ACM 201 Final

Wednesday 9th March, 2011

1. Please write down your solutions clearly and concisely, bind pages in order.

2. Please hand in your copy to Mulin at 216 Firestone by 6:00pm Wed 16th Mar, 2011.

3. The exam is CLOSED BOOK with the exceptions noted in what follows. The only materials that may be used are:
   - The required textbook by Evans.
   - Class notes.
   - Problem Sets 1-5 of this term.

4. The exam is NOT TIMED, so work at your own pace. On average, it can be finished in 6-8 hours.

5. Extensions are ONLY granted in medical or extreme cases. Please contact Prof. Hou for extension.

6. Honor Code is in effect.
   - (a) No Collaborations. The only persons who you can talk with about the final are Prof. Hou and TA. To be fair to the class, I will reply any email to all students.
   - (b) No Internet Searching.

7. Problems are not placed in order of difficulty, so start with ones you feel comfortable with first and come back to the sticky ones.

8. Total Points: 100.

9. Good Luck.
Problem 1

Hopf-Lax Formula

(20 points) In this problem, we complete some missing steps in the proof of the Hopf-Lax formula. Suppose the Lagrangian \( L : \mathbb{R}^n \rightarrow \mathbb{R} \) is convex.

1. (10 points) Show that the convexity implies \( L \) is continuous.

2. (10 points) If \( L \) satisfies

\[
\lim_{|q| \rightarrow \infty} \frac{L(q)}{|q|} = +\infty
\]

The infimum in the Hopf-Lax formula is really a minimum, i.e.,

\[
u(x, t) = \inf_{y \in \mathbb{R}^n} \left\{ tL \left( \frac{x-y}{t} \right) + g(y) \right\} = \min_{y \in \mathbb{R}^n} \left\{ tL \left( \frac{x-y}{t} \right) + g(y) \right\}
\]

Problem 2

Characteristics

(10 points)

1. (5 points) Show that the solution of the nonlinear equation

\[
u_x + \nu_y = \nu^2
\]

passing through the initial curve \( x = t, \ y = -t, \ u = t \), becomes infinite along the hyperbola

\[x^2 - y^2 = 4\]

2. (5 points) Explain why there are no solutions of the linear equation

\[
u_x + \nu_y = \nu
\]

which pass through the straight line \( x = t, \ y = t, \ u = 1\).

Problem 3

Reduced Wave Operator

(40 points) Let the differential operator \( L = \Delta + c \) be the reduced wave operator in \( n = 3 \) dimensions where \( c \) is a real number.

1. (5 points) Find all solutions of \( Lu = 0 \) with spherical symmetry.

2. (15 points) Write the fundamental solution for \( L \) at pole \( \xi \),

\[
K(x, \xi) = -\frac{\cos(\sqrt{c}|x-\xi|)}{4\pi|x-\xi|}
\]

Please note that \( \cos(ix) = \cosh(x) \) and \( \sin(ix) = i\sinh(x) \), so the above formula is well-defined even if \( c < 0 \). Show that for a solution \( u \) of \( Lu = 0 \) of class \( C^2(\Omega) \), we have

\[
u(\xi) = -\int_{\partial\Omega} \left( K(x, \xi) \frac{\partial u}{\partial n(x)}(x) - u(x) \frac{\partial K(x, \xi)}{\partial n(x)} \right) dS(x)
\]

where \( \Omega \) is a bounded open set with \( C^1 \) boundary in \( \mathbb{R}^3 \) and \( n(x) \) is the outward unit normal vector at \( x \in \partial\Omega \).
3. (10 points) Show that a solution $u$ of $Lu = 0$ in the ball $|x - \xi| \leq \rho$ with $\sin(\sqrt{c}\rho) \neq 0$ has the modified mean value property

$$u(\xi) = \frac{\sqrt{c}\rho}{\sin(\sqrt{c}\rho)} \frac{1}{4\pi \rho^2} \int_{|x - \xi| = \rho} u(x) \, dS(x)$$

(8)

4. (5 points) Assume $\Omega$ is connected. Show that a solution $u$ of $Lu = 0$ of class $C^2(\bar{\Omega})$ vanishing on $\partial \Omega$ necessarily vanishes in $\Omega$ provided $c < 0$.

5. (5 points) Show that for $c > 0$ there are solutions vanishing on a sphere but not in the interior.

Problem 4

Wave Equation in Five Dimensions

(30 points) We solved the initial value problem for the wave equation in two and three dimensions in class. In higher dimensions, the wave equation can be solved essentially similarly, which is the goal of this problem. To avoid technical difficulties, we restrict ourselves in five dimensions. Specifically, consider the initial-value problem for the wave equation in $n = 5$ dimensions,

$$\begin{cases}
\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 & x \in \mathbb{R}^n, \ t > 0 \\
u(x, 0) = f(x) \\
\frac{\partial u}{\partial t}(x, 0) = g(x)
\end{cases}$$

(9)

Write $M_u(x, r, t)$ the spherical means of $u(x, t)$ as a function of $x$, i.e.,

$$M_u(x, r, t) = \frac{1}{\partial B(x, r)} \int_{\partial B(x, r)} u(y, t) \, dS(y)$$

Similarly, we write $M_f$ and $M_g$.

1. (8 points) For $h \in C^2(\mathbb{R}^n)$, derive Darbou’s equation,

$$\left( \frac{\partial^2}{\partial r^2} + \frac{n - 1}{r} \frac{\partial}{\partial r} \right) M_h(x, r) = \Delta_x M_h(x, r)$$

(10)

2. (2 points) Write down Euler-Poisson-Darbox Equation in 5 dimensions.

3. (5 points) Set

$$N(x, r, t) = \left( r^2 \frac{\partial}{\partial r} + 3r \right) M_u(x, r, t)$$

Show that $N(x, r, t)$ is a solution of one-dimensional wave equation, i.e.,

$$N_{tt} = c^2 N_{rr}$$

4. (5 points) Solve $N$ from its initial data in terms of $M_f$ and $M_g$.

5. (10 points) Show that

$$u(x, t) = \lim_{r \to 0} \frac{N(x, r, t)}{3r} = \left( \frac{1}{3} t^2 \frac{\partial}{\partial t} + t \right) M_g(x, ct) + \frac{\partial}{\partial t} \left( \frac{1}{3} t^2 \frac{\partial}{\partial t} + t \right) M_f(x, ct)$$