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HOMOGENIZATION OF INCOMPRESSIBLE EULER EQUATIONS *1)

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Dedicated to Professor Zhong-ci Shi on the occasion of his 70th birthday

Abstract

In this paper, we perform a nonlinear multiscale analysis for incompressible Euler equations with rapidly oscillating initial data. The initial condition for velocity field is assumed to have two scales. The fast scale velocity component is periodic and is of order one. One of the important questions is how the two-scale velocity structure propagates in time and whether nonlinear interaction will generate more scales dynamically. By using a Lagrangian framework to describe the propagation of small scale solution, we show that the two-scale structure is preserved dynamically. Moreover, we derive a well-posed homogenized equation for the incompressible Euler equations. Preliminary numerical experiments are presented to demonstrate that the homogenized equation captures the correct averaged solution of the incompressible Euler equation.

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1. Introduction

In this paper, we study homogenization of the incompressible Euler equation with highly oscillating initial velocity field. The understanding of scale interactions for 3-D incompressible Euler and Navier-Stokes equations has been a major challenge. For high Reynolds number flows, the degrees of freedom are so high that it is almost impossible to resolve all small scales by direct numerical simulations. Deriving an effective equation for the large scale solution is very useful in engineering applications. The nonlinear and nonlocal nature of the Euler equations makes it difficult to construct a properly-posed multiscale solution. If one does not make the correct assumption in the asymptotic expansion of the multiscale solution, one may not be able to derive a well-posed homogenized equation.

The homogenization of the incompressible Euler equation with oscillating data was first studied by McLaughlin, Papanicolaou and Pironneau (MPP for short) in 1985 [8]. To construct

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a multiscale expansion for the solution of the Euler equation, they made a critical assumption that the oscillation is convected by the mean flow. Using multiscale expansion techniques, MPP obtained a periodic cell problem for the velocity field and the pressure. However, it is not clear whether the resulting cell problem has a solution that is periodic in both the fast space variable $\mathbf{y}$ and the fast time variable τ. Even if such solution exists, it may not be unique. Additional assumptions were imposed on the solution of the cell problem in order to derive a variant of the $k - \epsilon$ model.

Our study shows that the small scale oscillations are actually convected by the full oscillatory velocity field. The multiscale structure of the solution becomes apparent when we formulate the Euler equations in vorticity-stream function formulation and use the Lagrangian flow map to describe the propagation of oscillations. Vorticity preserves naturally the multiscale structure of its initial data via the Lagrangian formulation. Velocity can be constructed using the vorticity-stream function formulation. By using a Lagrangian description, we characterize the nonlinear convection of small scales exactly and turn a convection dominated transport problem into an elliptic problem for the stream function. Thus, traditional homogenization result for elliptic problems [1] can be used to obtain multiscale expansions for the stream function and the flow map respectively. Using the insight we obtain from the homogenization theory in the Lagrangian frame, we also derive the corresponding homogenized equation in the Eulerian frame, which can be used more effectively for computational purpose. The effect of viscosity can be included in our analysis. For the sake of simplicity in our presentation, we will not include the viscous effect in the present analysis.

The homogenization theory of the incompressible Euler equation provides a useful guideline in designing effective multiscale computational methods for incompressible flows. In particular, we can use the homogenization theory to examine the role of the so-called Reynolds stress [10, 4]. By making appropriate assumption on the homogeneity of the flow, we can propose new coarse grid model for the large scale solution which is dynamically coupled to a subgrid cell problem. Our multiscale analysis also reveals that without external forcing and viscosity effect, we need to remove certain resonant velocity component in the cell velocity field in order to avoid some secular growth in the multiscale expansion. This resonant cell velocity field corresponds to the non-mixable part of the velocity field. Removing this non-mixable velocity component is equivalent to adding a high frequency forcing term to the incompressible Euler equation, which has the effect of accelerating the flow mixing.

We have performed some preliminary numerical experiments to confirm the convergence of our multiscale analysis. For practical purpose, it is important to generalize our two-scale analysis to problems that do not have scale separation. We can do this either in the physical space or in the Fourier space. From the preliminary computations we present in this paper, we found that the numerical solution obtained from the homogenized equation gives an accurate approximation to the corresponding well-resolved solution of the incompressible Euler equation. Small scale velocity field can be reconstructed from the large scale solution and the subgrid cell solution. Furthermore, we found that there is no need to use a projection method to remove the resonant velocity component if the initial velocity field does not contain such resonant component.

The organization of the rest of the paper is as follows. In Section 2, we will present the formulation of the Euler equations with rapidly oscillating initial data. We will also review the previous work by MPP in some details. Section 3 is devoted to developing the multiscale analysis of the Euler equations in the Lagrangian formulation. We will use the insight gained in Section 3 to perform a multiscale analysis in the Eulerian formulation in Section 4. In Section 5, we will present some numerical examples to demonstrate the convergence of our multiscale analysis.

