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Multi-Hop Relay Protocols for Fast Consensus Seeking

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***Abstract*— Consensus protocols in coordinated multi-agent systems are distributed algorithms. Just using local information available to each single agent, all agents converge to an identical consensus state and the convergence speed is determined by the algebraic connectivity of the communication topology. In order to achieve a faster consensus seeking, we propose multi-hop relay protocols based on the current "nearest neighbor rules" consensus protocols. By employing multiple-hop paths in the network, more information is passed around and each agent enlarges its "available" neighborhood. We demonstrate that these relay protocols can increase the algebraic connectivity without physically adding or changing any edges in the graph. Moreover, time delay sensitivity of relay protocols are discussed in detail. We point out that a trade off exists between convergence performance and time delay robustness. Simulation results are also provided to verify the efficiency of relay protocols.**

***Index Terms*— Coordinated multi-agent systems, consensus protocol, multi-hop relay protocol, distributed algorithms, convergence speed, time delay.**

I. INTRODUCTION

Collective behaviors of coordinated multiple agents using nearest neighbor rules have attracted attentions of researchers from different disciplines. One of them is the consensus behavior, i.e., the states of all agents convergence to an identical value. Vicsek *et al.* proposed a simple but popular model for multiple agents in which each agent updates its headings based on the average of its own heading and its neighbors' [1]. Using simulation results, they showed that all agents move in the same direction eventually. A theoretical explanation for Vicsek's model is given by Jadbabaie *et al.* in [2]. Olfati-Saber and Murray proposed a simple consensus protocol and showed that, for balanced directed graphs, this protocol could solve the average consensus problem. Moreover, consensus seeking under general connected directed graphs were studied in [3], [4].

When agents' dynamics are considered, the consensus behavior was treated as the synchronization problem for coupled dynamical systems. Different approaches were employed such as Lyapunov's direct method in [5] and Laplacian matrix decomposition method in [6], [7]. Also, sufficient conditions for multiple dynamical systems synchronization over general connected directed graphs were discussed in [8].

Average consensus seeking has many applications in peer-to-peer networks [9], sensor fusion [10], and distributed Kalman filter [11]. The consensus convergence speed is very important. Xiao and Boyd treated a consensus process as an optimal linear iteration problem and increased the convergence speed by finding the optimal weights associated with each edge [12]. Olfati-Saber proposed a "random rewiring" procedure to boost the convergence speed for large scale graphs. However, physically changing the topology may be difficult in some applications. The question is can we get a better convergence speed without changing edges and their weights?

Fortunately, the answer is yes. In this paper, we propose multi-hop relay consensus protocols that use multi-hop paths in a graph to improve the convergence performance. The idea is simple: each vertex could get more information by passing its neighbors states to other neighbors. The improvement with two-hop relay protocol is given explicitly. Since relay protocols do not change the topology, it is easy to be implemented in practice. Furthermore, the effect of communication time delays are considered. An effective method to find the delay margin of two-hop relay protocol is given. A trade off between convergence speed and delay sensitivity is discussed.

The remainder of this paper is organized as follows. In section II, a brief review of concepts in algebraic graph theory and some preliminary results about consensus protocols are presented. We then propose multi-hop relay protocols for fast consensus seeking in section III and emphasize on the two-hop relay protocol. Section IV is devoted to investigating the stability of the relay protocols with communication time delays. Explicit result of delay margins is presented. Examples and simulation results are provided in section V and conclusions are listed in section VI.

II. Consensus Protocols and Consensus State

We use a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ to represent the interaction topology in a multi-agent system where $\mathcal{V}$ is a set of vertices and $\mathcal{E} \subseteq \mathcal{V}^2$ is a set of edges. Each edge of the graph is denoted by (v_i, v_j) and represents that agent v_i has access to the state of agent v_j. For any edge (v_i, v_j), we call v_i the *head* and v_j the *tail*. The directed graph $\mathcal{G}$ is called *symmetric* if, whenever $(v_i, v_j) \in \mathcal{E}$, then $(v_j, v_i) \in \mathcal{E}$ also. In a directed graph, the number of edges whose head is v_i is called *out-degree* of node v_i; the number of edges whose tail is v_i is called *in-degree* of node v_i. If edge $(v_i, v_j) \in \mathcal{E}$, then v_i is one of the *parent vertices* of v_j. The set of neighbors of vertex v_i is denoted by $N(v_i) = \{v_j \in \mathcal{V} : (v_i, v_j) \in \mathcal{E}\}$.

