Feedback Stabilization of Steady-State and Hopf Bifurcations: the Multi-Input Case*

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Abstract
Classification of stabilizability via smooth state feedback is obtained for multi-input nonlinear dynamical systems possessing a steady-state or Hopf bifurcation with the critical mode being linearly uncontrollable and all other modes being linearly controllable.

1. Introduction
Much research has been done to analyze the effects of state feedbacks on the criticality of bifurcations (see [1, 2, 3, 4, 5, 7, 8, 10] and the references therein). In [9], explicit classification of stabilizability was obtained under certain nondegeneracy conditions for a single-input nonlinear system possessing a steady-state or Hopf bifurcation with the critical mode being linearly uncontrollable. But one of the key steps in [9] is to put the linear part of the linearly controllable subsystem into the controller canonical form. Since controller canonical forms are not necessarily unique for multi-input LTI systems, the procedure in [9] might not be applicable to the multi-input case. In this paper, we solve the classification of stabilizability for the multi-input case without using controller canonical forms. The results in [9] and those in this paper are different from the previous research in that they are explicit necessary and sufficient conditions. Also, the stabilizing controllers can be explicitly constructed (see [9]).

2. Steady-State Bifurcation
Consider the following nonlinear multi-input system
\[
\dot{y} = f_\mu(y, u),
\]
where \( y \in \mathbb{R}^{n+1} (n \geq 1) \) is the state variable, \( \mu \in \mathbb{R} \) is a bifurcation parameter, and \( u \in \mathbb{R}^m (m \geq 1) \) is the control input. We assume \( n \geq 1 \) in this paper since the case when \( n = 0 \) is trivial. Throughout this section we assume all the assumptions are valid for \( \mu \) in the region \([-\bar{\mu}, \bar{\mu}]\). We make the following assumptions:

**AS-1** \( f_\mu(y, u) \) is at least \( C^4 \) with respect to \((y, u)\) and \( C^2 \) with respect to \( \mu \).

**AS-2** For \( u = 0 \), there exists a nominal equilibrium solution \( y = y_0(\mu) \) such that \( f_\mu(y_0(\mu), 0) = 0 \).

**AS-3** \( \lambda(\mu) \) is a simple real eigenvalue of \( \frac{\partial f_\mu}{\partial y}(y_0(\mu), 0) \) and satisfies \( \lambda(0) = 0 \), \( \frac{\partial \lambda}{\partial \mu}(0) \neq 0 \).

**AS-4** The eigenspace associated with \( \lambda(\mu) \) is linearly uncontrollable, and all other eigenspaces are linearly controllable.

We first expand \( f_\mu(y, u) \) into Taylor series around \( (y_0(\mu), 0) \), and use a linear transformation to linearly decouple the uncontrollable eigenspace from the controllable eigenspaces. Then we evaluate all the terms except the bifurcating eigenvalue at \( \mu = 0 \). The resulting

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system is given by
\[\dot{x} = \begin{bmatrix} dx & + & q_{11}x^2 & + & q_{12}x & + & q_{13}ux & + & \tilde{x}\dot{q}_{22}\dot{x} \\ + & \tilde{x}Tq_{23}u & + & uTq_{33}u & + & c_{111}x^3 & + & c_{112}x^2 \\ + & c_{113}ux^2 & + & \tilde{x}Tc_{122}\dot{x} & + & \tilde{x}Tc_{133}ux \\ + & uTc_{133}ux & + & \tilde{q}_{22}(\tilde{x},\tilde{x},\dot{x}) & + & \tilde{q}_{23}(\tilde{x},\dot{x},u) \\ + & c_{233}(\tilde{x},u,u) & + & c_{333}(u,u,u) & + & \cdots \end{bmatrix}, \tag{2}\]

where \(x \in \mathbb{R}, \dot{x} \in \mathbb{R}^n\), and all the coefficients are real tensors with appropriate dimensions. We assume the tensors are symmetric with respect to identical subscripts.

