CNOIDAL SOLUTIONS TO THE pTH-ORDER KORTEWEG-DE VRIES EQUATION

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Abstract. For the pth-order Korteweg-de Vries equation,

$$u_t + u^p\, u_x + u_{xxx} = 0,$$

we determine all travelling-wave solutions which may be expressed in terms of a single Jacobi *cosinus amplitudinus* (cn) function. This development includes the known cnoidal solutions for $p = 1$ and the soliton solutions for all p. For $p = 2$, we present a new class of cnoidal solutions with an arbitrary period. Moreover, we show that for odd p, there are solutions in terms of the secant function.

1. Simplifying the equation

We begin with the pth-order Korteweg-de Vries equation,

$$\begin{cases} u_t + u^p\, u_x + u_{xxx} = 0, \\ u(x,0) = u_0(x). \end{cases}$$

We shall consider the travelling-wave ansatz $u(x,t) = \psi(x - dt)$, which transforms the original PDE into the ODE

$$-du_x + u^p\, u_x + u_{xxx} = 0.$$

A single integration yields

$$-du + \tfrac{1}{p+1} u^{p+1} + u_{xx} = A.$$

Multiply this equation by u_x and integrate again to reach

$$\tfrac{1}{2}(u_x)^2 + \tfrac{1}{(p+1)(p+2)} u^{p+2} - \tfrac{1}{2} du^2 = Au + B. \tag{1.1}$$

The constants A and B lie at our disposal.

Define $y = x - dt$, and set

$$\varphi(x,t) = a + b\, \mathrm{cn}^q\,(cy; m).$$

Throughout, the shorthand cn^q will replace $\mathrm{cn}^q\,(cy; m)$. For each p, we shall find all exponents q and attendant constraints on the parameters

Date: 20 November 2001.

a, b, c, d and m so that φ satisfies Equation 1.1 for some pair of constants A and B.

2. Analysis of terms by order of exponent

Set $u = \varphi$, and consider the terms on the left-hand side of Equation 1.1. The first term

$$\begin{aligned}\tfrac{1}{2}(\varphi_x)^2 &= \tfrac{1}{2}(bcq \operatorname{cn}^{q-1}(cy;m) \operatorname{sn}(cy;m) \operatorname{dn}(cy;m))^2 \\ &= \tfrac{1}{2}b^2c^2q^2 \operatorname{cn}^{2(q-1)}(1-\operatorname{cn}^2)((1-m^2)+m^2\operatorname{cn}^2) \\ &= \tfrac{1}{2}b^2c^2q^2\left\{(1-m^2)\operatorname{cn}^{2(q-1)} + (2m^2-1)\operatorname{cn}^{2q} - m^2\operatorname{cn}^{2(q+1)}\right\},\end{aligned}$$

using the properties of the Jacobi elliptic functions. (We denote by sn and dn the *sinus amplitudinis* and *delta amplitudinis* functions.) The next term

$$\tfrac{1}{(p+1)(p+2)}\varphi^{p+2} = \tfrac{1}{(p+1)(p+2)}\sum_{k=0}^{p+2}\binom{p+2}{k}a^{p+2-k}b^k \operatorname{cn}^{kq},$$

and the final term

$$-\tfrac{1}{2}d\varphi^2 = -\tfrac{1}{2}a^2d - abd\operatorname{cn}^q - \tfrac{1}{2}bd\operatorname{cn}^{2q}.$$

Meanwhile, the right-hand side of Equation 1.1 is given by

$$A\varphi + B = (aA + B) + bA\operatorname{cn}^q.$$

Now, let us collect the terms from the foregoing expressions in Table 1. Since terms of distinct orders are linearly independent, we obtain a separate equation for each order. *Nota bene* that some of the expressions in the leftmost column may not be distinct, which point is the crux of the analysis.

3. The case $p = 1$

Notice that the selection $p = 1$ simplifies Table 1 substantially, so that there are no terms beyond those of order $3q$. Moreover, if none of the other rows have the same order as $3q$, then

$$\tfrac{p}{3!}a^{p-1}b^3 = 0.$$

It follows that $b = 0$, which implies that φ is constant. We can avoid this infelicitous conclusion by requiring that one of the other expressions in the left-hand column of Table 1 equal $3q$. Naturally, we forbid $q = 0$ to ensure that φ is nontrivial, so the remaining choices are $2(q-1) = 3q$ or $2(q+1) = 3q$. That is, $q = \pm 2$. Let us pursue each of these options separately.

