Beyond Myopic Best Response (in Cournot Competition)

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Abstract

A Nash Equilibrium is a joint strategy profile at which each agent myopically plays a best response to the other agents' strategies, ignoring the possibility that deviating from the equilibrium could lead to an avalanche of successive changes by other agents. However, such changes could potentially be beneficial to the agent, creating incentive to act non-myopically, so as to take advantage of others' responses.

To study this phenomenon, we consider a non-myopic Cournot competition, where each firm selects whether it wants to maximize profit (as in the classical Cournot competition) or to maximize revenue (by masquerading as a firm with zero production costs).

The key observation is that profit may actually be higher when acting to maximize revenue, (1) which will depress market prices, (2) which will reduce the production of other firms, (3) which will gain market share for the revenue maximizing firm, (4) which will, overall, increase profits for the revenue maximizing firm. Implicit in this line of thought is that one might take other firms' responses into account when choosing a market strategy. The Nash Equilibria of the non-myopic Cournot competition capture this action/response issue appropriately, and this work is a step towards understanding the impact of such strategic manipulative play in markets.

We study the properties of Nash Equilibria of *non-myopic* Cournot competition with linear demand functions and show existence of pure Nash Equilibria, that simple best response dynamics will produce such an equilibrium, and that for some natural dynamics this convergence is within linear time. This is in contrast to the well known fact that best response dynamics need not converge in the standard myopic Cournot competition.

Furthermore, we compare the outcome of the non-myopic Cournot competition with that of the standard myopic standard Cournot competition. Not surprisingly, perhaps, prices in the non-myopic game are lower and the firms, in total, produce more and have a lower aggregate utility.

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1 Introduction

Understanding competition between firms is a fundamental problem in economics. One of the oldest and most studied models in this area is the Cournot competition [3]. In a Cournot competition there is a single divisible good, each firm has a certain production cost per unit to manufacture the good, and each firm must select a quantity of the good to produce. The price is then set as a function of the total quantity produced by all of the firms. Naturally, as the quantity increases the price decreases, and thus the firms face a tradeoff between the amount produced and the market price.

The Cournot competition model highlights some potential problems with treating the Nash equilibrium as the inevitable outcome of competitive play. Consider the following example: There are two oil producing firms, Wildcat Drillers and W. Petroleum. Wildcat Drillers has a production cost of $0.5 USD per mega-barrel; W. Petroleum has a production cost of $0.3 USD per mega-barrel. If the price per mega-barrel decreases linearly, specifically, if price = (1 - total supply in mega-barrels), then the Cournot competition equilibrium price is $0.6. At this equilibrium price, both firms are producing and no firm can benefit by unilaterally changing its production quantity, assuming that the other firm does not change its production quantity. (In our toy example the price drops down to zero when the world supply is one mega-barrel of oil.)

If W. Petroleum were to increase its production such that the price dropped below $0.5, Wildcat Drillers would be producing at a loss. The inherent assumption in the Cournot-Nash equilibrium is that if this happened Wildcat Drillers would indeed continue producing at the same level as before, despite this loss, or that W. Petroleum would never manipulate the market in this manner. However, W. Petroleum may hypothesize that by driving the price down, Wildcat Drillers will in fact cease production, rather than continuing production a loss. This hypothesis seems rather natural, but its predictions are not captured by traditional Cournot-Nash equilibria.

The impetus for our work is a sense of unease about the assumption that agents act myopically and ignore responses to their own actions. In the context of competition, it seems natural that firms should be able to predict something about the behavior of other firms, as a function of changes in pricing.

To further understand this issue, we propose an abstract meta-game of the Cournot competition. In our meta-game, the firms can select between maximizing their profit (selecting action PM) or maximizing their revenue (selecting action RM). When a firm selects PM it simply tries to maximize it profits (similar to the Cournot competition). However, when a firm selects RM, it ignores its production cost, and attempts to maximize its revenue. After each selecting one of these two strategies, firms participate in a Cournot competition, where the PM firms use their true production costs to determine production levels and the RM firms use a production cost of zero to decide how much to produce. As in the standard Cournot competition, firms in the meta-game experience utilities as determined by their true production costs. The major difference between the meta-game and the underlying Cournot competition is that when a firm changes its action in the meta-game, it results in a change in the production quantities of the other firms (by converging to an equilibrium of the underlying Cournot competition). We refer to this meta-game as the PM/RM *game*.

We show that the PM/RM game always has a pure Nash equilibrium, and that the resulting equilibrium price of the PM/RM game is at most the Cournot competition market price and at least half of it. On the other hand, the aggregate utility of the firms participating in the competition might be significantly lower in the PM/RM game. Conceptually, we show that in our model, strategizing about others' responses increases competition, reduces prices, and improves social welfare, all while reducing corporate profits.

We are also interested in the dynamics underlying the Cournot competition and the PM/RM game. Interestingly, a single change of strategy in the PM/RM game may result in a dynamic cascade of best response moves in the underlying Cournot competition. For example, if W. Pertoleum increases production, then the market price will go down, and if it goes down enough then some firms may drop out of the market (e.g., Wildcat Drillers might stop production). As firms drop out of the market, the total supply goes down, and — possibly — firms that previously were not producing anything (say, a new company called Texas Oil) suddenly start production.[1]

We show that best response dynamics in the PM/RM game always converge to a pure Nash equilibrium. We also demonstrate simple dynamics that converge in a linear number of updates, and thus such an equilibrium is polytime-computable.

We consider two important special cases of the PM/RM game, in which we give a complete characterization of the pure Nash equilibria: (a) only two firms in the game and (b) all firms have the same production cost (the symmetric case). In the symmetric case it is interesting to observe that there are non-symmetric pure Nash equilibria. In fact, for any choice of i firms selecting PM and $m - i$ firms selecting RM, there is a cost c for which this strategy profile is in equilibrium.

We also extend our two-action meta-game to a continuous PM/RM game, where firms may bid an arbitrary real value, which is interpreted as their perceived cost. We show that in the continuous PM/RM game, the firms' utilities are concave in the relevant region, which implies that there is always a pure Nash equilibrium.

Related Work

Cournot competition assumes a so-called *conjectural variation model*, [2], *i.e.*, the Cournot conjectured variation is that if one firm changes it's production level then other firms will not adjust their production level accordingly. Under this assumption, the Cournot competition is a Nash Equilibrium, in this setting the Nash equilibrium is sometimes referred to as a Cournot-Nash equilibrium.

This Cournot conjectured variation is a subject of much debate and criticism in the economics literature. With conflicting conclusions: To quote Abreu, [1], "In recent times this model has been criticized for being too static, and thereby yielding predictions which are misleadingly competitive". Abreu then goes on to describe how the threat of punishment in an extended game could support higher prices than the Cournot equilibrium prices. In this setting, the market prices can be higher than the prices of the Cournot-Nash equilibrium.

Contrawise, Riordan, [7], considers a setting with imperfect information where firms only see the prices they receive. In a multi stage game, a firm could increase it's output to lower the market clearing price, this causes rival firms to think that the demand curve has shifted down, and hence induces them to lower their outputs in the future. Thus, the market price will be lower than that projected by the Cournot competition prices.

Like Abreu and Riordan, we consider firms that act non-myopically, firms assume that other firms with adapt to changes in the environment. We reach a conclusion rather similar to that of [7], qualitatively, and give quantitative projections as well.

[1] The dynamics described above are the dynamics of the underlying Cournot competition, and can be inferred as a consequence of actions in the PM/RM game. In the PM/RM game, there may also be meta-level cascading effects; for example, firms may move from maximizing profit to maximizing revenue, and then, after other firms respond (in the PM/RM game), they may go back to maximizing profit.

The best response dynamics of the linear Cournot competition are known to converge for two firm (see [6]) and possibly diverge for four or more firms [8]. The regret minimization dynamics are converge for linear Cournot competition [4].

