Complex Equiangular Tight Frames

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ABSTRACT

A complex equiangular tight frame (ETF) is a tight frame consisting of N unit vectors in $\mathbb{C}^d$ whose absolute inner products are identical. One may view complex ETFs as a natural geometric generalization of an orthonormal basis. Numerical evidence suggests that these objects do not arise for most pairs (d, N).

The goal of this paper is to develop conditions on (d, N) under which complex ETFs can exist. In particular, this work concentrates on the class of harmonic ETFs, in which the components of the frame vectors are roots of unity. In this case, it is possible to leverage field theory to obtain stringent restrictions on the possible values for (d, N).

Keywords: Equiangular lines, Grassmannian packing, Hadamard matrix, tight frames, Welch bound

1. INTRODUCTION

The central object of this paper is a geometric object called an *equiangular tight frame* (ETF). These objects can be viewed as collections of lines through the origin of a Euclidean space, where the angle between each distinct pair of lines is as large as possible.

Let us proceed with a formal definition. The usual Hermitian inner product will be denoted by $\langle \cdot,\ \cdot \rangle$ and the associated norm by $\|\cdot\|_2$. The symbol $\mathbf{I}$ will indicate a conformal identity matrix.

Definition 1. *Let $\boldsymbol{X}$ be a $d \times N$ matrix whose columns are $\boldsymbol{x}_1, \dots, \boldsymbol{x}_N$. The matrix $\boldsymbol{X}$ is called an equiangular tight frame if it satisfies three conditions.*

1. *Each column has unit norm: $\|\boldsymbol{x}_n\|_2 = 1$ for $n = 1, \dots, N$.*
2. *The columns are equiangular. For some nonnegative μ, we have*

$$|\langle \boldsymbol{x}_m,\ \boldsymbol{x}_n \rangle| = \mu \qquad \text{for } m \neq n \text{ and } m, n = 1, \dots, N.$$

3. *The matrix forms a tight frame. In symbols, $\boldsymbol{X}\boldsymbol{X}^* = (N/d)\,\mathbf{I}$.*

If the entries of $\boldsymbol{X}$ are complex (resp. real) numbers, we refer to $\boldsymbol{X}$ as a complex ETF *(resp.* real ETF*).*

Numerical evidence suggests that complex ETFs do not exist for most pairs (d, N). In brief, the goal of this paper is to understand for what pairs they do.

The concept of an equiangular tight frame can be traced back at least as far as Welch's work.[1] In consequence of his results, ETFs are sometimes referred to as Maximal Welch-Bound-Equality sequences.[2] ETFs have also been studied in other recent papers, where they are referred to as *optimal Grassmannian frames*[3] and as *2-uniform frames*.[4] In addition, ETFs are closely linked to objects that arise in other mathematical fields, including coding theory, graph theory, experimental design, and numerical analysis.[3,4] Equiangular tight frames have potential applications in communication[3] and in sparse approximation.[5] We would also like to mention that there is a deep connection between ETFs and the calculation of n-widths in finite-dimensional Banach spaces.[6]

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1.1. Basic Examples

There are two essential (if boring) examples of ETFs.

1. (Orthonormal Bases). When $N = d$, the sole examples of ETFs are unitary (and orthogonal) matrices. Evidently, the absolute inner product μ between distinct vectors is zero.
2. (Simplices). When $N = d + 1$, every ETF can be viewed as the vertices of a regular simplex centered at the origin.[7] The easiest way to realize this configuration is to project the canonical coordinate basis in $\mathbb{R}^{d+1}$ onto the orthogonal complement of the vector $\mathbf{1} = [1, 1, \ldots, 1]$ and renormalize the projections. The absolute inner product μ between distinct vectors is $1/d$.

Of course, these trivial cases are not the end of the story. The first genuine example of a real ETF consists of six vectors in $\mathbb{R}^3$. It can be constructed by choosing six nonantipodal vertices from a regular icosahedron centered at the origin.[8] The first genuine example of a complex ETF contains four vectors in $\mathbb{C}^2$ with absolute inner product $\mu = 1/\sqrt{3}$.

$$\boldsymbol{X} = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{3} & -1 & -1 & -1 \\ 0 & \sqrt{2} & \zeta\sqrt{2} & \zeta^2\sqrt{2} \end{bmatrix}.$$

Here, ζ is a (primitive) third root of unity. Note that all the negative entries in the top row may be replaced by $+1$ to obtain a distinct ETF. Another example of a complex ETF consists of nine equiangular vectors in $\mathbb{C}^3$ with absolute inner product $\mu = 1/2$.

$$\boldsymbol{X} = \frac{1}{2} \begin{bmatrix} 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & \mathrm{i}\sqrt{3} & -\mathrm{i}\sqrt{3} & +1 & +1 & +1 & -1 & -1 & -1 \\ 0 & 0 & 0 & \sqrt{2} & \zeta\sqrt{2} & \zeta^2\sqrt{2} & \sqrt{2} & \zeta\sqrt{2} & \zeta^2\sqrt{2} \end{bmatrix}.$$

We use i for the imaginary unit. Once again, all the negative entries in the top row may be replaced by $+1$ to obtain a distinct ETF.

