Complex Equiangular Tight Frames

Joel A. Tropp

jtropp@umich.edu

Department of Mathematics
The University of Michigan

With contributions from
Inderjit Dhillon, Robert Heath Jr., Nick Ramsey,
Thomas Strohmer, and Mátyás Sustik

Equiangular Tight Frames

- Let $\{\boldsymbol{x}_m\}$ be a collection of N unit vectors in $\mathbb{C}^d$ with $N \geq d$
- A lower bound on the maximum correlation between a pair of vectors:

$$\max_{m \neq n} |\langle \boldsymbol{x}_m, \; \boldsymbol{x}_n \rangle| \quad \geq \quad \sqrt{\frac{N-d}{d\,(N-1)}} \quad \stackrel{\text{def}}{=} \quad \mu(d, N)$$

- The bound is met if and only if
  1. The vectors are *equiangular*
  2. The vectors form a *tight frame*
- If the bound is met, we refer to the matrix $\boldsymbol{X} = \begin{bmatrix} \boldsymbol{x}_1 & \dots & \boldsymbol{x}_N \end{bmatrix}$ as a (d, N) *equiangular tight frame* or *ETF*

Examples of ETFs

- When $N = d$: An *orthonormal basis*

- When $N = d + 1$: The vertices of a *regular simplex*

- Four vectors in $\mathbb{C}^2$:

$$\frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{3} & 1 & 1 & 1 \\ 0 & \sqrt{2} & \mathrm{e}^{2\pi\mathrm{i}/3}\sqrt{2} & \mathrm{e}^{4\pi\mathrm{i}/3}\sqrt{2} \end{bmatrix}$$

- Six vectors in $\mathbb{R}^3$: Six nonantipodal vertices of a *regular icosahedron*

ETFs are Sporadic

	d				
N	2	3	4	5	6
3	$\mathbb{R}$	$\mathbb{R}$	..	..	..
4	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{R}$	..	..
5	..	.	$\mathbb{R}$	$\mathbb{R}$	..
6	..	$\mathbb{R}$	.	$\mathbb{R}$	$\mathbb{R}$
7	..	$\mathbb{C}$	$\mathbb{C}$	.	$\mathbb{R}$
8	..	.	$\mathbb{C}$	.	.
9	..	$\mathbb{C}$	.	.	$\mathbb{C}$
10	..	..	.	$\mathbb{R}$	.
11	..	..	.	$\mathbb{C}$	$\mathbb{C}$
12	..	..	.	.	$\mathbb{C}$
13	..	..	$\mathbb{C}$	.	.
14	..	..	.	.	.
15	..	..	.	.	.
16	..	..	$\mathbb{C}$	.	$\mathbb{R}$
17	..	..	..	.	.
18	..	..	..	.	.
19	..	..	..	.	.

	d				
N	2	3	4	5	6
20	..	..	..	.	.
21	..	..	..	$\mathbb{C}$	.
22	..	..	..	.	.
23	..	..	..	.	.
24	..	..	..	.	.
25	..	..	..	$\mathbb{C}$	.
26	..	..	..	..	.
27	..	..	..	..	.
28	..	..	..	..	.
29	..	..	..	..	.
30	..	..	..	..	.
31	..	..	..	..	$\mathbb{C}$
32	..	..	..	..	.
33	..	..	..	..	.
34	..	..	..	..	.
35	..	..	..	..	.
36	..	..	..	..	$\mathbb{C}$

Reference: [JAT–Dhillon–Heath–Strohmer 2005]

Classical Upper Bound

Theorem 1. *If there exists a complex* (d, N) *ETF, then*

$$N \leq d^2$$
$$N \leq (N - d)^2.$$

If there exists a real (d, N) *ETF, then*

$$N \leq \tfrac{1}{2}\, d\,(d + 1)$$
$$N \leq \tfrac{1}{2}\,(N - d)\,(N - d + 1).$$

- Reference: [van Lint–Seidel 1966]
- Other proofs: [Conway et al. 1996, Sustik et al. 2003]

Integrality Condition for Real ETFs

Theorem 2. [SuTDH 2004] *Assume that* $N \neq 2d$. *If there exists a real* (d, N) *ETF, then*

$$\sqrt{\frac{d\,(N-1)}{N-d}} \equiv \sqrt{\frac{(N-1)(N-d)}{d}} \equiv 1 \pmod 2.$$