TABLE 1. Analysis of terms by order.

Exponent on cn	Left-hand side	Right-hand side
0	$\frac{1}{(p+1)(p+2)}a^{p+2} - \frac{1}{2}a^2 d$	$aA + B$
q	$\frac{1}{p+1}a^{p+1}b - abd$	bA
$2(q-1)$	$\frac{1}{2}b^2c^2q^2(1-m^2)$	0
$2q$	$\frac{1}{2}b^2c^2q^2(2m^2-1) - \frac{1}{2}bd + \frac{1}{2}a^p b$	0
$2(q+1)$	$-\frac{1}{2}b^2c^2q^2m^2$	0
$3q$	$\frac{p}{3!}a^{p-1}b^3$	0
$4q$	$\frac{p(p-1)}{4!}a^{p-2}b^4$	0
$\dots$	$\dots$	$\dots$
$(p+2)q$	$\frac{1}{(p+1)(p+2)}b^{p+2}$	0

3.1. **A cnoidal solution.** When $q = +2$, Table 1 reduces to the following system.

$$\frac{1}{6}a^3 - \frac{1}{2}a^2 d - aA = B \tag{3.1}$$
$$\frac{1}{2}a^2 b - abd + 2b^2c^2(1-m^2) = bA \tag{3.2}$$
$$2b^2c^2(2m^2-1) - \frac{1}{2}bd + \frac{1}{2}ab^2 = 0 \tag{3.3}$$
$$\frac{1}{6}b^3 - 2b^2c^2m^2 = 0 \tag{3.4}$$

Solving Equations 3.3 and 3.4 yields the constraints

$$b = 12c^2m^2 \quad \text{and} \tag{3.5}$$
$$d = 12c^2m^2(8c^2m^2 - 4c^2 + a). \tag{3.6}$$

Since A and B are arbitrary, Equations 3.1 and 3.2 do not place any additional restrictions on the coefficients. In consequence,

$$\varphi(x,t) = a + b\ \mathrm{cn}^2\left\{c(x-dt);m\right\} \tag{3.7}$$

satisfies Equation 1.1 so long as Conditions 3.5 and 3.6 hold. This is the standard cnoidal solution to the 1st-order KdV equation, and it includes soliton solutions as a special case when $m = 1$.

3.2. **A reciprocal cnoidal wave.** When $q = -2$ the resulting system of equations looks slightly different.

$$\begin{aligned} \tfrac{1}{6}a^3 - \tfrac{1}{2}a^2 d - aA &= B && (3.8) \\ \tfrac{1}{3}a^2 b - abd - 2b^2c^2m^2 &= bA && (3.9) \\ 2b^2c^2(2m^2-1) - \tfrac{1}{2}bd + \tfrac{1}{2}ab^2 &= 0 && (3.10) \\ \tfrac{1}{6}b^3 + 2b^2c^2(1-m^2) &= 0 && (3.11) \end{aligned}$$

Equations 3.11 and 3.10 provide constraints

$$\begin{aligned} b &= 12c^2(1-m^2) \qquad \text{and} && (3.12) \\ d &= 12c^2(1-m^2)(8c^2m^2 - 4c^2 + a) && (3.13) \end{aligned}$$

As before, the other two equations do not place additional restrictions. Thus

$$\varphi(x,t) = a + b\ \mathrm{cn}^{-2}\left\{c(x-dt); m\right\} \tag{3.14}$$

satisfies Equation 1.1 so long as Conditions 3.12 and 3.13 obtain.

4. The general case

Let $p \geq 2$. As before, we require a second term to have the same order as the $(p+2)q$ term to ensure that b is nonzero. If q is nonzero, the only feasible options are $2(q-1) = (p+2)q$ and $2(q+1) = (p+2)q$, which yield the condition $q = \pm 2/p$. When q has this form, there is a single term of order $3q$, which leads to the equation

$$\tfrac{p}{3!}a^{p-1}b^3 = 0.$$

We must accept the conclusion that $a = 0$ to prevent φ from becoming constant. That is, nontrivial cnoidal solutions to KdV admit an additive constant only when $p = 1$.

Selecting $a = 0$ simplifies the problem tremendously. In fact, the only terms which remain bear exponents of $2(q-1), 2q, 2(q+1)$ and $(p+2)q$. Let us consider four separate cases.

