On the existence of equiangular tight frames

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Abstract

An equiangular tight frame (ETF) is a $d \times N$ matrix that has unit-norm columns and orthogonal rows of norm $\sqrt{N/d}$. Its key property is that the absolute inner products between pairs of columns are (i) identical and (ii) as small as possible. ETFs have applications in communications, coding theory, and sparse approximation. Numerical experiments indicate that ETFs arise for very few pairs (d, N), and it is an important challenge to develop restrictions on the pairs for which they can exist. This article uses field theory to provide detailed conditions on real and complex ETFs. In particular, it describes restrictions on harmonic ETFs, a specific type of complex ETF that appears in applications. Finally, the article offers empirical evidence that these conditions are sharp or nearly sharp, especially in the real case.

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1 Introduction

The focus of this paper is a geometric object called an *equiangular tight frame* (ETF). An ETF can be viewed as a set of unit vectors in a Hilbert space with the property that the absolute inner products between pairs of vectors are (i) identical and (ii) minimal. As a result, ETFs generalize the geometric properties of an orthonormal basis. The details of this interpretation appear in Section 2.1.

We begin with a more formal definition. The usual Hermitian inner product on $\mathbb{C}^d$ is denoted by $\langle \cdot,\ \cdot \rangle$ and the associated norm by $\|\cdot\|_2$. The symbol $\mathbf{I}$ indicates an identity matrix whose dimensions are determined by context.

Definition 1 *Let* $\boldsymbol{S}$ *be a* $d \times N$ *matrix whose columns are* $\boldsymbol{s}_1, \dots, \boldsymbol{s}_N$*. The matrix* $\boldsymbol{S}$ *is called an* equiangular tight frame *if it satisfies three conditions.*

(1) Each column has unit norm: $\|\boldsymbol{s}_n\|_2 = 1$ *for* $n = 1, \dots, N$*.*
(2) The columns are equiangular. For some nonnegative α*, we have*

$$|\langle \boldsymbol{s}_m,\ \boldsymbol{s}_n \rangle| = \alpha \qquad \text{when } 1 \le m < n \le N.$$

(3) The columns form a tight frame. That is, $\boldsymbol{S}\boldsymbol{S}^* = (N/d)\,\mathbf{I}$*.*

If the entries of $\boldsymbol{S}$ *are real (resp. complex) numbers, we refer to* $\boldsymbol{S}$ *as a* real ETF *(resp.* complex ETF*). Note that condition (3) implies* $N \ge d$*.*

Equiangular line systems—which satisfy conditions (1) and (2)—first appeared in the literature on discrete geometry [22,12]. The earliest results on systems that meet all three conditions appear in Welch's work [24]. More recently, ETFs have found applications in communications, coding theory, and sparse approximation [17,19]. In particular, Holmes and Paulsen have shown that an ETF provides an error-correcting code that is maximally robust against two erasures [9]. ETFs are sometimes called *Maximum Welch-Bound-Equality Sequences*, or *optimal Grassmannian frames*, or *two-uniform frames*. We prefer the more descriptive term "equiangular tight frame."

1.1 Summary of Results

The numerical evidence suggests that ETFs do not arise for most pairs (d, N). See [21, Sec. V-C] for details of this investigation. The goal of the present paper is to develop theoretical results that provide more insight on the pairs for which ETFs can and cannot exist. This section summarizes the major results of this paper, and it offers a unified picture of the current state of knowledge.

The first major contribution of this paper is to provide precise information on the possible pairs (d, N) for which real ETFs can exist. The following theorem encapsulates Theorem 15, Theorem 17, and Corollary 16 of the sequel. It improves substantially on earlier work of Holmes and Paulsen [9, Thm. 3.3].

Theorem A *Suppose that $1 < d < N - 1$. When $N \neq 2d$, a real $d \times N$ ETF can exist only if*

$$\sqrt{\frac{d\,(N-1)}{N-d}} \qquad \textit{and} \qquad \sqrt{\frac{(N-d)\,(N-1)}{d}} \qquad \textit{are odd integers.}$$

In particular, N is an even number. Furthermore, if a real $d \times 2d$ ETF exists, then d is an odd number and $(2d-1)$ is the sum of two squares.

The cases $N = d$ and $N = d + 1$ are both trivial, as explained in Section 2.2. We obtain the first part of this result by a direct application of field theory, which is a new approach. The second part follows from an equivalence between real ETFs and certain graphs, a method developed in [17,9].

We have been able to construct a real ETF for each $N \leq 100$ where the conditions of Theorem A hold and the classical bound of Theorem C is satisfied. Our calculations lead us to conjecture that a real ETF exists whenever these conditions permit it. See Section 6 for details of the numerical work.

Our second goal is to determine which pairs (d, N) admit a complex ETF. We have restricted our attention to *unital ETFs of degree p*. These ETFs have the additional property that each entry of the scaled matrix $d^{1/2}\boldsymbol{S}$ is a pth root of unity. Unital ETFs arise frequently in electrical engineering applications [15]. A very important subclass is the set of *harmonic ETFs*, which are formed by re-scaling a row submatrix of the $N \times N$ discrete Fourier transform matrix [17]. The following result, which contains Corollary 19 and Theorem 20 of the sequel, describes restrictions on unital and harmonic ETFs.

Theorem B *Fix $d > 1$, and suppose that there exists a $d \times N$ unital ETF of degree p. We always have the upper bound $N \leq d^2 - d + 1$. Moreover, setting*

$$\gamma = \frac{d\,(N-d)}{N-1},$$

we have the following requirements.

$$\begin{array}{lll} \textit{When} \quad p = 2: & \sqrt{\gamma} \in \mathbb{Z}. & \\ \phantom{\textit{When}} \quad p = 3: & \gamma = a^2 - ab + b^2 & \textit{for some } a, b \in \mathbb{Z}. \\ \phantom{\textit{When}} \quad p = 4: & \gamma = a^2 + b^2 & \textit{for some } a, b \in \mathbb{Z}. \\ \phantom{\textit{When}} \quad p \geq 5: & \gamma \in \mathbb{Z}. & \end{array}$$

In particular, if a $d \times N$ harmonic ETF exists, then γ is an integer.