If there exists a real $(d, 2d)$ *ETF, then*

$$d \equiv 1 \pmod 2 \qquad \textit{and} \qquad 2d-1 = a^2 + b^2 \quad \textit{where } a, b \in \mathbb{Z}.$$

- Equivalence between real ETFs and strongly regular graphs
- Related results: [Holmes–Paulsen 2004]

Harmonic ETFs

A *(d, N) harmonic ETF* over the p-th roots of unity has the form

$$\frac{1}{\sqrt{d}} \left[\quad \exp\left\{\tfrac{2\pi \mathrm{i}}{p}\, a_{jn}\right\} \quad \right] \qquad \text{where } a_{jn} \in \mathbb{Z}$$

References: [König 1999, Strohmer–Heath 2003, Xia et al. 2005]

Examples of Harmonic ETFs

- When $N = d$ and $p = 2$: Hadamard matrices
- When $N = d$ and $p = 4$: Complex Hadamard matrices

- 7 vectors in $\mathbb{C}^3$ with $p = 7$:

$$\frac{1}{\sqrt{3}} \exp \cdot \frac{2\pi \mathrm{i}}{7} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 3 & 6 & 2 & 5 & 1 & 4 \end{bmatrix}$$

- 7 vectors in $\mathbb{C}^4$ with $p = 7$:

$$\frac{1}{\sqrt{3}} \exp \cdot \frac{2\pi \mathrm{i}}{7} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 2 & 4 & 6 & 1 & 3 & 5 \\ 0 & 4 & 1 & 5 & 2 & 6 & 3 \end{bmatrix}$$

Integrality for Harmonic ETFs

Theorem 3. [JAT] *Suppose that there exists a (d, N) harmonic ETF over the p-th roots of unity. Define*

$$\gamma = \frac{d\,(N-d)}{N-1}.$$

We have the following consequences.

$$\begin{aligned} \textit{When } p = 2: \qquad & \sqrt{\gamma} \in \mathbb{Z} \\ p = 3: \qquad & \gamma = a^2 + ab + b^2 \quad \textit{where } a, b \in \mathbb{Z} \\ p = 4: \qquad & \gamma = a^2 + b^2 \quad \textit{where } a, b \in \mathbb{Z} \end{aligned}$$

Moreover, $\gamma \in \mathbb{Z}$ whenever the (unnormalized) entries of the ETF are roots of unity. In all these cases, $N \le d^2 - d + 1$.

Harmonic ETFs with $N \geq d+2$

d	N	$p=2$	3	4	Other
3	7			N	7
4	7			N	7
	13				13
5	11		N		11
	21			N	21
6	11		N	N	11
	16	Y	N	Y	Y
	31			N	31
7	15	N	N	N	15
	22			?	?
	43				?
8	15	N	N	?	15
	29				?
	57		?		57

d	N	$p=2$	3	4	Other
9	13		N		13
	19		?	?	?
	25				?
	37		?		37
	73			?	73
10	16	Y	?	Y	Y
	19			?	?
	31		?		?
	46			?	?
	91	?	?	?	91
11	23				?
	56	?	?	?	?
	111			?	?

Maximal Complex ETFs

- Numerical evidence strongly suggests that there is a (d, d^2) complex ETF for each $d = 1, 2, 3, 4, \ldots$.
- Explicit constructions exist for $d = 1, 2, 3, 4, 5, 6, 8$.
- The Integrality Theorem rules out harmonic ETFs as a possible source.

Open Questions:

- Prove that maximal ETFs always exist.
- Provide explicit constructions.

Related Papers and Contact Information

- T. "Constructing packings in projective spaces and Grassmannian spaces via alternating projection." ICES Report 04-23, May 2004.
- SuTDH. "On the existence of equiangular tight frames." UTCS TR-04-32, July 2004. (In revision.)
- T. "Topics in Sparse Approximation." Ph.D. Dissertation, August 2004.
- TDHSt. "Designing Structured Tight Frames via Alternating Projection." *IEEE Trans. Info. Theory*, January 2005.
- T. "Complex Equiangular Tight Frames." Wavelets XI, August 2005.

All papers available from `http://www.umich.edu/~jtropp`

E-mail: `jtropp@umich.edu`