2 The Model

2.1 Standard (Myopic) Linear Cournot Competition

We consider a set of m firms, $M = \{1, \dots m\}$, producing an identical good, where firm i has production cost c_i per unit of production. Every firm chooses a production level $x_i \in [0, 1]$. Let $x = \langle x_1, x_2, \dots, x_m \rangle$ be the joint production levels of all m firms. The linear Cournot model we consider here assumes the market price is a linearly decreasing function of the production levels, that is,

$$p(x) = 1 - \sum_{i=1}^{m} b_i x_i, \tag{1}$$

for strictly positive constants $b_1, b_2, \dots, b_m$. The profit (utility) of firm $i \in M$ is the profit per unit of production times the quantity produced, *i.e.*,

$$u_i(x) = (p(x) - c_i) \cdot x_i.$$

Consider a linear Cournot competition with firms $i \in M = \{1, \dots, m\}$ and production costs c_i. A Cournot-Nash equilibrium is a joint production level, $x^{\mathrm{eq}} = \langle x_1^{\mathrm{eq}}, x_2^{\mathrm{eq}}, \dots, x_m^{\mathrm{eq}} \rangle$, where for each firm i, x_i^{eq} maximizes the utility for firm i, given x_{-i}^{eq}.[2] That is,

$$x_i^{\mathrm{eq}} \in \mathrm{argmax}_x u_i(x, x_{-i}^{eq}) \qquad \text{for all } 1 \le i \le m.$$

The following proposition, and variants thereof, are well known. We give the proof only for the sake of completeness.

Proposition 1 *Given a linear Cournot competition of m firms with production levels x^{eq} at Cournot-Nash equilibrium, let $N \subseteq M = \{1, \dots, m\}$ be the set of firms with strictly positive production levels at equilibrium, i.e., $N = \{i \in M \mid x_i^{\mathrm{eq}} > 0\}$, and let $n = |N|$.*

The Cournot-Nash equilibrium has the following characteristics:

1. *For any firm $i \in N$ (with strictly positive production levels), we have*

$$x_i^{\mathrm{eq}} = \frac{p(x^{\mathrm{eq}}) - c_i}{b_i}. \tag{2}$$

2. *The market clearing price at equilibrium is*

$$p(x^{\mathrm{eq}}) = \frac{1 + \sum_{i \in N} c_i^{\mathrm{eq}}}{n + 1} = p_{\mathrm{eq}}(c). \tag{3}$$

[2]We denote by x_{-i} the vector x except for the i-th component, and by (x_i, a) the vector x where the i-th component is replaced by a.

3. *The utility of non-producing firms ($j \notin N$) is zero, and the utility of producing firms ($i \in N$) is*

$$u_i(x^{\text{eq}}) = \frac{(p_{\text{eq}}(c) - c_i)^2}{b_i}. \tag{4}$$

Proof: To compute the Cournot-Nash equilibrium we take the derivative of $u_i(x) = (p(x) - c_i) \cdot x_i$ with respect to x_i. It follows from Equation (1) that

$$\frac{\partial}{\partial x_i} u_i(x) = (p(x) - c_i) - b_i x_i. \tag{5}$$

It follows from Equation (5) that $b_i x_i^{\text{eq}} = p(x^{\text{eq}}) - c_i = p_{\text{eq}}(c) - c_i$. Note that in equilibrium a firm $i \in M$ has $x_i^{\text{eq}} > 0$ iff $c_i < p_{\text{eq}}(c)$. Taking the sum over all the firms $N \subseteq M$ with strictly positive production levels we have

$$|N| p_{\text{eq}}(c) - \sum_{j \in N} c_j = \sum_{j \in N} b_j x_j^{\text{eq}} = 1 - p_{\text{eq}}(c) \ ,$$

where the second equality follows from the definition of the market price in a linear Cournot competition (Equation (1)). This implies that the market clearing price at equilibrium is

$$p(x^{\text{eq}}) = p_{\text{eq}}(c) = \frac{1 + \sum_{j \in N} c_j}{n+1}.$$

Thus, the utility of a firm $i \in N$, at equilibrium, is $(p_{\text{eq}}(c) - c_i) \cdot x_i^{\text{eq}} = (p_{\text{eq}}(c) - c_i)^2 / b_i$. □

2.2 The PM/RM Game

To address the issue that actions of one firm may impact the actions of another, resulting in an outcome other than a Cournot-Nash equilbrium, we introduce a meta-game. In this new game, which we refer to as the PM/RM game, a firm selects between two strategies (we consider other variants later on):

1. PM (profit maximization), and
2. RM (revenue maximization).

In this PM/RM *game*, as in the Cournot competition, we have a set of M firms $\{1, \dots m\}$, and each firm i has a production cost c_i. Each firm selects an action in $\{\text{PM}, \text{RM}\}$. Let $g(c, \text{RM}) = 0$ and $g(c, \text{PM}) = c$. Given a joint action $z \in \{\text{PM}, \text{RM}\}^m$, we define a virtual cost vector $y(z)$ such that $y_i(z) = g(c_i, z_i)$.

One can interpret playing PM as though the board of directors tells the CEO to maximize profit in a (standard) Cournot competition. Choosing strategy RM can be viewed as though the board of directors instructs the CEO to ignore production costs. Effectively, the board determines a virtual cost, which could be either the true production cost or zero. In both cases, the CEO takes this virtual production cost and chooses a production level corresponding to that production cost in the standard Cournot competition. When production costs are zero, profit and revenue are identical, and thus we can consider such an action as revenue maximizing.

We now consider the Cournot-Nash equilibrium of this *virtual* Cournot competition, played with virtual production costs $y_i(z) = g(c_i, z_i)$ rather than c_i. For this Cournot-Nash equilibrium we have production levels $x^{\text{eq}}(y(z))$, and price $p_{eq}(y(z))$. It follows from Equation (2) that the production levels derived from the virtual Cournot competition are as follows:

1. If firm i chooses profit maximization (PM) then the production level is $x_i^{\mathrm{eq}}(y(z)) = (p_{eq}(y(z)) - c_i)/b_i$[3].

2. If firm i chooses revenue maximization (RM) then the production level $x^{\mathrm{eq}}(z) = x_i^{\mathrm{eq}}(y(z)) = p_{eq}(z)/b_i$.

Similar to the state of affairs for a (myopic) Cournot competition, the utility of firm $i \in M$ in the PM/RM game is $u_i(z) = (p_{eq}(z) - c_i)x_i(z)$. Note that a firm's utility in the PM/RM game is determined using the true production costs, *not* the virtual production costs.

In this model, market prices will always be positive, *i.e.*, $p_{eq}(z) \geq 0$. Similarly, the production level of any firm is always non-negative: $x_i \geq 0, \forall i$. Let $N_{eq}(z)$ be the set of firms with strictly positive production levels, given the joint action z of the PM/RM game. Let PM(z) be set of PM players with strictly positive production levels at joint action z, $\mathrm{PM}(z) = \{r : z_r = \mathrm{PM}, c_r < p_{eq}(z)\}$, and let RM$(z)$ be set of RM players at z, $\mathrm{RM}(z) = \{r : z_r = \mathrm{RM}\}$.

A firm i that selects $z_i = \mathrm{PM}$ is guaranteed a non-negative utility: Either it does not produce ($x_i(z) = 0$) or it produces ($x_i(z) = (p_{eq}(y(z)) - c_i)/b_i > 0$), and in both cases $u_i(z) = b_i x_i^2(z)$. A firm that chooses to maximize revenue always has strictly positive production level, and may find itself with negative utility. However, in the equilibria of the PM/RM game, all firms have non-negative utility (since all firms always have the option of playing PM).

