is an easy exercise to use this proposition to establish a one-to-one correspondence between the eigenvalues of $Q(U)$ and $D(U)$ to obtain Proposition 4.10.

In dimension two, the Poincaré duality establishes a simple correspondence between $Q(U)$ and $D(-U)$.

PROPOSITION 4.12 *For $d = 2$, one has*

$$(4.64) \quad Q(U) = {}^t P D(-U) P,$$

where P stands for the rotation matrix

$$(4.65) \quad P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Indeed, by the Poincaré duality, one has $\mathcal{F}_{\text{sol}} = \{P\nabla f : f \in C^\infty(\mathbb{T}^d)\}$, and Proposition 4.12 follows from the definition of $Q(U)$. Proposition 4.9 is then a direct consequence of Proposition 4.12, and one deduces from (4.53) that if $D(U) = D(-U)$, then

$$(4.66) \quad \lambda_{\max}(D(U))\lambda_{\min}(D(U)) = \left(\int_{\mathbb{T}^d} \exp(2U) dx \int_{\mathbb{T}^d} \exp(-2U) dx \right) I_d,$$

which leads to (3.17) of Theorem 3.13 by [37, theorem 3.1].

4.2 Subdiffusive Behavior from Homogenization on Infinitely Many Scales

4.2.1 Anomalous Behavior of the Exit Times: Theorems 3.19 and 3.22

4.2.1.1 In this subsection we will prove the asymptotic anomalous behavior of the mean exit times $\mathbb{E}_x[\tau(0, r)]$, defined as weak solutions of $L_V f = -1$ with Dirichlet conditions on $\partial B(0, r)$. Here $U_n \in C^1(\mathbb{T}^d)$; nevertheless, we will assume first that those functions are smooth and prove quantitative anomalous estimates on $\mathbb{E}_x[\tau(0, r)]$ depending only on the values of $D(V_0^n)$, K_0 , and K_α . Then, using standard estimates on the Green functions associated to divergence form elliptic operators (see, for instance, [41]), it is easy to check that the exit times $\mathbb{E}_x[\tau(0, r)]$ are continuous with respect to a perturbation of V in the $L^\infty(B(0, r))$ -norm. Using the density of smooth functions on $\overline{B(0, r)}$ in the set of bounded functions, we will then deduce that our estimates are valid for $U_n \in C^\alpha(\mathbb{T}^d)$.

Thus we can see the exit times as those associated to the solution of (3.23) and take advantage of the Ito formula.

The central lemma of the proof is Lemma 3.18, which will be proven in paragraph 4.2.1.2.

Letting $m_U^r(dx) = e^{-2U(x)} dx \left(\int_{B(0,r)} e^{-2U(x)} dx \right)^{-1}$, we will prove in paragraph 4.2.1.3 that for $P \in C^\infty(\overline{B(0, r)})$,

$$(4.67) \quad \begin{aligned} & \int_{B(0,r)} \mathbb{E}_x^{U+P}[\tau(0, r)] m_{U+P}^r(dx) \\ & \leq e^{2\text{Osc}_{B(0,r)}(P)} \int_{B(0,r)} \mathbb{E}_x^U[\tau(0, r)] m_{U+P}^r(dx) \\ & \geq e^{-2\text{Osc}_{B(0,r)}(P)} \int_{B(0,r)} \mathbb{E}_x^U[\tau(B(0, r))] m_{U+P}^r(dx). \end{aligned}$$

We give here the outline of the proof (see [37] for $d = 1$). A perpetual homogenization process takes place over the infinite number of scales $0, 1, \dots, n, \dots$, and the idea is still to distinguish, when one tries to estimate (3.29), the smaller scales that have already been homogenized ($0, 1, \dots, n_{\text{ef}}$, called *effective scales*), the bigger scales that have not had a visible influence on the diffusion ($n_{\text{dri}}, \dots, \infty$, called *drift scales* because they will be replaced by a constant drift in the proof), and some intermediate scales that manifest the particular geometric structure of their associated potentials in the behavior of the diffusion ($n_{\text{ef}} + 1, \dots, n_{\text{dri}} - 1 = n_{\text{ef}} + n_{\text{per}}$, called *perturbation scales* because they will enter in the proof as a perturbation of the homogenization process over the smaller scales).

