Spikes & Sines

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Spikes & Sines

- Work in $\mathbb{C}^n$

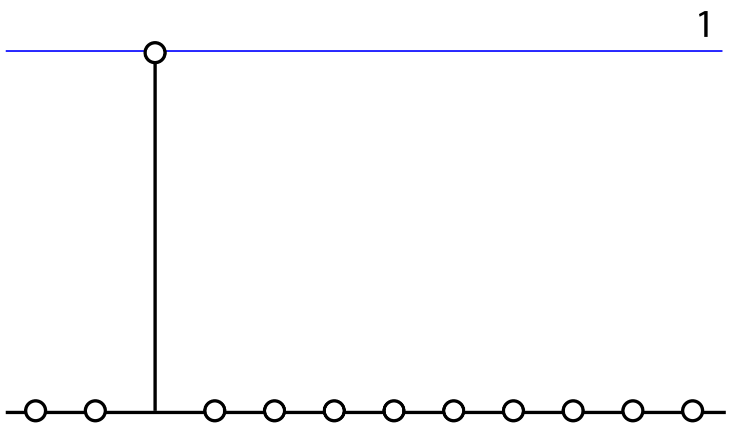
- Define *spike basis* $\{\mathbf{e}_j : j = 1, 2, \dots, n\}$

$$\mathbf{e}_j(t) = \begin{cases} 1, & t = j \\ 0, & t \neq j \end{cases} \qquad t = 1, 2, \dots, n$$

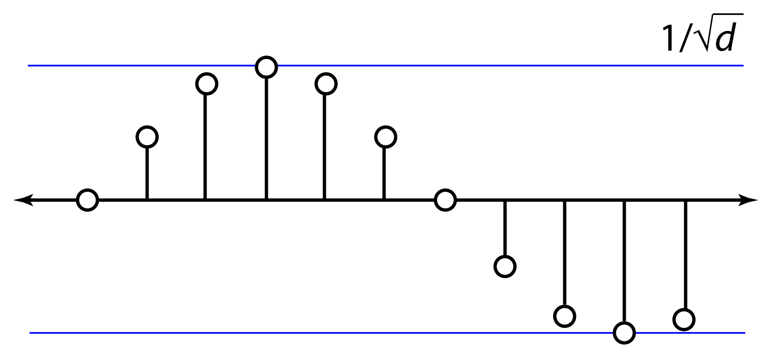
- Define *sine basis* $\{\mathbf{f}_j : j = 1, 2, \dots, n\}$

$$\mathbf{f}_j(t) = \frac{1}{\sqrt{n}} \mathrm{e}^{2\pi \mathrm{i} j t / n} \qquad t = 1, 2, \dots, n$$

In Captivity...



Spikes



Sines

The DFT matrix

- Define the *unitary DFT matrix*

$$\mathbf{F} = \begin{bmatrix} \text{———} & \mathbf{f}_1^* & \text{———} \\ \text{———} & \mathbf{f}_2^* & \text{———} \\ & \vdots & \\ \text{———} & \mathbf{f}_n^* & \text{———} \end{bmatrix}$$

- Suppose T and Ω are subsets of $\{1, 2, \dots, n\}$
- Write $\mathbf{F}_{\Omega T}$ for the submatrix of $\mathbf{F}$ with rows in Ω and columns in T
- Note that $\|\mathbf{F}_{\Omega T}\| \leq 1$

Linear Independence and the DFT

- Consider a collection of spikes and sines:

$$\mathscr{X}(T, \Omega) = \{\mathbf{e}_j : j \in T\} \cup \{\mathbf{f}_j : j \in \Omega\} \subset \mathbb{C}^n$$

- The Gram matrix of this collection is

$$\boldsymbol{G} = \begin{bmatrix} \mathbf{I}_{|\Omega|} & \mathbf{F}_{\Omega T} \\ (\mathbf{F}_{\Omega T})^* & \mathbf{I}_{|T|} \end{bmatrix}$$

- $\mathscr{X}(T, \Omega)$ is linearly independent if and only if $\boldsymbol{G}$ is nonsingular
- The extreme eigenvalues of $\boldsymbol{G}$ are $1 \pm \|\mathbf{F}_{\Omega T}\|$
- Thus $\mathscr{X}(T, \Omega)$ is linearly independent if and only if $\|\mathbf{F}_{\Omega T}\| < 1$

The Donoho–Stark Bound

Theorem 1. [Donoho–Stark 1989] *If* $|T| \cdot |\Omega| < n$ *then* $\|\mathbf{F}_{\Omega T}\| < 1$.

