Exercise 6.4 (Integral feedback for rejecting constant disturbances) Consider a linear system of the form

\[
\frac{dx}{dt} = Ax + Bu +Fd, \quad y = Cx
\]

where \( u \) is a scalar and \( d \) is a disturbance that enters the system through a disturbance vector \( F \in \mathbb{R}^n \). Assume that the matrix \( A \) is invertible and \( CA^{-1}B \neq 0 \). Show that integral feedback can be used to compensate for a constant disturbance by giving zero steady-state output error even when \( d \neq 0 \).

**Solution.** Consider first the steady state solution given by

\[
Ax + Bu + Fd = 0.
\]

It is clear that this equation does not have a solution which is zero unless the vectors \( B \) and \( F \) are parallel. When this is not the case we cannot expect to have a solution with zero steady state \( x \) no matter what control law is used. We therefore have to have a more modest goal, namely that the projection \( y = Cx \) of the state is zero in steady state.

A candidate for a controller which drives \( y \) to zero is the proportional-integral (PI) controller

\[
u = -Kx - k_i \int_0^t y(\tau) d\tau
\]

where \( y = Cx \). To design the controller we augment the state \( x \) by the output \( z = Cx \) which we would like to drive to zero. The augmented system is

\[
\frac{dx}{dt} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} x + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} F \\ 0 \end{bmatrix} d, \quad u = -Kx - k_i z.
\]

We will first explore the conditions for the augmented system to be reachable by forming the reachability matrix:

\[
W_r = \begin{bmatrix} B & AB & \ldots & A^{n-1}B \\ 0 & CB & \ldots & CA^{n-1}B \end{bmatrix}.
\]

To find the conditions for \( W_r \) to be of full rank, the matrix will be transformed by making column operations. Let \( a_i \) be the coefficients of the characteristic polynomial of the matrix \( A \). Multiplying the first column by \( a_n \), the second by \( a_{n-1} \) and the \( (n-1) \)th column by \( a_1 \) and adding the sum to the last column the matrix \( W_r \), it follows from the Cayley-Hamilton theorem that the transformed matrix becomes

\[
W_r = \begin{bmatrix} B & AB & \ldots & 0 \\ 0 & CB & \ldots & b_n \end{bmatrix},
\]

where

\[
b_n = C(A^{n-1}B + a_1A^{n-2}B + \ldots + a_{n-1}B)
= CA^{-1}(A^n + a_1A^{n-1} + \ldots + a_{n-1}A)B = -a_nCA^{-1}B = -(\det A)CA^{-1}B.
\]

Notice that we have used the Cayley-Hamilton theorem to obtain the last inequality. The parameter \( a_n \) is not zero because it was assumed that the matrix \( A \) was invertible. It also follows from the given assumptions that the number \( a_n \) is not zero and together these conditions imply that the system is reachable.

The closed loop system with the PI controller is

\[
\frac{d}{dt} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} A - BK & Bk_i \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ F \end{bmatrix} d.
\]

Since the system is reachable we can find controller gains \( K \) and \( k_i \) that assigns arbitrary eigenvalues to the closed loop system. Provided that the eigenvalues are in the left plane the closed loop system has an
equilibrium \( x_e, z_e \) where
\[
\begin{bmatrix}
A - BK & Bk_i \\
C & 0
\end{bmatrix}
\begin{bmatrix}
x_e \\
z_e
\end{bmatrix}
+ \begin{bmatrix}
F \\
0
\end{bmatrix}
d.
\]
The last line of this matrix equation implies that \( z = Cx_e = 0 \) which means that the output \( y \) is zero in steady state. Notice however that the state \( x_e, z_e \) is in general different from zero except for the case where \( F \) and \( B \) are parallel. If \( F = \alpha B \) the equation above has the solution
\[
x_e = 0, \quad z_e = \frac{\alpha}{k_i}d.
\]
In this case, the state \( x_e \) and the output \( y_e \) are zero but the augmented state \( z_e \) is different to compensate for the disturbance.