We define the best response correspondence of a firm i as $BR_i(z_{-i})$ to include all the best response actions, given that the other firms actions is z_{-i}. Since we have only two actions, we sometimes abuse the notation and talk about the best response action, when it is unique. A *best response sequence* is a sequence of joint actions $z^1, \ldots, z^k$, in which each joint action z^{j+1} is derived from the preceding joint action z^j by a single firm doing a best response move.

3 Nash Equilbria and Dynamics of the PM/RM game

In this section, we study the properties of the PM/RM meta-game and establish the existence of pure Nash equilibria.

3.1 Market price vs. Production cost

The next lemma plays an essential role in understanding the structure of Nash equilibria of the meta-game. It states that when a firm switches from profit maximization to revenue maximization, the price increases (and therefore the number of producing firms can only decrease if the switching firm was already producing).

Lemma 2 *Let z_{-i} be a joint action of all firms except of some firm i, and consider the two joint actions $z^{pm} = (z_{-i}, \mathrm{PM})$ and $z^{rm} = (z_{-i}, \mathrm{RM})$ in which firm i has action* PM *and* RM*, respectively. Let $n_{pm} = |N(z^{pm})|$ and $n_{rm} = |N(z^{rm})|$ denote the number of producing firms in the two joint actions and let the corresponding market prices be $p_{pm} = p_{eq}(z^{pm})$ and $p_{rm} = p_{eq}(z^{rm})$. Then*

1. $p_{pm} > p_{rm}$, and

2. if firm i produces at z^{pm}, then $n_{pm} \geq n_{rm}$.

[3]As $y(z)$ is a function of z we will use the notation $p(z)$ and $p(y(z))$ indistinguisably, and do likewise for arbitrary other functions of $y(z)$.

Proof: For claim 1, we can derive p_{pm} from p_{rm} by doing the computation in two stages. In the first stage, we consider the increase in the price as firm i changes its action from RM to PM while the other firms do not react; in the second stage, the other firms react to the price change and the price drops. We will argue that the price will stay above the original level.

In the first stage, after firm i changes from RM to PM, the price increases regardless of whether firm i keeps producing or stops producing. Specifically, if it keeps producing, the price increases by $\frac{c_i}{1+n_{rm}}$, and if it stops producing, the number of producers decreases by 1, and the price increases by a factor of $\frac{1+n_{rm}}{n_{rm}}$.

In the second phase, some firms that were not producing at price p_{rm} start producing. This affects the price by increasing the numerator by the sum of the production cost of the new producers; the denominator increases by the number of new producers. The crucial observation is that the new producers have production cost at least p_{rm} (since they were not producing at this price). It follows that the changes in the numerator and the denominator of the price will leave the price above p_{rm}.

Claim 2 follows directly from claim 1: Since the price goes up, every firm who produces before the change keeps producing after the price increase; the only exception may be firm i which changed its strategy to PM, but the premise is that firm i produces. □

The next lemma bounds the effect on the price when a firm switches from PM to RM.

Lemma 3 *With the same premises of Lemma 2 and the additional assumption that firm i produces at z^{pm}, we have*

$$p_{rm} + \frac{c_i}{1+n_{pm}} \le p_{pm} \le p_{rm} + \frac{c_i}{1+n_{rm}}.$$

Proof: Let $C = \sum_{y \in \mathrm{PM}(z_{rm})} c_y$ and $C' = \sum_{y \in \mathrm{PM}(z_{pm})} c_y$. By the premise of the lemma $i \in \mathrm{PM}(z_{pm})$, hence c_i is one of the terms in C'. The difference $C' - C - c_i$ is the sum of the production costs of the firms which start producing when i switches from RM to PM. There are $(n_{pm} - n_{rm})$ such firms, which by the previous lemma is non-negative. Since each of these firms has production cost between p_{rm} and p_{pm}, we have

$$(n_{pm} - n_{rm})p_{rm} \le C' - C - c_i \le (n_{pm} - n_{rm})p_{pm}$$

According to the definition of $p_{eq}(z)$ we have:

$$p_{rm} = \frac{1+C}{1+n_{rm}}; \qquad p_{pm} = \frac{1+C'}{1+n_{pm}}.$$

Combining these two equations, we have $p_{pm}(1+n_{pm}) - p_{rm}(1+n_{rm}) = C' - C$, which implies that

$$c_i + (n_{pm} - n_{rm})p_{rm} \le p_{pm}(1+n_{pm}) - p_{rm}(1+n_{rm}) \le c_i + (n_{pm} - n_{rm})p_{pm},$$

and the lemma follows. □

Lemma 4 *With the premises of Lemma 3 and the extra assumption that $c_i \le p_{rm}$:*

1. *If firm i prefers* PM *to* RM, *then* $c_i \ge p_{rm}\left(1 - \frac{1}{n_{rm}^2}\right)$.
2. *If firm i prefers* RM *to* PM, *then* $c_i \le p_{rm}\left(1 - \frac{1}{n_{pm}^2}\right)$.

Proof: The utilities of firm i in z^{pm} and z^{rm} are $u_i(z^{pm}) = \frac{1}{b_i}(p_{pm} - c_i)^2$ and $u_i(z^{rm}) = \frac{1}{b_i}p_{rm}(p_{rm} - c_i)$. If firm i prefers PM to RM, we have $u_i(z^{pm}) \geq u_i(z^{rm})$,

$$
\begin{aligned}
\frac{p_{rm}}{b_i}(p_{rm} - c_i) &\leq \frac{1}{b_i}(p_{pm} - c_i)^2; \\
p_{rm}(p_{rm} - c_i) &\leq (p_{pm} - c_i)^2; && (6) \\
p_{rm}(p_{rm} - c_i) &\leq (p_{rm} + \frac{c_i}{1 + n_{rm}} - c_i)^2, && (7)
\end{aligned}
$$

where inequality (7) follows from inequality (6) using Lemma 3 and the fact that the terms in the right-hand side inside the square are non-negative; this follows immediately from the premise of the lemma that firm i produces at z^{pm}. By simplifying the last inequality, we get the first part of the lemma.

The second part is similar. Since firm i prefers RM to PM, we have $u_i(z^{pm}) \leq u_i(z^{rm})$. Therefore

$$
\begin{aligned}
(p_{rm} - c_i)\frac{p_{rm}}{b_i} &> \frac{(p_{pm} - c_i)^2}{b_i}; \\
(p_{rm} - c_i)p_{rm} &> (p_{pm} - c_i)^2; && (8) \\
(p_{rm} - c_i)p_{rm} &> (p_{rm} + \frac{c_i}{1 + n_{pm}} - c_i)^2, && (9)
\end{aligned}
$$

where inequality (9) follows from inequality (8) using Lemma 3. Again, the right-hand side terms inside the squares are positive, and this is guaranteed by the extra assumption that $c_i \leq p_{rm}$. The last inequality is equivalent to the second inequality of the lemma. □

3.2 Existence of pure Nash Equilibrium

We first relate the price after the best response move to the cost of the firms.

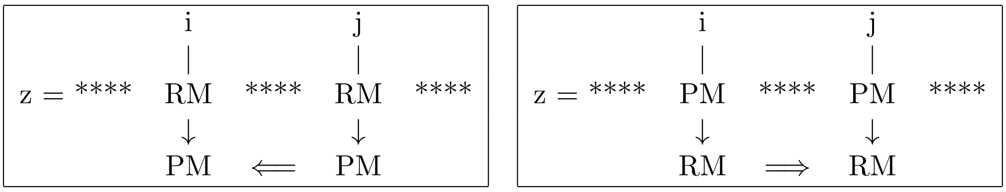
Observation 5 *Consider firms i and j with production costs $c_i > c_j$. Consider a joint action z where $z_i = z_j =$ RM. Let p' be the price if j changes to PM from z, let p'' be the price if i changes to PM from z. Then, $p' \leq p''$.*

Proof: We argue that $p' \leq p''$. One can view the cost change of firm i in two stages. In first stage it increases its cost by c_j, thus setting price p' in the system (it can be the case that x does not produce at p'). At the second change, firm i completes its cost change by increasing it by remaining $c_i - c_j$ (in case that i does not produce after first stage we have $p'' = p'$). Since the price is monotone in the cost, we get $p' \leq p''$. □

We now show that if firm j prefers to switch from RM to PM in the joint action z, then any firm i with higher production cost, and that plays RM in z, would also prefer to switch to PM.