A *path* in a directed graph is a sequence $u_0, \cdots, u_r$ of distinct vertices such that $(u_{i-1}, u_i) \in \mathcal{E}$ for i from 1 to r. A path is also called a *n-hop path* if there are n edges in it. A

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weak path is a sequence $u_0, \cdots, u_r$ of distinct vertices such that either (u_{i-1}, u_i) or (u_i, u_{i-1}) belongs to $\mathcal{E}$. A directed graph is *weakly connected* if any two vertices in the graph can be joined by a weak path, and it is *strongly connected* if any two vertices can be joined by a n-hop path. If a directed graph is neither strongly connected nor weakly connected, it is *disconnected*. For same vertices set, Fig. 1 reveals the relationships between these concepts. In some literature, a symmetric, connected directed graph is just called *connected* when only symmetric graphs were considered. In this paper, we focus on symmetric connected graphs.

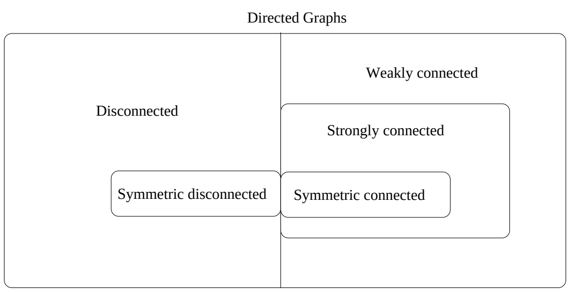

Fig. 1. Classification of directed graphs

An *adjacency* matrix $A = \{\alpha_{ij}\}$ of $\mathcal{G}$ with n vertices is defined as:

$$\alpha_{ij} = \begin{cases} 1, & (v_i, v_j) \in \mathcal{E} \\ 0, & \text{otherwise.} \end{cases}$$

More generally, a *weighted adjacency* matrix $\mathcal{A} = \{a_{ij}\}$ of a *weighted directed graph* $\mathcal{G}$ is defined as:

$$a_{ij} = \alpha_{ij} \cdot w_{ij}$$

where $w_{ij} > 0$ is the weight associated with vertices pair (v_i, v_j). For a weighted directed graph, the out-degree of node v_i is $\sum_j a_{ij}$; the in-degree of node v_i is $\sum_j a_{ji}$. Let D be the diagonal matrix with the out-degree of each vertex along the diagonal, then the *Laplacian* matrix L is defined by $L = D - \mathcal{A}$.

Let x_i denote the state of agent v_i. A multi-agent system is called to reach a *consensus* if $x_i = x_j$ for all v_i and $v_j \in \mathcal{V}$. This common value is called the *consensus state* which is depicted by η. A consensus protocol using nearest neighbor rules is represented by :

$$\dot{x}_i = -\sum_{j \in N(v_i)} w_{ij}(x_i - x_j), \tag{1}$$

and the multi-agent system can be presented by

$$\dot{X} = -LX \tag{2}$$

where $X = [x_1, \cdots, x_n]'$ and L is the Laplacian matrix.

For a symmetric connected graph, protocol (1) solves the average consensus problem and $\eta = \sum x_i(0)/n$.

III. Multi-Hop Replay Protocols

According to [13], for a symmetric connected directed graph, suppose the eigenvalues of L are denoted by $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$, the consensus convergence speed is bounded by the second smallest eigenvalue λ_2. Clearly, the value of λ_2 is determined by the topology and the weight associated with each edge. If we only consider the aspect of topology, the more links there are in the graph, the bigger the algebraic connectivity is, and the faster the convergence is. An upper bound of the convergence speed can be found for consensus protocol (1).

Lemma 3.1: The maximum value of the second smallest eigenvalue λ_2 of L for a symmetric connected directed graph $\mathcal{G}$ with n vertices is $\sum_{i \neq j} w_{ij}/(n-1)$.

Proof: We know that 0 is a simple eigenvalue of L associated with eigenvector $\mathbf{1} = [1, \cdots, 1]$ and all the other eigenvalues are real positive since L is symmetric.