The goal is to find a sufficiently smooth feedback with Taylor series expansion
\[u = K_1\dot{x} + K_2x + K_3x^2 + K_4\dot{x}x + K_5(\dot{x},\dot{x}) + \cdots, \tag{3}\]

and
\[K_1, K_2, K_3 \in \mathbb{R}^{m \times 1}, \quad K_4 \in \mathbb{R}^{m \times n}, \quad K_5 \in \mathbb{R}^{m \times n \times n}, \]
such that the equilibrium \((0,0)\) for the closed loop system is asymptotically stable at the bifurcation point, which means the bifurcation is a supercritical pitchfork bifurcation.

A simple but important fact is that the stabilizability of the bifurcation is not changed by a state feedback. Without loss of generality, we assume that \(A^{-1}\) exists. If not, then we use a feedback \(u = \tilde{v} + K_1\dot{x}\) such that \(A + BK_1\) is invertible. Define
\[Y_1 = q_{13} + q_{12}(-A)^{-1}B, \tag{4}\]
\[Y_2 = q_{33} + B^T(-A)^{-T}q_{22}(-A)^{-1}B + \frac{1}{2} \cdot \begin{bmatrix} [B^T(-A)^{-T}q_{23} + q_{23}^T(-A)^{-1}B] \end{bmatrix}, \tag{5}\]
\[\alpha_0 = c_{111} + q_{12}(-A)^{-1}\tilde{q}_{11}. \tag{6}\]

Since \(Y_2 = Y_2^T \in \mathbb{R}^{m \times m}\), there exists an orthonormal matrix \(U \in \mathbb{R}^{m \times m}\) such that
\[\begin{bmatrix} \hat{Y}_2 \\ \hat{Y}_1 \end{bmatrix} = U^T \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \]
\[= \text{Diag} \left[ \begin{bmatrix} \hat{Y}_1^1 \\ \hat{Y}_1^2 \\ \cdots \hat{Y}_1^m \end{bmatrix}, \begin{bmatrix} \hat{Y}_2^1 \\ \hat{Y}_2^2 \\ \cdots \hat{Y}_2^m \end{bmatrix} \right], \tag{7}\]
\[\hat{Y}_1 = Y_1U = \begin{bmatrix} \hat{Y}_1^1 \\ \hat{Y}_1^2 \\ \cdots \hat{Y}_1^m \end{bmatrix}, \tag{8}\]

Let \(m = \{1, 2, \cdots, m\}\), and defines index set \(I_j (j = 1, \cdots, 4) \in m\) as
\[I_1 = \{i \in m: \hat{Y}_1^i > 0\}, \tag{9}\]
\[I_2 = \{i \in m: \hat{Y}_1^i < 0\}, \tag{10}\]
\[I_3 = \{i \in m: \hat{Y}_1^i = 0, \hat{Y}_2^i \neq 0\}, \tag{11}\]
\[I_4 = \{i \in m: \hat{Y}_1^i = \hat{Y}_2^i = 0\}. \tag{12}\]

It is clear that \(I_j \cap I_j = \emptyset (i \neq j, i, j = 1, \cdots, 4)\), and \(I_1 \cup I_2 \cup I_3 \cup I_4 = m\). Define
\[K_a = U \begin{bmatrix} \hat{Y}_2^1 \\ \hat{Y}_2^2 \\ \cdots \hat{Y}_2^m \end{bmatrix}, \tag{13}\]

\[P(K) = \alpha_0 + \alpha_1K + K^T\alpha_2K + \alpha_3(K, K, K), \tag{15}\]

