Axisymmetric type II blowup solutions to the three dimensional Keller-Segel system

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Abstract

We construct axisymmetric solutions to the three-dimensional parabolic-elliptic Keller-Segel system that blows up in finite time. In particular, the singularity is of type II, which admits locally a leading order profile of the rescaled stationary solution of the two-dimensional system. Additionally, mass concentration occurs along a one-dimensional ring in the plane. In the analysis, we rely on an approximate solution of the eigenproblem associated with the linearized operator around the stationary solution as well as the modulation dynamics to control the perturbation function and derive the accurate blowup rate.

1 Introduction

1.1 Setting of the Problem

We consider the three-dimensional Keller-Segel system

$$\begin{cases} \partial_t u(\mathbf{x},t) = \nabla \cdot (\nabla u(\mathbf{x},t) - u(\mathbf{x},t)\nabla \Phi_u(\mathbf{x},t)) & (\mathbf{x},t) \in \mathbb{R}^3 \times \mathbb{R}, \\ -\Delta \Phi_u(\mathbf{x},t) = u(\mathbf{x},t), \end{cases} \tag{3dKS}$$

where the 3D Poisson field is written as $\Phi_u = \frac{1}{4\pi|\mathbf{x}|} * u$. More generally, one can also consider the d-dimensional Keller-Segel system, which we will discuss soon. The Keller-Segel system is a mathematical model of *Chemotaxis*, a biological phenomenon describing the motion of organisms induced by chemical signals, for example, the motions of slime mold Dictyostelium discoideum and the bacteria Escherichia coli. It was first established by Patlak [28] and Keller & Segel [22]. We refer to [21] and [7] for a survey of this model as well as related mathematical problems. Since the Keller-Segel system (in general dimension d) takes a divergence form, its strong solutions preserve the total mass:

$$\int_{\mathbb{R}^d} u(\mathbf{x},t)\, d\mathbf{x} = \int_{\mathbb{R}^d} u(\mathbf{x},0)\, d\mathbf{x} := M \quad \forall\, t > 0.$$

In addition, the solutions admit an important scaling symmetry (besides the translation symmetries in space and time): if $u(\mathbf{x},t)$ is a solution, then so is

$$u_\lambda(\mathbf{x},t) := \lambda^2 u(\lambda \mathbf{x}, \lambda^2 t),$$

for any $\lambda > 0$. We say that the solution u blows up in finite time T, if

$$\limsup_{t\to T} \|u(t)\|_{L^\infty(\mathbb{R}^d)} = +\infty.$$

We say that the blowup at time T is of type I, if there exists some constant $C > 0$, such that

$$\limsup_{t\to T}\, (T-t)\|u(t)\|_{L^\infty(\mathbb{R}^d)} \le C.$$

Otherwise, the blowup is called type II. The aim of this work is to construct a type II finite time blowup solution to the 3D system (3dKS) with its mass concentrating along a ring on the plane, i.e. $\{(x_1,x_2,0) : x_1^2 + x_2^2 = R^2\}$ with some $R > 0$.

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1.2 Previous Results

There are abundant results on both the well-posedness and singularity formation of Keller-Segel system in dimension d. In particular, the blowup mechanisms can vary significantly in different dimensions. The case $d = 2$ is called L^1-critical, as the scaling transformation $u \mapsto u_\lambda$ preserves the L^1-norm of u. On the other hand, the case $d \geq 3$ is called L^1-supercritical, and the scaling transformation preserves the $L^{d/2}$-norm.

The case $d = 2$. In the study of the 2D Keller-Segel system, the stationary state solution plays a fundamental role:

$$U(x_1, x_2) := \frac{8}{(1 + x_1^2 + x_2^2)^2}, \tag{1.1}$$

whose Poisson field is $\Psi_U = -2\log(x_1^2 + x_2^2)$. It turns out that $\int U = 8\pi$ is the critical mass threshold that distinguishes between the global existence and finite time blowup. For $M < 8\pi$, there is global existence of solutions that diffuse to zero, for example, see [4, 1]. For $M = 8\pi$, there exit infinite time blowup as well as global regularity results [3, 14, 2]. For $M > 8\pi$, there are various concrete examples of finite time blowup results. A well-known stable single blowup takes the form

$$u(\mathbf{x}, t) = \frac{1}{\lambda(t)}(U + \tilde{u}(t)) \left(\frac{\mathbf{x} - \mathbf{x}^*(t)}{\lambda(t)} \right), \quad \lambda(t) = \sqrt{T - t}\, e^{-\sqrt{\frac{|\log(T-t)|}{2}} + \mathcal{O}(1)}$$

with $\tilde{u} \to 0$ and $\mathbf{x}^*(t) \to \mathbf{x}^*$ as $t \to T$ in certain topology. Formal asymptotics and rigorous proofs can be found in [20, 35, 31, 9, 8, 5]. Our work is closely related to this line of research. Indeed, the 3D axisymmetric Keller-Segel system resembles a 2D one near the center of the blowup ring, which allows us to locally recover the same blowup mechanism (in particular, the same blowup rate). See the next section for a detailed discussion. There are other blowup scenarios, for example, the unstable ones in [8], as well as the multiple collapsing blowup [33, 11]. It is worth mentioning that for the 2D Keller-Segel system there is no type I blowup (for example, see *Theorem 10* in [37]).

The case $d \geq 3$. Similar to the 2D case, there is a threshold on $\|u(0)\|_{L^{d/2}}$ that distinguishes between global existence in time and finite time blowup. For small initial data, global existence results can be found in [13, 36]. Different from the $d = 2$ case, for $d \geq 3$ there exist type I blowups, see [19, 34, 27, 12]. There is also a type II radial collapsing sphere blowup, which was first formally constructed in [18] and then proved rigorously in [10]. In this scenario, the blowup profile is a traveling wave solution of the viscous Burgers' equation.

1.3 Statement of the Result

In (3dKS), we consider the axisymmetric setting and adopt the cylindrical coordinate

$$u = u(r, z, t), \quad r = \sqrt{x_1^2 + x_2^2}, \quad z = x_3.$$

By extension with 0, we can view $u(r, z)$ as a 2D function, i.e., $u(r, z) : \mathbb{R}^2 \to \mathbb{R}$. Denote $B(l) := \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < l^2\}$ to be the ball with radius $l > 0$ in $\mathbb{R}^2$. For any function $f(x_1, x_2) : \mathbb{R}^2 \to \mathbb{R}$, we define the norm

$$\|f\|_{\mathcal{E}} := \|f\|_{H^1(B(2))} + \|f(x_1, x_2)(1 + x_1^2 + x_2^2)^{\frac{3}{4}}\|_{L^\infty(\mathbb{R}^2 \setminus B(1))}.$$

The solutions we construct lie in the following function space:

$$\mathcal{E} := \{u : \mathbb{R}^2 \to \mathbb{R} \mid \|u\|_{\mathcal{E}} < +\infty\}.$$

Note, in particular, that $\mathcal{E} \subset L^p(\mathbb{R}^2)$ for any $p > \frac{4}{3}$.

Theorem 1 (Axisymetric type II blow up for the 3D Keller-Segel system)**.** *For any $T > 0$, there exists initial data u_0 in the function space $\mathcal{E}$ and $R_0 > 0$, such that the following holds for the associate solution to* (3dKS)*. It blows up at finite time T according to the dynamic*

$$u(r, z, t) = \frac{1}{\lambda^2(t)} (U + \tilde{u}(t)) \left(\frac{r - R(t)}{\lambda(t)}, \frac{z}{\lambda(t)} \right) \text{ with } u(r, z, 0) = u_0(r - R_0, z),$$

such that:

- *Law for the blowup scale:*

$$\lambda(t) = \sqrt{T-t}e^{-\sqrt{\frac{|\log(T-t)|}{2}}+\mathcal{O}(1)} \text{ as } t \to T; \tag{1.2}$$

- *Convergence to the stationary state profile:*

$$\|\tilde{u}(t)\|_{\mathcal{E}} \to 0 \text{ as } t \to T;$$

- *Convergence of the blow up radius: there exists $R_* = R_*(T, u_0, R_0) > 0$, such that $R(t) \to R_*$ as $t \to T$.*

Comments on the result.
(i) A new blowup scenario for the Keller-Segel system. To the best of our knowledge, this is the first blowup result of this kind for the Keller-Segel system, whose leading order geometry is nonradial with a 1-dimensional singular set. The solution we construct here converges in distribution to a Dirac measure supported on a 1-dimensional circle on the $x_3 = 0$ plane. It is worth noting that similar blowup phenomenons occur in other systems, for example, harmonic map flow into $\mathbb{S}^2$ [16] and supercritical heat equation [15]. In our case, the partial mass technique does not work and the linearized operator is essentially nonlocal. Indeed, in the analysis we need to derive a sharp control on the sizes of the perturbation in different regions so that they will not interfere with each other in order to close the bootstrap argument. It is interesting to see that while the leading order dynamics near the blowup ring resembles the 2D Keller-Segel system, the one away from it is still a 3D one and should be dealt with separately. This work provides a method of lifting a lower dimensional blowup to a higher dimensional space, which may be applied to other systems.
(ii) Simplification of the spectral analysis. The spectral information of the linearized operator $\mathscr{L}_\nu^\zeta$ (defined in (1.6)) in the radial sector played an essential role in the analysis of [9, 8]. However, as shown in [9], the precise construction of the eigenfunctions can be a heavy task. Since the eigenfunctions are only used to construct an approximate solution, it suffices to only solve the eigen problems approximately, which greatly simplifies our analysis. Indeed, through a simple asymptotic matching procedure, we obtain the first two approximate eigenfunctions of $\mathscr{L}_\nu^\zeta$ with small enough generated errors, which are sufficient for our analysis. See Proposition 1 for details. It is worth noting that a similar technique has been applied in a recent work [11] to construct a finite time singularity formed by the collision of two collapsing solitons for the 2D Keller-Segel system.
(iii) A robust approach. In our analysis, we completely avoid using the partial mass setting and control both the redial and nonradial parts of the perturbation at the same time. We remark that the analysis in [8] crucially used the partial mass setting for which the nonlocal operator $\mathscr{L}_\nu^\zeta$ was transformed into a local one which is self-adjoint in a weighted L^2 space. In our case, by enforcing suitable local orthogonality conditions, we are able to obtain equivalence of norms as well as coercivity of the linearized operator for the whole perturbation function. See Sect.2 for the discussion. Since the strategy provided here is simple and not restricted to the radial sector, we expect it can be implemented to other problems.
(iv) Adapted inner product and coercivity of the linearized operator. The coercivity of the linearized operator plays a crucial role in the control of the perturbation around the blowup ring. In this region, we deal with a two-scale problem – the larger parabolic scale and the smaller soliton scale. The linearized operator $\mathscr{L}_\nu^\zeta$ has different limits in these two scales (partly due to the presence of the scaling term), each of which has its own coercivity structure. Therefore, in order to obtain coercivity in both scales (i.e. the "global" coercivity) we design a mixed inner product (see (2.22)) that is compatible with both structures according to the idea of asymptotic matching. Moreover, it preserves norm equivalence for functions with the local orthogonality conditions, which is important for the energy estimates. See Proposition 2 for details.
(v) Topological argument and stability restriction. In this work, we are not able to obtain stability results. This relates to the fact that we only use rough information about the spectrum of the linearized operator. Specifically, the ODE for certain modulation parameter is unstable, which can be controlled only with a careful choice of initial data (i.e., an topological argument). See Sect.4 for details. With more refined

analysis (for example, that in [9, 8]), it is possible to establish stability at least on the axisymmetric level, which is intuitively true as the leading order dynamics around the blowup ring is a 2D one which has already been proved stable. This stability restriction can be viewed as "the price we pay for the simplification". It, however, remains unclear whether there is stability for general non-axisymmetric perturbation. This interesting open question can be left as a future work.

(vi) Connection with the Nonlinear Schrödinger equations (NLS). Finally, we remark on the connection between the Keller-Segel system and the Nonlinear Schrödinger equations:

$$i\partial_t\psi(\mathbf{x},t) + \Delta\psi(\mathbf{x},t) + \psi(\mathbf{x},t)|\psi(\mathbf{x},t)|^{p-1} = 0, \quad \mathbf{x} \in \mathbb{R}^d. \tag{NLS}$$

Here we summarize some blowup phenomenons of NLS that shares similarity with the Keller-Segel system. The case $p-1=\frac{4}{d}$ in (NLS) is called (L^2-)critical and blowup occurs once the mass (i.e., $\|\psi\|_{L^2}$) is above certain threshold. A stable blowup mechanism in this case enjoying the so-called "loglog" law can be found in [29, 24, 23, 26]. In the supercritical cases $p-1>\frac{4}{d}$, there exist standing ring (referring to the sphere in this context) blowup solutions (see [30, 32]) as well as collapsing ring blowup solutions (see [25]). It is worth noting that all the blowup solutions mentioned above converge in a certain sense to some 1-dimensional ground state solutions of (NLS). We also recommend [17] for a comprehensive review on NLS.

Notations. Unless otherwise specified, differential operators such as Δ, ∇ and $\nabla\cdot$ are understood as 2D ones, and $\int$ denotes the integration on $\mathbb{R}^2$. We denote the right half plane in $\mathbb{R}^2$ as $\mathbb{H} := \{(x_1,x_2)\in\mathbb{R}^2 : x_1 > 0\}$. For any function $f(r,z):\mathbb{H}\to\mathbb{R}$, we can interpret it as a 3D axisymmetric function via $\tilde{f}(x_1,x_2,x_3) := f(\sqrt{x_1^2+x_2^2}, x_3)$, and define its 3D Poisson field

$$\Phi_f := \frac{1}{4\pi|(x_1,x_2,x_3)|} * \tilde{f},$$

where $|(x_1,\dots,x_n)| := \sqrt{x_1^2+\cdots+x_n^2}$ denotes the standard Euclidean norm on $\mathbb{R}^n$. On the other hand, we can extend f to some $\bar{f}:\mathbb{R}^2\to\mathbb{R}$ via a small modification on the boundary, for example, the one described in (A.1). Then, we can define the 2D Poisson field for such function as

$$\Psi_f := -\frac{1}{2\pi}\log(|(x_1,x_2)|) * \bar{f}.$$

With a slight abuse of notation, we also use Φ and Ψ to denote the Poisson fields for functions on $\mathbb{R}^3$ and $\mathbb{R}^2$ (with suitable decay), respectively. We define the difference of the 2D and 3D Poisson fields as

$$\Theta_f := \Phi_f - \Psi_f.$$

Now for $\nu > 0$, we denote

$$U_\nu(x_1,x_2) := \frac{1}{\nu^2}U(x_1/\nu, x_2/\nu),$$

where U is the stationary solution defined in (1.1), and the 2D differential operator

$$\Lambda f(x_1,x_2) := -\frac{d}{d\nu}\bigg|_{\nu=1}\frac{1}{\nu^2}f(x_1/\nu,x_2/\nu) = 2f + x_2\partial_{x_1}f + x_2\partial_{x_2}f = \nabla\cdot((x_1,x_2)f).$$

Define $\chi\in C_c^\infty(\mathbb{R}^2)$ to be a radially symmetric positive cutoff function with:

$$\chi(\mathbf{x}) = \begin{cases} 1 \text{ for } |\mathbf{x}|\le 1, \\ 0, \text{ for } |\mathbf{x}|\ge 2. \end{cases}$$

With a little abuse of notation, we will denote $\chi = \chi(|\mathbf{x}|)$. Given two fixed constants $0<\zeta_*\ll 1\ll\zeta^*<+\infty$ (which will be specified later in the analysis) and a small parameter $\nu>0$, we denote ($\zeta := |\mathbf{x}|$)

$$\begin{aligned} &\chi_*(\zeta) := \chi(\zeta/\zeta_*), && \chi^*(\zeta) := \chi(\zeta/\zeta^*), \\ &\chi_\nu(\zeta) := \chi(\zeta/|\log\nu|), && \bar{\chi}_\nu(\zeta) := \chi(\zeta\nu/|\log\nu|). \end{aligned} \tag{1.3}$$

We define the norms:

$$\|f\|_{\text{in}}^2 := \int_{\mathbb{R}^2} \frac{\nu^2 f^2 \chi_\nu^2 \varrho_\nu}{U_\nu}, \quad \|f\|_{L^2(U_\nu)}^2 := \int_{\mathbb{R}^2} \frac{\nu^2 f^2}{U_\nu},$$

where ϱ_ν is the exponential weight function defined in (2.20). We denote the standard $L^2(\mathbb{R}^2)$ inner product as $\langle f\,,\,g\rangle := \int_{\mathbb{R}^2} fg$. For any two positive quantities, $A_1 \lesssim A_2$ means that there exists some universal (independent of any parameters in this problem) constant $C > 0$, such that $A_1 \le CA_2$. Similarly, $A_1 \approx A_2$ means that there exists universal $C > 0$, such that $\frac{1}{C}A_1 \le A_2 \le CA_1$. Universal constants will be denoted generically as C or δ, the specific values of which may change from line to line. We use brackets to specify the dependence of constants on other quantities. For example, $C(A_1, A_2)$ will denote (generically) a constant depending only on A_1 and A_2, the specific value of which may also vary in different places. The expression $A_1 = \mathcal{O}(A_2)$ means that there exists a universal C, such that $|A_1| \le CA_2$. We use $\sim$ to denote "asymptotically equivalent" under certain limiting process (which will always be specified), i.e., $A_1 \sim A_2$ means $\lim A_1/A_2 \approx 1$.

1.4 Strategy of the Proof

Now we briefly describe the main steps of the proof of Theorem 1.

Step 1: Renormalization and linearization of the problem. By the scaling invariance of the solutions, we first make change of variables according to the parabolic scaling and the standing ring scenario:

$$u(r,z,t) = \frac{1}{\mu^2} w\left(\frac{r-R(t)}{\mu}, \frac{z}{\mu}, \tau\right), \quad \bar{r} := \frac{r-R(t)}{\mu}, \quad \bar{z} := \frac{z}{\mu}, \quad \tau = \int_0^t \frac{1}{\mu^2(\tilde{t})}\, d\tilde{t},$$

where $\mu(t) := \sqrt{T-t}$ is the parabolic scale with blowup time $T > 0$. Define $\zeta := \sqrt{\bar{r}^2 + \bar{z}^2}$. Then, the system for (w, Φ_w) is

$$\begin{aligned} &\partial_\tau w = \nabla\cdot(\nabla w - w\nabla\Phi_w) + \frac{1}{\bar{r}+R/\mu}(\partial_{\bar{r}} w - w\partial_{\bar{r}}\Phi_w) - \beta\Lambda w + \frac{R_\tau}{\mu}\partial_{\bar{r}} w \quad \beta := -\frac{\mu_\tau}{\mu} = \frac{1}{2}, \\ &-\left(\partial_{\bar{r}}^2 + \partial_{\bar{z}}^2 + \frac{1}{\bar{r}+R/\mu}\partial_{\bar{r}}\right)\Phi_w = w. \end{aligned} \tag{1.4}$$

Since the blowup solutions we construct are of type II, there exists a smaller scale $\nu(t) \to 0$ as $t \to T$, beyond the parabolic scale. Such scale, named the soliton scale (or the blowup scale) will serve as a crucial asymptotically small parameter in our analysis. Specifically, we consider the soliton change of variables:

$$w(\bar{r},\bar{z},\tau) = \frac{1}{\nu^2} v\left(\frac{\bar{r}}{\nu}, \frac{\bar{z}}{\nu}, s\right), \quad \rho := \frac{\bar{r}}{\nu}, \quad \xi := \frac{\bar{z}}{\nu}, \quad \frac{ds}{d\tau} = \frac{1}{\nu^2}.$$

Define $\gamma := \sqrt{\rho^2+\xi^2}$. Note that $\Phi_v(\rho,\xi) = \Phi_w(\bar{r},\bar{z})$. Then, the system for (v, Φ_v) is

$$\begin{aligned} &\partial_s v = \nabla\cdot(\nabla v - v\nabla\Phi_v) + \frac{1}{\rho+R/(\mu\nu)}(\partial_\rho v - v\partial_\rho\Phi_v) - \left(\eta+\nu^2\beta\right)\Lambda v + \nu\frac{R_\tau}{\mu}\partial_\rho v \quad \eta := -\frac{\nu_s}{\nu}, \\ &-\left(\partial_\xi^2 + \partial_\rho^2 + \frac{1}{\rho+R/(\mu\nu)}\partial_{\bar{r}}\right)\Phi_v = v. \end{aligned} \tag{1.5}$$

The 3D system (1.4) can be approximated by a 2D one in the parabolic scale (in a sense that will soon become clear), i,e., in the region around the blowup ring (and away from the axis of symmetry). Linearization of the system (1.4) around the stationary solution U_ν gives the leading order linearized operator:

$$\mathscr{L}_\nu^\zeta f = \mathscr{L}_{0,\nu}^\zeta f - \frac{1}{2}\Lambda f = \nabla\cdot(U_\nu\nabla(\mathscr{M}_\nu^\zeta f)) - \frac{1}{2}\Lambda f, \quad \text{with} \quad \mathscr{M}_\nu^\zeta f := \frac{f}{U_\nu} - \Psi_f. \tag{1.6}$$

Equivalently, the linearized operator around U for the system (1.5) is

$$\mathscr{L} f = \mathscr{L}_0 f - \frac{1}{2}\nu^2\Lambda f = \nabla\cdot(U\nabla(\mathscr{M} f)) - \frac{1}{2}\nu^2\Lambda f, \quad \text{with} \quad \mathscr{M} f := \frac{f}{U} - \Psi_f.$$

The core of the analysis lies within the parabolic scale (which includes the smaller soliton scale). We remark that while most of the time our analysis is done using the parabolic variables, it is equivalent and sometimes more convenient to work with soliton variables.

Step 2: Construction of approximate solution.

The blowup solutions we construct converge locally to the steady state profile U around the blowup ring. More specifically, we will show that w in (1.4) decomposes as U_ν plus some controllable perturbation. However, due to the instability of the linearized operator in certain directions, orthogonality conditions need to be imposed on the perturbation. This is done by the introduction of modulation parameters, the dynamics of which gives us the blowup rate.

First of all, we construct the first two approximate eigenfunctions of the linearized operator $\mathscr{L}_\nu^\zeta$, corresponding to the positive and "almost zero" eigenvalues, respectively:

$$\mathscr{L}_\nu^\zeta \varphi_{i,\nu} = \left(1 - i + \frac{1}{2\log(\nu)}\right)\varphi_{i,\nu} + R_i, \quad i = 0, 1,$$

where R_i are some small errors. Then, exploiting the cancellation of $\varphi_{1,\nu} - \varphi_{0,\nu}$, we decompose the solution as

$$w = U_\nu + a(\varphi_{1,\nu} - \varphi_{0,\nu}) + \varepsilon,$$

where $a = a(\tau)$ is a modulation parameter. Thus, we can write out the evolution for ε:

$$\partial_\tau \varepsilon = \mathscr{L}_\nu^\zeta \varepsilon + L(\varepsilon) + NL(\varepsilon) + E, \tag{1.7}$$

where both the extra linear term $L(\varepsilon)$ and the nonlinear term $NL(\varepsilon)$ will be small in certain sense. It is essential that ε satisfies the local orthogonality conditions (We recall the definition of χ_* in (1.3)):

$$\int_{\mathbb{R}^2} \varepsilon \chi_*(\zeta)\, d\bar{r} d\bar{z} = \int_{\mathbb{R}^2} \varepsilon \Lambda U_\nu \chi_*(\zeta)\, d\bar{r} d\bar{z} = \int_{\mathbb{R}^2} \varepsilon \nabla U_\nu \chi_*(\zeta)\, d\bar{r} d\bar{z} = 0, \tag{1.8}$$

which is preserved by the modulation parameters together with the even symmetry in $\bar{z}$-direction of the solution. Through a study of the linearized operator, we can define an adapted inner product $\langle \cdot, \cdot \rangle_{\nu,*}$ (see (2.22)), such that its corresponding norm is equivalent to a weighted L^2 norm for any function satisfying the orthogonality conditions (1.8). Moreover, for such functions, $\mathscr{L}_\nu^\zeta$ (up to a slight modification) will be coercive under the adapted inner product, in the sense of (2.37). This coercivity is crucial in the energy estimates of ε.

Step 3: Modulation dynamics and energy estimates. The generated error in (1.7), consisting mainly of the modulation parameters, admits a further decomposition:

$$E = \mathrm{Mod}_0 \varphi_{0,\nu} + \mathrm{Mod}_1 \varphi_{1,\nu} + \frac{R_\tau}{\mu} \partial_{\bar{r}} U_\nu + \tilde{E},$$

where Mod_i and $\tilde{E}$ are terms defined in Proposition 4. Note that we have three modulation parameters at hand, namely $\nu(\tau), a(\tau)$ and $\frac{R_\tau(\tau)}{\mu}$, which correspond precisely to the three orthogonality conditions in (1.8). Projecting (1.7) onto these three directions together with the bootstrap assumptions on ε yields the modulation equations:

$$\begin{cases} |\mathrm{Mod}_0| = \left| a_\tau - 2a\beta\left(1 + \frac{1}{2\log(\nu)}\right) - 16\nu^2\left(\frac{\nu_\tau}{\nu} - \beta\right)\right| = \mathcal{O}\left(\frac{\nu^2}{|\log \nu|}\right), \\ |\mathrm{Mod}_1| = \left| -a_\tau + \frac{a(\tau)\beta}{\log(\nu)}\right| = \mathcal{O}\left(\frac{\nu^2}{|\log \nu|^2}\right), \\ \left|\frac{R_\tau}{\mu}\right| = \mathcal{O}\left(\frac{\nu}{|\log \nu|}\right), \end{cases}$$

the solution of which yields the blowup law given in (1.2). The energy estimates for ε are done separately in the inner zone (i.e., $\bar{r}^2 + \bar{z}^2 \lesssim 1$) and the outer zone (i.e., $\bar{r}^2 + \bar{z}^2 \gtrsim 1$). In the inner zone, a weighted H^1-norm of ε is controlled, thanks to the coercivity of $\mathscr{L}_\nu^\zeta$. In the outer zone, we come back to the original

3D structure and control a weighted L^∞-norm of ε via the dissipative structure of the system. The estimates of the two zones communicate in the intermediate area via an H^2-control of ε resulting from the parabolic regularity of the system. Finally, combining the modulation equations and the energy estimates, we are able to close the bootstrap argument for both the modulation parameters and the energy norms of ε, and the global-in-time (referring to the renormalized time variable τ) control of ε implies the finite time blowup of the solution of the original Keller-Segel system.

This work is organized as follows. In Sect.2, we construct the first two approximate eigenfunctions of the linearized operator, and then explore its coercivity properties. Sect.3 is the heart of the analysis, including the setup of the bootstrap assumptions, derivation of the modulation equations, and energy estimates. Finally, we close the bootstrap argument in Sect.4 and conclude the proof of the main theorem.

2 Properties of the linearized operator

This section is devoted to the study of the linearized operator

$$\mathscr{L}_\nu^\zeta f = \nabla \cdot \big(U_\nu \nabla \mathscr{M}_\nu^\zeta f\big) - \beta \Lambda f = \mathscr{L}_{0,\nu}^\zeta f - \beta \Lambda f.$$

First of all, we construct the first two approximate eigenfunctions of $\mathscr{L}_\nu^\zeta$ in Proposition 1, which will be important building blocks of the approximate solution of the Keller-Segel system. We also describe their asymptotic behaviors and generated errors, which will be helpful in the energy estimates. Next, we study the operator $\mathscr{M}_\nu^\zeta$, the appearance of which is natural from a linearization of the free energy functional associated to the two-dimensional Keller-Segel equation:

$$\mathcal{F}(f) = \int_{\mathbb{R}^2} f\Big(\log f - \frac{1}{2}\Psi_f\Big)\, d\mathbf{x}.$$

Its definiteness and norm equivalence properties will motivate our definition of the adapted inner product, with which we are able to prove the crucial coercivity result for $\mathscr{L}_\nu^\zeta$ (Proposition 2). We will adopt the soliton variables (ρ, ξ) instead of the parabolic ones $(\bar{r}, \bar{z})$ when it is more convenient, though these two settings are equivalent in terms of analysis.

2.1 Two approximate eigenfunctions

In [9], the authors used the partial mass setting to derive a complete description of the spectrum of $\mathscr{L}_\nu^\zeta$ in the radial setting. Here we derive only rough information of the spectrum via a simple asymptotic matching procedure, which is sufficient for our purpose of constructing blowup solutions.

Proposition 1 (Two approximate eigenfunctions)**.** *Consider $\beta > 0$ and $0 < \nu \ll 1$ to be fixed. There are two smooth radial functions $\varphi_{0,\nu}$ and $\varphi_{1,\nu}$, with supports in $\{\zeta :\ \zeta \le 2|\log \nu|\}$, that solve*

$$\mathscr{L}_\nu^\zeta \varphi_{i,\nu} = 2\beta\Big(1 - i + \frac{1}{2\log(\nu)}\Big)\varphi_{i,\nu} + R_i, \quad i = 0, 1.$$

(i) (Approximate eigenfunctions)

$$\varphi_{i,\nu}(\zeta) = -\frac{1}{16\nu^4}\varphi_i^{\mathrm{in}}(\zeta/\nu)\chi_m + \varphi_i^{\mathrm{ex}}(\zeta)(1-\chi_m)\chi_\nu = -\frac{1}{16\nu^4}\Lambda U(\zeta/\nu)\chi_\nu + \tilde{\varphi}_i(\zeta),$$

where φ_i^{in} and φ_i^{ex} are defined by (2.4) and (2.8), and the cutoff function $\chi_m(\zeta) = \chi(\zeta/\zeta_m)$ is defined at the beginning of the proof. In particular, we have the pointwise estimates for $k = 0, 1, 2$:

$$\begin{aligned} |\partial_\zeta^k \tilde{\varphi}_0(\zeta)| + |\partial_\zeta^k \nu\partial_\nu \tilde{\varphi}_0(\zeta)| &\lesssim \left(\frac{\nu^2\zeta^{2-k}\log^2(2+\zeta/\nu)}{(\nu+\zeta)^6} + \frac{\zeta^{2-k}}{|\log\nu|(\nu+\zeta)^4}\right)(1 + \log(\zeta)\mathbb{1}_{\{\zeta>1\}}), \\ |\partial_\zeta^k \tilde{\varphi}_1(\zeta)| + |\partial_\zeta^k \nu\partial_\nu \tilde{\varphi}_1(\zeta)| &\lesssim \frac{\zeta^{2-k}}{(\nu+\zeta)^4}(1 + \log(\zeta)\mathbb{1}_{\{\zeta>1\}}), \end{aligned} \tag{2.1}$$

and the improved estimate near the origin,

$$|\partial_\zeta^k \nu\partial_\nu(\varphi_{1,\nu}-\varphi_{0,\nu})| \lesssim \left(\frac{\nu^2\zeta^{2-k}\log^2(2+\zeta/\nu)}{(\nu+\zeta)^6} + \frac{1}{|\log(\nu)|}\cdot\frac{\zeta^{2-k}}{(\nu+\zeta)^2(1+\zeta)^2}\right)(1+\log(\zeta)\mathbb{1}_{\{\zeta>1\}}), \quad k=0,1,2,$$
$$|\partial_\zeta^k(\varphi_{1,\nu}-\varphi_{0,\nu})| \lesssim \frac{\zeta^{2-k}}{(\nu+\zeta)^4}(1+\log(\zeta)\mathbb{1}_{\{\zeta>1\}}), \quad k=0,1,2. \tag{2.2}$$

(ii) (Pointwise estimates of R_i)

$$\left|\partial_\zeta^k R_i(\zeta)\right| \lesssim \frac{\zeta^{2-k}}{(\nu+\zeta)^2(1+\zeta)^2}\frac{|\log(\nu+\zeta)|}{|\log(\nu)|} + \frac{\nu^2\zeta^{2-k}\log^2(2+\zeta/\nu)}{(\nu+\zeta)^4} \quad k=0,1,2, \qquad \left|\int_{\zeta<1} R_i(\zeta)\zeta d\zeta\right| \lesssim \frac{1}{|\log\nu|}. \tag{2.3}$$

Proof of Proposition 1. We proceed as follows: First, we construct the inner approximate eigenfunctions by iterate inversions of the linearized operator. Second, we solve the outer approximate eigen problems whose eigenfunctions are well known. Third, we match the inner eigenfunctions with the outer ones by specifying the $\mathcal{O}(|\log\nu|^{-1})$ part of the approximate eigenvalues. Finally, the pointwise estimates follow directly from the explicit construction of the (global) eigenfunctions.

The construction of φ_i^{in}: Fix a small constant $\zeta_m > 0$ (the subscript "m" stands for "matching"), and $\zeta = \zeta_m$ will be our matching spot. Denote $\chi_m(\zeta) := \chi(\zeta/\zeta_m)$. Now, consider the inner region, i.e., $\zeta \in (0, \zeta_m)$ or $\gamma \in (0, \zeta_m/\nu)$. The inner eigenproblem, in the soliton variables, is equivalent to

$$\mathscr{L}_0\varphi_i^{\text{in}} := \nabla\cdot\left(U\nabla\left(\frac{\varphi_i^{\text{in}}}{U} - \Psi_i^{\text{in}}\right)\right) = \nu^2\beta\Lambda\varphi_i^{\text{in}} + 2\nu^2\beta(1-i+\tilde{\alpha}_{i,\nu})\varphi_i^{\text{in}},$$

where

$$-(\partial_\gamma^2 + \frac{1}{\gamma}\partial_\gamma)\Psi_i^{\text{in}} = \varphi_i^{\text{in}},$$

and $\tilde{\alpha}_{i,\nu}$ is a next-order part of the approximate eigenvalues to be solved. We look for an approximate solution which takes the form:

$$\varphi_i^{\text{in}} = \Lambda U + \nu^2 c_{i,2}V_2 + \nu^2\tilde{c}_{i,2}\tilde{V}_2 + \nu^4 d_{i,4}V_{4,\#} + \nu^4\tilde{d}_{i,4}\tilde{V}_{4,\#} + \nu^4 c_{i,4}V_4 + \nu^4\tilde{c}_{i,4}\tilde{V}_4, \tag{2.4}$$

where

$$\mathscr{L}_0 V_2 = \Lambda U, \qquad \mathscr{L}_0\tilde{V}_2 = \Lambda^2 U, \quad \mathscr{L}_0 V_{4,\#} = \Lambda V_2,$$
$$\mathscr{L}_0\tilde{V}_{4,\#} = \Lambda\tilde{V}_2, \quad \mathscr{L}_0 V_4 = V_2, \qquad \mathscr{L}_0\tilde{V}_4 = \tilde{V}_2,$$

and $c_{i,2}, d_{i,4}, c_{i,4}, \tilde{c}_{i,2}, \tilde{d}_{i,4}, \tilde{c}_{i,4}$ are constants (may depend on $\tilde{\alpha}_{i,\nu}$) that will be chosen to improve the approximation and matching errors. The building block functions above can be solved explicitly (as the corresponding second order ODE admits explicit solutions). Their asymptotic behaviors, as $\gamma \to \infty$ are:

$$V_2 = \frac{4}{\gamma^2} + \mathcal{O}\left(\frac{\log^2(\gamma)}{\gamma^4}\right), \qquad \tilde{V}_2 = -\frac{8}{\gamma^2} + \mathcal{O}\left(\frac{\log^2(\gamma)}{\gamma^4}\right), \qquad V_{4,\#} = 1 + \mathcal{O}\left(\frac{\log^2(\gamma)}{\gamma^2}\right),$$
$$\tilde{V}_{4,\#} = -2 + \mathcal{O}\left(\frac{\log^2(\gamma)}{\gamma^2}\right), \quad V_4 = \log(\gamma) - \frac{5}{4} + \mathcal{O}\left(\frac{\log^2(\gamma)}{\gamma^2}\right), \quad \tilde{V}_4 = -2\log(\gamma) + \frac{7}{2} + \mathcal{O}\left(\frac{\log^2(\gamma)}{\gamma^2}\right). \tag{2.5}$$

and their asymptotic behavior as $\gamma \to 0^+$ are (with order of derivative $k = 0, 1, 2$):

$$V_2^{(k)} \sim \gamma^{2-k}, \quad \tilde{V}_2^{(k)} \sim \gamma^{2-k}, \quad V_{4,\#}^{(k)} \sim \gamma^{4-k},$$
$$\tilde{V}_{4,\#}^{(k)} \sim \gamma^{4-k}, \quad V_4^{(k)} \sim \gamma^{2-k}, \quad \tilde{V}_4^{(k)} \sim \gamma^{2-k}. \tag{2.6}$$

We remark that although these building block functions are not linearly independent at the leading order, some of them are in fact necessary in order to obtain cancellations in the generated error. This again (as we have already seen in the formal asymptotic matching in the previous section) emphasizes the idea of "iterative inversions of $\mathscr{L}_0$", which is a natural way of improving the generated error in the soliton scale. The inner generated error in the soliton variable is defined as

$$\begin{aligned}
R_i^\gamma &:= \mathscr{L}_0\varphi_i^{\text{in}} - \beta\nu^2\Lambda\varphi_i^{\text{in}} - 2\beta\nu^2(1-i+\tilde{\alpha}_{i,\nu})\varphi_i^{\text{in}} \\
&= \nu^2\left(c_{i,2}\Lambda U + \tilde{c}_{i,2}\Lambda^2 U - \beta\Lambda^2 U - 2\beta(1-i+\tilde{\alpha}_{i,\nu})\Lambda U\right) \\
&\quad + \nu^4\Big(d_{i,4}\Lambda V_2 + \tilde{d}_{i,4}\Lambda\tilde{V}_2 + c_{i,4}V_2 + \tilde{c}_{i,4}\tilde{V}_2 - \beta c_{i,2}\Lambda V_2 - \beta\tilde{c}_{i,4}\Lambda\tilde{V}_2 \\
&\quad - 2\beta c_{i,2}(1-i+\tilde{\alpha}_{i,\nu})V_2 - 2\beta\tilde{c}_{i,2}(1-i+\tilde{\alpha}_{i,\nu})\tilde{V}_2\Big) \\
&\quad - \beta\nu^6\left(d_{i,4}\Lambda V_{4,\#} + \tilde{d}_{i,4}\Lambda\tilde{V}_{4,\#} - c_{i,4}\Lambda V_4 - \tilde{c}_{i,4}\Lambda\tilde{V}_4\right) \\
&\quad - 2\beta\nu^6(1-i+\tilde{\alpha}_{i,\nu})\left(d_{i,4}V_{4,\#} + \tilde{d}_{i,4}\tilde{V}_{4,\#} - c_{i,4}V_4 - \tilde{c}_{i,4}\tilde{V}_4\right).
\end{aligned}$$

To cancel out the $\mathcal{O}(\nu^2)$ terms in R_i^γ, we choose

$$c_{i,2} = 2\beta(1-i+\tilde{\alpha}_{i,\nu}), \quad \tilde{c}_{i,2} = \beta.$$

Similarly, to cancel out the $\mathcal{O}(\nu^4)$ terms, we choose

$$\begin{aligned}
d_{i,4} &= 2\beta^2(1-i+\tilde{\alpha}_{i,\nu}), \quad \tilde{d}_{i,4} = \beta^2, \\
c_{i,4} &= 4\beta^2(1-i+\tilde{\alpha}_{i,\nu})^2, \quad \tilde{c}_{i,4} = 2\beta^2(1-i+\tilde{\alpha}_{i,\nu}),
\end{aligned}$$

and the error becomes

$$\begin{aligned}
R_i^\gamma = &- \beta^3\nu^6\left(\Lambda\tilde{V}_{4,\#} + 2(1-i+\tilde{\alpha}_{i,\nu})\Lambda V_{4,\#} + 2(1-i+\tilde{\alpha}_{i,\nu})\Lambda\tilde{V}_4 + 4(1-i+\tilde{\alpha}_{i,\nu})^2\Lambda V_4\right) \\
&- 2\beta^3\nu^6(1-i+\tilde{\alpha}_{i,\nu})\left(\tilde{V}_{4,\#} + 2(1-i+\tilde{\alpha}_{i,\nu})V_{4,\#} + 2(1-i+\tilde{\alpha}_{i,\nu})\tilde{V}_4 + 4(1-i+\tilde{\alpha}_{i,\nu})^2 V_4\right).
\end{aligned} \tag{2.7}$$

The construction of φ_i^{ex}: Then, we work in the outer region, i.e., $\zeta \in (\zeta_m, +\infty)$. In this region, we have $U_\nu(\zeta) \lesssim \nu^2$ and $\partial_\zeta \Psi_{U_\nu}(\zeta) \sim -\frac{4}{\zeta}$, hence, the operator $\mathscr{L}_\nu^\zeta$ behaves like the Hermite operator in dimension 6:

$$\mathscr{H} = \partial_\zeta^2 + \frac{5}{\zeta}\partial_\zeta - \beta\Lambda,$$

and we formally have $\mathscr{L}_\nu^\zeta = \mathscr{H} + \mathcal{O}(\nu^2)$ when $\zeta \geq \zeta_m$. Thus, we consider the approximate of the outer eigenfunction of the form

$$\varphi_i^{\text{ex}}(\zeta) = \Omega_i(\zeta) + \tilde{\varphi}_i^{\text{ex}}(\zeta), \tag{2.8}$$

for some lower order term $\tilde{\varphi}_i^{\text{ex}}(\zeta) \sim \mathcal{O}(\tilde{\alpha}_{i,\nu})$ and the leading term Ω_i solves

$$(\mathscr{H} - 2\beta(1-i))\,\Omega_i(\zeta) = 0, \quad \text{with} \quad \Omega_i(\zeta) \sim \frac{1}{\zeta^4} \text{ as } \zeta \to 0.$$

The solutions without exponential growth at infinity are

$$\Omega_0(\zeta) = \frac{1}{\zeta^4}, \quad \Omega_1(\zeta) = \frac{1}{\zeta^4} + \frac{\beta}{2\zeta^2}.$$

We remark that the eigen problem for the operator $\mathscr{H}$ actually determines our eigenvalues to the leading order, i.e., $\mathscr{H}f = \lambda f$ has solutions in the class of functions with suitable decay when $\lambda = 2\beta(1-i)$ $(i = 0, 1)$. Next, we consider the next order which solves:

$$(\mathscr{H} - 2\beta(1-i))\,\tilde{\varphi}_i^{\text{ex}} = 2\beta\tilde{\alpha}_{i,\nu}\Omega_i.$$

The solutions (without exponential growth and homogeneous modes) are

$$
\begin{aligned}
(2\beta\tilde{\alpha}_{0,\nu})^{-1}\tilde{\varphi}_0^{\mathrm{ex}}(\zeta) &= -\frac{\log(\zeta)}{\beta\zeta^4} - \frac{1}{\beta^2\zeta^6} - \frac{\beta\zeta^2-2}{\beta^2\zeta^4}e^{\frac{\beta\zeta^2}{2}}\int_\zeta^{+\infty}\frac{1}{r^3}e^{-\frac{\beta r^2}{2}}\,dr, \\
(2\beta\tilde{\alpha}_{1,\nu})^{-1}\tilde{\varphi}_1^{\mathrm{ex}}(\zeta) &= -\frac{1}{\zeta^4}e^{\frac{\beta\zeta^2}{2}}\int_\zeta^{+\infty}\frac{(\beta r^2+2)^2}{2\beta^2 r^3}e^{-\frac{\beta r^2}{2}}\,dr + \left(-\frac{\log(\zeta)}{\beta}+\frac{1}{\beta^2\zeta^2}\right)\frac{\beta\zeta^2+2}{2\zeta^4}.
\end{aligned}
\tag{2.9}
$$

Their asymptotic behaviors, as $\zeta \to 0^+$, are

$$
\begin{aligned}
(2\beta\tilde{\alpha}_{0,\nu})^{-1}\tilde{\varphi}_0^{\mathrm{ex}}(\zeta) &= -\frac{1}{4\zeta^2} + \frac{\beta}{32}\left(1-2\mathbf{E}-2\log(\frac{\beta}{2})-4\log(\zeta)\right) + \mathcal{O}(\zeta^2\log(\zeta)), \\
(2\beta\tilde{\alpha}_{1,\nu})^{-1}\tilde{\varphi}_1^{\mathrm{ex}}(\zeta) &= -\frac{1}{4\zeta^2} + \frac{\beta}{32}\left(-3+2\mathbf{E}+2\log(\frac{\beta}{2})+4\log(\zeta)\right) + \mathcal{O}(\zeta^2\log(\zeta)),
\end{aligned}
\tag{2.10}
$$

where $\mathbf{E}$ is the Euler constant. One remark: certain cancellation occurs in (2.9) as $\zeta \to 0^+$, so that the leading order behavior is $1/\zeta^2$ rather that $\log(\zeta)/\zeta^4$ or $1/\zeta^6$, which may seem likely given the expression of (2.10).