For a directed graph $\mathcal{G}$ with n vertices, we have

$$(n-1)\lambda_2 \leq \sum_{i=1}^{n} \lambda_i = tr(L) \leq \sum_{i \neq j} \alpha_{ij} w_{ij}$$

where $tr(L)$ is the trace of L. Thus,

$$\lambda_2 \leq \sum_{i \neq j} w_{ij}/(n-1).$$

The equation holds only when $\mathcal{G}$ is complete and all weights w_{ij} are identical. ■

The consensus protocol (1) reaches its maximum convergence speed if we configure the topology to be a complete graph with uniform weights. In this section, we extend protocol (1) to multi-hop relay protocols which employ multi-hop paths in the graph instead of changing edges and weights.

A. Two-Hop Relay Protocol

The distributed two-hop relay protocol is described as

$$\dot{x}_i = -\sum_{j \in N(v_i)} w_{ij}\Big((x_i - x_j) + \sum_{k \in N(v_j)} w_{jk}(x_i - x_k)\Big). \tag{3}$$

In two-hop relay protocol, what each vertex sends to its parent vertices is not only its own state, but also a collection of its instantaneous neighbors' states. It is equal to adding virtual "two-hop" paths as additional edges to original graph. For a directed graph $\mathcal{G}$, a *two-hop directed graph* $\hat{\mathcal{G}} = (\mathcal{V}, \hat{\mathcal{E}})$ is a graph that has the same vertex set and all the edges are "two-hop" paths of $\mathcal{G}$. Fig. 2 shows an example of the two-hop directed graph.

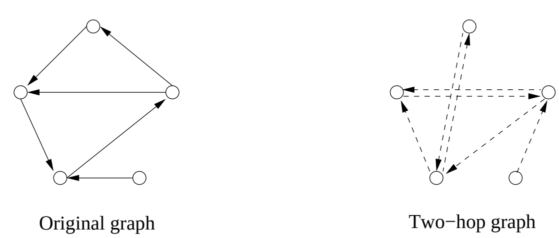

Fig. 2. A directed graph and its two-hop directed graph

For $\mathcal{G}$, there may exist self-loops in two-hop graph $\hat{\mathcal{G}}$, i.e., the head and tail of an edge are same. This is very common when $\mathcal{G}$ is symmetric. However, according to the relay protocol (3), these self-loops have no contributions to the dynamics. Thus, these self-loops are omitted if we don't consider communication delays. Moreover, between any pair of vertices in $\mathcal{G}$, multiple two-hop paths may exist. We consider them as one edge in $\hat{\mathcal{G}}$ and the weight associated with it is equal to the sum of two-hop paths' weights. Thus, the adjacency matrix $\hat{\mathcal{A}} = \{\hat{a}_{ik}\}$ of $\hat{\mathcal{G}}$ are:

$$\hat{a}_{ik} = \begin{cases} \sum_{j \in \mathcal{V}} w_{ij} w_{jk}, & (v_i, v_k) \in \hat{\mathcal{E}} \\ 0, & \text{otherwise.} \end{cases}$$

The corresponding out-degree diagonal and Laplacian matrices are denoted by $\hat{D}$ and $\hat{L}$ respectively.

Proposition 3.2: For a directed graph $\mathcal{G}$ with two-hop relay protocol (3), the dynamics of the whole system is described as

$$\dot{X} = -\tilde{L}_2 X. \tag{4}$$

where $\tilde{L}_2 = L + \hat{L}$.

Proof: Consider the joint graph $\tilde{\mathcal{G}} = \mathcal{G} \bigcup \hat{\mathcal{G}} = (\mathcal{V}, \mathcal{E} \bigcup \hat{\mathcal{E}})$, it is obvious that the two-hop relay protocol is a consensus protocol of $\tilde{\mathcal{G}}$. ■