1.2. ETFs are Sporadic

Extensive numerical experiments[9] for small values of d indicate that ETFs do not exist for most pairs (d, N). We reproduce the following table, in which the notation $\mathbb{R}$ means that a real ETF exists and $\mathbb{C}$ means that a complex ETF exists (but not a real ETF). A single period (.) indicates that no real ETF exists while two periods (..) indicate that no complex ETF exists.

	d						d				
N	2	3	4	5	6	N	2	3	4	5	6
3	$\mathbb{R}$	$\mathbb{R}$	..	..	..	20	..	..	..	.	.
4	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{R}$	..	..	21	..	..	..	$\mathbb{C}$	.
5	..	.	$\mathbb{R}$	$\mathbb{R}$	..	22	..	..	..	.	.
6	..	$\mathbb{R}$	.	$\mathbb{R}$	$\mathbb{R}$	23	..	..	..	.	.
7	..	$\mathbb{C}$	$\mathbb{C}$	.	$\mathbb{R}$	24	..	..	..	.	.
8	..	.	$\mathbb{C}$	.	.	25	..	..	..	$\mathbb{C}$	.
9	..	$\mathbb{C}$	.	.	$\mathbb{C}$	26	..	..	..	..	.
10	..	..	.	$\mathbb{R}$	.	27	..	..	..	..	.
11	..	..	.	$\mathbb{C}$	$\mathbb{C}$	28	..	..	..	..	.
12	..	..	.	.	$\mathbb{C}$	29	..	..	..	..	.
13	..	..	$\mathbb{C}$	.	.	30	..	..	..	..	.
14	..	..	.	.	.	31	..	..	..	..	$\mathbb{C}$
15	..	..	.	.	.	32	..	..	..	..	.
16	..	..	$\mathbb{C}$	.	$\mathbb{R}$	33	..	..	..	..	.
17	..	..	..	.	.	34	..	..	..	..	.
18	..	..	..	.	.	35	..	..	..	..	.
19	..	..	..	.	.	36	..	..	..	..	$\mathbb{C}$

As the table indicates, real ETFs occur rarely. This observation has been confirmed by theoretical results that rule out the existence of real ETFs except in limited circumstances.[10] These results permit the existence of a (nontrivial) real ETF for precisely 27 pairs (d, N) with $N \leq 100$. In each of these 27 cases, it is possible to construct a real ETF from a regular 2-graph[4] or, equivalently, from a strongly regular graph.[10]

The table also demonstrates that complex ETFs are more common than real ETFs. Nevertheless, both types of ETFs appear to exist only for sporadic pairs (d, N). The aim of this paper is to investigate this phenomenon. There are two basic avenues of pursuit. First, one could demonstrate the existence of specific complex ETFs via constructive (or nonconstructive) means. Second, one could attempt to rule out the possibility that a complex ETF exists for a specific pair (d, N). We will follow the latter route.

Unfortunately, we have not been able to develop detailed conditions that rule out the existence of general complex ETFs. Instead, we focus on an important subcategory, the *harmonic ETFs*, in which the elements of the matrix $\boldsymbol{X}$ are restricted to be roots of unity. We have been able to use field theory to develop concrete results on the existence of harmonic ETFs. We hope that these partial results will stimulate additional research on these fascinating objects.

2. PROPERTIES OF ETFS

Some basic properties of ETFs can be obtained quickly from the definition. This section offers a selection of valuable results that have appeared in the literature. The theory in this section applies to both real and complex ETFs.

2.1. Structural Results

We begin with the fundamental algebraic and analytic properties of ETFs. First, it is clear that the ETF property is invariant under some basic operations.[4]

PROPOSITION 1 (INVARIANCE). *Suppose that $\boldsymbol{X}$ is an ETF. The following transformations preserve the ETF property.*

1. *Left-multiplication of $\boldsymbol{X}$ by a unitary matrix.*
2. *Reordering the columns of $\boldsymbol{X}$.*
3. *Multiplying each column of $\boldsymbol{X}$ by a scalar of absolute value one.*

Two ETFs are called *frame equivalent* if one can be transformed into the other by a sequence of these basic operations.[7] We write $[\boldsymbol{X}]$ for the frame equivalence class of $\boldsymbol{X}$.