4.1. **Another cnoidal solution.** Setting $p = 2$ and $q = 1$, we obtain the following system of equations.

$$\begin{aligned} \tfrac{1}{2}b^2c^2(1-m^2) &= B && (4.1) \\ \tfrac{1}{2}b^2c^2(1-m^2) + \tfrac{1}{2}b^2c^2(2m^2-1) - \tfrac{1}{2}bd &= 0 && (4.2) \\ \tfrac{1}{12}b^4 - \tfrac{1}{2}b^2c^2m^2 &= 0 && (4.3) \end{aligned}$$

Equations 4.2 and 4.3 yield the constraints

$$\begin{aligned} b &= m|c|\sqrt{6} \qquad \text{and} && (4.4) \\ d &= m^3|c^3|\sqrt{6}. && (4.5) \end{aligned}$$

No additional restrictions fall from Equation 4.1. Therefore,

$$\varphi(x,t) = b\,\mathrm{cn}\,\{c(x-dt);m\} \tag{4.6}$$

satisfies Equation 1.1 whenever Conditions 4.4 and 4.5 are fulfilled. We see that the 2nd-order KdV equation also has a cnoidal solution. As before, the selection of $m=1$ reduces Solution 4.6 to a soliton.

4.2. **Soliton solutions to the pth-order equation.** Now, consider the case where $p \geq 3$ and $q = +2/p$. These choices yield another system of equations.

$$\begin{aligned} 2p^{-2}b^2c^2(1-m^2) &= 0 && (4.7)\\ 2p^{-2}b^2c^2(2m^2-1) - \tfrac{1}{2}bd &= 0 && (4.8)\\ \tfrac{1}{(p+1)(p+2)}b^{p+2} - 2p^{-2}b^2c^2m^2 &= 0 && (4.9) \end{aligned}$$

Since $b=0$ and $c=0$ yield trivial waves, Equation 4.7 implies that $m=1$. Then, Equations 4.8 and 4.9 become

$$\begin{aligned} b &= \left(\frac{2c^2(p+1)(p+2)}{p^2}\right)^{1/p} \quad \text{and} && (4.10)\\ d &= \frac{4c^2}{p^2}\left(\frac{2c^2(p+1)(p+2)}{p^2}\right)^{1/p}. && (4.11) \end{aligned}$$

If Conditions 4.10 and 4.11 hold, then

$$\varphi(x,t) = b\,\mathrm{sech}^{2/p}\,\{c(x-dt);m\} \tag{4.12}$$

satisfies Equation 1.1. And so the pth-order KdV equation has a soliton solution.

4.3. **Another reciprocal cnoidal wave.** The case when $p=2$ and $q=-1$ also distinguishes itself. Equation 1.1 reduces to the following equations.

$$\begin{aligned} -\tfrac{1}{2}b^2c^2m^2 &= B && (4.13)\\ \tfrac{1}{2}b^2c^2(2m^2-1) - \tfrac{1}{2}bd &= 0 && (4.14)\\ \tfrac{1}{12}b^4 + \tfrac{1}{2}b^2c^2(1-m^2) &= 0 && (4.15) \end{aligned}$$

We obtain two conditions from Equations 4.14 and 4.15.

$$\begin{aligned} b &= |c|\sqrt{1-m^2}\sqrt{6} && (4.16)\\ d &= |c^3|(2m^2-1)\sqrt{1-m^2}\sqrt{6} && (4.17) \end{aligned}$$

Equation 4.13 does not develop into a constraint. Thus,

$$\varphi(x,t) = b\ \mathrm{cn}^{-1}\,\{c(x-dt);m\} \tag{4.18}$$

solves Equation 1.1 when $p = 2$ and Conditions 4.16 and 4.17 both hold. When $m = 0$, Solution 4.18 is a travelling secant curve.

4.4. **More secant solutions.** It turns out that KdV equations of odd orders have secant solutions. Let p be an odd number no less than 3, and put $q = -2/p$. We must solve the following system.

$$-2p^{-2}b^2c^2m^2 = 0 \tag{4.19}$$

$$2p^{-2}b^2c^2(2m^2-1) - \tfrac{1}{2}bd = 0 \tag{4.20}$$

$$\tfrac{1}{(p+1)(p+2)}b^{p+2} + 2p^{-2}b^2c^2(1-m^2) = 0 \tag{4.21}$$

It follows from Equation 4.19 that $m = 0$. Then Equations 4.20 and 4.21 yield

$$b = -\left(\frac{2c^2(p+1)(p+2)}{p^2}\right)^{1/p} \quad \text{and} \tag{4.22}$$

$$d = \frac{2c^2}{p^2}\left(\frac{2c^2(p+1)(p+2)}{p^2}\right)^{1/p}. \tag{4.23}$$

Therefore, every odd-order KdV equation has a travelling secant solution of the form

$$\varphi(x,t) = b\sec^{2/p}\{c(x-dt)\}, \tag{4.24}$$

where the parameters fulfill Conditions 4.22 and 4.23.