Proof. The matrix $\mathbf{F}_{\Omega T}$ has $|\Omega|$ rows and $|T|$ columns; its entries have magnitude $n^{-1/2}$. Thus

$$\begin{aligned} \|\mathbf{F}_{\Omega T}\|^2 \leq \|\mathbf{F}_{\Omega T}\|_{\mathrm{F}}^2 &= \langle \mathbf{F}_{\Omega T},\ \mathbf{F}_{\Omega T} \rangle \\ &\leq \|\mathbf{F}_{\Omega T}\|_{1,1} \|\mathbf{F}_{\Omega T}\|_{\infty,\infty} \\ &= \frac{|\Omega|}{\sqrt{n}} \cdot \frac{|T|}{\sqrt{n}} \\ &< 1. \end{aligned}$$

Corollary 2. *If* $|T| + |\Omega| < 2\sqrt{n}$ *then* $\|\mathbf{F}_{\Omega T}\| < 1$.

The Donoho–Stark Uncertainty Principle

❧ Define $\operatorname{supp}(\boldsymbol{\alpha}) = \{j : \alpha_j \neq 0\}$ and $\|\boldsymbol{\alpha}\|_0 = |\operatorname{supp}(\boldsymbol{\alpha})|$

Corollary 3. [Discrete UP] *Let $\boldsymbol{x}$ be a vector in $\mathbb{C}^n$. Consider its representations in the spike and sine bases:*

$$\boldsymbol{x} = \sum\nolimits_{j=1}^{n} \alpha_j \mathbf{e}_j \qquad \textit{and} \qquad \boldsymbol{x} = \sum\nolimits_{j=1}^{n} \beta_j \mathbf{f}_j.$$

Then $\|\boldsymbol{\alpha}\|_0 \cdot \|\boldsymbol{\beta}\|_0 \geq n$.

Proof. Set $T = \operatorname{supp}(\boldsymbol{\alpha})$ and $\Omega = \operatorname{supp}(\boldsymbol{\beta})$. Note that

$$\sum\nolimits_{j \in T} \alpha_j \mathbf{e}_j - \sum\nolimits_{j \in \Omega} \beta_j \mathbf{f}_j = \mathbf{0}.$$

Therefore, $\mathscr{X}(T, \Omega)$ is linearly dependent and $|T|\,|\Omega| \geq n$.

The Dirac Comb

- Let n be a square number
- Set $T = \Omega = \{\sqrt{n}, 2\sqrt{n}, 3\sqrt{n}, \dots, n\}$
- The Poisson summation formula gives

$$\sum_{j \in T} \mathbf{e}_j = \sum_{j \in \Omega} \mathbf{f}_j.$$

- Thus the Donoho–Stark results are all sharp
- Reason: $\mathbb{Z}/\mathbb{Z}_n$ has nontrivial subgroups for composite n

Tao Uncertainty Principle

Idea: Counterexamples don't exist for prime n

Theorem 4. [Tao 2004] *Suppose n is prime.*

- *If $|T| + |\Omega| \le n$, then $\|\mathbf{F}_{\Omega T}\| < 1$.*
- *If $|T| + |\Omega| \ge n + 1$, then $\|\mathbf{F}_{\Omega T}\| = 1$.*

- Proof uses algebraic methods
- Some submatrices are very badly conditioned

Analytic Principle of the Large Sieve

Idea: Counterexamples have rigid structure

Define the *spread* of a set:

$$\Delta(\Omega) = \min\{|j - k \bmod n| : j, k \in \Omega, j \neq k\}$$

Theorem 5. [Large Sieve Inequality] *Suppose T has the form*

$$T = \{m + 1, m + 2, \dots, m + |T|\} \quad \textit{for an integer } m.$$

If $|T| + n/\Delta(\Omega) < n + 1$, then $\|\mathbf{F}_{\Omega T}\| < 1$.