Another issue is how much extra communication bandwidth the two-hop relay protocol needs. We assume that the graph is static and link weights associated with instantaneous neighbors are prior knowledge of each agent. We rewrite the protocol (1) as:

$$\dot{x}_i = -x_i \sum_{j \in N(v_i)} w_{ij} + \sum_{j \in N(v_i)} w_{ij} x_j \tag{5}$$

and the two-hop relay protocol (3) as

$$\begin{aligned} \dot{x}_i \;=\; & -x_i \textstyle\sum_{j \in N(v_i)} w_{ij} (1 + \sum_{k \in N(v_j)} w_{jk}) \\ & + \textstyle\sum_{j \in N(v_i)} w_{ij} \big(x_j + \sum_{k \in N(v_j)} w_{jk} x_k\big). \end{aligned} \tag{6}$$

For protocol (1), what link (v_i, v_j) transmits is the value of x_j. For protocol (3), what link (v_i, v_j) transmits is the value of x_j, $\sum w_{jk} x_k$, and $\sum w_{jk}$. However, for a static graph, $\sum w_{jk}$ is a constant and only need to be transmitted once. Thus, two-hop relay protocol needs double communication bandwidth except at the beginning.

B. Performance of the two-hop relay protocol

Suppose $\mathcal{G}$ is symmetric and connected, then L is symmetric and positive semi-definite, the two-hop graph $\hat{\mathcal{G}}$ is symmetric, and the joint graph $\tilde{\mathcal{G}}$ is also symmetric and connected.

Theorem 3.3: Assume a directed graph $\mathcal{G}$ is connected and symmetric, then

$$\lambda_2(L) \le \lambda_2(\tilde{L}) \tag{7}$$

Proof: For any vector x, it is true that

$$\begin{aligned} x^T \tilde{L} x \;&=\; x^T L x + x^T \hat{L} x \\ &=\; \textstyle\sum_{(v_i, v_j) \in \mathcal{E}} w_{ij}^2 (x_i - x_j)^2 \\ &\quad + \textstyle\sum_{(v_i, v_j) \in \hat{\mathcal{E}}} w_{ij}^2 (x_i - x_j)^2. \end{aligned}$$

According to [14], if x a unit vector and orthogonal to $\mathbf{1}$,

$$\frac{x^T L x}{x^T x} = \frac{\sum_{(v_i, v_j) \in \mathcal{E}} (x_i - x_j)^2}{\sum_{v_i \in \mathcal{V}} x_i^2} \ge \lambda_2(L)$$

and the equation holds only when x is an eigenvector associated with $\lambda_2(L)$.

Combine these two results, if we take x to be a unit eigenvector of $\tilde{L}$, orthogonal to $\mathbf{1}$, associated with eigenvalue $\lambda_2(\tilde{L})$, then we have

$$\lambda_2(\tilde{L}) = \frac{x^T \tilde{L} x}{x^T x} = \frac{x^T (L + \hat{L}) x}{x^T x} \ge \lambda_2(L) + \frac{x^T \hat{L} x}{x^T x}. \tag{8}$$

■

Theorem 3.3 shows that two-hop relay protocol increases the convergence speed. The improvement depends on the topology of $\hat{\mathcal{G}}$. Obviously, $\hat{\mathcal{G}}$ is symmetric too. It also can be shown that the edge set of $\hat{\mathcal{G}}$ is not empty if the original graph has more than two vertices,

Proposition 3.4: If $\hat{\mathcal{G}}$ is connected,

$$\lambda_2(\tilde{L}) \ge \lambda_2(L) + \lambda_2(\hat{L}). \tag{9}$$

Proof: When $\hat{\mathcal{G}}$ is connected, $x^T \hat{L} x / x^T x \ge \lambda_2(\hat{L}) > 0$. Thus, two-hop relay protocol improves the algebraic connectivity by at least $\lambda_2(\hat{L})$. ■

However, it is not true that $\hat{\mathcal{G}}$ is always connected. Fig. 3 shows a simple example. The original graph on the left is symmetric and connected, but the two-hop graph on the right is composed by two disconnected subgraphs.

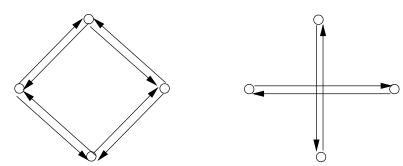

Fig. 3. An example for disconnected two-hop directed graph

C. Multi-hop relay protocol

It is possible to extend the two-hop relay protocol to multi-hop relay protocol. The protocol for m-hop relay protocol can be written as

$$\dot{x}_i = \underbrace{-\sum_j w_{ij} \big((x_i - x_j) + \sum_k w_{jk} ((x_i - x_k) + \cdots)\big)}_{m \text{ layers}}. \tag{10}$$

Clearly it adds more virtual edges to the original graph and enforces the convergence speed. However, there are three drawbacks. First, the worst case computation complexity of m-hop relay protocol on each agent is $O(n^{m-1})$. For large scale networks, it will be quickly infeasible as m increases. Second, m-times communication bandwidth are needed. Third, communication delays will accumulate along m-hop paths and that makes the protocol very sensitive to communication latency.

