Multiscale Computation of Isotropic Homogeneous Turbulent Flow

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Abstract. In this article we perform a systematic multi-scale analysis and computation for incompressible Euler equations and Navier-Stokes Equations in both 2D and 3D. The initial condition for velocity field has multiple length scales. By reparameterizing them in the Fourier space, we can formally organize the initial condition into two scales with the fast scale component being periodic. By making an appropriate multiscale expansion for the velocity field, we show that the two-scale structure is preserved dynamically. Moreover, we derive a well-posed homogenized equation for the incompressible Euler equations in the Eulerian formulations. Numerical experiments are presented to demonstrate that the homogenized equations indeed capture the correct averaged solution of the incompressible Euler and Navier Stokes equations. Moreover, our multiscale analysis reveals some interesting structure for the Reynolds stress terms, which provides a theoretical base for establishing an effective LES type of model for incompressible fluid flows.

1. Numerical Experiments

In previous sections we perform multiscale analysis on flows that the initial velocity field has two scales. However, practical fluid flows have infinitely non-separable scales. We have developed a scheme to transform a general initial condition with many non-separable scales into the form of a two-scale function so that we can use the two-scale analysis developed earlier to study more general multiscale problems [?]. In this section, we will perform detailed numerical experiments to check the accuracy of the multiscale analysis in both two and three space dimensions.

1.1. Numerical Experiments for the 2D Navier-Stokes Equations. We test the multiscale model for 2-D incompressible flow in a doubly periodic box of size $2\pi \times 2\pi$. For the homogenized problem, we solve the homogenized Euler equations given in Section 4.1 of [?].
\( (a) \quad \bar{\partial}_t \bar{u} + (\bar{u} \cdot \nabla_x) \bar{u} + \nabla_x \bar{p} + \nabla_x \cdot (w \otimes w) = \nu \Delta \bar{u}, \)

\[(1.1)\]

\( (b) \quad \nabla_x \cdot \bar{u} = 0, \)

\( (c) \quad \bar{u}_{|t=0} = U, \)

where \( \nu \) is viscosity. Since \( \nu \) is very small, a minor modification to the multiscale analysis can be made to accommodate the small viscous effect in the homogenized equations. We choose \( \nu = 10^{-4} \) in our computations. Note that the viscous term only affects the mean velocity equation, and the cell problem is still inviscid.

For the cell problem, we solve the simplified equations,

\( (a) \quad -\nabla_z^2 \cdot \left( D_x \theta^{-T} D_x \theta^{-1} \nabla_z^1 \psi^{(1)} \right) = \sigma_0(\theta, z + \theta^{(1)}), \)

\( (b) \quad \sigma_0(x, z) = \nabla_z^1 \cdot W(x, z). \)

where \( z \) is given by equation

\( (1.3) \quad z = \theta / \epsilon \)

and \( \theta^{(1)} \) is given by

\( (a) \quad \partial_z \theta^{(1)} - (I + D_z \theta^{(1)}) \nabla_z^1 \psi^{(1)} = 0, \)

\( (b) \quad \theta^{(1)}_{|t=0} = 0. \)

The leading order oscillating velocity field, \( w \), is given by:

\( (1.5) \quad w = -B \nabla_z^1 \psi^{(1)}, \)

where \( B = (-\nabla^1 \theta_2, \nabla^1 \theta_1) = D \theta^{-1} \) and \( \theta \) satisfies

\( (a) \quad \bar{\partial}_t \theta + (\bar{u} \cdot \nabla_x) \theta = 0, \)

\( (b) \quad \theta_{|t=0} = x. \)

The pseudo-spectral method is used to solve both the large-scale homogenized equations and the small-scale cell problem. We use the second-order Runge-Kutta method to discretize the equation in time. In our computations, we solve the homogenized equation using a coarse grid and large time step, \( \Delta t \). For each coarse grid point within the time interval from \( t_n \) to \( t_{n+1} \) with \( t_n = n \Delta t \), we solve the cell problem with the periodic boundary condition in the \( z \) variable from \( \tau = t_n / \epsilon \) to \( \tau = t_{n+1} / \epsilon \) using a subgrid \( \Delta z \) and a subgrid time step \( \Delta \tau \). We then average the cell solution from \( \tau = t_n / \epsilon \) to \( \tau = t_{n+1} / \epsilon \) to evaluate the Reynolds stress term and update the mean velocity at \( t_{n+1} \). To eliminate the aliasing error in the pseudo-spectral method, we use a 15th order Fourier smoothing function to damp the high frequency modes.