Lemma 6 *Consider firms i and j with production costs $c_i > c_j$. Consider a joint action z where $z_i = z_j =$ RM. If in z firm j prefers PM, i.e., $BR_j(z_{-j}) =$ PM, then firm i also prefers PM, i.e., $BR_j(z_{-i}) =$ PM. (See Figure 1(a).)*

Proof: Let $p = p_{eq}(z)$. If $c_i > p$ then clearly i prefers PM (since it has a negative utility when playing RM). For the rest of the proof we assume that $c_i \leq p$. Consider joint actions $z' = (z_{-j}, \text{PM})$,



(a) (b)

Figure 1: Consider joint action z with firms i, j such that $c_i > c_j$. Figure 1(a) corresponds to Lemma 6, Figure 1(b) corresponds to Lemma 7.

$z'' = (z_{-i}, \text{PM})$ with market prices p' and p'' respectively. The utility of firm j in joint action z is $u_j(z) = p(p - c_j)/b_j$, and the utility of firm j in joint action z' is $u_j(z') = (p' - c_j)^2/b_j$. The utility of firm i in joint action z is $u_i(z) = p(p - c_i)/b_i$ and the utility of firm i in joint action z'' is $u_i(z'') = (p'' - c_i)^2/b_i$. By assumption, j prefers to switch to PM when the joint action is z, so $u_j(z) < u_j(z')$, i.e.,

$$p(p - c_j)/b_j < (p' - c_j)^2/b_j \ , \tag{10}$$

and we want to show that $u_i(z) < u_i(z')$, i.e.,

$$p(p - c_i)/b_i < (p'' - c_i)^2/b_i \ . \tag{11}$$

Let n, n', n'' be the number of firms with non-zero production levels in z, z', z'', respectively.

Using Lemma 4, since j prefers PM, we have $c_j > p(1 - \frac{1}{n^2})$.

For fixed p and p', define $f(r) = (p' - r)^2 - p(p - r)$. Rearranging equation (10), we have $f(c_j) > 0$. We will complete the proof by showing that $f(r)$ is an increasing function in the range $r > p(1 - \frac{1}{n^2})$. Given that, since $c_i > c_j > p(1 - \frac{1}{n^2})$ and $f(c_j) > 0$, we will conclude $f(c_i) > 0$, and thus $p(p - c_i) < (p' - c_i)^2$. Finally, from Observation 5 we have $p' \le p''$ and hence $p(p - c_i) < (p'' - c_i)^2$, which will complete the proof.

We now show that f is increasing in the desired range. The derivative of f is $f'(r) = 2(r - p') + p$. From Lemma 3, $p' \le p + \frac{c_j}{1+n}$. For $r \ge c_j$ we get

$$\begin{aligned}
f'(r) &\ge 2r - 2(p + \frac{r}{1+n}) + p \\
&= 2r\frac{n}{n+1} - p \\
&\ge 2p\left(1 - \frac{1}{n^2}\right)\frac{n}{n+1} - p \\
&\ge p(\frac{2n-2}{n} - 1) \\
&= p\frac{n-2}{n} \ge 0 \ .
\end{aligned}$$

□

We now show that if firm i prefers to switch from PM to RM in the common action z, then any firm j with lower production cost that plays PM in z would also prefer to switch to RM.

Lemma 7 *Consider firms i and j with production costs $c_i > c_j$. Suppose $z_i = z_j = \text{PM}$. If in joint action z firm i prefers* RM, *i.e.,* $BR_j(z_{-i}) = \text{RM}$, *then firm j would also prefer to switch to* PM *from z, i.e.,* $BR_j(z_{-j}) = \text{PM}$. *(See Figure 1(b).)*

Proof: Let $p = p_{eq}(z)$. We have $z_i = z_j = \text{PM}$. Consider joint actions $z' = (z_{-j}, \text{RM})$, $z'' = (z_{-i}, \text{RM})$ with market prices p' and p''. The utility of firm j in the joint action z is $u_j(z) = p(p - j)/b_j$, and the utility of firm j in the joint action z' is $u_j(z') = (p' - c_j)^2/b_j$. The utilities of firm i are $u_i(z) = p(p - c_i)/b_i$ and $u_i(z'') = (p'' - c_i)^2/b_i$, respectively.

By assumption, i prefers RM, so $u_i(z) < u_i(z')$. Assume by way of contradiction that firm j prefers PM,i.e.,

$$(p - c_j)^2 > p'(p' - c_j) \ . \tag{12}$$

We will show that in this case firm i would also prefer PM, i.e.,

$$(p - c_i)^2 > p''(p'' - c_i) \ . \tag{13}$$

For fixed p and p'', again define $f(r) = (p - r)^2 - p'(p' - r)$. Rearranging equation (12), we get $f(c_j) > 0$. We will show that $f(r)$ is an increasing function in range $r > c_j$. Given that, since $f(c_j) > 0$ and $c_i > c_j$, we can conclude $f(c_i) > 0$, and thus $(p - c_i)^2 > p'(p' - c_i)$.

We now show that f is increasing in the desired range. The derivative $f'(r) = 2(r - p) + p' \geq 2c_j - 2p + p'$. Using Lemma 3, we have $p \leq p' + \frac{c_j}{1+n'}$. According to Lemma 4, since firm j prefers PM, we have $c_j > p'(1 - \frac{1}{n'^2})$. Therefore,

$$\begin{aligned}
f'(r) &\geq 2c_j - 2p' - 2\frac{c_j}{1+n'} + p' \\
&\geq \frac{2c_j}{n'+1}(n' + 1 - 1) - p' \\
&\geq 2p'\frac{(n'-1)(n'+1)}{n'^2}\frac{n'}{n'+1} - p' \\
&\geq p'\frac{2n'-2}{n'} - p' \\
&\geq p'\frac{2n'-2-n'}{n'} \\
&\geq p'\frac{n'-2}{n'} \ ,
\end{aligned}$$

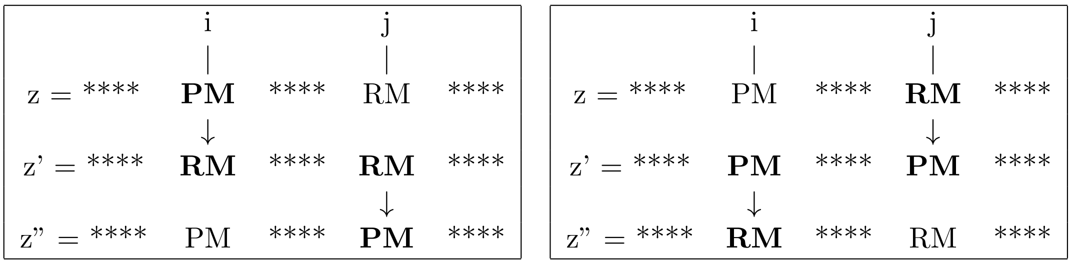
and therefore $f(r)$ is a non-decreasing function for $n' \geq 2$.

We have established that, assuming firm j prefers PM, then $(p - c_i)^2 > p'(p' - c_i)$.