where
\[\alpha_1 = c_{112}(-A)^{-1}B + c_{113} + q_{12}(-A)^{-1}\hat{Y}_1 + \hat{q}_{11}^T(-A)^{-T} \begin{bmatrix} 2q_{22}(-A)^{-1}B + q_{23} \end{bmatrix}, \]
\[\alpha_2 = \frac{1}{2} (\alpha_{21} + \alpha_{22}), \]
\[\alpha_{21} = B^T(-A)^{-T}c_{122}(-A)^{-1}B + c_{133} + B^T(-A)^{-T}c_{123} + q_{12}(-A)^{-1}\hat{Y}_2 + [2B^T(-A)^{-T}q_{22} + q_{23}^T(-A)^{-1}\hat{Y}_1, \]
\[\alpha_{22} = \frac{1}{6} \left[ c_{33}(X, Y, Z) + c_{33}(X, Z, Y) + c_{33}(Y, X, Z) + c_{33}(Z, X, Y) + c_{33}(Z, Y, X) \right], \]
\[\alpha_{31} = c_{222}(-A)^{-1}B_1, \quad \alpha_{32} = c_{223}(-A)^{-1}B_2, \quad \alpha_{33} = c_{233}(-A)^{-1}B_3, \]
\[\alpha_{34} = \frac{1}{4} \left[ c_{323}(-A)^{-1}B_1, (-A)^{-1}B_1 \right], \]
\[\alpha_{31} = \alpha_{32} = \alpha_{33} = \alpha_{34} = \{\}, \quad \alpha_{35} = \{\}, \quad \alpha_{36} = \{\}, \quad \alpha_{37} = \{\}, \quad \alpha_{38} = \{\}, \quad \alpha_{39} = \{\}, \quad \alpha_{40} = \{\}, \]
\[\hat{Y}_1 = \hat{q}_{33} + \hat{q}_{12}(-A)^{-1}B, \tag{16}\]
\[\hat{Y}_2 = \hat{q}_{22}((-A)^{-1}B_1, (-A)^{-1}B_2), \tag{17}\]
where \( \alpha_0 := \alpha_0(X,Y,Z) \), \( \alpha_m := \alpha_m(X,Y,Z) \), \( \tilde{\gamma}_2 := \tilde{\gamma}_2(Y,Z) \), and \( X,Y,Z \in \mathbb{R}^m \).

If \( q_1 \neq 0 \), \( I_3 = \emptyset \), but \( I_1 \neq \emptyset \), then define
\[
K_b = U \xi_b, \quad \xi_b \in \mathbb{R}^{m \times 1},
\]
where
\[
\xi_b^i = \begin{cases} 
\frac{-\tilde{\gamma}_1}{2 \tilde{\gamma}_2}, & i \in I_1, \\
0, & i \in I_4.
\end{cases}
\] (18)

If \( q_1 \neq 0 \), \( I_3 = \emptyset \), but \( I_2 \neq \emptyset \), then define \( K_b = U \xi_b, \xi_b \in \mathbb{R}^{m \times 1} \),
where
\[
\xi_b^i = \begin{cases} 
\frac{-\tilde{\gamma}_1}{2 \tilde{\gamma}_2}, & i \in I_2, \\
0, & i \in I_4.
\end{cases}
\] (19)

If \( q_1 = 0 \), and \( I_1 = \emptyset \) or \( I_2 = \emptyset \), then we define \( K_b = 0 \).

Now let \( f_1, \ldots, f_l \) be a set of basis of \( \text{Ker} \left[ \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \right] \), then any \( K_e \in \text{Ker} \left[ \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \right] \), we have \( K_e = FX \), where \( F = \begin{bmatrix} f_1 & f_2 & \cdots & f_l \end{bmatrix} \in \mathbb{R}^{m \times l} \) and \( X \in \mathbb{R}^{l \times 1} \).

Define
\[
\alpha_{0r} = \alpha_0 + \alpha_1 K_b + K_b^T \alpha_2 K_b + \alpha_3 (K_b, K_b, K_b) \in \mathbb{R},
\]
\[
\alpha_{1r} = \left[ \alpha_1 + 2 K_b^T \alpha_2 + 3 \alpha_3 (K_b, K_b) \right],
\]
\[
F \in \mathbb{R}^{l \times 1},
\]
\[
\alpha_{2r} = F^T \left[ \alpha_2 + 3 \alpha_3 \cdot K_b \right] F \in \mathbb{R}^{l \times l},
\]
\[
\alpha_{3r} = \alpha_3 (FX, FX, FX) \in \mathbb{R}^{l \times l},
\]
where \( \alpha_{3r} := \alpha_{3r}(X,X,X) \).