If, in addition, $p = q^s$ for prime q, then q divides N.

As described in Section 6, we have performed exhaustive computer searches which demonstrate that unital ETFs of low degree rarely exist. On the other hand, it is frequently possible to construct a harmonic ETF when the conditions of Theorem B permit it. Nevertheless, there remain pairs (d, N) where the conditions hold but we have no construction.

Finally, we mention a classical restriction on the size of an ETF. The result below is originally due to Gerzon [12], and it was subsequently extended by Holmes and Paulsen [9, Prop. 3.4]. See Theorem 5 of [20] for a very direct proof based on matrix theory.

Theorem C *Suppose that $1 < d < N - 1$, and assume that a $d \times N$ ETF exists. Then, for real ETFs,*

$$N \leq \min\{\tfrac{1}{2}\, d\,(d+1),\ \tfrac{1}{2}(N-d)\,(N-d+1)\}.$$

For complex ETFs,

$$N \leq \min\{d^2,\ (N-d)^2\}.$$

Note that the upper bound in Theorem B is more restrictive than the classical upper bound for complex ETFs. A consequence is that some complex ETFs are not unital.

1.2 Outline

Section 2 offers an introduction to the core properties of ETFs, and it provides some specific examples. Section 3 presents basic facts from field theory that underlie our proofs. In Section 4, we develop conditions on the existence of real ETFs using field theory and graph theory. Section 5 uses field theory to provide restrictions on unital and harmonic ETFs. Finally, in Section 6, we appraise how well our conditions describe the pairs (d, N) where various types of ETFs actually exist.

2 A Primer on ETFs

To prove the results in this article and to place them in context, we require some basic facts about ETFs. We calculate the angle between the columns of an ETF, and we establish that ETFs solve a certain geometric extremal problem. The basic examples of ETFs are described. Then we define unital and harmonic ETFs rigorously, and we give examples from these two classes.

By associating each column of an ETF with its one-dimensional span, we may view an ETF as a collection of lines through the origin. The number α in the definition represents the cosine of the acute angle between each pair of lines. Connected with this geometric interpretation are some facts that will be critical in the sequel.

Proposition 2 *Fix $d > 1$, and suppose that $\boldsymbol{S}$ is a $d \times N$ matrix with unit-norm columns. Then*

$$\max_{i \neq j} |\langle \boldsymbol{s}_i,\ \boldsymbol{s}_j \rangle| \geq \sqrt{\frac{N-d}{d\,(N-1)}}.$$

This bound is attained if and only if $\boldsymbol{S}$ is an ETF.

The inequality is originally due to Welch [24]. Strohmer and Heath offer a direct argument that also gives the second statement [17]. A very insightful proof appears in [8]. For reference, we give a short proof that echoes the approach of Strohmer and Heath.

PROOF. Let $\boldsymbol{S}$ be a $d \times N$ matrix with unit-norm columns, and let $\boldsymbol{G} = \boldsymbol{S}^* \boldsymbol{S}$ be its Gram matrix. Note that $\boldsymbol{G}$ has a unit diagonal. By squaring and summing the inequalities of the form $\max_{i \neq j} |\langle \boldsymbol{s}_i,\ \boldsymbol{s}_j \rangle| \geq |\langle \boldsymbol{s}_m,\ \boldsymbol{s}_n \rangle|$ over all $m \neq n$ we derive:

$$\begin{aligned} \max_{m \neq n} |\langle \boldsymbol{s}_m,\ \boldsymbol{s}_n \rangle| &\geq \left[\frac{1}{N(N-1)} \sum_{m \neq n} |\langle \boldsymbol{s}_m,\ \boldsymbol{s}_n \rangle|^2 \right]^{1/2} \\ &= \left[\frac{1}{N(N-1)} \left(\|\boldsymbol{G}\|_{\mathrm{F}}^2 - N \right) \right]^{1/2}. \end{aligned} \tag{1}$$

This bound holds with equality if and only if the absolute inner products between columns are identical. That is, ETF Condition (2) is in force.

Let $\lambda_1, \ldots, \lambda_d$ be the d largest eigenvalues of $\boldsymbol{S}\boldsymbol{S}^*$. They are also the eigenvalues of $\boldsymbol{G}$, so they satisfy $\sum_{j=1}^d \lambda_j^2 = \|\boldsymbol{S}\boldsymbol{S}^*\|_{\mathrm{F}}^2 = \|\boldsymbol{G}\|_{\mathrm{F}}^2$ and $\sum_{j=1}^d \lambda_j = \operatorname{trace}(\boldsymbol{G}) = N$. We can find a lower bound for the right-hand side of relation (1) by solving the optimization problem

$$\min \sum_{j=1}^d \lambda_j^2 \qquad \text{subject to} \qquad \sum_{j=1}^d \lambda_j = N.$$

The unique minimum occurs when $\lambda_j = N/d$ for $j = 1, 2, \ldots d$. In this case, the eigenvalues of the positive-definite matrix $\boldsymbol{S}\boldsymbol{S}^*$ are identical, which implies that $\boldsymbol{S}\boldsymbol{S}^* = (N/d)\,\mathbf{I}$. In other terms, ETF Condition (3) is in force.

We conclude that

$$\max_{m \neq n} |\langle \boldsymbol{s}_m,\ \boldsymbol{s}_n \rangle| \geq \left[\frac{1}{N(N-1)} \left(\sum_{j=1}^{d} \lambda_j^2 - N \right) \right]^{1/2}$$
$$\geq \left[\frac{1}{N(N-1)} \left(d \cdot \frac{N^2}{d^2} - N \right) \right]^{1/2}.$$

Simplify this expression to complete the proof of the bound. □

2.2 Basic Examples

There are two families of ETFs that arise for every dimension d and one family in dimension one with arbitrary number of vectors.