Every ETF implicitly contains a dual ETF, which is unique modulo frame equivalence.[4]

PROPOSITION 2 (DUALITY). *Suppose that $\boldsymbol{X}$ is a $d \times N$ ETF. Then we may associate to $\boldsymbol{X}$ a unique dual ETF with dimensions $(N-d) \times N$. This duality correspondence is an involution on the set of equivalence classes of ETFs. That is, $[\boldsymbol{X}]$ is the dual of the dual of $[\boldsymbol{X}]$.*

Proof. The d rows of $\boldsymbol{X}$ are orthogonal, and each one has squared norm (N/d). Therefore, we can form a scaled $N \times N$ unitary matrix by appending $(N-d)$ orthogonal rows with squared norm (N/d). A direct calculation establishes that the new rows also form an ETF. The row span of the new ETF is completely determined by the row span of $\boldsymbol{X}$, which yields the remaining claims. □

Another fundamental fact is that the value of μ is completely determined by the dimensions of the ETF.[3]

PROPOSITION 3 (SIZE OF ANGLES). *Suppose that $\boldsymbol{X}$ is a $d \times N$ ETF. Then the mutual absolute inner product $\mu = \mu(d, N)$ between distinct columns of $\boldsymbol{X}$ satisfies*

$$\mu = \sqrt{\frac{N-d}{d\,(N-1)}}.$$

Proof. Let $\boldsymbol{G} = \boldsymbol{X}^*\boldsymbol{X}$ be the Gram matrix of the ETF. The diagonal entries of $\boldsymbol{G}$ all equal one, while its off-diagonal entries all equal μ in absolute value. So the squared Frobenius norm of the Gram matrix is

$$\|\boldsymbol{G}\|_{\mathrm{F}}^2 = N + N\,(N-1)\,\mu^2.$$

Since $\boldsymbol{X}$ is a tight frame, its Gram matrix has exactly d nonzero eigenvalues, which all equal N/d. Thus,

$$\|\boldsymbol{G}\|_{\mathrm{F}}^2 = d\left(\frac{N}{d}\right)^2 = \frac{N^2}{d}.$$

These two expressions for the norm are evidently equal. We solve for μ to complete the argument. ▯

In fact, it is impossible to construct a sequence of unit vectors whose mutual inner products are all smaller than this value of μ.

Proposition 4 (Welch Bound). *Suppose that $\boldsymbol{X}$ is a $d \times N$ matrix with unit-norm columns. Then*

$$\max_{m \neq n} |\langle \boldsymbol{x}_m,\ \boldsymbol{x}_n \rangle| \geq \sqrt{\frac{N-d}{d\,(N-1)}}.$$

Moreover, if this bound is attained, then $\boldsymbol{X}$ is an ETF.

The first part of this result is originally due to Welch.[1] Strohmer and Heath offer a more direct argument that gives both conclusions.[3] The most insightful proof is probably due to Conway et al.[8] See also Chapter 7 of the author's dissertation.[11]

2.2. Geometric Properties

Equiangular tight frames have (at least) two gorgeous geometric characterizations. The rest of this section introduces these ideas.

2.2.1. Line packing

The $(d-1)$-dimensional complex projective space $\mathbb{P}^{d-1}(\mathbb{C})$ is the collection of all one-dimensional subspaces (i.e., lines) through the origin of $\mathbb{C}^d$. We will model the projective space by identifying each nonzero vector in $\mathbb{C}^d$ with its linear span. That is, two nonzero vectors $\boldsymbol{x}, \boldsymbol{y}$ in $\mathbb{C}^d$ are equivalent whenever $\boldsymbol{x} = \alpha\,\boldsymbol{y}$ for a complex scalar α. We form $\mathbb{P}^{d-1}(\mathbb{C})$ into a metric space with the distance function

$$\operatorname{dist}(\boldsymbol{x}, \boldsymbol{y}) = \arccos\left(\frac{|\langle \boldsymbol{x},\ \boldsymbol{y}\rangle|}{\|\boldsymbol{x}\|_2\ \|\boldsymbol{y}\|_2}\right).$$

So the distance between two lines is the acute angle between them.

An *optimal packing problem* in $\mathbb{P}^{d-1}(\mathbb{C})$ is to find N lines for which the closest pair is as far apart as possible. In symbols,

$$\max_{\boldsymbol{x}_1,\dots,\boldsymbol{x}_N \neq \boldsymbol{0}}\ \min_{m \neq n}\ \operatorname{dist}(\boldsymbol{x}_m, \boldsymbol{x}_n).$$

On account of Proposition 4, the minimum distance cannot be larger than

$$\arccos\sqrt{\frac{N-d}{d\,(N-1)}}.$$

Therefore, an ETF solves the problem of optimal packing in complex projective space. Analogous results hold for real projective space. The classical references for these ideas are due to Seidel et al.[12, 13]

2.2.2. Matrix constellations

Suppose that $\boldsymbol{X}$ is an ETF. Let us map the ETF into the space of $d \times d$ Hermitian matrices equipped with the Frobenius norm by sending each column $\boldsymbol{x}_n$ to the matrix $\boldsymbol{x}_n \otimes \boldsymbol{x}_n$. The image of the ETF is evidently a collection of positive semi-definite matrices with rank one and trace one. It is also easy to check that

$$\|\boldsymbol{x}_m \otimes \boldsymbol{x}_m - \boldsymbol{x}_n \otimes \boldsymbol{x}_n\|_{\mathrm{F}}^2 = 2 - 2\,|\langle \boldsymbol{x}_m,\ \boldsymbol{x}_n \rangle|^2.$$

So the image of the ETF is a set of equidistant matrices, i.e., the vertices of a regular simplex.