Matching φ_i^{in} and φ_i^{ex}: Now in the matching region $\nu \ll \zeta \ll 1$, we are going to match the normalized inner solution $-\frac{1}{16\nu^4}\varphi_i^{\mathrm{in}}(\zeta/\nu)$ with the outer solution $\varphi_i^{\mathrm{ex}}(\zeta)$. By the asymptotic (2.5) and the choice of constants above we have

$$
\begin{aligned}
-\frac{1}{16\nu^4}\varphi_i^{\mathrm{in}}(\zeta/\nu) = \frac{1}{\zeta^4} + \frac{\beta(i-\tilde{\alpha}_{i,\nu})}{2\zeta^2} - \frac{\beta^2}{16}\Big(&2(1-2i)+(10i-1)\tilde{\alpha}_{i,\nu}+(4-8i)\tilde{\alpha}_{i,\nu}\log(\zeta) \\
&-(4-8i)\tilde{\alpha}_{i,\nu}\log(\nu)+4\tilde{\alpha}_{i,\nu}^2(\log(\zeta)-\log(\nu)-\frac{5}{4})\Big) + \mathcal{O}(\nu^2), \quad \forall\, \zeta \gtrsim 1,
\end{aligned}
\tag{2.11}
$$

and by the asymptotic (2.10) we have

$$
\begin{aligned}
\varphi_i^{\mathrm{ex}}(\zeta) = \frac{1}{\zeta^4} + \frac{\beta(i-\tilde{\alpha}_{i,\nu})}{2\zeta^2} + \frac{\beta^2\tilde{\alpha}_{i,\nu}(1-2i)}{16}&\left(1+2i-2\mathbf{E}-2\log(\beta)+2\log(2)-4\log(\zeta)\right) \\
&+\mathcal{O}(\tilde{\alpha}_{i,\nu}\zeta^2\log(\zeta)), \quad \forall\, \zeta \ll 1.
\end{aligned}
\tag{2.12}
$$

Note that the first two terms of (2.11) and (2.12) already match. Now we are ready to determine $\tilde{\alpha}_{i,\nu}$. First of all, due to the $\tilde{\alpha}_{i,\nu}\log(\nu)$ term in (2.11), we must have $\tilde{\alpha}_{i,\nu} = \mathcal{O}(1/|\log\nu|)$ in order to minimize the matching error of (2.11) and (2.12). Furthermore, since the third term in (2.12) is of size $\mathcal{O}(1/|\log\nu|)$, in order to improve the matching error by $|\log\nu|^{-1}$ the $\mathcal{O}(1)$ parts in the third term of (2.11) must be canceled:

$$
2(1-2i)-(4-8i)\tilde{\alpha}_{i,\nu}\log(\nu) = \mathcal{O}(1/|\log\nu|),
$$

and one simple choice is

$$
\tilde{\alpha}_{i,\nu} = \frac{1}{2\log(\nu)} \quad i = 1,2.
$$

With this choice of $\tilde{\alpha}_{i,\nu}$ we obtain the matching error:

$$
-\frac{1}{16\nu^4}\varphi_i^{\mathrm{in}}(\zeta/\nu) - \varphi_i^{\mathrm{ex}}(\zeta) = \mathcal{O}\left(\frac{1}{\log(\nu)}\right), \quad \forall\, \frac{1}{4}\zeta_m \le \zeta \le 4\zeta_m.
\tag{2.13}
$$

Finally, using φ_i^{in} and φ_i^{ex}, we construct the global approximate eigenfunctions as

$$
\begin{aligned}
\varphi_{i,\nu}(\zeta) &:= -\frac{1}{16\nu^4}\varphi_i^{\mathrm{in}}(\zeta/\nu)\chi_m(\zeta) + (1-\chi_m(\zeta))\chi_\nu(\zeta)\varphi_i^{\mathrm{ex}}(\zeta) \\
&:= -\frac{1}{16\nu^4}\Lambda U(\zeta/\nu)\chi_\nu(\zeta) + \tilde{\varphi}_i(\zeta).
\end{aligned}
$$

Pointwise estimates: As for the pointwise estimate, we first note by (2.9) that

$$|\partial_\zeta^k \varphi_0^{\mathrm{ex}}(\zeta)| \lesssim \frac{1}{\zeta^{6-k}} + \frac{\log(\zeta)}{\zeta^{4+k}}, \quad |\partial_\zeta^k \varphi_1^{\mathrm{ex}}(\zeta)| \lesssim \frac{\log(\zeta)}{\zeta^{2+k}} + \frac{1}{\zeta^{6-k}}, \quad \forall\, \zeta \gg 1.$$

Then, (2.1) follows from this far field estimate and the asymptotics (2.6)(2.11). Note that the pointwise value of $\tilde{\varphi}_{0,\nu}$ is $\mathcal{O}(|\log\nu|^{-1})$ smaller than that of $\tilde{\varphi}_{1,\nu}$ in the region $\zeta \approx 1$, because when $i = 0$ there is a gain of $1/|\log\nu|$ in the $\mathcal{O}(1/\zeta^2)$ term in (2.11). This is important in the upcoming derivation of modulation estimates. To derive the improved estimate near the origin, we note that for $V(\gamma) = V_2(\gamma)$ or $\tilde{V}_2(\gamma)$:

$$\left|\nu\partial_\nu\left(\frac{1}{\nu^2}V(\zeta/\nu)\right)\right| = \left|\frac{1}{\nu^2}(\Lambda_\gamma V)(\zeta/\nu)\right| \lesssim \frac{\nu^2\zeta^2\log^2(1+\zeta/\nu)}{(\nu+\zeta)^6},$$

Besides, for $V(\gamma) = V_4(\gamma)$ or $\tilde{V}_4(\gamma)$ (the leading order of which is log growth at infinity), note that

$$|\partial_\zeta^k \nu\partial_\nu(\tilde{\alpha}_{1,\nu}V(\zeta/\nu))| \lesssim \frac{1}{|\log(\nu)|}\frac{\zeta^{2-k}}{(\nu+\zeta)^2} + \frac{\nu^2\zeta^2\log^2(1+\zeta/\nu)}{(\nu+\zeta)^4}, \quad k = 0,1,2.$$

Combining these facts gives us (2.2). Now we estimate the generated error:

$$\begin{aligned}
R_i &:= \mathscr{L}_\nu^\zeta \varphi_{i,\nu}(\zeta) - 2\beta(1-i+\tilde{\alpha}_{i,\nu})\varphi_{i,\nu}(\zeta)\\
&= -\frac{1}{16\nu^4}\chi_m(\zeta)\left(\mathscr{L}_\nu^\zeta - 2\beta(1-i+\tilde{\alpha}_{i,\nu})\right)\varphi_i^{\mathrm{in}}(\zeta/\nu) + (1-\chi_m(\zeta))\chi_\nu(\zeta)\left(\partial_\zeta^2 + \frac{1}{\zeta}\partial_\zeta - \partial_\zeta\Psi_{U_\nu}(\zeta)\cdot\partial_\zeta\right)\varphi_i^{\mathrm{ex}}(\zeta)\\
&\quad - \partial_\zeta U_\nu(\zeta)\cdot\partial_\zeta\Psi_{i,m}^{\mathrm{ex}}(\zeta) + 2(1-\chi_m(\zeta))\chi_\nu(\zeta)U_\nu(\zeta)\varphi_i^{\mathrm{ex}}(\zeta) - (1-\chi_m(\zeta))\chi_\nu(\zeta)\beta\Lambda\varphi_i^{\mathrm{ex}}(\zeta)\\
&\quad - \frac{1}{8\nu^5}\partial_\gamma\varphi_i^{\mathrm{in}}(\zeta/\nu)\chi_m'(\zeta) - \frac{1}{16\nu^4}\varphi_i^{\mathrm{in}}(\zeta/\nu)\chi_m''(\zeta) - \frac{1}{16\nu^4\zeta}\varphi_i^{\mathrm{in}}(\zeta/\nu)\chi_m'(\zeta) + \frac{1}{16\nu^4}\varphi_i^{\mathrm{in}}(\zeta/\nu)\chi_m'(\zeta)\partial_\zeta\Psi_{U_\nu}(\zeta)\\
&\quad - \left(\partial_\zeta\Psi_{i,m}^{\mathrm{in}}(\zeta) - \chi_m(\zeta)\partial_\zeta\Psi_{i,*}^{\mathrm{in}}(\zeta)\right)\partial_\zeta U_\nu(\zeta) + \beta\zeta\chi_m'(\zeta)\frac{1}{16\nu^4}\varphi_i^{\mathrm{in}}(\zeta/\nu)\\
&\quad - 2\chi_m'\chi_\nu\partial_\zeta\varphi_i^{\mathrm{ex}} - \chi_m''\chi_\nu\varphi_i^{\mathrm{ex}} - \frac{1}{\zeta}\chi_m'\chi_\nu\varphi_i^{\mathrm{ex}} + \chi_m'\chi_\nu\varphi_i^{\mathrm{ex}}(\zeta)\partial_\zeta\Psi_{U_\nu}(\zeta) + \beta\zeta\chi_m'\chi_\nu\varphi_i^{\mathrm{ex}}(\zeta)\\
&\quad + 2(1-\chi_m)\chi_{|\log\nu|}'\partial_\zeta\varphi_i^{\mathrm{ex}} + (1-\chi_m)\chi_{|\log\nu|}''\varphi_i^{\mathrm{ex}} + \frac{1}{\zeta}(1-\chi_m)\chi_{|\log\nu|}'\varphi_i^{\mathrm{ex}}\\
&\quad - (1-\chi_m)\chi_{|\log\nu|}'\varphi_i^{\mathrm{ex}}(\zeta)\partial_\zeta\Psi_{U_\nu}(\zeta) - \beta\zeta(1-\chi_m)\chi_{|\log\nu|}'\varphi_i^{\mathrm{ex}}(\zeta)\\
&:= R_i^{\mathrm{in}} + R_i^{\mathrm{ex}} + R_i^{bd},
\end{aligned}$$

where we denote

$$\begin{aligned}
&-(\partial_\zeta^2 + \frac{1}{\zeta}\partial_\zeta)\Psi_{i,m}^{\mathrm{in}}(\zeta) = -\frac{1}{16\nu^4}\varphi_i^{\mathrm{in}}(\zeta/\nu)\chi_m(\zeta), \qquad -(\partial_\zeta^2 + \frac{1}{\zeta}\partial_\zeta)\Psi_{i,*}^{\mathrm{in}}(\zeta) = -\frac{1}{16\nu^4}\varphi_i^{\mathrm{in}}(\zeta/\nu),\\
&-(\partial_\zeta^2 + \frac{1}{\zeta}\partial_\zeta)\Psi_{i,m}^{\mathrm{ex}}(\zeta) = \varphi_i^{\mathrm{ex}}(\zeta)(1-\chi_m(\zeta)).
\end{aligned}$$

We assume that the supports of χ_m' and $\chi_{|\log\nu|}'$ are disjoint so that the terms containing $\chi_m'\chi_{|\log\nu|}'$ are all zero which we do not write out in the expression of R_i. Note that for the outer solutions, we treat $(1-\chi_m)\chi_\nu\varphi_i^{\mathrm{ex}}$ as a whole, which is different from the case of the inner solutions. This is because the singularity of φ_i^{ex} at the origin only allows the existence of the Poisson field of $(1-\chi_m)\varphi_i^{\mathrm{ex}}$, not of φ_i^{ex}. As for the inner error, note that

$$R_i^{\mathrm{in}}(\zeta) = -\frac{16}{\nu^6}R_i^\gamma(\zeta/\nu)\chi_m(\zeta).$$

By (2.7), we have, for $\zeta \in [0,\zeta_m]$,

$$\left|\partial_\zeta^k\left(\frac{1}{\nu^6}R_i^\gamma(\zeta/\nu)\right)\right| \lesssim \frac{\nu^2\zeta^{2-k}\log^2(2+\zeta/\nu)}{(\nu+\zeta)^4} + \frac{\zeta^{2-k}|\log(\nu+\zeta)|}{(\nu+\zeta)^2|\log\nu|}.$$

As for the outer error, we calculate that

$$
\begin{aligned}
R_i^{\rm ex} &= (1-\chi_m)\chi_\nu \left(\partial_\zeta^2 + \frac{1}{\zeta}\partial_\zeta - \partial_\zeta \Psi_{U_\nu}\cdot\partial_\zeta\right)\varphi_i^{\rm ex} - \partial_\zeta U_\nu \cdot \partial_\zeta \Psi_{i,m}^{\rm ex} \\
&\quad + 2(1-\chi_m)\chi_\nu U_\nu \varphi_i^{\rm ex} - (1-\chi_m)\chi_\nu \beta\Lambda\varphi_i^{\rm ex} \\
&= (1-\chi_m)\chi_\nu(\mathscr{H} - 2\beta(1-i+\tilde{\alpha}_{i,\nu}))\varphi_i^{\rm ex}(\zeta) + (1-\chi_m)\chi_\nu\left(\frac{4\zeta}{\zeta^2+\nu^2} - \frac{4}{\zeta}\right)\partial_\zeta\varphi_i^{\rm ex} \\
&\quad + \frac{32\nu^2\zeta}{(\zeta^2+\nu^2)^3}\partial_\zeta\Psi_{i,m}^{\rm ex} + \frac{16\nu^2}{(\zeta^2+\nu^2)^2}(1-\chi_m)\chi_\nu\varphi_i^{\rm ex}(\zeta) \\
&= \tilde{\alpha}_{i,\nu}(1-\chi_m)\chi_\nu\tilde{\varphi}_i^{\rm ex}(\zeta) - \frac{4\nu^2}{\zeta(\zeta^2+\nu^2)}(1-\chi_m)\chi_\nu\partial_\zeta\varphi_i^{\rm ex}(\zeta) \\
&\quad + \frac{32\nu^2\zeta}{(\zeta^2+\nu^2)^3}\partial_\zeta\Psi_{i,m}^{\rm ex} + \frac{16\nu^2}{(\zeta^2+\nu^2)^2}(1-\chi_m)\chi_\nu\varphi_i^{\rm ex}(\zeta).
\end{aligned}
$$

Thus, by the asymptotic behavior at $\zeta\to\infty$ of the outer solutions as well as their Poisson fields, we have

$$
|\partial_\zeta^k R_i^{\rm ex}(\zeta)| \lesssim \frac{1}{\log^2(\nu)}\cdot\frac{\log(1+\zeta)}{\zeta^2}, \quad \forall \zeta\geq\zeta_m,\ \forall\, k=0,1,2.
$$

Now we come to the boundary error R_i^{bd}. First, note that $\operatorname{supp}\chi_m' \subset [\zeta_m, 2\zeta_m]$, by the matching condition (2.13), the terms involving the derivatives of χ_m cancel in pairs. For example,

$$
\left|\partial_\zeta^k\left(-\frac{1}{8\nu^4}\partial_\zeta\varphi_i^{\rm in}(\zeta/\nu)\chi_m'(\zeta) - 2\partial_\zeta\varphi_i^{\rm ex}(\zeta)\chi_m'(\zeta)\right)\right| \lesssim \frac{1}{|\log\nu|},
$$

and the rest are similar. Second, we note that $\operatorname{supp}\chi_{|\log\nu|}' \subset [|\log\nu|, 2|\log\nu|]$, and we have the decay property

$$
|\varphi_i^{\rm ex}(\zeta)| \lesssim \frac{\log(\zeta)}{(1+\zeta)^2}.
$$

Then, any term involving the derivative of χ_ν is of size at most $\mathcal{O}(1/|\log\nu|)$. It then remains to estimate the terms involving Poisson fields. Note that for a 2D radial symmetric Poisson problem with Neumann boundary condition (we assume S to have certain regularity which is the case in our problem):

$$
-(\partial_\zeta^2 + \frac{1}{\zeta}\partial_\zeta)\Psi(\zeta) = S(\zeta),
$$

the Poisson field satisfies

$$
\partial_\zeta\Psi(\zeta) = \frac{1}{\zeta}\int_0^\zeta rS(r)\,dr.
$$

Using this, we obtain that

$$
\partial_\zeta\Psi_{i,m}^{\rm in}(\zeta) - \chi_m(\zeta)\partial_\zeta\Psi_{i,*}^{\rm in}(\zeta) \equiv 0, \quad \forall\,\zeta\in[0,\zeta_m],
$$

and

$$
\partial_\zeta\Psi_{i,m}^{\rm in}(\zeta) - \chi_m(\zeta)\partial_\zeta\Psi_{i,*}^{\rm in}(\zeta) \equiv \text{const.}, \quad \forall\,\zeta\in[2\zeta_m,+\infty).
$$

Besides, for $\zeta>\zeta_m$, $\partial_\zeta U_\nu \lesssim \nu^2/\zeta^5$, which finishes the pointwise estimate for the boundary error. Finally, the estimate of the partial mass follows directly from the pointwise estimate of the generated error. ☐

2.2 Coercivity of the Linearized Operator

The main goal of this section is to establish the coercivity of the linearized operator $\mathscr{L}_\nu^\zeta$ (after a slight modification) under certain adapted inner products.

2.2.1 Properties of the Operator $\mathscr{M}_\nu^\zeta$

To begin with, we collect several properties of $\mathscr{M}_\nu^\zeta$, in particular, boundedness and definiteness.

Lemma 1. *Let f be a function on $\mathbb{R}^2$ with $\int_{\mathbb{R}^2}(1+|\mathbf{y}|^2)f^2 + \int_{\mathbb{R}^2}\frac{|\nabla f|^2}{U} < +\infty$. Then, there exist universal constants $c, C > 0$, such that*

$$\int_{\mathbb{R}^2} U|\nabla \mathscr{M} f|^2 \geq c\int_{\mathbb{R}^2}\frac{|\nabla f|^2}{U} - C\left(|\langle f\,,\,\Lambda U\rangle|^2 + |\langle f\,,\,\nabla U\rangle|^2\right).$$

In the parabolic variables, the inequality equivalently becomes

$$\int U_\nu|\nabla \mathscr{M}_\nu^\zeta f|^2 \geq c\int \frac{|\nabla f|^2}{U_\nu} - C\left(\nu^2|\langle f\,,\,\Lambda U_\nu\rangle|^2 + \nu^4|\langle f\,,\,\nabla U_\nu\rangle|^2\right).$$

Proof. The proof follows the same tactic as in [31]. First of all, by Hardy's inequality and the estimates of Poisson fields (A.14)(A.15), we have the a priori bounds

$$\begin{aligned}
&\int_{\mathbb{R}^2}(1+|\mathbf{y}|^2)g^2 \lesssim \int_{\mathbb{R}^2}(1+|\mathbf{y}|^2)^2|\nabla g|^2 = 8\int_{\mathbb{R}^2}\frac{|\nabla g|^2}{U},\\
&\int_{\mathbb{R}^2}\frac{|\nabla\Psi_g|^2}{1+|\mathbf{y}|^2} \lesssim \int_{\mathbb{R}^2}(1+|\mathbf{y}|^2)g^2 \lesssim \int_{\mathbb{R}^2}(1+|\mathbf{y}|^4)|\nabla g|^2,\\
&\int_{\mathbb{R}^2}\frac{|\Psi_g|^2}{1+|\mathbf{y}|^4} \lesssim \int_{\mathbb{R}^2}(1+|\mathbf{y}|^2)g^2 \lesssim \int_{\mathbb{R}^2}(1+|\mathbf{y}|^4)|\nabla g|^2.
\end{aligned} \tag{2.14}$$

Moreover, note that

$$\int_{\mathbb{R}^2} U|\nabla\mathscr{M} f|^2 = \int_{\mathbb{R}^2} U\left|\nabla\left(\frac{f}{U}\right) - \nabla\Psi_f\right|^2 \geq \frac{1}{2}\int_{\mathbb{R}^2} U\left|\nabla\left(\frac{f}{U}\right)\right|^2 - \int_{\mathbb{R}^2} U|\nabla\Psi_f|^2,$$

and through integration by parts,

$$\begin{aligned}
\int_{\mathbb{R}^2} U\left|\nabla\left(\frac{f}{U}\right)\right|^2 &= \int_{\mathbb{R}^2} U\left|\frac{\nabla f}{U} - \frac{f\nabla\Psi_U}{U}\right|^2\\
&= \int_{\mathbb{R}^2}\frac{|\nabla f|^2}{U} + \frac{|\nabla\Psi_U|^2 f^2}{U} - \int_{\mathbb{R}^2}\frac{2f\nabla f\cdot\nabla\Psi_U}{U}\\
&= \int_{\mathbb{R}^2}\frac{|\nabla f|^2}{U} + \frac{|\nabla\Psi_U|^2 f^2}{U} + \int_{\mathbb{R}^2} f^2\nabla\cdot\left(\frac{\nabla\Psi_U}{U}\right)\\
&= \int_{\mathbb{R}^2}\frac{|\nabla f|^2}{U} - f^2.
\end{aligned}$$

Then, by Poisson field estimates, we obtain the sub-coercivity estimate:

$$\int_{\mathbb{R}^2} U|\nabla\mathscr{M} f|^2 \gtrsim \int_{\mathbb{R}^2}\frac{|\nabla f|^2}{U} - C\int_{\mathbb{R}}(1+|\mathbf{y}|)f^2. \tag{2.15}$$

Assume, by contradiction, that there exists a sequence of functions $\{f_n\}$ that satisfies

$$\int_{\mathbb{R}^2}(1+|\mathbf{y}|^2)f_n^2 < +\infty, \quad \int_{\mathbb{R}^2}\frac{|\nabla f_n|^2}{U} = 1, \quad \langle f_n\,,\,\Lambda U\rangle = \langle f_n\,,\,\nabla U\rangle = 0, \quad \text{and} \lim_{n\to+\infty}\int_{\mathbb{R}^2} U|\nabla\mathscr{M} f_n|^2 = 0.$$

Then, by (2.14) and $\int|\Delta\Psi_{f_n}|^2 = \int f_n^2 < +\infty$, we know from Sobolev embedding that there exist some f and Ψ, such that (up to a subsequence)

$$\begin{aligned}
&f_n \rightharpoonup f, \ \text{ in } H^1(\mathbb{R}^2), \quad \text{and} \quad f_n \to f, \ \text{ in } L^2_{\text{loc}}(\mathbb{R}^2),\\
&\Psi_{f_n} \to \Psi, \ \text{ in } H^1_{\text{loc}}(\mathbb{R}^2).
\end{aligned}$$

In particular, the convergence above holds in the sense of distribution (i.e., in $\mathcal{D}'(\mathbb{R}^2)$), and $-\Delta\Psi = f$ in $\mathcal{D}'(\mathbb{R}^2)$. Since $\int_{\mathbb{R}^2} U|\nabla \mathscr{M} f_n|^2 \to 0$, $\nabla \mathscr{M} f_n \to 0$ in $\mathcal{D}'(\mathbb{R}^2)$ so that $\nabla\left(\frac{f}{U} - \Psi\right) = 0$ in $\mathcal{D}'(\mathbb{R}^2)$. In summary, we have

$$\begin{cases} -\Delta\Psi = f, \\ \nabla\left(\frac{f}{U} - \Psi\right) = 0, \end{cases} \quad \text{in } \mathcal{D}'(\mathbb{R}^2).$$

From standard lower semi-continuity estimates, we have

$$\int_{\mathbb{R}^2} \frac{|\nabla f|^2}{U} \le 1, \quad \int_{\mathbb{R}^2} \frac{|\nabla \Psi|^2}{1+|\mathbf{y}|^2} + \frac{\Psi^2}{1+|\mathbf{y}|^4} \lesssim 1.$$

Then, by elliptic regularity which is bootstrapped by the relation $\nabla\left(\frac{f}{U} - \Psi\right) = 0$, we know that $(f, \Psi) \in C^\infty(\mathbb{R}^2)$, and in particular $\Psi = \Psi_f$. By *Lemma 2.1* in [31], we obtain

$$f \in \operatorname{Span}\{\Lambda U, \partial_{y_1} U, \partial_{y_2} U\}.$$

Since the orthogonality conditions pass to f, i.e., $\langle f\,,\, \Lambda U\rangle = \langle f\,,\, \nabla U\rangle = 0$, we deduce that $f \equiv 0$. On the other hand, since by assumption $\int f_n^2(1+|\mathbf{y}|^2)$ are uniformly bounded, by local strong convergence we have

$$\lim_{n\to+\infty} \int_{\mathbb{R}^2} f_n^2(1+|\mathbf{y}|) = \int_{\mathbb{R}^2} f^2(1+|\mathbf{y}|).$$

Then, by sub-coercivity (2.15),

$$\int_{\mathbb{R}^2} f^2(1+|\mathbf{y}|) \gtrsim \lim_{n\to+\infty} \frac{1}{C}\left(\int_{\mathbb{R}^2} \frac{|\nabla f_n|^2}{U} - \int_{\mathbb{R}^2} U|\nabla \mathscr{M} f_n|^2\right) = \frac{1}{C},$$

which contradicts the fact that $f \equiv 0$. In summary, there exists some $c > 0$ such that

$$\int_{\mathbb{R}^2} U|\nabla \mathscr{M} f|^2 \ge c \int_{\mathbb{R}^2} \frac{|\nabla f|^2}{U}, \tag{2.16}$$

for any f with $\int \frac{|\nabla f|^2}{U} < +\infty$ satisfying the orthogonality conditions. Finally, for general f, define

$$F := f - \frac{\langle f\,,\, \Lambda U\rangle}{\langle \Lambda U\,,\, \Lambda U\rangle}\Lambda U - \sum_{i=1,2} \frac{\langle f\,,\, \partial_{y_i} U\rangle}{\langle \partial_{y_i} U\,,\, \partial_{y_i} U\rangle}\partial_{y_i} U.$$

Applying (2.16) to F completes the proof. ☐

The following lemma implies norm equivalence on some finite codimensional function space, which will motivates our design of the adapted inner product.

Lemma 2. *The quadratic form $(f, g) \mapsto \int_{\mathbb{R}^2} f \mathscr{M}_\nu^\zeta g$ is symmetric. Moreover, for any f such that $\int_{\mathbb{R}^2} f^2/U_\nu < +\infty$ and $\int_{\mathbb{R}^2} |\nabla f|^2/U_\nu < +\infty$, we have the estimates*

$$\int_{\mathbb{R}^2} U_\nu |\mathscr{M}_\nu^\zeta f|^2 \lesssim \int_{\mathbb{R}^2} \frac{f^2}{U_\nu}, \tag{2.17}$$

$$\int_{\mathbb{R}^2} U_\nu |\nabla \mathscr{M}_\nu^\zeta f|^2 \lesssim \int_{\mathbb{R}^2} \frac{|\nabla f|^2}{U_\nu}, \tag{2.18}$$

and

$$\int_{\mathbb{R}^2} f \mathscr{M}_\nu^\zeta f \ge \frac{1}{C}\int_{\mathbb{R}^2} \frac{f^2}{U_\nu} - C\left(\nu^4 |\langle f\,,\, \Lambda U_\nu\rangle|^2 + \nu^6 |\langle f\,,\, \nabla U_\nu\rangle|^2 + |\langle f\,,\, 1\rangle|^2\right), \tag{2.19}$$

for some universal $C > 0$. In addition, if $\int_{\mathbb{R}^2} f = 0$, we have the definiteness

$$\int_{\mathbb{R}^2} f \mathscr{M}_\nu^\zeta f \ge 0.$$

Proof. The symmetry of the quadratic form follows from the symmetry:

$$\langle f\,,\,\Psi_g\rangle = \langle \Psi_f\,,\,g\rangle = -\frac{1}{2\pi}\int\int \log|\mathbf{x}-\mathbf{y}| f(\mathbf{x})g(\mathbf{y})\, d\mathbf{x}d\mathbf{y}.$$

By (A.16) (taking $\alpha = 1$) and Hardy's inequality,

$$\int U_\nu |\nabla \Psi_f|^2 \lesssim \int \frac{\nu^2(1+\log(\zeta/\nu))}{(\nu+\zeta)^6}\int f^2(\nu+\zeta)^2 \lesssim \frac{1}{\nu^2}\int |\nabla f|^2(\nu+\zeta)^4 \lesssim \int \frac{|\nabla f|^2}{U_\nu}.$$

This completes the proof of (2.18). As for the rest, see *Lemma 2.1* and *Proposition 2.3* in [31]. □

2.2.2 Adapted Inner Product and Coercivity

Define the weight functions

$$\varrho_\nu(\zeta) := e^{-\frac{\beta\zeta^2}{2}}, \quad \varrho(\gamma) := e^{-\frac{\beta\nu^2\gamma^2}{2}}, \tag{2.20}$$

Observe the following two decompositions of the linearized operator (written in soliton variables):

$$\mathscr{L} f = \nabla\cdot(U\nabla\mathscr{M} f) - b\Lambda f = \frac{1}{\omega}\nabla\cdot(\omega\nabla f) + 2(U-b)f - \nabla U\cdot\nabla\Psi_f, \tag{2.21}$$

where we denote $b := \beta\nu^2$ and $\omega := \frac{1}{U}\varrho$. In the near field, i.e., $\gamma \ll \frac{1}{\nu}$, according to the first decomposition in (2.21) the scaling term $b\Lambda f$ becomes negligible, and the coercivity of $\nabla\mathscr{M}$ (Lemma 1) leads to the coercivity of $\mathscr{L}$ with some appropriate inner product in this domain. In the far field, i.e., $\gamma \gg \frac{1}{\nu}$, the terms $\nabla U\cdot\nabla\Psi_f$ and Uf become negligible due to the fast decay of ∇U and U, according to the second decomposition in (2.21). Therefore, $\mathscr{L}$ will be coercive with the weighted L^2-inner product (with ω as the weight function) in this domain. In order to obtain coercivity in the whole domain, we define the mixed inner product which adapts to both coercivity structures:

$$\langle f\,,\,g\rangle_* := \int_{\mathbb{R}^2} fg\bar{\chi}_\nu^2\omega - \int_{\mathbb{R}^2}\sqrt{\varrho}\bar{\chi}_\nu f\Psi_{\sqrt{\varrho}\bar{\chi}_\nu g} = \int_{\mathbb{R}^2}\sqrt{\varrho}\bar{\chi}_\nu f\mathscr{M}(\sqrt{\varrho}\bar{\chi}_\nu g),$$

or equivalently, in the parabolic variables

$$\langle \varepsilon\,,\,\vartheta\rangle_{\nu,*} := \int_{\mathbb{R}^2}\sqrt{\varrho_\nu}\chi_\nu\varepsilon\mathscr{M}_\nu^\zeta(\sqrt{\varrho_\nu}\chi_\nu\vartheta), \tag{2.22}$$

where we recall $\bar{\chi}_\nu(\gamma) := \chi(\gamma\nu/|\log\nu|)$ and $\chi_\nu(\zeta) := \chi(\zeta/|\log\nu|)$. One remark: thanks to Lemma 2, we know that for any f satisfying the orthogonality condition

$$\int_{\mathbb{R}^2} f\Lambda U\chi(2\gamma\nu/c_0)\sqrt{\varrho} = \int_{\mathbb{R}^2} f\nabla U\chi(2\gamma\nu/c_0)\sqrt{\varrho} = \int_{\mathbb{R}^2} f\chi(\gamma\nu/\zeta^*) = 0,$$

where $c_0 > 0$ is some fixed constants and b is small enough, there holds the equivalence of norms:

$$\frac{1}{C}\int f^2\bar{\chi}_\nu^2\omega \le \langle f\,,\,f\rangle_* \le C\int f^2\bar{\chi}_\nu^2\omega.$$

Besides, we need to modify the linearized operator a little bit to adapt to the inner product:

$$\tilde{\mathscr{L}} f := \Delta f - \nabla U\cdot\nabla\Psi_{\sqrt{\varrho}\bar{\chi}_\nu f} - \nabla f\cdot\nabla\Psi_U + 2Uf - b\Lambda f,$$

or equivalently,

$$\tilde{\mathscr{L}}_\nu^\zeta\varepsilon := \Delta\varepsilon - \nabla U_\nu\cdot\nabla\tilde{\Psi}_\varepsilon - \nabla\varepsilon\cdot\nabla\Psi_{U_\nu} + 2U_\nu\varepsilon - \beta\Lambda\varepsilon,$$

where $\tilde{\Psi}_\varepsilon := \Psi_{\chi_\nu\sqrt{\varrho_\nu}\varepsilon}$. Applying the aforementioned ideas, we are able to prove the following proposition.