(1) (Orthonormal Bases). When $N = d$, the sole examples of ETFs are unitary (and orthogonal) matrices. Evidently, the absolute inner product α between distinct vectors is zero.
(2) (Simplices). When $N = d+1$, every ETF can be viewed as the vertices of a regular simplex centered at the origin [7,17]. The easiest way to realize an instance of this configuration is to compute the orthogonal projection of the standard coordinate basis in $\mathbb{R}^{d+1}$ onto the orthogonal complement of the vector $[1, 1, \ldots, 1]^T \in \mathbb{R}^{d+1}$. Afterward, the projected vectors must be re-scaled so that they have unit norm. Note that the configuration lies in a d-dimensional subspace of the ambient vector space.
(3) (Degenerate Frame). When $d = 1$, an ETF is a $1 \times N$ row vector whose entries all have absolute value one.

The simplex and degenerate ETF have a "dual" relationship. Let $\boldsymbol{S}$ be the $(N-1) \times N$ real frame matrix of a simplex. Select a vector $\boldsymbol{s}$ that is orthogonal to the $N-1$ rows of $\boldsymbol{S}$, and note that the entries of $\boldsymbol{s}$ must have the same magnitude on account of ETF Condition (3). Rescale $\boldsymbol{s}$ to obtain a degenerate ETF with $N-1$ vectors.

This dual relationship extends to other pairs of ETFs. Let $\boldsymbol{S}$ be a $d \times N$ ETF. Up to a scaling factor, the rows of $\boldsymbol{S}$ are orthonormal, so we can augment $\boldsymbol{S}$ with $(N-d)$ more rows to form a scaled $N \times N$ unitary matrix. It can be shown that, up to scaling, the new rows form a $(N-d) \times N$ ETF. This "dual ETF" is unique, modulo the operations under which the ETF property is invariant: permutation of the columns, the change of phase for any column or an application of a left-rotation. See Section 3 of [18] for a detailed discussion of invariance and duality.

An important consequence of this construction is that the existence of a $d \times N$ ETF implies the existence of an $(N-d) \times N$ ETF. Therefore, we can restrict our attention to the case where $d \leq N/2$.

The first genuine example of a real ETF consists of six vectors in $\mathbb{R}^3$. It can be constructed by choosing six non-antipodal[1] vertices of an icosahedron centered at the origin [8]. For example,

$$\boldsymbol{S} = \frac{1}{\sqrt{1+\varphi^2}} \begin{bmatrix} 0 & 0 & 1 & 1 & \varphi & -\varphi \\ 1 & 1 & \varphi & -\varphi & 0 & 0 \\ \varphi & -\varphi & 0 & 0 & 1 & 1 \end{bmatrix} \qquad \text{where } \varphi = \frac{1+\sqrt{5}}{2}.$$

The first genuine example of a complex ETF contains four vectors in $\mathbb{C}^2$.

$$\boldsymbol{S} = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{3} & -1 & -1 & -1 \\ 0 & \sqrt{2} & \zeta\sqrt{2} & \zeta^2\sqrt{2} \end{bmatrix} \qquad \text{where } \zeta = \mathrm{e}^{2\pi\mathrm{i}/3}.$$

Another interesting complex ETF consists of nine vectors in $\mathbb{C}^3$:

$$\boldsymbol{S} = \frac{1}{2} \begin{bmatrix} 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & \mathrm{i}\sqrt{3} & -\mathrm{i}\sqrt{3} & +1 & +1 & +1 & -1 & -1 & -1 \\ 0 & 0 & 0 & \sqrt{2} & \zeta\sqrt{2} & \zeta^2\sqrt{2} & \sqrt{2} & \zeta\sqrt{2} & \zeta^2\sqrt{2} \end{bmatrix} \qquad \text{where } \zeta = \mathrm{e}^{2\pi\mathrm{i}/3}.$$

Both complex ETFs meet the upper bound of Theorem C. That is, $N = d^2$.

2.3 Unital and Harmonic ETFs

In this paper, we focus on a class of complex ETFs that arises in electrical engineering applications. Choose a primitive pth root of unity ζ_p, and suppose that $\boldsymbol{S}$ is a $d \times N$ ETF of the form

$$\boldsymbol{S} = \frac{1}{\sqrt{d}} \boldsymbol{X} \tag{2}$$

where the entries of $\boldsymbol{X}$ are drawn from the set $\{1, \zeta_p, \zeta_p^2, \ldots, \zeta_p^{p-1}\}$. We say that $\boldsymbol{S}$ is a *unital ETF of degree p*.

Harmonic ETFs are an especially important type of unital ETF. A $d \times N$

[1] Two points x and y are antipodal if $x = -y$.

harmonic ETF is a unital ETF of degree N that has the form

$$\boldsymbol{S} = \frac{1}{\sqrt{d}} \begin{bmatrix} 1 & \omega_1 & \omega_1^2 & \dots & \omega_1^{N-1} \\ 1 & \omega_2 & \omega_2^2 & \dots & \omega_2^{N-1} \\ & & & \dots & \\ 1 & \omega_d & \omega_d^2 & \dots & \omega_d^{N-1} \end{bmatrix}$$

where $\omega_1, \dots, \omega_d$ are distinct Nth roots of unity. In words, the matrix $\boldsymbol{S}$, apart from the scaling factor, is a submatrix of the $N \times N$ discrete Fourier transform. The literature contains several references to harmonic frames and ETFs [7,17,13,25].

Although most extant examples of complex ETFs are in fact unital, we will demonstrate that certain complex ETFs cannot be unital. Therefore, a complete elaboration of the complex case must extend beyond the unital ETFs.