We now will argue, similar to Observation 5, that $p' \geq p''$. One can view the cost change of firm i in two stages. In the first stage the cost decrease by c_j, thus setting price p' in the system (since utility of j is positive at p', we have $c_j \leq p'$, therefore j produces at p'). In the second stage, firm i decreases the price by the remaining $c_i - c_j$. Since the price is monotone in the cost, we get $p' \geq p''$. Therefore $(p - c_i)^2 > p''(p'' - c_i)$, contradicting our assumption that i prefers RM. □

We now use the above lemmas to show that certain sequences of joint actions cannot be part of any best response sequence.

Lemma 8 *Consider joint action z with $z_i = \text{PM}$, $z_j = \text{RM}$ and $c_i > c_j$. In addition, consider following joint actions: $z' = (z_{-i}, \text{RM})$, $z'' = (z'_{-j}, \text{PM})$. Then the sequnce of joint actions z, followed by z', followed by z'' cannot be best response sequence. (See Figure 2(a).)*



(a) (b)

Figure 2: Impossible series of best response moves with firms i, j such that $c_i > c_j$. Figure 2(a) corresponds to Lemma 8, Figure 2(b) corresponds to Lemma 9.

Proof: If z'' is a best response to z', then $u_j(z'') > u_j(z')$. From Lemma 6 follows that it should also hold $u_i(z') > u_i(z)$ in contradiction to z followed by z' is a best response sequence. □

Lemma 9 *Consider joint action z with $z_i = \mathrm{PM}$, $z_j = \mathrm{RM}$ and $c_i > c_j$. In addition, consider following joint actions: $z' = (z_{-j}, \mathrm{PM})$ and $z'' = (z'_{-i}, \mathrm{RM})$. Then sequence of joint actions z, followed by z', followed by z'' cannot be best response sequence. (See Figure 2(b)).*

Proof: If z'' is a best response to z', then $u_i(z'') > u_i(z')$. From Lemma 7 follows that it should also hold $u_j(z') > u_j(z)$ in contradiction to z followed by z' is a best response move. □

The following lemma will play a central role in showing that any best response dynamics converges to a pure Nash equilibrium. The lemma shows that if there is a sequence of firms switching from RM to PM, then in the initial joint action, the lowest cost firm among them would have a best response to switch from RM to PM.

Lemma 10 *Let z be a joint action with both firms i and j playing* RM. *Let n be the number of producers at z, such that $n \geq 3$. Consider a best response move of firm i followed by a best response move of j both changing their strategy from* RM *to* PM. *If $c_i > c_j$, a best response action of j to z_{-j} is* PM.

Proof: Consider joint actions $\hat{z} = (z_{-j}, \mathrm{PM})$, $\check{z} = (z_{-i}, \mathrm{PM})$, and joint action $\bar{z}$ that differ from z by actions of both firms i and j, i.e., $\bar{z}_{-i,-j} = z_{-i,-j}$ and $\bar{z}_i = \bar{z}_j = \mathrm{PM}$. Let p, $\hat{p}$, $\check{p}$ and $\bar{p}$ be market prices, and let number of producers be n, $\hat{n}$, $\check{n}$ and $\bar{n}$, respectively.

By assumption of the lemma we have $n \geq 3$. If $c_j > p$, then firm's utility $u_j(z) < 0$, thus it prefers $\hat{z}$ where its utility is nonnegative. For the rest of this lemma we consider $c_j < p$.

By the assumption of the lemma, j prefers $\bar{z}$ to $\check{z}$. Using Lemma 4 we have: $c_j \geq \check{p}(1 - \frac{1}{\check{n}^2})$. Assume j prefers z to $\hat{z}$. Using Lemma 4 we have: $j \leq p(1 - \frac{1}{\hat{n}^2})$. Combining together, we have

$$\check{p}(1 - \frac{1}{\check{n}^2}) \leq c_j \leq p(1 - \frac{1}{\hat{n}^2}). \tag{14}$$

According to Lemma 2 we have $\check{p} > p$. For inequality (14) to hold, we need

$$\begin{aligned} 1 - \frac{1}{\check{n}^2} &< 1 - \frac{1}{\hat{n}^2}; \\ \check{n} &< \hat{n}. \end{aligned}$$

We can have $\check{n} < \hat{n}$ only if $c_i > \check{p}$ and i stops producing when it changes from RM in z to PM in $\check{z}$.

From Observation 5 we have $\check{p} > \hat{p}$, therefore $\mathrm{PM}(\hat{z}) \setminus \{j\} \subseteq \mathrm{PM}(\check{z})$. Clearly, $\mathrm{RM}(z) = \mathrm{RM}(\hat{z}) \cup \{j\} = \mathrm{RM}(\check{z}) \cup \{i\}$. Hence, $\hat{n} \leq \check{n} + 1$. Combining together, we get

$$\check{n} < \hat{n} \leq \check{n} + 1,$$

which holds only for $\hat{n} = \check{n}+1$, therefore $\mathrm{PM}(\hat{z}) = \mathrm{PM}(\check{z}) \cup \{j\}$. We also have $|\mathrm{PM}(\check{z})| - |\mathrm{PM}(z)| = \hat{n} - n$. In addition, each firm i that produces at $\check{z}$ and not in z has production cost $c_i \geq p$. Using the above, we get

$$p = \frac{1 + \sum_{y \in \mathrm{PM}(z)} c_y}{1+n} < \frac{1 + \sum_{y \in \mathrm{PM}(\check{z})} c_y}{1+\hat{n}} = \frac{1 + \sum_{y \in \mathrm{PM}(\check{z})} c_y}{\hat{n}} \frac{\hat{n}}{1+\hat{n}} = \check{p}\frac{\hat{n}}{1+\hat{n}},$$

Therefore ,

$$\frac{p}{\check{p}} < \frac{\hat{n}}{1+\hat{n}}.$$

Using Inequality (14) we obtain,

$$\begin{aligned}
\check{p}(1 - \frac{1}{\check{n}^2}) &\leq p(1 - \frac{1}{\hat{n}^2}); \\
\check{p}\frac{(\check{n}^2-1)}{\check{n}^2} &\leq \check{p}\frac{\hat{n}}{1+\hat{n}} \frac{(\hat{n}^2-1)}{\hat{n}^2}; \\
\frac{(\check{n}^2-1)}{\check{n}^2} &\leq \frac{(\hat{n}-1)}{\hat{n}}; \\
1 - \frac{1}{\check{n}^2} &\leq 1 - \frac{1}{\hat{n}}; \\
\check{n}^2 &\leq \hat{n}.
\end{aligned}$$

Since, $\hat{n} = \check{n} + 1$, we have $(\hat{n}-1)^2 \leq \hat{n}$, that holds only for $\hat{n} \leq 2$. Since $n \leq \hat{n}$ it contradicts the assumption of the lemma that $n \geq 3$. □

The following theorem establishes that any sequence of best response moves converges to a pure Nash equilibrium.

Theorem 11 *Any sequence of best response moves in the* PM/RM *game converges to a pure Nash equilibrium.*

Proof: Suppose that game does not converge to Nash equilibrium, so there is a sequence of best response moves that cycles. Consider firm j with highest cost on the cycle. Let P the maximal chain of RM $\to$ PM moves that includes j.

Consider P of length at most 2. If j is the first best response move of P, then it contradicts to Lemma 8 (Figure 2(a)). If j is the last best response move of P, then it contradicts to Lemma 9 (Figure 2(b)).

We are left with P of size at least 3. Let z be a joint action in the beginning of P. Applying Lemma 10 recursively, we get that $BR_j(z_{-j}) = \mathrm{PM}$. Let i be firm that made best response move before P (it has to be $BR_i(z_{-i}) = \mathrm{RM}$). Using Lemma 8 we get contradiction that a cycle exists. □

3.3 Best Response Dynamics Converge to Nash Equilibrium

Consider a current joint action z in the PM/RM game. If z is not a Nash equilibrium then at least one of the firms prefers to switch strategy. We will show that a particular order of changing strategies leads quickly to a Nash equilibrium.