Proposition 2 (Coercivity estimate)**.** *There exist constants $\delta, \zeta_*, C, b_* > 0$, such that for any $0 < b < b_*$, and any ε satisfying $\int \frac{\varepsilon^2+|\nabla\varepsilon|^2}{U} < +\infty$ and*

$$\int_{\mathbb{R}^2} \varepsilon \Lambda U \chi(\gamma\nu/\zeta_*) = \int_{\mathbb{R}^2} \varepsilon \nabla U \chi(\gamma\nu/\zeta_*) = 0,$$

we have

$$\left\langle \tilde{\mathscr{L}}\varepsilon\,,\,\varepsilon \right\rangle_* \leq -\delta \left(\int_{\mathbb{R}^2} \frac{|\nabla\varepsilon|^2 \bar{\chi}_\nu^2}{U} \varrho + b \int_{\mathbb{R}^2} \frac{\varepsilon^2 \bar{\chi}_\nu^2}{U} \varrho \right) + C\nu^{100} \|\varepsilon\|^2_{L^\infty(\gamma \geq |\log\nu|/\nu)}$$

Proof. Define $\bar{\chi}_0(\gamma) := \chi(\gamma\nu/\zeta_0)$, and decompose

$$\varepsilon = \bar{\chi}_0 \varepsilon + (1-\bar{\chi}_0)\varepsilon := \varepsilon_1 + \varepsilon_2,$$

where $0 < \zeta_0 = \zeta_0(\varepsilon) \ll 1$ is a parameter to be determined. Though the specific value of ζ_0 depends on ε, we will see that there exists a universal constant $\zeta_* > 0$ such that $\zeta_* < \zeta_0$ for any ε. Thus,

$$\left\langle \tilde{\mathscr{L}}\varepsilon\,,\,\varepsilon \right\rangle_* = \left\langle \tilde{\mathscr{L}}\varepsilon_1\,,\,\varepsilon_1 \right\rangle + \left\langle \tilde{\mathscr{L}}\varepsilon_2\,,\,\varepsilon_2 \right\rangle + \left\langle \tilde{\mathscr{L}}\varepsilon_1\,,\,\varepsilon_2 \right\rangle + \left\langle \tilde{\mathscr{L}}\varepsilon_2\,,\,\varepsilon_1 \right\rangle.$$

For brevity, in the following we denote

$$\|f\|_{L^2_\omega} := \left(\int \frac{f^2 \varrho}{U} \right)^{\frac{1}{2}}.$$

Coercivity of $\left\langle \tilde{\mathscr{L}}\varepsilon_1\,,\,\varepsilon_1 \right\rangle_*$: Since $\bar{\chi}_\nu \varepsilon_1 = \varepsilon_1$ and

$$\sqrt{\varrho}\bar{\chi}_\nu \tilde{\mathscr{L}}\varepsilon_1 = \mathscr{L}(\sqrt{\varrho}\varepsilon_1) + (1-\sqrt{\varrho}\bar{\chi}_\nu)\nabla\Psi_{\sqrt{\varrho}\varepsilon_1} \cdot \nabla U + [\bar{\chi}_\nu\sqrt{\varrho},\, \mathscr{L} + \nabla U \cdot \nabla\Psi.]\varepsilon_1,$$

by integration by parts, we have

$$\begin{aligned} \left\langle \tilde{\mathscr{L}}\varepsilon_1\,,\,\varepsilon_1 \right\rangle_* = -\int U|\nabla\mathscr{M}(\varepsilon_1\sqrt{\varrho})|^2 + b\int \sqrt{\varrho}\varepsilon_1 \mathbf{y} \cdot \nabla\mathscr{M}(\sqrt{\varrho}\varepsilon_1) + \int (1-\sqrt{\varrho}\bar{\chi}_\nu)\nabla\Psi_{\sqrt{\varrho}\varepsilon_1} \cdot \nabla U \mathscr{M}(\sqrt{\varrho}\varepsilon_1) \\ + \int [\bar{\chi}_\nu\sqrt{\varrho},\, \mathscr{L} + \nabla U \cdot \nabla\Psi.]\varepsilon_1 \mathscr{M}(\sqrt{\varrho}\varepsilon_1). \end{aligned}$$

By Lemma 1, we have

$$-\int U|\nabla\mathscr{M}(\varepsilon_1\sqrt{\varrho})|^2 \leq -c\int \frac{|\nabla(\varepsilon_1\sqrt{\varrho})|^2}{U} + C\left(|\langle \varepsilon_1\sqrt{\varrho}\,,\,\Lambda U\rangle|^2 + |\langle \varepsilon_1\sqrt{\varrho}\,,\,\nabla U\rangle|^2 \right).$$

By the local orthogonality conditions

$$\int_{\mathbb{R}^2} \varepsilon \Lambda U \chi(2\gamma\nu/\zeta_*) = \int_{\mathbb{R}^2} \varepsilon \nabla U \chi(2\gamma\nu/\zeta_*) = 0$$

where $\zeta_* < \zeta_0$, we obtain

$$\begin{aligned} |\langle \varepsilon_1\sqrt{\varrho}\,,\,\Lambda U\rangle| &= \left| \int \varepsilon_1(\sqrt{\varrho} - \chi(2\gamma\nu/\zeta_*))\Lambda U \right| \\ &\leq \left(\int \frac{\varepsilon_1^2 \varrho}{U} \right)^{\frac{1}{2}} \left(\int (\Lambda U)^2 U \rho^{-1} (\sqrt{\varrho} - \chi(2\gamma\nu/\zeta_*))^2 \right)^{\frac{1}{2}} \\ &\lesssim \nu^2 \left(\int \frac{\varepsilon_1^2 \varrho}{U} \right)^{\frac{1}{2}} \lesssim \nu \left(\int \frac{|\nabla\varepsilon_1|^2 \varrho}{U} \right)^{\frac{1}{2}} \end{aligned}$$

where we use the Poincaré inequality (taking $\alpha = 2$ in this case):

$$\int \varepsilon_1^2 \varrho (1+\gamma)^{2+\alpha} \le C \frac{\zeta_0^\alpha}{\nu^\alpha} \int \varepsilon_1^2 (1+\gamma)^2 \le C^2 \int |\nabla \varepsilon_1|^2 (1+\gamma)^4 \le C^3 \frac{\zeta_0^\alpha}{\nu^\alpha} \int \frac{|\nabla \varepsilon_1|^2 \varrho}{U}, \quad \forall\, \alpha \ge 0, \tag{2.23}$$

for some universal $C > 0$ when ζ_0 is sufficiently small. Similar estimate holds for $|\left\langle \varepsilon_1 \sqrt{\varrho}\,,\, \nabla U \right\rangle|$. Thus, when b is sufficiently small (recall that $b := \beta\nu^2$), there exists a universal $\delta > 0$ such that

$$-\int U |\nabla \mathscr{M}(\varepsilon_1 \sqrt{\varrho})|^2 \le -\delta \int \frac{|\nabla \varepsilon_1|^2 \varrho}{U}. \tag{2.24}$$

Note, by (2.23), that

$$\int \frac{|\varepsilon_1 \nabla(\sqrt{\varrho})|^2}{U} = \int \frac{b^2 |\mathbf{y}|^2 \varepsilon_1^2 \varrho}{4U} \lesssim \zeta_0^4 \int \frac{|\nabla \varepsilon_1|^2 \rho}{U}.$$

Therefore, when ζ_0 is small enough,

$$\frac{1}{2} \int \frac{|\nabla \varepsilon_1|^2 \varrho}{U} \le \int \frac{|\nabla(\sqrt{\varrho}\varepsilon_1)|^2}{U} \le 2 \int \frac{|\nabla \varepsilon_1|^2 \varrho}{U}.$$

Besides, by Lemma 2 and Poincaré inequality,

$$\left| b \int \sqrt{\varrho}\varepsilon_1 \mathbf{y} \cdot \mathscr{M}(\sqrt{\varrho}\varepsilon_1) \right| \lesssim b \left(\int \frac{\varrho \varepsilon_1^2 |\mathbf{y}|^2}{U} \right)^{\frac{1}{2}} \left(\int \frac{|\nabla(\sqrt{\varrho}\varepsilon_1)|^2}{U} \right)^{\frac{1}{2}} \lesssim \zeta_0^2 \int \frac{|\nabla \varepsilon_1|^2 \varrho}{U}.$$

Thus, when b and ζ_0 are sufficiently small, we have

$$-\int U |\nabla \mathscr{M}(\varepsilon_1 \sqrt{\varrho})|^2 + b \int \sqrt{\varrho}\varepsilon_1 \mathbf{y} \cdot \nabla \mathscr{M}(\sqrt{\varrho}\varepsilon_1) \le -\delta \int \frac{|\nabla \varepsilon_1|^2 \varrho}{U}$$

for some universal $\delta > 0$. Next, by (A.12) and (A.13), we know that

$$|\Psi_{\varepsilon_1\sqrt{\varrho}}(\mathbf{y})|^2 \lesssim \log^2(4 + |\mathbf{y}|) \int \frac{\varepsilon_1^2 \varrho}{U}, \quad |\nabla \Psi_{\varepsilon_1\sqrt{\varrho}}(\mathbf{y})|^2 \lesssim \int \frac{|\nabla(\sqrt{\varrho}\varepsilon_1)|^2}{U}.$$

Therefore, by the above pointwise estimates of the Poisson field and (2.17),

$$\begin{aligned}
&\left| \int (1 - \sqrt{\varrho}\bar{\chi}_\nu) \nabla \Psi_{\sqrt{\varrho}\varepsilon_1} \cdot \nabla U \mathscr{M}(\sqrt{\varrho}\varepsilon_1) \right| \\
&\le \left(\int \frac{|\nabla(\sqrt{\varrho}\varepsilon_1)|^2}{U} \right)^{\frac{1}{2}} \left(\int (1 - \sqrt{\varrho}\bar{\chi}_\nu)^2 \frac{|\nabla U|^2}{U} \right)^{\frac{1}{2}} \left(\int U |\nabla \mathscr{M}(\sqrt{\varrho}\varepsilon_1)|^2 \right)^{\frac{1}{2}} \\
&\le \left(\int \frac{|\nabla(\sqrt{\varrho}\varepsilon_1)|^2}{U} \right)^{\frac{1}{2}} \left(\int (1 - \sqrt{\varrho}\bar{\chi}_\nu)^2 \frac{|\nabla U|^2}{U} \right)^{\frac{1}{2}} \left(\int \frac{\varepsilon_1^2 \rho}{U} \right)^{\frac{1}{2}}.
\end{aligned}$$

Note that when $\sqrt{b}\gamma \ll 1$, $1 - \sqrt{\varrho} = \frac{b\gamma^2}{4} + \mathcal{O}(b^2\gamma^4)$. Then, we have the estimates

$$\left| \int (1 - \sqrt{\varrho}\bar{\chi}_\nu)^2 \frac{|\nabla U|^2}{U} \right| \lesssim \left| \int_{\{|\mathbf{y}| < b^{-\frac{1}{3}}\}} \frac{b^2\gamma^4}{(1+\gamma)^6} \right| + \left| \int_{\{|\mathbf{y}| \ge b^{-\frac{1}{3}}\}} \frac{1}{(1+\gamma)^6} \right| \lesssim b^{\frac{5}{4}}.$$

Combining with the previous estimate and (2.23), we obtain

$$\left| \int (1 - \sqrt{\varrho}\bar{\chi}_\nu) \nabla \Psi_{\sqrt{\varrho}\varepsilon_1} \cdot \nabla U \mathscr{M}(\sqrt{\varrho}\varepsilon_1) \right| \lesssim b^{\frac{1}{8}} \int \frac{|\nabla \varepsilon_1|^2 \varrho}{U}. \tag{2.25}$$

For the remaining terms, since $\nabla(\bar{\chi}_\nu)\varepsilon_1 = 0$ (because of their disjoint supports),

$$[\bar{\chi}_\nu\sqrt{\varrho},\, \mathscr{L} + \nabla U\cdot\nabla\Psi.]\varepsilon_1 = -2\nabla(\sqrt{\varrho})\cdot\nabla\varepsilon_1 - \Delta(\sqrt{\varrho})\varepsilon_1 + \nabla(\sqrt{\varrho})\cdot\nabla\Psi_U\varepsilon_1 + b\mathbf{y}\cdot\nabla(\sqrt{\varrho})\varepsilon_1.$$

Therefore, by (2.23) we have

$$\int \frac{|[\bar{\chi}_\nu\sqrt{\varrho},\, \mathscr{L} + \nabla U\cdot\nabla\Psi.]\varepsilon_1|^2}{U} \lesssim \int \frac{b^2\gamma^2|\nabla\varepsilon_1|^2\varrho + \left(b^2 + b^4\gamma^4 + \frac{b^2\gamma^2}{(1+\gamma)^2}\right)\varepsilon_1^2\varrho}{U} \lesssim \zeta_0^2 b\int \frac{|\nabla\varepsilon_1|^2\varrho}{U}.$$

Then, by (2.17), (2.23) and Cauchy's inequality, we obtain

$$\left|\int [\bar{\chi}_\nu\sqrt{\varrho},\, \mathscr{L} + \nabla U\cdot\nabla\Psi.]\varepsilon_1\mathscr{M}(\sqrt{\varrho}\varepsilon_1)\right| \lesssim \zeta_0^2\int \frac{|\nabla\varepsilon_1|^2\varrho}{U}. \tag{2.26}$$

Finally, combining (2.24), (2.25), (2.26) and Poincaré inequality (2.23), it holds, when ζ_0 and b are sufficiently small, that

$$\left\langle \tilde{\mathscr{L}}\varepsilon_1\,,\,\varepsilon_1\right\rangle_* \le -\delta\left(\int \frac{|\nabla\varepsilon_1|^2\varrho}{U} + b\int\frac{\varepsilon_1^2\varrho}{U}\right). \tag{2.27}$$

for some universal $\delta > 0$.

<u>Coercivity of $\left\langle \tilde{\mathscr{L}}\varepsilon_2\,,\,\varepsilon_2\right\rangle_*$</u>: First, integrate by parts and we have

$$\begin{aligned}\left\langle \tilde{\mathscr{L}}\varepsilon_2\,,\,\varepsilon_2\right\rangle_* = &-\int |\nabla\varepsilon_2|^2\bar{\chi}_\nu^2\omega + \int \varepsilon_2^2\nabla\cdot(\omega\bar{\chi}_\nu\nabla\bar{\chi}_\nu) + \int 2(U-b)\varepsilon_2^2\bar{\chi}_\nu^2\omega\\ &- \int \nabla U\cdot\nabla\Psi_{\bar{\chi}_\nu\sqrt{\varrho}\varepsilon_2}\varepsilon_2\bar{\chi}_\nu^2\omega - \int \sqrt{\varrho}\bar{\chi}_\nu\tilde{\mathscr{L}}\varepsilon_2\Psi_{\sqrt{\varrho}\bar{\chi}_\nu\varepsilon_2}.\end{aligned}$$

As for the second term above, observe that $\nabla\bar{\chi}_\nu$ is supported in $\{|\log\nu|/\nu \le \gamma \le 2|\log\nu|/\nu\}$ and

$$e^{-\frac{b\gamma^2}{2}} \le e^{-\frac{\beta|\log\nu|^2}{2}} \lesssim \nu^N, \quad \forall\ \frac{|\log\nu|}{\nu} \le \gamma \le \frac{2|\log\nu|}{\nu}$$

for any fixed $N \gg 1$ when ν is sufficiently small. Thus, we have the estimate

$$\left|\int \varepsilon_2^2\nabla\cdot(\omega\bar{\chi}_\nu\nabla\bar{\chi}_\nu)\right| \le \nu^{100}\|\varepsilon\|^2_{L^\infty(\gamma\ge|\log\nu|/\nu)}. \tag{2.28}$$

Besides, since $2(U-b) \le -b$ when $\gamma > \zeta_0/\nu$ and ν is small enough, we have

$$\int 2(U-b)\varepsilon_2^2\bar{\chi}_\nu^2\omega \le -b\int \varepsilon_2^2\bar{\chi}_\nu^2\omega. \tag{2.29}$$

For the fourth term, we use the 2D Hardy–Littlewood–Sobolev (HLS) inequality (A.20):

$$\|\nabla\Psi_{\sqrt{\varrho}\bar{\chi}_\nu\varepsilon_2}\|_{L^4} \lesssim \left(\int \frac{\varepsilon_2^2\bar{\chi}_\nu^2\varrho}{U}\right)^{\frac12} = \|\varepsilon_2\bar{\chi}_\nu\|_{L^2_\omega},$$

and estimate by Cauchy's inequality that

$$\left|\int \nabla U\cdot\nabla\Psi_{\bar{\chi}_\nu\sqrt{\varrho}\varepsilon_2}\varepsilon_2\bar{\chi}_\nu^2\omega\right| \le \left(\int \bar{\chi}_\nu^2\varepsilon_2^2\omega\right)^{\frac12}\left(\int_{\{\gamma\ge\zeta_0/\nu\}}|\nabla U|^4\omega^2\right)^{\frac14}\|\nabla\Psi_{\sqrt{\varrho}\bar{\chi}_\nu\varepsilon_2}\|_{L^4} \lesssim b^{\frac54}\int \varepsilon_2^2\bar{\chi}_\nu^2\omega. \tag{2.30}$$

For the remaining terms, we divide them into three groups:

$$\begin{aligned}\int \sqrt{\varrho}\bar{\chi}_\nu\tilde{\mathscr{L}}\varepsilon_2\Psi_{\sqrt{\varrho}\bar{\chi}_\nu\varepsilon_2} = &\int \sqrt{\varrho}\bar{\chi}_\nu\Psi_{\sqrt{\varrho}\bar{\chi}_\nu\varepsilon_2}\Delta\varepsilon_2 + \int \sqrt{\varrho}\bar{\chi}_\nu\Psi_{\sqrt{\varrho}\bar{\chi}_\nu\varepsilon_2}(-\nabla\varepsilon_2\cdot\nabla\Psi_U - b\nabla\cdot(\mathbf{y}\varepsilon_2) + 2\varepsilon_2 U)\\ &- \int \sqrt{\varrho}\bar{\chi}_\nu\Psi_{\sqrt{\varrho}\bar{\chi}_\nu\varepsilon_2}\nabla\Psi_{\sqrt{\varrho}\bar{\chi}_\nu\varepsilon_2}\cdot\nabla U\end{aligned}$$

As before, by (A.12), we have the pointwise estimate of the Poisson field

$$|\Psi_{\sqrt{\varrho}\bar{\chi}_\nu\varepsilon_2}(\mathbf{y})|^2 \lesssim \log(4+|\mathbf{y}|)\int \varepsilon_2^2\bar{\chi}_\nu^2\omega.$$

Integrating by parts, we have

$$\int \sqrt{\varrho}\bar{\chi}_\nu\Psi_{\sqrt{\varrho}\bar{\chi}_\nu\varepsilon_2}\Delta\varepsilon_2 = \int \Delta(\sqrt{\varrho}\bar{\chi}_\nu\Psi_{\sqrt{\varrho}\bar{\chi}_\nu\varepsilon_2})\varepsilon_2.$$

By chain rule, when the derivative hits $\bar{\chi}_\nu$, we use the L^∞-control of ε_2 as before, and when the derivative hits elsewhere we use either the pointwise estimate of $\Psi_{\sqrt{\varrho}\bar{\chi}_\nu\varepsilon_2}$ or the L^4-estimate of $\nabla\Psi_{\sqrt{\varrho}\bar{\chi}_\nu\varepsilon_2}$:

$$\begin{aligned}
&\left|\int (2\nabla\bar{\chi}_\nu\cdot\nabla(\sqrt{\varrho}\Psi_{\sqrt{\varrho}\bar{\chi}_\nu\varepsilon_2}) + \Delta\bar{\chi}_\nu\sqrt{\varrho}\Psi_{\sqrt{\varrho}\bar{\chi}_\nu\varepsilon_2})\varepsilon_2\right| \lesssim \nu^{100}\|\varepsilon_2\bar{\chi}_\nu\|_{L^2_\omega}\cdot\|\varepsilon_2\|_{L^\infty(\gamma\geq|\log\nu|/\nu)},\\
&\left|\int \nabla\sqrt{\varrho}\cdot\nabla\Psi_{\sqrt{\varrho}\bar{\chi}_\nu\varepsilon_2}\varepsilon_2\right| \lesssim b\|\gamma\sqrt{U}\|_{L^4(\gamma\geq\zeta_0/\nu)}\|\nabla\Psi_{\sqrt{\varrho}\bar{\chi}_\nu\varepsilon_2}\|_{L^4}\|\varepsilon_2\bar{\chi}_\nu\|_{L^2_\omega} \lesssim b^{\frac54}\|\varepsilon_2\bar{\chi}_\nu\|^2_{L^2_\omega},\\
&\left|\int \Delta(\sqrt{\varrho})\Psi_{\sqrt{\varrho}\bar{\chi}_\nu\varepsilon_2}\varepsilon_2\right| \lesssim \|(b+b^2\gamma^2)\log(4+\gamma)\sqrt{U}\|_{L^2(\gamma\geq\zeta_0/\nu)}\|\varepsilon_2\bar{\chi}_\nu\|^2_{L^2_\omega} \lesssim b^{\frac54}\|\varepsilon_2\bar{\chi}_\nu\|^2_{L^2_\omega},\\
&\left|\int \bar{\chi}_\nu^2\varrho\varepsilon_2^2\right| \lesssim \frac{b^2}{\zeta_0^4}\|\varepsilon_2\bar{\chi}_\nu\|^2_{L^2_\omega}.
\end{aligned}$$

In summary, we obtain

$$\left|\int \Delta(\sqrt{\varrho}\bar{\chi}_\nu\Psi_{\sqrt{\varrho}\bar{\chi}_\nu\varepsilon_2})\varepsilon_2\right| \lesssim \nu^{100}\|\varepsilon_2\bar{\chi}_\nu\|_{L^2_\omega}\cdot\|\varepsilon_2\|_{L^\infty(\gamma\geq|\log\nu|/\nu)} + b^{\frac54}\|\varepsilon_2\bar{\chi}_\nu\|^2_{L^2_\omega}. \tag{2.31}$$

The estimate of the second group is more direct:

$$\begin{aligned}
&\left|\int \sqrt{\varrho}\bar{\chi}_\nu\Psi_{\sqrt{\varrho}\bar{\chi}_\nu\varepsilon_2}(-\nabla\varepsilon_2\cdot\nabla\Psi_U - b\nabla\cdot(\mathbf{y}\varepsilon_2) + 2\varepsilon_2 U)\right|\\
&\lesssim \|\varepsilon\|^2_{L^2_\omega}\left(\int_{\{\zeta_0/\nu\leq\gamma\leq|\log\nu|/\nu\}}(U^2+b^2)U\log^2(4+\gamma)\right)^{\frac12}\\
&\quad + \|\varepsilon_2\bar{\chi}_\nu\|_{L^2_\omega}\|\bar{\chi}_\nu\nabla\varepsilon_2\|_{L^2_\omega}\left(\int_{\{\zeta_0/\nu\leq\gamma\leq|\log\nu|/\nu\}}(b^2\gamma^2+|\nabla\Psi_U|^2)U\log^2(4+\gamma)\right)^{\frac12}\\
&\lesssim b^{\frac54}\|\varepsilon_2\bar{\chi}_\nu\|^2_{L^2_\omega} + b^{\frac14}\|\bar{\chi}_\nu\nabla\varepsilon_2\|^2_{L^2_\omega}.
\end{aligned} \tag{2.32}$$

For the last term, Cauchy's inequality yields

$$\begin{aligned}
\left|\int \sqrt{\varrho}\bar{\chi}_\nu\Psi_{\sqrt{\varrho}\bar{\chi}_\nu\varepsilon_2}\nabla\Psi_{\sqrt{\varrho}\bar{\chi}_\nu\varepsilon_2}\cdot\nabla U\right| &\lesssim \|\varepsilon_2\bar{\chi}_\nu\|_{L^2_\omega}\|\Psi_{\sqrt{\varrho}\bar{\chi}_\nu\varepsilon_2}\|_{L^4}\|\log(4+\gamma)\nabla U\|_{L^{\frac43}(\gamma\geq\zeta_0/\nu)}\\
&\lesssim b^{\frac32}\|\varepsilon_2\bar{\chi}_\nu\|^2_{L^2_\omega}.
\end{aligned} \tag{2.33}$$

Then, combine (2.31), (2.32) and (2.33), and we obtain

$$\left|\int \sqrt{\varrho}\bar{\chi}_\nu\mathscr{L}\varepsilon_2\Psi_{\sqrt{\varrho}\bar{\chi}_\nu\varepsilon_2}\right| \lesssim b^{\frac54}\|\varepsilon_2\bar{\chi}_\nu\|^2_{L^2_\omega} + b^{\frac14}\|\bar{\chi}_\nu\nabla\varepsilon_2\|^2_{L^2_\omega} + \nu^{100}\|\varepsilon_2\|^2_{L^\infty(\gamma\geq|\log\nu|/\nu)}. \tag{2.34}$$

Finally, combining (2.28), (2.29), (2.30) and (2.34), we have

$$\left\langle \mathscr{L}\varepsilon_2\,,\,\varepsilon_2\right\rangle_* \leq -\delta\left(\|\bar{\chi}_\nu\nabla\varepsilon_2\|^2_{L^2_\omega} + b\|\bar{\chi}_\nu\varepsilon_2\|^2_{L^2_\omega}\right) + C\nu^{100}\|\varepsilon_2\|^2_{L^\infty(\gamma\geq|\log\nu|/\nu)}, \tag{2.35}$$

for some universal $\delta, C > 0$, when ζ_0 is sufficiently small.

Estimates of $\left\langle \tilde{\mathscr{L}}\varepsilon_1\,,\,\varepsilon_2\right\rangle_* + \left\langle \tilde{\mathscr{L}}\varepsilon_2\,,\,\varepsilon_1\right\rangle_*$: The methods to estimate these interaction terms are the same as the previous ones. We remark that the interaction happens only in a relatively small region as ζ_0 is meant to be small. We will later exploit this smallness to control the interaction terms. Through integration by parts,

$$\left\langle \tilde{\mathscr{L}}\varepsilon_1\,,\,\varepsilon_2\right\rangle_* + \left\langle \tilde{\mathscr{L}}\varepsilon_2\,,\,\varepsilon_1\right\rangle_* = -2\int \omega\nabla\varepsilon_1\cdot\nabla\varepsilon_2 + 4\int (U-b)\varepsilon_1\varepsilon_2\omega$$
$$-\int \nabla U\cdot\nabla\Psi_{\sqrt{\varrho}\varepsilon_1}\varepsilon_2\omega - \int\sqrt{\varrho}\bar{\chi}_\nu\tilde{\mathscr{L}}\varepsilon_1\Psi_{\sqrt{\varrho}\bar{\chi}_\nu\varepsilon_2} - \int \nabla U\cdot\nabla\Psi_{\sqrt{\varrho}\varepsilon_2}\varepsilon_1\omega - \int\sqrt{\varrho}\bar{\chi}_\nu\tilde{\mathscr{L}}\varepsilon_2\Psi_{\sqrt{\varrho}\bar{\chi}_\nu\varepsilon_1}.$$

Note that the terms on the second line above are all lower-order terms, as we estimated in the previous steps. For the second term on the right-hand side,

$$4\int (U-b)\varepsilon_1\varepsilon_2 \leq -2b\int \bar{\chi}_0(1-\bar{\chi}_0)\varepsilon^2 \leq 0,$$

which has the desired sign. Thus, it remains to estimate the first term. Since $\nabla\varepsilon_1 = \bar{\chi}_0\nabla\varepsilon + \frac{\nu}{\zeta_0}(\nabla\chi)(\gamma\nu/\zeta_0)\varepsilon$ and $\nabla\varepsilon_2 = (1-\bar{\chi}_0)\nabla\varepsilon - \frac{\nu}{\zeta_0}(\nabla\chi)(\gamma\nu/\zeta_0)\varepsilon$, we have

$$\left|\int \omega\nabla\varepsilon_1\cdot\nabla\varepsilon_2\right| = \left|\int_{\{\zeta_0/\nu\leq\gamma\leq 2\zeta_0/\nu\}}\omega\nabla\varepsilon_1\cdot\nabla\varepsilon_2\right| \lesssim \int_{\{\zeta_0/\nu\leq\gamma\leq 2\zeta_0/\nu\}}|\nabla\varepsilon|^2\omega + \int_{\{\zeta_0/\nu\leq\gamma\leq 2\zeta_0/\nu\}}\frac{\varepsilon^2}{(1+\gamma)^2}\omega.$$

In summary, we have the estimate

$$\left|\left\langle \tilde{\mathscr{L}}\varepsilon_1\,,\,\varepsilon_2\right\rangle_* + \left\langle \tilde{\mathscr{L}}\varepsilon_2\,,\,\varepsilon_1\right\rangle_*\right| \lesssim \int_{\{\zeta_0/\nu\leq\gamma\leq 2\zeta_0/\nu\}}|\nabla\varepsilon|^2\omega + \int_{\{\zeta_0/\nu\leq\gamma\leq 2\zeta_0/\nu\}}\frac{\varepsilon^2}{(1+\gamma)^2}\omega + \text{l.o.t.}, \tag{2.36}$$

where the lower order terms (l.o.t.) can be absorbed into other terms when b is small enough.

Global coercivity: Now we are ready to derive the full coercivity based on the established estimates. First,by (2.27), (2.35) and (2.36), when b is sufficiently small, there exist universal constants $\delta, c, C > 0$ such that

$$\left\langle \tilde{\mathscr{L}}\varepsilon\,,\,\varepsilon\right\rangle_* \leq -\delta\int\left(b\varepsilon^2+|\nabla\varepsilon|^2\right)\bar{\chi}_\nu^2\omega + C\int_{\{\zeta_0/\nu\leq\gamma\leq 2\zeta_0/\nu\}}\left(|\nabla\varepsilon^2| + \frac{\varepsilon^2}{(1+\gamma)^2}\right)\omega$$
$$+C\nu^{100}\|\varepsilon_2\|^2_{L^\infty(\gamma\geq|\log\nu|/\nu)}.$$

By the Hardy-Poincaré type inequality (A.18), there exists $C' > 0$ such that

$$\int_{\{\gamma\leq\frac{1}{\nu}\}}\frac{\varepsilon^2}{(1+\gamma)^2}\omega \leq \int\frac{\varepsilon^2\chi^2(\gamma\nu)}{(1+\gamma)^2}\omega \leq C'\int_{\{\gamma\leq\frac{2}{\nu}\}}(b\varepsilon^2+|\nabla\varepsilon|^2)\omega.$$

We choose an integer $N_0 > 0$ such that

$$N_0 > \frac{4CC'}{\delta},$$

and fix $0 < \zeta_* \ll 1$ small enough such that $\zeta_* 2^{N_0}$ will satisfy all the smallness requirements for ζ_0 in the previous steps. Now we apply the following dyadic argument:

$$\frac{\delta}{2}\int\left(b\varepsilon^2+|\nabla\varepsilon|^2\right)\bar{\chi}_\nu^2\omega \geq \frac{\delta}{2}\int_{\{\gamma\leq\frac{2}{\nu}\}}\left(b\varepsilon^2+|\nabla\varepsilon|^2\right)\omega \geq \frac{\delta}{4C'}\int_{\{\gamma\leq\frac{1}{\nu}\}}\left(|\nabla\varepsilon|^2+\frac{\varepsilon^2}{(1+\gamma)^2}\right)\omega$$
$$\geq \sum_{j=0}^{N_0-1}\frac{\delta}{4C'}\int_{\{\frac{\zeta_* 2^j}{\nu}\leq\gamma\leq\frac{\zeta_* 2^{j+1}}{\nu}\}}\left(|\nabla\varepsilon|^2+\frac{\varepsilon^2}{(1+\gamma)^2}\right)\omega$$
$$\geq \frac{N_0\delta}{4C'}\min_{0\leq j\leq N_0}\left\{\int_{\{\frac{\zeta_* 2^j}{\nu}\leq\gamma\leq\frac{\zeta_* 2^{j+1}}{\nu}\}}\left(|\nabla\varepsilon|^2+\frac{\varepsilon^2}{(1+\gamma)^2}\right)\omega\right\}.$$

Let $0 \leq n_0 \leq N_0$ be the integer such that

$$\int_{\{\frac{\zeta_* 2^{n_0}}{\nu} \leq \gamma \leq \frac{\zeta_* 2^{n_0+1}}{\nu}\}} \left(|\nabla \varepsilon|^2 + \frac{\varepsilon^2}{(1+\gamma)^2}\right)\omega = \min_{0\leq j\leq N_0} \left\{\int_{\{\frac{\zeta_* 2^{j}}{\nu} \leq \gamma \leq \frac{\zeta_* 2^{j+1}}{\nu}\}} \left(|\nabla \varepsilon|^2 + \frac{\varepsilon^2}{(1+\gamma)^2}\right)\omega\right\},$$

and define $\zeta_0 = \zeta_* 2^{n_0}$. It follows by the choice of ζ_0 and the definition of N_0 that

$$C\int_{\{\zeta_0/\nu\leq\gamma\leq 2\zeta_0/\nu\}} \left(|\nabla \varepsilon^2| + \frac{\varepsilon^2}{(1+\gamma)^2}\right)\omega < \frac{\delta}{2}\int \left(b\varepsilon^2 + |\nabla\varepsilon|^2\right)\bar{\chi}_\nu^2\omega.$$

This completes the proof of the proposition. ☐

The coercivity result can be stated equivalently in the parabolic variables:

Corollary 1. *There exist constants $\delta, \zeta_*, C, \nu_* > 0$, such that for any $0 < \nu < \nu_*$ and any ε satisfying $\int \frac{\varepsilon^2+|\nabla\varepsilon|^2}{U_\nu} < +\infty$ and the orthogonality conditions*

$$\int_{\mathbb{R}^2} \varepsilon \Lambda U_\nu \chi(\zeta/\zeta_*)\sqrt{\varrho_\nu} = \int_{\mathbb{R}^2} \varepsilon \nabla U_\nu \chi(\zeta/\zeta_*)\sqrt{\varrho_\nu} = 0,$$

we have

$$\left\langle \tilde{\mathscr{L}}_\nu^\zeta \varepsilon\,,\, \varepsilon\right\rangle_{\nu,*} \leq -\delta\left(\int_{\mathbb{R}^2} \frac{|\nabla\varepsilon|^2\chi_\nu^2}{U_\nu}\varrho_\nu + \int_{\mathbb{R}^2}\frac{\varepsilon^2\chi_\nu^2}{U_\nu}\varrho_\nu\right) + C\nu^{100}\|\varepsilon\|^2_{L^\infty(\zeta\geq 1/\nu)}. \tag{2.37}$$

At the end of this section, we introduce a higher-order coercivity result of the linearized operator $\mathscr{L}_0$, which will be used in the H^1 energy estimate in the inner region.

Proposition 3 (Higher order dissipation structure)**.** *There exists $\delta > 0$, such that for any ε that satisfies $\int \frac{\varepsilon^2+|\nabla\varepsilon|^2+|\nabla^{(2)}\varepsilon|^2}{U} < +\infty$ and the orthogonal conditions*

$$\langle \varepsilon\,,\, \Lambda U\rangle = \langle \varepsilon\,,\, \partial_\rho U\rangle = \langle \varepsilon\,,\, \partial_\xi U\rangle = 0,$$

it holds that

$$\int \frac{|\mathscr{L}_0\varepsilon|^2}{U} > \delta\left(\int (1+\gamma)^4|\Delta\varepsilon|^2 + \int (1+\gamma)^2|\nabla\varepsilon|^2 + \int\varepsilon^2 + \int\frac{|\nabla\Psi_\varepsilon|^2}{(1+\gamma)^4}\right).$$

Proof. See *Proposition 2.8* in [31]. We remark that although the orthogonality conditions there are different, the proof remains valid as long as ε lies in some subspace whose intersection with Span $\{\Lambda U, \partial_\rho U, \partial_\xi U\}$ is $\{0\}$. In particular, it can be applied here. ☐

3 Construction of Blowup Solutions

In this section, we start constructing the finite time blowup solution. We first decompose the whole solution into an approximate one plus a perturbation function, where we will introduce modulation parameters driving the evolution of the perturbation. We setup the bootstrap assumptions in Definition 1, and then derive modulation equations in Lemma 3. Finally, we perform a series of energy estimates in the inner region and outer region, respectively, for the perturbation.

