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002

CONVERGENCE OF A POINT VORTEX METHOD FOR VORTEX SHEETS*

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Abstract. Based on the observation that a point vortex approximation can be made spectrally accurate by using Van de Vooren's desingularization, short time convergence of the point vortex method for both vortex sheets and the Boussinesq approximation are proved using analytic data. The spectral accuracy of the method allows a very simple proof to be obtained without using the Cauchy-Kowalewski theorem.

Key words. vortex sheets, desingularization, spectral accuracy

AMS(MOS) subject classifications. 65M15, 76C05

1. Introduction. In this paper, we prove short time convergence of two vortex methods for vortex sheets and the Boussinesq approximation of the Rayleigh–Taylor problem, respectively. The first numerical method we consider is a point vortex method for vortex sheets using staggered grid [2], [9], [11]. This method does not differ much from the usual point vortex method formally, but it exhibits spectral accuracy in the space discretization when the vortex sheet is analytic. The analysis of Caflisch and Lowengrub [3], [8] is applicable in the case of this particular numerical method. However, we have found a proof which does not rely on the Cauchy-Kowaleski theorem and hence greatly simplifies the analysis.

The difficulty with vortex sheet calculations is due to the Kelvin–Helmholtz instability and singularity formation. Typically, the error $e(t)$ at time t will be amplified by a factor of $O(1/h)$ (h is the initial meshsize)

$$\frac{d}{dt} e(t) \leqq \frac{c}{h} e(t) + r(t),$$

where $r(t)$ is the truncation error of the method. By the Gronwall inequality, this would imply that

$$e(t) \leqq \int_0^t \exp(c(t-s)/h) r(s)\, ds.$$

Our observation is that the spectral accuracy, i.e., $r(s) \leqq \exp(-b/h)$ for some positive constant b and $0 \leqq s \leqq T$, can dominate instabilities for short times, i.e.,

$$e(t) \leqq \exp((-b+ct)/h) \int_0^t \exp(-cs/h)\, ds < \frac{h}{c} \exp((-b+ct)/h) \to 0 \quad \text{as } h \to 0$$

for $t < b/c$. This allows us to obtain a short time convergence result without using the Cauchy–Kowalewski theorem directly. It is worth noting that the method does not

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have to be spectrally accurate to apply this technique. It is enough that an approximate equation can be constructed so that the numerical method is spectrally accurate with respect to the approximate equation. It is based on this observation that we prove convergence of the corresponding time discrete method, which is only first-order accurate in time. This is in the spirit of Strang [14].

The second method we consider is for the Boussinesq approximation of the Rayleigh–Taylor problem [13]. Due to the presence of gravity and the density difference, circulation is generated in the process. This introduces additional difficulty in the convergence study. Here, we use a staggered point vortex approximation for the interface equation and a discrete spectral method for the circulation equation. Using a similar analysis to that in the vortex sheet problem, we show that the method is also spectrally accurate in the space discretization and converges with spectral accuracy for short times. We believe that our techniques here could also be useful in analyzing the full Rayleigh–Taylor problem.

2. A point vortex method for vortex sheets.

2.1. Derivation of the method. In the case of a periodic vortex sheet with period 2π, where

$$z = z(\gamma, t) = x(\gamma, t) + iy(\gamma, t)$$

is the sheet position and $z(\gamma+2\pi, t) = z(\gamma, t)$, the Birkhoff–Rott equation becomes

$$\frac{dz^*(\gamma, t)}{dt} = \frac{1}{4\pi i}\int_{-\pi}^{\pi} \cot\left(\frac{z(\gamma, t) - z(\gamma+\xi, t)}{2}\right) d\xi, \qquad z(\gamma, 0) = z_0(\gamma), \tag{1}$$

where z^* is the complex conjugation of z, γ is the Lagrangian circulation parametrization along the sheet, and the integral is a principal value integral. The analyticity of the sheet is guaranteed, for short times, by Theorem 1.

THEOREM 1 (existence and uniqueness). *Let* $z(\gamma, 0) = \gamma + s_0(\gamma)$, *where* $s_0(\gamma)$ *is analytic in the strip* $|\mathrm{Im}\,\gamma| \leqq \rho_0$. *If* $\|s_{0\gamma}(\cdot)\|_{\infty,\rho_0} \leqq \varepsilon$, *then there exists a unique solution* $z = \gamma + s(\gamma, t)$ *to* (1) *for* $0 \leqq t < T_c$ *and*

$$\|s_\gamma(\cdot, t)\|_{\infty,\rho(t)} \leqq \varepsilon$$

for $\rho(t) = \alpha\rho_0 - Ct$ *for any* $0 < \alpha < 1$, *where* $C = C(\varepsilon, \rho_0, \alpha)$, $T_c = \alpha\rho_0/C$ *and*

$$\|f(\gamma)\|_{\infty,\rho} = \sup_{|\mathrm{Im}\,\gamma| \leqq \rho} |f(\gamma)|.$$

We refer to Sulem et al. [15], Caflisch and Orellana [4], and Duchon and Robert [5] for a proof.

