# Constructing Equiangular Tight Frames with Alternating Projection

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- A lower bound on the maximum correlation between a pair of vectors:

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- References: [van Lint–Seidel 1966, Welch 1974]

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- The bound is met if and only if

  1. The vectors are equiangular
  2. The vectors form a tight frame

# Tight Frames vs. Equiangular Tight Frames

Tight frame:

$$\begin{bmatrix} 1.0000 & 0.2414 & -0.6303 & 0.5402 & -0.3564 & -0.3543 \\ 0.2414 & 1.0000 & -0.5575 & -0.4578 & 0.5807 & -0.2902 \\ -0.6303 & -0.5575 & 1.0000 & 0.2947 & 0.3521 & -0.2847 \\ 0.5402 & -0.4578 & 0.2947 & 1.0000 & -0.2392 & -0.5954 \\ -0.3564 & 0.5807 & 0.3521 & -0.2392 & 1.0000 & -0.5955 \\ -0.3543 & -0.2902 & -0.2847 & -0.5954 & -0.5955 & 1.0000 \end{bmatrix}$$

Equiangular tight frame:

$$\begin{bmatrix} 1.0000 & 0.4472 & -0.4472 & 0.4472 & -0.4472 & -0.4472 \\ 0.4472 & 1.0000 & -0.4472 & -0.4472 & 0.4472 & -0.4472 \\ -0.4472 & -0.4472 & 1.0000 & 0.4472 & 0.4472 & -0.4472 \\ 0.4472 & -0.4472 & 0.4472 & 1.0000 & -0.4472 & -0.4472 \\ -0.4472 & 0.4472 & 0.4472 & -0.4472 & 1.0000 & -0.4472 \\ -0.4472 & -0.4472 & -0.4472 & -0.4472 & -0.4472 & 1.0000 \end{bmatrix}$$

# The Gram Matrix

- Suppose $\{\boldsymbol{s}_j\}$ is an $N$-vector equiangular tight frame for $\mathbb{F}^d$
- Its *Gram matrix* *G* has entries

$$g_{jk} = \langle \boldsymbol{s}_k, \boldsymbol{s}_j \rangle$$

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- Suppose $G$ has Properties 1–4. Then $G = S^* S$, where the columns of $S$ form an $N$-vector equiangular tight frame for $\mathbb{F}^d$

# Constraint Sets

- Define the *structural constraint set*

$$\mathscr{H} \stackrel{\text{def}}{=} \{H \in \mathbb{F}^{N \times N} : H = H^*, \quad \operatorname{diag} H = \mathbf{e}, \quad \text{and} \quad |h_{jk}| \leq \mu \text{ for } j \neq k\}$$

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- Define the *spectral constraint set*

$$\mathscr{G} \stackrel{\text{def}}{=} \left\{G \in \mathbb{F}^{N \times N} : \boldsymbol{\lambda}(G) = [\, \underbrace{N/d \ \ldots \ N/d}_{d} \ 0 \ \ldots \ 0 \,]^T\right\}$$

