FLOW AROUND A SYMMETRIC OBSTACLE

JOEL A. TROPP

Abstract. In this article, we apply Schauder’s fixed point theorem to demonstrate the existence of a solution to a certain integral equation. This solution represents the tangential velocity of a fluid flowing around a symmetric obstacle.

1. The Flow Configuration

We are interested in determining the two-dimensional motion of an ideal fluid around an obstacle. For simplicity, we assume that the object is symmetric about the x-axis and that the flow is parallel with the x-axis as $x$ approaches $-\infty$.

We will denote the fluid domain by $\Omega$ and the boundary of the object by $\partial \Omega$. We will write $\mathbf{u}$ for the vector velocity field of the fluid. And the configuration will resemble Figure 1.

![Figure 1. The flow configuration.](image)

2. Potential Flow

To understand the nature of this example, we will briefly discuss the fluid mechanics which undergird the problem.
2.1. **The Euler Equations.** The following coupled set of PDEs is called the Euler Equations:

\[
\begin{align*}
\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} &= \frac{1}{\rho} \nabla p + \mathbf{F}_{\text{ext}} \\
\nabla \cdot \mathbf{u} &= 0
\end{align*}
\]

in $$\Omega$$,

where $$\rho$$ denotes the constant fluid density, $$p$$ indicates the scalar pressure field and $$\mathbf{F}_{\text{ext}}$$ are the external forces acting on the fluid, such as gravity.

The Euler Equations describes flows which are inviscid and incompressible. They are useful for determining the large-scale properties of a Newtonian fluid which is not moving too rapidly. The Euler Equations do not describe the finer structures in the fluid, since these result from viscous interactions between the fluid particles. They only apply to fluids which respond to shocks in a linear fashion. They cannot account for shock waves, and they cannot model rarefied fluids. In spite of these limitations, they fit a wide variety of real conditions.

The first equation is essentially the fluid-mechanical version of Newton’s Second Law, $$\mathbf{F} = \rho \mathbf{a}$$. In this connection, the momentum formulation is used instead:

\[\frac{d(\rho \mathbf{u})}{dt} = \mathbf{F}.\]

The terms on the lefthand side of the first equation represent momentum and the righthand side terms represent forces. In the Navier-Stokes equations, which are used to model viscous fluids, the righthand side contains an additional term to account for the shear stresses between fluid particles.

The second equation expresses the law of conservation of mass. For every fixed volume inside the flow domain, the influx of fluid must equal the efflux of fluid in each infinitesimal time step. This equation is called the continuity condition or the incompressibility equation.

2.2. **The Potential Formulation.** We consider only steady flows, i.e. those which do not change with time. As a result, the lefthand side of the momentum equation becomes zero. We need this equation only if we wish to calculate the pressure field.

Moreover, we assume that our flow is irrotational. This is a serious limitation, but it is appropriate in certain limited contexts. As a result of this supposition, there exists a scalar potential function $$\phi$$ which satisfies the relation $$\mathbf{u} = \nabla \phi$$. Substituting this formula into the incompressibility condition, we obtain

\[\Delta \phi = 0 \quad \text{in } \Omega.\]
Next, we recognize that the boundary is impenetrable. Therefore, the normal velocity of the fluid at surface of the obstacle is always zero. This fact yields the boundary condition

$$\nabla \varphi \cdot \mathbf{n} = 0 \quad \text{on } \partial \Omega.$$ 

For well-posedness, we must place one more condition on the flow, say

$$\lim_{x \to -\infty} \mathbf{u} = (U_\infty, 0).$$

The quantity $U_\infty$ is called the onset velocity.

Combining these results, we reach the following Laplace problem with Neumann boundary conditions:

$$\begin{cases} 
\Delta \varphi = 0 & \text{in } \Omega, \\
\nabla \varphi \cdot \mathbf{n} = 0 & \text{on } \partial \Omega \quad \text{and} \\
\nabla \varphi \to (U_\infty, 0) & \text{as } x \to -\infty.
\end{cases}$$

3. The Strategy

Since we are working in two dimensions, we can apply complex variable theory to simplify the problem. The essential idea is to map the flow domain conformally to a semicircle, on which the Laplace problem may be solved explicitly. Transforming back to the original domain, we obtain a potential solution to the problem (see Figure 2). This technique was pioneered by Joukowski who discovered a conformal mapping between a cambered airfoil and an ellipse. These mappings were frequently used in aeronautics in the earlier part of the century before numerical methods became tractable.