2.4 Examples of Unital ETFs

The simplest examples of unital ETFs come from real and complex Hadamard matrices. A *Hadamard matrix* is a $d \times d$ matrix $\boldsymbol{X}$ with entries in $\{\pm 1\}$ that satisfies $\boldsymbol{X}^* \boldsymbol{X} = d\,\mathbf{I}$. A long-standing conjecture in coding theory states that a Hadamard matrix exists whenever d is a multiple of four [5]. If $\boldsymbol{X}$ is a Hadamard matrix, then $d^{-1/2}\boldsymbol{X}$ is a unital ETF of degree two.

Similarly, a *complex Hadamard matrix* is a $d \times d$ matrix $\boldsymbol{X}$ with entries in $\{\pm 1, \pm \mathrm{i}\}$ that satisfies $\boldsymbol{X}^* \boldsymbol{X} = d\,\mathbf{I}$. If $\boldsymbol{X}$ is a complex Hadamard matrix, then $d^{-1/2}\boldsymbol{X}$ is a unital ETF of degree four.

It is also possible to construct unital ETFs that are rectangular. The smallest real example seems to be a 6×16 unital ETF of degree two:

$$\boldsymbol{S} = \frac{1}{\sqrt{6}} \begin{bmatrix} + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + \\ + & + & + & + & + & + & + & + & - & - & - & - & - & - & - & - \\ + & + & + & + & - & - & - & - & + & + & + & + & - & - & - & - \\ + & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - \\ + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - \\ + & - & - & + & - & + & + & - & - & + & + & - & + & - & - & + \end{bmatrix}.$$

For clarity, we have replaced the ± 1 entries of the ETF by their signs. It is possible to extend this matrix to a 16×16 Hadamard matrix, which can be verified by inspection of the tables in [16].

The smallest nontrivial example of a harmonic ETF has dimensions 3×7.

$$\boldsymbol{S} = \frac{1}{\sqrt{3}} \exp \cdot \frac{2\pi \mathrm{i}}{7} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 3 & 6 & 2 & 5 & 1 & 4 \end{bmatrix}.$$

The notation $\exp \cdot$ indicates the componentwise exponential of the matrix.

3 Algebraic Background

The results of this paper depend on some basic facts from field theory. Lang's textbook is a standard introduction to this material [11]. For the sake of completeness, we review the essential definitions.

The ring of integers and the field of rationals are denoted by $\mathbb{Z}$ and $\mathbb{Q}$ respectively. A polynomial whose coefficients are drawn from a subfield $\mathbb{F}$ of the complex numbers is referred to as a *polynomial over* $\mathbb{F}$. The complex number ω is *algebraic* over $\mathbb{F}$ if it is the root of some polynomial over $\mathbb{F}$. An *algebraic integer* is the root of a monic polynomial with integer coefficients.

Fact 3 *The set of algebraic integers in* $\mathbb{Q}$ *equals the set* $\mathbb{Z}$ *of ordinary integers.*

Fact 4 *The algebraic integers form a ring, i.e., they are closed under addition and multiplication.*

Fact 5 *The roots of a monic polynomial with algebraic integer coefficients are also algebraic integers.*

The *minimal polynomial* of ω over $\mathbb{F}$ is the (unique) lowest degree monic polynomial over $\mathbb{F}$ that contains ω among its roots.

Fact 6 *A minimal polynomial over* $\mathbb{F}$ *has simple roots.*

Two numbers that have the same minimal polynomial over $\mathbb{F}$ are called *algebraic conjugates* over $\mathbb{F}$.

Fact 7 *Suppose that* ω *and* ζ *are algebraic conjugates over* $\mathbb{F}$*. If* p *is a polynomial over* $\mathbb{F}$ *that has* ω *as a root with multiplicity* m*, then* ζ *is also a root of* p *with multiplicity* m*.*

With these facts at hand, we may prove the following key lemma.

Lemma 8 *Let $\boldsymbol{A}$ be an Hermitian matrix whose entries are algebraic integers. Then the eigenvalues of $\boldsymbol{A}$ are real algebraic integers.*

In addition, assume that the entries of $\boldsymbol{A}$ belong to a subfield $\mathbb{F}$ of the complex numbers. If $\boldsymbol{A}$ has an eigenvalue ω whose multiplicity differs from the multiplicity of every other eigenvalue, then ω belongs to $\mathbb{F}$.

PROOF. The matrix $\boldsymbol{A}$ is Hermitian, so its eigenvalues are real numbers. By definition, an eigenvalue of $\boldsymbol{A}$ is a root of the characteristic polynomial $t \mapsto \det(t\,\mathbf{I} - \boldsymbol{A})$. Since the entries of $\boldsymbol{A}$ are algebraic integers, Fact 4 implies that the characteristic polynomial is a monic polynomial with algebraic integer coefficients. Then Fact 5 shows that the eigenvalues of $\boldsymbol{A}$ are algebraic integers.

Assume that the entries of $\boldsymbol{A}$ belong to $\mathbb{F}$. Thus, the eigenvalues of $\boldsymbol{A}$ are algebraic over $\mathbb{F}$. Since ω has a different multiplicity from the other eigenvalues of $\boldsymbol{A}$, Fact 7 precludes the possibility that ω might have any algebraic conjugates over $\mathbb{F}$. Applying Fact 6, we see that the minimal polynomial of ω over $\mathbb{F}$ is $t \mapsto t - \omega$. Thus, ω belongs to $\mathbb{F}$. □

This type of field-theoretic argument appears frequently in the analysis of integer matrices. A similar technique was used by Lemmens and Seidel in their study of equiangular lines [12].

We also require some basic theory of cyclotomic fields [23]. Recall that $\mathbb{Q}(\omega)$ denotes the smallest field extension of $\mathbb{Q}$ that contains ω, while $\mathbb{Z}[\omega]$ is the smallest ring containing $\mathbb{Z}$ and ω.

Throughout this article, we use ζ_p to denote a primitive pth root of unity.