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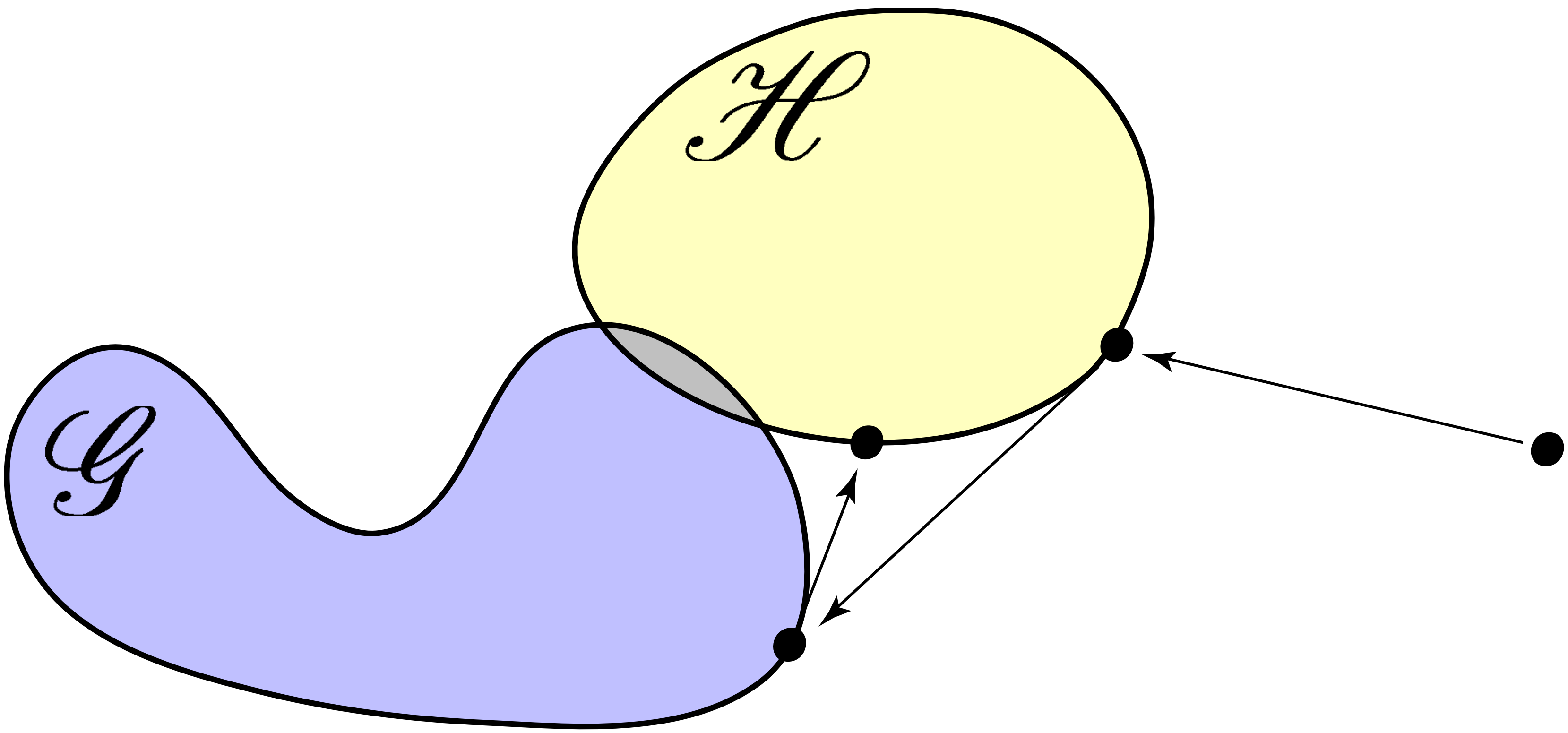
Goal: Find a matrix in $\mathscr{G} \cap \mathscr{H}$

# Alternating Projection



# Matrix Nearness Problems

**Proposition 1.** *Let $G$ be an Hermitian matrix. With respect to Frobenius norm, the unique matrix in $\mathscr{H}$ closest to $G$ has a unit diagonal and off-diagonal entries that satisfy*

$$h_{mn} = \begin{cases} g_{mn} & \text{if } |g_{mn}| \leq \mu, \text{ and} \\ \mu\, g_{mn} / |g_{mn}| & \text{otherwise.} \end{cases}$$

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**Proposition 2.** *Let H be an Hermitian matrix whose eigenvalue decomposition is $\sum_{n=1}^{N} \lambda_n \boldsymbol{u}_n \boldsymbol{u}_n^*$ with the eigenvalues decreasingly ordered. With respect to Frobenius norm, a matrix in $\mathscr{G}$ closest to H is given by*

$$(N/d) \sum_{n=1}^{d} \boldsymbol{u}_n \boldsymbol{u}_n^*.$$

# Global Convergence

**Theorem 1.** *Suppose that alternating projection generates an (infinite) sequence of iterates $\{(G_t, H_t)\}$. The sequence has at least one accumulation point.*

- *Every accumulation point lies in $\mathscr{G} \times \mathscr{H}$.*

- *Every accumulation point $(\overline{G}, \overline{H})$ satisfies*

$$\left\|\overline{G} - \overline{H}\right\|_{\mathrm{F}} = \lim_{t \to \infty} \|G_t - H_t\|_{\mathrm{F}} .$$

- *Every accumulation point $(\overline{G}, \overline{H})$ satisfies*

$$\left\|\overline{G} - \overline{H}\right\|_{\mathrm{F}} = \operatorname{dist}(\overline{G}, \mathscr{H}) = \operatorname{dist}(\overline{H}, \mathscr{G}).$$