It would take us too far afield to work through this derivation. More information is available in Batchelor or in Birkhoff and Zarantonello.
The end result, however, is the following integral equation:

\[ u(\sigma) = M \int_{\pi/2}^{\sigma} \nu(\tau)e^{-r(u(\tau))} d\tau, \]

where \( u \) is the tangential velocity of the fluid at the boundary; \( \sigma \) is an angle variable ranging between 0 and \( \pi \); \( M \) is a constant depending on the onset velocity \( U_\infty \); \( \nu \) is a function related to the shape of the boundary; and \( r \) is the harmonic conjugate of \( \theta \), where \( \theta \) is the harmonic extension of the function representing the angle of the tangent to the boundary. These definitions are somewhat hazy, but the purpose of this article is not to provide a detailed accounting of the physics.

We will call the integral operator on the righthand side \( F_M \). As a result, we obtain the equation \( u = F_M(u) \). And so we have cast our existence question in terms of a fixed point problem. It is still necessary, however, to ascertain whether a fixed point of this equation will satisfy the Laplace problem we are trying to solve. This is in fact the case, but that is another theorem for another day. Our primary concern is to illustrate the application of a fixed point theorem.

Incidentally, for any fixed point \( u \), we must have \( u(\pi/2) = 0 \). This corresponds to the physical notion of a stagnation point in the flow, a location at which the fluid has zero velocity. The derivation actually takes this assumption as granted.

4. The Details

We search for a fixed point in the Banach space \( X = C(0, \pi) \cap \{ u : u(\pi/2) = 0 \} \), which is the set of continuous velocity functions with a leading stagnation point.

We make two claims, namely that \( F_M(X) \) is precompact and that \( F_M \) is continuous. Let us examine the implications.

If \( F_M(X) \) is precompact, then the closed convex hull \( \overline{co}(F_M(X)) \) is both compact and convex. Moreover, it is clear that this set contains its image under \( F_M \), viz.

\[ F_M : \overline{co}(F_M(X)) \to F_M(X) \subset \overline{co}(F_M(X)). \]

If \( F_M \) is also continuous, then the Schauder theorem will furnish a fixed point of the operator \( F_M \), which is what we sought. Therefore, let us seek to substantiate our claims.

First, we introduce two auxiliary propositions.

**Proposition 1.** For any real \( x \) and \( y \), the following estimate holds:

\[ |e^{-x} - e^{-y}| \leq |x - y|e^{-x} + e^{-y}|. \]
Proof. We form a differential quotient and apply the mean value theorem to obtain a point $\xi$ between $x$ and $y$ such that

$$\frac{|e^{-x} - e^{-y}|}{|x - y|} = e^{-\xi}.$$ 

Next, we note that $e^{-\xi}$ is majorized by either $e^{-x}$ or $e^{-y}$. As a result, we may write

$$\frac{|e^{-x} - e^{-y}|}{|x - y|} \leq e^{-x} + e^{-y}.$$ 

And we multiply through by the factor $|x - y|$ to reach our conclusion.

Let us define the quantity

$$2\gamma = \max_{\tau} \theta - \min_{\tau} \theta.$$ 

It will develop in the analysis that we must have $2\gamma < \pi$. This condition means, more or less, that the object is convex because the boundary cannot double back on itself.

**Proposition 2.** If $\theta$ satisfies the aforesaid condition then the following estimate is in force for $p$ ranging between $0$ and $\pi/2\gamma$.

$$\int_0^{\pi} \cosh(p\tau(\gamma)) \, d\tau \leq \pi \sec p\gamma.$$ 

**Proof.** Use the Residue Theorem$^1$.

We are now prepared to demonstrate the foregoing claims.