Below we present numerical results on the decay of 2-D homogeneous turbulence, and compare the homogenization model with well resolved direct numerical simulation (DNS). The DNS uses a 512 \times 512 fine grid and the same viscosity as the homogenized equation. The simulation starts with random initial condition, where the initial distribution of the stream function in the Fourier space is,

\( (1.7) \quad |\hat{\psi}(k)| = \frac{k}{k^4 + \delta}, \quad k = |k|, \)

with random phases. This choice of initial velocity field is similar to the earlier work of Henshaw-Kreiss-Reyna [?]. In the computation, we choose \( \delta = 10^{-5} \). The
initial vorticity distribution is plotted in Figure 1. At $t = 5.0$, coherent vortices emerge from the random initial condition, which is denoted as 'vortex generation period'. At later stages, the flow is dominated by the mutual interactions of coherent vortices. The number of vortices decreases and the averaged vortex radius and circulation increase.

Since our multiscale analysis is developed for well-mixing flows, we start our multiscale computation using the homogenized equations starting at $t = 5.0$ when the flow completes the 'vortex generation period'. We use the technique presented in Section 5 to prepare the velocity obtained from the DNS at $t = 5.0$ in the form of a two-scale initial condition so that we can apply the homogenized equations (1.1) to solve the multiscale problem. The dimension of the coarse grid (slow variables) is $64 \times 64$ and that of small scale (fast variables) is $32 \times 32$. The time step for the homogenized equation is $\Delta t = 0.01$ and the subgrid time step for the cell problem is $\Delta \tau = 0.01$. Figure 2 compares the vorticity distribution of the homogenization result with DNS result at $t = 5.0$, 10.0 and 20.0. The plots show the stretching of vortex filament by the mean flow. In addition, the size of the vortices grows due to the merger of vorticity of the same sign, which is one of the mechanisms that contributes to the inverse energy cascade. The vorticity distribution from the homogenization is in excellent agreement with the DNS, suggesting that the multiscale model captures the vortex interactions at both large scales and small scales.

The accuracy of the solution can be examined by studying whether the solution dissipates the correct amount of energy and enstrophy, which are two statistical global quantities of turbulent flows. It is known that the temporal evolution of the
Mean kinetic energy and the mean enstrophy is governed by the following equations \[ \frac{d}{dt} \left( \frac{1}{2} |\mathbf{u}|^2 \right) = -\nu \langle \omega^2 \rangle, \tag{1.8} \]
\[ \frac{d}{dt} \left( \frac{1}{2} \omega^2 \right) = -\nu \langle \nabla \omega^2 \rangle. \tag{1.9} \]

From equation (1.9), we can see that the enstrophy is bounded by its initial value. Therefore in the limit of \( \nu \to 0 \), \( d\langle |\mathbf{u}|^2 \rangle/dt \to 0 \), i.e. the total kinetic energy is conserved. On the other hand, there exists a cascade of enstrophy from large scale to small scale \([?, ?]\), which causes the total enstrophy to decay in time. Specifically, vorticity gradients are amplified with the formation of thin filaments. These filaments are stretched until they reach the very small dissipation scales, so that the enstrophy and all positive-order vorticity moments decay. In Figure 3, we show the temporal evolution of the total kinetic energy and the total enstrophy using three different approaches, which are (i) the DNS, (ii) the simulation using the homogenized equations, and (iii) the simulation of the homogenized equations ignoring the Reynolds stress term. With very small viscosity, the energy decay from all 3 simulations is negligible. On the other hand, enstrophy decays continuously, with maximum decay rate during the initial vortex formation period. The decay rate becomes smaller during the vortex merger stage. We can see that the simulation with coarse grid using approach (iii) leads to much slower enstrophy decay because it could not capture the enstrophy cascade from large scales to the dissipation scale. On the other hand, the enstrophy decay rate of the simulation using the homogenized equations is very close to that of the DNS, suggesting that the dissipation mechanism is well resolved within each cell.