# Local Convergence

**Theorem 2.** *In addition, suppose there is an iteration $T$ during which $\|G_T - H_T\|_{\mathrm{F}} < N/(d\sqrt{2})$. We may conclude that*

- *The accumulation point $(\overline{G}, \overline{H})$ is a fixed point of the algorithm.*
- *The component sequences are asymptotically regular, i.e.,*

$$\|G_{t+1} - G_t\|_{\mathrm{F}} \to 0 \quad \textit{and} \quad \|H_{t+1} - H_t\|_{\mathrm{F}} \to 0.$$

- Either *the component sequences both converge in norm,*

$$\left\|G_t - \overline{G}\right\|_{\mathrm{F}} \to 0 \quad \textit{and} \quad \left\|H_t - \overline{H}\right\|_{\mathrm{F}} \to 0,$$

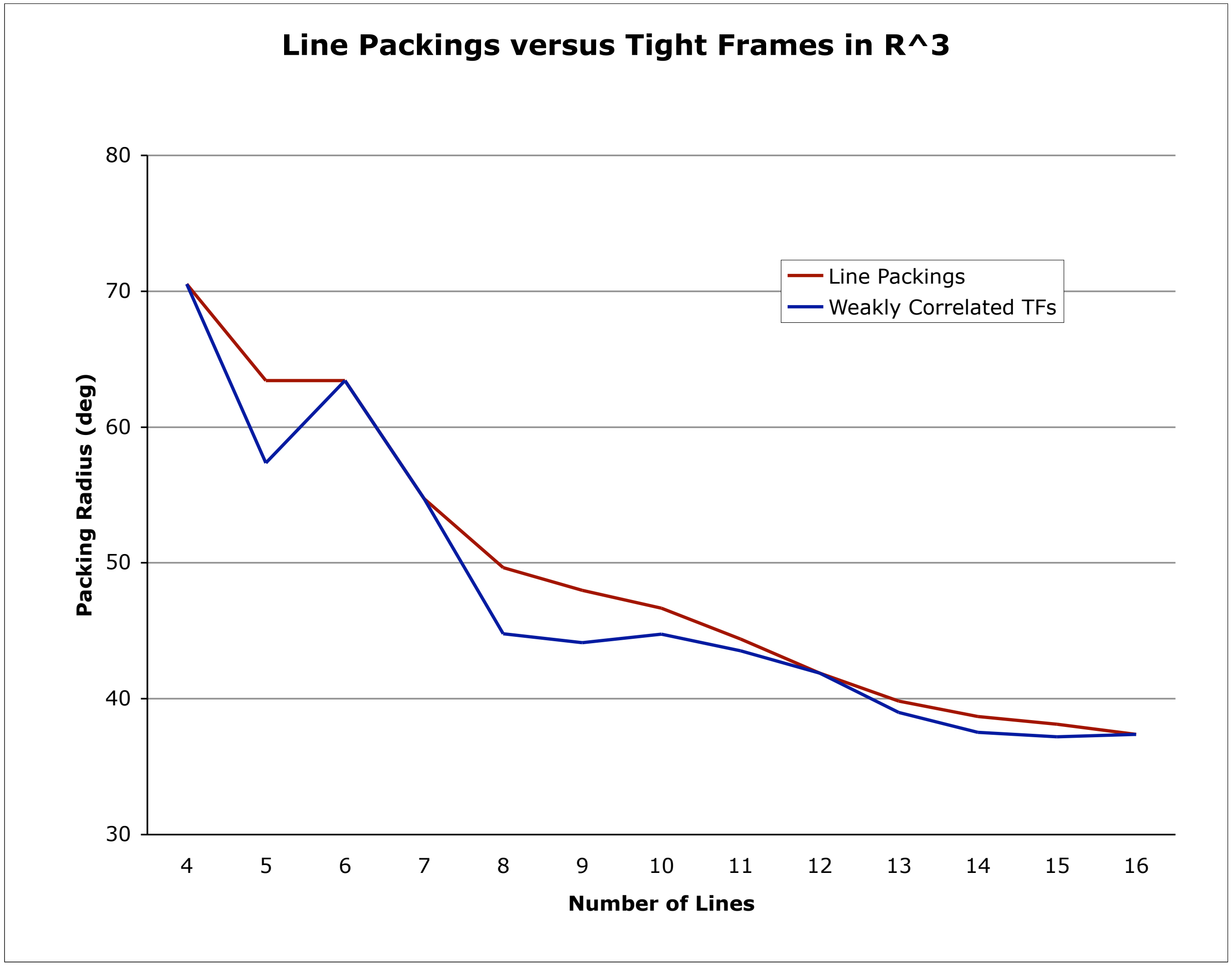
  or *the set of accumulation points forms a continuum.*

