Simultaneous Sparsity

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Simple Sparse Approximation

- Work in the d-dimensional, complex inner-product space $\mathbb{C}^d$
- Let $\{\boldsymbol{\varphi}_\omega : \omega \in \Omega\}$ be a collection of unit-norm elementary signals

- Choose T indices $\lambda_1, \dots, \lambda_T \in \Omega$
- Suppose we measure a noisy sparse signal

$$\boldsymbol{s} \quad = \quad \sum_{t=1}^{T} c_t \, \boldsymbol{\varphi}_{\lambda_t} \; + \; \boldsymbol{\nu}$$

- The *simple sparse approximation problem* asks
  1. Can we identify the indices $\lambda_1, \dots, \lambda_T$?
  2. Can we estimate the coefficients $c_1, \dots, c_T$?

Facts about Greedy Algorithms

A *greedy algorithm* for sparse approximation makes locally optimal choices in an effort to obtain a good global solution.

Advantages

- Fast
- Easy to implement
- Work well for many problems

Disadvantages

- Less robust than ℓ_1 methods
- Not effective for superresolution

Orthogonal Matching Pursuit (OMP)

Input: A signal $\boldsymbol{s}$ and the number of terms T
Output: Indices $\{\lambda_1, \dots, \lambda_T\}$ and coefficients $\{c_1, \dots, c_T\}$

1. Set the initial residual $\boldsymbol{r}_0 = \boldsymbol{s}$ and the counter $t = 1$
2. Find an index λ_t that solves

$$\max_{\omega \in \Omega} \; |\langle \boldsymbol{r}_{t-1}, \boldsymbol{\varphi}_\omega \rangle|$$

3. Determine the orthogonal projector P_t onto $\operatorname{span}\{\boldsymbol{\varphi}_{\lambda_1}, \dots, \boldsymbol{\varphi}_{\lambda_t}\}$
4. Calculate the new residual: $\boldsymbol{r}_t = \boldsymbol{s} - P_t \, \boldsymbol{s}$
5. Increment t, and repeat until $t = T$
6. The coefficient estimates appear in the expansion

$$P_T \, \boldsymbol{s} \quad = \quad \sum_{t=1}^{T} c_t \, \boldsymbol{\varphi}_{\lambda_t}$$

Simultaneous Sparse Approximation

Idea: More observations should make the problem easier

- Choose T indices $\lambda_1, \dots, \lambda_T \in \Omega$
- Suppose we measure K noisy sparse signals

$$\boldsymbol{s}_k \quad = \quad \sum_{t=1}^{T} c_{tk}\, \boldsymbol{\varphi}_{\lambda_t} \; + \; \boldsymbol{\nu}_k$$

- The *simultaneous sparse approximation problem* asks
  1. Can we identify the indices $\lambda_1, \dots, \lambda_T$?
  2. Can we estimate the set of coefficients $\{c_{tk}\}$?

Application: MIMO Communications

Transmit 1:	$\boldsymbol{\varphi}_{\lambda_1}$	$\sim\sim\sim$	Receive 1:	$\sum_t h_{t1}\, \boldsymbol{\varphi}_{\lambda_t} + \boldsymbol{\nu}_1$
Transmit 2:	$\boldsymbol{\varphi}_{\lambda_2}$	$\sim\sim\sim$	Receive 2:	$\sum_t h_{t2}\, \boldsymbol{\varphi}_{\lambda_t} + \boldsymbol{\nu}_2$
...			...	
Transmit t:	$\boldsymbol{\varphi}_{\lambda_t}$	$\sim\sim\sim$	Receive k:	$\sum_t h_{tk}\, \boldsymbol{\varphi}_{\lambda_t} + \boldsymbol{\nu}_k$

- The dimension d corresponds with the length of a transmission block
- Send one elementary signal on each of T transmit antennas
- Measure one superposition on each of K receive antennas

- The numbers h_{tk} are fading coefficients
- The vectors $\boldsymbol{\nu}_k$ are additive noise

Goal: Identify which elementary signals were transmitted

Simultaneous OMP

Input: A $d \times K$ signal matrix S and the number of terms T
Output: Indices $\{\lambda_1, \ldots, \lambda_T\}$