To do this, let us define $BRF(z)$ to be set of firms that prefer to switch strategy: $BRF(z) = \{i | z_i \notin BR_i(z_{-i})\}$. Intuitively, among the firms in PM(z), the one with lower production cost is more likely to prefer to switch strategy. Similarly, among the firms in RM(z), the one with the maximum production cost is more likely to switch strategy. We will consider the dynamics that take advantage of this intuition. Let us define minPM(z) to be the firm with minimum production cost among firms in PM(z) and maxRM(z) to be the firm with maximum production cost among firms in RM(z).

Consider the following natural best response dynamics:

While $BRF(z) \neq \emptyset$, perform one of the following actions

1. If minPM(z) $\in$ BRF(z), firm minPM(z) changes its strategy.
2. If maxRM(z) $\in$ BRF(z), firm maxRM(z) changes its strategy.

In the proof of the following lemma, we show that if $BRF(z) \neq \emptyset$ then one of the actions is applicable.

Lemma 12 *The above procedure converges to a Nash equilibrium in at most* $2m$ *steps.*

Proof: According to Lemma 7, if minPM(z) $\notin$ BRF(z) then no firm PM(z) is in BRF(z). Similarly if firm maxRM(z) $\notin$ BRF(z), then no firm in RM(z) is in BRF(z). Therefore, one of the two steps can be always performed while BRF(z) $\neq \emptyset$.

If we order the firms in decreasing order according to their production cost, the current joint action is a vector in $\{\mathrm{PM}, \mathrm{RM}\}^m$. The procedure either replaces the rightmost PM or the leftmost RM. Furthermore, the most recent action cannot be undone immediately (it wouldn't be best response otherwise).

The claim is that the procedure finishes in at most $2m$ steps. To see this, observe that at the beginning the procedure changes the leftmost RM or the rightmost PM until the vector consists of a sequence of PM's followed by a sequence of RM's. From that point on, all the PM's precede the RM's in the current vector. It follows that it takes at most m steps to reach the point where the PM's precede the RM's and at most m more steps to reach the final vector. □

Starting from a joint action in which all firms play PM, the above dynamics will converge in at most $2m$ steps to a Nash equilibrium (in fact, the proof shows that only m steps suffice). It follows that

Corollary 13 *There is a polynomial time computation to find some Nash equilibrium.*

One might mistakenly assume that the Nash equilibria are all of the RM for low production cost firms, PM for high production cost firms, the following shows that this is false:

Example 14 *Consider two firms that have costs 0.30 and 0.28. It is straightforward to see that* (RM, PM) *is a Nash equilibrium (and also* (PM, RM) *is an equilibrium).*

4 Comparison of the PM/RM game and Cournot competition

In this section we compare the outcome of the PM/RM game versus the myopic Cournot competition. We start by comparing the prices. For simplicity we assume that $b_i = 1$.

Assume z is a pure Nash equilibrium of the PM/RM game, where k firms are producing. Since this is an equilibrium, all other firms that select PM, do not produce and have zero utility. Any producing firm $i \in M$ has production cost $c_i < p(z)$, and any firm which has $c_i < p(z)$ is producing.

Consider the Cournot competition price, which is equivalent to having all firms play PM. In this case, we can think of computing the price in two steps, first, we let the firms that selected RM to switch to PM, and then let any firm that was not producing before (since its cost was higher than $p(z)$) to produce. After the first stage, the price is at least the previous price, and at most

$$p' \leq \frac{1 + kp(z)}{1 + k} = p(z) + \frac{1 - p(z)}{1 + k}.$$

After the second step, since we are adding firms with production cost $c_i \in [p(z), p']$, the price can only go down (but remains above the price of $p(z)$). We can also lower bound $p(z)$, since clearly $p(z) \geq \frac{1}{k+1}$.

We can now bound the difference between the Cournot competition price, p_c, and the PM/RM game price, as follows,

$$1 \leq \frac{p_c}{p(z)} \leq \frac{p'}{p(z)} = 1 + \frac{1 - p(z)}{p(z)(1 + k)} = 1 + \frac{1}{p(z)(1 + k)} - \frac{1}{(1 + k)} \leq 2 - \frac{1}{1 + k} \leq 2 - \frac{1}{1 + n},$$

where in the first inequality uses the fact that $p(z) \geq \frac{1}{k+1}$.

We can also show that the above bound is almost tight. Consider the case of symmetric firms with production cost of $c = \frac{1}{n} - \frac{1}{n^2}$. The pure Nash equilibrium in this case is all the firms selecting RM (see Section 5.2). The Cournot competition price is $\frac{1+nc}{1+n} = \frac{2-\frac{1}{n}}{1+n}$ while when all the firms select RM, which is the pure Nash equilibrium, the price is $\frac{1}{1+n}$. In this case the ratio between the prices is $2 - \frac{1}{n}$. We have established the following theorem.

Theorem 15 *Let p_c be the Cournot competition price and p_{pr} be the* PM/RM *game price. Then,*

$$1 \leq \frac{p_c}{p_{pr}} \leq 2 - \frac{1}{1 + n},$$

and there is a case where $\frac{p_c}{p_{pr}} = 2 - \frac{1}{n}$.

In our setting the price p defines the total production, since $\sum_{i \in M} x_i = 1 - p$. Therefore the total revenue (of all firms) is $p(1 - p)$, when the price is p. Since the price in the PM/RM game is at least half the price of the Cournot competition, the total revenue is at least half. (Note that the produced amount increases while the price decreases.)

We now can compare the utility of the firms in the two settings. We will show that the utility can be dramatically different. Consider again the case of symmetric firms with production cost of $c = \frac{1}{n} - \frac{1}{n^2}$. The utility of each firm in the Cournot competition is,

$$\left(\frac{2 - \frac{1}{n}}{1 + n} - \frac{1}{n} + \frac{1}{n^2} \right)^2 = \left(\frac{2n^2 - n - n(n + 1) + (n + 1)}{n^2(1 + n)} \right)^2 = \Theta(\frac{1}{n^2}).$$

The utility of each firm in the PM/RM game is,

$$\left(\frac{1}{1+n}-\frac{1}{n}+\frac{1}{n^2}\right)\frac{1}{1+n}=\left(\frac{n^2-n(1+n)+(1+n)}{n^2(1+n)}\right)\frac{1}{1+n}=\Theta(\frac{1}{n^4}).$$

This implies that the ratio in the utilities can be as large as n^2. Since all the utilities are identical, the same ratio holds for the sum of the utilities.

Theorem 16 *There is a case where the sum of the utilities in the Cournout competition is a factor of $\Theta(n^2)$ larger than the sum of the utilities in the* PM/RM *game.*

5 Instances of special interest for the PM/RM Game

In this section we consider two cases: when only two firms on the market and when all firms have the same cost.

5.1 PM/RM game: Two firms, complete characterization

In this section we give a complete characterization of all the pure Nash equilibria for the case of two firms. The characterization is divided to four cases, depending on the firms' costs, compared to 1/2.

Theorem 17 *For two firms in the* PM/RM *game, there is always at least one pure Nash equilibria, and the characterization when a joint action is a pure Nash equilibrium is as follows are:*

(RM,RM) *when* $c_1 < \frac{1}{4}$ *and* $c_2 < \frac{1}{4}$.

(PM,RM) *when either: 1.* $c_1 \in [\frac{1}{4}, \frac{1}{2}]$ *and* $c_2 < \frac{1+c_1}{4}$, *or 2.* $c_1 < \frac{3}{2}\sqrt{1-2c_2}-(1-2c_2)$ *and* $c_2 \le \frac{1}{2}$.

(RM,PM) *when either: 1.* $c_1 < \frac{1+c_2}{4}$ *and* $c_2 \in [\frac{1}{4}, \frac{1}{2}]$, *or 2.* $c_1 \le \frac{1}{2}$ *and* $c_2 < \frac{3}{2}\sqrt{1-2c_1}-(1-2c_1)$.