The positive semi-definite matrices form a closed, convex cone. The intersection between the hyperplane of trace-one matrices and the cone forms a compact base $\mathscr{B}$ for the cone. The extreme points of this base are precisely the rank-one, trace-one Hermitian matrices. Therefore, we may view the ETF as a regular simplex inscribed in the base $\mathscr{B}$. In fact, the converse is also true: Any regular simplex inscribed in the base $\mathscr{B}$ can be pulled back to an ETF. These results are essentially contained in the work of Conway et al.[8]

3. UPPER BOUNDS

Our first result shows that a complex ETF cannot exist unless the number N of columns is not much larger than the dimension d. This theorem is classical,[13] but the matrix-theoretic proof given here is very recent.[10]

THEOREM 5. *A $d \times N$ complex ETF can exist only when both $N \le d^2$ and $N \le (N-d)^2$.*

Proof. Suppose that $\boldsymbol{X}$ is a $d \times N$ complex ETF, and let $\boldsymbol{G} = \boldsymbol{X}^*\boldsymbol{X}$ be its Gram matrix. Recall that the Gram matrix is conjugate symmetric; it has a unit diagonal; and its off-diagonal entries have constant magnitude μ. Using $\cdot$ to denote the Hadamard (i.e., componentwise) product, we obtain

$$\boldsymbol{G} \cdot \boldsymbol{G}^T = \mu^2\,\mathbf{J} + (1-\mu^2)\,\mathbf{I}$$

where $\mathbf{J}$ is the $N \times N$ matrix of ones. The only nontrival eigenvalue of $\mathbf{J}$ equals N, and Proposition 3 furnishes the value of μ. These two facts allow us to conclude that $\boldsymbol{G} \cdot \boldsymbol{G}^T$ has (full) rank N.

Now we will bound the rank of $\boldsymbol{G} \cdot \boldsymbol{G}^T$ above using general properties of the Hadamard product. First, note that the Gram matrix of the ETF has rank d. Rank is submultiplicative with respect to the Hadamard product,[14] hence

$$\operatorname{rank}(\boldsymbol{G} \cdot \boldsymbol{G}^T) \le \operatorname{rank}(\boldsymbol{G}) \cdot \operatorname{rank}(\boldsymbol{G}^T) = d^2.$$

We have already calculated that $\boldsymbol{G} \cdot \boldsymbol{G}^T$ has rank N, and it follows that $N \le d^2$.

According to Proposition 2, there is an $(N-d) \times N$ complex ETF that is dual to $\boldsymbol{X}$. To obtain the second bound, just apply the first bound to the dual ETF. ∎

A more careful version of this argument yields a sharper bound for the real case.[10] This result states that a $d \times N$ real ETF can exist only when both $N \le \frac{1}{2} d\,(d+1)$ and $N \le \frac{1}{2}(N-d)\,(N-d+1)$.

4. HARMONIC ETFS

To develop more detailed conditions on the existence of complex ETFs, we must restrict our attention to the subcategory of *harmonic ETFs*.[3] The entries of a harmonic ETF are roots of unity, which allows us to exploit powerful methods from field theory. From the point of view of applications, harmonic ETFs may be more valuable than general ETFs because they are easier to generate. In addition, they fit better with the extensive literature on sequence design.[2]

In this section, we will also consider ETFs with harmonic Gram matrices. That is, the phases of the off-diagonal entries of the Gram matrix must be rational multiples of 2π. It is possible to develop very stringent conditions on this type of ETF.

4.1. Algebraic Background

The results in this section depend on some basic facts from field theory. Lang's textbook is a standard introduction to this material.[15] For the sake of completeness, we will review the essential definitions.

A polynomial whose coefficients are drawn from a subfield $\mathbb{F}$ of the complex numbers is referred to as a *polynomial over* $\mathbb{F}$. The complex number α is *algebraic* over $\mathbb{F}$ if it is the root of some polynomial over $\mathbb{F}$. An *algebraic integer* is the root of a monic polynomial with integer coefficients.

FACT 1. *The algebraic integers form a ring, i.e., they are closed under addition and multiplication.*

FACT 2. *The roots of a monic polynomial over the algebraic integers remain algebraic integers.*

The *minimal polynomial* of α over $\mathbb{F}$ is the (unique) lowest degree monic polynomial over $\mathbb{F}$ that contains α among its roots.

FACT 3. *A minimal polynomial over* $\mathbb{F}$ *has simple roots.*

Two numbers that have the same minimal polynomial over $\mathbb{F}$ are called *algebraic conjugates* over $\mathbb{F}$.

FACT 4. *Suppose that* α *and* β *are algebraic conjugates over* $\mathbb{F}$. *If* p *is a polynomial over* $\mathbb{F}$ *that has* α *as a root with multiplicity* m, *then* β *is also a root of* p *with multiplicity* m.

With these facts at hand, we may prove the following lemma.