2. Formulation

We consider the incompressible Euler equation

$$\mathbf{u}_t^\epsilon + (\mathbf{u}^\epsilon \cdot \nabla)\mathbf{u}^\epsilon = -\nabla p^\epsilon, \tag{1}$$

$$\nabla \cdot \mathbf{u}^\epsilon = 0, \tag{2}$$

with multiscale initial data $\mathbf{u}^\epsilon(\mathbf{x}, 0) = \mathbf{u}_0^\epsilon(\mathbf{x})$. Here $\mathbf{u}^\epsilon(t, \mathbf{x})$ and $p^\epsilon(t, \mathbf{x})$ are velocity and pressure respectively. Throughout the paper, we will use boldface letters to denote vector variables. For the time being, we only consider the infinite domain and assume that the solution decays to zero sufficiently fast at infinity. Further, we assume that $\mathbf{u}^\epsilon(\mathbf{x}, 0) = \mathbf{U}(\mathbf{x}) + \mathbf{W}(\mathbf{x}, \frac{\mathbf{x}}{\epsilon})$, where $\mathbf{W}(\mathbf{x}, \mathbf{y})$ is periodic in $\mathbf{y}$ and has mean zero. This is a typical assumption in homogenization theory [1]. The question of interest is how to derive a homogenized equation for the averaged velocity field for small but finite ϵ.

The homogenization of the Euler equation with oscillating data was first studied by McLaughlin-Papanicolaou-Pironneau (MPP for short) [8]. In [8], MPP made an important assumption that the small scale oscillation is convected by the mean flow. Based on this assumption, they made the following multiscale expansion for the velocity and pressure:

$$\mathbf{u}^\epsilon(t, \mathbf{x}) = \mathbf{u}(t, \mathbf{x}) + \mathbf{w}(t, \mathbf{x}, \frac{t}{\epsilon}, \frac{\boldsymbol{\theta}(t, \mathbf{x})}{\epsilon}) + \epsilon \mathbf{u}_1(t, \mathbf{x}, \frac{t}{\epsilon}, \frac{\boldsymbol{\theta}(t, \mathbf{x})}{\epsilon}) + \cdots$$

$$p^\epsilon(t, \mathbf{x}) = p(t, \mathbf{x}) + q(t, \mathbf{x}, \frac{t}{\epsilon}, \frac{\boldsymbol{\theta}(t, \mathbf{x})}{\epsilon}) + \epsilon p_1(t, \mathbf{x}, \frac{t}{\epsilon}, \frac{\boldsymbol{\theta}(t, \mathbf{x})}{\epsilon}) + \cdots$$

where $\mathbf{w}(t, \mathbf{x}, \tau, \mathbf{y})$, $\mathbf{u}_1(t, \mathbf{x}, \tau, \mathbf{y})$, q, and p_1 are assumed to be periodic in both $\mathbf{y}$ and τ, and the phase $\boldsymbol{\theta}$ is convected by the mean velocity field $\mathbf{u}$

$$\frac{\partial \boldsymbol{\theta}}{\partial t} + \mathbf{u} \cdot \nabla_{\mathbf{x}} \boldsymbol{\theta} = 0, \quad \boldsymbol{\theta}(0, \mathbf{x}) = \mathbf{x}\ . \tag{3}$$

By substituting the above multiscale expansions into the Euler equation and equating coefficients of the same order, MPP obtained a homogenized equation for $(\mathbf{u}, p)$, a periodic cell problem for $(\mathbf{w}(t, \mathbf{x}, \tau, \mathbf{y}),\ q(t, \mathbf{x}, \tau, \mathbf{y}))$. On the other hand, it is not clear whether the resulting cell problem for $\mathbf{w}$ and q has a unique solution that is periodic in both $\mathbf{y}$ and τ. Additional assumptions were imposed on the solution of the cell problem in order to derive a variant of the $k - \epsilon$ model.

The understanding of how small scale solution being propagated dynamically is clearly very important in deriving the homogenized equation. Our study shows that the small scale oscillations are convected by the full oscillatory velocity field, not just the mean velocity:

$$\frac{\partial \boldsymbol{\theta}^\epsilon}{\partial t} + \mathbf{u}^\epsilon \cdot \nabla_{\mathbf{x}} \boldsymbol{\theta}^\epsilon = 0, \quad \boldsymbol{\theta}^\epsilon(0, \mathbf{x}) = \mathbf{x}\ . \tag{4}$$

This is clear for the 2-D Euler equation since vorticity, ω^ϵ, is conserved along the characteristics:

$$\omega^\epsilon(t, \mathbf{x}) = \omega_0(\boldsymbol{\theta}^\epsilon(t, \mathbf{x}), \frac{\boldsymbol{\theta}^\epsilon(t, \mathbf{x})}{\epsilon}),$$

where $\omega_0(\mathbf{x}, \mathbf{x}/\epsilon)$ is the initial vorticity, which is of order $O(1/\epsilon)$. Similar conclusion can be drawn for the 3-D Euler equation. Now the multiscale structure of $\boldsymbol{\theta}^\epsilon(\mathbf{x}, t)$ is coupled to the multiscale structure of $\mathbf{u}^\epsilon$. In some sense, we embed multiscale structure within multiscale expansions. It is quite a challenge to unfold the multiscale solution structure. Naive multiscale expansion for $\boldsymbol{\theta}^\epsilon$ may lead to generation of infinite number of scales for $t > 0$.