# Experimental Results

| | $d$ | | | | |
|---|---|---|---|---|---|
| $N$ | 2 | 3 | 4 | 5 | 6 |
| 3 | $\mathbb{R}$ | $\mathbb{R}$ | .. | .. | .. |
| 4 | $\mathbb{C}$ | $\mathbb{R}$ | $\mathbb{R}$ | .. | .. |
| 5 | .. | . | $\mathbb{R}$ | $\mathbb{R}$ | .. |
| 6 | .. | $\mathbb{R}$ | . | $\mathbb{R}$ | $\mathbb{R}$ |
| 7 | .. | $\mathbb{C}$ | $\mathbb{C}$ | . | $\mathbb{R}$ |
| 8 | .. | . | $\mathbb{C}$ | . | . |
| 9 | .. | $\mathbb{C}$ | . | . | $\mathbb{C}$ |
| 10 | .. | .. | . | $\mathbb{R}$ | . |
| 11 | .. | .. | . | $\mathbb{C}$ | $\mathbb{C}$ |
| 12 | .. | .. | . | . | $\mathbb{C}$ |
| 13 | .. | .. | $\mathbb{C}$ | . | . |
| 14 | .. | .. | . | . | . |
| 15 | .. | .. | . | . | . |
| 16 | .. | .. | $\mathbb{C}$ | . | $\mathbb{R}$ |
| 17 | .. | .. | .. | . | . |
| 18 | .. | .. | .. | . | . |
| 19 | .. | .. | .. | . | . |

| | $d$ | | | | |
|---|---|---|---|---|---|
| $N$ | 2 | 3 | 4 | 5 | 6 |
| 20 | .. | .. | .. | . | . |
| 21 | .. | .. | .. | $\mathbb{C}$ | . |
| 22 | .. | .. | .. | . | . |
| 23 | .. | .. | .. | . | . |
| 24 | .. | .. | .. | . | . |
| 25 | .. | .. | .. | $\mathbb{C}$ | . |
| 26 | .. | .. | .. | .. | . |
| 27 | .. | .. | .. | .. | . |
| 28 | .. | .. | .. | .. | . |
| 29 | .. | .. | .. | .. | . |
| 30 | .. | .. | .. | .. | . |
| 31 | .. | .. | .. | .. | $\mathbb{C}$ |
| 32 | .. | .. | .. | .. | . |
| 33 | .. | .. | .. | .. | . |
| 34 | .. | .. | .. | .. | . |
| 35 | .. | .. | .. | .. | . |
| 36 | .. | .. | .. | .. | $\mathbb{C}$ |

# Weakly Correlated Tight Frames



# Packing in Projective Spaces

$\mathbb{P}^2(\mathbb{R})$

| | Packing Radii (Degrees) | | |
|---|---|---|---|
| $N$ | JAT | NJAS | Difference |
| 4 | 70.53 | 70.53 | 0.00 |
| 5 | 63.43 | 63.43 | 0.00 |
| 6 | 63.43 | 63.43 | 0.00 |
| 7 | 54.74 | 54.74 | 0.00 |
| 8 | 49.64 | 49.64 | 0.00 |
| 9 | 47.98 | 47.98 | 0.00 |
| 10 | 46.67 | 46.67 | 0.00 |
| 11 | 44.40 | 44.40 | 0.00 |
| 12 | 41.88 | 41.88 | 0.00 |
| 13 | 39.81 | 39.81 | 0.00 |
| 14 | 38.52 | 38.68 | 0.17 |
| 15 | 37.93 | 38.13 | 0.20 |
| 16 | 37.36 | 37.38 | 0.02 |
| 17 | 35.00 | 35.24 | 0.23 |
| 18 | 34.22 | 34.41 | 0.19 |
| 19 | 32.93 | 33.21 | 0.28 |
| 20 | 32.48 | 32.71 | 0.23 |

