THE CONE THEOREM

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Abstract. We prove a fixed point theorem for functions which are positive with respect to a cone in a Banach space.

1. Definitions

Definition 1. Let X be a real Banach space. A subset C of X is called a *cone* if the following are true:

1. C is nonempty and nontrivial (i.e. C contains a nonzero point);
2. $\lambda C \subset C$ for any nonnegative λ;
3. C is convex;
4. C is closed; and
5. $C \cap (-C) = \{0\}$.

These conditions have fairly intuitive meanings. The second one shows that a cone is a collection of rays emanating from the origin, while the first forces the cone to contain at least one ray. The third requirement ensures that the cone contains no holes. The fourth guarantees that the cone contains its boundaries. And the last condition makes sure that the cone is not too big; it must not take up more than half the space.

Examples of cones are easy to come by. In $\mathbb{R}^3$, the definition of a cone precisely matches our geometrical intuition, with the stipulation that the vertex must coincide with the origin. In $\mathbb{R}$, the nonnegative numbers form a cone. In $\mathbb{R}^2$, any wedge which extends to infinity from the origin is a cone (see Figure 1). We can also find more abstract examples. In any L^p space (including L^∞), the set $C = \{f : f \geq 0\}$ is a cone. Similarly, in ℓ^p spaces, the set of nonnegative sequences form a cone.

Next, we will define a partial ordering of the Banach space with respect to a cone.

Definition 2. We say that $y \leq x$ if and only if $x - y \in C$.

Proposition 1. *The relation $\leq$ defines a partial ordering on X.*

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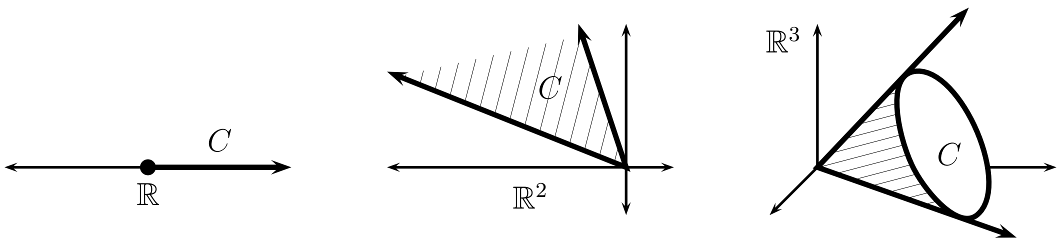

FIGURE 1. Examples of cones.

Proof. We simply check that the relation satisfies the necessary criteria.

Reflexivity: Since $0 \in C$, we see that $x \leq x$ for any point x.

Antisymmetry: If $x \leq y$ and $y \leq x$, then both $y - x \in C$ and $-(y-x) \in C$. It follows that $y - x = 0$, and so $x = y$.

Transitivity: If $x \leq y$ and $y \leq z$, then $y - x \in C$ and $z - y \in C$. By convexity and the ray property, $2\{\frac{1}{2}(y-x) + \frac{1}{2}(z-y)\} \in C$. We see that $z - x \in C$, and we may conclude that $x \leq z$.

So $\leq$ is a partial ordering. □

The partial ordering also satisfies several other properties, which follow immediately from the definition of a cone.

1. Multiplication by a nonnegative scalar preserves the ordering. Together, $x \leq y$ and $\lambda \geq 0$ imply that $\lambda x \leq \lambda y$.
2. Addition of a fixed vector preserves the ordering. The inequality $x \leq y$ implies that $x + z \leq y + z$ for any vector z.
3. The ordering preserves limits. If $y_n \to y$ and each $y_n \leq x$, then $y \leq x$.

It turns out that any partial ordering in a Banach space which satisfies these three additional properties will define a cone.

We make one final definition before we turn to our main theorem.

Definition 3. Let C be a cone contained in a Banach space X. We say that an operator $A : X \to X$ is *positive* with respect to the cone if $A(C) \subset C$.

2. The Cone Theorem

Theorem 4 (Krasnoselskii). *Let C be a cone in a Banach space X. Let $A : X \to C$ be continuous and compact. Suppose that for distinct, positive real numbers r and R, the following conditions hold for $x \in C$:*

1. *When $||x|| = r$, we have $Ax - x \notin C$.*
2. *When $||x|| = R$, we have $x - Ax \notin C$.*

Then we may conclude that A has a fixed point φ within the cone which, moreover, satisfies $||\varphi|| \in (r, R)$.

To understand the content of this theorem better, let us consider the case when $X = \mathbb{R}$ and C is the set of nonnegative numbers. Then, the cone theorem is a slightly specialized version of the intermediate value theorem (see Figure 2).

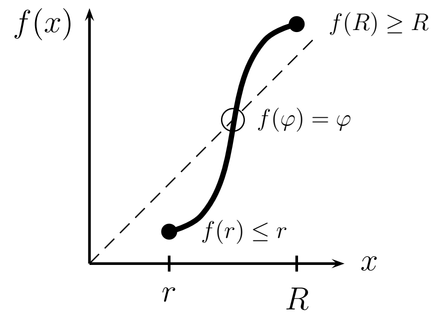

FIGURE 2. The Cone Theorem in $\mathbb{R}$.

It is more difficult to understand how this theorem applies to $\mathbb{R}^2$. The reader may wish to contemplate this situation. More or less, the theorem demonstrates that a function which always points outward (or inward) at the boundary of an annular sector has a fixed point within that sector (see Figure 3).