References: [Donoho–Logan 1992, Jameson 2006]

Random Sets

Idea: Generic collections of spikes and sines are linearly independent

- For an integer m, define class of index sets with cardinality m:

$$\mathscr{S}_m = \{S : S \subset \{1, 2, \dots, n\} \text{ and } |S| = m\}$$

- Let Ω be a uniformly random element of $\mathscr{S}_m$, i.e.,

$$\operatorname{Prob}\{\Omega = S\} = |\mathscr{S}_m|^{-1} \qquad \text{for each } S \in \mathscr{S}_m.$$

- Say "Ω is a random set with cardinality $|\Omega|$"

The Candès–Romberg Bound

Theorem 6. [Candès–Romberg 2006] *Suppose that*

$$|T| + |\Omega| \le \frac{cn}{\sqrt{\log n}}.$$

If T is an arbitrary *set with cardinality $|T|$ and Ω is a* random *set with cardinality $|\Omega|$, then*

$$\operatorname{Prob}\left\{ \|\mathbf{F}_{\Omega T}\|^2 \ge 0.5 \right\} \le n^{-1}.$$

❧ Proof uses the moment method and heavy-duty combinatorics

Sets are not Created Equal

Theorem 7. [T 2006] *Suppose that*

$$|T| \log n + |\Omega| \leq cn.$$

If T is an arbitrary *set with cardinality $|T|$ and Ω is a* random *set with cardinality $|\Omega|$, then*

$$\operatorname{Prob}\left\{ \|\mathbf{F}_{\Omega T}\|^2 \geq 0.5 \right\} \leq n^{-1}.$$

❧ Proof uses Rudelson's selection lemma

Reference: [*Random Subdictionaries*]

Restricted Isometry Consequences

Except with probability n^{-1}, a random set Ω with cardinality $|\Omega|$ satisfies

$$\frac{|\Omega|}{2n} \leq \|\mathbf{F}_{\Omega T}\|^2 \leq \frac{3\,|\Omega|}{2n} \qquad \text{for all } T \text{ where } |T| \leq \frac{cn}{\log^5 n}.$$

Corollary 8. [T 2007] *Except with probability n^{-1}, a random set Ω has the following property. For each set T whose cardinality*

$$|T| \leq \frac{cn}{\log^5 n},$$

it holds that $\|\mathbf{F}_{\Omega T}\|^2 \leq 0.5$.

References: [Candès–Tao 2006, Rudelson–Vershynin 2006, *Spikes & Sines*]

When Both Sets are Random

Theorem 9. [T 2007] *Fix* $\varepsilon > 0$*. Suppose that* $n \geq N(\varepsilon)$ *and that*

$$|T| + |\Omega| \leq c(\varepsilon) \cdot n.$$

Let T *and* Ω *be* random *sets with cardinalities* $|T|$ *and* $|\Omega|$*. Then*

$$\mathrm{Prob}\left\{ \|\mathbf{F}_{\Omega T}\|^2 \geq 0.5 \right\} \leq 4 \exp\{-n^{1/2-\varepsilon}\}.$$

Reference: [*Spikes & Sines*]

Near Optimality

- *Assume* it were possible to obtain

$$\text{Prob}\left\{\|\mathbf{F}_{\Omega T}\|^2 \geq 0.5\right\} \leq \exp\{-n^{1/2+\varepsilon}\}$$

- Consider case where n is a square number and $|T| + |\Omega| = 2\sqrt{n}$
- Only about $\exp\{n^{1/2}\log n\}$ ways to pick the sets
- Union bound $\Rightarrow$ no pair of sets yields $\|\mathbf{F}_{\Omega T}\| = 1$
- *Contradiction:* The Dirac comb has $\|\mathbf{F}_{\Omega T}\| = 1$

Proof Techniques

- Reduction to square case with independent coordinate model
- Rudelson–Vershynin theorem on spectral norm of random submatrix
  - Rudelson's selection lemma
  - Noncommutative Khintchine inequality
- Classical Khintchine inequality for $(1, 2)$ norm of random submatrix
- Bourgain and Tzafriri's extrapolation
  - Minimax property of Chebyshev polynomials

Reference: [T 2006, *Random Paving*]

Normalized Random Submatrices

- If $|\Omega| = \delta n$ then columns of $\mathbf{F}_{\Omega T}$ have ℓ_2 norm $\delta^{1/2}$
- Should normalize matrix by $\delta^{-1/2}$

Theorem 10. [T 2007] *Fix $\delta \in (0, 1)$. Suppose that $n \geq N(\delta)$ and that*