3. Multiscale Analysis in Lagrangian Frame

The key idea in our multiscale analysis for the Euler equation is to reformulate the problem using $\boldsymbol{\theta}^\epsilon$ as a new variable. This amounts to using a Lagrangian description of the Euler equation. Specifically, we introduce a change of variables from $\mathbf{x}$ to $\boldsymbol{\alpha}$ with $\boldsymbol{\alpha} = \boldsymbol{\theta}^\epsilon(t, \mathbf{x})$. It is easy to see that the inverse of this map, denoted as $\mathbf{x} = \mathbf{X}^\epsilon(t, \boldsymbol{\alpha})$, is the Lagrangian flow map:

$$\frac{\partial \mathbf{X}(t, \boldsymbol{\alpha})}{\partial t} = \mathbf{u}^\epsilon(t, \mathbf{X}(t, \boldsymbol{\alpha})), \quad \mathbf{X}(0, \boldsymbol{\alpha}) = \boldsymbol{\alpha}. \tag{5}$$

In terms of the $\boldsymbol{\alpha}$ variable, the vorticity of the 3-D Euler equation has a simple expression [2]:

$$\boldsymbol{\omega}^\epsilon(t, \mathbf{X}^\epsilon(t, \boldsymbol{\alpha})) = \frac{\partial \mathbf{X}^\epsilon}{\partial \boldsymbol{\alpha}}(t, \boldsymbol{\alpha}) \boldsymbol{\omega}_{\mathbf{0}}(\boldsymbol{\alpha}, \frac{\boldsymbol{\alpha}}{\epsilon}) \ , \tag{6}$$

where $\boldsymbol{\omega}^\epsilon = \nabla_{\mathbf{x}} \times \mathbf{u}^\epsilon$, and $\boldsymbol{\omega}_0$ is the initial vorticity. Velocity can be computed via the stream function $\boldsymbol{\psi}^\epsilon$, i.e. $\mathbf{u}^\epsilon = \nabla_{\mathbf{x}} \times \boldsymbol{\psi}^\epsilon$. Further, the stream function, $\boldsymbol{\psi}^\epsilon$, satisfies

$$- \triangle_{\mathbf{x}} \, \boldsymbol{\psi}^\epsilon = \boldsymbol{\omega}^\epsilon \ .$$

In terms of the $\boldsymbol{\alpha}$ variable, we have

$$-\nabla_{\boldsymbol{\alpha}} \cdot \left(\mathcal{A}\mathcal{A}^\top \nabla_{\boldsymbol{\alpha}} \boldsymbol{\psi}^\epsilon \right) = \frac{\partial \mathbf{X}^\epsilon}{\partial \boldsymbol{\alpha}}(t, \boldsymbol{\alpha}) \boldsymbol{\omega}_0(\boldsymbol{\alpha}, \frac{\boldsymbol{\alpha}}{\epsilon}) \ , \tag{7}$$

where $\mathcal{A} = \left(\frac{\partial \mathbf{X}^\epsilon}{\partial \boldsymbol{\alpha}}\right)^{-1}$, and $\mathcal{A}^\top$ is the transpose of $\mathcal{A}$. Since the flow is incompressible, we have $|\frac{\partial \mathbf{X}^\epsilon}{\partial \boldsymbol{\alpha}}| = 1$. Thus we can express $\mathcal{A}$ in terms of $\frac{\partial \mathbf{X}^\epsilon}{\partial \boldsymbol{\alpha}}$. We can see clearly that the small scale structure is propagated along the Lagrangian trajectory as a function of $\boldsymbol{\alpha}/\epsilon$.

We are interested in studying the mixing property and the large time behavior of the Euler equation. For this reason, we look for multiscale solutions of the form:

$$\begin{aligned} \boldsymbol{\psi}^\epsilon &= \boldsymbol{\psi}^{(0)}(t, \boldsymbol{\alpha}) + \epsilon \boldsymbol{\psi}^{(1)}(t, \boldsymbol{\alpha}, \tau, \mathbf{y}) + O(\epsilon^2) \ , \\ \mathbf{X}^\epsilon &= \mathbf{X}^{(0)}(t, \boldsymbol{\alpha}) + \epsilon \mathbf{X}^{(1)}(t, \boldsymbol{\alpha}, \tau, \mathbf{y}) + O(\epsilon^2), \end{aligned}$$

where $\mathbf{y} = \boldsymbol{\alpha}/\epsilon$, $\tau = t/\epsilon$, and $\boldsymbol{\psi}^{(1)}$ and $\mathbf{X}^{(1)}$ are periodic functions with respect to $\mathbf{y}$ and have zero mean.

By performing careful multiscale analysis, we can obtain the homogenized equations for both two and three space dimensional Euler equations. For simplicity, we only state the main result for the 2-D Euler equation. Detailed multiscale analysis for the 3-D Euler equation can be found in [7]. The homogenized equations for $\mathbf{X}^{(0)}$, $\mathbf{X}^{(1)}$, and $\psi^{(0)}$, $\psi^{(1)}$ are given as follows:

$$\partial_t \mathbf{X}^{(0)} - \left(\nabla_{\boldsymbol{\alpha}}^\perp \psi^{(0)} \cdot \nabla_{\boldsymbol{\alpha}} \right) \mathbf{X}^{(0)} = 0, \quad \mathbf{X}^{(0)}|_{t=0} = \boldsymbol{\alpha}, \tag{8}$$

$$\partial_\tau \mathbf{X}^{(1)} - \left(\nabla_{\boldsymbol{\alpha}} \mathbf{X}^{(0)} + \nabla_{\mathbf{y}} \mathbf{X}^{(1)} \right) \nabla_{\mathbf{y}}^\perp \psi^{(1)} = 0, \quad \mathbf{X}^{(1)}|_{\tau=0} = \mathbf{0}, \tag{9}$$