We will now derive a spectrally accurate numerical method based on Van de Vooren desingularization [16]. Van de Vooren noted that

$$\int_{-\pi}^{\pi} \cot\left(\frac{z(\gamma) - z(\gamma+\xi)}{2}\right) d\xi = \int_{-\pi}^{\pi} \left(\cot\left(\frac{z(\gamma) - z(\gamma+\xi)}{2}\right) + \frac{1}{z_\gamma(\gamma)}\cot\left(\frac{\xi}{2}\right)\right) d\xi \tag{2}$$

due to the fact that (2) is a principal value integral. The integral of the right side is now a bounded function with

$$\lim_{\xi\to 0}\left(\cot\left(\frac{z(\gamma) - z(\gamma+\xi)}{2}\right) + \frac{1}{z_\gamma(\gamma)}\cot\left(\frac{\xi}{2}\right)\right) = \frac{z_{\gamma\gamma}(\gamma)}{z_\gamma(\gamma)^2}. \tag{3}$$

In fact, we shall see later that the integrand is an analytic function of ξ. This is a crucial observation.

If we call the right side of (2)

$$I[z](\gamma, t)=\frac{1}{4\pi i}\int_{-\pi}^{\pi}\left(\cot\left(\frac{z(\gamma)-z(\gamma+\xi)}{2}\right)+\frac{1}{z_\gamma(\gamma)}\cot\left(\frac{\xi}{2}\right)\right)d\xi, \tag{4}$$

then a trapezoidal rule approximation of it is

$$J_h[z]\equiv\frac{h}{4\pi i}\sum_{\substack{j=-N\\ j\neq 0}}^{N-1}\cot\left(\frac{z(\gamma)-z(\gamma+jh)}{2}\right)+\frac{h}{4\pi i}\frac{z_{\gamma\gamma}(\gamma)}{z_\gamma(\gamma)^2}, \tag{5}$$

where $h=\pi/N$. The last term corresponds to $j=0$. This extra term produces difficulties for numerical schemes due to the presence of first and second derivatives. Fortunately, this extra term can be eliminated by an appropriate linear combination of two grid levels:

$$\begin{aligned}\tilde{I}_h &= 2J_h-J_{2h}\\ &=\frac{2h}{4\pi i}\sum_{\substack{j=-N\\ j\neq 0}}^{N-1}\cot\left(\frac{z(\gamma)-z(\gamma+h)}{2}\right)-\frac{2h}{4\pi i}\sum_{\substack{j=-N/2\\ j\neq 0}}^{N/2-1}\cot\left(\frac{z(\gamma)-z(\gamma+2jh)}{2}\right)\\ &=\frac{2h}{4\pi i}\sum_{j=-N/2}^{N/2-1}\cos\left(\frac{z(\gamma)-z(\gamma+(2j+1)h)}{2}\right),\end{aligned} \tag{6}$$

which is the same as the midpoint rule approximation of $I[z](\gamma, t)$ with meshsize $2h$. Let $z_k=z(\gamma_k)$ where $\gamma_k=kh$. Then the numerical method becomes

$$\frac{d\tilde{z}_k^*}{dt}=\frac{h}{2\pi i}\sum_j\cot\left(\frac{\tilde{z}_k-\tilde{z}_{k+2j+1}}{2}\right)\equiv\tilde{I}_h[\tilde{z}_k],\qquad \tilde{z}_k(0)=z_0(\gamma_k). \tag{7}$$

We denote by $I_h[z_k]$ the corresponding discretization evaluated at the exact trajectory positions z_k's:

$$I_h[z_k]=\frac{h}{2\pi i}\sum_j\cos\left(\frac{z_k-z_{k+2j+1}}{2}\right).$$

The ideas of using every other point and regularizing the integrand in a Cauchy principal integral have been used by several authors in the literature. Baker [2] used these ideas in water wave calculations, and he noted that the method was spectrally accurate if the removable singularity could be evaluated spectrally. Roberts [9] also used similar ideas to obtain a spectrally accurate method for interval waves. In the context of studying vortex sheet singularity by vortex methods, Shelley [11] was the first to use scheme (7) together with the filtering technique developed by Krasny [7]. Finally, the spectral accuracy of a midpoint rule approximation for a periodic singular integrand has been analyzed by Sidi and Israeli [12].

2.2. Summary of results. We first introduce some notation. Define

$$\|f\|_{\infty,\rho}=\sup_{|\mathrm{Im}\,\gamma|\leqq\rho}|f(\gamma)|,\qquad \|f\|_{l_\infty}=\max_k|f(\gamma_k)|,$$

$$|||f|||_\rho=\sum_{k=-\infty}^{\infty}|\hat{f}(k)|\,e^{\rho|k|},$$

where f is 2π-periodic and

$$\hat{f}(k)=\int_{-\pi}^{\pi}f(x)\,e^{ikx}\,dx$$

are the Fourier coefficients. Then we have the following main result.

Define $T_1 < T_c$ and $\rho = \alpha\rho_0 - CT_1$ so that $\rho > 0$, where C and α are defined as in Theorem 1.

THEOREM 2 (convergence of the method). *Let $z(\gamma, t) = \gamma + s(\gamma, t)$ be a solution of the continuous Birkhoff–Rott equation for $0 \leqq t < T_c$, where T_c is given by Theorem 1. If $\tilde{z}_k(t)$ is the solution of (7), then*

$$\|z(\cdot, t) - \tilde{z}(t)\|_{l_\infty} \leqq 4C(T_1)\, e^{-\pi\rho'/h}\, e^{2\tilde{C}(T_1)t/h}$$

for $t \leqq T = \min(T_1, (\pi\rho'/2\tilde{C}) - s)$ for any $\rho' < \rho$ and $s > 0$. The constants are defined as follows:

$$\tilde{C} = 4\left(\max_{|z|=r} \frac{|z|^2}{|\sin(z)^2|}\right) \cdot \sum_j \frac{1}{(2j+1)^2},$$

$$r = \pi(1 + \|s_\gamma\|_{\infty,\rho})/2 + h^2,$$

$$C = \max_\gamma \left\|\left| \cot\left(\frac{z(\gamma) - z(\gamma + \cdot)}{2}\right) + \frac{1}{z_\gamma(\gamma)} \cot\left(\frac{\cdot}{2}\right)\right\|\right|_\rho.$$

We follow the standard vortex method arguments, and so we need the following lemmas.