We also compare the spectra between the DNS and the simulation using the homogenized equations at \( t = 20.0 \). We reconstruct the fine grid velocity field by
\[ \mathbf{u}(t, \mathbf{x}) = \mathbf{u}(t, \mathbf{x}) + \mathbf{w}(t, \mathbf{x}, \theta(t, \mathbf{x}), \tau, \frac{\theta(t, \mathbf{x})}{\epsilon}), \tag{1.10} \]
where the fine grid phase function is obtained using the spectral interpolation. The agreement is very good at low wavenumbers. At high wavenumbers, the DNS spectra decay faster than the spectra of the simulation using the homogenized equations. This explains the difference in the enstrophy decay rate between the two simulations. The difference is partly due to neglecting the higher order terms in the homogenized equations and the cell problem. Another more important reason is that there is viscous dissipation in all the scales in the DNS, whereas we neglect viscosity in the cell problem in the homogenized equations. Nonetheless, the model accurately captures the dynamics of the large scale, as well as the averaged effect from the small scales.

1.2. Numerical Experiments for the 3D Navier-Stokes Equations. In this subsection, we perform careful direct numerical simulations (DNS) of the 3D Navier-Stokes equations to validate the multiscale analysis presented in the previous sections. Given the number of grid points and the size of the computational domain, the smallest resolved length scale or equivalently the largest wave number, \( k_{\max} \), is prescribed. In a three-dimensional turbulent flow the kinetic energy cascades in time to smaller, more dissipative scales. The scale at which viscous dissipation becomes dominant, and which represents the smallest scales of turbulence, is characterized
by the Kolmogorov length scale $\eta$. In a fully resolved DNS, the condition $k_{max}\eta < 1$ is necessary for the small scales to be adequately represented. Consequently, $k_{max}$ limits the highest achievable Reynolds number in a direct numerical simulation for a given computational box.

There are three main characteristic length scales in an isotropic turbulent flow: the integral scale $I$ characterizing the energy containing scales is defined as

$$I = \frac{3\pi}{4} \int_0^{k_{max}} \frac{E(k)}{k} dk$$

where $E(k)$ is the energy spectrum function at the scalar wave number $k$; the Kolmogorov microscale $\eta$ representative of dissipative scale is

$$\eta = \left( \frac{\nu^3}{\epsilon} \right)^{1/4}$$

where $\epsilon$ is the volume averaged energy-dissipation rate; and the Taylor microscale $\lambda$ characterizing the mixed energy dissipation scales is defined as

$$\lambda = \sqrt{\frac{u_{rms}}{\langle \partial u_1/\partial x_1 \rangle^2}}$$

where $u_{rms}$ is the root mean square value of each component of velocity defined as

$$u_{rms}^2 = \frac{2}{3} \int_0^{k_{max}} E(k) dk$$

The time scale of the energy-containing eddies is the large-eddy-turnover time $T$ defined as $T = I/u_{rms}$. The Taylor Reynolds number is

$$Re = \frac{u_{rms}\lambda}{\nu}$$

Forced isotropic turbulence in a periodic box can be considered as one of the most basic numerically simulated turbulent flows. Forced isotropic turbulence is achieved by applying isotropic forcing to the low wave number modes so that the turbulent cascade develops as the statistical equilibrium is reached. Statistical equilibrium is signified by the balance between the input of kinetic energy through the forcing and its output through the viscous dissipation. Isotropic forcing cannot be produced in a laboratory and therefore forced isotropic turbulence is an idealized flow configuration that can be achieved only via a controlled numerical experiment; nevertheless, forced isotropic turbulence represents an important test case for studying basic properties of turbulence in a statistical equilibrium.

In the statistically stationary state, the average rate of energy addition to the velocity field is equal to the average energy-dissipation rate. The Reynolds number attainable for a given size of simulation is substantially higher for forced turbulence than for the case of decaying turbulence.

We apply forcing over a spherical shell with shell walls of unit width centered at wave number one, such that the total energy injection rate is constant in time. This forcing procedure was used by (Misra and Pullin). The forcing amplitude is adjustable via the parameter $\delta$ while the phase of forcing is identical to that of the velocity components at the corresponding wave vectors. The Fourier coefficient of
the forcing term is written as

\[ \hat{f} = \frac{\delta}{N} \frac{\hat{u}}{\sqrt{u_k u_k}} \]

Here, \( \hat{f} \) and \( \hat{u} \) are the Fourier transforms of the forcing vector and velocity, respectively, \( N \) is the number of wave modes that are forced. The above form of forcing ensures that the energy injection rate, \( \sum \hat{f} \cdot \hat{u} \), is a constant which is equal to \( \delta \). We chose \( \delta = 0.1 \) for all of our runs. The forcing is added at the large scales (\( |k| \) between 1 and 2).