$$\nabla_{\boldsymbol{\alpha}}^\perp \cdot \left(\nabla_{\boldsymbol{\alpha}} X^{(0)^\top} \nabla_{\boldsymbol{\alpha}} X^{(0)} \nabla_{\boldsymbol{\alpha}}^\perp \psi^{(0)} \right) + \nabla_{\boldsymbol{\alpha}}^\perp \cdot \langle \mathcal{A}_0^\top \mathcal{A}_0 \nabla_{\mathbf{y}}^\perp \psi^{(1)} \rangle = \nabla_{\boldsymbol{\alpha}}^\perp \cdot \mathbf{U}, \tag{10}$$

$$\nabla_{\mathbf{y}}^\perp \cdot \left(\mathcal{A}_0^\top \mathcal{A}_0 \nabla_{\mathbf{y}}^\perp \psi^{(1)} \right) = \nabla_{\mathbf{y}}^\perp \cdot \mathbf{W}, \tag{11}$$

where $\nabla_{\boldsymbol{\alpha}}^\perp = (-\partial_{\alpha_2}, \partial_{\alpha_1})$, $\langle f \rangle$ stands for the average of f as a function $\mathbf{y}$ over one period, $\mathbf{U}$ and $\mathbf{W}$ are the initial mean and fluctuating velocity components respectively, and $\mathcal{A}_0 = \nabla_{\boldsymbol{\alpha}} \mathbf{X}^{(0)} + \nabla_{\mathbf{y}} \mathbf{X}^{(1)}$ is the leading order term in the expansion of the Jacobian matrix of $\mathcal{A}$. It can be shown that $|\mathcal{A}_0| \equiv 1$, which implies the well-posedness of the $\psi^{(1)}$-equation. Note that equation (11) is an elliptic equation for the first order corrector $\psi^{(1)}$ as a function of $\mathbf{y}$ with periodic boundary condition. It is similar to the first order corrector in the standard elliptic homogenization.

The elliptic problems (10)-(11) for $\psi^{(0)}$ and $\psi^{(1)}$ are clearly solvable with appropriate boundary condition. However, to avoid the secular growth of the multiscale expansion, we need to ensure that $\epsilon \mathbf{X}^1 \to 0$ as $\epsilon \to 0$.

Let $\mathbf{w} = \nabla_{\mathbf{y}}^\perp \psi^{(1)}$ be the cell velocity, and $\mathbf{Y}(\tau, \mathbf{y})$ be the cell characteristic, i.e. for each $\boldsymbol{\alpha}$ and t fixed, $\mathbf{Y}(\tau, \mathbf{y})$ satisfies

$$\frac{d\mathbf{Y}}{d\tau} = \mathbf{w}(\tau, \mathbf{Y}) \ , \quad \mathbf{Y}(0, \mathbf{y}) = \mathbf{y}.$$

If there is a non-mixable resonant component in the cell velocity field, we will have

$$\frac{1}{T}\int_0^T \mathbf{w}(t,\boldsymbol{\alpha},\tau,\mathbf{Y}(\tau,\mathbf{y}))d\tau \to \eta(t,\boldsymbol{\alpha},\mathbf{y}) \neq 0 \quad \text{as } T\to\infty.$$

In this case, $\epsilon\mathbf{X}^1$ will be of order $O(t)$ for $t>0$, i.e. the flow map will develop an $O(1)$ oscillation dynamically. To avoid this secular growth, we will use the following projection method to remove the resonant component from the cell velocity field $\mathbf{w}$:

$$\mathbf{w}(t,\boldsymbol{\alpha},\tau,\mathbf{y}) \quad \leftarrow \quad \mathbf{w}(t,\boldsymbol{\alpha},\tau,\mathbf{y}) - \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathbf{w}(t,\boldsymbol{\alpha},\tau,\mathbf{Y}(\tau,\mathbf{y}))d\tau\ .$$

For the Navier-Stokes equation, viscosity and random forcing play the role to eliminate the non-mixable component of the flow velocity. Eliminating this non-mixable component is essential for the flow to be fully mixed. For the inviscid Euler equation, the projection method is equivalent to adding a high frequency forcing to the momentum equation to eliminate the non-mixable component. This can be also viewed as an acceleration method to enhance the mixing of the flow.

4. Multiscale Analysis in the Eulerian Frame

In this section, we will derive the homogenized equation in the Eulerian formulation. In practice, it is more convenient to compute the homogenized equation in the Eulerian formulation. In deriving the multiscale analysis in the Eulerian frame, we use the phase variable $\boldsymbol{\theta}^\epsilon$ to describe the propagation of the small scale component of the velocity field, but use the Eulerian variable to describe the large scale averaged solution. One advantage of this approach is that we can characterize the nonlinear convection of small scales exactly using this phase variable and turn a convection dominated transport problem into an elliptic problem for the pressure. Thus, other multiscale computational methods for elliptic problems (e.g. [6]) can be applied to study the upscaling of the multiscale pressure equation.