CONSISTENCY LEMMA. *Let $z(\gamma, t) = \gamma + s(\gamma, t)$ be given by Theorem 1. Then*

$$\left|\frac{1}{4\pi i}\int_{-\pi}^{\pi} \cot\left(\frac{z(\gamma) - z(\gamma+\xi)}{2}\right) d\xi - \frac{h}{2\pi i}\sum_j \cot\left(\frac{z_k - z_{k+2j+1}}{2}\right)\right| \leqq 4C\, e^{-\pi\rho'/h} \quad \textit{for } \rho' < \rho,$$

i.e., the method is spectrally accurate.

STABILITY LEMMA. *Suppose that $\|z - \tilde{z}\|_{l_\infty} \leqq h^2$ for $t \leqq T^*$. Then for $t \leqq T^*$,*

$$\left|\frac{h}{2\pi i}\sum_j \left(\cot\left(\frac{z_k - z_{k+2j+1}}{2}\right) - \cot\left(\frac{\tilde{z}_k - \tilde{z}_{k+2j+1}}{2}\right)\right)\right| \leqq \frac{2}{h}\tilde{C}\|z - \tilde{z}\|_{l_\infty}.$$

We now present the proof of Theorem 2.

Proof. We assume the validity of the consistency and stability lemmas. Their proofs will be given in the next two sections. We have

$$\frac{d}{dt}(z_k - \tilde{z}_k)^* = I[z_k] - \tilde{I}_h[\tilde{z}_k] = (I[z_k] - I_h[z_k]) + (I_h[z_k] - \tilde{I}_h[\tilde{z}_k]).$$

Define T^* by

$$T^* = \sup\{t \mid 0 \leqq t \leqq T, \|z - \tilde{z}\|_{l_\infty} \leqq h^2\}.$$

It follows from the consistency lemma that

$$|I[z_k] - I_h[z_k]| \leqq 4C\, e^{-\pi\rho'/h}$$

for all $0 \leqq t \leqq T$. On the other hand, the stability lemma implies that

$$|I_h[z_k] - \tilde{I}_h[\tilde{z}_k]| \leqq \frac{2}{h}\tilde{C}\|z - \tilde{z}\|_{l_\infty}$$

for $t \leqq T^*$. Therefore, we obtain for $t \leqq T^*$ that

$$\frac{d}{dt}\|z - \tilde{z}\|_{l_\infty} \leqq 4C\, e^{-\pi\rho'/h} + \frac{2}{h}\tilde{C}\|z - \tilde{z}\|_{l_\infty}.$$

It follows from Gronwall's inequality that

$$\|z(t) - \tilde{z}(t)\|_{l_\infty} \leqq \frac{2Ch}{\tilde{C}}\, e^{-(\pi\rho' - 2\tilde{C}t)/h} \tag{8}$$

for $t \leqq T^*$. But $T^* \leqq T \leqq (\pi\rho'/2\tilde{C}) - s$ by the assumption of Theorem 2. This implies that

$$\|z(t) - \tilde{z}(t)\|_{l_\infty} \leqq \frac{2Ch}{\tilde{C}} e^{-2\tilde{C}s/h}.$$

Thus, for h small enough, we have

$$\|z(t) - \tilde{z}(t)\|_{l_\infty} \leqq h^2/2 < h^2.$$

Hence $T^* = T$ and (8) holds for $0 \leqq t \leqq T$, which completes the proof of the theorem.

2.3. Proof of the consistency lemma. The main idea of our consistency argument is to show that the integrand is analytic and that the trapezoidal rule is spectrally accurate for periodic analytic functions.

LEMMA 1. *If $z(\gamma, t) = \gamma + s(\gamma, t)$ is analytic function of γ in $|\mathrm{Im}\ \gamma| \leqq \rho$ and $\|s_\gamma\|_{\infty,\rho} \leqq \varepsilon \leqq \frac{1}{3}$, then*

$$\cot\left(\frac{z(\gamma) - z(\gamma+\xi)}{2}\right) + \frac{1}{z_\gamma(\gamma)} \cot\left(\frac{\xi}{2}\right)$$

is an analytic function in $|\mathrm{Im}\ \xi| \leq \rho$.

Proof. Note that for $|\xi| < \pi$,

$$\cot \xi = \frac{1}{\xi} - \frac{\xi}{3} - \frac{\xi^3}{45} - \cdots.$$

But

$$\left|\frac{z(\gamma) - z(\gamma+\xi)}{2}\right| \leqq \frac{\pi}{2}(1 + \|s_\gamma\|_{\infty,\rho}) \leqq \frac{\pi}{2}(1+\varepsilon) < \pi$$

for $|\xi| \leqq \pi$. Thus we have an expansion for

$$\cot\left(\frac{z(\gamma) - z(\gamma+\xi)}{2}\right) = \frac{2}{z(\gamma) - z(\gamma+\xi)} - \frac{z(\gamma) - z(\gamma+\xi)}{6} - \frac{(z(\gamma) - z(\gamma+\xi))^3}{45} - \cdots.$$