1. Set the initial residual $R_0 = S$ and the counter $t = 1$
2. Find an index λ_t that solves

$$\max_{\omega \in \Omega} \sum_{k=1}^{K} |\langle R_{t-1}\, \mathbf{e}_k, \boldsymbol{\varphi}_\omega \rangle|$$

3. Determine the orthogonal projector P_t onto $\operatorname{span}\{\boldsymbol{\varphi}_{\lambda_1}, \ldots, \boldsymbol{\varphi}_{\lambda_t}\}$
4. Calculate the new residual: $R_t = S - P_t\, S$
5. Increment t, and repeat until $t = T$

Experiments with S-OMP

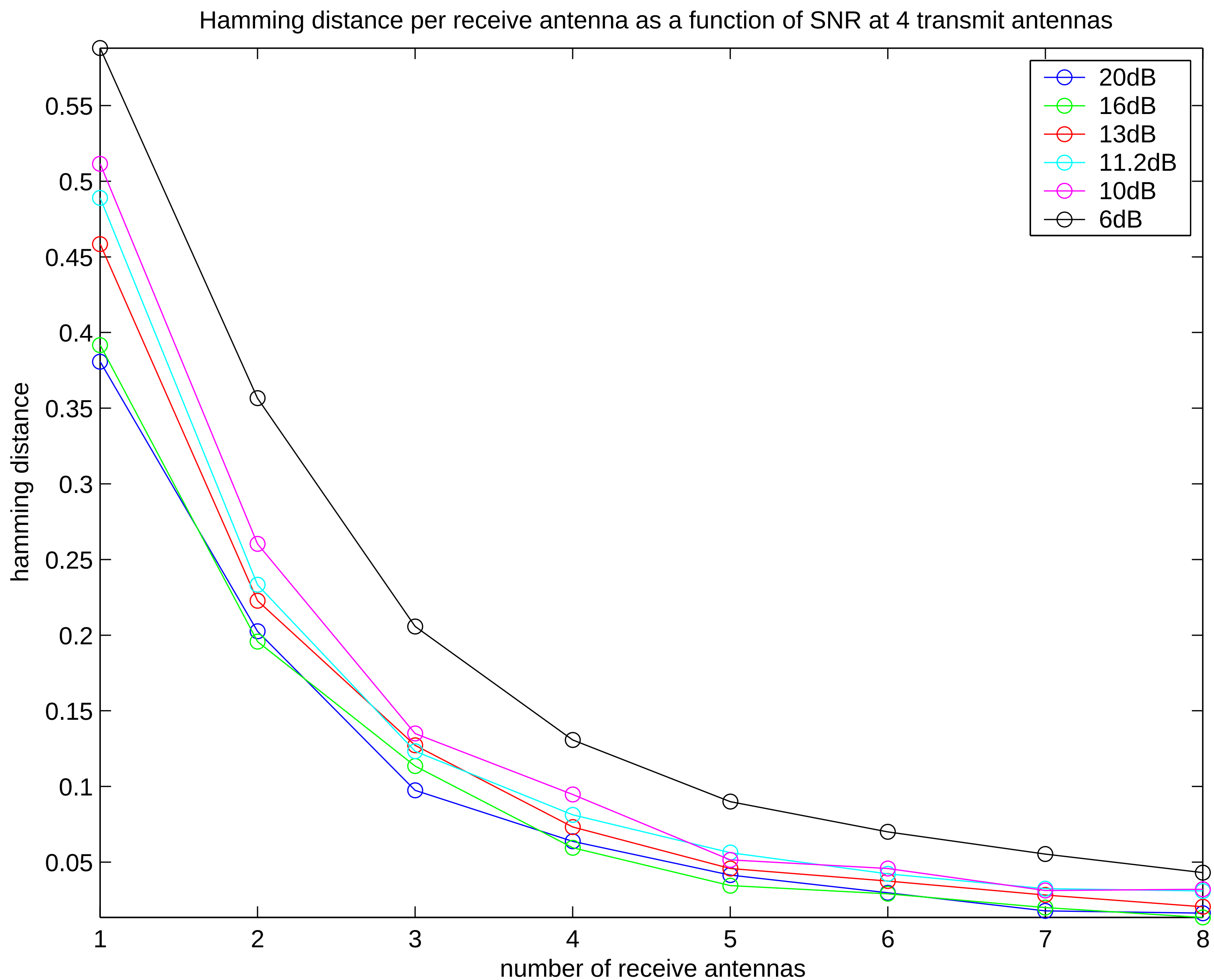
- The $\mathbb{Z}_4$ Kerdock code yields 64 elementary signals in $\mathbb{C}^8$
- Fix the number of transmit/receive antennas and the SNR