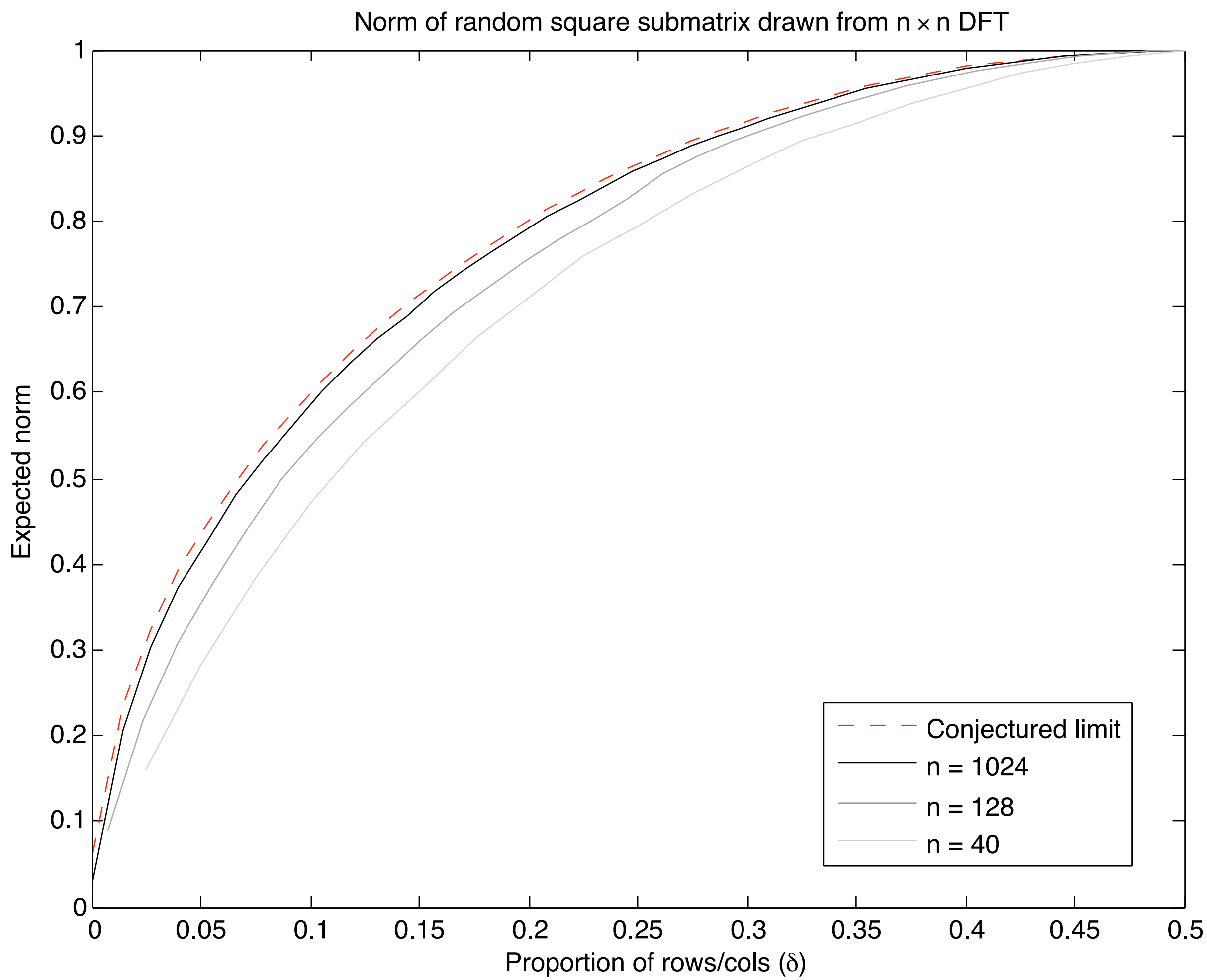
$$|\Omega| = |T| = \delta n.$$

Let T and Ω be random sets with cardinalities $|T|$ and $|\Omega|$. Then

$$\mathrm{Prob}\left\{\frac{1}{\sqrt{\delta}}\left\|\mathbf{F}_{\Omega T}\right\| \geq 10\right\} \leq n^{-1}.$$