Since $z(\gamma)$ is analytic, it follows that the only nonanalytic term is

$$\frac{2}{z(\gamma) - z(\gamma+\xi)}.$$

It can be verified easily that

$$\frac{1}{z(\gamma) - z(\gamma+\xi)} = -\frac{1}{(1+s_\gamma)\xi} + \frac{s_{\gamma\gamma}}{2(1+s_\gamma)^2} + O(\xi)$$

and

$$\frac{1}{z_\gamma} \cot(\xi/2) = \frac{2}{(1+s_\gamma)\xi} - \frac{\xi}{3(1+s_\gamma)} + O(\xi^3).$$

Therefore, we conclude that

$$\cot\left(\frac{z(\gamma) - z(\gamma+\xi)}{2}\right) + \frac{1}{z_\gamma(\gamma)} \cot\left(\frac{\xi}{2}\right) = \frac{s_{\gamma\gamma}}{(1+s_\gamma)^2} + O(\xi)$$

for ξ near zero. Therefore $\xi = 0$ is a removable singularity and hence the function is actually analytic in $|\mathrm{Im}\ \xi| \leqq \rho$.

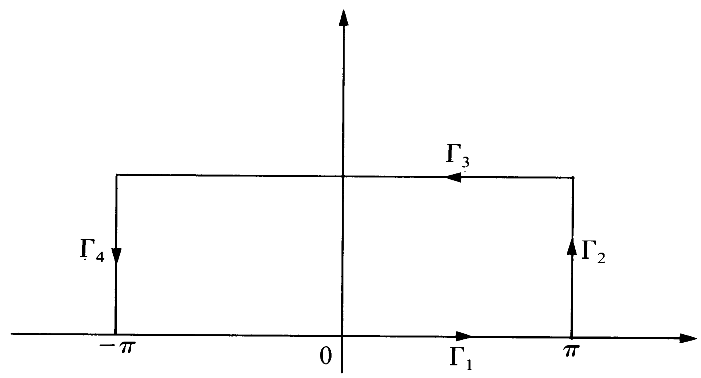
LEMMA 2. *Let $f(z)$ be 2π-periodic and analytic in $|\mathrm{Im}\, z| \leqq \rho$. Then we have*

$$|\hat{f}(k)| \leqq C\, e^{-\rho|k|}.$$

Proof. Recall that

$$\hat{f}(k) = \int_{-\pi}^{\pi} f(x)\, e^{ikx}\, dx.$$

Without loss of generality, we may assume $k > 0$. Let $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$.



Since $f(z)$ is analytic inside Γ by assumption, we have

$$\int_{\Gamma} e^{ikz} f(z)\, dz = 0,$$

by the Cauchy integral theorem [10]. Note that periodicity of f implies

$$\left(\int_{\Gamma_2} + \int_{\Gamma_4}\right) f(z)\, e^{ikz}\, dz = 0.$$

Therefore we arrive at

$$\hat{f}(k) = \int_{\Gamma_1} f(z)\, e^{ikz}\, dz = -\int_{\Gamma_3} f(z)\, e^{ikz}\, dz = e^{-k\rho} \int_{-\pi}^{\pi} f(x + i\rho)\, e^{ikx}\, dx.$$

But $f(z)$ is analytic, so $|f(x + i\rho)| \leqq C$ for $|x| \leqq \pi$. Therefore we have proved

$$|\hat{f}(k)| \leqq C\, e^{-\rho|k|}.$$

LEMMA 3. *Let f be analytic in $|\mathrm{Im}\, z| \leqq \rho$ and periodic with period 2π. Then,*

$$\left| \int_{-\pi}^{\pi} f(x)\, dx - h \sum_{j=-N}^{N-1} f(jh) \right| \leqq 2\pi |||f|||_{\rho'}\, e^{-2\pi\rho'/h}$$

for any $\rho' < \rho$, where $h = \pi/N$.

Proof. By the Poisson summation formula (see, e.g., [6], [10])

$$\int_{-\pi}^{\pi} f(x)\, dx - h \sum_{j=-N}^{N-1} f(jh) = 2\pi \sum_{k \neq 0} \hat{f}(2kN).$$

Moreover, we have

$$\begin{aligned} \left| \sum_{k \neq 0} \hat{f}(2kN) \right| &\leqq \sum_{k \neq 0} |\hat{f}(2kN)|\, e^{-\rho' 2N}\, e^{\rho'|k|2N} \\ &\leqq e^{-2\pi\rho'/h} \sum_{k \neq 0} |\hat{f}(2kN)|\, e^{\rho'|k|2N} \leqq |||f|||_{\rho'}\, e^{-2\pi\rho'/h}. \end{aligned}$$

We note that if f is analytic in $|\text{Im } z| \leqq \rho$, then $|||f|||_{\rho'} < \infty$ for any $\rho' < \rho$ by Lemma 2. This completes the proof of Lemma 3.

Proof of the consistency lemma. Direct applications of Lemmas 1-3 to $I[z] - J_h[z]$ and $I[z] - J_{2h}[z]$, respectively, prove the consistency lemma.