Fact 9 *Suppose that ζ_p is a primitive pth root of unity. The ring of algebraic integers in the field $\mathbb{Q}(\zeta_p)$ equals the ring $\mathbb{Z}[\zeta_p]$.*

Fact 10 *The set of real algebraic integers in $\mathbb{Z}[\zeta_p]$ coincides with the ring $\mathbb{Z}[2\,\mathrm{Re}\,\zeta_p]$.*

The minimal polynomial of ζ_p is called the pth *cyclotomic polynomial*, and it is denoted by $\Phi_p(t)$. The roots of $\Phi_p(t)$ are precisely the primitive pth roots of unity. Note that $\Phi_1(t) = t - 1$.

Fact 11 *The cyclotomic polynomials satisfy the identity*

$$\prod_{a|p} \Phi_a(t) = t^p - 1 \tag{3}$$

where the symbol $a|p$ means that a divides p.

4 Conditions on Real ETFs

Assume that $1 < d < N$, and suppose that $\boldsymbol{S}$ is a $d \times N$ real ETF. In this section, we will see that the pair (d, N) must satisfy rigid integrality conditions. Denote the absolute inner product between columns by α, and recall from Proposition 2 that

$$\alpha = \sqrt{\frac{N-d}{d(N-1)}}. \tag{4}$$

Next, construct the *signature matrix* of the ETF:

$$\boldsymbol{A} \quad = \quad \frac{1}{\alpha}\,(\boldsymbol{S}^*\boldsymbol{S} - \mathbf{I}).$$

Since the inner products between columns of $\boldsymbol{S}$ have magnitude α, the off-diagonal entries of $\boldsymbol{A}$ are 1 or -1. Therefore, this matrix completely encodes the pattern of phases in the inner products. It is identical with the signature matrix considered by Holmes and Paulsen [9, Def. 3.1].

Our primary analysis is based on a detailed study of the eigenvalues of $\boldsymbol{A}$ using methods of field theory. Observe that $\boldsymbol{A}$ is Hermitian; it has a zero diagonal; and its off-diagonal entries have unit modulus. Since an ETF satisfies the equation $\boldsymbol{S}\boldsymbol{S}^* = (N/d)\,\mathbf{I}$, it follows that the two eigenvalues of $\boldsymbol{A}$ are

$$\lambda_1 = -\frac{1}{\alpha} = -\sqrt{\frac{d(N-1)}{N-d}} \qquad \text{and} \qquad \lambda_2 = \frac{N-d}{d\,\alpha} = \sqrt{\frac{(N-1)(N-d)}{d}}$$

with respective multiplicities $N-d$ and d. The key idea in our proof is that λ_1 and λ_2 cannot take general real values because the entries of $\boldsymbol{A}$ are severely limited.

Following [9], we note that

$$\lambda_1\lambda_2 = -(N-1)$$

and that $\boldsymbol{A}$ satisfies the quadratic matrix equation [2]

$$\boldsymbol{A}^2 - (\lambda_1 + \lambda_2)\boldsymbol{A} - (N-1)\mathbf{I} = \mathbf{0}. \tag{5}$$

This point can be verified by a direct calculation.

[2] In fact the left hand side is the minimal polynomial of $\boldsymbol{A}$.

4.1 Weak Integrality Conditions

For a real ETF, the off-diagonal entries of the signature matrix $\boldsymbol{A}$ equal ± 1. Although we are only interested in real ETFs, it is more natural to consider the case where the entries of the signature matrix are roots of unity. In this setting, the possible values of λ_1 and λ_2 are already quite special.

Theorem 12 *Assume that $1 < d < N - 1$ and $N \neq 2d$. Suppose that $\boldsymbol{S}$ is a $d \times N$ ETF whose signature matrix $\boldsymbol{A}$ has entries in the ring $\mathbb{Z}[\zeta_p]$. Then the eigenvalues λ_1 and λ_2 of $\boldsymbol{A}$ both belong to $\mathbb{Z}[2\operatorname{Re}\zeta_p]$.*

PROOF. Since $N \neq 2d$, the two eigenvalues of $\boldsymbol{A}$ have different multiplicities. The entries of $\boldsymbol{A}$ are algebraic integers in $\mathbb{Q}(\zeta_p)$, so Lemma 8 implies that λ_1 and λ_2 belong to the set of real algebraic integers in $\mathbb{Q}(\zeta_p)$. Facts 9 and 10 identify this set as $\mathbb{Z}[2\operatorname{Re}\zeta_p]$. $\square$

As we have noted, a real ETF generates a matrix $\boldsymbol{A}$ whose off-diagonal entries equal ± 1. We may apply the previous theorem with $p = 1$ to obtain a first result for real ETFs.

Corollary 13 *Assume that $1 < d < N - 1$ and $N \neq 2d$. Suppose that $\boldsymbol{S}$ is a $d \times N$ real ETF. Then the eigenvalues λ_1 and λ_2 of the signature matrix $\boldsymbol{A}$ are ordinary integers.*

A simple consequence of this corollary is the weak integrality condition stated in the paper of Holmes and Paulsen [9, Thm. 3.3].

Corollary 14 (Holmes–Paulsen) *If $d < N$ and a real $d \times N$ ETF exists, then*

$$(N - 2d)\sqrt{\frac{N-1}{d\,(N-d)}} \qquad \text{is an integer.}$$

PROOF. When $d = 1$, $d = N-1$ or $N = 2d$, the result is obvious. Otherwise, we introduce the value of α from (4), and we find that $(\lambda_1 + \lambda_2)$ equals the stated expression. Since λ_1 and λ_2 are integers, the result follows instantly. $\square$

Holmes and Paulsen established this condition for real ETFs by looking at the components of the matrix equation (5). They do not appear to recognize that λ_1 and λ_2 must in fact be integers.

See the technical report [18] for applications of Theorem 12 to more general signature matrices.

The results of the last subsection can be sharpened significantly. Indeed, the possible choices for the eigenvalues λ_1 and λ_2 are even more limited. This is our major result for real ETFs.