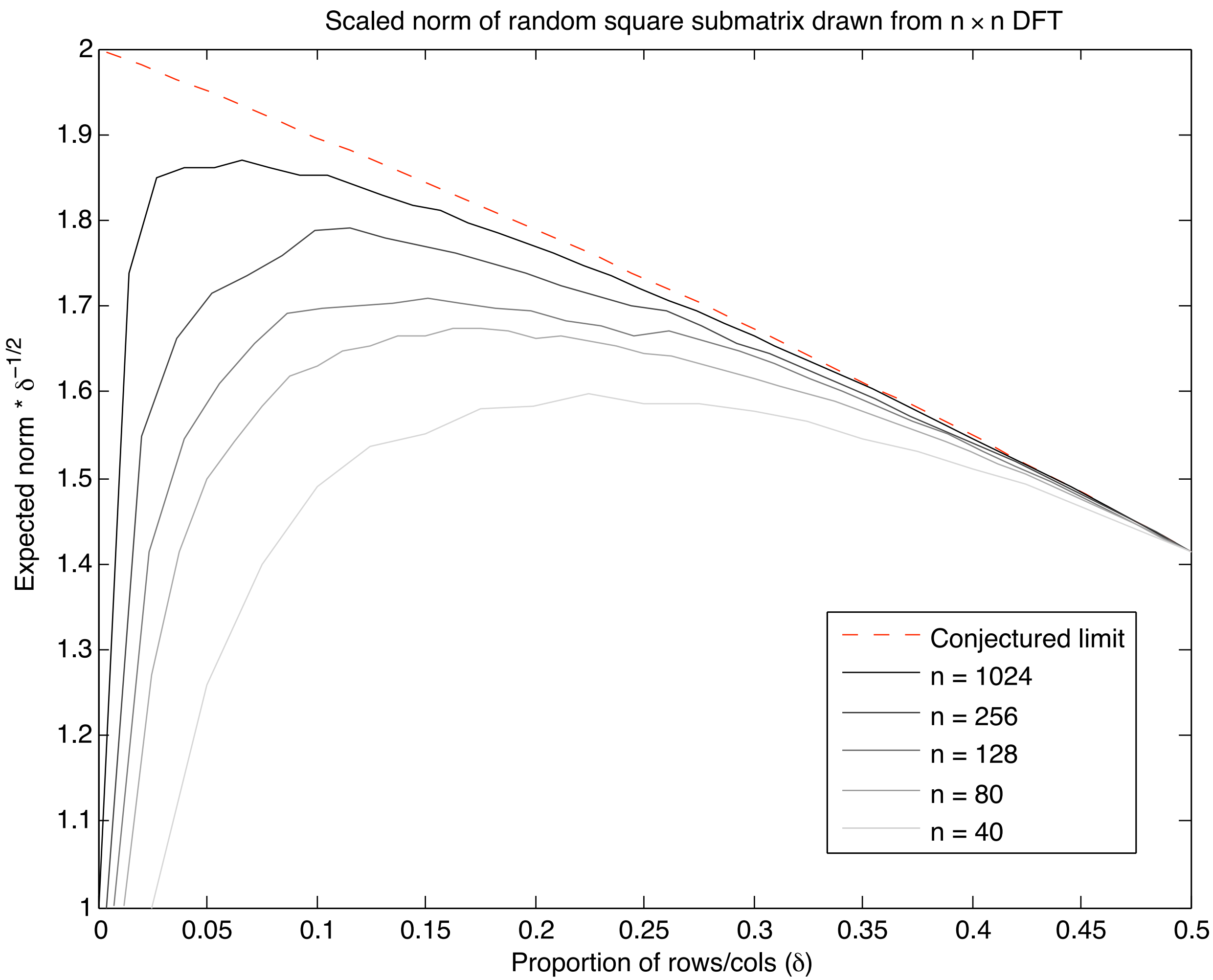
Reference: [*Spikes & Sines*]