2.4. Proof of the stability lemma. To prove the stability lemma, we consider

$$I_h[z_k] - \tilde{I}_h[\tilde{z}_k] = \frac{h}{2\pi i} \sum_j \left(\cot\left(\frac{z_k - z_{k+2j+1}}{2}\right) - \cot\left(\frac{\tilde{z}_k - \tilde{z}_{k+2j+1}}{2}\right)\right)$$
$$= \frac{h}{2\pi i} \sum_j \left(\cot\left(\frac{z_k - z_{k+2j+1}}{2}\right) - \cot\left(\frac{z_k - \tilde{z}_{k+2j+1}}{2}\right)\right)$$
$$+ \frac{h}{2\pi i} \sum_j \left(\cot\left(\frac{z_k - \tilde{z}_{k+2j+1}}{2}\right) - \cot\left(\frac{\tilde{z}_k - \tilde{z}_{k+2j+1}}{2}\right)\right).$$

Recall that $z(\gamma, t) = \gamma + s(\gamma, t)$. Note that

$$|z_k - z_{k+2j+1}| \geqq (2j+1)h(1 - \|s_\gamma\|_{\infty,\rho}).$$

Without the loss of generality, we may assume $\|s_{0\gamma}(\cdot)\|_{\infty,\rho_0} \leqq \frac{1}{3}$. Then Theorem 1 implies that $\|s_\gamma\|_{\infty,\rho} \leqq \frac{1}{3}$. Let $y_{i,j} = \theta(\tilde{z}_{k+2j+1} - z_{k+2j+1})$ with $|\theta| \leqq 1$. By assumption, $|y_{i,j}| \leqq h^2$ for $t \leqq T^*$. So we obtain

$$|z_k - z_{k+2j+1} + y_{i,j}|/2 \geqq 2(2j+1)h/6 - h^2/2 \geqq (2j+1)h/8 > 0$$

for h small enough. Thus we can apply the mean value theorem to get

$$\frac{h}{2\pi i} \sum_j \left(\cot\left(\frac{z_k - z_{k+2j+1}}{2}\right) - \cot\left(\frac{z_k - \tilde{z}_{k+2j+1}}{2}\right)\right)$$
$$= \frac{h}{2\pi i} \sum_j \frac{d}{dz} \cot\left(\frac{z_k - z_{k+2j+1} + y_{i,j}}{2}\right)(\tilde{z}_{k+2j+1} - z_{k+2j+1})$$
$$= \frac{h}{2\pi i} \sum_j \left(\frac{-1}{2\sin^2\left(\frac{z_k - z_{k+2j+1} + y_{i,j}}{2}\right)}\right)(\tilde{z}_{k+2j+1} - z_{k+2j+1}).$$

Furthermore, note that

$$\frac{z}{\sin(z)} = 1 + \frac{z^2}{6} + \cdots$$

is analytic for $|z| < \pi$. Therefore if $\|s_\gamma\|_{\infty,\rho} \leqq \frac{1}{3}$, then

$$\max_{|z| \leqq r} \frac{|z|}{|\sin(z)|} \leqq \max_{|z| = r} \frac{|z|}{|\sin(z)|} \equiv C_1$$

for $r = \pi(1 + \|s\|_{\infty,\rho})/2 + h^2 < \pi$. Now since

$$\frac{|z_k - z_{k+2j+1} + y_{i,j}|}{2} \leqq \pi(1 + \|s_\gamma\|_{\infty,\rho})/2 + h^2,$$

we get

$$\frac{h}{2\pi} \sum_j \frac{1}{2\left|\sin^2\left(\frac{z_k - z_{k+2j+1} + y_{i,j}}{2}\right)\right|} |\tilde{z}_{k+2j+1} - z_{k+2j+1}|$$

$$\leqq \|z_k - \tilde{z}_k\|_\infty \frac{h}{2\pi} \sum_j \frac{C_1^2}{\left|\frac{z_k - z_{k+2j+1} + y_{i,j}}{2}\right|^2}$$

$$\leqq \|z_k - \tilde{z}_k\|_\infty C_1^2 h \sum_j \frac{4}{(2j+1)^2 h^2}$$

$$\leqq \frac{4C_1^2}{h} \sum_j \frac{1}{(2j+1)^2} \|z - \tilde{z}\|_{l_\infty}.$$

Denote $\tilde{C} = 4C_1^2 \sum_j 1/(2j+1)^2$. Then, we have shown that

$$\left|\frac{h}{2\pi i} \sum_j \left(\cot\left(\frac{z_k - z_{k+2j+1}}{2}\right) - \cot\left(\frac{z_k - \tilde{z}_{k+2j+1}}{2}\right)\right)\right| \leqq \frac{1}{h} \tilde{C} \|z - \tilde{z}\|_{l_\infty}$$

for $t \leqq T^*$. Similarly, we can show that

$$\left|\frac{h}{2\pi i} \sum_j \left(\cot\left(\frac{z_k - \tilde{z}_{k+2j+1}}{2}\right) - \cot\left(\frac{\tilde{z}_k - \tilde{z}_{k+2j+1}}{2}\right)\right)\right| \leqq \frac{1}{h} \tilde{C} \|z - \tilde{z}\|_{l_\infty}$$

for $t \leqq T^*$. This completes the proof of the stability lemma.

2.5. Convergence of the time discrete method. Our convergence analysis for the semidiscrete method is based on the observation that the method is spectrally accurate. When we discretize the method in time, however, the method is of at most finite order accuracy in time. Thus, the analysis in previous sections does not apply directly to the time discrete method.

The way to get around this difficulty is to compare the time discrete method with an approximate equation to the Birkhoff-Rott equation. This approximate equation is chosen in such a way that the time discrete method approximates this equation with spectral accuracy. Thus, the techniques developed in the previous sections can be used to obtain convergence of the time discrete method. This approach is inspired by Strang's argument [14].