Theorem 15 *Assume that $1 < d < N - 1$ and $N \neq 2d$. Suppose that $\boldsymbol{S}$ is a real $d \times N$ ETF. Then the eigenvalues λ_1 and λ_2 of the signature matrix are both odd integers.*

When $N = d$ or $N = d + 1$, all ETFs arise from the two families described in Section 2.2. We will attend to the case $N = 2d$ in Section 4.3. Our proof adapts an argument of P. M. Neumann quoted in [12]. The basic idea is to derive from the signature matrix $\boldsymbol{A}$ another integer matrix with known eigenvalues. Then we apply the field-theoretic methods to see that these eigenvalues must lie in a discrete set.

PROOF. Let us form a new matrix $\boldsymbol{M}$ whose entries all equal zero or one:

$$\boldsymbol{M} = \tfrac{1}{2}\,(\mathbf{J} - \mathbf{I} - \boldsymbol{A})$$

where the symbol $\mathbf{J}$ denotes a conformal matrix of ones. We have ruled out the possibility that $N \leq d+1$, so the eigenvalue λ_1 of $\boldsymbol{A}$ has geometric multiplicity at least two. In consequence, the $(N-1)$-dimensional null space of $\mathbf{J}$ must intersect the invariant subspace of $\boldsymbol{A}$ associated with λ_1. Any vector in this intersection is an eigenvector of $\boldsymbol{M}$ with eigenvalue $\mu_1 = -\frac{1}{2}\,(1+\lambda_1)$. A similar argument establishes that $\mu_2 = -\frac{1}{2}\,(1+\lambda_2)$ is also an eigenvalue of $\boldsymbol{M}$.

Corollary 13 establishes that λ_1 and λ_2 are integers, so the eigenvalues μ_1 and μ_2 must be rational numbers. The entries of $\boldsymbol{M}$ are integers, so Lemma 8 proves that the eigenvalues of $\boldsymbol{M}$ are also algebraic integers. We conclude that μ_1 and μ_2 are rational integers. That is, λ_1 and λ_2 are odd. □

This result has another striking consequence.

Corollary 16 *If $1 < d < N-1$ and a real $d \times N$ ETF exists, then N is even.*

PROOF. When $N \neq 2d$ this point follows immediately from Theorem 15 and the fact that $\lambda_1 \lambda_2 = -(N-1)$. □

4.3 Graph-Theoretic Conditions

It was observed in [17,9,1] that real equiangular tight frames naturally give rise to regular two-graphs and *vice versa*. Theorem 3.10 of [9] provides complete details of this correspondence. It is also known that regular two-graphs naturally give rise to strongly regular graphs with certain parameter sets [6, Ch. 4]. In consequence, there is also a natural connection between real ETFs and certain strongly regular graphs [18].

The connection between real ETFs and graphs is already well known in the ETF literature. To our knowledge the full power of this correspondence has not yet been exploited. As an illustration, we state without proof a theorem which relies on the deep result that a certain class of strongly regular graphs can exist only when the number of vertices is the sum of two squares. See [6, Thm. 2.18] and [22] for background; a full proof appears in [18].

Theorem 17 *If there exists a real $d \times 2d$ ETF, then d is odd and $(2d-1)$ is the sum of two squares.*

A well-known result of Euler states that a natural number is the sum of two squares if and only if each prime factor having the form $(4k+3)$ occurs in the prime factorization with an even power.

5 Conditions on Complex ETFs

We now turn our attention to the existence of complex ETFs. Although numerical evidence indicates that complex ETFs exist for few pairs (d, N), we have been unable to develop conditions that hold in general. Instead, this section focuses on the subclass of *unital ETFs* and the further subclass of *harmonic ETFs*. From the point of view of applications, unital ETFs may be more valuable than general ETFs because they are easier to generate. In addition, they fit better with the extensive literature on sequence design [15].

5.1 Integrality Conditions I

Our primary contribution is to develop simple requirements on the admissible values of d and N in a unital ETF. The result relies on general properties of ETFs and the arithmetic closure of a number ring. The argument here extends work from [20].

Theorem 18 *Let $\mathscr{A}$ be a sub-ring of the algebraic integers, closed under complex conjugation. Fix $d > 1$. Suppose that $\boldsymbol{X}$ is a $d \times N$ matrix whose entries belong to $\mathscr{A}$, and assume that $d^{-1/2}\,\boldsymbol{X}$ is an ETF. It follows that $\mathscr{A}$ contains a number z for which*

$$|z| = \sqrt{\frac{d\,(N-d)}{N-1}}.$$

In particular,

$$\frac{d\,(N-d)}{N-1} \qquad \text{is an integer.}$$

As a consequence, we have the upper bound $N \leq d^2 - d + 1$.

Observe that this result holds in the case where the entries of $\boldsymbol{X}$ are mixed roots of unity. Indeed, the theorem even applies when $\mathscr{A}$ is the entire set of algebraic integers.

PROOF. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be the first two columns of $\boldsymbol{X}$. Since $\mathscr{A}$ is a ring, closed under complex conjugation, the inner-product $\langle \boldsymbol{x},\ \boldsymbol{y} \rangle$ and its complex conjugate both belong to $\mathscr{A}$. It follows that $|\langle \boldsymbol{x},\ \boldsymbol{y} \rangle|^2$ is also an element of $\mathscr{A}$, hence an algebraic integer.

Since $d^{-1/2}\,\boldsymbol{X}$ is an ETF,

$$|\langle \boldsymbol{x},\ \boldsymbol{y} \rangle| = \alpha\, d = \sqrt{\frac{d\,(N-d)}{N-1}}.$$

Therefore, $\mathscr{A}$ contains an element with magnitude $\alpha\, d$, which is the content of the first conclusion.

Squaring the foregoing equation yields

$$|\langle \boldsymbol{x},\ \boldsymbol{y} \rangle|^2 = \frac{d\,(N-d)}{N-1},$$

which is clearly a rational number. Since it is also an algebraic integer, the ratio must be an ordinary integer (Fact 3).