3.1 Decomposition of the Solution and Formulation of the Linearized Problem

Consider the following decomposition of the solution:

$$w(\bar{r},\bar{z},\tau) = U_\nu(\zeta) + P(\zeta,\tau) + \varepsilon(\bar{r},\bar{z},\tau) := W(\zeta,\tau) + \varepsilon(\bar{r},\bar{z},\tau), \tag{3.1}$$

where we denote

$$P(\zeta,\tau) := a(\tau)(\varphi_{1,\nu}(\zeta) - \varphi_{0,\nu}(\zeta)).$$

Now we study the evolution of the solution in the near field. Inserting the decomposition $w = U_\nu + P + \varepsilon$ into (1.4), we obtain the equation for ε

$$\partial_\tau \varepsilon = \mathscr{L}_\nu^\zeta \varepsilon + L(\varepsilon) + NL(\varepsilon) + E, \tag{3.2}$$

where the extra linear terms

$$\begin{aligned} L(\varepsilon) = &-\nabla\cdot(W\nabla\Theta_\varepsilon + P\nabla\Psi_\varepsilon + \varepsilon\nabla(\Phi_W - \Psi_{U_\nu})) \\ &+ \frac{1}{\bar r + R/\mu}(\partial_{\bar r}\varepsilon - \varepsilon\partial_{\bar r}\Phi_W - W\partial_{\bar r}\Phi_\varepsilon) + \frac{R_\tau}{\mu}\partial_{\bar r}\varepsilon, \end{aligned}$$

the nonlinear terms

$$NL(\varepsilon) = -\nabla\cdot(\varepsilon\nabla\Phi_\varepsilon) - \frac{1}{\bar r + R/\mu}\varepsilon\partial_{\bar r}\Phi_\varepsilon.$$

and the generated error

$$\begin{aligned} E = &-P_\tau + \mathscr{L}_\nu^\zeta P - \nabla\cdot(W\nabla\Theta_W + P\nabla\Psi_P) + \left(\frac{\nu_\tau}{\nu} - \beta\right)\Lambda U_\nu \\ &+ \frac{1}{\bar r + R/\mu}(\partial_{\bar r}W - W\partial_{\bar r}\Phi_W) + \frac{R_\tau}{\mu}\partial_{\bar r}W, \end{aligned}$$

where we can compute that

$$P_\tau = a_\tau(\varphi_{1,\nu}(\zeta) - \varphi_{0,\nu}(\zeta)) + +a(\tau)\frac{\nu_\tau}{\nu}\nu\partial_\nu(\varphi_{1,\nu}(\zeta) - \varphi_{0,\nu}(\zeta)).$$

We require ε to be even in z-variable (which is preserved by the evolution) and impose the local orthogonality conditions:

$$\int_{\mathbb{R}^2} \varepsilon\chi_*(\zeta)\, d\bar r d\bar z = \int_{\mathbb{R}^2} \varepsilon\Lambda U_\nu \chi_*(\zeta)\, d\bar r d\bar z = \int_{\mathbb{R}^2} \varepsilon\nabla U_\nu \chi_*(\zeta)\, d\bar r d\bar z = 0, \tag{3.3}$$

which are preserved by the modulation parameters $a(\tau), \nu(\tau), R_\tau/\mu$ and the even symmetry in z. Recall the definition of the inner norm

$$\|f\|_{\text{in}} := \left(\int_{\mathbb{R}^2} \frac{\nu^2 f^2\chi_\nu^2}{U_\nu} e^{-\frac{\beta\zeta^2}{2}}\right)^{\frac12}.$$

Proposition 4 (Decomposition of the generated error)**.** *The generated error can be decomposed as*

$$E = \text{Mod}_0\varphi_{0,\nu} + \text{Mod}_1\varphi_{1,\nu} + \frac{R_\tau}{\mu}\partial_{\bar r}U_\nu + \tilde E, \tag{3.4}$$

where

$$\begin{aligned} \text{Mod}_0 &= a_\tau - 2a\beta\left(1 + \frac{1}{2\log(\nu)}\right) - 16\nu^2\left(\frac{\nu_\tau}{\nu} - \beta\right), \\ \text{Mod}_1 &= -a_\tau + \frac{a(\tau)\beta}{\log(\nu)}. \end{aligned}$$

Then, we have the weighted L^2-estimate for the error:

$$\|\tilde E\|_{\text{in}} \lesssim \frac{\nu^2 + |a|}{|\log\nu|} + \left|\frac{\nu_\tau}{\nu}\right|\frac{|a|}{|\log\nu|} + |a|\sqrt{|\log\nu|}\cdot\left|\frac{R_\tau}{\mu}\right| + \frac{a^2}{\nu}.$$

In addition, we have the following estimates for the local L^2-projections of $\tilde{E}$ onto $1, \Lambda U_\nu, \nabla U_\nu$:

$$\begin{aligned} \left|\left\langle \tilde{E}\,,\,\chi_*\right\rangle\right| &\lesssim \frac{\nu^2+|a|}{|\log\nu|} + \left|\frac{\nu_\tau}{\nu}\right|\frac{|a|}{|\log\nu|} + a^2|\log\nu|,\\ \left|\left\langle \tilde{E}\,,\,\chi_*\Lambda U_\nu\right\rangle\right| &\lesssim 1 + \frac{a^2}{\nu^4} + \left|\frac{\nu_\tau}{\nu}\cdot\frac{a}{\nu^2}\right| + |a|,\\ \left|\left\langle \tilde{E}\,,\,\partial_{\bar r}U_\nu\chi_*\right\rangle\right| &\lesssim \left|\frac{aR_\tau}{\nu^4\mu}\right|, \quad \left|\left\langle \tilde{E}\,,\,\partial_{\bar z}U_\nu\chi_*\right\rangle\right| = 0. \end{aligned} \tag{3.5}$$

Proof. Recall, from Proposition 1, that

$$\mathscr{L}_\nu^\zeta\varphi_{i,\nu} = 2\beta\left(1-i+\frac{1}{2\log(\nu)}\right)\varphi_{i,\nu} + R_i,$$

and

$$\varphi_{i,\nu}(\zeta) = -\frac{1}{16\nu^2}\Lambda U_\nu(\zeta)\chi_* + \tilde{\varphi}_i(\zeta).$$

We obtain the decomposition (3.4), with

$$\begin{aligned} \tilde{E} &= -a(\tau)\frac{\nu_\tau}{\nu}\nu\partial_\nu(\varphi_{1,\nu}(\zeta)-\varphi_{0,\nu}(\zeta)) - \nabla\cdot(W\nabla\Theta_W + P\nabla\Psi_P)\\ &\quad + \left(\frac{\nu_\tau}{\nu}-\beta\right)(\Lambda U_\nu(\zeta)+16\nu^2\varphi_{0,\nu}(\zeta)) + \frac{1}{\bar r + R/\mu}(\partial_{\bar r}W - W\partial_{\bar r}\Phi_W) + \frac{R_\tau}{\mu}\partial_{\bar r}P + a(\tau)(R_1-R_0). \end{aligned} \tag{3.6}$$

The proof of the estimates relies on the pointwise estimates derived in Proposition 1.
Estimate of $\|\tilde{E}\|_{\text{in}}$: First, by (2.2),

$$\begin{aligned} \|\nu\partial_\nu(\varphi_{1,\nu}-\varphi_{0,\nu})\|_{\text{in}}^2 &\lesssim \int_0^{+\infty}\frac{\nu^4\zeta^5\log^4(2+\zeta/\nu)\log^2(4+\zeta)}{(\nu+\zeta)^8}e^{-\frac{\zeta^2}{2}}\,d\zeta + \frac{1}{\log^2(\nu)}\int_0^{+\infty}\zeta^5\log^2(4+\zeta)e^{-\frac{\zeta^2}{2}}\,d\zeta\\ &\lesssim \frac{1}{|\log\nu|^2}. \end{aligned}$$

Second, by (2.1)

$$\begin{aligned} \|\Lambda U_\nu + 16\nu^2\varphi_{0,\nu}\|_{\text{in}}^2 &= \|16\nu^2\tilde{\varphi}_{0,\nu}\|_{L^2_{\omega_\nu}}^2\\ &\lesssim \nu^4\int_0^{+\infty}\frac{\nu^4\zeta^5\log^4(2+\zeta/\nu)\log^2(4+\zeta)}{(\nu+\zeta)^8}e^{-\frac{\zeta^2}{2}}\,d\zeta + \nu^4\int_0^{+\infty}\frac{\zeta^5\log^2(4+\zeta)}{|\log\nu|^2(\nu+\zeta)^4}e^{-\frac{\zeta^2}{2}}\,d\zeta\\ &\lesssim \frac{\nu^4}{|\log\nu|^2}. \end{aligned}$$

Third, by (2.2),

$$\|\partial_{\bar r}P\|_{\text{in}}^2 = \left\|a(\tau)\frac{\bar r}{\zeta}\partial_\zeta(\varphi_{1,\nu}-\varphi_{0,\nu})\right\|_{\text{in}}^2 \lesssim a^2\int\frac{\zeta^2\log^2(4+\zeta)}{(\nu+\zeta)^4}e^{-\frac{\zeta^2}{2}}\,d\bar r d\bar z \lesssim a^2|\log\nu|.$$

Fourth, by (2.3), we have $|R_i(\zeta)(\nu+\zeta)^2e^{-\frac{\zeta^2}{4}}| \lesssim \frac{1}{|\log\nu|(1+\zeta)^4}$ for any $\zeta\geq 0$, and it follows that

$$\|R_i\|_{\text{in}} \lesssim \frac{1}{|\log\nu|}.$$

Next, we estimate the term $\nabla\cdot(P\nabla\Psi_P) = \partial_\zeta P\partial_\zeta\Psi - P^2$. By (2.2) we obtain the following pointwise estimates ($k=0,1,2$):

$$|\partial_\zeta^k P(\zeta)| \lesssim \frac{|a|\zeta^{2-k}\log(2+\zeta)}{(\nu+\zeta)^4}, \tag{3.7}$$

and for $\zeta = \mathcal{O}(1)$,

$$|\partial_\zeta \Psi_P(\zeta)| = \left| \frac{a(\tau)}{\zeta} \int_0^\zeta r(\varphi_{1,\nu}(r) - \varphi_{0,\nu}(r))\, dr \right| \lesssim \frac{|a|}{\zeta} \int_0^\zeta \frac{r^3}{(\nu + r)^4}\, dr \lesssim \frac{|a|}{\zeta} \log(1 + \zeta^4/\nu^4). \tag{3.8}$$

Then, we have

$$\|\partial_\zeta P \partial_\zeta \Psi_P\|_{\text{in}}^2 \lesssim a^4 \int_0^{+\infty} \frac{\zeta \log^2(4+\zeta)}{(\nu+\zeta)^4} \log^2(1+\zeta/\nu) e^{-\frac{\zeta^2}{2}}\, d\zeta \lesssim \frac{a^4}{\nu^2},$$

and

$$\|P^2\|_{\text{in}}^2 \lesssim a^4 \int_0^{+\infty} \frac{\zeta^9 \log^4(4+\zeta)}{(\nu+\zeta)^{12}} e^{-\frac{\zeta^2}{2}}\, d\zeta \lesssim \frac{a^4}{\nu^2}.$$

Finally, by Lemma 9, we know that

$$\nabla \Phi_W(\zeta) = \nabla \Psi_W(\zeta) + \mathcal{O}(\mu^s), \quad \forall\, \zeta > 0,$$

for some $s > 0$. It follows, in particular, that $\|\nabla \Theta_W\|_{L^\infty(\zeta \le \zeta_*)} = \mathcal{O}(\mu^s/\nu^l)$. Thus, the rest terms are all of lower orders (recall that $\mu = \mathcal{O}(\nu^k)$ for any $k > 0$). This, combined with the estimates above, concludes the local L^2-estimate of $\tilde{E}$. Similarly, based on pointwise estimates (2.1)(2.2)(2.3), we can derive the estimates of the local L^2-projections.

Estimate of $\left\langle \tilde{E}\,,\, \chi_* \right\rangle$: First, through integration by parts,

$$|\langle \nabla \cdot (P \nabla \Psi_P)\,,\, \chi_* \rangle| = \left| \int P \nabla \Psi_P \cdot \nabla \chi_* \right| \le a^2 |\log \nu|.$$

Second, by the eigenproblem equation, we note that

$$\begin{aligned} & \mathscr{L}_\nu^\zeta \varphi_{0,\nu} = 2\beta \left(1 + \frac{1}{2\log(\nu)}\right) \varphi_{0,\nu} + R_0, \\ \iff \quad & \tilde{\varphi}_0 = \varphi_{0,\nu} + \frac{1}{16\nu^2} \Lambda U_\nu(\zeta) \chi_\nu(\zeta) = \frac{1}{2\beta + \beta/\log(\nu)} \left(\mathscr{L}_\nu^\zeta \varphi_{0,\nu} - R_0 \right) + \frac{1}{16\nu^2} \Lambda U_\nu(\zeta) \chi_\nu(\zeta), \end{aligned}$$

where

$$\begin{aligned} \mathscr{L}_\nu^\zeta \varphi_{0,\nu} &= \nabla \cdot \left(U_\nu \nabla \left(\frac{\varphi_{0,\nu}}{U_\nu} - \Psi_{\varphi_{0,\nu}} \right) \right) - \beta \Lambda \varphi_{0,\nu} \\ &= \nabla \cdot \left(U_\nu \nabla \left(\frac{\tilde{\varphi}_0}{U_\nu} - \Psi_{\tilde{\varphi}_0} \right) \right) - \beta \Lambda \tilde{\varphi}_0 + \frac{\beta}{16\nu^2} \Lambda^2 U_\nu, \quad \forall\, \zeta \le \zeta_*. \end{aligned}$$

By the divergence structure and pointwise estimate (2.1), we have

$$\left| \int \chi_* \Lambda \tilde{\varphi}_0\, d\bar{r} d\bar{z} \right| = \left| \int \zeta \partial_\zeta \chi_* \tilde{\varphi}_0\, d\bar{r} d\bar{z} \right| \lesssim \frac{1}{|\log \nu|},$$

and

$$\begin{aligned} \left| \int \chi_* \nabla \cdot \left(U_\nu \nabla \left(\frac{\tilde{\varphi}_0}{U_\nu} - \Psi_{\tilde{\varphi}_0} \right) \right) \right| &= \left| \int \partial_\zeta \chi_* U_\nu \partial_\zeta \left(\frac{\tilde{\varphi}_0}{U_\nu} - \Psi_{\tilde{\varphi}_0} \right) \right| \\ &= \left| \int \partial_\zeta \chi_* \left(\partial_\zeta \tilde{\varphi}_0 - \tilde{\varphi}_0 \partial_\zeta \Psi_{U_\nu} - U_\nu \partial_\zeta \Psi_{\tilde{\varphi}_0} \right) \right| \lesssim \frac{1}{|\log \nu|}, \end{aligned}$$

since by (2.1),

$$|\partial_\zeta \Psi_{\tilde{\varphi}_0}(\zeta)| \lesssim \left| \frac{1}{\zeta} \int_0^\zeta \frac{r^3}{(\nu+r)^4 |\log \nu|} + \frac{\nu^2 r^3 \log(1+r/\nu)}{(\nu+r)^6} \, dr \right| \lesssim \frac{\log(1+\zeta/\nu)}{|\log \nu|\zeta}.$$

In addition, due to the cancellation $2U_\nu(\zeta)/\nu^2 + \Lambda U_\nu(\zeta)/\nu^2 = \mathcal{O}(\nu^2)$ for $\zeta \in [\zeta_*/2, \zeta_*]$,

$$\begin{aligned}\left| \int \chi_* \frac{1}{2+1/\log(\nu)} \cdot \frac{1}{16\nu^2} \Lambda^2 U_\nu + \frac{1}{16\nu^2} \Lambda U_\nu \right| &= \left| \int \frac{1}{2+1/\log(\nu)} \cdot \frac{\partial_\zeta \chi_*}{16\nu^2} \zeta \Lambda U_\nu + \frac{\partial_\zeta \chi_*}{16\nu^2} \zeta U_\nu \right| \\ &\lesssim \left| \int \frac{\partial_\zeta \chi_*}{32\nu^2} \zeta \Lambda U_\nu + \frac{\partial_\zeta \chi^*}{16\nu^2} \zeta U_\nu \right| + \frac{1}{|\log \nu|} \cdot \left| \int \frac{\partial_\zeta \chi_*}{\nu^2} \zeta \Lambda U_\nu \right| \lesssim \frac{1}{|\log \nu|},\end{aligned}$$

and by (2.3),

$$\left| \int \chi_* R_0 \, d\bar{r} d\bar{z} \right| \lesssim \frac{1}{|\log \nu|}.$$

Combining these estimates we obtain

$$\left| \left\langle \Lambda U_\nu + 16\nu^2 \varphi_{0,\nu} \, , \, \chi_* \right\rangle \right| \lesssim \frac{\nu^2}{|\log \nu|}.$$

Note that $\nu \partial_\nu (\nu^{-2} V(\zeta/\nu)) = -\nu^{-2} \Lambda V(\zeta/\nu)$, where $V = V_2$ or $V = \tilde{V}_2$ in the construction of $\varphi_{i,\nu}$ in Proposition 1. Exploiting this divergence structure, we have the estimate

$$\left| \left\langle a \frac{\nu_\tau}{\nu} \nu \partial_\nu (\varphi_{1,\nu} - \varphi_{0,\nu}) \, , \, \chi_* \right\rangle \right| \lesssim \left| a \frac{\nu_\tau}{\nu} \right| \left(\nu^2 + \int_0^{2\zeta_*} \frac{\zeta^3}{|\log \nu| (\nu+\zeta)^2} \, d\zeta \right) \lesssim \left| \frac{\nu_\tau}{\nu} \right| \frac{|a|}{|\log \nu|}.$$

The estimates of the rest terms are more straightforward

$$\begin{aligned}&|\langle a R_i \, , \, \chi_* \rangle| \lesssim |a| \int_0^{2\zeta_*} \frac{\zeta^3}{(\nu+\zeta)^2} \frac{\log(\zeta)}{|\log(\nu)|} + \frac{\nu \zeta^3}{(\nu+\zeta)^3} \, d\zeta \lesssim \frac{|a|}{|\log \nu|}, \\ &|\langle \partial_{\bar{r}} P \, , \, \chi_* \rangle| = 0,\end{aligned}$$

<u>Estimate of the rest terms:</u> The methods are similar, and we briefly summarize them below.
For $\left\langle \tilde{E} \, , \, \Lambda U_\nu \chi_* \right\rangle$:

$$\begin{aligned}&\left| \left\langle P^2 \, , \, \Lambda U_\nu \chi_* \right\rangle \right| \lesssim a^2 \int_0^{2\zeta_*} \frac{\nu^2 \zeta^5}{(\nu+\zeta)^{12}} \, d\zeta \lesssim \frac{a^2}{\nu^4}, \\ &\left| \left\langle \Lambda U_\nu + 16\nu^2 \varphi_{0,\nu} \, , \, \Lambda U_\nu \chi_* \right\rangle \right| \lesssim \frac{1}{|\log \nu|} \int_0^{2\zeta_*} \frac{\nu^4 \zeta^3}{(\nu+\zeta)^8} \, d\zeta + \nu^6 \int_0^{\zeta_*} \frac{\zeta^3 \log(1+\zeta/\nu)}{(\nu+\zeta)^{10}} \, d\zeta \lesssim 1, \\ &\left| \left\langle a \frac{\nu_\tau}{\nu} \nu \partial_\nu (\varphi_{1,\nu} - \varphi_{0,\nu}) \, , \, \Lambda U_\nu \chi_* \right\rangle \right| \lesssim \left| a \frac{\nu_\tau}{\nu} \right| \int_0^{2\zeta_*} \frac{\nu^4 \zeta^3}{(\nu+\zeta)^{10}} + \frac{\nu^2 \zeta^3}{|\log \nu| (\nu+\zeta)^6} \, d\zeta \lesssim \left| \frac{\nu_\tau a}{\nu^3} \right|, \\ &|\langle \partial_\zeta P \partial_\zeta \Psi_P \, , \, \Lambda U_\nu \chi_* \rangle| \lesssim a^2 \int_0^{2\zeta_*} \frac{\nu^2 \zeta \log(1+\zeta/\nu)}{(\nu+\zeta)^8} \, d\zeta \lesssim \frac{a^2}{\nu^4}, \\ &|\langle a R_i \, , \, \Lambda U_\nu \chi_* \rangle| \lesssim |a| \int_0^{2\zeta_*} \frac{\nu^2 \zeta^3 \log(\zeta)}{|\log \nu| (\nu+\zeta)^6} + \frac{\nu^3 \zeta^3}{(\nu+\zeta)^7} \, d\zeta \lesssim |a|, \\ &|\langle \partial_{\bar{r}} P \, , \, \Lambda U_\nu \chi_* \rangle| = 0.\end{aligned}$$

For $\left\langle \tilde{E} \, , \, \partial_{\bar{r}} U_\nu \chi_* \right\rangle$:

$$|\langle \partial_{\bar{r}} P \, , \, \partial_{\bar{r}} U_\nu \chi_* \rangle| \lesssim a \int \frac{\nu^2 \bar{r}^2 \chi_*}{(\nu+\zeta)^{10}} \, d\bar{r} d\bar{z} \lesssim \frac{a}{\nu^4},$$

and the other terms are all zero. □

3.2 Bootstrap Regime and Modulation Equations

As we will see in the energy estimates, it only suffices to estimate the gradient of ε in the region $\zeta \lesssim 1$. Thus, we define

$$\varepsilon^* := \chi^* \varepsilon.$$

Recall that $\chi^*(\zeta) = \chi(\zeta/\zeta^*)$ and

$$\|f\|^2_{L^2(U_\nu)} := \int \frac{\nu^2 f^2}{U_\nu}.$$

Now, we are ready to setup our bootstrap assumptions:

Definition 1 (Bootstrap)**.** *We say that a solution w of (1.4) lies in the bootstrap regime* $\mathrm{BS}(\tau_0, \tau_*, \zeta^*, M_0, \{K_i\}_{i=1}^7)$ *if it satisfies the following: on time interval $[\tau_0, \tau_*]$, w admits the decomposition (3.1) where the perturbation ε satisfies the locally orthogonal decomposition (3.3). In addition, the following estimates holds on $[\tau_0, \tau_*]$:*
(i) (Modulation parameters)

$$\begin{aligned}
\frac{1}{K_1} e^{-\sqrt{\beta\tau + M_0}} \leq & \nu(\tau) \leq K_1 e^{-\sqrt{\beta\tau+M_0}}, \\
& |a(\tau) - 8\nu^2(\tau)| \leq \frac{K_2 \nu^2}{|\log \nu|}, \\
& \left|\frac{R_\tau}{\mu}\right| \leq \frac{K_3 \nu}{|\log \nu|}
\end{aligned}$$

(ii) (Remainders)

$$\begin{aligned}
&\|\varepsilon\|_{\mathrm{in}} \leq K_4 \frac{\nu^2}{|\log \nu|}, \\
&\|\nabla \varepsilon^*\|_{L^2(U_\nu)} \leq K_5 \frac{\nu^2}{|\log \nu|}, \\
&\|\varepsilon\|_{H^2(\frac{1}{2}\zeta^* \leq \zeta \leq 4\zeta^*)} \leq K_6 \frac{\nu^2}{|\log \nu|}, \\
&\|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta \geq \zeta^*)} \leq \frac{K_7}{\sqrt{\beta\tau + M_0}} e^{-2\sqrt{\beta\tau+M_0}}.
\end{aligned}$$

Note that by the bootstrap assumptions on ν and $\|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta \geq \zeta^*)}$, we have

$$\|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta \geq \zeta^*)} \leq \frac{K_7 C(K_1)\nu^2}{|\log \nu|}.$$

The reason why we make such assumption on $\|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta \geq \zeta^*)}$ is a technical treatment to avoid the oscillatory behavior of ν in time when doing integration, the details of which can be found in Lemma 7.

Lemma 3 (Modulation equations)**.** *Assume that the solution is in the bootstrap regime* $\mathrm{BS}(\tau_0, \tau_*, \zeta^*, M_0, \{K_i\}_{i=1}^7)$ *defined in Definition 1. Then, the following estimates hold for any $\tau \in [\tau_0, \tau_*]$:*

$$\begin{aligned}
&|\mathrm{Mod}_0| = \left| a_\tau - 2a\beta\left(1 + \frac{1}{2\log(\nu)}\right) - 16\nu^2\left(\frac{\nu_\tau}{\nu} - \beta\right)\right| \leq C\left(\|\varepsilon\|_{\mathrm{in}} + \|\nabla\varepsilon^*\|_{L^2(U_\nu)}\right) + \frac{C(K_i)\nu^2}{|\log \nu|^2}, \\
&|\mathrm{Mod}_1| = \left| -a_\tau + \frac{a(\tau)\beta}{\log(\nu)}\right| \leq \frac{C(K_1, K_2, K_4, K_5)\nu^2}{|\log \nu|^2} + \frac{C(K_i)\nu^2}{|\log \nu|^3}, \\
&\left|\frac{R_\tau}{\mu}\right| \leq \frac{C(K_4, K_5)\nu}{|\log \nu|} + \frac{C(K_i)\nu}{|\log \nu|^2}.
\end{aligned} \tag{3.9}$$

Proof. The strategy of the proof is the following: Since the evolution of the modulation parameters is determined by the preservation of the (local) orthogonality conditions (3.3), we take time derivatives of the orthogonality equations and use energy bounds for ε to obtain the desired estimates.

Estimate of Mod_1 by projection to χ_*: By the orthogonality condition (3.3), we obtain

$$0 = \frac{d}{d\tau}\left\langle \varepsilon\,,\, \chi_* \right\rangle = \left\langle \partial_\tau \varepsilon\,,\, \chi_* \right\rangle = \left\langle \mathscr{L}_\nu^\zeta \varepsilon + L(\varepsilon) + NL(\varepsilon) + E\,,\, \chi_* \right\rangle. \tag{3.10}$$

Recall that (as $\langle \partial_{\tilde r} U_\nu\,,\, \chi_* \rangle = 0$)

$$\left\langle E\,,\, \chi_* \right\rangle = \mathrm{Mod}_0 \left\langle \varphi_{0,\nu}\,,\, \chi_* \right\rangle + \mathrm{Mod}_1 \left\langle \varphi_{1,\nu}\,,\, \chi_* \right\rangle + \left\langle \tilde{E}\,,\, \chi_* \right\rangle.$$

Then, by (2.4) (3.5) and the fact that $\left\langle \frac{1}{\nu^2}\Lambda U_\nu\,,\, \chi_* \right\rangle = \mathcal{O}(1)$, we have

$$\left| \left\langle \varphi_{0,\nu}\,,\, \chi_* \right\rangle \right| \lesssim 1, \quad \left| \left\langle \varphi_{1,\nu}\,,\, \chi_* \right\rangle \right| \gtrsim |\log \nu|, \quad \left| \left\langle \tilde{E}\,,\, \chi_* \right\rangle \right| \lesssim \frac{\nu^2 + |a|}{|\log \nu|} + \left| a \frac{\nu_\tau}{\nu} \right| + C(K_i)\nu^3. \tag{3.11}$$

We remark that the gain of $|\log \nu|$ in $\langle \varphi_{1,\nu}\,,\, \chi_* \rangle$ will enable us to control Mod_1 by Mod_0 and gain a $|\log \nu|$ smallness in the estimate of Mod_1. Next, we estimate the terms containing ε. Since the operator $\mathscr{M}_\nu^\zeta$ is self-adjoint in $(L^2(\mathbb{R}^2), \langle \cdot\,,\, \cdot \rangle)$, which follows from the self-adjointness of $(-\Delta)^{-1}$, we have

$$\begin{aligned} \left| \left\langle \mathscr{L}_\nu^\zeta \varepsilon\,,\, \chi_* \right\rangle \right| &= \left| \left\langle U_\nu \nabla \mathscr{M}_\nu^\zeta(\varepsilon) - \beta y \varepsilon\,,\, \nabla \chi_* \right\rangle \right| \\ &\le \left| \left\langle \mathscr{M}_\nu^\zeta(\varepsilon)\,,\, \nabla \cdot (U_\nu \nabla \chi_*) \right\rangle \right| + \left| \left\langle \beta y \varepsilon\,,\, \nabla \chi_* \right\rangle \right|. \end{aligned}$$

Since $\nabla \cdot (U_\nu \nabla \chi_*)$ is compactly supported in $[\zeta_*, 2\zeta_*]$ and is of size $\mathcal{O}(\nu^2)$, i.e., $|\nabla \cdot (U_\nu \nabla \chi_*)| \lesssim \nu^2 \mathbb{1}_{\{\zeta_* \le \zeta \le 2\zeta_*\}}$, by (2.17) we have the estimate:

$$\left| \left\langle \mathscr{M}_\nu^\zeta(\varepsilon)\,,\, \nabla \cdot (U_\nu \nabla \chi_*) \right\rangle \right| \lesssim \int_{\{\zeta_* \le \zeta \le 2\zeta_*\}} |\varepsilon| + \left| \left\langle \nabla \Psi_\varepsilon\,,\, U_\nu \nabla \chi_* \right\rangle \right| \lesssim \|\varepsilon\|_{\mathrm{in}} + \nu^2 \|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta \ge \zeta^*)},$$

where we use the pointwise estimate of the Poisson field by (A.14) and (A.15):

$$\|\nabla \Psi_\varepsilon\|_{L^\infty(\zeta_* \le \zeta \le 2\zeta_*)} \lesssim \left(\int \varepsilon^2 (1+\zeta)^{\frac{1}{3}} \right)^{\frac{1}{2}} \lesssim \frac{1}{\nu^2}\|\varepsilon\|_{\mathrm{in}} + \|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta \ge \zeta^*)}. \tag{3.12}$$

It follows that

$$\left| \left\langle \mathscr{L}_\nu^\zeta \varepsilon\,,\, \chi_* \right\rangle \right| \lesssim \|\varepsilon\|_{\mathrm{in}} + \nu^2 \|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta \ge \zeta_*)}. \tag{3.13}$$

Due to Lemma 9, we neglect the terms of order $\mathcal{O}(\mu^s)$ and estimate the rest terms in $L(\varepsilon)$. By (3.12) and the pointwise estimates (3.7) and (3.8), we have

$$\begin{aligned} \left| \left\langle -\nabla \cdot (P \nabla \Psi_\varepsilon + \varepsilon \nabla (\Psi_W - \Psi_{U_\nu})) + \frac{R_\tau}{\mu} \partial_{\tilde r} \varepsilon\,,\, \chi_* \right\rangle \right| &= \left| \left\langle P \nabla \Psi_\varepsilon + \varepsilon \nabla \Psi_P + \frac{R_\tau}{\mu} e_1 \varepsilon\,,\, \nabla \chi_* \right\rangle \right| \\ &\lesssim \left(\frac{|a|}{\nu^2} + \left| \frac{R_\tau}{\mu} \right| \right) \|\varepsilon\|_{\mathrm{in}} + |a| \cdot \|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta \ge \zeta_*)}. \end{aligned} \tag{3.14}$$

Finally, for the nonlinear terms, neglect the part of order $\mathcal{O}(\mu)$. By Cauchy's inequality and (3.12), we have:

$$\begin{aligned} \left| \left\langle -\nabla \cdot (\varepsilon \nabla \Psi_\varepsilon)\,,\, \chi_* \right\rangle \right| = \left| \left\langle \varepsilon \nabla \Psi_\varepsilon\,,\, \nabla \chi_* \right\rangle \right| &\le \|\varepsilon\|_{\mathrm{in}} \|\nabla \Psi_\varepsilon\|_{L^\infty(\zeta_* \le \zeta \le 2\zeta_*)} \\ &\le \left(\frac{1}{\nu^2}\|\varepsilon\|_{\mathrm{in}} + \|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta \ge \zeta^*)} \right) \|\varepsilon\|_{\mathrm{in}}. \end{aligned} \tag{3.15}$$

By (3.10) and collecting all the estimates (3.11)(3.13)(3.14)(3.15), we obtain

$$\begin{aligned}|\mathrm{Mod}_1| &\lesssim \frac{1}{|\log\nu|}\Bigg(|\mathrm{Mod}_0| + \left(1+\frac{|a|}{\nu^2}+\left|\frac{R_\tau}{\mu}\right|\right)\|\varepsilon\|_{\mathrm{in}} + (\nu^2+|a|)\|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta\geq\zeta^*)} \\ &\quad + \frac{1}{\nu^2}\|\varepsilon\|_{\mathrm{in}}^2 + \frac{1}{\nu^2}\|\varepsilon\|_{\mathrm{in}}\|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta\geq\zeta^*)} + \frac{\nu^2+|a|}{|\log\nu|} + \left|a\frac{\nu_\tau}{\nu}\right| + C(K_i)\nu^3\Bigg) \\ &\lesssim \frac{|\mathrm{Mod}_0|}{|\log\nu|} + \frac{\nu^2 C(K_1,K_2,K_4)}{|\log\nu|^2} + \frac{C(K_i)\nu^2}{|\log\nu|^3}. \end{aligned} \tag{3.16}$$

<u>Estimate of Mod_0 by projection to $\Lambda U_\nu\chi_*$</u>: Similarly, we compute

$$\begin{aligned} 0 &= \frac{d}{d_\tau}\langle\varepsilon\,,\,\Lambda U_\nu\chi_*\rangle = \langle\partial_\tau\varepsilon\,,\,\Lambda U_\nu\chi_*\rangle + \langle\varepsilon\,,\,\partial_\tau\Lambda U_\nu\chi_*\rangle \\ &= \left\langle\mathscr{L}_\nu^\zeta\varepsilon + L(\varepsilon)+NL(\varepsilon)+E\,,\,\Lambda U_\nu\chi_*\right\rangle + \left\langle\varepsilon\,,\,\frac{\nu_\tau}{\nu}\nu\partial_\nu\Lambda U_\nu\chi_*\right\rangle. \end{aligned} \tag{3.17}$$

Note, by (2.4) and (3.5), that

$$|\langle\varphi_{i,\nu}\,,\,\Lambda U_\nu\chi_*\rangle| \gtrsim \frac{1}{\nu^2}\langle\Lambda U_\nu\,,\,\Lambda U_\nu\chi_*\rangle \gtrsim \frac{1}{\nu^4}, \quad \left|\left\langle\tilde{E}\,,\,\Lambda U_\nu\chi_*\right\rangle\right| \lesssim 1+\frac{a^2}{\nu^4}+\frac{C(K_i)}{|\log\nu|}.$$

Through integration by parts and Cauchy's inequality,

$$\begin{aligned} \left|\left\langle\mathscr{L}_{0,\nu}^\zeta\varepsilon\,,\,\Lambda U_\nu\chi_*\right\rangle\right| &= |\langle\nabla\varepsilon-\varepsilon\nabla\Psi_{U_\nu}-U_\nu\nabla\Psi_\varepsilon\,,\,\nabla(\Lambda U_\nu\chi_*)\rangle| \\ &\leq \left(\|\varepsilon\|_{\mathrm{in}}+\|\nabla\varepsilon^*\|_{L^2(U_\nu)}\right)\left|\left\langle\frac{U_\nu}{\nu^2}\,,\,(\nabla(\Lambda U_\nu\chi_*))^2\right\rangle\right|^{\frac{1}{2}} + |\langle\nabla\Psi_\varepsilon U_\nu\,,\,\nabla(\Lambda U_\nu\chi_*)\rangle| \\ &\lesssim \frac{1}{\nu^4}\left(\|\varepsilon\|_{\mathrm{in}}+\|\nabla\varepsilon^*\|_{L^2(U_\nu)}\right) + \frac{1}{\nu^3}\|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta\geq\zeta^*)}, \end{aligned}$$

where we apply pointwise estimates of Poisson field (A.16) (A.17) to estimate

$$\begin{aligned} \left|\int\frac{\nu^4\nabla\Psi_\varepsilon}{(\nu+\zeta)^9}\right| &\lesssim \left|\int\frac{\nu^4(1+\zeta/\nu)}{(\nu+\zeta)^{10}}\left(\int\varepsilon^*(\nu+\zeta)^2\right)^{\frac{1}{2}}\right| + \left|\int\frac{\nu^4}{(\nu+\zeta)^9}\right|\cdot\|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta\geq\zeta^*)} \\ &\lesssim \frac{1}{\nu^4}\|\nabla\varepsilon^*\|_{L^2(U_\nu)} + \frac{1}{\nu^3}\|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta\geq\zeta^*)}. \end{aligned}$$

Besides,

$$|\langle\Lambda\varepsilon\,,\,\Lambda U_\nu\chi_*\rangle| \leq \left|\left\langle\frac{U_\nu}{\nu^2}\,,\,(\mathbf{y}\cdot\nabla(\Lambda U_\nu\chi_*))^2\right\rangle\right|^{\frac{1}{2}}\cdot\|\varepsilon\|_{\mathrm{in}} \lesssim \frac{1}{\nu^3}\|\varepsilon\|_{\mathrm{in}}.$$

For $L(\varepsilon)$, by (2.2) and similar methods as adapted above:

$$\begin{aligned} &|\langle\nabla P\cdot\nabla\Psi_\varepsilon\,,\,\Lambda U_\nu\chi_*\rangle| \lesssim \frac{|a|}{\nu^4}\|\nabla\varepsilon^*\|_{L^2(U_\nu)} + \frac{|a|}{\nu^3}\|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta\geq\zeta^*)}, \\ &|\langle P\varepsilon\,,\,\Lambda U_\nu\chi_*\rangle| \lesssim \left|\left\langle\frac{U_\nu}{\nu^2}\,,\,(P\Lambda U_\nu\chi_*)^2\right\rangle\right|^{\frac{1}{2}}\|\varepsilon\|_{\mathrm{in}} \leq \frac{|a|}{\nu^5}\|\varepsilon\|_{\mathrm{in}}, \\ &|\langle\nabla\varepsilon\cdot\nabla\Psi_P\,,\,\Lambda U_\nu\chi_*\rangle| \lesssim \left|\left\langle\frac{U_\nu}{\nu^2}\,,\,(\nabla\Psi_P\Lambda U_\nu\chi_*)^2\right\rangle\right|^{\frac{1}{2}}\|\nabla\varepsilon^*\|_{\mathrm{in}} \leq \frac{|a|}{\nu^4}\|\nabla\varepsilon^*\|_{\mathrm{in}}, \\ &\left|\left\langle\frac{R_\tau}{\mu}\partial_{\bar{r}}\varepsilon\,,\,\Lambda U_\nu\chi_*\right\rangle\right| \lesssim \frac{1}{\nu^3}\left|\frac{R_\tau}{\mu}\right|\|\nabla\varepsilon^*\|_{\mathrm{in}}. \end{aligned}$$

As for the nonlinear term,

$$|\langle \nabla\varepsilon \cdot \nabla\Psi_\varepsilon \,,\, \Lambda U_\nu \chi_* \rangle| \lesssim \|\nabla\varepsilon^*\|_{L^2(U_\nu)} \left(\int \frac{U_\nu}{\nu^2} (\Lambda U_\nu)^2 |\nabla \Psi_\varepsilon|^2 \right)^{\frac12}$$
$$\lesssim \frac{1}{\nu^4}\|\nabla\varepsilon^*\|_{\text{in}}^2 + \frac{1}{\nu^3}\|\nabla\varepsilon^*\|_{L^2(U_\nu)} \|\varepsilon(1+\zeta)^{\frac32}\|_{L^\infty(\zeta\geq\zeta^*)},$$

and

$$\left|\left\langle \varepsilon^2 \,,\, \Lambda U_\nu \chi_* \right\rangle\right| \lesssim \frac{1}{\nu^6}\|\varepsilon\|_{\text{in}}^2.$$

At last,

$$\left|\left\langle \varepsilon \,,\, \frac{\nu_\tau}{\nu} \nu \partial_\nu \Lambda U_\nu \chi_* \right\rangle\right| \lesssim \frac{1}{\nu^3}\left|\frac{\nu_\tau}{\nu}\right| \|\varepsilon\|_{\text{in}}.$$

Inserting all these estimates into (3.17), we obtain

$$|\text{Mod}_0| \lesssim |\text{Mod}_1| + \|\varepsilon\|_{\text{in}} + \|\nabla\varepsilon^*\|_{L^2(U_\nu)} + \frac{C(K_i)\nu^2}{|\log\nu|^2}.$$

Combining with (3.16), we can further refine

$$|\text{Mod}_0| \leq C\left(\|\varepsilon\|_{\text{in}} + \|\nabla\varepsilon^*\|_{L^2(U_\nu)}\right) + \frac{C(K_i)\nu^2}{|\log\nu|^2},$$
$$|\text{Mod}_1| \leq \frac{C(K_1,K_2,K_4,K_5)\nu^2}{|\log\nu|^2} + \frac{C(K_i)\nu^2}{|\log\nu|^3}.$$

<u>Estimate of $\frac{R_\tau}{\mu}$ by projection to $\partial_{\bar r} U_\nu \chi_*$</u>: As before, we compute $0 = \partial_\tau \langle \varepsilon \,,\, \partial_{\bar r} U_\nu\rangle$. Note that

$$\langle E \,,\, \partial_{\bar r} U_\nu \chi_* \rangle = \frac{R_\tau}{\mu} \langle \partial_{\bar r} U_\nu \,,\, \partial_{\bar r} U_\nu \chi_* \rangle + \left\langle \tilde E \,,\, \partial_{\bar r} U_\nu \chi_* \right\rangle,$$

where

$$\langle \partial_{\bar r} U_\nu \,,\, \partial_{\bar r} U_\nu \chi_* \rangle \gtrsim \frac{1}{\nu^4}, \quad \left|\left\langle \tilde E \,,\, \partial_{\bar r} U_\nu \chi_* \right\rangle\right| \lesssim \left|\frac{R_\tau}{\nu^2\mu}\right|.$$

The estimates of the scalar products with terms containing ε are similar, with everything amplified by $1/\nu$ compared to the scalar products with ΛU_ν (since $|\partial_{\bar r} U_\nu| \lesssim \frac1\nu |\Lambda U_\nu|$ for any $\zeta \leq 2\zeta^*$), and we will not repeat them here. To summarize, we have

$$\left|\frac{R_\tau}{\mu}\right| \leq \frac{C(K_4,K_5)\nu}{|\log\nu|} + \frac{C(K_i)\nu}{|\log\nu|^2}.$$

□

A direct consequence of the modulation estimates is the following control on $|\frac{\nu_\tau}{\nu}|$.