A natural choice of the approximate equation to the Birkhoff-Rott equation is the time discrete Birkhoff-Rott equation:

$$\text{(9)} \qquad \frac{v^*(\gamma, (n+1)\Delta t) - v^*(\gamma, n\Delta t)}{\Delta t} = \frac{1}{4\pi i} \int_{-\pi}^{\pi} \cot\left(\frac{v(\gamma, n\Delta t) - v(\gamma + \xi, n\Delta t)}{2}\right) d\xi,$$

$$v(\gamma, 0) = z_0(\gamma).$$

Again, short time existence of (9) is guaranteed by analyticity of initial data.

THEOREM 3. *Under the assumption of Theorem* 1, *there exists a unique solution* $v(\gamma, n\Delta t) = \gamma + \tilde{s}(\gamma, n\Delta t)$ *to* (9) *for* $0 \leqq t \leqq T_c$ *satisfying*

$$\|\tilde{s}_\gamma(\cdot, t)\|_{\infty, \rho(t)} \leqq \varepsilon,$$

and

$$\text{(10)} \qquad \|s(\cdot, n\Delta t) - \tilde{s}(\cdot, n\Delta t)\|_{\infty, \rho(t)} \leqq C\Delta t,$$

where $\rho(t)$, C, *and* T_c *are defined as in Theorem* 1, *and* $\gamma + s(\gamma, t)$ *is a solution of* (1).

The proof of Theorem 3 is almost identical to that of Theorem 1. We refer to [3] and [8].

THEOREM 4 (convergence of the time discrete method). *Let* $z(\gamma, t) = \gamma + s(\gamma, t)$ *be a solution of the continuous Birkhoff-Rott equation for* $0 \leqq t < T_c$, *and* $\tilde{z}_k^n$ *is a solution* *to the time discrete point vortex method*

$$\tilde{z}_k^{n+1*} = \tilde{z}_k^{n*} + \Delta t \frac{h}{2\pi i} \sum_j \cot\left(\frac{\tilde{z}_k - \tilde{z}_{k+2j+1}}{2}\right), \qquad \tilde{z}_k^0 = z_0(\gamma_k). \tag{11}$$

Under the assumption of Theorem 2, *we have*

$$\|z(\cdot, n\Delta t) - \tilde{z}^n\|_{l_\infty} \leqq C_1 e^{-\pi\rho'/h} e^{C_2 t/h} + C_3 \Delta t,$$

for $t \leqq T = \min(T_1, (\pi\rho'/2\tilde{C}) - s)$ *for any* $\rho' < \rho$ *and* $s > 0$.

Proof. Let $v(\gamma, t)$ be a solution of (9). Arguing exactly as in the proof of Theorem 2, we can show that

$$\|v(\cdot, n\Delta t) - \tilde{z}^n\|_{l_\infty} \leqq C_1 e^{-\pi\rho'/h} e^{C_2 t/h}. \tag{12}$$

On the other hand, it follows from Theorem 3 that

$$\|z(\cdot, n\Delta t) - v(\cdot, n\Delta t)\|_{l_\infty} \leqq C_3 \Delta t. \tag{13}$$

Therefore, (12) and (13) together prove convergence of the method.

3. The Boussinesq approximation of the Rayleigh–Taylor problem. We now show how to apply these ideas to a slightly different problem: the Boussinesq approximation of the Rayleigh–Taylor problem. Here there is an additional difficulty in that the force of gravity and density differences cause the generation of circulation.

It is well known that in the limit as the Attwood number $A \to 0$ and the gravity constant $g \to \infty$ with constraint $Ag = 1$, the equations governing the interface $z(\gamma, t)$ reduce to

$$\frac{dz^*(\alpha, t)}{dt} = \frac{1}{4\pi i} \int_{-\pi}^{\pi} \gamma(\alpha + \xi, t) \cot\left(\frac{z(\alpha, t) - z(\alpha + \xi, t)}{2}\right) d\xi, \tag{14}$$

$$\frac{d\gamma(\alpha, t)}{dt} = i(z_\alpha - z_\alpha^*). \tag{15}$$

Equations (14) and (15) are called the Boussinesq approximation [13]. Here we have made the analytic extension $z^*(\alpha, t) = \bar{z}(\bar{\alpha}, t)$ so that the equations are now analytic in α.

As a prerequisite to our convergence theorem, we first state the existence result for (14) and (15).

THEOREM 5. *Let* $\gamma(\alpha, 0) = 0$ *and* $z(\alpha, 0) = \alpha + s_0(\alpha)$, *where* s_0 *is analytic in the strip* $|\mathrm{Im}\,\gamma| \leqq \rho_0$. *If* $\|s_{0\alpha}\|_{\infty,\rho_0} \leqq \varepsilon$ *sufficiently small, then there exists a unique analytic solution* $\{z(\alpha, t), \gamma(\alpha, t)\}$ *for* $0 \leqq t \leqq T_c$ *satisfying* $\|s_\alpha\|_{\infty,\rho(t)} \leqq \varepsilon$ *and* $\|\gamma\|_{\infty,\rho(t)} \leqq \varepsilon$, *where* $\rho(t) = \beta\rho_0 - a^{-1}t$ *for any* $\beta < 1$ *and a depends only on* ε *and* ρ_0.