The DNS is performed using a 512³ mesh in a periodic 3D cube of sides \( L = 2\pi \). We solve the incompressible Navier-Stokes equation for the velocity field, using a pseudospectral code. A second-order explicit Runge-Kutta scheme is used for time-marching. To eliminate the aliasing error in the pseudo-spectral method, we use a 36th order Fourier smoothing function to damp the high frequency modes. Compared with the 2/3 dealiasing method, carefully designed Fourier smoothing scheme can resolve higher wavenumber with the same grid mesh (Li and Hou). The initial condition is a random field with a spectrum peaked around \( |k| = 30 \).

1.2.1. DNS Results. To present the results in a nondimensional form, we use the integral length scale \( I \) and the root mean square of velocity \( u_{rms} \). Throughout the forced simulations, these two quantities vary significantly; however, as equilibrium is approached, the integral length scale for the simulation approaches the value of approximately \( I \approx 0.56 \), and the root mean square of velocity is \( u_{rms} = 0.61 \). It follows, that the corresponding eddy-turnover time is \( T = I/u_{rms} = 0.91 \). The computations are continued for more than 30 eddy-turnover times. The plot of the total kinetic energy, and energy spectrum indicates that after 10 eddy-turnover time, the flow reach equilibrium state. At this stage, the equilibrium Taylor Reynolds number in this simulation is approximately 223, and the Reynolds number based on the integral length scale is 1056. The equivalent viscosity is 0.0005. Figure 6 shows the isosurface of the magnitude of the vorticity at equilibrium (\( t = 30 \)), at a value two standard deviations above the mean value. The intense vorticity is organized into tube-like structures.

1.2.2. The eddy viscosity model. Following the upscaling formulation derived in the previous sections. We can solve the homogenized Navier-Stokes equations,

\[ \begin{align*}
(a) & \quad \partial_t \theta + (\mathbf{u} \cdot \nabla_x) \theta + \nabla_q + \nabla_x \cdot (\mathbf{w} \otimes \mathbf{w}) = \nu \nabla \mathbf{u} + \mathbf{f}, \\
(b) & \quad \nabla \cdot \mathbf{u} = 0, \\
(c) & \quad \mathbf{u}|_{t=0} = \mathbf{U}(x),
\end{align*} \]

where \( \mathbf{w} \otimes \mathbf{w} = \mathbf{w} \mathbf{w}^\top \), and \( \nu \) is viscosity. At the mean time, we solve the averaged equations for the inverse of the Lagrangian map:

\[ \begin{align*}
(a) & \quad \partial_t \mathbf{x} + (\mathbf{u} \cdot \nabla_x) \mathbf{x} = 0, \\
(b) & \quad \mathbf{x}|_{t=0} = \mathbf{x}.
\end{align*} \]

For the cell equations we solve

\[ \begin{align*}
(a) & \quad \partial_t \mathbf{w} + (D_x \mathbf{w} \cdot \nabla_x) \mathbf{w} + D_x \mathbf{w}^\top \nabla_x q - \nu' \nabla_z \cdot (D_x \theta D_x \mathbf{w}^\top \nabla_z \mathbf{w}) = 0, \\
(b) & \quad (D_x \mathbf{w}^\top \nabla_z) \cdot \mathbf{w} = 0, \\
(c) & \quad \mathbf{w}|_{\tau=t=0} = \mathbf{W}(x, x/\epsilon),
\end{align*} \]
where $\nu' = \nu/\epsilon$ is the cell viscosity.

Since we only force the low wavenumbers, the forcing term does not enter the cell equations, which govern the flow of subgrid scales. The energy cascade between the resolved scales and the subgrid scales is caused by the convection terms. In the homogenized equation, it is through the subgrid stress term $(w \otimes w)$. The net effect of this term is to dissipate the energy in the large scale. In the cell equation, it is through the term $D_z w D_x \theta w$. The net effect of this term is to input energy into the small scales, which eventually is dissipated by the viscous effect.