We will now use (3.25) and (4.67) to prove Theorem 3.19. For that purpose, we will first fix the number of scales that one can consider as *homogenized* (we write “ef” for *effective*)

$$(4.68) \quad n_{\text{ef}}(r) = \sup \left\{ n \geq 0 : e^{(n+1)(9d+15)K_0} R_n^2 \leq \frac{C_1}{8C_d r^2} \right\} < \infty,$$

where C^1 and C_d are the constants appearing in the left term of (3.25). Next we fix the number of scales that will enter in the computation as a perturbation of the homogenization process (we write “per” for *perturbation*)

$$(4.69) \quad n_{\text{per}}(r) = \inf \{ n \geq 0 : R_{n+1} \geq r \} - n_{\text{ef}}(r).$$

For $r > C_{d, K_0, \rho_{\max}}$, $n_{\text{ef}}(r)$ and $n_{\text{per}}(r)$ are well defined. Let us choose $U = V_0^{n_{\text{ef}}(r)}$ and $P = V_{n_{\text{ef}}(r)+1}^\infty$ in (4.67); we will bound from above $\text{Osc}_{B(0,r)}(V_{n_{\text{ef}}(r)+1}^\infty)$ by

$$\text{Osc}(V_{n_{\text{ef}}(r)+1}^{n_{\text{ef}}(r)+n_{\text{per}}(r)}) + \|V_{n_{\text{ef}}(r)+n_{\text{per}}(r)+1}^\infty\|_\alpha r^\alpha.$$

For the lower bound of (4.67), when $x \in B(0, r/2)$, we will bound $\mathbb{E}_x^U[\tau(0, r)]$ from below by $\mathbb{E}_x^U[\tau(x, r/2)]$, and for the upper bound when $x \in B(0, r)$, we will bound it from above by $\mathbb{E}_x^U[\tau(x, 2r)]$. Then using (3.25) to control those exit times, one obtains

$$(4.70) \quad \begin{aligned} & \int_{B(0,r)} \mathbb{E}_x^V[\tau(B(0, r))] m_V^{B(0,r)}(dx) \\ & \leq C_d e^{C_{K_\alpha, \alpha} + 8n_{\text{per}}(r)K_0} \frac{r^2}{\lambda_{\max}(D(V^{0, n_{\text{ef}}(r)}))} \\ & \geq C_d e^{-C_{K_\alpha, \alpha} - 8n_{\text{per}}(r)K_0} \frac{r^2}{\lambda_{\max}(D(V^{0, n_{\text{ef}}(r)}))}. \end{aligned}$$

Theorem 3.19 follows directly from the last inequalities by using the estimates (3.9) on $D(V_0^n)$ and (2.11) on R_n and observing that

$$(4.71) \quad n_{\text{per}}(r) \leq \inf \left\{ m \geq 0 : \frac{R_{m+n_{\text{ef}}(r)+1}}{R_{n_{\text{ef}}(r)+1}} \geq C_d e^{(n_{\text{ef}}(r)+2)(9d+15)K_0/2} \right\}.$$

The proof of Theorem 3.22 follows similar lines, the stability result (4.67) being replaced by Condition 3.21 on stability.

4.2.1.2 It is sufficient to prove (3.25) for $x = 0$.

For $l \in \mathbb{S}^{d-1}$, let χ_l be the \mathbb{T}_R^d -periodic solution of the cell problem associated to L_U with $\chi_l(0) = 0$.

Let ϕ_l represent the \mathbb{T}_R^d -periodic solution of the ergodicity problem $L_U \phi_l = |l - \nabla \chi_l|^2 - {}^t l D(U) l$ with $\phi_l(0) = 0$, and let $F_l(x) = l \cdot x - \chi_l(x)$ and $\psi_l(x) = F_l^2(x) - \phi_l(x)$. Observe that since $L_U F_l^2 = |l - \nabla \chi_l|^2$, it follows that

$$(4.72) \quad L_U \psi_l = {}^t l D(U) l.$$

The following inequality will be used to show that $\sum_{i=1}^d \psi_{e_i}$ behaves like $|x|^2$:

$$(4.73) \quad C_1 |x|^2 - C_2 (\|\chi\|_\infty^2 + \|\phi\|_\infty) \leq \sum_{i=1}^d \psi_{e_i}(x) \leq C_3 (|x|^2 + \|\chi\|_\infty^2 + \|\phi\|_\infty).$$

Using Theorem 4.5 [40, theorem 5.4, chap. 5] to control F_l and ψ_l over one period (observing that $L_U F_l = 0$ and $L_U \psi_l = -1$) and using $\chi_l = l \cdot x - F_l$ and $\phi_l = F_l^2 - \psi_l$, one easily obtains that $\|\phi\|_\infty \leq C_d e^{(9d+13) \text{Osc}(U)} R^2$. Combining this estimate with (4.16), one obtains

$$(4.74) \quad \|\chi\|_\infty^2 + \|\phi\|_\infty \leq C_d e^{(9d+13) \text{Osc}(U)} R^2.$$