Let \( V \in \mathbb{R}^{l \times l} \) be an orthonormal matrix such that
\[
\alpha_{1r} = \alpha_{1r} V
\]
\[
= \begin{bmatrix} \tilde{\alpha}_{1r}^1 & \tilde{\alpha}_{1r}^2 & \cdots & \tilde{\alpha}_{1r}^l \end{bmatrix},
\] (20)
\[
\tilde{\alpha}_{1r} = V^T \alpha_{2r} V
\]
\[
= \text{Diag} \left[ \tilde{\alpha}_{2r}^1, \ldots, \tilde{\alpha}_{2r}^l \right],
\] (21)

Let \( l = \{ 1, 2, \ldots, l \} \), and define index sets
\[
I_{jr} (j = 1, \ldots, 4) \in 1 \text{ as}
\]
\[
I_{1r} = \{ i \in l : \tilde{\alpha}_{2r}^i > 0 \},
\] (22)
\[
I_{2r} = \{ i \in l : \tilde{\alpha}_{2r}^i < 0 \},
\] (23)
\[
I_{3r} = \{ i \in l : \tilde{\alpha}_{2r}^i = 0, \tilde{\alpha}_{1r}^i \neq 0 \},
\] (24)
\[
I_{4r} = \{ i \in l : \tilde{\alpha}_{2r}^i = \tilde{\alpha}_{1r}^i = 0 \}.
\] (25)

Define
\[
\hat{\alpha}_{or} := \alpha_{or} - \sum_{i \in I_{1r} \cup I_{2r}} \frac{（\tilde{\alpha}_{1r}^i）^2}{4 \tilde{\alpha}_{2r}^i}.
\] (26)

With above definitions, we have the following theorem.

**Theorem 1** A complete classification of stabilizability of a steady-state bifurcation via smooth feedback is given by the diagram in the next page. Each of the cases is a full path in the tree.

The degenerate cases are those when the criticality has to be determined by \( 4^k \), \( 5^k \), and higher order terms on the center manifold. It should be noted that the classification of stabilizability here for the multi-input case is more complicated than that for a single input system (see [9]). Theorem 1 can be proved via center manifold reduction (see [6]), which we omit here due to space constraint. Explicit construction of stabilizing control laws can be derived, which is also omitted here.

3. **Hopf Bifurcation**

Consider the following multi-input system
\[
\dot{y} = f_\mu(y, u),
\] (27)
where \( y \in \mathbb{R}^{n+2} \) is the state variable, \( \mu \in \mathbb{R} \) is a bifurcation parameter, and \( u \in \mathbb{R}^m \) is the control input. We assume \( n \geq 1 \) since the case \( n = 0 \) is trivial. Throughout this section we assume all the assumptions are valid for \( \mu \) in the region \([-\bar{\mu}, \bar{\mu}]\). We make the following assumptions:

**AH-1** \( f_\mu(y, u) \) is at least \( C^4 \) with respect to \((y, u)\) and \( C^2 \) with respect to \( \mu \).

**AH-2** For \( u = 0 \), there exists a steady-state solution \( y_0(\mu) \) such that \( f_\mu(y_0(\mu), 0) = 0 \).

**AH-3** \( \lambda_{1,2}(\mu) = \sigma(\mu) \pm i \omega(\mu) \) are a simple pair of eigenvalues of \( \frac{\partial f_\mu}{\partial y}(y_0(\mu), 0) \) and satisfy \( \sigma(0) = 0, \frac{\omega(0)}{\mu} \neq 0, \) and \( \omega(0) \neq 0 \).

**AH-4** The eigenspaces associated with \( \lambda_{1,2}(\mu) \) are linearly controllable, and all other eigenspaces are linearly uncontrollable.
Under these assumptions, we transform the system (1) into the standard form by the same procedure as the steady-state case. The resulting normal form is given by

\[
\dot{z} = (d\mu + i\omega)z + q_{11}z^2 + q_{12}z^2 + q_{13}\tilde{x}z \\
+ q_{14}uz + q_{22}u^2 + q_{23}\tilde{x}z + q_{24}uz^* \\
+ \tilde{x}^T q_{33}\tilde{x} + \tilde{x}^T q_{34}u + u^T q_{44}u + c_{111}\tilde{x}z \\
+ c_{112}z^2 \tilde{x} + c_{113}z \tilde{x}^2 + c_{114}\tilde{x} \tilde{x}^2 \\
+ c_{122}|z|^2 \tilde{x} + c_{123}|z|^2 \tilde{x}^2 \\
+ c_{124}|z|^2 \tilde{x}^2 + c_{125}u|z|^2 \\
+ \tilde{x}^T c_{233}\tilde{x} + \tilde{x}^T c_{234}u + u^T c_{444}uz \\
+ c_{222}\tilde{x}^3 + c_{223}\tilde{x}^2 \tilde{x} + c_{224}\tilde{x} \tilde{x}^2 \\
+ \tilde{x}^T c_{233}\tilde{x} + \tilde{x}^T c_{234}u + u^T c_{444}uz \\
+ c_{333}(\tilde{x}, \tilde{x}, \tilde{x}) + c_{334}(\tilde{x}, \tilde{x}, u) + \\
c_{344}(\tilde{x}, u, u) + c_{444}(u, u, u) + \cdots, \quad (28)
\]