LEMMA 6. *Let* $\boldsymbol{A}$ *be an Hermitian matrix whose entries are algebraic integers. Then the eigenvalues of* $\boldsymbol{A}$ *are real algebraic integers.*

In addition, assume that the entries of $\boldsymbol{A}$ *belong to a subfield* $\mathbb{F}$ *of the complex numbers. If* $\boldsymbol{A}$ *has an eigenvalue* α *whose multiplicity is different from that of the other eigenvalues, then* α *belongs to* $\mathbb{F}$.

Proof. The matrix $\boldsymbol{A}$ is Hermitian, so its eigenvalues are real numbers. By definition, an eigenvalue of $\boldsymbol{A}$ is a root of the characteristic polynomial $t \mapsto \det(t\,\mathbf{I} - \boldsymbol{A})$. Since the entries of $\boldsymbol{A}$ are algebraic integers, Fact 1 implies that the characteristic polynomial is a monic polynomial with algebraic integer coefficients. Then Fact 2 shows that the eigenvalues of $\boldsymbol{A}$ are algebraic integers.

Assume that the entries of $\boldsymbol{A}$ belong to $\mathbb{F}$. Thus, the eigenvalues of $\boldsymbol{A}$ are algebraic over $\mathbb{F}$. Since α has a different multiplicity from the other eigenvalues of $\boldsymbol{A}$, Fact 4 precludes the possibility that α might have any algebraic conjugates over $\mathbb{F}$. Applying Fact 3, we see that the minimal polynomial of α over $\mathbb{F}$ is linear. Thus, α belongs to $\mathbb{F}$. □

This type of field-theoretic argument appears frequently in the analysis of integer matrices. A similar technique was used by Lemmens and Seidel in their study of equiangular lines.[13]

We also require some fundamentals about cyclotomic fields.[16] Note that $\mathbb{Q}(\omega)$ denotes the smallest field extending $\mathbb{Q}$ that contains ω, while $\mathbb{Z}[\omega]$ is the smallest ring extending $\mathbb{Z}$ that contains ω.

FACT 5. *Suppose that* ζ_p *is a primitive* p*-th root of unity. The ring of algebraic integers in the field* $\mathbb{Q}(\zeta_p)$ *coincides with the ring* $\mathbb{Z}[\zeta_p]$.

FACT 6. *The set of real algebraic integers in* $\mathbb{Z}[\zeta_p]$ *coincides with the ring* $\mathbb{Z}[2\,\mathrm{Re}\,\zeta_p]$.

4.2. Harmonic ETFs

We begin with a formal definition of a harmonic ETF. Choose a primitive p-th root of unity ζ_p, and suppose that $\boldsymbol{X}$ is a $d \times N$ ETF of the form

$$\boldsymbol{X} = \frac{1}{\sqrt{d}}\,\boldsymbol{S} \tag{1}$$

where the entries of $\boldsymbol{S}$ are drawn from the set $\{1, \zeta_p, \zeta_p^2, \dots, \zeta_p^{p-1}\}$.

The class of harmonic ETFs includes both real and complex Hadamard matrices as special cases. Strohmer and Heath[3] have shown how to build other types of harmonic ETFs by exploiting methods of König.[17] To be precise, choose $d = q^\ell + 1$, where q is prime and ℓ is a natural number, and set $N = d^2 - d + 1$. Then there exists a $d \times N$ harmonic ETF over the N-th roots of unity.

Our contribution is to develop a simple requirement on the admissible values of d and N. The result relies on general properties of ETFs and the arithmetic closure of a number ring.

THEOREM 7 (HARMONIC ETFS). *Let $\boldsymbol{S}$ be a $d \times N$ matrix whose entries are p-th roots of unity, and assume that $\boldsymbol{S}$ gives rise to a harmonic ETF via* (1). *Then the ring $\mathbb{Z}[\zeta_p]$ must contain a complex number z such that*

$$|z| = \sqrt{\frac{d\,(N-d)}{N-1}}.$$

In particular, the ring $\mathbb{Z}[2\,\mathrm{Re}\,\zeta_p]$ must contain the number $d\,(N-d)/(N-1)$.

Note that the following argument applies to every ETF whose entries belong to $\mathbb{Z}[\zeta_p]$.

Proof. Consider the Gram matrix $\boldsymbol{A} = \boldsymbol{S}^*\boldsymbol{S}$. Since the ring $\mathbb{Z}[\zeta_p]$ is closed under complex conjugation, the entries of $\boldsymbol{A}$ are all elements of $\mathbb{Z}[\zeta_p]$. On the other hand, since $\boldsymbol{S}/\sqrt{d}$ is an ETF, the off-diagonal entries of $\boldsymbol{A}$ must have magnitude $\mu\, d$, where the value of μ is given by Proposition 3. These observations lead to a simple compatibility condition. The ring $\mathbb{Z}[\zeta_p]$ must contain a number with magnitude $\mu\, d$, which is the content of the theorem's first conclusion.