Since V has been assumed to be smooth (in a first step), we can use the Ito formula to obtain

$$(4.75) \quad \psi_l(y_t) = \int_0^t \nabla \psi_l(y_s) d\omega_s + {}^t l D(U) l t.$$

Now, with e_i being an orthonormal basis of \mathbb{R}^d , write M_t the local martingale as

$$(4.76) \quad M_t = \sum_{i=1}^d \psi_{e_i}(y_t) - \text{tr}(D(U)) t.$$

Define

$$\tau'(0, r) = \inf \left\{ t \geq 0 : \left| \sum_{i=1}^d \psi_{e_i}(y_t) \right| = r \right\}.$$

According to inequality (4.73), one has

$$(4.77) \quad \begin{aligned} \tau'(0, C_1 r^2 - C_2 (\|\chi\|_\infty^2 + \|\phi\|_\infty)) &\leq \tau(0, r) \\ &\leq \tau'(0, C_3 (r^2 + \|\chi\|_\infty^2 + \|\phi\|_\infty)). \end{aligned}$$

Since $M_{t \wedge \tau'(0, r)}$ is uniformly integrable (easy to prove by using inequalities (4.77)), one obtains

$$(4.78) \quad \mathbb{E}[\tau'(0, r)] = \frac{r}{\text{tr}(D(U))}.$$

Thus, by using inequality (4.74) and the Voigt-Reiss inequality $D(U) \geq e^{-2\text{Osc}(U)}$, one obtains (3.25).

4.2.1.3 The proof of the weak stability result (4.67) is based on the following obvious lemma that describes the monotonicity of Green functions as quadratic forms.

LEMMA 4.13 *Let Ω be a smooth, bounded, open subset of \mathbb{R}^d . Assume that M and Q are symmetric, smooth, coercive matrices on $\overline{\Omega}$. Assume $M \leq \lambda Q$ with $\lambda > 0$; then for all $f \in C^0(\Omega)$, indicating by G_Q the Green functions of $-\nabla Q \nabla$ with Dirichlet condition on $\partial\Omega$,*

$$(4.79) \quad \int_{\Omega} G_Q(x, y) f(y) f(x) dx dy \leq \lambda \int_{\Omega} G_M(x, y) f(y) f(x) dx dy .$$

PROOF: Let $f \in C^0(\overline{\Omega})$. Let ψ_M and ψ_Q be the solutions of $-\nabla M \nabla \psi_M = f$ and $-\nabla Q \nabla \psi_Q = f$ with Dirichlet conditions on $\partial\Omega$. Observe that ψ_M and ψ_Q are the unique minimizers of $I_M(h, f)$ and $I_Q(h, f)$ with

$$(4.80) \quad I_M(h, f) = \frac{1}{2} \int_{\Omega} {}^t \nabla h M \nabla h dx - \int_{\Omega} h(x) f(x) dx$$

and $I_M(\psi_M, f) = -\frac{1}{2} \int_{\Omega} \psi_M(x) f(x) dx$. Observe that since $M \leq \lambda Q$,

$$(4.81) \quad I_M(h, f) \leq \lambda I_Q\left(h, \frac{f}{\lambda}\right),$$

and the minimum of the right member in equation (4.81) is reached at ψ_Q/λ . It follows that $\int_{\Omega} \psi_Q(x) f(x) \leq \lambda \int_{\Omega} \psi_M(x) f(x)$, which proves the lemma. \square

Then, (4.67) follows directly from this lemma by choosing $Q = e^{-2(U+P)}$ and $M = e^{-2U}$, and observing that $\mathbb{E}_x^U[\tau(0, r)] = 2 \int_{B(0,r)} G_{e^{-2U} I_d}(x, y) e^{-2U(y)} dy$.

4.2.2 Anomalous Heat Kernel Tail: Theorem 3.24

4.2.2.1 From the pointwise anomaly of the hitting times of Theorem 3.22, one can deduce the anomalous heat kernel tail by adapting a strategy used by M. T. Barlow and R. Bass for the Sierpinski carpet. This strategy is described in detail in the proof of [7, theorem 3.11], so we will give only the main lines of its adaptation.

We will estimate $\mathbb{P}_x[\tau(x, r) < t]$ and use $\mathbb{P}_x[|y_t| > r] \leq \mathbb{P}_x[\tau(x, r) < t]$ to obtain Theorem 3.24.