The multiscale analysis provided a homogenization scheme to calculate the Reynolds stress terms from the cell equation. However, for 3D problem it is impractical to solve the cell equation and take average on each cell. Instead, we need to design numerical schemes that utilize the homogeneous property of turbulence, i.e., schemes that capture the large scale effect of the Reynolds stress from solving cell equations on limited number of cells. The homogenization scheme can reach the same accuracy as the DNS if we solve cell equations on each grid point. As the number of cells to solve is reduced, the order of accuracy decreases. As the first order approximation, cell equations are solved to obtain one coefficient to be used in the averaged equation. In this case, the coefficient to be calculated is the dynamics eddy viscosity, when we assume that the Reynolds stress tensors is aligned with the local rate of strain tensor of the filtered scale.

Let $B = (w \otimes w)$, which is the sub-grid stress term (SGS). If we assume that deviatoric part of the SGS tensor is locally aligned with the rate of strain tensor $D$ of the mean flow, where $D = \frac{1}{2}(\nabla u + \nabla u^T)$, i.e.

\begin{equation}
B = \frac{1}{3}kI - 2\nu_k D
\end{equation}

where $k = \text{tr}(B)$ is the SGS kinetic energy, and $\nu_k$ the eddy viscosity. The Smagorinsky model can be derived from the $k^{-5/3}$ spectra, and gives

\begin{equation}
k = c_I \Delta^2 ||D||^2 \quad \nu_k = c_D \Delta^2 ||D||,
\end{equation}

For isotropic turbulence, $c_I = 0.404$, $c_D = 0.042$.

The weakness of the Smagorinsky model is that the effect of the sub-grid flow on the mean flow is always dissipative, and is uniform both in time and space. To release these constraints, the dynamic model of Germano filters the governing equation a second time. By minimizing the residual error throughout the domain, $c_I$ and $c_D$ can be determined dynamically. They vary in time, and can be negative, but are uniform in space. Numerical simulation shows that at large Reynolds number, $c_I$ and $c_D$ become close to 0.4 and 0.04.

Use the two scale model, we can also design a dynamic model to determine the eddy viscosity. $c_I$ can be determined from

\begin{equation}
k = c_I \Delta^2 ||D||^2 = \text{tr}(B)
\end{equation}

and $c_D$ can be determined by minimizing the error term locally

\begin{equation}
e = ||B - \frac{1}{3}kI + 2c_D \Delta^2 |D|D||
\end{equation}
which leads to

\begin{equation}
    c_D = -\frac{B_D \otimes D}{2\Delta^2||D||^2}
\end{equation}

where \(B_D = B - \frac{1}{4}\text{tr}(B)I\), and \(B = \langle w \otimes w \rangle\) is obtained by solving the cell equations. Thus \(c_I\) and \(c_D\) can vary both in time and in space. In the current case of homogeneous isotropic turbulence, we assume that the eddy viscosity is uniform in space. We solve the cell equation at 4 grid points on each dimension, i.e., total of 64 cells. For each cell we solve on a grid of \(32^3\). The averaged equation is solved with grid number of \(64^3\). On each cell we calculate the \(c_I\) and \(c_D\) with equation (1.22) and (1.24). The overall eddy viscosity is taken to be the average of all cells, and is plug into the filtered equation. This process repeat at each large time step.

Figure 7 plot the evolution of \(c_D\). Since the turbulent flow is isotropic and homogeneous, the value of \(C_d\) relaxs to the Smagorinsky’s value of 0.4. Figure 8 plot the comparison of energy spectrum at \(t = 30.0\). An inertia range is established that has slope \(k^{-5/3}\). This improved eddy viscosity model shows good agreement with DNS at the resolved wave numbers.

1.2.3. Upscaling scheme with higher accuracy. To improve the accuracy of the Reynolds stress model. We retain the Reynolds stress tensor calculated from the cell equations. However, it is impractical to calculate all the cell equations at each time step. When the turbulent flow field approach the statistical stationary state, the variation of Reynolds stress tensor become small both in space and in time. Therefore it is not necessary to resolve every cell. Instead, we only need to solve those cells that is disturbed from the equilibrium state. From the cell equation (1.19), we notice that the disturbance to the cell equation comes from the term \(\nabla_x \theta\), which represent influence from the local grid of the mean flow. If the value of \(\nabla_x \theta\) remains constant, the cell will stay in equilibrium stage. However, if the value of \(\nabla_x \theta\) change due to shearing of the mean flow map, the cell will be disturbed away from the equilibrium stage, and will evolve until it reaches a new equilibrium state. Based on this analysis, we proposed a new numerical scheme that at each time step \(n\), we only update those cells where the L2 norm of \(\langle (\nabla_x \theta)_n - (\nabla_x \theta)_{n-1} \rangle\) are larger than other cells. We can decide the number of cells \(N\) to update at each time step. In the present study, we tested \(N = 32, 64, 128\). The cell equations are solved with \(32^3\) grid and the averaged equations are solved with \(64^3\) grid.