Corollary 2. *Assume that the solution is in the bootstrap regime given in Definition 1. Then, it holds that*

$$\left|\frac{\nu_\tau}{\nu}\right| \leq \frac{C(K_2,K_4,K_5)}{|\log\nu|}. \tag{3.18}$$

Proof. Inserting $|a - 8\nu^2| \leq \frac{K_2\nu^2}{|\log\nu|}$ into the estimate of $|\text{Mod}_0 + \text{Mod}_1|$ given by (3.9), the result follows. □

3.3 Energy Estimates

3.3.1 L^2 Inner Estimate

Now we establish an important L^2-monotonicity result for ε. One technical treatment is needed to avoid a loophole in the energy estimates: Due to an incompatibility between the decomposition of generated error and the local orthogonality conditions, the modulation estimates are not small enough to close the L^2 energy estimate. However, by projecting out the direction of the first approximate eigenfunction, we are able to get rid of this issue. This is possible thanks to the special structure of the adapted inner product as well as the slow decay of the stationary solution (e.g. U does not belong to L^1), which makes the aforementioned projection an acceptable modification to the original norm.
Note, by Lemma 2 and orthogonality conditions (3.3), we have

$$\int \frac{\varepsilon^2(\chi_\nu)^2\varrho_\nu}{U_\nu} - C\nu^2\int \frac{\varepsilon^2(\chi_\nu)^2\varrho_\nu}{U_\nu} \lesssim \int \varepsilon\chi_\nu\sqrt{\varrho_\nu}\mathscr{M}_\nu^\zeta(\varepsilon\chi_\nu\sqrt{\varrho_\nu}) \lesssim \int \frac{\varepsilon^2(\chi_\nu)^2\varrho_\nu}{U_\nu}.$$

Thus, $\langle \varepsilon\,,\,\varepsilon\rangle_{\nu,*} = \int \varepsilon\chi_\nu\sqrt{\varrho_\nu}\mathscr{M}_\nu^\zeta(\varepsilon\chi_\nu\sqrt{\varrho_\nu})$ is equivalent to the norm $\frac{1}{\nu^2}\|\varepsilon\|_{\rm in}^2$. Define

$$d_0 := \frac{\int \chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu}\mathscr{M}_\nu^\zeta(\varepsilon\chi_\nu\sqrt{\varrho_\nu})}{\int \chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu}\mathscr{M}_\nu^\zeta(\chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu})} \lesssim \frac{\|\varepsilon\|_{\rm in}}{|\log\nu|},$$

as we have

$$\left|\int \chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu}\mathscr{M}_\nu^\zeta(\varepsilon\chi_\nu\sqrt{\varrho_\nu})\right| \lesssim \frac{1}{\nu^2}\|\varepsilon\|_{\rm in}, \quad \left|\int \chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu}\mathscr{M}_\nu^\zeta(\varphi_{0,\nu}\chi_\nu\sqrt{\varrho_\nu})\right| \gtrsim \frac{|\log\nu|}{\nu^2}.$$

We project out the $\varphi_{0,\nu}$ direction of ε, and consider the evolution of

$$\begin{aligned}&\int \varepsilon\chi_\nu\sqrt{\varrho_\nu}\mathscr{M}_\nu^\zeta(\varepsilon\chi_\nu\sqrt{\varrho_\nu}) - d_0\int \varepsilon\chi_\nu\sqrt{\varrho_\nu}\mathscr{M}_\nu^\zeta(\chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu})\\ =&\int \varepsilon\chi_\nu\sqrt{\varrho_\nu}\mathscr{M}_\nu^\zeta(\varepsilon\chi_\nu\sqrt{\varrho_\nu}) - \frac{(\int \chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu}\mathscr{M}_\nu^\zeta(\varepsilon\chi_\nu\sqrt{\varrho_\nu}))^2}{\int \chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu}\mathscr{M}_\nu^\zeta(\chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu})} \sim \frac{1}{\nu^2}\|\varepsilon\|_{\rm in}^2.\end{aligned}$$

Recall that $\varphi_{0,\nu} = -\frac{1}{16\nu^2}\Lambda U_\nu\chi_\nu + \tilde\varphi_0$, and

$$\mathscr{M}_\nu^\zeta(\chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu}) = -\frac{1}{8\nu^2} + \mathscr{M}_\nu^\zeta(-\frac{1}{16\nu^2}\Lambda U_\nu(\chi_\nu\sqrt{\varrho_\nu}\chi_\nu - 1)) + \mathscr{M}_\nu^\zeta(\chi_\nu\sqrt{\varrho_\nu}\tilde\varphi_0).$$

We also recall the pointwise estimate

$$|\partial_\zeta^k\tilde\varphi_0(\zeta)| + |\partial_\zeta^k\nu\partial_\nu\tilde\varphi_0(\zeta)| \lesssim \left(\frac{\nu^2\zeta^{2-k}\log(1+\zeta/\nu)}{(\nu+\zeta)^6} + \frac{\zeta^{2-k}}{|\log\nu|(\nu+\zeta)^4}\right)(1+\log(\zeta)\mathbb{1}_{\{\zeta>1\}}).$$

In the following argument, since $\frac{1}{\nu^2}(\chi_\nu\sqrt{\varrho_\nu}\chi_\nu - 1)$ and $\tilde\varphi_0$ are always estimated together, for brevity we denote

$$\bar\varphi_0 := -\frac{1}{16\nu^2}\Lambda U_\nu(\chi_\nu\sqrt{\varrho_\nu}\chi_\nu - 1) + \chi_\nu\sqrt{\varrho_\nu}\tilde\varphi_0.$$

By the pointwise estimates, it is helpful to note that

$$\int \frac{|\bar\varphi_0|^2}{U_\nu} \lesssim \frac{1}{\nu^2}, \quad \int \frac{|\nabla\bar\varphi_0|^2}{U_\nu} \lesssim \frac{|\log\nu|}{\nu^2}.$$

Lemma 4 (Control of $\|\varepsilon\|_{\rm in}$)**.** *Let w be a solution in the bootstrap regime* $\mathrm{BS}(\tau_0,\tau_*,\zeta^*,M_0,\{K_i\}_{i=1}^7)$*. Then, the following estimate holds on* $[\tau_0,\tau_*]$*:*

$$\begin{aligned}&\frac{d}{d\tau}\left(\frac{1}{2}\int \varepsilon\chi_\nu\sqrt{\varrho_\nu}\mathscr{M}_\nu^\zeta(\varepsilon\chi_\nu\sqrt{\varrho_\nu}) - \frac{d_0}{2}\int \varepsilon\sqrt{\varrho_\nu}\chi_\nu\mathscr{M}_\nu^\zeta(\varphi_{0,\nu}\sqrt{\varrho_\nu}\chi_\nu)\right)\\ \le& -\frac{\delta_0}{\nu^2}\left(\|\varepsilon\|_{\rm in}^2 + \|\nabla\varepsilon\|_{\rm in}^2\right) + \frac{C\nu^2}{|\log\nu|^2} + \frac{C(K_i)\nu^2}{|\log\nu|^{\frac{7}{3}}}\end{aligned}$$

Proof. The first half of the proof estimates the main part (i.e., the leading order dynamics) $\frac{d}{d\tau}\frac{1}{2}\int \varepsilon\chi_\nu\sqrt{\varrho_\nu}\mathscr{M}_\nu^\zeta(\varepsilon\chi_\nu\sqrt{\varrho_\nu})$ which yields damping. Then, the second half deals with the correction term, which projects out the Mod_0 direction of the main part.

Step 1: Leading order dynamics

First of all, by (3.2) and recall the definition of $\langle\cdot\,,\,\cdot\rangle_{\nu,*}$ in (2.22), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{d\tau}\langle\varepsilon\,,\,\varepsilon\rangle_{\nu,*} &= \langle\partial_\tau\varepsilon\,,\,\varepsilon\rangle_{\nu,*} + \frac{1}{2}\left\langle\frac{\partial}{\partial_\tau}\left(\frac{\varrho_\nu\chi_\nu^2}{U_\nu}\right),\,\varepsilon^2\right\rangle - \left\langle\frac{\partial}{\partial_\tau}(\sqrt{\varrho_\nu}\chi_\nu)\,\varepsilon\,,\,\tilde{\Psi}_\varepsilon\right\rangle\\ &= \left\langle\tilde{\mathscr{L}}_\nu^\zeta\varepsilon\,,\,\varepsilon\right\rangle_{\nu,*} + \left\langle(\mathscr{L}_\nu^\zeta - \tilde{\mathscr{L}}_\nu^\zeta)\varepsilon\,,\,\varepsilon\right\rangle_{\nu,*} + \langle L(\varepsilon)\,,\,\varepsilon\rangle_{\nu,*} + \langle NL(\varepsilon)\,,\,\varepsilon\rangle_{\nu,*} + \langle E\,,\,\varepsilon\rangle_{\nu,*}\\ &\quad + \frac{1}{2}\left\langle\frac{\partial}{\partial_\tau}\left(\frac{\varrho_\nu\chi_\nu^2}{U_\nu}\right),\,\varepsilon^2\right\rangle - \left\langle\frac{\partial}{\partial_\tau}(\sqrt{\varrho_\nu}\chi_\nu)\,\varepsilon\,,\,\tilde{\Psi}_\varepsilon\right\rangle.\end{aligned}$$

Damping term: By the coercivity of the modified linearized operator (2.37), we have the damping

$$\left\langle\tilde{\mathscr{L}}_\nu^\zeta\varepsilon\,,\,\varepsilon\right\rangle_{\nu,*} \le -\delta\left(\int\frac{\varepsilon^2\chi_\nu^2\varrho_\nu}{U_\nu} + \int\frac{|\nabla\varepsilon|^2\chi_\nu^2\varrho_\nu}{U_\nu}\right) + C\nu^{100}\|\varepsilon\|^2_{L^\infty(\zeta\ge 1/\nu)},$$

for some universal $\delta, C > 0$.

Estimate of term $(\mathscr{L}_\nu^\zeta - \tilde{\mathscr{L}}_\nu^\zeta)\varepsilon$: Note that

$$(\mathscr{L}_\nu^\zeta - \tilde{\mathscr{L}}_\nu^\zeta)\varepsilon = \nabla U_\nu\cdot(\nabla\tilde{\Psi}_\varepsilon - \nabla\Psi_\varepsilon) = \nabla U_\nu\cdot\nabla\Psi_{(\sqrt{\varrho_\nu}\chi_\nu - 1)\varepsilon},$$

Integrating by parts, we have

$$\begin{aligned}\left\langle\nabla U_\nu\cdot\nabla\Psi_{(\sqrt{\varrho_\nu}\chi_\nu-1)\varepsilon}\,,\,\varepsilon\right\rangle_{\nu,*} &= \int\sqrt{\varrho_\nu}\chi_\nu\nabla U_\nu\cdot\nabla\Psi_{(1-\sqrt{\varrho_\nu}\chi_\nu)\varepsilon}\mathscr{M}_\nu^\zeta(\sqrt{\varrho_\nu}\chi_\nu\varepsilon)\\ &= \int\sqrt{\varrho_\nu}\chi_\nu U_\nu(1-\sqrt{\varrho_\nu}\chi_\nu)\varepsilon\mathscr{M}_\nu^\zeta(\sqrt{\varrho_\nu}\chi_\nu\varepsilon)\\ &\quad - \int U_\nu\nabla(\sqrt{\varrho_\nu}\chi_\nu)\cdot\nabla\Psi_{(1-\sqrt{\varrho_\nu}\chi_\nu)\varepsilon}\mathscr{M}_\nu^\zeta(\sqrt{\varrho_\nu}\chi_\nu\varepsilon)\\ &\quad - \int\sqrt{\varrho_\nu}\chi_\nu U_\nu\nabla\Psi_{(1-\sqrt{\varrho_\nu}\chi_\nu)\varepsilon}\cdot\nabla\mathscr{M}_\nu^\zeta(\sqrt{\varrho_\nu}\chi_\nu\varepsilon)\\ &=: I + II + III.\end{aligned}$$

By Cauchy's inequality and (2.17), we obtain

$$|I| \lesssim \left(\int\varrho_\nu\chi_\nu^2 U_\nu(1-\sqrt{\varrho_\nu}\chi_\nu)^2\varepsilon^2\right)^{\frac{1}{2}}\left(\int\frac{\varepsilon^2\chi_\nu^2\varrho_\nu}{U_\nu}\right)^{\frac{1}{2}}.$$

Then, applying $U_\nu(\zeta)(1-\sqrt{\varrho_\nu}\chi_\nu)^2 \lesssim (\nu+\zeta)^2$, inequality (A.19), the control of the outer norm, and Cauchy's inequality, we have the estimate

$$|I| \le \frac{\delta}{10}\left(\int\frac{(\varepsilon^2+|\nabla\varepsilon|^2)\chi_\nu^2\varrho_\nu}{U_\nu}\right) + C\nu^2\int\frac{\varepsilon^2\chi_\nu^2\varrho_\nu}{U_\nu} + C\nu^{100}\|\varepsilon\|^2_{L^\infty(\zeta\ge\zeta^*)}.$$

Next, by (A.14) and (A.15), we have the pointwise estimate of the Poisson field (taking $\alpha = \frac{1}{4}$):

$$\begin{aligned}\int_0^{2\pi}|\nabla\Psi_{(1-\sqrt{\varrho_\nu}\chi_\nu)\varepsilon}(\theta,\zeta)|^2\,d\theta &\lesssim \frac{1+\mathbb{1}_{\{\zeta\le 1\}}\log\zeta}{(1+\zeta)^{\frac{1}{2}}}\int(1-\sqrt{\varrho_\nu}\chi_\nu)^2\varepsilon^2(1+\zeta)^{\frac{1}{2}}\\ &\lesssim C(\zeta^*)\frac{1+\mathbb{1}_{\{\zeta\le 1\}}\log\zeta}{(1+\zeta)^{\frac{1}{2}}}\left(\|\varepsilon\|^2_{\mathrm{in}} + \|\varepsilon(1+\zeta)^{\frac{3}{2}}\|^2_{L^\infty(\zeta\ge\zeta^*)}\right),\end{aligned}\tag{3.19}$$

where we apply $(1-\sqrt{\varrho_\nu}\chi_\nu)^2(1+\zeta)^{\frac{1}{2}} \lesssim (\nu+\zeta)^4 \mathbb{1}_{\{\zeta<1\}} + (1+\zeta)^{\frac{1}{2}}\mathbb{1}_{\{\zeta\geq 1\}}$. Then, by Cauchy's inequality and (2.17),

$$
\begin{aligned}
|II| &\lesssim \frac{1}{\nu}\|\varepsilon\|_{\text{in}} \left(\int U_\nu |\nabla(\sqrt{\varrho_\nu}\chi_\nu)|^2 \cdot |\nabla\Psi_{(1-\sqrt{\varrho_\nu}\chi_\nu)\varepsilon}|^2 \right)^{\frac{1}{2}} \\
&\lesssim \frac{1}{\nu}\|\varepsilon\|_{\text{in}} \left(\int_0^{+\infty} U_\nu |\nabla(\sqrt{\varrho_\nu}\chi_\nu)|^2 \zeta \int_0^{2\pi} |\nabla\Psi_{(1-\sqrt{\varrho_\nu}\chi_\nu)\varepsilon}|^2 \, d\theta d\zeta \right)^{\frac{1}{2}} \\
&\lesssim C(\zeta^*)\frac{|\log\nu|}{\nu}\|\varepsilon\|_{\text{in}} \left(\|\varepsilon\|_{\text{in}} + \|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta\geq\zeta^*)} \right).
\end{aligned}
$$

For III, using (2.18), (A.19) and Cauchy's inequality, we have

$$
\begin{aligned}
|III| &\leq C\left(\int U_\nu \varrho_\nu \chi_\nu^2 \cdot |\nabla\Psi_{(1-\sqrt{\varrho_\nu}\chi_\nu)\varepsilon}|^2 \right)^{\frac{1}{2}} \left(\int \frac{|\nabla(\sqrt{\varrho_\nu}\chi_\nu\varepsilon)|^2}{U_\nu} \right)^{\frac{1}{2}} \\
&\leq \frac{\delta}{10}\left(\int \frac{(\varepsilon^2+|\nabla\varepsilon|^2)\chi_\nu^2\varrho_\nu}{U_\nu} \right) + C(\zeta^*)|\log\nu|^2 \left(\|\varepsilon\|_{\text{in}}^2 + \|\varepsilon(1+\zeta)^{\frac{3}{2}}\|^2_{L^\infty(\zeta\geq\zeta^*)} \right).
\end{aligned}
$$

Finally, combining the estimates of I, II, III above, we obtain

$$
\left| \left\langle (\mathscr{L}_\nu^\zeta - \tilde{\mathscr{L}}_\nu^\zeta)\varepsilon \,,\, \varepsilon \right\rangle_{\nu,*} \right| \leq \frac{\delta}{5}\left(\int \frac{(\varepsilon^2+|\nabla\varepsilon|^2)\chi_\nu^2\varrho_\nu}{U_\nu} \right) + C(\zeta^*)|\log\nu|^2 \left(\|\varepsilon\|_{\text{in}}^2 + \|\varepsilon(1+\zeta)^{\frac{3}{2}}\|^2_{L^\infty(\zeta\geq\zeta^*)} \right).
$$

<u>Estimate of term $L(\varepsilon)$:</u> In the following, we denote the $\mathcal{O}(\mu^s)$ terms (under the bootstrap assumption) as the lower order terms (l.o.t.), as $\mu^s = \mathcal{O}(\nu^k)$ for any fixed $k>0$ when ν is sufficiently small. By Lemma 9, we know that

$$
L(\varepsilon) = -\nabla\cdot(\varepsilon\nabla\Psi_P + P\nabla\Psi_\varepsilon) + \frac{R_\tau}{\mu}\partial_{\bar{r}}\varepsilon + \text{l.o.t.}.
$$

Integrating by parts, we obtain

$$
\begin{aligned}
\langle -\nabla\cdot(\varepsilon\nabla\Psi_P + P\Psi_\varepsilon) \,,\, \varepsilon\rangle_{\nu,*} &= \int \nabla(\sqrt{\varrho_\nu}\chi_\nu)\cdot(\varepsilon\nabla\Psi_P + P\nabla\Psi_\varepsilon)\mathscr{M}_\nu^\zeta(\sqrt{\varrho_\nu}\chi_\nu\varepsilon) \\
&\quad + \int \sqrt{\varrho_\nu}\chi_\nu(\varepsilon\nabla\Psi_P + P\Psi_\varepsilon)\cdot\nabla\mathscr{M}_\nu^\zeta(\sqrt{\varrho_\nu}\chi_\nu\varepsilon).
\end{aligned}
$$

By Lemma 2, inequality (A.19), and the control of the outer L^∞-norm, we first have

$$
\int U_\nu|\mathscr{M}_\nu^\zeta(\varepsilon\sqrt{\varrho_\nu}\chi_\nu)|^2 \lesssim \frac{1}{\nu^2}\|\varepsilon\|^2_{\text{in}}, \quad \int U_\nu|\nabla\mathscr{M}_\nu^\zeta(\varepsilon\sqrt{\varrho_\nu}\chi_\nu)|^2 \lesssim \frac{1}{\nu^2}\left(\|\varepsilon\|^2_{\text{in}} + \|\nabla\varepsilon\|^2_{\text{in}} + \nu^{100}\|\varepsilon\|^2_{L^\infty(\zeta\geq\zeta^*)} \right). \tag{3.20}
$$

Then, thanks to Cauchy's inequality, it remains to estimate

$$
\int (\varrho_\nu\chi_\nu^2 + |\nabla(\sqrt{\varrho}\chi_\nu)|^2)(|\varepsilon\nabla\Psi_P|^2 + |P\nabla\Psi_\varepsilon|^2)\frac{1}{U_\nu}.
$$

By the pointwise estimate $|\nabla\Psi_P(\zeta)| \lesssim |a(\tau)|\log(1+\zeta/\nu)/\zeta \lesssim \frac{|a|}{\nu}$ and (A.19), we have

$$
\int (\varrho_\nu\chi_\nu^2 + |\nabla(\sqrt{\varrho_\nu}\chi_\nu)|^2)\frac{|\varepsilon\nabla\Psi_P|^2}{U_\nu} \lesssim \frac{|a|^2}{\nu^4}\left(\|\varepsilon\|^2_{\text{in}} + \|\nabla\varepsilon\|^2_{\text{in}} + \nu^{100}\|\varepsilon\|^2_{L^\infty(\zeta\geq\zeta^*)} \right).
$$

For the other part, we consider the decomposition $\nabla\Psi_\varepsilon = \nabla\tilde{\Psi}_\varepsilon + \nabla\Psi_{(1-\sqrt{\varrho_\nu}\chi_\nu)\varepsilon}$. By (3.19),

$$
\int (\varrho_\nu\chi_\nu^2 + |\nabla(\sqrt{\varrho_\nu}\chi_\nu)|^2)\frac{|P\nabla\Psi_{(1-\sqrt{\varrho_\nu\chi_\nu})\varepsilon}|^2}{U_\nu} \leq \frac{C(\zeta^*)a^2}{\nu^2}\left(\|\varepsilon\|^2_{\text{in}} + \|\varepsilon(1+\zeta)^{\frac{3}{2}}\|^2_{L^\infty(\zeta\geq\zeta^*)} \right).
$$

By the Hardy-Littlewood-Sobolev inequality (A.21):

$$\|\nabla\tilde{\Psi}_\varepsilon\|_{L^4} \lesssim \frac{1}{\nu}\|\varepsilon\|_{\rm in}\|U_\nu\|_{L^2}^{\frac{1}{2}} \lesssim \frac{1}{\nu^{\frac{3}{2}}}\|\varepsilon\|_{\rm in},$$

we have

$$\int(\varrho_\nu\chi_\nu^2+|\nabla(\sqrt{\varrho_\nu}\chi_\nu)|^2)\frac{|P\nabla\tilde{\Psi}_\varepsilon|^2}{U_\nu} \lesssim \|\nabla\tilde{\Psi}_\varepsilon\|_{L^4}^2\left(\int\frac{(\varrho_\nu\chi_\nu^2+|\nabla(\sqrt{\varrho_\nu}\chi_\nu)|^2)^2P^4}{U_\nu^2}\right)^{\frac{1}{2}} \lesssim \frac{a^2}{\nu^5}\|\varepsilon\|_{\rm in}^2.$$

For the last term in $L(\varepsilon)$, by Cauchy's inequality,

$$\left|\frac{R_\tau}{\mu}\left\langle\varepsilon\,,\,\partial_{\bar r}\varepsilon\right\rangle_{\nu,*}\right| \lesssim \frac{1}{\nu^2}\left|\frac{R_\tau}{\mu}\right|\|\varepsilon\|_{\rm in}\|\nabla\varepsilon\|_{\rm in}.$$

Finally, collecting all the estimates above and by the bootstrap assumptions, we obtain

$$\begin{aligned}\left|\langle L(\varepsilon)\,,\,\varepsilon\rangle_{\nu,*}\right| &\leq C\left(\frac{|a|}{\nu^{\frac{3}{2}}}+\left|\frac{R_\tau}{\nu}\right|^{\frac{1}{2}}\right)\left(\int\frac{(\varepsilon^2+|\nabla\varepsilon|^2)\chi_\nu^2\varrho_\nu}{U_\nu}\right)+\frac{C|a|}{\nu^2}\|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta\geq\zeta^*)}^2+C(\zeta^*,K_i)\nu^3\\ &\leq\frac{\delta}{10}\left(\int\frac{(\varepsilon^2+|\nabla\varepsilon|^2)\chi_\nu^2\varrho_\nu}{U_\nu}\right)+\frac{C|a|}{\nu^2}\|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta\geq\zeta^*)}^2+C(\zeta^*,K_i)\nu^3,\end{aligned}$$

where the second inequality above holds when ν is sufficiently small.
Estimate of the nonlinear term $NL(\varepsilon)$: Again, the terms of order $\mathcal{O}(\mu^s)$ are treated as lower order terms. It then suffices to estimate the term $-\nabla\cdot(\varepsilon\nabla\Psi_\varepsilon)$. Integrate by parts, and we have

$$\langle-\nabla\cdot(\varepsilon\nabla\Psi_\varepsilon)\,,\,\varepsilon\rangle_{\nu,*}=\int\nabla(\sqrt{\varrho_\nu}\chi_\nu)\cdot(\varepsilon\nabla\Psi_\varepsilon)\mathscr{M}_\nu^\zeta(\sqrt{\varrho_\nu}\chi_\nu\varepsilon)+\int\sqrt{\varrho_\nu}\chi_\nu\varepsilon\nabla\Psi_\varepsilon\cdot\nabla\mathscr{M}_\nu^\zeta(\sqrt{\varrho_\nu}\chi_\nu\varepsilon).$$

As before, by Cauchy's inequality and (3.20), it suffices to estimate

$$\int(\varrho_\nu\chi_\nu^2+|\nabla(\sqrt{\varrho_\nu}\chi_\nu)|^2)\varepsilon^2|\nabla\Psi_\varepsilon|^2\frac{1}{U_\nu}.$$

To this end, we decompose $\nabla\Psi_\varepsilon=\nabla\Psi_{\chi^*\varepsilon}+\nabla\Psi_{(1-\chi^*)\varepsilon}$, where we recall $\chi^*(\zeta):=\chi(\zeta/\zeta^*)$. For the first part, We further decompose:

$$\begin{aligned}\int(\varrho_\nu\chi_\nu^2+|\nabla(\sqrt{\varrho_\nu}\chi_\nu)|^2)\varepsilon^2|\nabla\Psi_{\varepsilon\chi^*}|^2\frac{1}{U_\nu}&=\int(\varrho_\nu\chi_\nu^2+|\nabla(\sqrt{\varrho_\nu}\chi_\nu)|^2)\varepsilon^2(\chi^*)^2|\nabla\Psi_{\varepsilon\chi^*}|^2\frac{1}{U_\nu}\\&\quad+\int(\varrho_\nu\chi_\nu^2+|\nabla(\sqrt{\varrho_\nu}\chi_\nu)|^2)\varepsilon^2(1-(\chi^*)^2)|\nabla\Psi_{\varepsilon\chi^*}|^2\frac{1}{U_\nu}\\&:=T_1+T_2.\end{aligned}$$

By Cauchy's inequality, Sobolev embedding $H^1(\mathbb{R}^2)\hookrightarrow L^4(\mathbb{R}^2)$, and HLS inequality (A.20),

$$\begin{aligned}|T_1|&\lesssim\|\nabla\Psi_{\varepsilon\chi^*}\|_{L^4}^2\cdot\left(\int\frac{\varepsilon^4(\chi^*)^4(\nu+\zeta)^8}{\nu^4}\right)^{\frac{1}{2}}\lesssim\|\varepsilon\chi^*\|_{L^2}^2\cdot\left(\int\left|\nabla\left(\frac{\varepsilon\chi^*(\nu+\zeta)^2}{\nu}\right)\right|^2\right)\\&\lesssim\left(\int\frac{\varepsilon^2(\chi^*)^2(\nu+\zeta)^2}{\nu^2}\right)\cdot\left(\int\frac{|\nabla\varepsilon|^2(\chi^*)^2}{U_\nu}+\int\frac{\varepsilon^2(\nu+\zeta)^2(\chi^*)^2}{\nu^2}\right).\end{aligned}$$

By the L^∞ control of ε in the far field, Cauchy's inequality, and HLS inequality,

$$|T_2|\lesssim\frac{1}{\nu^2}\|\varepsilon\|_{L^\infty(\zeta\geq\zeta^*)}^2\|\nabla\Psi_{\varepsilon\chi^*}\|_{L^4}^2\lesssim\nu^2C(K_i)\int\frac{\varepsilon^2(\chi^*)^2(\nu+\zeta)^2}{\nu^2}.$$

In summary, by the inequality (A.19), we obtain

$$\int(\varrho_\nu\chi_\nu^2+|\nabla(\sqrt{\varrho_\nu}\chi_\nu)|^2)\varepsilon^2|\nabla\Psi_{\varepsilon\chi^*}|^2\frac{1}{U_\nu}\lesssim\left(\frac{1}{\nu^4}\|\varepsilon\|_{\rm in}^2+\nu^2C(K_i)+\int\frac{|\nabla\varepsilon|^2(\chi^*)^2}{U_\nu}\right)\int\frac{(|\nabla\varepsilon|^2+\varepsilon^2)\chi_\nu^2\sqrt{\varrho_\nu}}{U_\nu}$$
$$\lesssim\frac{C(K_i)}{|\log\nu|^2}\int\frac{(|\nabla\varepsilon|^2+\varepsilon^2)\chi_\nu^2\sqrt{\varrho_\nu}}{U_\nu}. \tag{3.21}$$

For the second part, we apply (A.17), and obtain (choosing $p=\frac{3}{2}$)

$$\|\nabla\Psi_{(1-\chi^*)\varepsilon}\|_{L^\infty}\lesssim\|(1-\chi^*)\varepsilon\|_{L^\infty}+\|(1-\chi^*)\varepsilon\|_{L^{\frac{3}{2}}}\lesssim\|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta\geq\zeta^*)}.$$

It follows that

$$\begin{aligned}&\int\frac{(\varrho_\nu\chi_\nu^2+|\nabla(\sqrt{\varrho_\nu}\chi_\nu)|^2)\varepsilon^2|\nabla\Psi_{\varepsilon(1-\chi^*)}|^2}{U_\nu}\\&\lesssim\|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta\geq\zeta^*)}^2\int\frac{(\varrho_\nu\chi_\nu^2+|\nabla(\sqrt{\varrho_\nu}\chi_\nu)|^2)\varepsilon^2}{U_\nu}\\&\lesssim\frac{1}{\nu^2}\|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta\geq\zeta^*)}^2\left(\|\varepsilon\|_{\rm in}^2+\|\nabla\varepsilon\|_{\rm in}^2+\nu^{100}\|\varepsilon\|_{L^\infty(\zeta\geq\zeta^*)}^2\right).\end{aligned} \tag{3.22}$$

Finally, combining (3.21) and (3.22), we obtain the nonlinear estimate

$$\begin{aligned}\left|\langle NL(\varepsilon)\,,\,\varepsilon\rangle_{\nu,*}\right|&\leq\frac{C(K_i)}{|\log\nu|}\int\frac{(|\nabla\varepsilon|^2+\varepsilon^2)\chi_\nu^2\sqrt{\varrho_\nu}}{U_\nu}+C(K_i)\nu^3\\&\leq\frac{\delta}{10}\int\frac{(|\nabla\varepsilon|^2+\varepsilon^2)\chi_\nu^2\sqrt{\varrho_\nu}}{U_\nu}+C(K_i)\nu^3,\end{aligned}$$

where the second inequality holds when ν is sufficiently small.
<u>Estimate of the generated error E:</u> Recall that

$$E=\mathrm{Mod}_0\varphi_{0,\nu}+\mathrm{Mod}_1\varphi_{1,\nu}+\frac{R_\tau}{\mu}\partial_{\bar r}U_\nu+\tilde{E}.$$

By the algebraic identity $\mathscr{M}_\nu^\zeta(\Lambda U_\nu)=-2$, the orthogonality conditions (3.3) and the decomposition $\varphi_{i,\nu}=-\frac{1}{16\nu^2}\Lambda U_\nu\chi_\nu+\tilde\varphi_i$, we have

$$\begin{aligned}\left|\langle\varepsilon\,,\,\mathrm{Mod}_i\varphi_{i,\nu}\rangle_{\nu,*}\right|&\lesssim|\mathrm{Mod}_i|\left(\left|\int\varepsilon\sqrt{\varrho_\nu}\chi_\nu\mathscr{M}_\nu^\zeta\Big(\frac{1}{\nu^2}(1-\sqrt{\varrho_\nu}\chi_\nu^2)\Lambda U_\nu\Big)\right|+\left|\int\varepsilon\sqrt{\varrho_\nu}\chi_\nu\mathscr{M}_\nu^\zeta(\chi_\nu\sqrt{\varrho_\nu}\tilde\varphi_i)\right|\right)\\&\lesssim\frac{\delta}{10\nu^2}\|\varepsilon\|_{\rm in}^2+\frac{|\mathrm{Mod}_i|^2}{\nu^2}\end{aligned}$$

Similarly, by the algebraic identity $\mathscr{M}_\nu^\zeta(\nabla U_\nu)=0$, we have

$$\begin{aligned}\left|\int\varepsilon\sqrt{\varrho_\nu}\chi_\nu\mathscr{M}_\nu^\zeta\Big(\frac{R_\tau}{\mu}\partial_{\bar r}U_\nu\sqrt{\varrho_\nu}\chi_\nu\Big)\right|&\lesssim\left|\int\varepsilon\sqrt{\varrho_\nu}\chi_\nu\mathscr{M}_\nu^\zeta\Big(\frac{R_\tau}{\mu}\partial_{\bar r}U_\nu(1-\sqrt{\varrho_\nu}\chi_\nu)\Big)\right|\\&\lesssim\frac{\delta}{10\nu^2}\|\varepsilon\|_{\rm in}^2+\nu^2|\log\nu|\cdot\left|\frac{R_\tau}{\mu}\right|^2.\end{aligned}$$

As for $\tilde E$, by Cauchy's inequality,

$$\left|\left\langle\tilde E\,,\,\varepsilon\right\rangle_{\nu,*}\right|\leq\frac{C}{\nu}\|\varepsilon\|_{\rm in}\left(\int\frac{\tilde E^2}{U_\nu}\right)^{\frac{1}{2}}\leq\frac{\delta}{10\nu^2}\|\varepsilon\|_{\rm in}^2+\frac{C(\nu^2+|a|)}{|\log\nu|^2}+\frac{C(K_i)\nu^2}{|\log\nu|^3}.$$

In summary, we have

$$\left|\langle\varepsilon\,,\,E-\mathrm{Mod}_0\varphi_{0,\nu}\rangle_{\nu,*}\right|\le\frac{3\delta}{10\nu^2}\|\varepsilon\|_{\mathrm{in}}^2+\frac{C}{\nu^2}|\mathrm{Mod}_1|^2+\frac{C(\nu^2+|a|)}{|\log\nu|^2}+\frac{C(K_i)\nu^2}{|\log\nu|^3}.$$

Estimate of time derivative terms: Once We note $\frac{\partial}{\partial_\tau}=\nu_\tau\frac{\partial}{\partial_\nu}$, then the estimates are straight forward from definition. First, we have

$$\left|\left\langle\frac{\partial}{\partial_\tau}\left(\frac{\varrho_\nu\chi_\nu^2}{U_\nu}\right)\,,\,\varepsilon^2\right\rangle\right|\lesssim\frac{\nu_\tau}{\nu}\left(\frac{1}{\nu^2}\|\varepsilon\|_{\mathrm{in}}^2+\nu^{100}\|\varepsilon\|_{L^\infty(\zeta\ge\zeta^*)}^2\right).$$

Second, by $\partial_\nu(\sqrt{\varrho_\nu}\chi_\nu)=\frac{\zeta\nu_\tau}{|\log\nu|^2\nu}\chi'(\zeta/|\log\nu|)\sqrt{\varrho_\nu}$ and (A.12), we have

$$\left|\left\langle\varepsilon\partial_\tau\left(\sqrt{\varrho_\nu}\chi_\nu\right)\,,\,\tilde{\Psi}_\varepsilon\right\rangle_{\nu,*}\right|\lesssim\nu^{100}\nu_\tau\|\varepsilon\|_{L^\infty(\zeta\ge\zeta^*)}\|\varepsilon\|_{\mathrm{in}}\lesssim\|\varepsilon\|_{\mathrm{in}}^2+\nu^{100}$$

Conclusion of Step 1: Finally, collecting all the estimates above, and we obtain, when ν is sufficiently small,

$$\begin{aligned}&\partial_\tau\frac{1}{2}\int\varepsilon\sqrt{\varrho_\nu}\chi_\nu\mathscr{M}_\nu^\zeta(\varepsilon\sqrt{\varrho_\nu}\chi_\nu)-\mathrm{Mod}_0\int\varepsilon\sqrt{\varrho_\nu}\chi_\nu\mathscr{M}_\nu^\zeta(\varphi_{0,\nu}\sqrt{\varrho_\nu}\chi_\nu)\\&\le-\frac{\delta}{4\nu^2}\left(\|\varepsilon\|_{\mathrm{in}}^2+\|\nabla\varepsilon\|_{\mathrm{in}}^2\right)+\frac{C}{\nu^2}|\mathrm{Mod}_1|^2+\frac{C(K_2)\nu^2}{|\log\nu|^2}+C(K_i)\nu^4.\end{aligned}\tag{3.23}$$

Step 2: Correction term estimate

Now we are to estimate the extra terms induced by $\partial_\tau(d_0\int\varepsilon\chi_\nu\sqrt{\varrho_\nu}\mathscr{M}_\nu^\zeta(\chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu}))$. We write

$$\begin{aligned}\partial_\tau\left(d_0\int\varepsilon\chi_\nu\sqrt{\varrho_\nu}\mathscr{M}_\nu^\zeta(\chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu})\right)&=2d_0\int\partial_\tau\varepsilon\chi_\nu\sqrt{\varrho_\nu}\mathscr{M}_\nu^\zeta(\chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu})\\&\quad+2d_0\int\varepsilon\partial_\tau(\chi_\nu\sqrt{\varrho_\nu})\mathscr{M}_\nu^\zeta(\chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu})+2d_0\int\varepsilon\chi_\nu\sqrt{\varrho_\nu}\mathscr{M}_\nu^\zeta(\partial_\tau\bar{\varphi}_0)\\&\quad+2d_0\int\varepsilon\chi_\nu\sqrt{\varrho_\nu}\bar{\varphi}_0\partial_\tau\left(\frac{1}{U_\nu}\right)+2d_0\partial_\tau\left(-\frac{1}{8\nu^2}\right)\int\varepsilon\chi_\nu\sqrt{\varrho_\nu}\\&\quad+\partial_\tau\left(\frac{1}{\int\chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu}\mathscr{M}_\nu^\zeta(\chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu})}\right)\left(\int\chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu}\mathscr{M}_\nu^\zeta(\chi_\nu\sqrt{\varrho_\nu}\varepsilon)\right)^2.\end{aligned}$$

Estimate of $d_0\int\partial_\tau\varepsilon\chi_\nu\sqrt{\varrho_\nu}\mathscr{M}_\nu^\zeta(\chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu})$: Plug in the evolution equation for ε:

$$\partial_\tau\varepsilon=\Delta\varepsilon-\nabla\cdot(U_\nu\nabla\Psi_\varepsilon+\varepsilon\nabla\Psi_{U_\nu})-\beta\Lambda\varepsilon+L(\varepsilon)+NL(\varepsilon)+E,$$

and we estimate term by term. First of all, through integration by parts and Cauchy's inequality,

$$\begin{aligned}\left|d_0\int\chi_\nu\sqrt{\varrho_\nu}\Delta\varepsilon\mathscr{M}_\nu^\zeta(\chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu})\right|&\lesssim\frac{1}{\nu^2}\left|d_0\int\chi_\nu\sqrt{\varrho_\nu}\Delta\varepsilon\right|+\left|d_0\int\chi_\nu\sqrt{\varrho_\nu}\Delta\varepsilon\mathscr{M}_\nu^\zeta(\bar{\varphi}_0)\right|\\&\lesssim\frac{|d_0|}{\nu^2}\left|\int(1+\zeta^2)\varepsilon\chi_\nu\sqrt{\varrho_\nu}\right|+\left|d_0\int(\chi_\nu\sqrt{\varrho_\nu}\nabla\varepsilon-\varepsilon\nabla(\chi_\nu\sqrt{\varrho_\nu}))\cdot\nabla\mathscr{M}_\nu^\zeta(\bar{\varphi}_0)\right|\\&\quad+\left|d_0\int\Delta(\chi_\nu\sqrt{\varrho_\nu})\varepsilon\mathscr{M}_\nu^\zeta(\bar{\varphi}_0)\right|\\&\lesssim\frac{1}{\nu^2|\log\nu|^{\frac{1}{3}}}(\|\varepsilon\|_{\mathrm{in}}^2+\|\nabla\varepsilon\|_{\mathrm{in}}^2)+\nu^{100}\|\varepsilon\|_{L^\infty(\zeta\ge\zeta^*)}\|\varepsilon\|_{\mathrm{in}}.\end{aligned}$$

where we use the estimate $|\int \varepsilon\chi_\nu\sqrt{\varrho_\nu}| \lesssim |\log\nu|^{\frac{1}{2}}(\int(\nu+\zeta)^2\varepsilon^2\chi_\nu\sqrt{\varrho_\nu})^{\frac{1}{2}} + (\int(\nu+\zeta)^4\varepsilon^2\chi_\nu\sqrt{\varrho_\nu})^{\frac{1}{2}}$, and (when ν is sufficiently small)

$$\int \frac{\zeta^4(\nu+\zeta)^4\varepsilon^2\chi_\nu\sqrt{\varrho_\nu}}{\nu^2} \lesssim \int_{\{\zeta\le|\log\nu|^{\frac{2}{3}}\}} \frac{\zeta^4(\nu+\zeta)^4\varepsilon^2\chi_\nu\sqrt{\varrho_\nu}}{\nu^2} + \int_{\{\zeta>|\log\nu|^{\frac{2}{3}}\}} \frac{\zeta^4(\nu+\zeta)^4\varepsilon^2\chi_\nu\sqrt{\varrho_\nu}}{\nu^2}$$
$$\lesssim \frac{|\log\nu|^{\frac{4}{3}}}{\nu^2}\|\nabla\varepsilon\|_{\mathrm{in}}^2 + \nu^{100}\|\varepsilon\|^2_{L^\infty(\zeta\ge\zeta^*)}.$$