(PM,PM) *when one of the cases below holds:*

1. $c_1 \in [\frac{1+c_2}{4}, \frac{1}{2}]$ *and* $c_2 \in [\frac{1+c_1}{4}, \frac{1}{2}]$.
2. $c_1 \le \frac{1}{2}$ *and* $c_2 \in [\frac{3}{2}\sqrt{1-2c_1}-(1-2c_1), \frac{1+c_1}{2}]$.
3. $c_1 \in [\frac{3}{2}\sqrt{1-2c_2}-(1-2c_2), \frac{1+c_2}{2}]$ *and* $c_2 \le \frac{1}{2}$.
4. $c_1 > \frac{1}{2}$, *and* $c_2 > \frac{1}{2}$.

Proof: We consider four cases, depending on the firms' costs, compared to 1/2.
Case 1 $\{c_1 \le 1/2$ and $c_2 \le 1/2\}$: This is the most interesting case, in which the two firms are producing, as we will see later. We first define the price as a function of the action of the firms.[4]

	RM	PM
RM	$\frac{1}{3}$	$\frac{1+c_2}{3}$
PM	$\frac{1+c_1}{3}$	$\frac{1+c_1+c_2}{3}$

[4]In all matrices that we use, row firm is firm 1 and column firm is firm 2.

Next we derive the production level $x_i(z)$, at each joint action.

	RM	PM
RM	$\left(\frac{1}{3b_1}, \frac{1}{3b_2}\right)$	$\left(\frac{1+c_2}{3b_1}, \frac{1-2c_2}{3b_2}\right)$
PM	$\left(\frac{1-2c_1}{3b_1}, \frac{1+c_1}{3b_2}\right)$	$\left(\frac{1+c_2-2c_1}{3b_1}, \frac{1+c_1-2c_2}{3b_2}\right)$

Each entry of production level matrix is non-negative for $c_i \leq 1/2$. Therefore, firm i that plays PM will produce. For two firms we have the following payoff matrix:

	RM	PM
RM	$\left(\frac{1}{3b_1}(\frac{1}{3} - c_1), \frac{1}{3b_2}(\frac{1}{3} - c_2)\right)$	$\left((\frac{1+c_2}{3} - c_1)\frac{1+c_2}{3b_1}, (\frac{1-2c_2}{3})^2\frac{1}{b_2}\right)$
PM	$\left((\frac{1-2c_1}{3})^2\frac{1}{b_1}, (\frac{1+c_1}{3} - c_2)\frac{1+c_1}{3b_2}\right)$	$\left((\frac{1+c_2-2c_1}{3})^2\frac{1}{b_1}, (\frac{1+c_1-2c_2}{3})^2\frac{1}{b_2}\right)$

To compute the pure Nash equilibria, we compute the preference of the firms. We start with firm 1.

- Firm 1 prefers (PM, RM) to (RM, RM) when $\frac{1}{3b_1}(\frac{1}{3} - c_1) < (\frac{1-2c_1}{3})^2\frac{1}{b_1}$, which holds for $c_1 > \frac{1}{4}$.
- Firm 1 prefers (PM, PM) to (RM, PM) when $(\frac{1+c_2}{3} - c_1)\frac{1+c_2}{3b_1} < (\frac{1+c_2-2c_1}{3})^2\frac{1}{b_1}$, which holds for $c_1 > \frac{1+c_2}{4}$.
- Firm 2 prefers (RM, PM) to (RM, RM) for $c_2 > \frac{1}{4}$.
- Firm 2 prefers (PM, PM) to (RM, PM) for $c_2 > \frac{1+c_1}{4}$.

The conditions for each joint actions to be a pure Nash equilibrium, are as follows:

	RM	PM
RM	$c_1 < \frac{1}{4}, c_2 < \frac{1}{4}$	$c_1 < \frac{1+c_2}{4}, c_2 > \frac{1}{4}$
PM	$c_1 > \frac{1}{4}, c_2 < \frac{1+c_1}{4}$	$c_1 > \frac{1+c_2}{4}, c_2 > \frac{1+c_1}{4}$

Case 2 $\{c_1 > \frac{1}{2}$ and $c_2 > \frac{1}{2}\}$: If both firms select RM then the price is 1/3 and they both have negative utility. Assume firm 1 selects RM. If firm 2 selects PM then the price is $p = \frac{1+c_2}{3}$. Since $c_2 > 1/2$, then $c_2 > p$, and firm 2 is not producing. If the firm 2 selects PM and is not producing then the price is $1/2 < c_1$, thus firm 1 has negative utility. Therefore, in this case both firms have PM as a dominating action.

Case 3 $\{c_1 \leq \frac{1}{2}$ and $c_2 > \frac{1}{2}\}$: If firm 2 selects RM, then the price is at most $\frac{1+c_1}{3} \leq \frac{1}{2}$. Therefore, in this case, action PM will be a strictly dominating action for firm 2.

Consider joint action (RM, PM). We have production level $x_2 = \frac{1+c_2}{3} - c_2 < 0$, thus firm 2 not producing. For joint action (PM, PM), firm 1 always produces (production level $x_1 \geq \min\{\frac{1+c_1+c_2}{3}, \frac{1+c_1}{2}\} - c_1 > 0$, so we have firm 2 produces if $p = \frac{1+c_1+c_2}{3} > c_2$.

In the case that $c_2 > \frac{1+c_1}{2}$ firm 1 produces alone, so her dominating action is PM, and her utility is $(\frac{1-c_1}{2})^2\frac{1}{b_1}$.

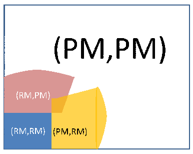


Figure 3: Characterization of two firms pure Nash Equliria

For $c_2 < \frac{1+c_1}{2}$, since firm 2 dominating action is PM, the payoff values are:

$$\begin{aligned} u_1(RM, PM) &= \left(\frac{1}{2} - c_1\right)\frac{1}{2b_1}; \\ u_2(PM, PM) &= \left(\frac{1+c_2-2c_1}{3}\right)^2\frac{1}{b_1}. \end{aligned}$$

Firm 1 will prefer action PM if $c_2 \in [\frac{3}{2}\sqrt{1-2c_1} - (1-2c_1), \frac{1+c_1}{2}]$, otherwise, when $c_2 < \frac{3}{2}\sqrt{1-2c_1} - (1-2c_1)$ firm 1 prefers RM.
Case 4 $\{c_1 > \frac{1}{2}$ and $c_2 \leq \frac{1}{2}\}$: Similar to the previous case, firm 1 will select PM. Firm 2 will prefer action PM if $c_1 \in [\frac{3}{2}\sqrt{1-2c_2} - (1-2c_2), \frac{1+c_2}{2}]$, otherwise, when $c_1 < \frac{3}{2}\sqrt{1-2c_2} - (1-2c_2)$ firm 2 prefers RM. □

In each of the four regions, we showed that for any value of the production cost, there exists a pure Nash equilibrium. (For some values there exists two pure Nash equilibria, see Example 14.) The different pure Nash equilibria can be observed in Figure 3.

5.2 Symmetric firms

We consider the case of m symmetric firms with cost c for each, playing the PM/RM game. Namely, each firm selects an action in $\{\text{RM}, \text{PM}\}$. The firms that select the action RM will be revenue maximizer (behave as though their production cost is zero). The firms that select PM would be profit maximizers. Note that since this is a symmetric case, in equilibrium all the firms would be producing, i.e., $n = m$. We assume that for each firm i, $b_i = 1$.