Motivated by our multiscale analysis in the Lagrangian frame, we look for multiscale expansions of the velocity field and the pressure of the following form:

$$\mathbf{u}^\epsilon(t,\mathbf{x}) = \mathbf{u}(t,\mathbf{x}) + \mathbf{w}(t,\boldsymbol{\theta}(t,\mathbf{x}),\tau,\mathbf{y}) + \epsilon\mathbf{u}^{(1)}(t,\boldsymbol{\theta}(t,\mathbf{x}),\tau,\mathbf{y}) + \cdots\ , \tag{12}$$

$$p^\epsilon(t,\mathbf{x}) = p(t,\mathbf{x}) + q(t,\boldsymbol{\theta}(t,\mathbf{x}),\tau,\mathbf{y}) + \epsilon p^{(1)}(t,\boldsymbol{\theta}(t,\mathbf{x}),\tau,\mathbf{y}) + \cdots\ , \tag{13}$$

where $\tau = t/\epsilon$ and $\mathbf{y} = \boldsymbol{\theta}^\epsilon(t,\mathbf{x})/\epsilon$. We assume that $\mathbf{w}$, and q have zero mean with respect to $\mathbf{y}$. The phase function $\boldsymbol{\theta}^\epsilon$ is defined in (4) and it has the following multiscale expansion:

$$\boldsymbol{\theta}^\epsilon = \boldsymbol{\theta}(t,\mathbf{x}) + \epsilon\boldsymbol{\theta}^{(1)}(t,\boldsymbol{\theta}(t,\mathbf{x}),\tau,\frac{\boldsymbol{\theta}^\epsilon}{\epsilon}) + \cdots\ . \tag{14}$$

This particular form of multiscale expansion was suggested by our Lagrangian multiscale analysis. If one tried to expand $\boldsymbol{\theta}^\epsilon$ naively as a function of $\mathbf{x}/\epsilon$ and t/ϵ, one would find that there is a generation of infinite number of scales at $t>0$ and would not be able to obtain a well-posed cell problem.

Expanding the Jacobian matrix, we get $\nabla_{\mathbf{x}}\boldsymbol{\theta}^\epsilon = \mathcal{B}^{(0)} + \epsilon\mathcal{B}^{(1)} + \cdots$. Substituting the expansion into the Euler equation and matching the terms of the same order, we obtain the following homogenized equation:

$$\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u} + \nabla_{\mathbf{x}}\cdot\langle\langle\mathbf{w}\mathbf{w}\rangle\rangle = -\nabla_{\mathbf{x}}p, \quad \mathbf{u}|_{t=0} = \mathbf{U}(\mathbf{x})\ , \tag{15}$$

$$\nabla_{\mathbf{x}}\cdot\mathbf{u} = 0, \tag{16}$$

where $\langle\langle\mathbf{w}\mathbf{w}\rangle\rangle$ stands for the space-time average in $(\mathbf{y},\tau)$. The equation for $\mathbf{w}$ is given by

$$\partial_\tau\mathbf{w} + {\mathcal{B}^{(0)}}^\top\nabla_{\mathbf{y}}q = 0, \quad \tau>0;$$

$$({\mathcal{B}^{(0)}}^\top\nabla_{\mathbf{y}})\cdot\mathbf{w} = 0, \quad \mathbf{w}|_{\tau=0} = \mathbf{W}(\mathbf{x},\mathbf{y}), \quad t=0\ .$$

Further, it can be shown that ${\mathcal{B}^{(0)}}^\top \nabla_{\mathbf{y}} q$ has zero mean in $\mathbf{y}$.

Moreover, we can derive the evolution equations for $\boldsymbol{\theta}$ and $\boldsymbol{\theta}^{(1)}$ as follows

$$\partial_t \boldsymbol{\theta} + (\mathbf{u}\cdot\nabla_{\mathbf{x}})\boldsymbol{\theta} = \mathbf{0}, \quad \boldsymbol{\theta}|_{t=0} = \mathbf{x}, \tag{17}$$

$$\partial_\tau \boldsymbol{\theta}^{(1)} + (\mathbf{w}\cdot\nabla_{\mathbf{x}})\boldsymbol{\theta} = 0, \quad \boldsymbol{\theta}^{(1)}|_{\tau=0} = \mathbf{0}. \tag{18}$$

From $\boldsymbol{\theta}$ and $\boldsymbol{\theta}^{(1)}$, we can compute the Jacobian matrix $\mathcal{B}^{(0)}$ as follows:

$$\mathcal{B}^{(0)} = (I - D_y \boldsymbol{\theta}^{(1)})^{-1} \nabla_{\mathbf{x}} \boldsymbol{\theta}\ . \tag{19}$$

As in the Lagrangian case, we also need to remove the non-mixable component of the cell velocity to avoid secular growth in $\epsilon\boldsymbol{\theta}^{(1)}$. For this purpose, we apply a projection method to $\mathbf{w}$ to remove the component of $\mathbf{w}$ that has nonzero mean in τ:

$$\mathbf{w} \ \leftarrow\ \mathbf{w} - \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{w}\, d\tau.$$

We remark that there is some related work on homogenization of the Navier-Stokes equation with large amplitude high frequency periodic forcing, see, e.g. [3, 9]. In these problems, the multiscale solution has a simple structure, $\mathbf{u}^\epsilon = \mathbf{u}(t,\mathbf{x}) + \mathbf{w}(t,\mathbf{x},\mathbf{x}/\epsilon) + \cdots$. The cell problem is diffusion dominated, based on the balance between the high frequency forcing and the cell diffusion.