It is possible to develop a weaker condition that, perhaps, is more useful. Suppose that $|z| = \mu\, d$ and $z \in \mathbb{Z}[\zeta_p]$. Since the ring is closed under complex conjugation, the real number $|z|^2 = z\bar{z}$ must also be an element of $\mathbb{Z}[\zeta_p]$. According to Fact 6, the set of real numbers in $\mathbb{Z}[\zeta_p]$ is precisely $\mathbb{Z}[2\ \mathrm{Re}\,\zeta_p]$. ∎

It seems necessary to check the conditions in Theorem 7 directly, since we do not know any simple characterization of the magnitudes that appear in $\mathbb{Z}[\zeta_p]$. For small values of p, this exercise leads to a nice corollary. To develop conditions for higher values of p, it may help to know some properties of trigonometric algebraic numbers[18] and cyclotomic fields.[16]

COROLLARY 8. *Suppose that $\boldsymbol{S}$ is a $d \times N$ matrix whose entries are p-th roots of unity, and assume that it gives rise to an ETF of the form* (1). *Write $\gamma = d\,(N-d)/(N-1)$. Then we have the following conditions.*

$$\begin{aligned}
\textit{When}\quad p=2: &\qquad \sqrt{\gamma} \in \mathbb{Z}.\\
p=3: &\qquad \gamma = a^2+ab+b^2 \quad \textit{for some } a,b\in\mathbb{Z}.\\
p=4: &\qquad \gamma = a^2+b^2 \quad \textit{for some } a,b\in\mathbb{Z}.\\
p=5: &\qquad \gamma\in\mathbb{Z}.\\
p=6: &\qquad \gamma\in\mathbb{Z}.\\
p=8: &\qquad \gamma\in\mathbb{Z}.
\end{aligned}$$

Observe that the case $p = 2$ yields an ETF with ± 1 entries, which might be viewed as a generalized Hadamard matrix. The cases $p = 3, 4$ lead to ETFs over the Eisenstein and Gaussian units. Meanwhile, $p = 8$ leads to an 8-PSK constellation.

For the case $p = 4$, the number γ must be expressible as a sum of two squares. A result of Euler states that this is possible if and only if each prime factor of γ with the form $(4k + 3)$ occurs with an even power.

Proof. When $p = 2$, it is clear that $\mathbb{Z}[\zeta_2] = \mathbb{Z}$. The claim follows immediately from the first conclusion of Theorem 7.

When $p = 3$, the elements of the set $\mathbb{Z}[\zeta_3]$ have the form $a + b\,\zeta_3$, where a, b are integers. The magnitude of this number is $|a + b\,\zeta_3| = \sqrt{a^2 + ab + b^2}$. Apply the first part of Theorem 7. The case $p = 4$ is similar.

The case $p = 5$ is a bit harder. Abbreviate $\omega = 2\,\mathrm{Re}\,\zeta_5$. A short calculation establishes that ω solves the quadratic equation $t^2 + t - 1 = 0$ and that $\omega = \frac{1}{2}(\sqrt{5} - 1)$. It follows that each number in $\mathbb{Z}[\omega]$ can be written as a $\mathbb{Z}$-linear combination of 1 and ω. (Use the relation $\omega^2 = 1 - \omega$ to reduce higher powers of ω.) By the second conclusion of Theorem 7, we must have $\gamma = a + b\,\omega$ for some integers a, b. But γ is rational, so $b = 0$. The cases $p = 6, 8$ follow the same pattern. ∎

4.3. Examples

To demonstrate what harmonic ETFs look like, we offer a selection of examples. These matrices were provided by M. Sustik.

A *Hadamard matrix* is a $d \times d$ matrix $\boldsymbol{A}$ whose entries are ± 1 and which satisfies $\boldsymbol{A}^* \boldsymbol{A} = d\,\mathbf{I}$. We see that a Hadamard matrix gives rise to a $d \times d$ harmonic ETF with $p = 2$ via the formula (1). Here are the signs of the entries in the (unique) 4×4 Hadamard matrix:

$$\boldsymbol{S} = \begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & - & + & - \\ + & - & - & + \end{bmatrix}.$$

Hadamard matrices can exist only when $d = 1, 2$ or when d is a multiple of four. A long-standing conjecture in coding theory states that Hadamard matrices exist *whenever* d is a multiple of four.[19]

It is also possible to construct rectangular harmonic ETFs over ± 1. Here is a 6×16 example:

$$\boldsymbol{S} = \begin{bmatrix} + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + \\ + & + & + & + & + & + & + & + & - & - & - & - & - & - & - & - \\ + & + & + & + & - & - & - & - & + & + & + & + & - & - & - & - \\ + & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - \\ + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - \\ + & - & - & + & - & + & + & - & - & + & + & - & + & - & - & + \end{bmatrix}.$$

It is possible to extend this matrix to a 16×16 Hadamard matrix, which can be verified by inspection.[20] Craigen has developed general results on extending ± 1 matrices to Hadamard matrices.[21]

A *complex Hadamard matrix* is a $d \times d$ matrix $\boldsymbol{A}$ whose entries are $\pm 1, \pm \mathrm{i}$ and which satisfies $\boldsymbol{A}^* \boldsymbol{A} = d\,\mathbf{I}$. In other words, a complex Hadamard matrix yields a harmonic ETF with $p = 4$ and $N = d$. Here is a 6×6 example.