where \( z \in \mathbb{C}, \tilde{x} \in \mathbb{R}^n \), and other coefficients are real or complex tensors with appropriate dimensions. As in the steady-state case, we also assume the tensors are symmetric with respect to identical subscripts.

The goal is to find a sufficiently smooth feedback with Taylor series expansion

\[
u = K_{1}\tilde{x} + K_{2}\tilde{x}^2 + K_{3}\tilde{x}^3 + K_{4}\tilde{x}^4 + K_{5}|z|^2 \\
+ K_{6}\tilde{x}^6 + K_{7}\tilde{x}^7 + K_{8}\tilde{x}^8 \\
+ K_{9}(\tilde{x}, \tilde{x}) + \cdots, \quad (30)
\]
and with
\[ K_1 \in \mathbb{R}^{m \times n}, \quad K_2 = K_3' \in \mathbb{C}^n, \]
\[ K_4 = K_4' \in \mathbb{C}^n, \quad K_5 \in \mathbb{R}^m, \]
\[ K_7 = K_8' \in \mathbb{R}^{m \times n}, \quad K_9 \in \mathbb{R}^{m \times n \times m}, \]

such that the equilibrium \((0,0)\) for the closed loop system is asymptotically stable at the bifurcation point, which implies that the bifurcation is a supercritical Hopf bifurcation.

Similar to the steady-state bifurcation case, a state feedback does not affect stabilizability of the system. So without loss of generality, we assume \((sI - A)^{-1}\) exists for \(s = 0, \pm i\omega, \pm 2i\omega\). We first select \(K_1\) such that \(A + BK_1\) is Hurwitz. Define
\[
C_0 = c_{112} - \frac{1}{i\omega}q_{11} q_{12} + q_{13} (-A)^{-1} q_{12},
\]
\[
\alpha_0 = \text{Re} C_0. \quad (31)
\]

Letting \(s, s_1, s_2 \in \mathbb{C}\), define
\[
\Phi_1(s) = q_{14} + q_{12} (sI - A)^{-1} B,
\]
\[
\Phi_2(s) = q_{24} + q_{23} (sI - A)^{-1} B,
\]
\[
\hat{\Phi}_1(s) = \hat{q}_{14} + \hat{q}_{13} (sI - A)^{-1} B,
\]
\[
\hat{\Phi}_2(s) = \hat{q}_{24} + \hat{q}_{23} (sI - A)^{-1} B,
\]
\[
\Psi(s_1, s_2) = 2B^T(sI - A)^{-T} q_{33} \cdot (s_2 I - A)^{-1} B
\]
\[+ q_{23}^T (2i\omega - A)^{-1} \hat{q}_{11},
\]
\[
\alpha_0 = \text{Re} C_0. \quad (31)
\]

where \(\hat{\Phi}(s_1, s_2) = \hat{\Phi}(s_1, s_2)(X,Y)\). We define
\[
\Theta_1 = \text{Re} \Phi_1(0), \quad \Theta_2 = \Phi_2(2i), \quad \Theta_3 = \Psi(i\omega, 0), \quad \Theta_4 = \Psi(-i\omega, 2i).
\]