Similarly, by the pointwise estimates of the Poisson field (A.16) (A.17), we have the estimate

$$\left|d_0\int \chi_\nu\sqrt{\varrho_\nu}\nabla\cdot(U_\nu\nabla\Psi_\varepsilon+\varepsilon\nabla\Psi_{U_\nu})\mathscr{M}_\nu^\zeta(\chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu})\right| \lesssim \frac{1}{|\log\nu|^{\frac{1}{2}}\nu^2}(\|\varepsilon\|_{\mathrm{in}}^2+\|\nabla\varepsilon\|_{\mathrm{in}}^2) + \|\varepsilon(1+\zeta)^{\frac{3}{2}}\|^2_{L^\infty(\zeta\ge\zeta^*)}.$$

and (the extra linear term is smaller, but here a rough estimate is enough)

$$\left|d_0\int \chi_\nu\sqrt{\varrho_\nu}L(\varepsilon)\mathscr{M}_\nu^\zeta(\chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu})\right| \le \frac{C(K_i)}{|\log\nu|^{\frac{1}{2}}\nu^2}(\|\varepsilon\|_{\mathrm{in}}^2+\|\nabla\varepsilon\|_{\mathrm{in}}^2) + C(K_i)\|\varepsilon(1+\zeta)^{\frac{3}{2}}\|^2_{L^\infty(\zeta\ge\zeta^*)} + \mathcal{O}(\mu^s).$$

As for the scaling term, the estimate is similar to the $\Delta\varepsilon$ term:

$$\left|d_0\int \chi_\nu\sqrt{\varrho_\nu}\beta\Lambda\varepsilon\mathscr{M}_\nu^\zeta(\chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu})\right| \lesssim \frac{1}{\nu^2|\log\nu|^{\frac{1}{3}}}\left(\|\varepsilon\|_{\mathrm{in}}^2+\|\nabla\varepsilon\|_{\mathrm{in}}^2\right) + \mathcal{O}(\nu^{100}).$$

For the nonlinear term, by the estimates of Poisson filed and Cauchy's inequality:

$$\left|d_0\int \chi_\nu\sqrt{\varrho_\nu}NL(\varepsilon)\mathscr{M}_\nu^\zeta(\chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu})\right| \lesssim \left|\frac{d_0}{\nu^2}\int\nabla(\chi_\nu\sqrt{\varrho_\nu})\cdot\nabla\Psi_\varepsilon\varepsilon\right| + \left|d_0\int\nabla(\chi_\nu\sqrt{\varrho_\nu})\cdot\nabla\Psi_\varepsilon\varepsilon\mathscr{M}_\nu^\zeta(\bar\varphi_0)\right|$$
$$+\left|d_0\int\chi_\nu\sqrt{\varrho_\nu}\varepsilon\nabla\Psi_\varepsilon\cdot\nabla\mathscr{M}_\nu^\zeta(\bar\varphi_0)\right|$$
$$\lesssim \frac{\|\varepsilon\|_{\mathrm{in}}^2+\|\nabla\varepsilon\|_{\mathrm{in}}^2}{\nu^2|\log\nu|^{\frac{1}{2}}}\left(\frac{1}{\nu}\|\nabla\varepsilon\|_{\mathrm{in}} + |\log\nu|\cdot\|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta\ge\zeta^*)}\right),$$

where we use the estimates

$$\left|\frac{d_0}{\nu^2}\int\chi_\nu\sqrt{\varrho_\nu}\zeta\partial_\zeta\Psi_\varepsilon\varepsilon\right| \lesssim \frac{|d_0|}{\nu}\left(\int\frac{\varepsilon^2\chi_\nu^2\varrho_\nu(\nu+\zeta)^4}{\nu^2}\right)^{\frac{1}{2}}\left(\int_{\{\zeta\le|2\log\nu|\}}\frac{\zeta^2}{(\nu+\zeta)^4}|\nabla\Psi_\varepsilon|^2\right)^{\frac{1}{2}}$$
$$\lesssim \frac{\|\varepsilon\|_{\mathrm{in}}^2}{\nu^2|\log\nu|}\left(\frac{1}{\nu}\|\nabla\varepsilon\|_{\mathrm{in}} + |\log\nu|^{\frac{1}{2}}\|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta\ge\zeta^*)}\right),$$
$$\left|d_0\int\zeta\chi_\nu\sqrt{\varrho_\nu}\partial_\zeta\Psi_\varepsilon\varepsilon\mathscr{M}_\nu^\zeta(\bar\varphi_0)\right| \lesssim |d_0|\left(\int\frac{\zeta^2(\nu+\zeta)^2\varepsilon^2\chi_\nu^2\varrho_\nu}{\nu^2}\right)^{\frac{1}{2}}\left(\int\frac{\nu^2}{(\nu+\zeta)^2}|\nabla\Psi_\varepsilon|^2|\mathscr{M}_\nu^\zeta(\bar\varphi_0)|^2\right)^{\frac{1}{2}}$$
$$\lesssim \frac{\|\varepsilon\|_{\mathrm{in}}^2}{\nu^2|\log\nu|}\left(\frac{1}{\nu}\|\nabla\varepsilon\|_{\mathrm{in}} + |\log\nu|\cdot\|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta\ge\zeta^*)}\right),$$
$$\left|d_0\int\chi_\nu\sqrt{\varrho_\nu}\varepsilon\nabla\Psi_\varepsilon\cdot\nabla\mathscr{M}_\nu^\zeta(\bar\varphi_0)\right| \lesssim |d_0|\left(\int\frac{(\nu+\zeta)^2\varepsilon^2\chi_\nu^2\varrho_\nu}{\nu^2}\right)^{\frac{1}{2}}\left(\int\frac{\nu^2}{(\nu+\zeta)^2}|\nabla\Psi_\varepsilon|^2|\nabla\mathscr{M}_\nu^\zeta(\bar\varphi_0)|^2\right)^{\frac{1}{2}}$$
$$\lesssim \frac{\|\varepsilon\|_{\mathrm{in}}\|\nabla\varepsilon\|_{\mathrm{in}}}{\nu^2|\log\nu|^{\frac{1}{2}}}\left(\frac{1}{\nu}\|\nabla\varepsilon\|_{\mathrm{in}} + |\log\nu|\cdot\|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta\ge\zeta^*)}\right).$$

Finally, for the generative error,

$$\left|d_0\int\chi_\nu\sqrt{\varrho_\nu}(E-\mathrm{Mod}_0\varphi_{0,\nu})\mathscr{M}_\nu^\zeta(\chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu})\right| \lesssim \frac{|\mathrm{Mod}_1|\cdot\|\varepsilon\|_{\mathrm{in}}}{\nu^2} + \frac{\|\varepsilon\|_{\mathrm{in}}}{|\log\nu|^2} + C(K_i)\nu\|\varepsilon\|_{\mathrm{in}},$$

where we use the estimate $|\int \tilde{E}\chi_\nu\sqrt{\varrho_\nu}| \lesssim \frac{\nu^2}{|\log\nu|} + C(K_i)\nu^3$ and the fact $\int \partial_{\bar{r}} U_\nu \chi_\nu\sqrt{\varrho_\nu}\mathscr{M}_\nu^\zeta(\varphi_{0,\nu}\chi_\nu\sqrt{\varrho_\nu})$=0. Finally, combining all these estimates, we obtain

$$\left| d_0 \int (\partial_\tau\varepsilon - \mathrm{Mod}_0\varphi_{0,\nu})\chi_\nu\sqrt{\varrho_\nu}\mathscr{M}_\nu^\zeta(\chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu}) \right| \lesssim C(K_i)\frac{\|\varepsilon\|_{\mathrm{in}}^2 + \|\nabla\varepsilon\|_{\mathrm{in}}^2}{\nu^2|\log\nu|^{\frac{1}{3}}} + C(K_i)\frac{\nu^2}{|\log\nu|^3}.$$

<u>Estimate of the rest time derivative terms:</u> By $\partial_\tau\chi_\nu\sqrt{\varrho_\nu} = \frac{\zeta}{|\log\nu|^2}\frac{\nu_\tau}{\nu}\chi'(\zeta/|\log\nu|)$ and the L^∞ estimate of ε in the far field, we have

$$\left| d_0 \int \varepsilon\partial_\tau(\chi_\nu\sqrt{\varrho_\nu})\mathscr{M}_\nu^\zeta(\chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu}) \right| \le C(K_i)\nu^{100},$$

when ν is sufficiently small. Similarly, by Cauchy's inequality, we have

$$\left| d_0 \int \varepsilon\chi_\nu\sqrt{\varrho_\nu}\mathscr{M}_\nu^\zeta(\partial_\tau(\bar{\varphi}_0) \right| + \left| d_0 \int \varepsilon\chi_\nu\sqrt{\varrho_\nu}\bar{\varphi}_0\partial_\tau\left(\frac{1}{U_\nu}\right) \right| + \left| d_0\partial_\tau\left(-\frac{1}{8\nu^2}\right)\int \varepsilon\chi_\nu\sqrt{\varrho_\nu} \right| \le C(K_i)\frac{\|\varepsilon\|_{\mathrm{in}}^2 + \|\nabla\varepsilon\|_{\mathrm{in}}^2}{\nu^2|\log\nu|}.$$

For the last term,

$$\left| \partial_\tau\left(\frac{1}{\int \chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu}\mathscr{M}_\nu^\zeta(\chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu})} \right) \right| \lesssim \frac{|\partial_\tau(\nu^{-2}\int\bar{\varphi}_0)| + |\partial_\tau\int\bar{\varphi}_0\mathscr{M}_\nu^\zeta(\bar{\varphi}_0)|}{(\int \chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu}\mathscr{M}_\nu^\zeta(\chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu}))^2} \lesssim \frac{\nu^2}{|\log\nu|}.$$

Then, we obtain

$$\left| \partial_\tau\left(\frac{1}{\int \chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu}\mathscr{M}_\nu^\zeta(\chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu})} \right)\left(\int \chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu}\mathscr{M}_\nu^\zeta(\chi_\nu\sqrt{\varrho_\nu}\varepsilon) \right)^2 \right| \lesssim \frac{C(K_i)\|\varepsilon\|_{\mathrm{in}}^2}{\nu^2|\log\nu|}.$$

Finally, all the estimates above yields

$$\partial_\tau\left(d_0\int \varepsilon\chi_\nu\sqrt{\varrho_\nu}\mathscr{M}_\nu^\zeta(\chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu}) \right) = 2\mathrm{Mod}_0\int \chi_\nu\sqrt{\varrho_\nu}\varphi_{0,\nu}\mathscr{M}_\nu^\zeta(\varepsilon\chi_\nu\sqrt{\varrho_\nu}) + \mathcal{O}\left(C(K_i)\frac{\nu^2}{|\log\nu|^{\frac{7}{3}}} \right). \quad (3.24)$$

Step 3: Conclusion:
Combining (3.23) and (3.24) yields the final result. ☐

3.3.2 H^1 Inner Estimate

By orthogonality conditions (3.3) and Lemma 1, we know that there exists a universal $C>0$, such that

$$\frac{1}{C}\int \frac{|\nabla\varepsilon^*|^2}{U_\nu} - C\|\varepsilon\|_{\mathrm{in}}^2 < -\int \mathscr{L}_{0,\nu}^\zeta(\varepsilon^*)\mathscr{M}_\nu^\zeta(\varepsilon^*) = \int U_\nu|\nabla\mathscr{M}_\nu^\zeta(\varepsilon^*)|^2 < C\int\frac{|\nabla\varepsilon^*|^2}{U_\nu}.$$

Thus, $\int U_\nu|\nabla\mathscr{M}_\nu^\zeta(\varepsilon^*)|^2$ is equivalent to the norm $\|\nabla\varepsilon^*\|_{L^2(U_\nu)}$ (with some negligible error).

Lemma 5 (Control of $\|\nabla\varepsilon^*\|_{L^2(U_\nu)}$)**.** *Let w be a solution in the bootstrap regime* $\mathrm{BS}(\tau_0,\tau_*,\zeta^*,M_0,\{K_i\}_{i=1}^7)$*. Then, the following estimate holds on $[\tau_0,\tau_*]$:*

$$\frac{1}{2}\frac{d}{d\tau}\int U_\nu|\nabla\mathscr{M}_\nu^\zeta(\varepsilon^*)|^2 \le \frac{C}{\nu^2}\left(\|\nabla\varepsilon\|_{\mathrm{in}}^2 + \|\varepsilon\|_{\mathrm{in}}^2\right) + \frac{C}{\nu^2}\|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta\ge\zeta^*)}^2 + \frac{C\nu^2}{|\log\nu|^2} + \frac{C(\zeta^*,K_i)\nu^2}{|\log\nu|^3},$$

for some universal C, some $C(\zeta^*,K_i)$ dependent on $\{\zeta^*\}\cup\{K_i\}_{i=1}^7$, and any constant $K>0$.

Proof. First of all, the evolution of ε^* is:

$$\frac{d}{d\tau}\varepsilon^* = \mathscr{L}^\zeta_{0,\nu}\varepsilon^* + [\chi^*, \mathscr{L}^\zeta_{0,\nu}]\varepsilon + \chi^*\sqrt{\varrho_\nu}\left(-\beta\Lambda\varepsilon + L(\varepsilon) + NL(\varepsilon) + E\right),$$

based on which we compute that

$$\begin{aligned}
\frac{1}{2}\frac{d}{d\tau}\int U_\nu|\nabla\mathscr{M}^\zeta_\nu(\varepsilon^*)|^2 &= \int U_\nu\nabla\mathscr{M}^\zeta_\nu(\varepsilon^*)\cdot\nabla\mathscr{M}^\zeta_\nu(\partial_\tau\varepsilon^*) + \frac{1}{2}\int(\partial_\tau U_\nu)|\nabla\mathscr{M}^\zeta_\nu(\varepsilon^*)|^2 \\
&\quad + \int U_\nu\nabla\mathscr{M}^\zeta_\nu(\varepsilon^*)\cdot\nabla\left(\varepsilon\partial_\tau\left(\frac{1}{U_\nu}\right)\right) \\
&= -\int\mathscr{L}^\zeta_{0,\nu}\varepsilon^*\mathscr{M}^\zeta_\nu(\mathscr{L}^\zeta_{0,\nu}\varepsilon^*) + \int U_\nu\nabla\mathscr{M}^\zeta_\nu(\varepsilon^*)\cdot\nabla\mathscr{M}^\zeta_\nu\left(\chi^*\left(-\beta\Lambda\varepsilon + L(\varepsilon) + NL(\varepsilon)\right)\right) \\
&\quad + \int U_\nu\nabla\mathscr{M}^\zeta_\nu(\varepsilon^*)\cdot\nabla\mathscr{M}^\zeta_\nu\left(\chi^*E + [\chi^*, \mathscr{L}^\zeta_{0,\nu}]\varepsilon\right) \\
&\quad + \frac{1}{2}\int(\partial_\tau U_\nu)|\nabla\mathscr{M}^\zeta_\nu(\varepsilon^*)|^2 + \int U_\nu\nabla\mathscr{M}^\zeta_\nu(\varepsilon^*)\cdot\nabla\left(\varepsilon^*\partial_\tau\left(\frac{1}{U_\nu}\right)\right).
\end{aligned}$$

Coercivity in H^2: Denote $\varepsilon_2 := \mathscr{L}^\zeta_{0,\nu}\varepsilon^*$, and decompose

$$\varepsilon_2 = a_0\Lambda U_\nu + a_1\partial_{\bar r}U_\nu + \tilde\varepsilon_2,$$

where

$$a_0 = \frac{\langle\varepsilon_2\,,\,\Lambda U_\nu\rangle}{\langle\Lambda U_\nu\,,\,\Lambda U_\nu\rangle},\quad a_1 = \frac{\langle\varepsilon_2\,,\,\partial_{\bar r}U_\nu\rangle}{\langle\partial_{\bar r}U_\nu\,,\,\partial_{\bar r}U_\nu\rangle}.$$

By (A.16) and Cauchy's inequality, We have the estimate

$$|a_0| \lesssim \nu^2\int|(\nabla\varepsilon^* - U_\nu\nabla\Psi_{\varepsilon^*} - \varepsilon^*\nabla\Psi_{U_\nu})\cdot\nabla(\Lambda U_\nu)| \lesssim \frac{1}{\nu^2}\|\nabla\varepsilon^*\|_{L^2} \lesssim \frac{C(K_i)}{|\log\nu|}.$$

Similarly, we have

$$|a_1| \lesssim \frac{C(K_i)\nu}{|\log\nu|}.$$

Then, by the algebraic identities $\mathscr{M}^\zeta_\nu(\Lambda U_\nu) = -2$, $\mathscr{M}^\zeta_\nu(\nabla U_\nu) = 0$, the divergence form of ε_2, and (2.19), we have

$$\int\varepsilon_2\mathscr{M}^\zeta_\nu(\varepsilon_2) = \int\tilde\varepsilon_2\mathscr{M}^\zeta_\nu(\tilde\varepsilon_2) \approx \int\frac{|\tilde\varepsilon_2|^2}{U_\nu} \approx \int\frac{|\varepsilon_2|^2}{U_\nu} + \frac{C(K_i)}{|\log\nu|^2},$$

where we denote for two non-negative quantities $A \approx B$, if there exists a universal constant $c > 0$, such that $cA \le B \le \frac{1}{c}A$. Then, we are to show that $\int\frac{|\varepsilon_2|^2}{U_\nu}$ is equivalent to certain weighted H^2-norm for ε^*. Define

$$\tilde\varepsilon^* := \varepsilon^* - \frac{\langle\varepsilon^*\,,\,\Lambda U_\nu\rangle}{\langle\Lambda U_\nu\,,\,\Lambda U_\nu\rangle}\Lambda U_\nu - \frac{\langle\varepsilon^*\,,\,\nabla U_\nu\rangle}{\langle\nabla U_\nu\,,\,\nabla U_\nu\rangle}\cdot\nabla U_\nu := \varepsilon^* - c_1\Lambda U_\nu - \mathbf{c}_2\cdot\nabla U_\nu.$$

Thus, we have

$$\langle\tilde\varepsilon^*\,,\,\Lambda U_\nu\rangle = \langle\tilde\varepsilon^*\,,\,\nabla U_\nu\rangle = 0.$$

By the local orthogonality conditions (3.3), we estimate that

$$|c_1| \lesssim \nu^2\left|\int\varepsilon^*\Lambda U_\nu\right| \lesssim \nu^4\|\varepsilon^*\|_{L^2},\quad |\mathbf{c}_2| \lesssim \nu^4\left|\int\varepsilon^*\nabla U_\nu\right| \lesssim \nu^6\|\varepsilon^*\|_{L^2}.$$

Then, by the estimates above and Proposition 3 (in the parabolic variables),

$$\begin{aligned}\int \frac{|\mathscr{L}^\zeta_{0,\nu}\varepsilon^*|^2}{U_\nu} = \int \frac{|\mathscr{L}^\zeta_{0,\nu}\tilde{\varepsilon}^*|^2}{U_\nu} &\geq \delta\left(\int \frac{|\Delta\tilde{\varepsilon}^*|^2}{U_\nu} + \frac{1}{\nu^2}\int(\nu+\zeta)^2|\nabla\tilde{\varepsilon}^*|^2 + \frac{1}{\nu^2}\int \tilde{\varepsilon}^{*2}\right)\\ &\geq \delta\left(\int \frac{|\Delta\varepsilon^*|^2}{U_\nu} + \frac{1}{\nu^2}\int(\nu+\zeta)^2|\nabla\varepsilon^*|^2 + \frac{1}{\nu^2}\int \varepsilon^{*2}\right) - C\nu^4\|\varepsilon^*\|^2_{L^2}\\ &\geq \frac{\delta}{2}\left(\int \frac{|\Delta\varepsilon^*|^2}{U_\nu} + \frac{1}{\nu^2}\int(\nu+\zeta)^2|\nabla\varepsilon^*|^2 + \frac{1}{\nu^2}\int \varepsilon^{*2}\right),\end{aligned}$$

when ν is small enough. Since ε^* is compactly supported, integration by parts yields the following control (one can, for example, apply the density argument by considering the functions in $\mathcal{D}(\mathbb{R}^2)$ first):

$$\int |\nabla^{(2)}\varepsilon^*|^2\zeta^{2p} \leq C(p)\left(\int |\Delta\varepsilon^*|^2\zeta^{2p} + \int |\nabla\varepsilon^*|^2\zeta^{2p-2}\right), \quad p=1,2.$$

It follows that there exists some $\delta'>0$, such that

$$\int \frac{|\mathscr{L}^\zeta_{0,\nu}\varepsilon^*|^2}{U_\nu} \geq \delta'\left(\int \frac{|\nabla^{(2)}\varepsilon^*|^2(\nu+\zeta)^4}{\nu^2} + \frac{1}{\nu^2}\int(\nu+\zeta)^2|\nabla\varepsilon^*|^2 + \frac{1}{\nu^2}\int \varepsilon^{*2}\right).$$

For brevity, in the following we denote

$$\|\varepsilon^*\|^2_{H^2_\#} := \int \frac{|\nabla^{(2)}\varepsilon^*|^2(\nu+\zeta)^4}{\nu^2} + \frac{1}{\nu^2}\int(\nu+\zeta)^2|\nabla\varepsilon^*|^2 + \frac{1}{\nu^2}\int \varepsilon^{*2}.$$

Finally, gather all the results above, and we obtain

$$\begin{aligned}-\int \mathscr{L}^\zeta_{0,\nu}\varepsilon^*\mathscr{M}^\zeta_\nu(\mathscr{L}^\zeta_{0,\nu}\varepsilon^*) &\leq -\delta'\|\varepsilon^*\|^2_{H^2_\#} + \frac{C}{\nu^2}\left(\int(\nu+\zeta)^2|\nabla\varepsilon^*|^2 + \int\frac{(\nu+\zeta)^2(\varepsilon^*)^2}{\nu^2}\right)\\ &\leq -\delta'\|\varepsilon^*\|^2_{H^2_\#} + \frac{C}{\nu^4}\left(\|\varepsilon\|^2_{\text{in}} + \|\nabla\varepsilon\|^2_{\text{in}}\right) \leq -\delta'\|\varepsilon^*\|^2_{H^2_\#} + \frac{C(K_i)}{|\log\nu|^2},\end{aligned}$$

and

$$-\int \mathscr{L}^\zeta_{0,\nu}\varepsilon^*\mathscr{M}^\zeta_\nu(\mathscr{L}^\zeta_{0,\nu}\varepsilon^*) \leq -\delta\int\frac{\tilde{\varepsilon}_2^2}{U_\nu},$$

for some universal $\delta,\delta',C>0$. In other words, it means that there is a partial H^2-damping (i.e., damping in a certain finite codimensional subspace) and a full H^2-damping with an error of size $\frac{1}{|\log\nu|^2}$.

Estimate of the scaling term $-\beta\Lambda\varepsilon$: Note that

$$-\beta\int U_\nu\nabla\mathscr{M}^\zeta_\nu(\varepsilon^*)\cdot\nabla\mathscr{M}^\zeta_\nu(\chi^*\Lambda\varepsilon) = -2\beta\int U_\nu|\nabla\mathscr{M}^\zeta_\nu(\varepsilon^*)|^2 - \beta\int U_\nu\nabla\mathscr{M}^\zeta_\nu(\varepsilon^*)\cdot\nabla\mathscr{M}^\zeta_\nu(\chi^*\mathbf{y}\cdot\nabla\varepsilon),$$

where the first term of the right-hand side has the desirable sign. As for the second term, by the identities $\frac{\mathbf{y}\cdot\nabla\varepsilon^*}{U_\nu} = \mathbf{y}\cdot\nabla\left(\frac{\varepsilon^*}{U_\nu}\right) + \frac{\mathbf{y}\cdot\nabla U_\nu\varepsilon^*}{U_\nu^2}$ and $\Psi_{\mathbf{y}\cdot\nabla\varepsilon^*} = \mathbf{y}\cdot\nabla\Psi_{\varepsilon^*} - 2\Psi_{\varepsilon^*}$ (since ε^* is compacted supported, the Poisson fields are all well defined, so that this identity can be verified by computing the Laplacian on the right-hand side), we have

$$\mathscr{M}^\zeta_\nu(\chi^*\mathbf{y}\cdot\nabla\varepsilon) = \mathbf{y}\cdot\nabla\mathscr{M}^\zeta_\nu(\varepsilon^*) - \mathscr{M}^\zeta_\nu(\mathbf{y}\cdot\nabla(\chi^*)\varepsilon) + \frac{\mathbf{y}\cdot\nabla U_\nu\varepsilon^*}{U_\nu^2} + 2\mathscr{M}^\zeta_\nu(\varepsilon^*) - \frac{2\varepsilon^*}{U_\nu}.$$

Then, through integration by parts, Cauchy's inequality and (2.18), we obtain

$$\left|\beta\int U_\nu\nabla\mathscr{M}^\zeta_\nu(\varepsilon^*)\cdot\nabla\mathscr{M}^\zeta_\nu(\chi^*\mathbf{y}\cdot\nabla\varepsilon)\right| \leq C\left(\frac{1}{\nu^2}\|\varepsilon\|^2_{\text{in}} + \frac{1}{\nu^2}\|\nabla\varepsilon\|^2_{\text{in}}\right),$$

for some $C = C(\zeta^*)$. It follows that

$$-\beta \int U_\nu \nabla \mathscr{M}_\nu^\zeta(\varepsilon^*) \cdot \nabla \mathscr{M}_\nu^\zeta(\chi^* \Lambda \varepsilon) \leq C \left(\frac{1}{\nu^2} \|\varepsilon\|_{\text{in}}^2 + \frac{1}{\nu^2} \|\nabla \varepsilon\|_{\text{in}}^2 \right).$$

Estimate of $L(\varepsilon)$ As before, we neglect the terms of order $\mathcal{O}(\mu^s)$, thanks to (A.3). Thus,

$$L(\varepsilon) = -\nabla \cdot (\varepsilon \nabla \Psi_P + P \nabla \Psi_\varepsilon) + \frac{R_\tau}{\mu} \partial_{\bar{r}} \varepsilon + \text{l.o.t.}.$$

Recall the pointwise estimates $|\nabla \Psi_P(\zeta)| \lesssim \frac{|a|}{\nu}$ and $|\partial_\zeta^k P(\zeta)| \lesssim \frac{|a|\zeta^{2-k} \log(4+\zeta)}{(\nu+\zeta)^4}$. The following estimate relies on the structure of $\mathscr{M}_\nu^\zeta$, specifically $\mathscr{M}_\nu^\zeta \Lambda U_\nu = -2$ and $\mathscr{M}_\nu^\zeta \nabla U_\nu = 0$. First, we note that

$$\begin{aligned}\int \varepsilon_2 \mathscr{M}_\nu^\zeta(\chi^* \nabla \cdot (\varepsilon \nabla \Psi_P + P \nabla \Psi_\varepsilon)) &= \int \nabla \cdot (\varepsilon^* \nabla \Psi_P + P \chi^* \nabla \Psi_\varepsilon) \mathscr{M}_\nu^\zeta(\tilde{\varepsilon}_2) \\ &\quad - \int U_\nu \nabla \mathscr{M}_\nu^\zeta(\varepsilon^*) \cdot \nabla \mathscr{M}_\nu^\zeta(\nabla \chi^* \cdot \nabla \Psi_P \varepsilon + \nabla \chi^* \cdot \nabla \Psi_\varepsilon P),\end{aligned}$$

where we use the decomposition $\varepsilon_2 = \tilde{\varepsilon}_2 + a_0 \Lambda U_\nu + a_1 \partial_{\bar{r}} U_\nu$ and $\int \nabla \cdot (\varepsilon^* \nabla \Psi_P + P \chi^* \nabla \Psi_\varepsilon) = 0$. By Cauchy's inequality, we obtain

$$\begin{aligned}\left| \int \nabla \cdot (\varepsilon^* \nabla \Psi_P + P \chi^* \nabla \Psi_\varepsilon) \mathscr{M}_\nu^\zeta(\tilde{\varepsilon}_2) \right| &\leq \frac{\delta}{10} \int \frac{\tilde{\varepsilon}_2^2}{U_\nu} + C(\zeta^*) \int \frac{|\nabla \cdot (\varepsilon^* \nabla \Psi_P + P \chi^* \nabla \Psi_\varepsilon)|^2}{U_\nu} \\ &\leq \frac{\delta}{10} \int \frac{\tilde{\varepsilon}_2^2}{U_\nu} + C(\zeta^*) \left(\frac{a^2}{\nu^4} \|\nabla \varepsilon\|_{\text{in}}^2 + \frac{a^2}{\nu^2} \|\varepsilon (1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta \geq \zeta^*)}^2 \right),\end{aligned}$$

and

$$\left| \int U_\nu \nabla \mathscr{M}_\nu^\zeta(\varepsilon^*) \cdot \nabla \mathscr{M}_\nu^\zeta(\nabla \chi^* \cdot \nabla \Psi_P \varepsilon + \nabla \chi^* \cdot \nabla \Psi_\varepsilon P) \right| \leq C(\zeta^*) \frac{|a|}{\nu^3} \left(\|\varepsilon\|_{\text{in}}^2 + \|\nabla \varepsilon\|_{\text{in}}^2 + \|\varepsilon (1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta \geq \zeta^*)}^2 \right),$$

where we use the elliptic regularity

$$\int_{\zeta^* \leq \zeta \leq 2\zeta^*} |\nabla^{(2)} \Psi_\varepsilon|^2 \leq C(\zeta^*) \int_{\frac{1}{2}\zeta^* \leq \zeta \leq 4\zeta^*} |\nabla \Psi_\varepsilon|^2 + \varepsilon^2. \tag{3.25}$$

With the same argument, we can estimate

$$\left| \int \varepsilon_2 \mathscr{M}_\nu^\zeta(\chi^* \frac{R_\tau}{\mu} \partial_{\bar{r}} \varepsilon) \right| \leq \frac{\delta}{10} \int \frac{\tilde{\varepsilon}_2^2}{U_\nu} + \left| \frac{R_\tau}{\mu} \right|^2 \frac{C}{\nu^2} (\|\varepsilon\|_{\text{in}}^2 + \|\nabla \varepsilon\|_{\text{in}}^2).$$

In summary, we obtain

$$\left| \int \varepsilon_2 \mathscr{M}_\nu^\zeta(\chi^* L(\varepsilon)) \right| \leq \frac{\delta}{5} \int \frac{\tilde{\varepsilon}_2^2}{U_\nu} + \frac{C(\zeta^*, K_i)}{\nu} \left(\|\varepsilon\|_{\text{in}}^2 + \|\nabla \varepsilon\|_{\text{in}}^2 \right).$$

Estimate of the nonlinear term $NL(\varepsilon)$: We neglect the term of order $\mathcal{O}(\mu^s)$, and as before, rely on the structure of $\mathscr{M}_\nu^\zeta$. First, we write

$$\int \varepsilon_2 \mathscr{M}_\nu^\zeta(\chi^* \nabla \cdot (\varepsilon \nabla \Psi_\varepsilon)) = \int \tilde{\varepsilon}_2 \mathscr{M}_\nu^\zeta(\nabla \cdot (\varepsilon^* \nabla \Psi_\varepsilon)) - \int U_\nu \nabla \mathscr{M}_\nu^\zeta(\varepsilon^*) \cdot \nabla \mathscr{M}_\nu^\zeta(\nabla \chi^* \cdot \nabla \Psi_\varepsilon \varepsilon). \tag{3.26}$$

Then, we are to derive L^∞-bounds for $\nabla \Psi_\varepsilon$ and ε. Denote $q^*(\gamma) = \nu^2 \varepsilon^*(\nu \gamma)$. Then, $\nabla \Psi_{q^*}(\gamma) = \nu \nabla \Psi_{\varepsilon^*}(\nu \gamma)$. By Sobolev embedding and the pointwise estimates of the Poisson field (A.14)(A.15), we have

$$\|\nabla \Psi_{q^*}\|_{L^\infty} \lesssim \|\nabla q^*\|_{L^2} + \|\nabla \Psi_{q^*}\|_{L^2} \leq C(\zeta^*) \|\nabla \varepsilon^*\|_{L^2(U_\nu)} \leq \frac{C(\zeta^*, K_i) \nu^2}{|\log \nu|}.$$

Thus,

$$\|\nabla\Psi_{\varepsilon^*}\|_{L^\infty} \leq \frac{C(\zeta^* K_i)\nu}{|\log\nu|}.$$

Moreover, by (A.17), we have

$$\|\nabla\Psi_{(1-\chi^*)\varepsilon}\|_{L^\infty} \lesssim \|(1-\chi^*)\varepsilon\|_{L^\infty} + \|(1-\chi^*)\varepsilon\|_{L^{\frac{3}{2}}} \lesssim \|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta\geq\zeta^*)} \leq C(K_i)\nu^2.$$

In summary, we've shown that

$$\|\nabla\Psi_\varepsilon\|_{L^\infty} \leq \frac{C(\zeta^*, K_i)\nu}{|\log\nu|}.$$

Similarly, Sobolev embedding yields

$$\|q^*\|_{L^\infty} \lesssim \|q^*\|_{H^2} \lesssim \nu^2\|\varepsilon^*\|_{H^2_\#} \lesssim \nu^2 \int \varepsilon_2 \mathscr{M}_\nu^\zeta(\varepsilon_2) + \frac{C(K_i)\nu^2}{|\log\nu|}.$$

It then follows that

$$\|\varepsilon^*\|_{L^\infty} \lesssim \int \varepsilon_2 \mathscr{M}_\nu^\zeta(\varepsilon_2) + \frac{C(K_i)}{|\log\nu|} = \int \tilde{\varepsilon}_2 \mathscr{M}_\nu^\zeta(\tilde{\varepsilon}_2) + \frac{C(K_i)}{|\log\nu|}$$

Now coming back to (3.26), by Cauchy's inequality, we obtain

$$\left|\int \varepsilon_2 \mathscr{M}_\nu^\zeta(\nabla\cdot(\varepsilon^*\nabla\Psi_\varepsilon))\right| \leq \left(\frac{\delta}{20} + \frac{C\|\varepsilon\|_{\text{in}}}{\nu}\right)\int \frac{\tilde{\varepsilon}_2^2}{U_\nu} + \frac{C(\zeta^*, K_i)}{|\log\nu|^2\nu^2}\left(\|\varepsilon\|_{\text{in}}^2 + \||\nabla\varepsilon\|_{\text{in}}^2\right),$$

and

$$\left|\int U_\nu \nabla\mathscr{M}_\nu^\zeta(\varepsilon^*)\cdot\nabla\mathscr{M}_\nu^\zeta(\nabla\chi^*\cdot\nabla\Psi_\varepsilon\varepsilon)\right| \leq \frac{C(\zeta^*, K_i)}{\nu}\|\nabla\varepsilon\|_{\text{in}}^2 + \frac{C(\zeta^*)}{\nu^3}(\|\varepsilon\|_{\text{in}}^4 + \|\varepsilon\|_{L^\infty(\zeta\geq\zeta^*)}^4),$$

where we again use (3.25). In summary, by the bootstrap assumptions, we have the estimate

$$\left|\int \varepsilon_2 \mathscr{M}_\nu^\zeta(\chi^* NL(\varepsilon))\right| \leq \frac{\delta}{4}\int \frac{\tilde{\varepsilon}_2^2}{U_\nu} + \frac{C(\zeta^*, K_i)\nu^2}{|\log\nu|^3} + \mathcal{O}(\mu^s),$$

when ν is sufficiently small.

Estimate of the rest terms : We write

$$[\chi^*, \mathscr{L}_{0,\nu}^\zeta]\varepsilon = -2\nabla\chi^*\cdot\nabla\varepsilon - \varepsilon\Delta\chi^* + \varepsilon\nabla\chi^*\cdot\nabla\Psi_{U_\nu} + \nabla\cdot(U_\nu\nabla\Psi_{\varepsilon^*} - \chi^* U_\nu\nabla\Psi_\varepsilon) + U_\nu\nabla\chi^*\cdot\nabla\Psi_\varepsilon.$$

First of all, by Cauchy's inequality,

$$\left|\int U_\nu\nabla\mathscr{M}_\nu^\zeta(\varepsilon^*)\cdot\nabla\mathscr{M}_\nu^\zeta(-2\nabla\chi^*\cdot\nabla\varepsilon - \varepsilon\Delta\chi^* + \varepsilon\nabla\chi^*\cdot\nabla\Psi_{U_\nu})\right| \leq \frac{C}{\nu^2}\left(\|\nabla\varepsilon\|_{\text{in}}^2 + \|\varepsilon\|_{\text{in}}^2\right).$$

Second, by the pointwise estimates of the Poisson field, the bootstrap assumptions and inequality (A.17),

$$\begin{aligned}
\left|\int \varepsilon_2\mathscr{M}(\nabla\cdot(U_\nu\nabla\Psi_{\varepsilon^*} - \chi^* U_\nu\nabla\Psi_\varepsilon))\right| &= \left|\int \tilde{\varepsilon}_2\mathscr{M}(\nabla\cdot(U_\nu\nabla\Psi_{\varepsilon^*} - \chi^* U_\nu\nabla\Psi_\varepsilon))\right| \\
&\leq \left|\int \tilde{\varepsilon}_2\mathscr{M}(\nabla\cdot(U_\nu(1-\chi^*)\nabla\Psi_{\varepsilon^*})\right| + \left|\int \tilde{\varepsilon}_2\mathscr{M}(\nabla\cdot(U_\nu\chi^*\nabla\Psi_{\varepsilon(1-\chi^*)})\right| \\
&\leq \frac{\delta}{10}\int\frac{\tilde{\varepsilon}_2^2}{U_\nu} + \frac{C}{\nu^2}\|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta\geq\zeta^*)}^2 + C(\zeta^*, K_i)\nu^3.
\end{aligned}$$

Similarly, we have the estimate

$$\left|\int U_\nu \nabla \mathscr{M}_\nu^\zeta(\varepsilon^*) \cdot \nabla \mathscr{M}_\nu^\zeta(U_\nu \nabla\chi^* \cdot \nabla\Psi_\varepsilon)\right| \le \frac{C}{\nu^2}\left(\|\nabla\varepsilon\|_{\text{in}}^2 + \|\varepsilon\|_{\text{in}}^2\right) + C(\zeta^*, K_i)\nu^3.$$

To summarize, we have

$$\left|\int \varepsilon_2 \mathscr{M}_\nu^\zeta([\chi^*, \mathscr{L}_{0,\nu}^\zeta]\varepsilon)\right| \le \frac{\delta}{10}\int \frac{\tilde{\varepsilon}_2^2}{U_\nu} + \frac{C}{\nu^2}\left(\|\nabla\varepsilon\|_{\text{in}}^2 + \|\varepsilon\|_{\text{in}}^2\right) + \frac{C}{\nu^2}\|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta \ge \zeta^*)}^2 + C(\zeta^*, K_i)\nu^3.$$

The estimate of the generated error relies on the structure of the operator $\mathscr{M}_\nu^\zeta$. Note that

$$\begin{aligned}
&-\int U_\nu \nabla \mathscr{M}_\nu^\zeta(\varepsilon^*) \cdot \nabla \mathscr{M}_\nu^\zeta(\chi^* E) = \int \varepsilon_2 \mathscr{M}^\zeta(\chi^* E) \\
&= -\sum_{i=0,1} \text{Mod}_i \int U_\nu \nabla \mathscr{M}_\nu^\zeta(\varepsilon^*) \cdot \nabla \mathscr{M}_\nu^\zeta\left(-\frac{1}{16\nu^2}(\chi^*-1)\Lambda U_\nu + \tilde{\varphi}_i\right) + a_0\int \chi^*\tilde{E} + \int \tilde{\varepsilon}_2 \mathscr{M}_\nu^\zeta(\chi^*\tilde{E}).
\end{aligned}$$

Then, by Cauchy's inequality, the estimates of $\tilde{E}$ and the estimate for $|\text{Mod}_i|$ in (3.9), we obtain

$$\left|\int \varepsilon_2 \mathscr{M}_\nu^\zeta(\chi^* E)\right| \le \frac{\delta}{10}\int \frac{\tilde{\varepsilon}_2^2}{U_\nu} + \frac{C}{\nu^2}\left(\|\varepsilon\|_{\text{in}}^2 + \|\nabla\varepsilon\|_{\text{in}}^2\right) + \frac{C(\nu^2+|a|)}{|\log\nu|^2} + \frac{C(K_i)\nu^2}{|\log\nu|^3}.$$

Finally, for the extra time derivative terms, note that $\partial_\tau = \frac{\nu_\tau}{\nu}\nu\partial_\nu$, and the estimates are straightforward:

$$\frac{1}{2}\int (\partial_\tau U_\nu)|\nabla \mathscr{M}_\nu^\zeta(\varepsilon^*)|^2 + \int U_\nu \nabla \mathscr{M}_\nu^\zeta(\varepsilon^*) \cdot \nabla\left(\varepsilon^* \partial_\tau\left(\frac{1}{U_\nu}\right)\right) \le \frac{C(K_i)}{\nu^2|\log\nu|}\left(\|\varepsilon\|_{\text{in}}^2 + \|\nabla\varepsilon\|_{\text{in}}^2\right).$$

<u>Conclusion:</u> Collecting all the estimates above, we obtain:

$$\begin{aligned}
\frac{1}{2}\frac{d}{d\tau}\int U_\nu |\nabla \mathscr{M}_\nu^\zeta(\varepsilon^*)|^2 \le{}& \frac{C}{\nu^2}\left(\|\nabla\varepsilon\|_{\text{in}}^2 + \|\varepsilon\|_{\text{in}}^2\right) + \frac{C}{\nu^2}\|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta\ge\zeta^*)}^2 \\
&+ \frac{C(\nu^2+|a|)}{|\log\nu|^2} + \frac{C(\zeta^*, K_i)\nu^2}{|\log\nu|^3},
\end{aligned}$$

for some universal constants $\delta', C > 0$ and constant $C(K_i)$ depending on K_i $(1 \le i \le 7)$. □

3.3.3 Higher Order Estimates in the Middle Range

Lemma 6 (H^2 control of ε in the middle range)**.** *Let w be a solution in the bootstrap regime* $\text{BS}(\tau_0, \tau_*, \zeta^*, M_0, \{K_i\}_{i=1}^7)$*. Then, for any $0 < \zeta_1 < \zeta_2$ and $\tau \in [\tau_0, \tau_*]$ we have the following estimate*

$$\frac{d}{d\tau}\|\varepsilon\|_{H_*^2(\zeta_1,\zeta_2)}^2 \le -\delta(\zeta_1,\zeta_2)\|\varepsilon\|_{H_*^2(\zeta_1,\zeta_2)}^2 + \frac{C(\zeta_1,\zeta_2)K_4\nu^2}{|\log\nu|}|\text{Mod}_0| + \frac{C(\zeta_1,\zeta_2)K_4^2\nu^4}{|\log\nu|^2} + \frac{C(\zeta_1,\zeta_2,K_i)\nu^4}{|\log\nu|^3},$$

with the norm $\|\cdot\|_{H_*^2(\zeta_1,\zeta_2)}$ defined in (3.32), and positive constants $\delta(\zeta_1,\zeta_2)$ and $C(\zeta_1,\zeta_2)$ depending only on ζ_1, ζ_2.