$\mathbb{P}^3(\mathbb{R})$

| | Packing Radii (Degrees) | | |
|---|---|---|---|
| $N$ | JAT | NJAS | Difference |
| 5 | 75.52 | 75.52 | 0.00 |
| 6 | 70.53 | 70.53 | 0.00 |
| 7 | 67.02 | 67.02 | 0.00 |
| 8 | 65.53 | 65.53 | 0.00 |
| 9 | 64.26 | 64.26 | 0.00 |
| 10 | 64.26 | 64.26 | 0.00 |
| 11 | 60.00 | 60.00 | 0.00 |
| 12 | 60.00 | 60.00 | 0.00 |
| 13 | 55.46 | 55.46 | 0.00 |
| 14 | 53.63 | 53.84 | 0.21 |
| 15 | 52.07 | 52.50 | 0.43 |
| 16 | 50.97 | 51.83 | 0.85 |
| 17 | 50.66 | 50.89 | 0.23 |
| 18 | 50.28 | 50.46 | 0.18 |
| 19 | 49.65 | 49.71 | 0.06 |
| 20 | 49.11 | 49.23 | 0.12 |
| 21 | 48.48 | 48.55 | 0.07 |

# Packing in Grassmannian Spaces

$\mathbb{G}(2, \mathbb{R}^4)$, Chordal Distance

| | Squared Packing Radii | | |
|---|---|---|---|
| $N$ | JAT | NJAS | Difference |
| 3 | 1.5000 | 1.5000 | 0.0000 |
| 4 | 1.3333 | 1.3333 | 0.0000 |
| 5 | 1.2500 | 1.2500 | 0.0000 |
| 6 | 1.2000 | 1.2000 | 0.0000 |
| 7 | 1.1656 | 1.1667 | 0.0011 |
| 8 | 1.1423 | 1.1429 | 0.0005 |
| 9 | 1.1226 | 1.1231 | 0.0004 |
| 10 | 1.1111 | 1.1111 | 0.0000 |
| 11 | 0.9981 | 1.0000 | 0.0019 |
| 12 | 0.9990 | 1.0000 | 0.0010 |
| 13 | 0.9996 | 1.0000 | 0.0004 |
| 14 | 1.0000 | 1.0000 | 0.0000 |
| 15 | 1.0000 | 1.0000 | 0.0000 |
| 16 | 0.9999 | 1.0000 | 0.0001 |
| 17 | 1.0000 | 1.0000 | 0.0000 |
| 18 | 0.9992 | 1.0000 | 0.0008 |

$\mathbb{G}(2, \mathbb{R}^5)$, Chordal Distance

| | Squared Packing Radii | | |
|---|---|---|---|
| $N$ | JAT | NJAS | Difference |
| 3 | 1.7500 | 1.7500 | 0.0000 |
| 4 | 1.6000 | 1.6000 | 0.0000 |
| 5 | 1.5000 | 1.5000 | 0.0000 |
| 6 | 1.4400 | 1.4400 | 0.0000 |
| 7 | 1.4000 | 1.4000 | 0.0000 |
| 8 | 1.3712 | 1.3714 | 0.0002 |
| 9 | 1.3464 | 1.3500 | 0.0036 |
| 10 | 1.3307 | 1.3333 | 0.0026 |
| 11 | 1.3069 | 1.3200 | 0.0131 |
| 12 | 1.2973 | 1.3064 | 0.0091 |
| 13 | 1.2850 | 1.2942 | 0.0092 |
| 14 | 1.2734 | 1.2790 | 0.0056 |
| 15 | 1.2632 | 1.2707 | 0.0075 |
| 16 | 1.1838 | 1.2000 | 0.0162 |
| 17 | 1.1620 | 1.2000 | 0.0380 |
| 18 | 1.1589 | 1.1909 | 0.0319 |

# For more information. . .

## Reports

- JAT with Dhillon, Heath, and Strohmer. *Designing Structured Tight Frames via Alternating Projection*. ICES Report 03-50, December 2003.

- JAT. *Constructing Packings in Projective Spaces and Grassmannian Spaces via Alternating Projection*. ICES Report 04-23, May 2004.

- JAT with Dhillon, Heath, M. Sustik. *Necessary Conditions for the Existence of Equiangular Tight Frames*. In preparation, May 2004.

## Addresses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