5. Numerical Experiments

The above multiscale analysis can be generalized to problems with general multiscale initial data without scale separation and periodic structure. This can be done either in the physical space or in the Fourier space. We found it more effective to work on the Fourier space. We restrict our computation to 2-D incompressible fluids with oscillatory initial data in a doubly periodic box of size $2\pi \times 2\pi$. We apply the pseudo-spectral method to compute both the large-scale homogenized equation for $\mathbf{u}$ and the small-scale cell problem for $\mathbf{w}$. The 4th-order Adamas-Bashforth time discretization method is used for both the homogenized equation and the cell problem. To obtain a more effective discretization of the cell problem, we reformulate the cell problem by making the following change of variables in $\mathbf{y}$ and $\mathbf{w}$:

$$\mathbf{z} \ = \ \mathbf{y} - \boldsymbol{\theta}_1(t, \boldsymbol{\theta}(t,\mathbf{x}), \tau, \mathbf{y}), \tag{20}$$

$$\widetilde{\mathbf{w}} \ = \ (\nabla_{\mathbf{x}}\boldsymbol{\theta})\mathbf{w}. \tag{21}$$

Then, the cell problem for $\mathbf{w}$ becomes

$$\partial_\tau \widetilde{\mathbf{w}} + (\widetilde{\mathbf{w}}\cdot\nabla_{\mathbf{z}})\widetilde{\mathbf{w}} + (\nabla_{\mathbf{x}}\boldsymbol{\theta})(\nabla_{\mathbf{x}}\boldsymbol{\theta}^\top)\nabla_{\mathbf{z}} q = 0, \tag{22}$$

$$\nabla_{\mathbf{z}}\cdot\widetilde{\mathbf{w}} = 0, \tag{23}$$

$$\widetilde{\mathbf{w}}|_{\tau=0} = \mathbf{W}(\mathbf{x},\mathbf{z}), \quad t = 0. \tag{24}$$

In our computations, we solve the homogenized equation (15) and the cell problem (22) together within each coarse grid time step. For example, we update the mean velocity by an explicit time marching method using the mean velocity at time $t_n = n\Delta t$ and the cell velocity $\mathbf{w}$ from the previous time step, $\tau_n = t_n/\epsilon$. We then update the cell solution $\mathbf{w}$ in the fast time variable τ from τ_n to τ_{n+1} using the mean velocity at t_n and the cell velocity $\mathbf{w}$ at τ_n.

To check the accuracy of our homogenized equation, we compare the computational result obtained by solving the homogenized equation with that obtained by a well resolved direct numerical simulation (DNS). Further, we use the first two terms in the multiscale expansion for the velocity field to reconstruct the fine grid velocity field. More specifically, we use the spectral interpolation of the phase function, and approximate the velocity field by

$$\mathbf{u}^\varepsilon(t,\mathbf{x}) \sim \mathbf{u}(\mathbf{t},\mathbf{x}) + \mathbf{w}(\mathbf{t}, \boldsymbol{\theta}(\mathbf{t},\mathbf{x}), \tau, \frac{\boldsymbol{\theta}^\epsilon(\mathbf{t},\mathbf{x})}{\varepsilon}). \tag{25}$$

In setting up initial data, we sample the stream function for both the mean flow and the fluctuation from the same Fourier distribution $\hat{\psi}(\mathbf{k})$, e.g.,

$$|\hat{\psi}(k)| = \frac{k}{k^\alpha + \delta}, \qquad k \equiv |\mathbf{k}|, \quad k \le k_{\max}, \tag{26}$$

with random phases and properly chosen α and δ. This choice of initial velocity field is inspired by the earlier work of Henshaw-Kreiss-Reyna [5]. In the computation, we choose $\alpha = 4$ and $\delta = 10^{-5}$. The Fourier transform of the large scale components of the stream function is given by

$$\hat{\psi}_<(\mathbf{k}) = \hat{\psi}(\mathbf{k}), \qquad k \le k_{\text{cutoff}}, \tag{27}$$

and the Fourier transform of the small scale components of the stream function is given in terms of $\psi_>(\mathbf{y})$ with $\mathbf{y} = \mathbf{x}/\epsilon$, where $\psi_>(\mathbf{y})$ is homogeneous with no dependence on slow variable and is given by

$$\hat{\psi}_>(\mathbf{k}) = \frac{1}{\varepsilon^2}\hat{\psi}(\frac{\mathbf{k}}{\varepsilon}), \qquad \frac{k}{\varepsilon} \le k_{\max}. \tag{28}$$

One can now generate the initial velocity field for the mean, $\mathbf{U}(\mathbf{x})$, and the fluctuation, $\mathbf{W}(\mathbf{y})$ from $\psi_<(\mathbf{x})$ and $\psi_>(\mathbf{y})$ respectively. For this particular initial condition, $\mathbf{U}$ has a maximum value equal to 1, and $\mathbf{W}$ has a maximum value 0.8. We plot the Fourier spectrum of the initial horizontal velocity field in Figure 4a. As we can see, there is no scale separation in this initial condition.