**Lemma 1.** The set $F_M(X)$ is precompact.

**Proof.** The well-known theorem of Ascoli and Arzelà provides a sufficient condition for the precompactness of a set of continuous functions, namely that the set must be equicontinuous and equibounded.

To prove equicontinuity, we select from $X$ a function $u$. We must show that the modulus of continuity of $F_M(u)$ is independent of $u$. We choose two points $x$ and $y$ in the interval $[0, \pi]$. Then we may write

$^{1}$Better you than me.
the following chain of inequalities:

\[ |F_M(u)(x) - F_M(u)(y)| \leq M \left| \int_x^y \nu(\tau)e^{-r(u(\tau))} \, d\tau \right| \]

\[ \leq M \|\nu\|_\infty \left\{ \int_x^y \left| e^{-r(u(\tau))} \right| \, d\tau \right\}^{1/q} \left\{ \int_x^y e^{-pr(u)} \, d\tau \right\}^{1/p} \]

\[ \leq M \|\nu\|_\infty |x - y|^{1/q} \left\{ 2 \int_x^y \cosh pr(u) \, d\tau \right\}^{1/p} \]

\[ \leq M \|\nu\|_\infty (2\pi \sec p\gamma)^{1/p} |x - y|^{1/q}, \]

where we have chosen \( p \) from the interval \((1, \pi/2)\), and \( q \) is the Hölder conjugate of \( p \). Since the estimate does not depend on the function \( u \) or the points \( x \) and \( y \), we discover that the set is equicontinuous.

Using the inequality developed above, it is easy to see that \( F_M(X) \) is also equibounded. Given a function \( u \) in \( X \) and a point \( x \) in the interval \([0, \pi]\),

\[ |F_M(u)(x)| \leq 2M \|\nu\|_\infty (2\pi \sec p\gamma)^{1/p} \pi^{1/q}, \]

using the triangle inequality. Equiboundedness follows. \( \square \)

And finally, we use similar arguments to see that \( F_M \) is continuous.

**Lemma 2.** The operator \( F_M \) is continuous on \( X \).

**Proof.** We select arbitrary functions \( u \) and \( v \) in \( X \), and we estimate the norm

\[ \|F_M(u) - F_M(v)\|_X \]

\[ \leq M \left| \sup_{\sigma} \int_{\pi/2}^\sigma \nu(\tau)(e^{-r(u(\tau))} - e^{-r(v(\tau))}) \, d\tau \right| \]

\[ \leq M \|\nu\|_\infty \int_0^\pi \left| e^{-r(u)} - e^{-r(v)} \right| \, d\tau \]

\[ \leq M \|\nu\|_\infty \int_0^\pi |r(u) - r(v)||e^{-r(u)} + e^{-r(v)}| \, d\tau \]

\[ \leq M \|\nu\|_\infty \left\{ \int_0^\pi |e^{-pr(u)} + e^{-pr(v)}| \, d\tau \right\}^{1/p} \left\{ \int_0^\pi |r(u - v)|q \, d\tau \right\}^{1/q}, \]

where \( p \) and \( q \) are Hölder conjugate indices as before. We apply Minkowski’s inequality to the first integral and bound the result using our proposition. Then, we estimate the last integral using a theorem from Zygmund which relates harmonic functions to their harmonic conjugates: \( \|r\|_q \leq \||\theta||_p \). (Or something.) This result follows from
Young’s inequality applied to Poisson’s formula. And so we reach
\[
\|F_M(u) - F_M(v)\|_X \leq M\|\nu\|_\infty (2\pi \sec p\gamma)^{1/p} \|\theta(u - v)\|_p \\
\leq M\|\nu\|_\infty (2\pi \sec p\gamma)^{1/p} \|\theta(u - v)\|_\infty \\
\leq M\|\nu\|_\infty \|\theta\|_\infty (2\pi \sec p\gamma)^{1/p} \|u - v\|_\infty.
\]
As \( v \) approaches \( u \) in \( X \), the righthand side tends to zero. And we may conclude that \( F_M \) is continuous on \( X \).

From these lemmata follows our main theorem.

**Theorem 1.** The operator \( F_M \) has a fixed point in \( X \).