This point further implies that there is an integer k for which

$$k\,(N-1) = d\,(N-d)$$

Divide through by $(N-1)$ and distribute on the right-hand side to reach

$$k = d - \frac{d\,(d-1)}{N-1}.$$

But the fraction must be an integer, which is impossible unless $N - 1 \leq d\,(d-1)$. This relation yields the upper bound. $\square$

Although this result is straightforward to derive, it leads to stringent conditions on unital ETFs for every value of p.

Corollary 19 (Unital ETFs) *Fix $d > 1$, and suppose that there exists a $d \times N$ unital ETF of degree p. Then we have the following restrictions on the number $\gamma = d\,(N-d)/(N-1)$.*

$$\begin{array}{lll} \textit{When} & p = 2: & \sqrt{\gamma} \in \mathbb{Z}. \\ & p = 3: & \gamma = a^2 - ab + b^2 \quad \textit{for some } a, b \in \mathbb{Z}. \\ & p = 4: & \gamma = a^2 + b^2 \quad \textit{for some } a, b \in \mathbb{Z}. \\ & p \geq 5: & \gamma \in \mathbb{Z}. \end{array}$$

In all these cases, the upper bound $N \leq d^2 - d + 1$ is in force.

Observe that the case $p = N$ covers harmonic ETFs.

PROOF. Suppose that $\boldsymbol{S} = d^{-1/2}\,\boldsymbol{X}$ is a $d \times N$ unital ETF of degree p. By construction, the entries of $\boldsymbol{X}$ belong to the ring $\mathbb{Z}[\zeta_p]$. Fact 9 states that $\mathbb{Z}[\zeta_p]$ is a sub-ring of the algebraic integers. Since $\overline{\zeta}_p = \zeta_p^{p-1}$, the ring is closed under complex conjugation. Apply Theorem 18 to see that the ratio $\gamma = d\,(N-d)/(N-1)$ must be an integer.

For small values of p, it is possible to refine this conclusion. When $p = 2$, it is clear that $\mathbb{Z}[\zeta_2] = \mathbb{Z}$. Apply the first conclusion of Theorem 18 to see that $\sqrt{\gamma}$ is an integer.

When $p = 3$, the elements of the set $\mathbb{Z}[\zeta_3]$ have the form $a + b\,\zeta_3$, where a, b are integers. We calculate the magnitude of the elements by using the identities $\zeta_3^3 = 1$ and $\zeta_3^2 + \zeta_3 + 1 = 0$ and $\overline{\zeta}_3 = \zeta_3^2$:

$$|a + b\,\zeta_3| = \sqrt{(a + b\,\zeta_3)(a + b\,\overline{\zeta}_3)} = \sqrt{a^2 - ab + b^2}.$$

Apply the first part of Theorem 18.

In case $p = 4$, the primitive roots of unity are $\pm\mathrm{i}$. The elements of $\mathbb{Z}[\mathrm{i}]$ are of the form $a + b\mathrm{i}$, and so $|a + b\mathrm{i}| = \sqrt{a^2 + b^2}$. Apply Theorem 18 again. $\square$

In Section 2.2, we exhibited a 2×4 complex ETF. The upper bound stated in Corollary 19 makes it impossible to construct a unital ETF with these dimensions. We conclude that the class of complex ETFs is strictly larger than the class of unital ETFs.

This subsection demonstrates that N, the number of columns of the unital ETF, may be restricted severely by the degree of the ETF.

Theorem 20 *Fix $d > 1$, and suppose that there exists a $d \times N$ unital ETF of degree $p = q^s$, where q is a prime. Then q divides N.*

PROOF. Suppose that $d^{-1/2}\boldsymbol{X}$ is a $d \times N$ unital ETF of degree p. Let $\boldsymbol{x}^T$ and $\boldsymbol{y}^T$ be the first two rows of $\boldsymbol{X}$. By ETF Condition (3), the two rows are orthogonal: $\langle \boldsymbol{x},\ \boldsymbol{y} \rangle = 0$. Since the entries of $\boldsymbol{x}$ and $\boldsymbol{y}$ are all powers of ζ_p, it follows that their inner product is a sum of N powers of ζ_p, not necessarily distinct. Therefore,

$$\langle \boldsymbol{x},\ \boldsymbol{y} \rangle = \sum_{k=0}^{p-1} c_k \, \zeta_p^k = 0$$

where $\{c_k\}$ is a set of nonnegative integers that sum to N.

Define the polynomial $u(t) = \sum_{k=0}^{p-1} c_k t^k$. We have established that $u(\zeta_p) = 0$ and that $u(1) = N$. A consequence of Fact 7 is that the minimal polynomial Φ_p of ζ_p divides the polynomial u. In particular, $\Phi_p(1)$ divides $u(1)$. We can complete the proof by showing that $\Phi_p(1) = q$ whenever $p = q^s$ for a prime number q and positive integer s.

We prove by induction on s that $\Phi_{q^s}(1) = q$. For the base case $s = 1$, we invoke Fact 11:

$$\Phi_1(t)\Phi_q(t) = t^q - 1 = (t-1)(t^{q-1} + t^{q-2} + \cdots + t + 1).$$

Since $\Phi_1(t) = t - 1$, the polynomial $\Phi_q(t)$ must be equal to $t^{q-1} + t^{q-2} + \cdots + t + 1$. Substitute $t = 1$ to complete the argument.