In the computation, we choose $\varepsilon = 0.05$, $k_{\max} = 200$ and accordingly $k_{\text{cutoff}} = 1/\varepsilon$. The dimension of the coarse grid (slow variable) is 64×64 and that of small scale (fast variable) is 32×32. The time-step for the homogenized equation is $\Delta t = 10^{-2}$ and the subgrid time-step for the cell problem is $\Delta\tau = 10^{-2}$. The time average is applied over each interval of length $\Delta t/\varepsilon$. In Figure 1a, we plot the initial horizontal velocity field in the fine mesh. The corresponding coarse grid velocity field is plotted in Figure 1b. We compare the computation obtained by the homogenized equation with that obtained by DNS at $t = 0.5$ in Figure 2. We use the spectral interpolation to reconstruct the fine grid velocity field as a sum of the homogenized solution $\mathbf{u}$ and the cell velocity field $\mathbf{w}$. We can see that the reconstructed velocity field (plotted only on the coarse grid) captures very well the fine grid velocity field obtained by DNS using a 512×512 grid. We also compare the accuracy in the Fourier space, which is given in Figure 4b. The agreement is excellent in both low frequencies and high frequencies. Further, we compare the mean velocity field obtained by the homogenized equation with that obtained by direct simulation using a low pass filter. The results are plotted in Figure 3 and Figure 6 respectively. We can see that the agreement between the two calculations is very good up to $t = 1.0$. The low pass filter we used is a Gaussian-type filter which has a spectral symbol $\exp(-sk^2)$, where s is known as filter scale and k is the module of Fourier modes. We use $s = 0.01$ and $s = 0.005$ respectively in our calculations. The choice of $s = 0.01$ seems to give a better result. We are currently performing convergence study of long time calculations. To avoid excessive deformation in the phase variable, we need to reinitialize the phase variable from time to time. This will be reported elsewhere in the future.

Our objective is to use the homogenization theory to design an effective coarse grid model that can capture correctly the large scale behavior but with a computational cost comparable to the traditional Large Eddy Simulation (LES) models [10, 4]. To achieve this, we need to take into account the special structures in the fully mixed flow, such as homogeneity and possible local self-similarity of the flow. When the flow is fully mixed, we expect that the Reynolds stress term, i.e. $\langle \mathbf{ww} \rangle$, will reach to a statistical equilibrium relatively fast. As a consequence, we may need to solve for the cell problem in τ for only a small number of time steps after updating the effective velocity in one coarse grid time step. Moreover, for homogeneous flow, we need not solve the cell problem for every coarse grid. It should be sufficient to solve one or a few representative cell problems for fully mixed flow and use the solution of these representative cell solutions to compute the Reynolds stress term in the homogenized velocity equation. If this can be achieved, it will offer a significant computational saving.

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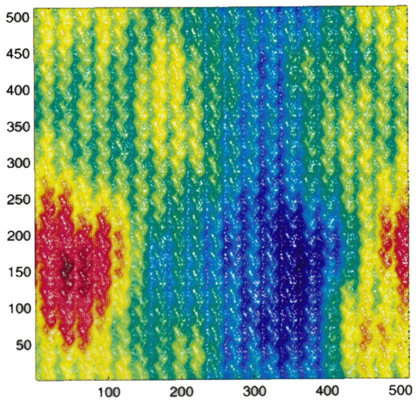
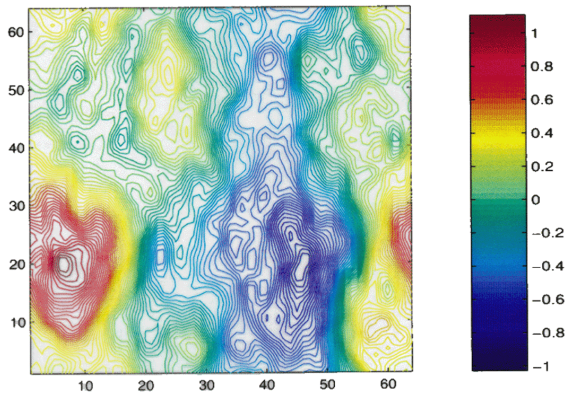
Fig. 1a



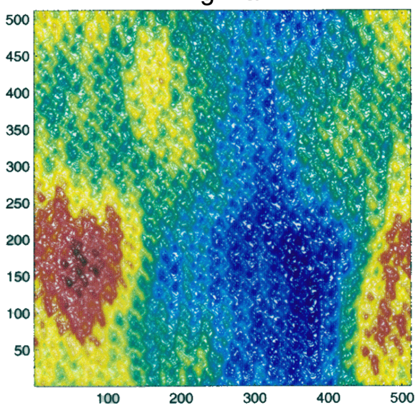
t=0 U+W (fine grid)

Fig. 1b



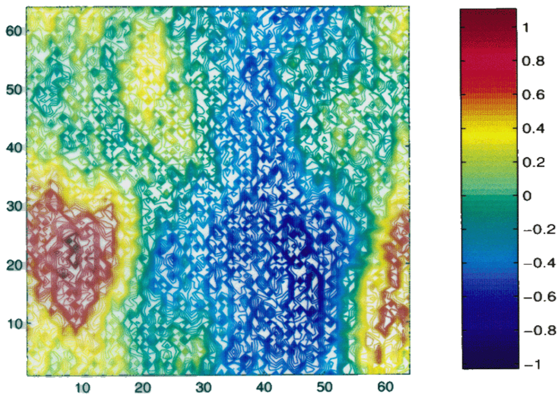
t=0 U (coarse grid)