IV. Two-Hop Relay Protocols with Time Delays

For communication delay in relay protocols, we consider the transfer function associated with edge (v_i, v_j) with latency τ_{ij} is $h_{ij}(s) = e^{-\tau_{ij} s}$. Delays will be accumulated in two-hop relay protocol. We focus on the simplest case where all delays are identical, i.e., $\tau_{ij} = \tau$ for any $(v_i, v_j) \in \mathcal{E}$. The protocol (1) can be written as:

$$\dot{x}_i = -\sum_{j \in N(v_i)} w_{ij}(x_i(t-\tau) - x_j(t-\tau)) \tag{11}$$

and the two-hop relay protocol is:

$$\begin{aligned}\dot{x}_i \quad = \quad & -\textstyle\sum_{j \in N(v_i)} w_{ij}\Big((x_i(t-\tau) - x_j(t-\tau)) \\ & + \textstyle\sum_{k \in N(v_j)} w_{jk}(x_i(t-2\tau) - x_k(t-2\tau))\Big).\end{aligned} \tag{12}$$

Equations (2) and (4) change to

$$\dot{X} = -LX(t-\tau) \tag{13}$$

and

$$\dot{X} = -LX(t-\tau) - \hat{L}X(t-2\tau) \tag{14}$$

respectively.

Let $Z = V^{-1}X$ where

$$V^{-1} = \begin{bmatrix} 1 & \mathbf{1} \\ \mathbf{1} & -I \end{bmatrix}. \tag{15}$$

For two-hop relay protocol, we have to

$$\dot{Z} = -V^{-1}LVZ(t-\tau) - V^{-1}\hat{L}VZ(t-2\tau). \tag{16}$$

Note that

$$V^{-1}LV = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & L_{22} \end{bmatrix} \text{ and } V^{-1}\hat{L}V = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \hat{L}_{22} \end{bmatrix}$$

where L_{22} is full rank.

Let us assume that $X(t) = 0$ for any $t < 0$. Then the system asymptotically converges to $\eta = \sum x_i(0)/n$ if and only if the partition characteristic polynomial

$$p_{22}(s, e^{-\tau s}) = \det\left(sI + L_{22}e^{-\tau s} + \hat{L}_{22}e^{-2\tau s}\right) \tag{17}$$

has no zero in the closed right half plane (RHP). This condition is equivalent to that the characteristic polynomial

$$p(s, e^{-\tau s}) = \det\left(sI + Le^{-\tau s} + \hat{L}e^{-2\tau s}\right) \tag{18}$$

has no zero in the closed RHP except the simple zero at the origin. In [15], $p(s, e^{-\tau s})$ is also called a real quasipolynomials of s. A similar result also holds for (13).

One of the essential properties of quasipolynomials is the continuity of the zeros with respect to delay. In other words, when τ increases, zeros in left half plane (LHP) move to RHP. Time delay does not affect the zero $s = 0$. Thus, we need to find minimum value of τ such that the first stable zero crosses the imaginary axis. Besides, the conjugate symmetry property of quasipolynomials make it possible to calculate the critical value of time delay and the corresponding crossing frequency.

Definition 4.1: Given initial value $X(0)$ and assumption $X(t) = 0$ for any $t < 0$, the smallest value of τ such that the system cannot converge to a consensus is determined as

$$\tau^* = \min\{\tau > 0 \,|\, p(j\omega, e^{-j\tau\omega}) = 0 \text{ and } \omega \neq 0\}. \tag{19}$$

We call τ^* the *delay margin* of the consensus protocol. For any $\tau \in [0, \tau^*)$, the system of (13) or (14) converges to the consensus state $\eta = \sum x_i(0)/n$.