Although we do not need to update every cell solution at each time step, we need to save the flow field of each cell as the initial condition when that cell needs to be updated at later time. In this case the memory usage is equivalent to a grid of \(64^3 \times 64^3 = 4096^3\), which exceeds the capacity of the cluster we use. To deliviate this problem, after solving the cell equation and obtained the Reynolds stresses, we only save the first 16 Fourier mode in each direction. When the flow field of a cell needs to be updated, we interpolate the saved modes into 64 modes in each direction. In addition, the turbulence within a cell has its own time scale, the eddy turn over time \(t_{cell}\), which is about 1/64 of the eddy-turn-over-time of the averaged flow, \(T\). Numerical experiments shows that it is enough to solve the cell equations for \(5t_{cell}\), and the cell flow field will reach equilibrium state. This scheme has substantial saving in computation time compared with DNS, and has accuracy comparable with DNS.

Figure 9 plot the comparison of energy spectrum at \(T = 20.0\). It is shown that the energy spectra obtained with \(N = 64\) and 128 agree very well with the
DNS, whereas the $N = 32$ solution has developed instability, and results in energy build-up at the cut-off wave number.

The comparison in energy spectrum only shows that the SGS model capture the total energy dissipation at the cutoff wavenumber. It is more important to study basic structural properties of subgrid stresses with respect to the large scale characteristics of the flow field. A comprehensive knowledge of the fine scale motions is essential in the development of a proper turbulent theory and any turbulent model. These coherent fine scale structures have been observed in other types of turbulent flows such as turbulent mixing layers and turbulent channel flows where they exhibited similar characteristics.

Three-dimensional measurement techniques has been used to study the alignment of the eigenvectors of actual and modeled components of the subgrid stresses as well as the alignments between eigenvectors of the rate of strain tensor and vorticity vector. The studies confirmed the preferred local alignment of the eigenvector of the rate of strain tensor corresponding to the intermediate eigenvalue with the vorticity vector, which was previously observed using pointwise DNS data. There is also a preferred relative angle between the most compressive eigendirection of the rate of strain tensor and the most extensive eigendirection of the SGS tensor. Following the previous work of experimental study and LES, we will discuss the statistics of alignment between the eigenvectors of the SGS tensor $\tau$, and both the eigenvectors of the rate of strain tensor $D$, and the unit vorticity vector, $\omega$.

We denote the eigenvectors of $D$ by $[e_1, e_2, e_3]$, ordered according to the corresponding eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$, with $\lambda_1 > \lambda_2 > \lambda_3$. The eigenvectors of $\tau$ are $(t_1, t_2, t_3)$ with eigenvalues $(\gamma_1, \gamma_2, \gamma_3)$, such that $\gamma_1 > \gamma_2 > \gamma_3$. Thus, for example, we refer to $e_1$ as the most extensional eigendirection of $D$, and to $t_3$ as the most compressive eigendirection of $\tau$.

We first compute the distribution of the energy transfer function, $\epsilon_{sgs} = \tau : D$, between the filtered scales and subgrid scales. The segment with $\epsilon_{sgs} < 0$ corresponds to energy backscatter. Figure 10 shows that the LES computation with 64 cells capture the pdf distribution accurately. The computation with 32 cells is less accurate, however, it still capture some amount of energy backscattering.

Using the same data sets we computed the distributions of the alignment, represented by the cosine of the angles, between the eigenvectors of the rate of strain tensor and vorticity vector as well as the SGS tensor. Figure 11 and figure 12 indicate that both the DNS and the LES demonstrate the preferential alignment between the vorticity vector and the eigendirection of the rate of strain tensor corresponding to the intermediate eigenvalue. In addition, figure 11 and figure 12 present the alignment between the eigenvectors of the SGS stress tensor and the rate of strain tensor $D$ of the resolved scale. The eigenvectors of the subgrid tensor $\tau$ solved from the cell equations display qualitatively similar alignment as that computed from DNS data. In particular, the experimentally observed preferred angle between $e_3$ and $t_1$ is evident.