Letting \(K = [I - K_1(i\omega - A)^{-1} B]^{-1} K_2 = K_R + iK_I\), we define
\[
F_{112} = 3c_{333} ((i\omega - A)^{-1} BKK^*) + 2c_{334} ((i\omega - A)^{-1} BK),
\]
\[
\gamma_j = \text{Re} \{F_{112} + F_{122}\} (K_R, K_R, K_R),
\]
\[
\gamma_j = \text{Im} \{F_{112} + F_{122}\} (K_R, K_R, K_R),
\]
\[
\gamma_j = \text{Re} \{F_{112} + F_{122}\} (K_R, K_R, K_R),
\]
\[
\gamma_j = \text{Im} \{F_{112} - F_{122}\} (K_R, K_R, K_R),
\]
\[
\gamma_j = \text{Re} \{F_{112} - F_{122}\} (K_R, K_R, K_R),
\]
\[
\gamma_j = \text{Im} \{F_{112} - F_{122}\} (K_R, K_R, K_R),
\]

where \(\gamma_j (j = 1, \cdots, 4)\) are defined as
\[
\gamma_1 = \gamma_1 (K_R, K_R, K_R),
\]
\[
\gamma_2 = \gamma_2 (K_R, K_R, K_R),
\]
\[
\gamma_3 = \gamma_3 (K_R, K_I, K_I),
\]
\[
\gamma_4 = \gamma_4 (K_I, K_I, K_I).
\]
Define $\bar{K} = \begin{bmatrix} K_R \\ K_I \end{bmatrix}$. Also define

$$\gamma_a = \gamma_1(K_R, R, K_R) + \gamma_2(K_R, K_I, K_I) + \gamma_3(K_R, K_I, K_I) + \gamma_4(K_I, K_I, K_I),$$

where $\gamma_a := \gamma_a(\bar{K}, \tilde{K}, \tilde{K})$ and define $\gamma$ as the symmetrization of $\gamma_a$, i.e., for any $i, j, k \in 2m := \{1, \ldots, 2m\}$, we have

$$\gamma_{ij} = \frac{1}{6} \left( \gamma_{ij}^* + \gamma_{ij}^k + \gamma_{ij}^k + \gamma_{ik}^j + \gamma_{ik}^j + \gamma_{jk}^i \right).$$

Define the norm of $\gamma$ as the infinity norm

$$\|\gamma\| = \max_{i, j, k \in 2m} |\gamma_{ij}|,$$

and $\|\Theta\| (j = 1, \ldots, 4)$ are defined similarly. Define

$$\Theta = [\|\Theta_1\| \cdots \|\Theta_4\| \|\gamma\|].$$

Let

$$D_1 = c_{123}(i\omega - A)^{-1}B + c_{124}$$

$$+ q_{13}(A)^{-1}\Phi_2(i\omega)$$

$$+ q_{23}(2i\omega - A)^{-1}\Phi_1(i\omega)$$

$$+ q_{12}(2i\omega - A)^{-T}.$$ (32)

$$[q_{33}(i\omega - A)^{-1}B + q_{34}]$$

$$- \frac{1}{i\omega} q_{11} \Phi_2(i\omega),$$

$$D_2 = c_{113}(-i\omega - A)^{-1}B + c_{114}$$

$$+ q_{13}(-A)^{-1}\Phi_1(i\omega)$$

$$+ q_{12}(2i\omega - A)^{-T}.$$ (33)

$$[q_{33}(i\omega - A)^{-1}B + q_{34}]$$

$$- \frac{1}{i\omega} (q_{11} + q_{12}) \Phi_1(-i\omega)$$

$$- \frac{1}{3i\omega} q_{22} \Phi_2(-i\omega),$$

$$E_{11} = B^T(i\omega - A)^{-T}c_{233}(i\omega - A)^{-1}B$$

$$+ B^T(i\omega - A)^{-T}c_{234} + c_{244}$$

$$+ [2B^T(i\omega - A)^{-T}q_{33} + q_{34}^T](A)^{-1}\Phi_2(i\omega)$$

$$+ \frac{1}{2} q_{33}(2i\omega - A)^{-1}\Phi(i\omega, i\omega) + \frac{1}{2i\omega} [q_{12} \Psi(i\omega, i\omega) - 2\Phi^T(i\omega)\Phi_2(i\omega)],$$

$$E_{12} = 2B^T(-i\omega - A)^{-T}c_{133}(i\omega - A)^{-1}B$$

$$+ c_{134}(i\omega - A)^{-T}B$$

$$+ B^T(-i\omega - A)^{-T}c_{134} + 2c_{144}$$

$$+ \Phi^T(-i\omega)(A)^{-T}.$$ (34)