Fig. 2a



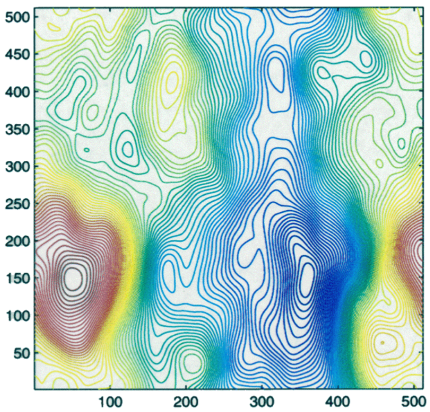
t=0.5 U+W (DNS, fine grid)

Fig. 2b



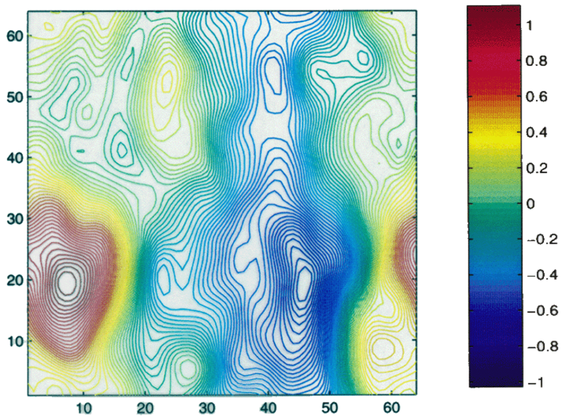
t=0.5 U+W (interpolated on coarse grid)

Fig. 3a



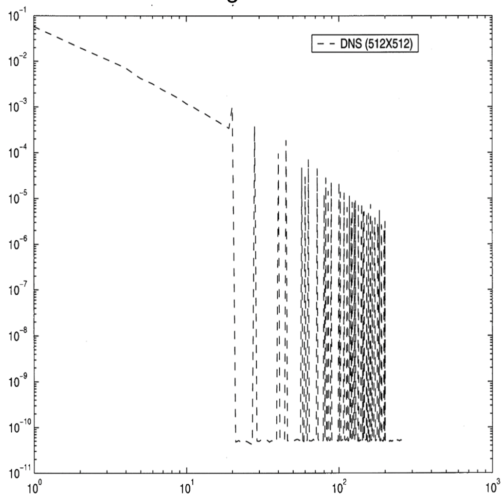
t=1.0 mean flow U (DNS, fine grid)
filter k=0.01

Fig. 3b



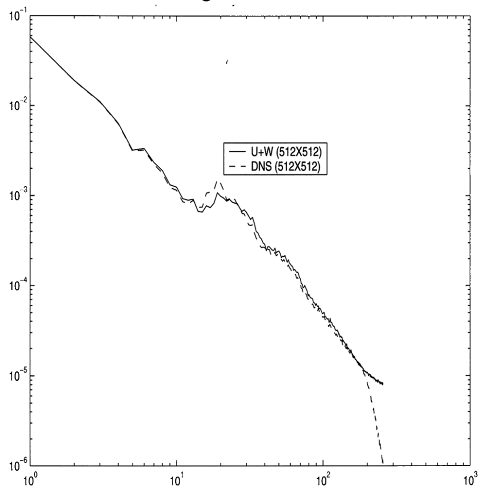
t=1.0 mean flow U (coarse grid)
filter k=0.01

Fig. 4a



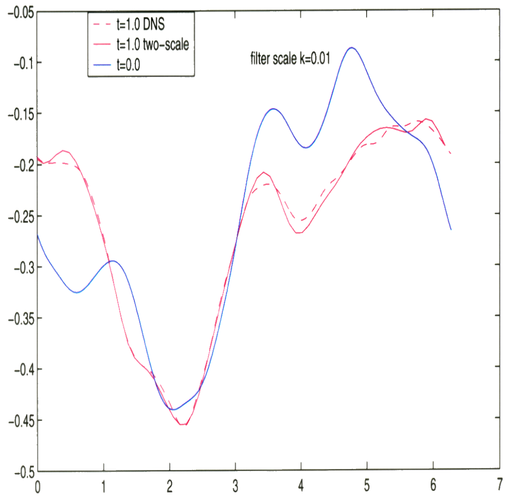
t=0 spectrum of velocity

Fig. 4b



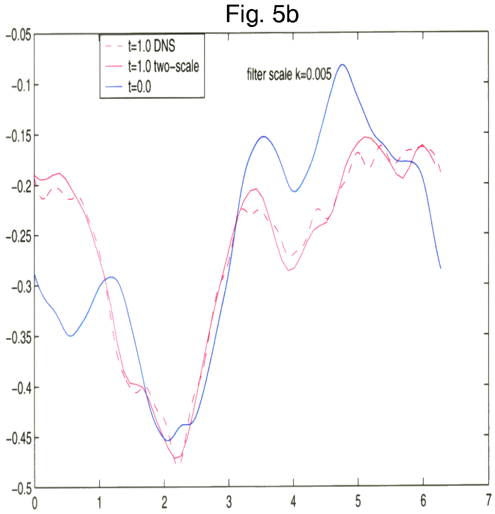
t=0.5 spectrum of velocity

Fig. 5a



t=1.0, cross-section of mean flow U
filter scale = 0.01

Fig. 5b



t=1.0, cross-section of mean flow U
filter scale = 0.005