Using the notation from Theorem 3.22 and $M := (d, K_0, K_\alpha, \alpha, \mu, \lambda_{\max})$, we will show in paragraph 4.2.2.2 that for $r > C(M, \rho_{\max})$, one has

$$(4.82) \quad \mathbb{P}_x[\tau(x, r) \leq t] \leq \frac{t}{r^{2+\sigma(r)(1+\gamma)} C_{19}(M)} + 1 - C_{20}(M) r^{-2\gamma\sigma(r)} .$$

Now we will use [7, lemma 3.14] given below (this is also [8, lemma 1.1]).

LEMMA 4.14 *Let $\xi_1, \xi_2, \dots, \xi_n, V$ be nonnegative random variables such that $V \geq \sum_{i=1}^n \xi_i$. Suppose that for some $p \in (0, 1)$, $a > 0$, and $t > 0$,*

$$\mathbb{P}(\xi_i \leq t : \sigma(\xi_1, \xi_2, \dots, \xi_{i-1})) \leq p + at ;$$

then

$$\ln \mathbb{P}(V \leq t) \leq 2 \left(\frac{ant}{p} \right)^{1/2} - n \ln \frac{1}{p} .$$

Let $n \geq 1$ and $g = r/n$. Define the stopping times $S_i, i \geq 0$, by $S_0 = 0$ and

$$S_{i+1} = \inf \{ t \geq S_i : |y_t - y_{S_i}| \geq g \} .$$

Let $\xi_i = S_i - S_{i-1}$ for $i \geq 1$. Let \mathcal{F}_t be the filtration of y_t , and let $\mathcal{G}_i = \mathcal{F}_{S_i}$. Then it follows from equation (4.82) that for $r/n > C(M, \rho_{\max})$,

$$(4.83) \quad \begin{aligned} \mathbb{P}_x[\xi_{i+1} \leq t | (G)_i] &= \mathbb{P}_{y_{S_i}}[\tau(y_{S_i}, g) \leq t] \\ &\leq C_{21}(M) \frac{t}{g^{2+\sigma(r)(1+\gamma)}} + 1 - C_{20}(M) g^{-2\sigma(r)\gamma} . \end{aligned}$$

Since $|y_{S_i} - y_{S_{i+1}}| = g$, it follows that $\mathbb{P}_x |x - y_{S_n}| \leq r$. Thus

$$S_n = \sum_{i=1}^n \xi_i \leq \tau(x, r) ,$$

and by Lemma 4.14 with

$$a = C_{21}(M) \left(\frac{n}{r} \right)^{2+\sigma(r)(1+\gamma)} , \quad p = 1 - C_{20}(M) \left(\frac{n}{r} \right)^{2\sigma(r)\gamma} ,$$

one obtains

$$(4.84) \quad \ln \mathbb{P}_x[\tau(x, r) \leq t] \leq 2 \left(\frac{nt C_{21} \left(\frac{n}{r} \right)^{2+\sigma(r)(1+\gamma)}}{1 - C_{20} \left(\frac{n}{r} \right)^{2\sigma(r)\gamma}} \right)^{1/2} - n \ln \frac{1}{1 - C_{20} \left(\frac{n}{r} \right)^{2\sigma(r)\gamma}} .$$

Minimizing the right term in (4.84) over n under the constraint (4.83) and the assumptions (3.37) and $\rho_{\min} > C_{6,M}$, one obtains Theorem 3.24.

4.2.2.2 Equation (4.82) is an adaptation of [7, lemma 3.16]. Observe that

$$\begin{aligned} \mathbb{E}_x[\tau(x, r)] &\leq t + \mathbb{E}_x[1(\tau(x, r) > t) \mathbb{E}_{y_t}[\tau(x, r) - t]] \\ &\leq t + \mathbb{P}_x[1(\tau(x, r) > t)] \sup_{y \in B(x, r)} \mathbb{E}_y[\tau(x, r)] . \end{aligned}$$

Using $\forall y \in B(x, r)$, \mathbb{P}_y a.s. $\tau(x, r) \leq \tau(y, 2r)$, one has by Theorem 3.22 for $r > C(M, \rho_{\max})$

$$C_{33}(M) r^{2+\sigma(r)(1-\gamma)} \leq \mathbb{E}_x[\tau(x, r)] \leq t + \mathbb{P}_x[\tau(x, r) > t] C_{34}(M) r^{2+\sigma(r)(1+\gamma)} ,$$

which leads to (4.82).

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