Proof. The main idea of the proof is the parabolic regularity together with pointwise estimates in the middle range. To start with, from the control of the inner norm of ε, we have

$$\|\varepsilon\|_{L^2(\frac{1}{8}\zeta_1 \le \zeta \le 8\zeta_2)} \le C(\zeta_1,\zeta_2)\|\varepsilon\|_{\text{in}} \le C(\zeta_1,\zeta_2)\frac{K_4\nu^2}{|\log\nu|}.$$

The evolution of ε can be written as

$$\partial_\tau\varepsilon = \Delta\varepsilon + \mathcal{G}\cdot\nabla\varepsilon + \mathcal{F}\varepsilon - \nabla W \cdot \nabla\Psi_\varepsilon - \nabla\cdot(\varepsilon\nabla\Psi_\varepsilon) + E + \text{l.o.t},$$

where we recall $W = U_\nu + P$ and

$$\mathcal{F} := 2W - 2\beta, \quad \mathcal{G} := -\nabla\Psi_W - \beta\mathbf{y} + \frac{R_\tau}{\mu}\mathbf{e}_1.$$

Here l.o.t. (lower order terms) denotes the terms that, up to second derivatives, can be estimated in the middle range with order $\mathcal{O}(\mu^s)$, and hence negligible in our analysis (we will omit it in the rest of this proof). Note that in the middle range $\frac{1}{8}\zeta_1 \le \zeta \le 8\zeta_2$, we have for $k = 0, 1, 2$,

$$|\partial_\zeta^{(k)} W| \le C(\zeta_1, \zeta_2)\left(\nu^2 + |a(\tau)|\right), \quad |\partial_\zeta^{(k+1)}\Psi_W| \le C(\zeta_1, \zeta_2)\left(1 + \frac{|a|}{\nu}\right).$$

Also, by (3.4), (3.6), (2.1), and Lemma 3, we have for any $\zeta \in [\frac{1}{8}\zeta_1, 8\zeta_2]$ and any $k \ge 0$,

$$|\partial_\zeta^{(k)} E| \le C(\zeta_1, \zeta_2)|\mathrm{Mod}_0| + \frac{C(\zeta_1, \zeta_2)\nu^2}{|\log\nu|} + \frac{C(\zeta_1, \zeta_2, K_i)\nu^2}{|\log\nu|^2}.$$

For $j = 0, 1, 2$, we define a family of cutoff functions:

$$\chi_j(\zeta) = \begin{cases} 1, & \frac{1}{2^{2-j}}\zeta_1 \le \zeta \le 2^{2-j}\zeta_2, \\ 0, & \zeta \in [0, \frac{1}{2^{3-j}}\zeta_1] \cup [2^{3-j}\zeta_2, +\infty). \end{cases}$$

For brevity, we denote $C(K_i) := C(\zeta_1, \zeta_2, K_i : 1 \le i \le 7)$.
<u>L^2-evolution</u>: Compute that

$$\frac{1}{2}\frac{d}{d\tau}\|\varepsilon\chi_0\|_{L^2}^2 = \langle \chi_0\left(\Delta\varepsilon + \mathcal{G}\cdot\nabla\varepsilon + \mathcal{F}\varepsilon - \nabla W\cdot\nabla\Psi_\varepsilon - \nabla\cdot(\varepsilon\nabla\Psi_\varepsilon) + E\right), \chi_0\varepsilon\rangle.$$

First of all, by Cauchy's inequality,

$$\langle \chi_0\Delta\varepsilon, \chi_0\varepsilon\rangle = -\|\chi_0\nabla\varepsilon\|_{L^2}^2 - \langle \nabla\varepsilon, 2\chi_0\varepsilon\nabla\chi_0\rangle \le -\frac{1}{2}\|\chi_0\nabla\varepsilon\|_{L^2}^2 + C(\zeta_1, \zeta_2)\frac{K_4^2\nu^4}{|\log\nu|^2}.$$

Next, due to the pointwise estimates $|\mathcal{F}| + |\mathcal{G}| \lesssim 1$, we obtain

$$\langle \varepsilon\chi_0, \chi_0(\mathcal{F}\varepsilon + \mathcal{G}\cdot\nabla\varepsilon)\rangle \le \frac{1}{8}\|\chi_0\nabla\varepsilon\|_{L^2}^2 + C(\zeta_1, \zeta_2)\frac{K_4^2\nu^4}{|\log\nu|^2}.$$

By Hardy-Littlewood-Sobolev (HLS) inequality, (A.12) and (A.17), we have the following estimates of Poisson field:

$$\|\nabla\Psi_{\varepsilon\chi^*}\|_{L^4} \lesssim \frac{1}{\nu^{\frac{3}{2}}}\|\varepsilon\|_{\mathrm{in}} \lesssim \frac{K_4\sqrt{\nu}}{|\log\nu|}, \quad \|\nabla\Psi_{\varepsilon(1-\chi^*)}\|_{L^\infty} \lesssim \|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta\ge\zeta^*)} \lesssim K_7\nu^2. \tag{3.27}$$

Thus, by (3.27),

$$-\langle \varepsilon\chi_0, \chi_0\nabla W\cdot\nabla\Psi_\varepsilon\rangle \lesssim \frac{C(K_i)\nu^{\frac{9}{2}}}{|\log\nu|^2}.$$

As for the nonlinear term, using (3.27) and the Sobolev embedding $W^{1,p}(\mathbb{R}^2) \hookrightarrow L^{\frac{2p}{2-p}}(\mathbb{R}^2)$ ($p < 2$), we have

$$-\langle \varepsilon\chi_0, \chi_0\nabla\cdot(\varepsilon\nabla\Psi_\varepsilon)\rangle \lesssim \|\varepsilon\|_{L^2(\frac{1}{8}\zeta_1\le\zeta\le8\zeta_2)}\|\nabla(\chi_0\varepsilon)\|_{L^2}^2 + \frac{C(\zeta_1, \zeta_2, K_i)}{|\log\nu|}\|\chi_0\nabla\varepsilon\|_{L^2}^2 + \frac{C(\zeta_1, \zeta_2, K_i)\nu^4}{|\log\nu|^3}.$$

Finally, by the pointwise estimate of E,

$$\langle \varepsilon\chi_0, E\chi_0\rangle \le \frac{C(\zeta_1, \zeta_2)K_4\nu^2}{|\log\nu|}|\mathrm{Mod}_0| + \frac{C(\zeta_1, \zeta_2)\nu^4}{|\log\nu|^2} + \frac{C(\zeta_1, \zeta_2, K_i)\nu^4}{|\log\nu|^3}.$$

In summary, we have

$$\frac{d}{d\tau}\|\chi_0\varepsilon\|_{L^2}^2 \le -\frac{1}{4}\|\chi_0\nabla\varepsilon\|_{L^2}^2 + C(\zeta_1,\zeta_2)\frac{K_4\nu^2}{|\log\nu|}|\mathrm{Mod}_0| + \frac{C(\zeta_1,\zeta_2)\nu^4}{|\log\nu|^2} + \frac{C(\zeta_1,\zeta_2,K_i)\nu^4}{|\log\nu|^3}. \tag{3.28}$$

Evolution of first derivatives: Denote by ∂ either $\partial_{\bar r}$ or $\partial_{\bar z}$. Then,

$$\frac{1}{2}\frac{d}{d\tau}\|\partial\varepsilon\chi_1\|_{L^2}^2 = \langle \chi_1\partial\left(\Delta\varepsilon + \mathcal{G}\cdot\nabla\varepsilon + \mathcal{F}\varepsilon - \nabla W\cdot\nabla\Psi_\varepsilon - \nabla\cdot(\varepsilon\nabla\Psi_\varepsilon) + E\right),\, \chi_1\partial\varepsilon\rangle.$$

Similarly,

$$\langle \chi_1\partial\Delta\varepsilon\,,\,\partial\varepsilon\chi_1\rangle = -\|\chi_1\nabla\partial\varepsilon\|_{L^2}^2 - \langle\nabla\partial\varepsilon\,,\,2\chi_1\nabla\chi_1\partial\varepsilon\rangle \le -\frac{1}{2}\|\chi_1\nabla\partial\varepsilon\|_{L^2}^2 + C\|\partial\varepsilon\chi_0\|_{L^2}^2,$$

and

$$\langle\partial\varepsilon\chi_1\,,\,\chi_1\partial(\mathcal{G}\cdot\nabla\varepsilon + \mathcal{F}\varepsilon)\rangle \le \frac{1}{16}\|\chi_1\nabla\partial\varepsilon\|_{L^2}^2 + C(\zeta_1,\zeta_2)\|\partial\varepsilon\chi_0\|_{L^2}^2 + C(\zeta_1,\zeta_2)\frac{K_4^2\nu^4}{|\log\nu|^2}.$$

Next, through integration by parts and estimates of the Poisson field,

$$\begin{aligned}
-\langle\chi_1\partial\varepsilon\,,\,\chi_1\partial(\nabla W\cdot\nabla\Psi_\varepsilon)\rangle &= \left\langle\chi_1^2\partial^2\varepsilon\,,\,\nabla W\cdot\nabla\Psi_\varepsilon\right\rangle + \langle 2\partial\chi_1\chi_1\partial\varepsilon\,,\,\nabla W\cdot\nabla\Psi_\varepsilon\rangle\\
&\le \frac{1}{8}\|\chi_1\partial^2\varepsilon\|_{L^2}^2 + C\|\partial\varepsilon\chi_0\|_{L^2}^2 + \frac{C(\zeta_1,\zeta_2,K_i)\nu^5}{|\log\nu|^2}.
\end{aligned}$$

As for the nonlinear term, by the Sobolev embedding $H^2(\mathbb{R}^2)\hookrightarrow L^\infty(\mathbb{R}^2)$, we have

$$\|\varepsilon\chi_1\|_{L^\infty} \lesssim \|\varepsilon\chi_0\|_{L^2} + \|\chi_0\nabla\varepsilon\|_{L^2} + \|\chi_1\nabla^{(2)}\varepsilon\|_{L^2}.$$

Then, through integration by parts, the L^∞ estimate above, Sobolev embedding and (3.27), we obtain

$$\begin{aligned}
-\langle\chi_1\partial\varepsilon\,,\,\chi_1\partial\left(\nabla\cdot(\varepsilon\nabla\Psi_\varepsilon)\right)\rangle &= \left\langle\chi_1^2\partial^2\varepsilon\,,\,\nabla\varepsilon\cdot\nabla\Psi_\varepsilon - \varepsilon^2\right\rangle + \left\langle 2\partial\chi_1\chi_1\partial\varepsilon\,,\,\nabla\varepsilon\cdot\nabla\Psi_\varepsilon - \varepsilon^2\right\rangle\\
&\le \frac{C(\zeta_1,\zeta_2,K_i)}{|\log\nu|}\|\partial^2\varepsilon\chi_1\|_{L^2}^2 + \frac{C(\zeta_1,\zeta_2,K_i)}{|\log\nu|}\|\partial\varepsilon\chi_0\|_{L^2}^2 + \frac{C(\zeta_1,\zeta_2)K_4^2\nu^4}{|\log\nu|^2}.
\end{aligned}$$

Finally, through the integration by parts,

$$\langle\chi_1\partial\varepsilon\,,\,\partial E\chi_1\rangle = -\left\langle\varepsilon\,,\,\partial(\chi_1^2\partial E)\right\rangle \lesssim C(\zeta_1,\zeta_2)\frac{K_4\nu^2}{|\log\nu|}|\mathrm{Mod}_0| + \frac{C(\zeta_1,\zeta_2)K_4\nu^4}{|\log\nu|^2} + \frac{C(\zeta_1,\zeta_2,K_i)\nu^4}{|\log\nu|^3}.$$

In summary, when ν is sufficiently small, we have

$$\begin{aligned}
\frac{d}{d\tau}\|\chi_1\nabla\varepsilon\|_{L^2}^2 \le& -\frac{1}{4}\|\chi_1\nabla^{(2)}\varepsilon\|_{L^2}^2 + C(\zeta_1,\zeta_2)\|\nabla\varepsilon\chi_0\|_{L^2}^2 + \frac{C(\zeta_1,\zeta_2)K_2\nu^2}{|\log\nu|}|\mathrm{Mod}_0|\\
&+ \frac{C(\zeta_1,\zeta_2)K_4^2\nu^4}{|\log\nu|^2} + \frac{C(\zeta_1,\zeta_2,K_i)\nu^4}{|\log\nu|^3}.
\end{aligned} \tag{3.29}$$

Evolution of second derivatives: Denote generically ∂^2 a second order partial derivative (e.g. $\partial_{\bar r}\partial_{\bar r}, \partial_{\bar r}\partial_{\bar z}$). Then,

$$\frac{1}{2}\frac{d}{d\tau}\|\partial^2\varepsilon\chi_2\|_{L^2}^2 = \left\langle\chi_2\partial^2\left(\Delta\varepsilon + \mathcal{G}\cdot\nabla\varepsilon + \mathcal{F}\varepsilon - \nabla W\cdot\nabla\Psi_\varepsilon - \nabla\cdot(\varepsilon\nabla\Psi_\varepsilon) + E\right),\, \chi_2\partial^2\varepsilon\right\rangle.$$

The estimates of the first three terms are identical:

$$\left\langle\chi_2\partial^2\Delta\varepsilon\,,\,\partial^2\varepsilon\chi_2\right\rangle = -\|\chi_2\nabla\partial^2\varepsilon\|_{L^2}^2 - \left\langle\nabla\partial^2\varepsilon\,,\,2\chi_2\nabla\chi_2\partial^2\varepsilon\right\rangle \le -\frac{1}{2}\|\chi_2\nabla\partial^2\varepsilon\|_{L^2}^2 + C\|\partial^2\varepsilon\chi_1\|_{L^2}^2,$$

and

$$\left\langle \partial^2\varepsilon\chi_2\,,\,\chi_2\partial^2(\mathcal{G}\cdot\nabla\varepsilon+\mathcal{F}\varepsilon)\right\rangle \le \frac{1}{16}\|\chi_2\nabla\partial^2\varepsilon\|_{L^2}^2 + C(\zeta_1,\zeta_2)\|\partial^2\varepsilon\chi_1\|_{L^2}^2 + C(\zeta_1,\zeta_2)\|\partial\varepsilon\chi_0\|_{L^2}^2 + \frac{C(\zeta_1,\zeta_2)K_4^2\nu^4}{|\log\nu|^2}.$$

Next, through integration by parts once,

$$-\left\langle \chi_2\partial^2\varepsilon\,,\,\chi_2\partial^2\left(\nabla W\cdot\nabla\Psi_\varepsilon\right)\right\rangle = \left\langle \chi_2^2\partial^3\varepsilon + 2\chi_2\partial\chi_2\partial^2\varepsilon\,,\,\nabla(\partial W)\cdot\nabla\Psi_\varepsilon + \nabla W\cdot\nabla(\partial\Psi_\varepsilon)\right\rangle.$$

The estimates are the same as before, except for the $\nabla W\cdot\nabla(\partial\Psi_\varepsilon)$ term, which is done in the following way: by elliptic regularity, HLS inequality, (A.14) and (A.15),

$$\begin{aligned}\int_{\{\frac{1}{2}\zeta_*\le\zeta\le 2\zeta^*\}}|\nabla^{(2)}\Psi_\varepsilon|^2 &\le C(\zeta_1,\zeta_2)\int_{\{\frac14\zeta_1\le\zeta\le 4\zeta_2\}}\varepsilon^2 + C(\zeta_1,\zeta_2)\int_{\{\frac14\zeta_1\le\zeta\le 4\zeta_2\}}|\nabla\Psi_\varepsilon|^2 \\ &\le \frac{C(\zeta_1,\zeta_2)}{\nu^3}\|\varepsilon\|_{\text{in}}^2 + C(\zeta_1,\zeta_2)\|\varepsilon(1+\zeta)^{\frac32}\|_{L^\infty(\zeta\ge\zeta^*)}^2.\end{aligned} \tag{3.30}$$

Thus, we have

$$\begin{aligned}-\left\langle \chi_2\partial^2\varepsilon\,,\,\chi_2\partial^2\left(\nabla W\cdot\nabla\Psi_\varepsilon\right)\right\rangle &\le \frac18\|\chi_2\partial^3\varepsilon\|_{L^2}^2 + C(\zeta_1,\zeta_2)\|\chi_1\nabla^{(2)}\varepsilon\|_{L^2}^2 + C(\zeta_1,\zeta_2)\|\chi_0\nabla\varepsilon\|_{L^2}^2 \\ &\quad + \frac{C(\zeta_1,\zeta_2,K_i)\nu^5}{|\log\nu|^2} + \frac{C(\zeta_1,\zeta_2)|a|^2}{\nu^3}\frac{K_4^2\nu^4}{|\log\nu|^2}.\end{aligned}$$

As for the nonlinear terms, integrate by parts once:

$$-\left\langle \chi_2\partial^2\varepsilon\,,\,\chi_2\partial^2\left(\nabla\cdot(\varepsilon\nabla\Psi_\varepsilon)\right)\right\rangle = \left\langle \chi_2^2\partial^3\varepsilon + 2\chi_2\partial\chi_2\partial^2\varepsilon\,,\,\nabla(\partial\varepsilon)\cdot\nabla\Psi_\varepsilon + \nabla\varepsilon\cdot\nabla(\partial\Psi_\varepsilon) - 2\varepsilon\partial\varepsilon\right\rangle.$$

Note that all the local terms (i.e., terms not involving the Poisson field) together with the term $\nabla(\partial\varepsilon)\cdot\nabla\Psi_\varepsilon$ can be estimated in the same way as before. It then remains to deal with the term $\nabla\varepsilon\cdot\nabla(\partial\Psi_\varepsilon)$. By the Sobolev embedding $H^2(\mathbb{R}^2)\hookrightarrow L^\infty(\mathbb{R}^2)$,

$$\|\nabla\varepsilon\chi_2\|_{L^\infty} \lesssim \|\nabla^{(3)}\varepsilon\chi_2\|_{L^2} + \|\nabla^{(2)}\varepsilon\chi_1\|_{L^2} + \|\nabla\varepsilon\chi_0\|_{L^2}.$$

This, together with (3.30), gives

$$\|\nabla\varepsilon\cdot\nabla(\partial\Psi_\varepsilon)\|_{L^2} \lesssim \frac{C(\zeta_1,\zeta_2,K_i)}{|\log\nu|}\left(\|\nabla^{(3)}\varepsilon\chi_2\|_{L^2} + \|\nabla^{(2)}\varepsilon\chi_1\|_{L^2} + \|\nabla\varepsilon\chi_0\|_{L^2}\right).$$

Therefore, we obtain the nonlinear estimate

$$-\left\langle \chi_2\partial^2\varepsilon\,,\,\chi_2\partial^2\left(\nabla\cdot(\varepsilon\nabla\Psi_\varepsilon)\right)\right\rangle \le \frac{C(\zeta_1,\zeta_2,K_i)}{|\log\nu|}\|\partial^3\varepsilon\chi_2\|_{L^2}^2 + C(\zeta_1,\zeta_2)\left(\|\nabla^{(2)}\varepsilon\chi_1\|_{L^2}^2 + \|\nabla\varepsilon\chi_0\|_{L^2}^2\right) + \frac{CK_4^2\nu^4}{|\log\nu|^2}$$

The estimate for E is the same:

$$\left\langle \chi_2\partial^2\varepsilon\,,\,\partial^2E\chi_2\right\rangle = \left\langle \varepsilon\,,\,\partial^2(\chi_2^2\partial^2E)\right\rangle \le \frac{C(\zeta_1,\zeta_2)K_4\nu^2}{|\log\nu|}|\text{Mod}_0| + \frac{C(\zeta_1,\zeta_2)K_4\nu^4}{|\log\nu|^2} + \frac{C(\zeta_1,\zeta_2,K_i)\nu^4}{|\log\nu|^3}$$

In summary, when ν is sufficiently small we have

$$\begin{aligned}\frac{d}{d\tau}\|\chi_2\nabla^{(2)}\varepsilon\|_{L^2}^2 &\le -\frac14\|\chi_2\nabla^{(3)}\varepsilon\|_{L^2}^2 + C(\zeta_1,\zeta_2)\left(\|\nabla^{(2)}\varepsilon\chi_1\|_{L^2}^2 + \|\nabla\varepsilon\chi_0\|_{L^2}^2\right) \\ &\quad + \frac{C(\zeta_1,\zeta_2)K_4\nu^2}{|\log\nu|}|\text{Mod}_0| + \frac{C(\zeta_1,\zeta_2)K_4^2\nu^4}{|\log\nu|^2} + \frac{C(\zeta_1,\zeta_2,K_i)\nu^4}{|\log\nu|^3}.\end{aligned} \tag{3.31}$$

Conclusion: Combining (3.28) (3.29) and (3.31), we know that there exists $C_0 = C_0(\zeta_1, \zeta_2) > 0$ sufficiently large, such that once we define

$$\|\varepsilon\|^2_{H^2_*(\zeta_1,\zeta_2)} := \|\varepsilon\chi_0\|^2_{L^2} + \frac{1}{C_0}\|\nabla\varepsilon\chi_1\|^2_{L^2} + \frac{1}{C_0^2}\|\nabla^{(2)}\varepsilon\chi_2\|^2_{L^2}, \tag{3.32}$$
$$\|\varepsilon\|^2_{H^3_*(\zeta_1,\zeta_2)} := \|\nabla\varepsilon\chi_0\|^2_{L^2} + \frac{1}{C_0}\|\nabla^{(2)}\varepsilon\chi_1\|^2_{L^2} + \frac{1}{C_0^2}\|\nabla^{(3)}\varepsilon\chi_2\|^2_{L^2},$$

it holds that

$$\frac{d}{d\tau}\|\varepsilon\|^2_{H^2_*(\zeta_1,\zeta_2)} \le -\frac{1}{8}\|\varepsilon\|^2_{H^3_*(\zeta_1,\zeta_2)} + \frac{C(\zeta_1,\zeta_2)K_4\nu^2}{|\log\nu|}|\mathrm{Mod}_0| + \frac{C(\zeta_1,\zeta_2)K_4^2\nu^4}{|\log\nu|^2} + \frac{C(\zeta_1,\zeta_2,K_i)\nu^4}{|\log\nu|^3}.$$

Finally, the result follows form the Poincaré inequality

$$\|\varepsilon\|_{H^2_*(\zeta_1,\zeta_2)} \le C'\|\varepsilon\|_{H^3_*(\zeta_1,\zeta_2)} + \frac{C(\zeta_1,\zeta_2)K_4\nu^2}{|\log\nu|}.$$

□

3.3.4 Far Field Estimate

Lemma 7 (L^∞ control of ε in the far field)**.** *There exists $M > 0$, such that for any $M_0 \ge M$ and $\zeta^* > M$, the following holds. Let w be a solution in the bootstrap regime $\mathrm{BS}(\tau_0, \tau_*, \zeta^*, M_0, \{K_i\}_{i=1}^7)$. Then, for any $\tau \in [\tau_0, \tau_*]$ we have the following estimate*

$$\|\varepsilon(\tau)(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta\ge\zeta^*)} \le C(\zeta^*, K_1)\frac{\nu^3(\tau)}{\nu^3(\tau_0)}\left(\|\varepsilon(\tau_0)(1+\zeta(\tau_0))^{\frac{3}{2}}\|_{L^\infty(\zeta\ge\zeta^*)} + \|\varepsilon(\tau_0)\|_{L^\infty(\zeta\ge\frac{1}{2}\zeta^*)}\right)$$
$$+ \frac{CK_7}{\zeta^*}\frac{e^{-2\sqrt{\beta\tau+M_0}}}{\sqrt{\beta\tau+M_0}} + \frac{C(\zeta^*, K_1, K_4, K_5, K_6)e^{-2\sqrt{\beta\tau+M_0}}}{\sqrt{\beta\tau+M_0}} + \frac{C(\zeta^*, K_i)e^{-2\sqrt{\beta\tau+M_0}}}{\beta\tau+M_0}. \tag{3.33}$$

Proof. To derive the far field L^∞-control, we go all the way back to the original 3D system (3dKS), where we exploit the dissipating structure of the heat operator. Consider the following decomposition of the solution:

$$u(\mathbf{x}, t) = \frac{1}{\mu^2}W(\frac{r-R}{\mu}, \frac{z}{\mu}, \tau) + \frac{1}{\mu^2}\varepsilon(\frac{r-R}{\mu}, \frac{z}{\mu}, \tau),$$

where we recall $r = |(x_1, x_2)|$, $z = x_3$, $\mathbf{x} = (x_1, x_2, x_3)$, $\tau = -\log(T-t) + \log(T)$, and $\mu = \sqrt{T-t} = e^{-\tau/2+\log(T)/2}$ (without loss of generality, we can assume $T = 1$ in the following). Besides, recall that

$$W(\bar{r}, \bar{z}, \tau) = U_\nu(\bar{r}, \bar{z}) + P(\bar{r}, \bar{z}, \tau),$$

where we denote $\bar{r} := (r-R)/\mu$ and $\bar{z} := z/\mu$ as before. One remark: since R, μ depend on time, so do $\bar{r}, \bar{z}$, and we will specify their time dependence whenever necessary. Let $\eta(\bar{r}, \bar{z})$ be a smooth cutoff function such that

$$\eta(\bar{r}, \bar{z}) \equiv 1 \text{ when } \sqrt{\bar{r}^2+\bar{z}^2} \ge \zeta^*, \quad \eta(\bar{r}, \bar{z}) \equiv 0 \text{ when } \sqrt{\bar{r}^2+\bar{z}^2} < \frac{1}{2}\zeta^*.$$

Denote $\varepsilon_*(\bar{r}, \bar{z}, \tau) := \varepsilon(\bar{r}, \bar{z}, \tau)\eta(\bar{r}, \bar{z})$. Then, the evolution equation for ε_* can be written as

$$\partial_\tau\varepsilon_* = \left(\partial_{\bar{r}}^2 + \partial_{\bar{z}}^2 + \frac{1}{\bar{r}+R/\mu}\partial_{\bar{r}}\right)\varepsilon_* - \frac{1}{2}\Lambda\varepsilon_* + \frac{R_\tau}{\mu}\partial_{\bar{r}}\varepsilon_* + S(\bar{r}, \bar{z}, \tau), \tag{3.34}$$

where

$$
\begin{aligned}
S(\bar r,\bar z,\tau)&=\eta\Big(\Delta W-\nabla\cdot(\varepsilon\nabla\Phi_W+W\nabla\Phi_\varepsilon+\varepsilon\nabla\Phi_\varepsilon+W\nabla\Phi_W)-\frac{1}{\bar r+R/\mu}(\varepsilon\partial_{\bar r}\Phi_W+W\partial_{\bar r}\Phi_\varepsilon+\varepsilon\partial_{\bar r}\Phi_\varepsilon\\
&\quad+W\partial_{\bar r}\Phi_W)+\frac{1}{\bar r+R/\mu}\partial_{\bar r}W-\frac12\Lambda W+\frac{R_\tau}{\mu}\partial_{\bar r}W-\partial_\tau W\Big)-\Delta\eta\varepsilon-2\nabla\varepsilon\cdot\nabla\eta-\frac{1}{\bar r+R/\mu}\partial_{\bar r}\eta\varepsilon\\
&\quad+\frac12\left((\bar r,\bar z)\cdot\nabla\eta\right)\varepsilon-\frac{R_\tau}{\mu}\partial_{\bar r}\eta\varepsilon\\
&=\eta\Big(-\nabla\cdot(\varepsilon\nabla\Phi_W+W\nabla\Phi_\varepsilon+\varepsilon\nabla\Phi_\varepsilon)-\frac{1}{\bar r+R/\mu}(\varepsilon\partial_{\bar r}\Phi_W+W\partial_{\bar r}\Phi_\varepsilon+\varepsilon\partial_{\bar r}\Phi_\varepsilon)+\mathrm{Mod}_0\varphi_{0,\nu}\\
&\quad+\mathrm{Mod}_1\varphi_{1,\nu}+\tilde E\Big)-\Delta\eta\varepsilon-2\nabla\varepsilon\cdot\nabla\eta-\frac{1}{\bar r+R/\mu}\partial_{\bar r}\eta\varepsilon+\frac12\left((\bar r,\bar z)\cdot\nabla\eta\right)\varepsilon-\frac{R_\tau}{\mu}\partial_{\bar r}\eta\varepsilon \qquad (3.35)
\end{aligned}
$$

Now, due to the parabolic scaling, it is natural to relate (3.34) to the standard heat equation. Denote

$$
\overline{u}(\mathbf{x},t):=\frac{1}{\mu^2(\tau)}\varepsilon_*\Big(\frac{r-R(\tau)}{\mu(\tau)},\frac{z}{\mu(\tau)},\tau\Big),\quad \mathbf{x}\in\mathbb{R}^3. \tag{3.36}
$$

Then, $\overline{u}$ solves the following heat equation with an axisymmetric force:

$$
\partial_t\overline{u}(\mathbf{x},t)=\Delta^{(3)}\overline{u}(\mathbf{x},t)+\frac{1}{\mu^4}S\Big(\frac{r-R}{\mu},\frac{z}{\mu},-\log(T-t)\Big). \tag{3.37}
$$

Now, we consider the evolution of (3.34) in the time interval $[\tau_0,\tau]$, or equivalently, that of (3.37) in $[t_0,t]$. Given the initial data $\overline{u}(\mathbf{x},t_0)=\overline{u}_0(\mathbf{x})$, the solution of (3.37) can be expressed by the convolution of the 3D heat kernel:

$$
\begin{aligned}
&\overline{u}(\mathbf{x},t)=\int_{\mathbb{R}^3}H(\mathbf{x}-\tilde{\mathbf{x}},t-t_0)\overline{u}_0(\tilde{\mathbf{x}})\,d\mathbf{x}\\
&+\int_0^{t-t_0}\int_{\mathbb{R}^3}H(\mathbf{x}-\tilde{\mathbf{x}},t-t_0-s)\frac{1}{\mu^4(s+t_0)}S\left(\frac{|(\tilde x_1,\tilde x_2)|-R(s+t_0)}{\mu(s+t_0)},\frac{\tilde x_3}{\mu(s+t_0)},-\log(T-t_0-s)\right)\,d\tilde{\mathbf{x}}ds,
\end{aligned}
$$

where

$$
H(\mathbf{x},t)=\frac{1}{(4\pi t)^{\frac32}}e^{-\frac{|\mathbf{x}|^2}{4t}}.
$$

Thus, we obtain an explicit expression for ε_* through the relation (3.36) ($\varepsilon_{*,0}(\bar r,\bar z):=\varepsilon_*(\bar r,\bar z,\tau_0)$):

$$
\begin{aligned}
&\varepsilon_*(\bar r,\bar z,\tau)=\\
&\frac{\mu^2(\tau)}{\mu^2(\tau_0)}\int_{-\infty}^{+\infty}\int_{-\frac{R(\tau_0)}{\mu(\tau_0)}}^{+\infty}\int_0^{2\pi}\frac{\mu^3(\tau_0)(\tilde p+\frac{R(\tau_0)}{\mu(\tau_0)})}{(4\pi(e^{-\tau_0}-e^{-\tau}))^{\frac32}}\exp\Bigg(-\frac{\mu^2(\tau_0)}{4(e^{-\tau_0}-e^{-\tau})}\Big[\Big(\frac{\mu(\tau)}{\mu(\tau_0)}(\bar r+\frac{R(\tau)}{\mu(\tau)})\\
&\quad-(\tilde p+\frac{R(\tau_0)}{\mu(\tau_0)})\cos(\theta)\Big)^2+\big(\tilde p+\frac{R(\tau_0)}{\mu(\tau_0)}\big)^2\sin^2(\theta)+\big(\frac{\mu(\tau)}{\mu(\tau_0)}\bar z-\tilde q\big)^2\Big]\Bigg)\varepsilon_{*,0}(\tilde p,\tilde q)\,d\theta d\tilde p d\tilde q\\
&+\int_{\tau_0}^{\tau}\frac{\mu^2(\tau)}{\mu^2(\tilde\tau)}d\tilde\tau\int_{-\infty}^{+\infty}\int_{-\frac{R(\tilde\tau)}{\mu(\tilde\tau)}}^{+\infty}\int_0^{2\pi}\frac{\mu^3(\tilde\tau)(\tilde p+\frac{R(\tilde\tau)}{\mu(\tilde\tau)})}{(4\pi(e^{-\tilde\tau}-e^{-\tau}))^{\frac32}}\exp\Bigg(-\frac{\mu^2(\tilde\tau)}{4(e^{-\tilde\tau}-e^{-\tau})}\Big[\Big(\frac{\mu(\tau)}{\mu(\tilde\tau)}(\bar r+\frac{R(\tau)}{\mu(\tau)})\\
&\quad-(\tilde p+\frac{R(\tilde\tau)}{\mu(\tilde\tau)})\cos(\theta)\Big)^2+\big(\tilde p+\frac{R(\tilde\tau)}{\mu(\tilde\tau)}\big)^2\sin^2(\theta)+\big(\frac{\mu(\tau)}{\mu(\tilde\tau)}\bar z-\tilde q\big)^2\Big]\Bigg)S(\tilde p,\tilde q,\tilde\tau)\,d\theta d\tilde p d\tilde q\\
&=:\frac{\mu^2(\tau)}{\mu^2(\tau_0)}\int_{\mathbb{R}^2}\widehat{H}(\bar r,\bar z,\tilde p,\tilde q,\tau,\tau_0)\varepsilon_{*,0}(\tilde p,\tilde q)\,d\tilde p d\tilde q+\int_{\tau_0}^{\tau}\frac{\mu^2(\tau)}{\mu^2(\tilde\tau)}d\tilde\tau\int_{\mathbb{R}^2}\widehat{H}(\bar r,\bar z,\tilde p,\tilde q,\tau,\tilde\tau)S(\tilde p,\tilde q,\tilde\tau)\,d\tilde p d\tilde q\\
&=:I_1(\bar r,\bar z,\tau)+I_2(\bar r,\bar z,\tau).
\end{aligned}
$$

The expression above is nothing but the convolution with the heat kernel written in cylindrical coordinates, i.e., $\widehat{H}$. In the following estimates, we make use of the two key properties of the heat kernel: total mass 1 (in $\mathbb{R}^3$) and exponential decay. As before, denote the time-dependent variable $\zeta := \sqrt{\bar{r}^2 + \bar{z}^2}$. Observe that

$$0 < \sigma(\tau, \tau_0) := \frac{e^{-\tau_0} - e^{-\tau}}{\mu^2(\tau_0)} = 1 - e^{\tau_0 - \tau} < 1.$$

First of all, when $\zeta(\tau) \leq 2\zeta^* \frac{\mu(\tau_0)}{\mu(\tau)}$, since the heat kernel has total mass 1, we have

$$|I_1(\bar{r}, \bar{z}, \tau)| \leq \frac{\mu^2(\tau)}{\mu^2(\tau_0)} \|\varepsilon_{*,0}\|_{L^\infty}.$$

Thus, in this domain we have

$$\|I_1(\bar{r}, \bar{z}, \tau)(1 + \zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta^* \leq \zeta \leq 2\zeta^* \mu(\tau_0)/\mu(\tau))} \leq 4\zeta^{*\frac{3}{2}} \frac{\mu^{\frac{1}{2}}(\tau)}{\mu^{\frac{1}{2}}(\tau_0)} \|\varepsilon_{*,0}\|_{L^\infty}. \tag{3.38}$$

Second, when $\zeta(\tau) > 2\zeta^* \frac{\mu(\tau_0)}{\mu(\tau)}$, i.e., $\zeta(\tau)\frac{\mu(\tau)}{\mu(\tau_0)} \geq 2\zeta^*$, denoting $B(\delta) \subset \mathbb{R}^2$ to be the ball centered at $(\frac{\mu(\tau)}{\mu(\tau_0)}\bar{r}, \frac{\mu(\tau)}{\mu(\tau_0)}\bar{z})$ with radius δ (to be determined), we split the integral into two parts:

$$\begin{aligned} I_1(\bar{r}, \bar{z}, \tau) &= \frac{\mu^2(\tau)}{\mu^2(\tau_0)} \int_{B(\delta)} \widehat{H}(\bar{r}, \bar{z}, \tilde{p}, \tilde{q}, \tau, \tau_0) \varepsilon_{*,0}(\tilde{p}, \tilde{q})\, d\tilde{p} d\tilde{q} + \frac{\mu^2(\tau)}{\mu^2(\tau_0)} \int_{\mathbb{R}^2 \setminus B(\delta)} \widehat{H}(\bar{r}, \bar{z}, \tilde{p}, \tilde{q}, \tau, \tau_0) \varepsilon_{*,0}(\tilde{p}, \tilde{q})\, d\tilde{p} d\tilde{q} \\ &=: J_1(\bar{r}, \bar{z}, \tau) + J_2(\bar{r}, \bar{z}, \tau). \end{aligned}$$

By the decay property of ε_*, we have the estimate

$$|J_1(\bar{r}, \bar{z}, \tau)| \leq \frac{\mu^2(\tau)}{\mu^2(\tau_0)} \frac{\|\varepsilon_*(\tau_0)(1 + \zeta(\tau_0))^{\frac{3}{2}}\|_{L^\infty(\zeta \geq \zeta^*)}}{(1 + \frac{\mu(\tau)}{\mu(\tau_0)}\zeta(\tau) - \delta)^{\frac{3}{2}}}. \tag{3.39}$$

Meanwhile, by the exponential decay of $\widehat{H}$, we have

$$|J_2(\bar{r}, \bar{z}, \tau)| \lesssim \frac{\mu^2(\tau)}{\mu^2(\tau_0)} \frac{\delta}{\sqrt{\sigma(\tau, \tau_0)}} e^{-\frac{\delta^2}{\sigma(\tau, \tau_0)}} \|\varepsilon_*(\tau_0)\|_{L^\infty}. \tag{3.40}$$

Therefore, when ζ^* is large enough, choosing $\delta := \sqrt{\zeta(\tau)\frac{\mu(\tau)}{\mu(\tau_0)}}$ and combining (3.39)(3.40), we have

$$|I_1(\bar{r}, \bar{z}, \tau)| \lesssim \frac{1}{(1 + \zeta(\tau))^{\frac{3}{2}}} \frac{\mu^{\frac{1}{2}}(\tau)}{\mu^{\frac{1}{2}}(\tau_0)} \left(\|\varepsilon(\tau_0)(1 + \zeta(\tau_0))^{\frac{3}{2}}\|_{L^\infty(\zeta \geq \zeta^*)} + \|\varepsilon_*(\tau_0)\|_{L^\infty} \right). \tag{3.41}$$

Besides, we note that for large $M_0 > 0$, by the bootstrap assumption it holds that

$$\begin{aligned} \frac{\mu(\tau)}{\mu(\tau_0)} = e^{-\frac{1}{2}(\tau - \tau_0)} &= \left(e^{-(\sqrt{\tau/2 + M_0} - \sqrt{\tau_0/2 + M_0})} \right)^{\sqrt{\tau/2 + M_0} + \sqrt{\tau_0/2 + M_0}} \\ &< \left(e^{-(\sqrt{\tau/2 + M_0} - \sqrt{\tau_0/2 + M_0})} \right)^{100} \leq C(K_1) \frac{\nu^{100}(\tau)}{\nu^{100}(\tau_0)}. \end{aligned} \tag{3.42}$$

In summary, by (3.38)(3.41), we have when ζ^* is sufficiently large:

$$\|I_1(\bar{r}, \bar{z}, \tau)(1 + \zeta(\tau))^{\frac{3}{2}}\|_{L^\infty(\zeta \geq \zeta^*)} \leq C(K_1) \frac{\nu^3(\tau)}{\nu^3(\tau_0)} \left(\|\varepsilon(\tau_0)(1 + \zeta(\tau_0))^{\frac{3}{2}}\|_{L^\infty(\zeta \geq \zeta^*)} + \|\varepsilon_*(\tau_0)\|_{L^\infty} \right),$$

where C is some universal constant. This completes the estimate for I_1.