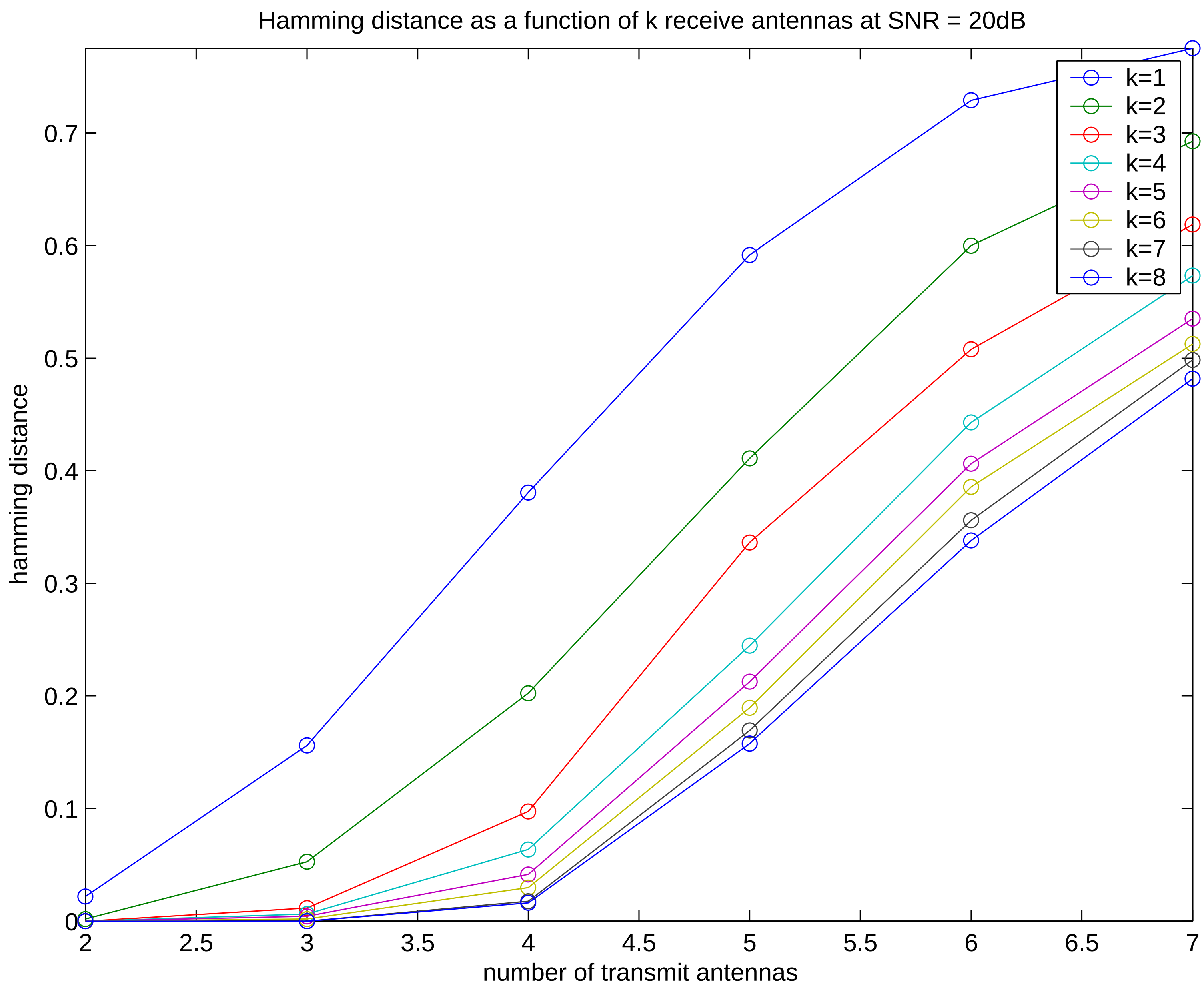
- For each trial, we construct K signals