Theorem 5 can be proved by a straightforward application of the abstract Cauchy-Kowalewski theorem [1] to the system:

$$\frac{d}{dt}\begin{pmatrix} s_\alpha^*(\alpha, t) \\ \gamma(\alpha, t) \end{pmatrix} = \begin{pmatrix} \partial_\alpha \dfrac{1}{4\pi i} \displaystyle\int_{-\pi}^{\pi} \gamma(\alpha + \xi, t) \cot\left(\frac{s(\alpha, t) - s(\alpha + \xi, t) - \xi}{2}\right) d\xi \\ i(s_\alpha - s_\alpha^*) \end{pmatrix}$$

with the norm

$$\left|\left|\left|\begin{pmatrix} u \\ v \end{pmatrix}\right|\right|\right| = |||u|||_\rho + |||v|||_\rho.$$

The estimates to show that this system satisfies the hypotheses of the Cauchy-Kowalewski theorem are similar to those found in [3]. We omit the proof.

To introduce our numerical scheme for the Boussinesq approximation, we first define the discrete Fourier transform:

$$\hat{f}^d(k)=\frac{1}{2N}\sum_{j=-N}^{N-1} f(jh)\, e^{-ikjh},$$

where f is assumed to be 2π-periodic, $h=\pi/N$, and $k=-N, -N+1, \cdots, N-1$. It is easy to check the following inversion formula:

$$f(jh)=\sum_{k=-N}^{N-1} \hat{f}^d(k)\, e^{ikjh}.$$

We now define our numerical method, which is of spectral accuracy. Let $w_k(t) \sim z(\alpha_k, t)$ and $\eta_k(t) \sim \gamma(\alpha_k, t)$. Our method is

$$\frac{d}{dt} w_k(t)^* = \frac{h}{2\pi i}\sum_j \eta_{k+2j+1}(t) \cot\left(\frac{w_k(t)-w_{k+2j+1}(t)}{2}\right), \tag{16}$$

$$\frac{d}{dt}\eta_k(t)=\sum_{j=-N}^{N-1} i\cdot j(\hat{w}^d(j)-\hat{w}^{*d}(j))\, e^{ijkh}. \tag{17}$$

THEOREM 6. *Suppose* $\{z(\alpha_k, t), \gamma(\alpha_k, t)\}$ *is an analytic solution of the Boussinesq approximation* (14)–(15) *for* $t \leqq T_c$, *and* $\{w_k(t), \eta_k(t)\}$ *is a solution of* (16)–(17). *Under the assumption of Theorem* 5, *we have*

$$\|w(t)-z(\cdot, t)\|_{l_\infty} \leqq C(T_1)\, e^{-\pi\rho'/h}\, e^{2\tilde{C}(T_1)t/h},$$

$$\|\eta(t)-\gamma(\cdot, t)\|_{l_\infty} \leqq C(T_1)\, e^{-\pi\rho'/h}\, e^{2\tilde{C}(T_1)t/h}$$

for $t \leqq T=\min(T_1, (\pi\rho'/2\tilde{C})-s)$ *for any* $\rho'<\rho$ *and* $s>0$. *Here* $T_1<T_c$, *and* ρ *is defined as in Theorem* 5.

Proof. First, we observe that

$$\int_{-\pi}^{\pi} \gamma(\alpha+\xi, t)\cot\left(\frac{z(\alpha, t)-z(\alpha+\xi)}{2}\right) d\xi$$
$$=\int_{-\pi}^{\pi}\left(\gamma(\alpha+\xi, t)\cot\left(\frac{z(\alpha, t)-z(\alpha+\xi)}{2}\right)+\frac{\gamma(\alpha)}{z_\alpha(\alpha)}\cot\left(\frac{\alpha}{2}\right)\right) d\xi,$$

since the left side is a principal value integral. It is easy to see that the integrand in the right side is analytic in ξ. Thus we can show by arguing exactly as in the proof of the consistency lemma in § 2.3 that

$$\left|\frac{1}{4\pi i}\int_{-\pi}^{\pi}\gamma(\alpha_k+\xi)\cot\left(\frac{z(\alpha_k)-z(\alpha_k+\xi)}{2}\right) d\xi-\frac{h}{4\pi i}\sum_j \gamma_{k+2j+1}\cot\left(\frac{z_k-z_{k+2j+1}}{2}\right)\right|$$
$$\leqq 4C\, e^{-\pi\rho'/h}.$$

Next, we note that

$$|\hat{z}(k)-\hat{z}^d(k)|=\left|\hat{z}(k)-\frac{1}{2N}\sum_{j=-N}^{N-1} z(jh)\, e^{-ikjh}\right|$$
$$=\left|\hat{z}(k)-\frac{1}{2N}\sum_{l=-\infty}^{\infty}\hat{z}(l)\left(\sum_{j=-N}^{N-1} e^{ij(l-k)h}\right)\right|.$$

Since

$$\sum_{j=-N}^{N-1} e^{ij(k-l)h} = \begin{cases} 2N & \text{if } l = k \bmod 2N, \\ 0 & \text{otherwise,} \end{cases} \tag{18}$$

we get

$$|\hat{z}(k) - \hat{z}^d(k)| \leqq \sum_{\substack{l=-\infty \\ l \neq 0}}^{\infty} |\hat{z}(k+2lN)| \leqq \|\|z\|\|_{\rho'} e^{-2\pi\rho'/h} \tag{19}$$

for $\rho' < \rho$ by Lemma 2. Therefore, we have

$$\begin{aligned} \left| z_\alpha(kh) - \sum_{j=-N}^{N-1} i \cdot j\hat{z}^d(j)\, e^{ijkh} \right| &= \left| \sum_{j=-\infty}^{\infty} i \cdot j\hat{z}(j)\, e^{ijkh} - \sum_{j=-N}^{N-1} i \cdot j(\hat{z}^d(j)\, e^{ijkh} \right| \\ &\leqq \left| \sum_{j=-N}^{N-1} i \cdot j(\hat{z}(j) - \hat{z}^d(j))\, e^{ikjh} \right| + \sum_{|j| \geqq N-1} |\hat{z}(k)| \\ &\leqq C\, e^{-2\pi\rho'/h} \quad \text{for } \rho' < \rho, \end{aligned}$$

where we have used Lemma 2 and (19) in the last inequality. A similar result applies to the term z_α^*. This proves the spectral accuracy of the method.