Suppose that k firms select the action PM and $n-k$ firms select the action RM. In this case the price is $p^k = \frac{1+kc}{n+1}$. Firms that select PM will produce $x_p^k = p^k - c = \frac{1-(n+1-k)c}{n+1}$, and firms that will select RM will produce $x_r^k = p^k = \frac{1+kc}{n+1}$. The utility of firms that select PM is $u_p^k = x_p^k(p^k - c) = (\frac{1-(n+1-k)c}{n+1})^2$, while the utility of firms that select RM will be $u_r^k = x_r^k(p^k - c) = \frac{1+kc}{n+1}(\frac{1-(n+1-k)c}{n+1})$.

We will now compute for which costs c is it a Nash equilibrium to have k firms selecting PM and $n-k$ firms selecting RM.

Theorem 18 *Let* $a^k = \frac{n-1}{n(n-k)+k+n-1}$ *for* $k \in [1, n]$, $a^0 = 0$, $b^k = \frac{n-1}{n(n-k)+k}$ *for* $k \in [0, n-1]$ *and* $b^n = 1$. *If players' cost* $c \in [a^k, b^k]$ *then there is a pure Nash with* k *firms selecting* PM *and* $n-k$ *firms selecting* RM.

Proof: If a firm selecting RM deviates to PM ($k \leq n-1$), then its new utility would be

$u_p^{k+1} = (\frac{1-(n-k)c}{n+1})^2$. Action RM will be a Best Response if,

$$\begin{aligned}
\frac{1+kc}{n+1}\left(\frac{1-(n+1-k)c}{n+1}\right) &\geq \left(\frac{1-(n-k)c}{n+1}\right)^2 \\
(1+kc)(1-(n+1-k)c) &\geq (1-(n-k)c)^2 \\
1-(n+1)c+2kc-k(n+1-k)c^2 &\geq 1-2(n-k)c+(n-k)^2c^2 \\
(n-1)c &\geq ((n-k)^2+k(n+1-k))c^2 \\
\frac{n-1}{n(n-k)+k} &\geq c
\end{aligned}$$

If a firm selecting PM deviates to RM ($k \geq 1$), then its new utility would be $u_r^{k-1} = \frac{1+(k-1)c}{n+1}(\frac{1-(n+2-k)c}{n+1})$. Action PM will be a Best Response if,

$$\begin{aligned}
\left(\frac{1-(n+1-k)c}{n+1}\right)^2 &\geq \frac{1+(k-1)c}{n+1}\left(\frac{1-(n+2-k)c}{n+1}\right) \\
(1-(n+1-k)c)^2 &\geq (1+(k-1)c)(1-(n+2-k)c) \\
1-2(n+1-k)c+(n+1-k)^2c^2 &\geq 1-(n-2k+3)c-(k-1)(n+2-k)c^2 \\
((n+1-k)^2+(k-1)(n+2-k))c^2 &\geq (n-1)c \\
(n(n-k)+k+n-1)c &\geq (n-1) \\
c &\geq \frac{n-1}{n(n-k)+k+n-1}
\end{aligned}$$

This implies that for $c^k \in [a^k, b^k]$ there is a pure Nash with k firms selecting PM and $n-k$ firms selecting RM, where $a^k = \frac{n-1}{n(n-k)+k+n-1}$ for $k \in [1, n]$ and $b^k = \frac{n-1}{n(n-k)+k}$ for $k \in [0, n-1]$. Note that $a^k = b^{k-1}$, and let $a^0 = 0$ and $b^n = 1$. This covers the entire range of symmetric production costs. □

6 The continuous game

In this section we extend the PM/RM game, where each firm selects between two actions, to a *continuous* PM/RM *game*. In the continuous PM/RM game, each firm i selects a bid $z_i \in [0, 1]$. Let $z_1, \ldots, z_m$ be the reported bids. Given the bids z, the price $p(z)$ is set using a Cournot competition assuming costs z, i.e., we have $p(z) = p_{eq}(z)$. Let $N(z) = \{i \mid z_i < p(z)\}$ and $n(z) = |N(z)|$. Recall, the price as a function of firm i's bid is $p(z_{-i}, y) = \frac{1+\sum_{j \in N(z_{-i},y)} z_j}{n(z_{-i},y)+1}$. In our analysis we fix the bids of the other firms, i.e., z_{-i}. Therefore we can write the functions as a dependent on y, while they have an implicit dependency, for example, $p(y)$ for $p(z_{-i}, y)$.

Firm i's utility is

$$U_i(z_{-i}, y) = \begin{cases} (p(y)-y)(p(y)-c_i), & \text{for } y < p(y) \\ 0, & \text{for } y \geq p(y) \end{cases}$$

again, to simplify the notation, we refer to $U_i(z_{-i}, y)$ as $U_i(y)$.

Lemma 19 *$p(y)$ and $U_i(y)$ are continuous functions in firm i's bid.*

Proof: Clearly, the price is a continuous function when it takes a value not equal to any firm's bid. Now consider a price p such that k firms' bids take value p. It is straightforward to show that the addition of these firms to the set $N(z_{-i}, y)$ does not cause a jump in price: The price p before those firms join is some $p = \frac{A}{1+n}$, where n is the number of producing firms. The price after they join is $\frac{A+k\cdot p}{1+n+k} = \frac{(1+n)\cdot p+k\cdot p}{1+n+k} = p$. Thus, the price function is continuous.

Player i's utility function is continuous (taking value zero) when i's bid matches the price or falls above it. Elsewhere, the utility is the multiplication of two continuous functions and thus also continuous. □

When we fix z_{-i}, U_i is continuous over a compact space $[0, 1]$. Thus, (quasi) convexity is sufficient to establish existence of pure Nash equilibria according to Debreu-Fan-Glicksberg theorem (see, [5]).

Lemma 20 *For any fixed z_{-i} the function $U_i(y)$ is convex when it is non-negative.*

Proof: Note that the derivative of U_i exists except at points where the value of $n(y)$ changes. In regions where y is greater than or equal to the price, the derivative of U_i is zero. In regions where $n(y)$ is fixed but y is less than the price, the derivative of U_i is monotonically decreasing in y.

In order to complete the proof that U_i is convex, we will show that the limit from below of the derivative at a point where $n(y)$ changes is strictly greater than the limit from above of the derivative at that point.

Consider the derivative of U_i (between points where $N(y)$ changes):

$$\begin{aligned}\frac{\partial}{\partial y}U_i(y) &= 2p(y)p'(y) - p(y) - (y + c_i)p'(y) + c_i \\ &= p'(y)(2p(y) - y - c_i) - p(y) + c_i \ .\end{aligned}$$

Assume that for $y \in [\alpha-\epsilon, \alpha)$ we have a price $p_-(y)$ and n_- firms producing, and for $y \in (\alpha, \alpha+\epsilon]$ we have $p_+(y)$ and n_+ firms producing. Now, when y increases, $p(y)$ increases, and when p increases $n(y)$ increases. Since the price $p(y)$ is continuous (even when the number of producing firms changes), we have $p_+(\alpha) = p_-(\alpha)$. Let $p(\alpha)$ be that value. We can also assume that $p(\alpha) \geq \alpha$ otherwise the utility is zero, and that $p(\alpha) \geq c_i$, otherwise the utility is not positive. This implies that $2p(\alpha) - \alpha - c_i \geq 0$. Since $n_- < n_+$ we have that $p'_-(\alpha) > p'_+(\alpha)$. This implies that

$$\frac{\partial}{\partial y_-}U_i(\alpha) = p'_-(y)(2p(\alpha) - \alpha - c_i) - p(\alpha) + c_i > p'_+(y)(2p(\alpha) - \alpha - c_i) - p(\alpha) + c_i = \frac{\partial}{\partial y_+}U_i(\alpha)$$

which completes the proof. □

Since firms can always achieve non-negative utility, any Nash equilibrium of the game will reflect non-negative utility for all firms, so the above lemma is sufficient to establish existence of pure Nash equilibria for the continuous game.

Theorem 21 *Any continuous* PM/RM *game has a pure Nash equilibrium.*

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