$$\boldsymbol{s}_k \quad = \quad \sum_{t=1}^{T} h_{tk}\, \boldsymbol{\varphi}_{\lambda_t} \;+\; \boldsymbol{\nu}_k$$

  where h_{tk} are Gaussian variables and $\boldsymbol{\nu}_k$ are Gaussian vectors
- S-OMP is used to pick T elementary signals
- Calculate the fraction correctly identified

- Average over 1000 trials


Hamming distance per receive antenna as a function of SNR at 4 transmit antennas
hamming distance
number of receive antennas
0.55
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
1
2
3
4
5
6
7
8
20dB
16dB
13dB
11.2dB
10dB
6dB


Hamming distance as a function of k receive antennas at SNR = 20dB
hamming distance
number of transmit antennas
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
2
2.5
3
3.5
4
4.5
5
5.5
6
6.5
7
k=1
k=2
k=3
k=4
k=5
k=6
k=7
k=8

A Theoretical Result for S-OMP

Claim: Each result for simple sparse approximation has an analog for simultaneous sparse approximation.

Define the *coherence parameter* $\mu = \max_{\lambda \neq \omega} |\langle \boldsymbol{\varphi}_\lambda, \boldsymbol{\varphi}_\omega \rangle|$

Theorem 1. *Suppose that $T\,\mu \leq \frac{1}{3}$. Let S be a signal matrix. After T iterations, S-OMP calculates a T-term approximation A_T of the signal matrix that satisfies*

$$\|S - A_T\|_{\mathrm{F}} \quad \leq \quad \sqrt{1 \; + \; 6\,KT} \; \|S - A_{\mathrm{opt}}\|_{\mathrm{F}}$$

where A_{opt} is the optimal T-term approximation of S in Frobenius norm.

Convex Relaxation for SSA

- Can view simultaneous sparse approximation as a combinatorial optimization problem

$$\min_{C} \quad \#\text{ nonzero rows of } C \quad \text{subject to} \quad \left\| S - \Phi\, C \right\|_{\mathrm{F}} \leq \varepsilon$$

- Can replace this combinatorial problem with a related convex program

$$\min_{C} \quad \sum_{\omega} \max_{k} \; |c_{\omega k}| \quad \text{subject to} \quad \left\| S - \Phi\, C \right\|_{\mathrm{F}} \leq \delta$$

- One can prove the two problems often have similar solutions

Related Work

- Çetin, Malioutov, Willsky (Algorithms, Applications)
- Chen, Huo (Theory)
- Cotter, Engan, Kreutz-Delgado, Rao, et al. (Algorithms, Applications)
- Gribonval, Nielsen (Theory, Applications)
- Leviatan, Lutoborsky, Temlyakov (Theory)

Publications and Preprints

- TGS, "Simultaneous sparse approximation via greedy pursuit," ICASSP 2005.
- TGS, "Algorithms for simultaneous sparse approximation. Part I: Greedy pursuit," submitted November 2004.
- T, "Algorithms for simultaneous sparse approximation. Part II: Convex relaxation," submitted November 2004.
- GT, "Applications of sparse approximation in communications," submitted January 2005.
- TG, "Signal recovery from partial information via Orthogonal Matching Pursuit," submitted March 2005.

Papers available from `http://www.umich.edu/~jtropp/`.