$$[q_{33}(i\omega - A)^{-1}B + q_{34}]$$

$$+ [2B^T(-i\omega - A)^{-T}q_{33} + q_{34}^T](2i\omega - A)^{-1}\Phi_1(i\omega)$$

$$+ q_{13}(-A)^{-1}\Psi(-i\omega, i\omega)$$

$$- \frac{1}{i\omega} (2q_{11} + q_{12}) \Psi(-i\omega, i\omega),$$

$$E_{22} = \frac{1}{i\omega} \Phi^T(-i\omega)\Phi_2^T(-i\omega)$$

$$- \frac{1}{3i\omega} q_{22}^T \Phi_2(-i\omega).$$ (35)

Define

$$\alpha_1 = \text{Re}\{D_1 + D_2\},$$

$$\alpha_2 = \text{Im}\{D_2 - D_1\},$$

$$\delta_1 = \frac{1}{2} \text{Re}\{E_{11} + E_{12} + E_{22}\}$$

$$+ \text{Re}\{E_{11} + E_{12} + E_{22}^T\},$$

$$\delta_2 = \frac{1}{2} \text{Im}\{E_{22} - E_{11}\}$$

$$+ \text{Im}\{E_{22} - E_{11}^T\},$$

$$\delta_3 = \frac{1}{2} \text{Re}\{E_{12} - E_{11} - E_{22}\}$$

$$+ \text{Re}\{E_{12} - E_{11} - E_{22}^T\},$$

Let $U \in \mathbb{R}^{2m \times 2m}$ be an orthonormal matrix such that

$$\Delta := U^T \begin{bmatrix} \delta_1 & \delta_2 \\ \delta_2 & \delta_3 \end{bmatrix} U = \text{Diag} [\Delta^1, \Delta^2, \ldots, \Delta^{2m}] ,$$

$$\Gamma := [\alpha_1 \alpha_2] U = [\Gamma^1 \cdots \Gamma^{2m}],$$

Define

$$I_1 = \{j: \Delta^j > 0\},$$

$$I_2 = \{j: \Delta^j < 0\},$$

$$I_3 = \{j: \Delta^j = 0, \Gamma^j \neq 0\},$$

$$I_4 = \{j: \Delta^j = 0, \Gamma^j = 0\},$$

$$\hat{\alpha}_0 = \alpha_0 - \sum_{j \in I_1} (\Gamma^j)^2 / 4 \Delta^j.$$

(37)
Given the above notations, we have the following theorem.

**Theorem 2** A complete classification of stabilizability of a Hopf bifurcation via smooth feedback is given in the following table, where $\alpha_0$, $\Theta$, $I_2$, $I_3$, and $\alpha_0$ are given by (31), (32), (34), (35), and (37), respectively.

<table>
<thead>
<tr>
<th>Cases</th>
<th>$\alpha_0$</th>
<th>$\Theta$</th>
<th>$I_2 \cup I_3$</th>
<th>$\alpha_0$</th>
</tr>
</thead>
<tbody>
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<td>Stabilizable</td>
<td>$&lt; 0$</td>
<td></td>
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The degenerate cases occur when the criticality of the Hopf bifurcation is determined by the 5th and higher order terms. Due to the space limitations, here we will not give the full proof of the theorem. The procedure here is very similar to the steady-state bifurcation case, except that we also use normal form reduction in addition to the center manifold reduction (see [6]). It can be seen that the classification is essentially the same as the single-input case (see [9]). The explicit construction of stabilizing control laws is also similar to that in [9], which we omit here.

4. Conclusions and Future Work

We have provided a complete explicit classification of smooth feedback stabilization of a steady-state or a Hopf bifurcation for multi-input nonlinear systems under certain nondegeneracy conditions. Explicit forms of stabilizing control laws could also be constructed.

A possible future work might be the classification of stabilizability for the degenerate cases. We expect the framework in this paper to work for certain degenerate scenarios. Another possible future work is study of stabilization of nonlinear control systems for which multiple critical modes are linearly unstabilizable.

**References**