$$\boldsymbol{S} = \begin{bmatrix} +1 & +1 & +1 & +1 & +1 & +1 \\ +1 & +1 & +\mathrm{i} & -1 & -1 & -\mathrm{i} \\ +1 & +\mathrm{i} & -\mathrm{i} & +\mathrm{i} & -\mathrm{i} & -1 \\ +1 & -1 & +1 & -\mathrm{i} & -1 & +\mathrm{i} \\ +1 & -1 & -1 & +1 & +\mathrm{i} & -\mathrm{i} \\ +1 & -\mathrm{i} & -1 & -1 & +1 & +\mathrm{i} \end{bmatrix}.$$

Note that there is no real Hadamard matrix of this order.

It is natural to extend the idea of complex Hadamard matrices beyond the Gaussian integers. Suppose that $\boldsymbol{A}$ is a $d \times d$ matrix whose entries are $\{1, \zeta_p, \dots, \zeta_p^{p-1}\}$ and which satisfies $\boldsymbol{A}^* \boldsymbol{A} = d\,\mathbf{I}$. These matrices also lead to square harmonic ETFs. Here are two examples of these Hadamard-like matrices with $p = 3$:

$$\boldsymbol{S} = \exp \cdot \frac{2\pi \mathrm{i}}{3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \qquad \boldsymbol{S} = \exp \cdot \frac{2\pi \mathrm{i}}{3} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 1 & 2 & 0 & 2 & 1 \\ 0 & 2 & 1 & 2 & 0 & 1 \\ 0 & 2 & 2 & 1 & 1 & 0 \end{bmatrix}.$$

The notation $\exp \cdot$ indicates the componentwise exponential of a matrix, so the matrix displays the powers of the roots of unity. Finally, we offer an example of a 5×5 Hadamard-like matrix with $p = 5$:

$$\boldsymbol{S} = \exp \cdot \frac{2\pi \mathrm{i}}{5} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 1 & 3 \\ 0 & 3 & 1 & 4 & 2 \\ 0 & 4 & 3 & 2 & 1 \end{bmatrix}.$$

It is also possible to construct rectangular harmonic ETFs. Here are two examples with $p = 7$:

$$\boldsymbol{S} = \exp \cdot \frac{2\pi \mathrm{i}}{7} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 3 & 6 & 2 & 5 & 1 & 4 \end{bmatrix} \qquad \boldsymbol{S} = \exp \cdot \frac{2\pi \mathrm{i}}{7} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 2 & 4 & 6 & 1 & 3 & 5 \\ 0 & 4 & 1 & 5 & 2 & 6 & 3 \end{bmatrix}.$$

The first ETF can be constructed using König's approach,[3,17] but the second one cannot. The latter two examples are also intriguing because there exist no real ETFs with the same dimensions.[10] We conclude by displaying a 4×13 harmonic ETF with $p = 13$.

$$\boldsymbol{S} = \exp \cdot \frac{2\pi \mathrm{i}}{13} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 0 & 3 & 6 & 9 & 12 & 2 & 5 & 8 & 11 & 1 & 4 & 7 & 10 \\ 0 & 9 & 5 & 1 & 10 & 6 & 2 & 11 & 7 & 3 & 12 & 8 & 4 \end{bmatrix}.$$

The latter ETF can also be constructed with König's technique.[3,17]

4.4. ETFs with Harmonic Gram Matrices

The Gram matrix of a real ETF has (at most) two distinct off-diagonal entries: $+\mu$ and $-\mu$. As a generalization, one might also study ETFs whose Gram matrices contain (scaled) roots of unity. We will say that these ETFs have *harmonic Gram matrices*. Let us demonstrate that the dimensions of these ETFs must satisfy very stringent requirements. This discussion follows the technical report.[10]

Suppose that $\boldsymbol{X}$ is a $d \times N$ ETF, and let μ have the value stated in Proposition 3. Suppose that the matrix

$$\boldsymbol{A} = \frac{1}{\mu}(\boldsymbol{X}^* \boldsymbol{X} - \mathbf{I}) \tag{2}$$

contains p-th roots of unity in its off-diagonal entries. (The diagonal, of course, is zero.) An example of this type of ETF occurs when $d = 3$, $N = 9$, and $p = 6$:

$$\boldsymbol{A} = \exp \cdot \frac{2\pi \mathrm{i}}{6} \begin{bmatrix} \star & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \star & 5 & 5 & 5 & 3 & 1 & 1 & 1 \\ 0 & 1 & \star & 5 & 1 & 5 & 5 & 3 & 1 \\ 0 & 1 & 1 & \star & 5 & 5 & 1 & 5 & 3 \\ 0 & 1 & 5 & 1 & \star & 5 & 3 & 1 & 5 \\ 0 & 3 & 1 & 1 & 1 & \star & 5 & 5 & 5 \\ 0 & 5 & 1 & 5 & 3 & 1 & \star & 5 & 1 \\ 0 & 5 & 3 & 1 & 5 & 1 & 1 & \star & 5 \\ 0 & 5 & 5 & 3 & 1 & 1 & 5 & 1 & \star \end{bmatrix}.$$

The stars serve as a reminder that the diagonal of $\boldsymbol{A}$ is zero. This example was also provided by M. Sustik.