5. Integrability of reciprocal cnoidal waves

Let us establish a lemma which will enable us to determine when negative powers of the cnoidal function are locally integrable.

Lemma 5.1. *The function ψ defined by*

$$\psi(x) = \mathrm{cn}^{-2/p}(x;m)$$

is locally integrable if and only if $p \geq 3$ or $m = 1$.

Proof. Since $\mathrm{cn}(x,1) = \mathrm{sech}(x)$, it is immediate that $\mathrm{cn}^{-2/p}(x,1) = \cosh^{2/p}(x)$. The hyperbolic cosine is everywhere continuous, so it is integrable on compact sets. Therefore, we need only consider the case where $m < 1$.

The function ψ is continuous except at a discrete set of points, so it is necessary and sufficient to compute its integral in the neighborhood of a single discontinuity. If z is a zero of the cn function, then

$$\mathrm{cn}\,(z+x;m) = \frac{\sqrt{1-m^2}\,\mathrm{sn}\,(x;m)}{\mathrm{dn}\,(x;m)},$$

so we may as well investigate the behavior of the right-hand side for small x. Form the integral

$$\int_{z-\varepsilon}^{z+\varepsilon} \psi(x)\,\mathrm{d}x = (1-m^2)^{-1/p} \int_{-\varepsilon}^{\varepsilon} \frac{\mathrm{dn}^{2/p}(x;m)}{\mathrm{sn}^{2/p}(x;m)}\,\mathrm{d}x,$$

where ε is a small positive number. Since $m < 1$, the *delta amplitudinis* function satisfies the inequality

$$0 < B \leq \mathrm{dn}\,(x;m) \leq C < 2$$

for all x. The fact that $\mathrm{sn}^{2/p}$ is even and nonnegative allows us to bound the integral above and below.

$$B \int_0^{\varepsilon} \frac{\mathrm{d}x}{\mathrm{sn}^{2/p}\,(x;m)} \leq \int_{z-\varepsilon}^{z+\varepsilon} \psi(x)\,\mathrm{d}x \leq C \int_0^{\varepsilon} \frac{\mathrm{d}x}{\mathrm{sn}^{2/p}\,(x;m)},$$

where the constants have been combined. The *sinus amplitudinis* function is analytic, and it has Taylor series development

$$\mathrm{sn}\,z = z - \tfrac{1}{6}(1+m^2)z^3 + R_5(z)$$

about the point $z = 0$. Since ε is microscopic, we may use the first terms of the series to continue the estimate:

$$B \int_0^{\varepsilon} \frac{\mathrm{d}x}{x^{2/p}} \leq \int_{z-\varepsilon}^{z+\varepsilon} \psi(x)\,\mathrm{d}x \leq C \int_0^{\varepsilon} \frac{\mathrm{d}x}{\left\{x - \frac{1}{6}(1+m^2)x^3\right\}^{2/p}}.$$

Noticing that the right-hand integral satisfies the relation

$$\int_0^{\varepsilon} \frac{\mathrm{d}x}{\left\{x - \frac{1}{6}(1+m^2)x^3\right\}^{2/p}} \leq \{1 - \tfrac{1}{6}(1+m^2)\varepsilon^2\}^{-2/p} \int_0^{\varepsilon} x^{-2/p}\,\mathrm{d}x,$$

we obtain

$$B \int_0^{\varepsilon} x^{-2/p}\,\mathrm{d}x \leq \int_{z-\varepsilon}^{z+\varepsilon} \psi(x)\,\mathrm{d}x \leq C \int_0^{\varepsilon} x^{-2/p}\,\mathrm{d}x.$$

It follows that the integral of ψ converges if and only if $-2/p > -1$, *viz.* $p \geq 3$. □

Using this lemma, it is a simple matter to show that Solutions 3.14 and 4.18, the reciprocal cnoidal waves obtained when $p = 1, 2$, are not locally integrable. In consequence, they cannot correspond to a (locally) finite quantity of fluid. We must question the physical interpretation of these waves. On the other hand, the travelling secant waves corresponding to Solution 4.24 are locally integrable, which suggests that they might actually model some physical phenomenon.