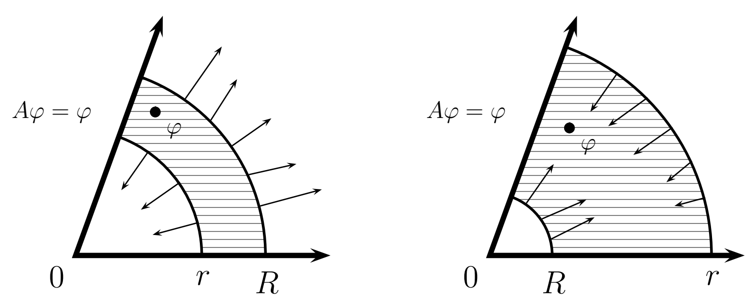

FIGURE 3. The Cone Theorem in $\mathbb{R}^2$.

That being said, we proceed to the proof of the theorem, which is an easy application of degree theory.

Proof. First, we simplify the problem by restricting our considerations to the topological space C endowed with the relative topology. In the sequel, it must be understood that any set which extends beyond the cone should be interpreted as the intersection of that set with the cone. This does not affect the degree theory, except insofar as it changes the definition of a boundary point.

For simplicity, we shall also assume that $r < R$. We can handle the opposite case easily once we have completed the current demonstration.

Using the homotopy property, we shall determine the degree of zero within $B_r(0)$ and within $B_R(0)$ with respect to the function $I - A$. Then, we shall apply the additivity property to determine the degree of zero within the annular sector $C_r^R = B_R(0) \setminus B_r(0)$. This degree, being odd, will imply the existence of a fixed point within the annular sector (see Figure 4).

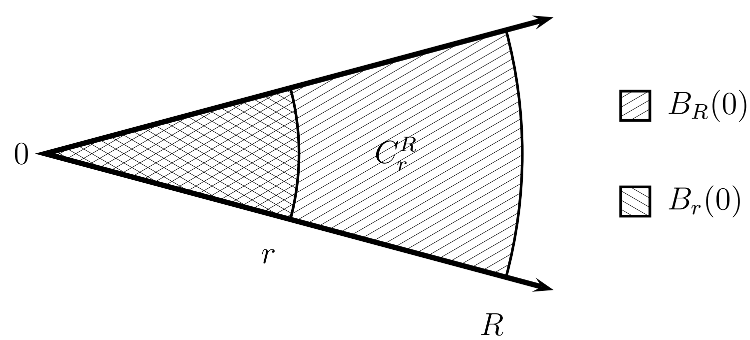

FIGURE 4. Dividing the cone to determine the degree of zero inside the annular sector C_r^R.

We define a function $\Phi = I - A$, which we shall examine using the degree theory.

First, we wish to determine the degree of the regular value zero within the set $B_r = B_r(0)$. To this end, we define the homotopy $\Phi_t(x) = (I - tA)(x)$ for $0 \leq t \leq 1$. We need to ensure that the homotopy is valid. It is clearly a continuous function of t. Now, we fix a bounded set E. Since A is compact, the set $[0, 1] \times A(E)$ is precompact. Scalar multiplication is continuous, so the image of this product space under scalar multiplication is also precompact, which is what we needed to check.

We must also check that the homotopy is admissible for calculating the degree of zero within B_r. That is, no zeros may enter the set through the boundary during the transition from $t = 0$ to $t = 1$. To the contrary, suppose that $\Phi_t(x) = 0$ for some t and some point x on the boundary of B_r. Since we are working in the relative topology, the boundary contains only the points on $S_r(0)$; we do not need to consider the frontier between the cone and the remainder of the space. Now, we may rewrite our supposition as $x = tAx$. Since $tAx \in C$ and $t \leq 1$, we have $tAx \leq Ax$. Consequently $x \leq Ax$ for some $x \in S_r$, which contradicts the lower cone condition.

Now, we apply the homotopy property to see that $\deg(\Phi_0, B_r, 0) = \deg(\Phi_1, B_r, 0)$. In other words, $\deg(I, B_r, 0) = \deg(I - A, B_r, 0)$. It is obvious that the lefthand member equals one. And we conclude that

$$\deg(I - A, B_r, 0) = 1.$$

Next, we wish to determine the degree of zero within the set $B_R = B_R(0)$. First, we note that the set $(I - A)(B_R)$ is bounded. Since the cone is unbounded, we may select a point w from the portion of the cone which is not contained in $\overline{(I - A)(B_R)}$. Now, we define a new homotopy $\Phi_t(x) = (I - A - tw)(x)$. The fixed part $(I - A)$ is invertible, and the set $\cup_t \{tw\}$ is evidently compact. Thus, the homotopy is valid.

We must check that the new homotopy is admissible. To the contrary, we assume that there exists a t and a boundary point x for which $\Phi_t(x) = 0$. Then, we have $x - Ax = tw$, which implies that $x - Ax \in C$. Due to the topology, the only boundary points of B_R lie on the sphere $S_R(0)$. But then we reach a contradiction to the upper cone condition.

Therefore, $\deg(I - A, B_R, 0) = \deg(I - A - w, B_R, 0)$. We have chosen w in such a way that the function $(I - A - w)$ never vanishes on B_R, so the righthand member is zero. We conclude that

$$\deg(I - A, B_R, 0) = 0.$$

Using the additivity property, we see that

$$\deg(I - A, C_r^R, 0) = \deg(I - A, B_R, 0) - \deg(I - A, B_r, 0) = -1.$$

In consequence, there exists at least one regular point $\varphi \in C_r^R$ at which $(I - A)(\varphi) = 0$. So A has a fixed point in C_r^R.

If instead we repeat the proof with the assumption that $R < r$, we reach

$$\deg(I - A, C_r^R, 0) = \deg(I - A, B_r, 0) - \deg(I - A, B_R, 0) = +1.$$

Once again, we discover that A has a fixed point with norm between r and R. □