Lemma 4.2: Let τ^* and $\tilde{\tau}^*$ indicate the delay margin of (13) and (14) respectively, then $\tau^* \geq \tilde{\tau}^*$.

Proof: First, let us find τ^*. According to Schur theorem, there exists a unitary matrix T such that $U = T^{-1}LT$ is an upper triangular with the eigenvalues along the diagonal. So

$$\begin{aligned}\det(sI + Le^{-\tau s}) &= \det(sI + TUT^{-1}e^{-\tau s}) \\ &= \det(T(sI + Ue^{-\tau s})T^{-1}) \\ &= s \cdot \Pi_{i=2}^{n}(s + \lambda_i(L)e^{-\tau s}).\end{aligned}$$

We need to find the smallest $\tau > 0$ such that the first stable zero reaches the imaginary axis. Let $s = j\omega$ and we have have

$$j\omega = -e^{\tau j\omega}\lambda_i(L) \tag{20}$$

Solving this equation gives us

$$\begin{cases} \omega = \lambda_i(L) \neq 0 \\ \tau = \pi/2\lambda_i(L). \end{cases} \tag{21}$$

So the delay margin

$$\tau^* = \min \pi/2\lambda_i(L) = \pi/2\lambda_n(L). \tag{22}$$

Then we consider $\tilde{\tau}^*$. The approach above fails for $\det(sI + Le^{-\tau s} + \hat{L}e^{-2\tau s})$. However, it is obvious that $\tilde{\tau}^*$ should be no bigger than the delay margin for $\det(sI + (L + \hat{L})e^{-\tau s})$, which is $\pi/2\lambda_n(\tilde{L})$. Moreover, $\lambda_n(\tilde{L}) \geq \lambda_n(L)$ according to [14]. So we have

$$\tau^* = \pi/(2\lambda_n(L)) \geq \pi/(2\lambda_n(\tilde{L})) \geq \tilde{\tau}^*.$$

■

Lemma 4.2 just shows us the delay sensitivity of two-hop relay protocol is no better than consensus protocol (1). Following theorem gives us explicit results on $\tilde{\tau}^*$ by using frequency-sweeping test.

Theorem 4.3: For system (14), let rank$(\hat{L}) = q$ and define

$$\bar{\tau}_i = \min_{1 \leq k \leq n-1} \theta_k^i / \omega_k^i$$

when the generalized eigenvalues $\lambda_i(G(s), H)$ satisfy the following equation:

$$\lambda_i(G(j\omega_k^i), H) = e^{-j\theta_k^i}$$

for some $\omega_k^i \in (0, \infty)$ and $\theta_k^i \in [0, 2\pi)$, where

$$G(s) = \begin{bmatrix} 0 & I \\ -sI & -L_{22} \end{bmatrix} \text{ and } H = \begin{bmatrix} I & 0 \\ 0 & \hat{L}_{22} \end{bmatrix}.$$

Then the consensus delay margin of (14) is

$$\tilde{\tau}^* = \min_{1 \leq i \leq n+q-1} \bar{\tau}_i.$$

Proof: Finding generalized eigenvalues for matrix pair (A, B) is the problem of finding λ_i and non-zero vector y such that $Ay = \lambda_i By$.

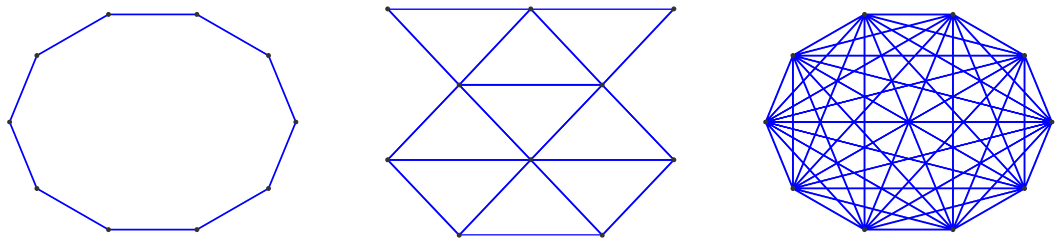

Fig. 4. Three examples with different topologies: $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$