To obtain stability estimates for the method, we use a discrete l_2 norm:

$$\|z\|_{l_2} = \left(h \sum_{k=-N}^{N-1} z_k \overline{z_k} \right)^{1/2}.$$

Clearly, we have $\|z\|_{l_\infty} \leqq h^{-1/2} \|z\|_{l_2}$. Define

$$T^* = \sup\{t \mid t \leqq T_c, \|z(\cdot, t) - w(t)\|_{l_\infty} \leqq h^2, \|\gamma(\cdot, t) - \eta(t)\|_{l_\infty} \leqq h^2\}.$$

Splitting the stability error in z-equation into two terms, we have

$$\begin{aligned} &\frac{h}{2\pi i} \sum_j \gamma_{k+2j+1} \cot\left(\frac{z_k - z_{k+2j+1}}{2}\right) - \frac{h}{2\pi i} \sum_j \eta_{k+2j+1} \cot\left(\frac{w_k - w_{k+2j+1}}{2}\right) \\ &= \frac{h}{2\pi i} \sum_j \gamma_{k+2j+1} \left(\cot\left(\frac{z_k - z_{k+2j+1}}{2}\right) - \cot\left(\frac{w_k - w_{k+2j+1}}{2}\right) \right) \\ &\quad + \frac{h}{2\pi i} \sum_j (\gamma_{k+2j+1} - \eta_{k+2j+1}) \cot\left(\frac{w_k - w_{k+2j+1}}{2}\right) \equiv \mathrm{I}_k + \mathrm{II}_k. \end{aligned}$$

Using Young's inequality and arguing almost exactly as in the proof of the stability lemma in § 2.4, we can show that for $t < T^*$

$$\|\mathrm{I}\|_{l_2} \leqq \frac{C}{h} \|z(\cdot, t) - w(t)\|_{l_2}.$$

For the term II_k, since

$$\left| \cot\left(\frac{w_k - w_{k+2j+1}}{2}\right) \right| \geqq C/h \quad \text{for } t < T^*,$$

Young's inequality implies

$$\begin{aligned} \|\mathrm{II}\|_{l_2} &\leqq \|\gamma(\cdot, t) - \eta(t)\|_{l_2} \max_k \left(\frac{h}{2\pi} \sum_{j=-N}^{N-1} \left| \cot\left(\frac{w_k - w_{k+2j+1}}{2}\right) \right| \right) \\ &\leqq \frac{C}{h} \|\gamma(\cdot, t) - \eta(t)\|_{l_2}. \end{aligned}$$

We now turn to estimating stability errors in the γ-equation. Denote $e_k = z_k - w_k$. The definition of the l_2 norm gives

$$\text{III} \equiv \left\| \sum_{j=-N}^{N-1} i \cdot j \hat{e}_j^d e^{ijkh} \right\|_{l_2}^2 = \frac{1}{2N} \sum_{k=-N}^{N-1} \left(\sum_{j=-N}^{N-1} i \cdot j \hat{e}_j^d e^{ijkh} \right) \left(\sum_{j=-N}^{N-1} (-i \cdot j) \overline{\hat{e}_j^d} e^{-ijkh} \right)$$

$$= \frac{1}{2N} \sum_{k=-N}^{N-1} \sum_{-N<j,l<N-1} j \cdot \hat{e}_j^d \overline{\hat{e}_l^d} e^{ik(j-l)h}.$$

Interchanging the order of summations, we obtain

$$\text{III} = \frac{1}{2N} \sum_{-N<j,l<N-1} j \cdot l \hat{e}_j^d \overline{\hat{e}_l^d} \sum_{k=-N}^{N-1} e^{ik(j-l)h}.$$

Using (18), we get

$$\text{III} = \sum_{-N<j<N-1} j \cdot j \hat{e}_j^d \overline{\hat{e}_j^d} \leqq N^2 \sum_{j=-N}^{N-1} \hat{e}_j^d \overline{\hat{e}_j^d}.$$

The discrete Plancherel identity [6] (which can be verified directly using (18))

$$\sum_{j=-N}^{N-1} \hat{e}_j^d \overline{\hat{e}_j^d} = \frac{1}{2N} \sum_{j=-N}^{N-1} e_j \overline{e_j}$$

then implies

$$\text{III} \leqq N^2 \|e\|_{l_2}^2.$$

Therefore we have proved

$$\left\| \sum_{j=-N}^{N-1} i \cdot j \hat{e}_j^d e^{ijkh} \right\|_{l_2} \leqq \frac{C}{h} \|e\|_{l_2}.$$

This completes the stability estimates. Thus Theorem 6 follows from the consistency and stability estimates in the same way as in the proof of Theorem 2.

Remark. Convergence of the time discrete method can be proved in a similar fashion as in § 2.5. We believe that the techniques developed here can also be used to prove convergence of the vortex method for the full Rayleigh–Taylor problem.

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