1.3. Conclusion and discussion. Using homogenization theory, we derived upscaling scheme to solve Euler equations and Navier-Stokes equations. Numerical experiments demonstrated that the current scheme not only obtained good agreement with high resolution DNS result in energy spectrum and kinetic energy evolution, but also correctly captured the higher order turbulence statistics, including propability distribution function of subgrid energy transfer, correlation
between the eigen vectors of the SGS tensor, the rate of the strain tensor, and the unit vorticity vector.

We can compare our current scheme with the available LES models. All LES schemes are based on some assumed correlation about the subgrid stresses and the local rate of strain of the resolved flow. However, DNS result and more recent high resolution experiment results did not show any obvious correlation between those two tensors. The advantage of the current scheme is that the homogenized equations and the cell equations are derived directly from the governing equations, so the mathematical formulation is exact except that higher order terms are ignored in the cell equation. In numerical implementation, to reduce the number of degree of freedom, we rely on general assumptions of the homogeneity of flow and universality of inertia range.

In summary, compared with LES, the current scheme is based on accurate mathematical models and more generally accepted and sound assumptions on turbulence structure. Numerical results demonstrate that the current scheme higher accuracy and flexibility. On the other hand, it is more computationally economic than DNS. DNS computes all the scales equally. However, when the turbulence reaches the state of statistical equilibrium, most subgrid scales will remain statistical stationary, therefore do not need to be calculated. However, due to turbulence intermittency, some subgrid scales will be disturbed by the large scale flow. At this moment, the disturbed scales need to be recalculated. This is how the current scheme is saving computation time. We only need to compute those subgrid cells that are disturbed away from equilibrium. We noticed that the large scale flow disturbs the cell flow through the convection term $D_z \mathbf{w} D_x \mathbf{w}$, and regions with large value of $D_z \mathbf{w} D_x \mathbf{w}$ coincide with regions of high shears. Therefore the accuracy of the current scheme is comparable with that of DNS, but requires much less computation resources. For example, in the numerical test, we solved 64 cells with $32^3$ grid points, and obtained excellent agreement with the DNS results using $512^3$ grid points. To this end, the current scheme enables a 128 fold saving in the number of grid points.

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Figure 2. Contour plot of vorticity. Contour levels: max=1.4, min=-0.9.
Figure 3. Temporal evolution of kinetic energy $(||u_1||_{L^2}^2 + ||u_2||_{L^2}^2)^{\frac{1}{2}}$ and enstrophy $||\omega||_{L^2}$, DNS(N=512); (○), homogenization; (△), without Reynolds stress terms.

Figure 4. Spectrum of velocity component $u_1$. (---), $t = 0.0$; (-----), DNS, $t = 20.0$; (-----), homogenization, $t = 20.0$. 
Figure 5. Temporal evolution of total kinetic energy.

Figure 6. Iso-surface plot of vorticity.
Figure 7. Temporal evolution of computed eddy viscosity $C_D$.

Figure 8. The comparison of energy spectra between DNS and the averaged equation at $t = 30$. 
Figure 9. Comparison of energy spectrum at $t = 30.0$. (blue), DNS. (red), $N=128$; (green), $N=64$; (yellow), $N=32$.

Figure 10. Probability distribution of $\epsilon_{sgs} = \tau ; D$; solid line: forced DNS; dashed line: $N=128$; dotted line: $N=64$. 
Figure 11. Probability distribution of $\omega \cdot e_\alpha$ from DNS; solid line: $\omega \cdot e_1$; dashed line: $\omega \cdot e_2$; dotted line: $\omega \cdot e_3$.

Figure 12. Probability distribution of $\omega \cdot e_\alpha$ from LES; solid line: $\omega \cdot e_1$; dashed line: $\omega \cdot e_2$; dotted line: $\omega \cdot e_3$. 
Figure 13. Probability distribution of $e_\alpha \cdot t_3$ from DNS; solid line: $e_3 \cdot t_1$; dashed line: $e_2 \cdot t_2$; dotted line: $e_1 \cdot t_3$

Figure 14. Probability distribution of $e_\alpha \cdot t_3$ from DNS; solid line: $e_3 \cdot t_1$; dashed line: $e_2 \cdot t_2$; dotted line: $e_1 \cdot t_3$