Using Lemma 6, it is not hard to develop necessary conditions on ETFs with harmonic Gram matrices.

Theorem 9. *Assume that $N \neq 2d$. Let $\boldsymbol{A}$ be a hollow $N \times N$ matrix whose off-diagonal entries are p-th roots of unity, and suppose that $\boldsymbol{A}$ arises from a $d \times N$ ETF via* (2). *Then*

$$\sqrt{\frac{(N-1)(N-d)}{d}} \in \mathbb{Z}[2 \operatorname{Re} \zeta_p] \qquad \textit{and} \qquad \sqrt{\frac{d\,(N-1)}{N-d}} \in \mathbb{Z}[2 \operatorname{Re} \zeta_p].$$

In case $N = 2d$, it seems necessary to invoke deeper methods, such as the Bruck–Ryser–Chowla Theorem.

Proof. Since $\boldsymbol{X}$ is an ETF, the eigenvalues of $\boldsymbol{X}^*\boldsymbol{X}$ are (N/d) with multiplicity d and zero with multiplicity $(N-d)$. It follows that the two eigenvalues of $\boldsymbol{A}$ are

$$\lambda_1 = \frac{N-d}{d\,\mu} \qquad \text{and} \qquad \lambda_2 = -\frac{1}{\mu}$$

with respective multiplicities d and $(N-d)$. Substituting the value of μ yields

$$\lambda_1 = \sqrt{\frac{(N-1)(N-d)}{d}} \qquad \text{and} \qquad \lambda_2 = -\sqrt{\frac{d\,(N-1)}{N-d}}.$$

Since $N \neq 2d$, the two eigenvalues have different multiplicities.

By construction, the matrix $\boldsymbol{A}$ is Hermitian. It has a zero diagonal, and its off-diagonal entries are all p-th roots of unity. Therefore, the entries of $\boldsymbol{A}$ are algebraic integers in the field $\mathbb{Q}(\zeta_p)$. Lemma 6 implies that the eigenvalues of $\boldsymbol{A}$ are real algebraic integers in the field $\mathbb{Q}(\zeta_p)$. Facts 5 and 6 show that the set of real algebraic integers in $\mathbb{Q}(\zeta_p)$ coincides with the ring $\mathbb{Z}[2\,\mathrm{Re}\,\zeta_p]$. The theorem follows. ▯

As before, we can draw a corollary that outlines the situation when p is small.

Corollary 10. *Assume that $N \neq 2d$. Let $\boldsymbol{A}$ be a hollow $N \times N$ matrix whose off-diagonal entries are p-th roots of unity, and suppose that $\boldsymbol{A}$ arises from a $d \times N$ ETF via (2). Let $\beta_1 = \sqrt{(N-1)(N-d)/d}$ and $\beta_2 = \sqrt{d\,(N-1)/(N-d)}$. Then the following conditions are in force.*

When
$$\begin{aligned}
p = 2: &\quad \beta_1, \beta_2 \in \mathbb{Z}.\\
p = 3: &\quad \beta_1, \beta_2 \in \tfrac{1}{2}\mathbb{Z}.\\
p = 4: &\quad \beta_1, \beta_2 \in \mathbb{Z}.\\
p = 5: &\quad \beta_1, \beta_2 \in \mathbb{Z} \cup \tfrac{1}{2}(\sqrt{5}-1)\mathbb{Z}.\\
p = 6: &\quad \beta_1, \beta_2 \in \tfrac{1}{2}\mathbb{Z}.\\
p = 8: &\quad \beta_1, \beta_2 \in \mathbb{Z} \cup \sqrt{2}\,\mathbb{Z}.
\end{aligned}$$

We omit the proof, which is similar to that of Corollary 8. Note that when $p = 2$, the corollary refers to the case of real ETFs. Stronger results are possible in this setting.[10]

5. OPEN PROBLEMS

We conclude this paper with several important open problems. First, we recall the question that motivated this paper.

Open Question 1. *Find conditions on (d, N) that rule out the existence of general complex ETFs.*

In the other direction, constructions of ETFs are always interesting and valuable.

Open Question 2. *Develop constructions of (harmonic) ETFs for pairs (d, N) where they exist.*

Every real ETF can be constructed from a strongly regular graph.[4, 10] It is not known whether a similar process is possible in the complex case.

Open Question 3. *Are complex ETFs equivalent to some type of graph or combinatorial object?*

Finally, the most vexing challenge concerns the existence of *maximal complex ETFs*. The upper bound of Theorem 5 shows that a complex ETF can exist only when $N \leq d^2$. Meanwhile, numerical experiments suggest that for each dimension d there exists a complex ETF containing the maximal number of vectors.[22]

Open Question 4. *For each natural number d, does there exist a complex ETF containing $N = d^2$ vectors? How can it be constructed?*

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