For the inductive step, assume the statement is true for each positive integer $j < s$. We invoke Fact 11 again to obtain

$$\prod_{j=0}^{s} \Phi_{q^j}(t) = t^{q^s} - 1 = (t-1)(t^{q^s-1} + t^{q^s-2} + \cdots + 1).$$

Cancel the factor $\Phi_1(t) = t - 1$ from both sides to arrive at the identity:

$$\prod_{j=1}^{s} \Phi_{q^j}(t) = t^{q^s-1} + t^{q^s-2} + \cdots + 1.$$

Substitute $t = 1$ to obtain $\prod_{j=1}^{s} \Phi_{q^j}(1) = q^s$. By the induction hypothesis, the left-hand side equals $q^{s-1}\Phi_{q^s}(1)$. We conclude that $\Phi_{q^s}(1) = q$ for each positive integer s. $\square$

d	N	d	N	d	N	d	N
3	6	13	26	21	42	33	66
5	10	15	30	23	46	41	82
6	16	15	36	25	50	43	86
7	14	19	38	27	54	45	90
7	28	19	76	28	64	45	100
9	18	20	96	31	62	49	98

Table 1
The pairs (d, N) with $N \leq 100$ and $d \leq N/2$ for which a real ETF exists. The restriction $d \leq N/2$ is motivated by the arguments of Section 2.2. This table excludes the trivial cases $d = 1$, $N = d$ and $N = d + 1$ where an ETF always exists.

One can imagine developing results for other values of p by using the same strategy. We omit a detailed discussion.

6 How Good are the Conditions?

The results in this paper are all negative in the sense that they prohibit the existence of ETFs in certain circumstances. When our conditions do not rule out the existence of an ETF, it is natural to ask whether an ETF actually exists. The easiest way to answer this question is to supply a construction. This section appraises the gap between known constructions and our necessary conditions.

6.1 Existence of Real ETFs

In the real case, the conditions stated in Theorem A and Theorem C appear to describe completely the pairs (d, N) where real ETFs exist. Table 1 lists all the pairs (d, N) with $d + 2 \leq N \leq 100$ that meet these conditions. In each of these cases, we were able to construct a real ETF from a strongly regular graph [18]. On account of this coincidence, we conjecture that the conditions stated in these two results are both necessary and sufficient for the existence of a real ETF. For more information on strongly regular graphs, the closely related two-graphs, and tables of these graphs, we refer the reader to [3,14,4].

$p = 2$																	
d	2	3	4	5	6	6	7	8	9	10	10	11	11	12	13	13	14
N	2	4	4	6	6	16	8	8	10	10	16	12	56	12	14	40	14
	Y	Y	Y	N	N	Y	Y	Y	N	N	Y	Y	?	Y	?	?	N

$p = 3$																	
d	2	3	5	5	6	7	8	8	8	9	11	12	12	13	14	14	14
N	3	3	6	21	6	15	9	15	57	9	12	12	45	27	15	27	183
	Y	Y	Y	N	Y	N	Y	N	?	Y	Y	Y	?	Y	N	Y	?

$p = 4$																	
d	2	3	4	5	6	6	7	7	8	9	10	10	10	11	11	12	12
N	2	4	4	6	6	16	8	22	8	10	10	16	46	12	56	12	34
	Y	Y	Y	Y	Y	Y	Y	N	Y	Y	Y	Y	?	Y	?	Y	?

Table 2
Small pairs (d, N) where Corollary 19 and Theorem 20 permit a unital ETF of degree $p \leq 4$. The symbols "Y" and "N" indicate whether or not a particular ETF exists according to our exhaustive search. A "?" means that the status of this pair is open. For $p = 2, 3$ we list all the allowed pairs with $d \leq 14$. For $p = 4$ we show the allowed pairs with $d \leq 12$.

6.2 Existence of Unital ETFs

We have performed a comprehensive computer search for unital ETFs of small degree. The evidence indicates that these objects rarely exist, even when the necessary conditions of Corollary 19 admit the possibility. Table 2 summarizes our findings.

6.3 Existence of Harmonic ETFs

The situation with harmonic ETFs is somewhat more interesting. Strohmer and Heath observe that harmonic ETFs can be constructed from combinatorial objects called difference sets [17]. As a specific example, they explain how to build harmonic ETFs from a family of difference sets identified by König [10]. This construction succeeds for any d of the form $d = q^m + 1$ where q is prime and m is a natural number, and it yields a harmonic ETF with $N = d^2 - d + 1$ columns, the maximum number possible according to Corollary 19.

Somewhat later, Xia et al. [25] noticed the connection between harmonic ETFs

d	N	Exists?
3	7	Y
4	7	Y
	13	Y
5	11	Y
	21	Y
6	11	Y
	16	?

d	N	Exists?
6	31	Y
7	15	Y
	22	?
	43	?
8	15	Y
	29	Y
	57	Y

d	N	Exists?
9	13	Y
	19	?
	25	?
	37	Y
	73	Y

Table 3
All nontrivial pairs (d, N) with $d \leq 9$ where conditions permit the existence of a harmonic ETF. The notation "Y" indicates that a construction is known, while a "?" means that the status of this pair is open.

and difference sets, and they proved that harmonic ETFs and difference sets are in one-to-one correspondence. Their article provides a short list of pairs (d, N) where a harmonic ETF can be constructed.

Table 3 exhibits all nontrivial pairs (d, N) with $d \leq 9$ where our conditions permit the existence of a harmonic ETF, and we indicate whether a construction is known [5,16,22].

6.4 Existence of Maximal ETFs

A *maximal ETF* is one that meets the upper bounds of Theorem C. In other words, a maximal real ETF has dimensions $d \times \frac{1}{2}d(d+1)$, and a maximal complex ETF has dimensions $d \times d^2$. Curiously, maximal complex ETFs exist far more often than real ones. This subsection offers a glimpse of this phenomenon.

Theorem 15 allows us to conclude that maximal real ETFs can appear only when $\sqrt{d+2}$ is odd. This condition requires that $d = 7, 23, 47, \ldots$. It is indeed possible to construct maximal real ETFs for $d = 7, 23$ from the E8 and Leech lattices. See [12] for more information.

The literature contains analytic constructions of maximal complex ETFs only when $d = 1, 2, 3, 4, 5, 7, 8$. But numerical evidence suggests that these objects exist for *every* natural number d. Resolving this conjecture is currently the major open problem in the study of ETFs. See [2] and its references for the most current work.

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