Numerics I

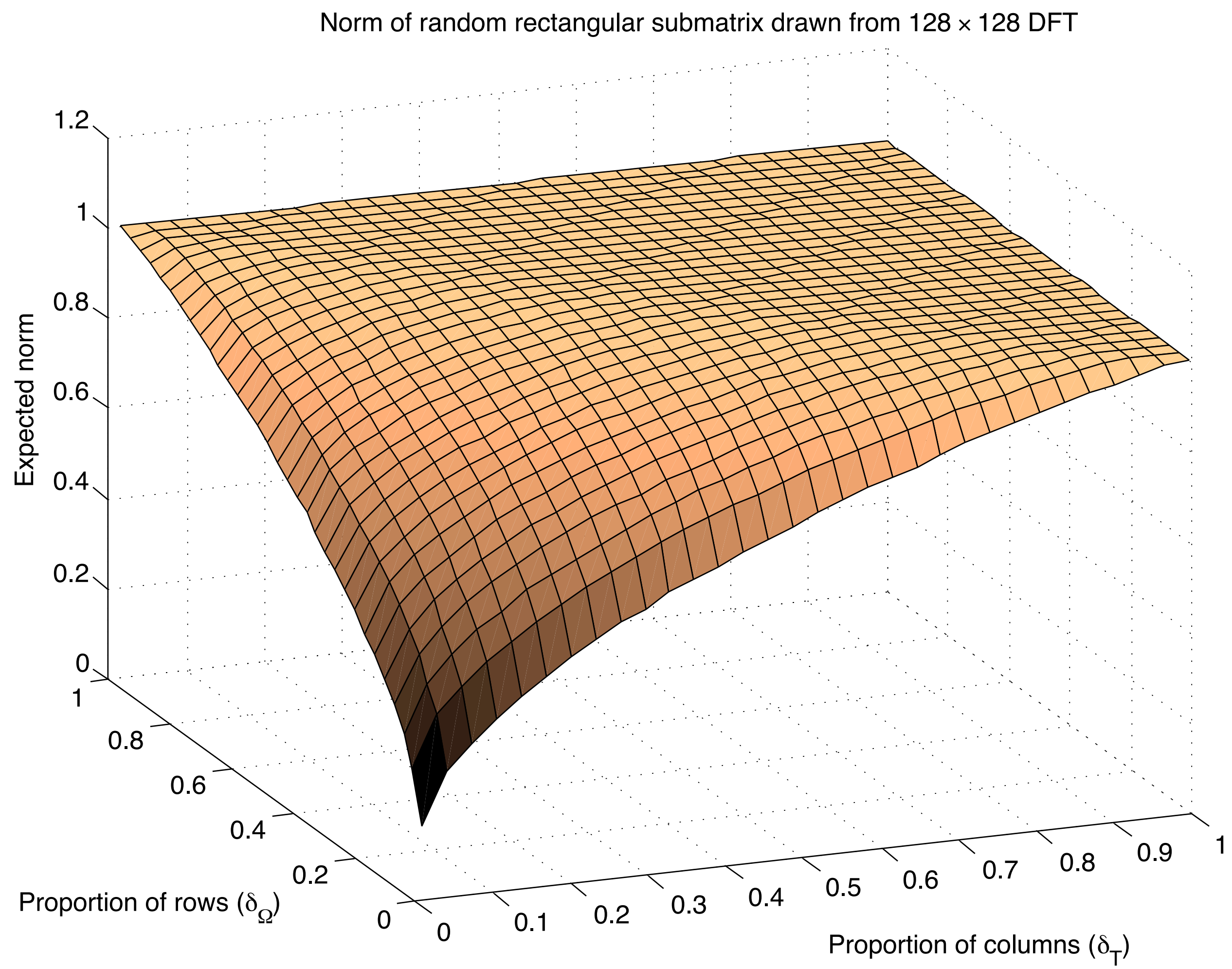


Norm of random square submatrix drawn from n × n DFT
Expected norm
Proportion of rows/cols (δ)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
Conjectured limit
n = 1024
n = 128
n = 40

Numerics II


Scaled norm of random square submatrix drawn from n × n DFT
Expected norm * δ^−1/2
Proportion of rows/cols (δ)
2
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
Conjectured limit
n = 1024
n = 256
n = 128
n = 80
n = 40

Numerics III



Open Questions

Conjecture 11. [Quartercircle Law] *Suppose that*

$$\delta = \frac{|T| + |\Omega|}{2n} \leq \frac{1}{2}.$$

If T and Ω are random sets with cardinalities $|T|$ and $|\Omega|$, then

$$\mathbb{E}\,\|\mathbf{F}_{\Omega T}\| \leq 2\sqrt{\delta(1-\delta)}.$$

The inequality becomes an equality as $n \to \infty$.

❧ What is the correct tail behavior?

❧ Study behavior of $\sigma_{\min}(\mathbf{F}_{\Omega T})$ for random T, Ω

To learn more...

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Web: `http://www.umich.edu/~jtropp`

Papers:

- "Random subdictionaries of general dictionaries," 2006
- "The random paving property for uniformly bounded matrices," 2006
- "On the linear independence of spikes and sines," 2007