According to the aforementioned similarity transform, if the system (14) converges to the average consensus is determined by if the system

$$\begin{bmatrix} \dot{z}_2 \\ \vdots \\ \dot{z}_n \end{bmatrix} = -L_{22} \begin{bmatrix} \dot{z}_2(t-\tau) \\ \vdots \\ \dot{z}_n(t-\tau) \end{bmatrix} - \hat{L}_{22} \begin{bmatrix} \dot{z}_2(t-2\tau) \\ \vdots \\ \dot{z}_n(t-2\tau) \end{bmatrix}$$

is asymptotically stable. Since none of the generalized eigenvalues of $(G(j\omega), H)$ can be strictly larger than 1 for all $\omega \in (0, \infty)$, the result follows by adopting the frequency-sweeping test for multiple commensurate delay systems in [15]. ■

V. Examples and Simulation Results

In order to verify the efficiency of two-hop relay consensus protocol, we test it on three graphs. Fig. 4 shows the topologies of $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ from left to right, where $\mathcal{G}_1$ is a 2-regular graph, $\mathcal{G}_2$ is a net in which each vertex connects to neighbors located inside a certain range, and $\mathcal{G}_3$ is a complete graph. All of them have ten vertices. They are symmetric and connected. Each pair of edges (v_i, v_j) and (v_j, v_i) belong to those graphs is denoted by a single link and we assume that $w_{ij} = 1$ for any vertices pair.

Fig. 5 to Fig. 8 show the simulation results of $\mathcal{G}_2$ with same initial conditions and different delays. Note that, even the system can become unstable, the sum of the states keeps constant. Table I shows the algebraic connectivities and delay margins for all three graphs with or without two-hop relay protocols. Delay margins without relay protocols are calculated according to equation (22) in Lemma 4.2. Delay margins with relay are computed using frequency sweep method mentioned in Theorem 4.3. Note that the magnitudes of generalized eigenvalues inevitably exceed 1 after a certain ω, the computation needs to be done only over a finite frequency interval [15]. We actually run the computation twice. First time we try to find an appropriate frequency interval. Second time we use a much smaller frequency step over the interval in order to find more accurate value of delay margin.

For each graph, relay protocol improves the convergence speed. However, time-delay robustness is impaired due to the delay accumulation along the two-hop paths. Moreover, along the columns of the table, we can tell that algebraic connectivity increases and delay margin decreases when the graph includes more links. We put these data in Fig. 9. For each bar, the right lower point corresponds to protocol (1) and left upper point corresponds to two-hop relay protocol. It is true that relay protocols actually boost up algebraic connectivity by sacrificing delay margin.

TABLE I

Performance V.S. Robustness for Relay Protocols

	Algebraic connectivity λ_2		**Delay margin τ^***	
	Without relay	With relay	Without relay	With relay
$\mathcal{G}_1$	0.382	1.7639	0.3927	0.1796
$\mathcal{G}_2$	0.9118	7.3846	0.2167	0.0396
$\mathcal{G}_3$	10	90	0.1571	0.0095

VI. Conclusions and Future Work

In this paper, we propose multi-hop relay protocols for fast consensus seeking and emphasize on the two-hop relay protocol. When the topology is symmetric and connected, using relay protocols can efficiently improve the convergence speed without physically changing the topology. The cost we need to pay is that extra communication bandwidth.

A trade off between the convergence performance and robustness of communication time delay are shown by investigating three typical topologies with relay protocols. The more edges the graph includes, the faster the convergence speed is, while the more sensitive the protocol is to the time delay. Moreover, frequency sweep method can efficiently find the delay margin with multiple commensurate delays and it is a power tool to study the time delay sensitivity of multi-hop relay protocols.

Future work will include studying large group of graphs and putting their performance/robustness data into Fig. 9. Relationship between patterns and topology characters should be carefully examined. Comprehensively describe and deeply understand this trade off will benefit us for topology and protocol design for multi-agent networks.