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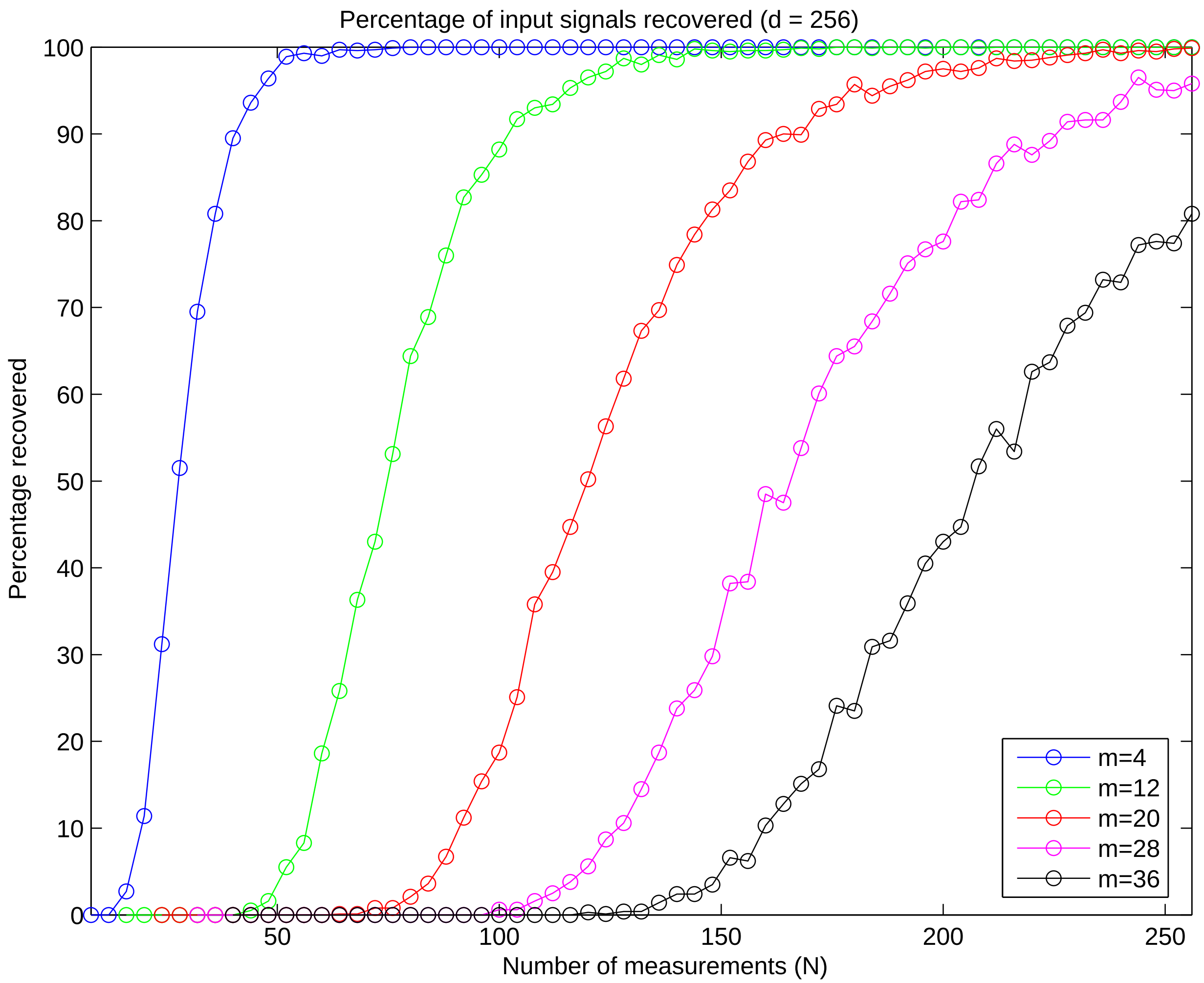
Greed: Still Good



Greed: Still Good

Theorem 2. *Suppose that* $\boldsymbol{s}$ *is an arbitrary* m*-sparse signal in* $\mathbb{R}^d$*. Given* $\mathrm{K}_p\, m \log d$ *random linear measurements of* $\boldsymbol{s}$*, OMP can recover* $\boldsymbol{s}$ *with probability* $1 - O(d^{-p})$*.*

This theorem is more or less equivalent with results for ℓ_1 minimization due to Candès–Tao, Donoho, and Rudelson–Vershynin.