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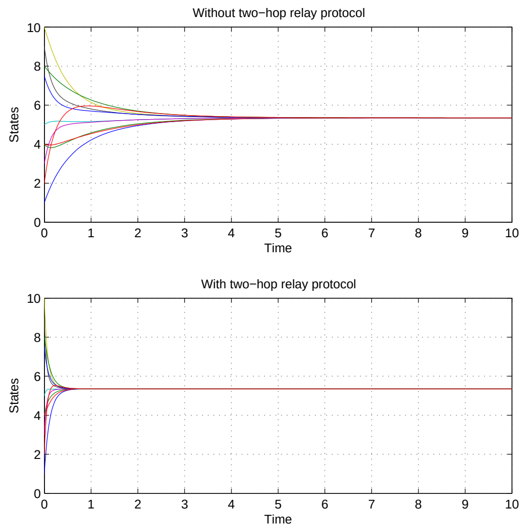

Fig. 5. States of graph $\mathcal{G}_2$ with no delay.

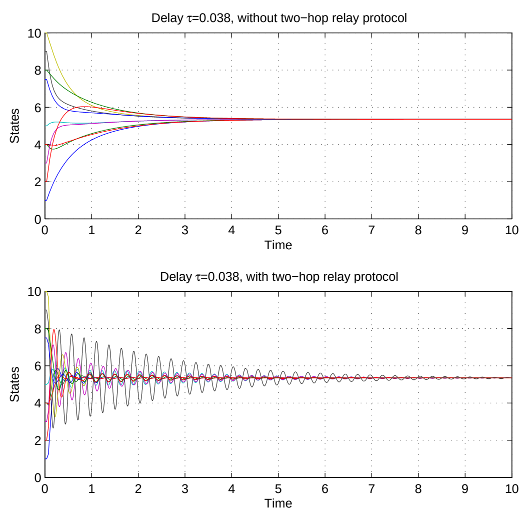

Fig. 6. States of graph $\mathcal{G}_2$ with delay $\tau = 0.038$.

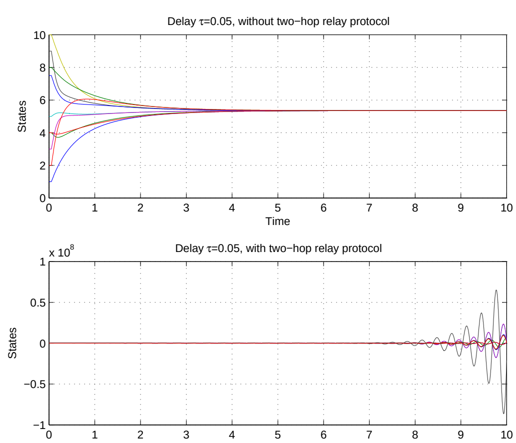

Fig. 7. States of graph $\mathcal{G}_2$ with delay $\tau = 0.05$.

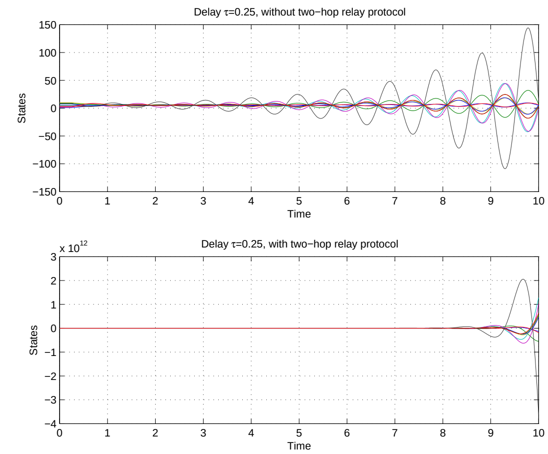

Fig. 8. States of graph $\mathcal{G}_2$ with delay $\tau = 0.25$.

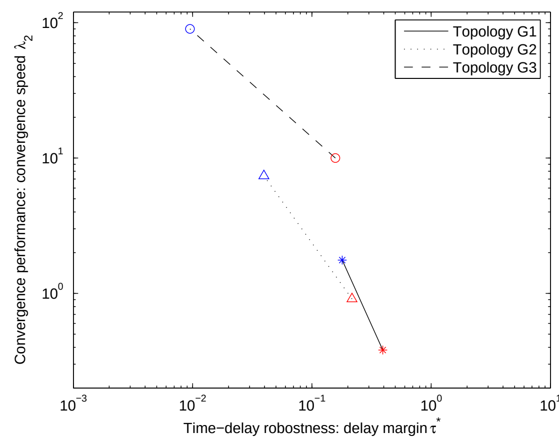

Fig. 9. Trade off between λ_2 and τ^*

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