Next, according to (3.35), we can decompose the source term as

$$S = \eta\nabla\cdot S_1 + \frac{1}{\bar{r}+R/\mu}S_2 + S_3 - 2\nabla\cdot(\varepsilon\nabla\eta),$$

where (plugging in the definition of $\tilde{E}$)

$$\begin{aligned}
S_1 &= -\varepsilon\nabla\Phi_W - W\nabla\Phi_\varepsilon - \varepsilon\nabla\Phi_\varepsilon - W\nabla\Theta_W - P\nabla\Psi_P,\\
S_2 &= -\eta(\varepsilon\partial_{\bar{r}}\Phi_W + W\partial_{\bar{r}}\Phi_\varepsilon + \varepsilon\partial_{\bar{r}}\Phi_\varepsilon) - \eta(1-\chi(\zeta\nu))(\partial_{\bar{r}}W + W\partial_{\bar{r}}\Phi_W) - \partial_{\bar{r}}\eta\varepsilon,\\
S_3 &= \eta\mathrm{Mod}_0\varphi_{0,\nu} + \eta\mathrm{Mod}_1\varphi_{1,\nu} - \eta a\nu_\tau\nu\partial_\nu(\varphi_{1,\nu}-\varphi_{0,\nu}) + \eta\left(\frac{\nu_\tau}{\nu}-\beta\right)(\Lambda U_\nu + 16\nu^2\varphi_{0,\nu})\\
&\quad + \eta\frac{R_\tau}{\mu}\partial_{\bar{r}}W + \eta a(R_1-R_0) + \Delta\eta\varepsilon + \frac{1}{2}((\bar{r},\bar{z})\cdot\nabla\eta)\varepsilon - \frac{R_\tau}{\mu}\partial_{\bar{r}}\eta\varepsilon + \frac{\eta\chi(\zeta\nu)}{\bar{r}+R/\mu}(\partial_{\bar{r}}W + W\partial_{\bar{r}}\Phi_W).
\end{aligned}$$

By the bootstrap assumption, the pointwise estimates in Proposition 1, the Poisson field estimates (A.3)(A.4) (A.16)(A.17), and the L^∞ control on the boundary $\|\varepsilon\|_{L^\infty(\zeta^*\le\zeta\le2\zeta^*)} \lesssim \|\varepsilon\|_{H^2(\frac{1}{2}\zeta^*\le\zeta\le4\zeta^*)} \lesssim C(\zeta^*)\frac{K_6\nu^2}{|\log\nu|}$, we obtain the pointwise estimates

$$\begin{aligned}
&\|S_1(\bar{r},\bar{z},\tau)(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta\ge\zeta^*)} \le \frac{C}{\zeta^*}\|\varepsilon(\tau)(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta\ge\zeta^*)} + C(K_i)\nu(\tau)^3,\\
&\|S_2(\bar{r},\bar{z},\tau)(1+\zeta)^{\frac{3}{2}}\|_{L^\infty} \le \frac{C}{\zeta^*}\|\varepsilon(\tau)(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta\ge\zeta^*)} + C(\zeta^*)\frac{K_6\nu^2}{|\log\nu|} + C(K_i)\nu(\tau)^3,\\
&\|S_3(\bar{r},\bar{z},\tau)(1+\zeta)^{\frac{3}{2}}\|_{L^\infty} \le \frac{C|\mathrm{Mod}_0|}{\zeta^*} + C(\zeta^*)\frac{K_6\nu^2}{|\log\nu|} + \frac{C(K_i)\nu(\tau)^2}{|\log\nu(\tau)|^2}.
\end{aligned} \tag{3.43}$$

Through integration by parts,

$$\begin{aligned}
I_2(\bar{r},\bar{z},\tau) &= \int_{\tau_0}^{\tau}\frac{\mu^{\frac{1}{4}}(\tau)}{\mu^{\frac{1}{4}}(\tilde{\tau})}\cdot\frac{\mu^{\frac{7}{4}}(\tau)}{\mu^{\frac{7}{4}}(\tilde{\tau})}\,d\tilde{\tau}\int_{\mathbb{R}^2}\widehat{H}(S_3-\nabla\eta\cdot S_1)\,d\tilde{p}d\tilde{q}\\
&\quad + \int_{\tau_0}^{\tau}\frac{\mu^{\frac{1}{4}}(\tau)}{\sqrt{\sigma(\tau,\tilde{\tau})}\mu^{\frac{1}{4}}(\tilde{\tau})}\cdot\frac{\mu^{\frac{7}{4}}(\tau)}{\mu^{\frac{7}{4}}(\tilde{\tau})}\,d\tilde{\tau}\int_{\mathbb{R}^2}\frac{\sqrt{\sigma(\tau,\tilde{\tau})}\widehat{H}}{\tilde{p}+R(\tilde{\tau})/\mu(\tilde{\tau})}S_2 - \sqrt{\sigma(\tau,\tilde{\tau})}\nabla\widehat{H}\cdot(\eta S_1 - 2\nabla\eta\varepsilon)\,d\tilde{p}d\tilde{q}\\
&=: \int_{\tau_0}^{\tau}\frac{\mu^{\frac{1}{4}}(\tau)}{\mu^{\frac{1}{4}}(\tilde{\tau})}I_{2,a}(\bar{r},\bar{z},\tau,\tilde{\tau})\,d\tilde{\tau} + \int_{\tau_0}^{\tau}\frac{\mu^{\frac{1}{4}}(\tau)}{\sqrt{\sigma(\tau,\tilde{\tau})}\mu^{\frac{1}{4}}(\tilde{\tau})}I_{2,b}(\bar{r},\bar{z},\tau,\tilde{\tau})\,d\tilde{\tau}.
\end{aligned}$$

Observe that kernels $\frac{\sqrt{\sigma(\tau,\tilde{\tau})}\widehat{H}}{\tilde{p}+R(\tilde{\tau})/\mu(\tilde{\tau})}$ and $\sqrt{\sigma(\tau,\tilde{\tau})}\nabla\widehat{H}$ share similar properties with $\widehat{H}$: bounded total mass and exponential decay, which are all we need in deriving the estimate for I_1. Thus, by (3.42) and (3.43), with a similar argument we can show that

$$\begin{aligned}
&\|I_{2,a}(\bar{r},\bar{z},\tau,\tilde{\tau})(1+\zeta(\tau))^{\frac{3}{2}}\|_{L^\infty(\zeta\ge\zeta^*)}\\
&\le Ce^{4\sqrt{\beta\tilde{\tau}+M_0}-4\sqrt{\beta\tau+M_0}}\left(\|S_3(\tilde{\tau})(1+\zeta(\tilde{\tau}))^{\frac{3}{2}}\|_{L^\infty} + \|S_1(\tilde{\tau})(1+\zeta(\tilde{\tau}))^{\frac{3}{2}}\|_{L^\infty(\zeta\ge\zeta^*)}\right)\\
&\le \frac{CK_7}{\zeta^*}\frac{e^{-2\sqrt{\beta\tau+M_0}}}{\sqrt{\beta\tau+M_0}} + \frac{C(\zeta^*,K_1,K_4,K_5,K_6)e^{-2\sqrt{\beta\tau+M_0}}}{\sqrt{\beta\tau+M_0}} + \frac{C(\zeta^*,K_i)e^{-2\sqrt{\beta\tau+M_0}}}{\beta\tau+M_0},
\end{aligned}$$

and

$$\begin{aligned}
&\|I_{2,b}(\bar{r},\bar{z},\tau,\tilde{\tau})(1+\zeta(\tau))^{\frac{3}{2}}\|_{L^\infty(\zeta\ge\zeta^*)}\\
&\le Ce^{4\sqrt{\beta\tilde{\tau}+M_0}-4\sqrt{\beta\tau+M_0}}\left(\|S_2(\tilde{\tau})(1+\zeta(\tilde{\tau}))^{\frac{3}{2}}\|_{L^\infty} + \|(\eta S_1(\tilde{\tau})-2\nabla\eta\varepsilon(\tilde{\tau}))(1+\zeta(\tilde{\tau}))^{\frac{3}{2}}\|_{L^\infty}\right)\\
&\le \frac{CK_7}{\zeta^*}\frac{e^{-2\sqrt{\beta\tau+M_0}}}{\sqrt{\beta\tau+M_0}} + \frac{C(\zeta^*,K_1,K_6)e^{-2\sqrt{\beta\tau+M_0}}}{\sqrt{\beta\tau+M_0}} + C(K_i)\nu^3.
\end{aligned}$$

Finally, it remains to estimate the time integrals:

$$\int_{\tau_0}^{\tau} \frac{\mu^{\frac{1}{4}}(\tau)}{\mu^{\frac{1}{4}}(\tilde{\tau})}\, d\tilde{\tau} = \int_{\tau_0}^{\tau} e^{-\frac{1}{8}(\tau-\tilde{\tau})}\, d\tilde{\tau} < 8,$$

and

$$\begin{aligned}\int_{\tau_0}^{\tau} \frac{\mu^{\frac{1}{4}}(\tau)}{\sqrt{\sigma(\tau,\tilde{\tau})}\mu^{\frac{1}{4}}(\tilde{\tau})}\, d\tilde{\tau} = \int_{\tau_0}^{\tau} \frac{e^{-\frac{1}{8}(\tau-\tilde{\tau})}}{\sqrt{1-e^{\tilde{\tau}-\tau}}}\, d\tilde{\tau} &= \int_{t_0}^{t} \frac{(T-t)^{\frac{1}{8}}}{(T-\tilde{t})^{\frac{5}{8}}} \frac{1}{\sqrt{t-\tilde{t}}}\, d\tilde{t} \\ &\leq \int_{t-1}^{t} \frac{(T-t)^{\frac{1}{8}}}{(T-\tilde{t})^{\frac{5}{8}}} \frac{1}{\sqrt{t-\tilde{t}}}\, d\tilde{t} \leq 10 + \mathcal{O}\left((T-t)^{\frac{1}{8}}\right).\end{aligned}$$

This completes the proof of the Lemma. □

4 Existence of Blowup Solutions

Now with the energy estimates and modulation equations, we are ready to prove the existence of blowup solutions. It suffices to show that there exist certain initial data $(\varepsilon_0, \nu_0, a_0, R_0)$ and parameters $\zeta^*, K_1, \ldots, K_7$, such that the evolution (3.2) will be trapped in some bootstrap regime $\mathrm{BS}(K_i : 1 \leq i \leq 7)$ for all $\tau \in [0, +\infty)$. Roughly speaking, ζ^* is chosen first which depends on some of the universal constants in the estimates, then K_i (the order of dependence among which will be specified shortly), and finally ν_0 (or equivalently, M_0 in the bootstrap statement), so that $C(\zeta^*, K_i)/|\log \nu|$ will have the smallness we want.

The following lemma will help us close the bootstrap for ε.

Lemma 8. *Let $f(\tau) \geq 0$ be a differentiable function in τ. Let $\nu(\tau)$ be the parameter in the Bootstrap regime $\mathrm{BS}(\tau_0, \tau_*, \zeta^*, M_0, \{K_i\}_{i=1}^7)$. Suppose we have the following differential inequality*

$$f'(\tau) \leq -\delta f(\tau) + \frac{K\nu^k(\tau)}{|\log \nu(\tau)|^2},$$

for some constants $\delta, K > 0$ and $k > 1$. Then, there exists constant $C(\delta, k), \bar{M} > 0$, such that for any $\log(\nu(0))^2 = M_0 > \bar{M}$, we have

$$|f(\tau)| \leq \frac{K\nu^k(\tau)}{\delta|\log \nu|^2} + \frac{KC(\delta, k, K_1, K_2, K_4, K_5)\nu^k}{|\log \nu|^3} + f(0)e^{-\delta\tau}, \tag{4.1}$$

holds for any $\tau \in [0, \tau_*]$.

Proof. The proof applies Gronwall's inequality and integration by parts. First, we have by Gronwall's inequality,

$$f(\tau) \leq e^{-\delta\tau} f(0) + e^{-\delta\tau} \int_0^{\tau} \frac{K\nu^k(s)}{|\log \nu(s)|^2} e^{\delta s}\, ds. \tag{4.2}$$

Through integration by parts, we have

$$\int_0^{\tau} \frac{K\nu^k(s)}{|\log \nu(s)|^2} e^{\delta s}\, ds = \frac{e^{\delta\tau}}{\delta} \frac{K\nu^k(\tau)}{|\log \nu(\tau)|^2} - \frac{K\nu^k(0)}{|\log \nu(0)|^2} - \frac{K}{\delta} \int_0^{\tau} \frac{\nu'(s)}{\nu(s)} \left(\frac{k\nu^k(s)}{|\log \nu(s)|^2} - \frac{2\nu^k(s)}{(\log \nu(s))^3}\right) e^{\delta s}\, ds. \tag{4.3}$$

Then, by the estimate of $|\frac{\nu'}{\nu}|$ in (3.18) and the bootstrap assumption for ν, we have

$$\left|\frac{K}{\delta} \int_0^{\tau} \frac{\nu'(s)}{\nu(s)} \left(\frac{k\nu^k(s)}{|\log \nu(s)|^2} - \frac{2\nu^k(s)}{(\log \nu(s))^3}\right) e^{\delta s}\, ds\right| \leq \frac{KkC(K_1, K_2, K_4, K_5)}{\delta} \int_0^{\tau} \frac{e^{\delta s - k\sqrt{\beta s + M_0}}}{(\beta s + M_0)^{\frac{3}{2}}}\, ds \tag{4.4}$$

By change of variables ($x := \sqrt{\beta s + M_0}$) and integration by parts,

$$\begin{aligned}\int_0^\tau \frac{e^{\delta s - k\sqrt{\beta s + M_0}}}{(\beta s + M_0)^{\frac{3}{2}}}\, ds &= \int_{\sqrt{M_0}}^{\sqrt{\beta\tau + M_0}} \frac{2}{\beta x^2} e^{\frac{\delta}{\beta}x^2 - kx - \frac{\delta}{\beta}M_0}; dx \\ &= \frac{e^{\frac{\delta}{\beta}x^2 - kx - \frac{\delta}{\beta}M_0}}{\delta x^2 (x - \frac{\beta k}{2\delta})}\Bigg|_{x=\sqrt{M_0}}^{\sqrt{\beta\tau+M_0}} - \int_{\sqrt{M_0}}^{\sqrt{\beta\tau+M_0}} \left(\frac{1}{\delta x^2(x - \frac{\beta k}{2\delta})}\right)' e^{\frac{\delta}{\beta}x^2 - kx - \frac{\delta}{\beta}M_0}\, dx \\ &\le \frac{Ce^{\delta\tau - k\sqrt{\beta\tau + M_0}}}{\delta(\beta\tau + M_0))^{\frac{3}{2}}} + \frac{C}{M_0\delta}\int_{\sqrt{M_0}}^{\sqrt{\beta\tau+M_0}} \frac{2}{\beta x^2} e^{\frac{\delta}{\beta}x^2 - kx - \frac{\delta}{\beta}M_0}; dx\end{aligned}$$

Therefore, when M_0 is sufficiently large, we have

$$\int_0^\tau \frac{e^{\delta s - k\sqrt{\beta s + M_0}}}{(\beta s + M_0)^{\frac{3}{2}}}\, ds \le \frac{Ce^{\delta\tau - k\sqrt{\beta\tau + M_0}}}{\delta(\beta\tau + M_0))^{\frac{3}{2}}}. \tag{4.5}$$

Finally, inserting (4.5), (4.4) and (4.3) back into (4.2), we obtain the result. □

Now we are ready to prove the main proposition, which will conclude the proof of Theorem 1.

Proposition 5. *There exist a choice of parameters $(\zeta^*, M_0, \{K_i\}_{i=1}^7)$ and initial data for w, such that the solution w of (1.4) will be trapped in the bootstrap regime* $\mathrm{BS}(0, +\infty, \zeta^*, M_0, \{K_i\}_{i=1}^7)$.

Proof. The proof proceeds as follows. First by specifying the dependence on the parameters $(\zeta^*, \{K_i\}_{i=1}^\nu, M_0)$, we exploit the energy estimates to show that the remainder ε will always be trapped in the bootstrap regime, given sufficiently small initial data. Then, as for the modulation parameters, the main part is to apply a topological argument to show the existence of an initial $a(0)$ such that the parameter $a(\tau)$ will remain trapped in the bootstrap regime for all time. The rest part (ν and R_τ/μ) follows directly from the $|\mathrm{Mod}_i|$ estimates and time integration.

Trapping ε: Suppose w is a solution trapped in some bootstrap regime $\mathrm{BS}(\tau_0, \tau_*, \zeta^*, M_0, \{K_i\}_{i=1}^7)$. Since the parameter M_0 is chosen at last to make $C(\zeta^*, K_i)/|\log\nu|$ arbitrarily small, it suffices to keep track of only the leading order terms in the energy estimates. In the following, when we say "M_0" is large enough, it means M_0 is chosen large depending on ζ^* and all K_i's. First of all, choose $\zeta^* \gg C$ and $K_7 \gg \zeta^* C(\zeta^*, K_1, K_4, K_5, K_6)$ for the constants $C(\zeta^*, K_1, K_4, K_5, K_6)$ in (3.33). Then, for small enough initial $\varepsilon(0)$ (e.g., $\|\varepsilon(0)(1+\zeta^{\frac{3}{2}})\|_{L^\infty(\zeta\ge\zeta^*)} + \|\varepsilon(0)\|_{L^\infty(\zeta\ge\frac{1}{2}\zeta^*)} \lesssim \nu^3(0)$ suffices) that is even in z-variable and satisfies the orthogonality conditions (3.3) and large enough M_0, by (3.33), we have

$$\|\varepsilon(\tau)(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta\ge\zeta^*)} \le \frac{2CK_7}{\zeta^*}\frac{e^{-2\sqrt{\beta\tau+M_0}}}{\sqrt{\beta\tau+M_0}} \le \frac{K_7 e^{-2\sqrt{\beta\tau+M_0}}}{2\sqrt{\beta\tau+M_0}}, \quad \forall\, \tau \in [0, \tau_*]. \tag{4.6}$$

As for the middle range H^2-estimate, by (4.1), $|\mathrm{Mod}_0|$ estimate (3.9) and Lemma 6 (taking $\zeta_1 = \frac{1}{4}\zeta^*$ and $\zeta_2 = 2\zeta^*$), we have

$$\|\varepsilon\|_{H^2(\frac{1}{4}\zeta^*\le\zeta\le2\zeta^*)} \le C(\zeta^*)\|\varepsilon\|_{H^2_*(\frac{1}{4}\zeta^*, 2\zeta^*)} \le \frac{C(\zeta^*)(K_4 + \sqrt{K_4K_5})\nu^2}{|\log\nu|} \le \frac{K_6\nu^2}{2|\log\nu|} \quad \forall\, \tau \in [0, \tau_*], \tag{4.7}$$

once we take $K_6^2 \gg C(\zeta^*)(K_4^2 + K_4K_5)$ and $\varepsilon(0)$ to be small enough. For the H^1 estimate, define

$$\|\varepsilon\|_{H^1_\nu}^2 := K^2\left(\frac{1}{2}\int \varepsilon\chi_\nu\sqrt{\varrho_\nu}\mathscr{M}_\nu^\zeta(\varepsilon\chi_\nu\sqrt{\varrho_\nu}) - \frac{d_0}{2}\int \varepsilon\sqrt{\varrho_\nu}\mathscr{M}_\nu^\zeta(\varphi_{0,\nu}\sqrt{\varrho_\nu}\chi_\nu)\right) + \frac{1}{2}\int U_\nu|\nabla\mathscr{M}_\nu^\zeta(\varepsilon^*)|^2,$$

where $K \gg 1$ is to be determined. Then, by Lemma 4, Lemma 5 ,(4.6), and the equivalence of norms, we have

$$\begin{aligned}\frac{d}{d\tau}\|\varepsilon\|_{H^1_\nu}^2 &\le -\frac{(\delta_0K^2 - C)}{\nu^2}\left(\|\varepsilon\|_{\mathrm{in}}^2 + \|\nabla\varepsilon\|_{\mathrm{in}}^2\right) + \frac{CK^2\nu^2}{|\log\nu|^2} + \frac{C(K_1)K_7^2}{(\zeta^*)^2}\frac{\nu^2}{|\log\nu|^2} \\ &\le -\frac{(\delta_0K^2 - C)}{C(\zeta^*)}\|\varepsilon\|_{H^1_\nu}^2 + \frac{CK^2\nu^2}{|\log\nu|^2} + \frac{C(K_1)K_7^2}{(\zeta^*)^2}\frac{\nu^2}{|\log\nu|^2}.\end{aligned}$$

Thus, we can choose $K^2 = C(K_1)K_7^2/K_4^2 \gg 1$, and by (4.1) we obtain

$$\frac{1}{\nu^2}\left(K^2\|\varepsilon\|_{\text{in}}^2 + \|\nabla\varepsilon^*\|_{L^2(U_\nu)}^2\right) \le C\|\varepsilon\|_{H^1_\nu}^2 \le \frac{C(\zeta^*)\nu^2}{\delta_0|\log\nu|^2} + \frac{C(\zeta^*)K_4^2\nu^2}{\delta_0|\log\nu|^2} \quad \forall\,\tau\in[0,\tau_*],$$

when M_0 is sufficiently large and $\varepsilon(0)$ is sufficiently small. Thus, choosing $K_5^2 \gg \frac{C(\zeta^*)K_4^2}{\delta_0}$ and $K_4^2 \gg \frac{C(\zeta^*)}{\delta_0}$ above, we obtain

$$\|\varepsilon\|_{\text{in}}^2 \le \frac{K_4^2\nu^4}{2|\log\nu|^2}, \quad \|\nabla\varepsilon^*\|_{L^2(U_\nu)}^2 \le \frac{K_5^2\nu^4}{2|\log\nu|^2}, \quad \forall\,\tau\in[0,\tau_*]. \tag{4.8}$$

Collecting (4.6), (4.7) and (4.8), we see that all the bootstrap constants are improved by a factor of $\frac{1}{2}$. As a summary, it is helpful to recap the dependence of the parameters:

$$\begin{aligned} K_7 \gg C(\zeta^*, K_1, K_4, K_5, K_6) \Longrightarrow K_5 \gg C(\zeta^*)K_6 \Longrightarrow K_6 \gg C(\zeta^*)\sqrt{K_4K_5} \Longrightarrow K_5 \gg \frac{C(\zeta^*)}{\sqrt{\delta_0}}K_4 \\ \Longrightarrow K_4 \gg \frac{C(\zeta^*)}{\sqrt{\delta_0}} \Longrightarrow \zeta^* \gg C. \end{aligned} \tag{4.9}$$

Trapping modulation parameters: First of all, simply choosing $K_3 \gg C(K_4, K_5)$ in (3.9), we have

$$\left|\frac{R_\tau}{\mu}\right| \le \frac{C(K_4,K_5)\nu}{|\log\nu|} \le \frac{K_3\nu}{2|\log\nu|}.$$

Denote $\tilde a := a - 8\nu^2$, and the bootstrap assumption on a is equivalent to $|\tilde a| \le \frac{K_2\nu^2}{|\log\nu|}$. Then, inserting the decomposition into the Mod_1-equation gives

$$\left|\frac{\nu_\tau}{\nu} - \frac{\beta}{2\log\nu} + \frac{\tilde a_\tau}{16\nu^2}\right| \le \frac{C(K_2,K_4,K_5)}{|\log\nu|^2}.$$

It follows that

$$\frac{d}{d\tau}(\log^2\nu) = \beta - \frac{\tilde a_\tau \log\nu}{8\nu^2} + \mathcal{O}\left(\frac{C(K_2,K_4,K_5)}{|\log\nu|}\right).$$

Integrating in $[0,\tau]$, we have

$$(\log\nu(\tau))^2 = (\log\nu(0))^2 + \beta\tau - \int_0^\tau \frac{\tilde a_\tau(s)\log\nu(s)}{8\nu^2(s)}\,ds + \mathcal{O}\left(\int_0^\tau \frac{C(K_2,K_4,K_5)}{|\log\nu|}\,ds\right). \tag{4.10}$$

Note, through integration by parts, that

$$\begin{aligned} \int_0^\tau \frac{\tilde a_\tau(s)\log\nu(s)}{8\nu^2(s)}\,ds &= \frac{\tilde a(\tau)\log\nu(\tau)}{8\nu^2(\tau)} - \frac{\tilde a(0)\log\nu(0)}{8\nu^2(0)} - \int_0^\tau \tilde a(s)\left(\frac{\log\nu(s)}{8\nu^2(s)}\right)'\,ds \\ &= \mathcal{O}(K_2 + C(K_2,K_4,K_5)\sqrt{\beta\tau}). \end{aligned}$$

where we use (3.18) and

$$\int_0^\tau \frac{1}{|\log\nu(s)|}\,ds \lesssim \sqrt{\beta\tau + M_0} - \sqrt{M_0} \lesssim \sqrt{\beta\tau},$$

when M_0 is large enough. Therefore, once we choose $K_1 \gg C(K_2,K_4,K_5)$, by (4.10) and the estimate above we obtain

$$\frac{2}{K_1}e^{-\sqrt{\beta\tau+M_0}} \le e^{-\sqrt{\beta\tau+M_0+\mathcal{O}(C(K_2,K_4,K_5)(1+\sqrt{\beta\tau}))}} \le \frac{K_1}{2}e^{-\sqrt{\beta\tau+M_0}}.$$

Finally, we control $a(\tau)$. By the $|\text{Mod}_0|$-estimate in (3.9) and the decomposition $a = 8\nu^2 + \tilde a$, we have

$$|\tilde a_\tau - 2\beta\tilde a| \le \frac{C(K_4,K_5)\nu^2}{|\log\nu|}. \tag{4.11}$$

We choose $K_2 \gg C(K_4, K_5)$, and claim that there exits an initial $\tilde{a}(0) \in [-\frac{K_2\nu^2(0)}{|\log \nu(0)|}, \frac{K_2\nu^2(0)}{|\log \nu(0)|}]$, such that a will be trapped for all time. Before proving this claim, we summarize the dependence of these parameters:

$$K_1 \gg C(K_2, K_4, K_5) \Longrightarrow K_2 \gg C(K_4, K_5),\ K_3 \gg C(K_4, K_5). \tag{4.12}$$

Combining (4.9) and (4.12), it is clear that there exist parameters ζ^* and $\{K_i\}_{i=1}^7$ that satisfy all these constraints. Now we prove the claim. First of all, we fix the parameters $(\zeta^*, \{K_i\}_{i=1}^7)$ and other initial conditions according to the aforementioned discussion. Then, suppose, for contradiction, that for any $a_0 := \tilde{a}(0) \in [-\frac{K_2\nu^2(0)}{|\log \nu(0)|}, \frac{K_2\nu^2(0)}{|\log \nu(0)|}]$, the corresponding solution $w_{\tilde{a}}$ will exit the bootstrap regime in finite time. We denote the supremum of time that w_{a_0} stays in the bootstrap regime by τ_{a_0}. Note that when $\tau = \tau_{a_0}$, since all other bootstrap constants are improved by $\frac{1}{2}$, we have $|\tilde{a}(\tau_{a_0})| = \frac{K_2\nu^2(\tau_{a_0})}{|\log \nu(\tau_{a_0})|}$. Denote $I := [-K_2, K_2]$, then we obtain a map $\psi : I \to \{-K_2, K_2\}$, defined as

$$\psi : \frac{a_0|\log \nu(0)|}{\nu^2(0)} \mapsto \frac{\tilde{a}(\tau_{a_0})|\log \nu(\tau_{a_0})|}{\nu^2(\tau_{a_0})}.$$

First of all, this map is continuous due to the standard continuous dependence of the solutions on the initial data. Second, if $|\tilde{a}(\tau)| = \frac{K_2\nu^2(\tau)}{|\log \nu(\tau)|}$ for any time τ, by (4.11), we have

$$\tilde{a}_\tau(\tau) = 2\beta\tilde{a}(\tau) + \mathcal{O}\left(\frac{C(K_4, K_5)\nu^2(\tau)}{|\log \nu(\tau)|}\right).$$

Thus, $\tilde{a}(\tau)$ is nonzero and have the same sign as $\tilde{a}(\tau)$ (since we choose $K_2 \gg C(K_4, K_5)$). Then, for any $\tau' > \tau$ (and $|\tau' - \tau|$ suitably small), we will have $|\tilde{a}(\tau')| > \frac{K_2\nu^2(\tau')}{|\log \nu(\tau')|}$, existing the bootstrap regime. As a consequence, ψ is the identity map when restricting on $\partial I = \{-K_2, K_2\}$. However, now since $I = \psi^{-1}(\{K_2\}) \cap \psi^{-1}(\{-K_2\})$ which is the union of two disjoint nonempty (as $\psi(\pm K_2) = \pm K_2$) open sets, this gives a contradiction as I is topologically connected (this is in fact a special case of the Brouwer fixed point theorem). ☐

Theorem 1 is directly implied by Proposition 5, except for the part $\|\tilde{u}(t)\|_{\mathcal{E}} \to 0$ as $t \to T$. This is because the inner region (which is controlled by an H^1 norm) and outer region (which is controlled by an L^∞ norm) for the perturbation ε in the bootstrap assumptions is divided at the parabolic scale ($\zeta \sim 1$), while the definition of $\|\cdot\|_{\mathcal{E}}$ makes such division in the soliton scale ($\zeta \sim \nu$). However, the former can easily imply the latter. To see this, note that by Lemma 6 (and the proof therein) and Lemma 8, we have $\|\varepsilon\|_{H^2(\nu\le\zeta\le\zeta^*)} = \mathcal{O}(|\log \nu|^{-1})$. Thus, by the Sobolev embedding $H^2 \hookrightarrow L^\infty$ in 2D, we have $\|\varepsilon\|_{L^\infty(\nu\le\zeta\le\zeta^*)} = \mathcal{O}(|\log \nu|^{-1})$. Combining with the far field L^∞ control of ε, we obtain $\|\varepsilon(1+\zeta)^{\frac{3}{2}}\|_{L^\infty(\zeta\ge\nu)} = \mathcal{O}(|\log \nu|^{-1})$. It follows that $\|q(1+\gamma)^{\frac{3}{2}}\|_{L^\infty(\gamma\ge1)} = \mathcal{O}(\sqrt{\nu}/|\log \nu|) \to 0$ as $t \to T$ (we recall $q(\rho, \xi, t) = \nu^2\varepsilon(\nu\rho, \nu\xi, t)$), so that we actually have the weighted L^∞ control of $\tilde{u}$ (the notation in Theorem 1) from the soliton scale, i.e., $\gamma \ge 1$. This completes the proof of Theorem 1.

A Appendix: Estimates and inequalities

Estimates of the Poisson Fields

On $\mathbb{R}^3$, we denote the cylindrical coordinate by (r, θ, z), and denote $\tilde{r} := r - R/\mu$ where we assume $0 < \mu \ll 1$ to be a small parameter tending to zero. Let $u = u(\tilde{r}, z)$ be an axisymmetric function centered around the ring with radius R/μ on the plane $z = 0$. Let $\Phi_u = \frac{1}{4\pi|\mathbf{x}|} * u$ be its 3D Poisson field. Then, we are to define a 2D Poisson field as an approximation to the 3D one in a sense that will soon become cleer. Define a function on $(\tilde{r}, z) \in \mathbb{R}^2$ by

$$\bar{u}(\tilde{r}, z) := \begin{cases} u(\tilde{r}, z), & \tilde{r} \ge -R/\mu, \\ u(-2R/\mu - \tilde{r}, z), & \tilde{r} < -R/\mu. \end{cases}$$

Let $\eta(x)$ be a smooth cutoff function in 1D, such that

$$0 \le \eta(x) \le 1, \quad \eta(x) \equiv 1 \quad \forall\, x \in [0, +\infty), \quad \text{and } \eta(x) \equiv 0 \quad \forall\, x \in (-\infty, -1].$$

Now, define $u_*(\tilde{r}, z) := \eta(r)\bar{u}(\tilde{r}, z)$, which is a smooth extension of u to $\mathbb{R}^2$, and

$$\Psi_u := -\frac{1}{2\pi} \log |(\tilde{r}, z)| * u_*. \tag{A.1}$$

One remark is that the choice of definition for the 2D Poisson field of u is not unique. In the following analysis, we will see that what matters is that Ψ_u solves the Poisson equation on the half plane:

$$-(\partial_{\tilde{r}}^2 + \partial_z^2)\Psi(\tilde{r}, z) = u(\tilde{r}, z), \quad \forall\, (\tilde{r}, z) \in [-R/\mu, +\infty) \times \mathbb{R},$$

and $\nabla\Phi_u$ has certain decay property. In general, it works if we extend u to a function on the whole $\mathbb{R}^2$ with suitably small modification, and then consider its convolution with the 2D fundamental solution to the Laplace equation. In our case, since u_* and u differ little in L^p norms when viewed as $\mathbb{R}^2$-functions (with an extension by zero for u) estimates of Ψ_u follow from those of u. The following lemma illustrates that the difference between $\nabla\Psi_u(\tilde{r}, z)$ and $\nabla\Phi_u(\tilde{r}, z)$ can be controlled pointwisely when $u(\bar{r}, z)$ satisfies certain decay property.

Lemma 9 (Difference of 2D and 3D Poisson fields)**.** *Assume $u(r, z)$ to be a function with suitable decay in $\mathbb{R}^2$ and denote its 2D Poisson field by $\Psi_u := -\frac{1}{2\pi} \log |(r, z)| * u$. We can also interpret $u(r, z)$, $(r, z) \in \mathbb{H} := \{(r, z) : r > 0\}$, to be an axisymmetric function in $\mathbb{R}^3$, in which case we can define its 3D Poisson field by $\Phi_u := \frac{1}{4\pi|\mathbf{x}|} * u$ ($\mathbf{x} \in \mathbb{R}^3$). Denote by $B_{(x,y)}(l) \subset \mathbb{R}^2$ the ball centered at (x, y) with radius $l > 0$. Assume the following decay property of $u(r, z)$:*

$$\begin{aligned} &\|u(r, z)(1 + (r - R/\mu)^2 + z^2)^{\frac{3}{4}}\|_{L^\infty(\mathbb{R}^2 \backslash B_{(R/\mu,0)})(\zeta^*)} \le L_\infty, \\ &\|u(r, z)\|_{L^2(B_{(R/\mu,0)}(\zeta^*))} \le L_2, \quad \|\nabla u(r, z)\|_{L^2(B_{(R/\mu,0)}(\zeta^*))} \le L_2'. \end{aligned} \tag{A.2}$$

where $\zeta^* > 0$ is fixed, and L_∞, L_2, $L_2' > 0$ are some constants. Then, there exists $\mu^* > 0$ such that for any $0 < \mu < \mu^*$, it holds that given any $\frac{6}{7} < s < 1$ we have the following estimates on the gradients of Poisson fields:
(i) (Near field approximation)

$$|\nabla\Psi_u(r, z) - \nabla\Phi_u(r, z)| \le C_1(L_\infty + L_2 + L_2')\mu^\kappa, \quad \forall\, (r, z) \in B_{(R/\mu,0)}(\mu^{-s}), \tag{A.3}$$

(ii) (Far-field control)

$$|\nabla\Phi_u(r, z)| \le C_2(L_\infty + L_2)\mu^\kappa, \quad \forall\, (r, z) \in \mathbb{H} \backslash B_{(R/\mu,0)}(\mu^{-s}). \tag{A.4}$$

Here $\kappa > 0$, and κ, C_1, C_2 are all universal constants depending only on s.

Proof. The strategy of the proof is to exploit the explicit expressions of the Poisson fields, especially its decay behaviors in the far field. A key observation is that: away from the axis of symmetry the fundamental solution of the 3D axisymmetric Poisson equation is asymptotically close to that of the 2D Poisson equation. We also note that, by our construction, u_* shares the same control (up to a universal constant) of u as in (A.2).
Proof of (i). First of all, the Poisson fields are expressed as

$$\nabla\Psi_u(r, z) = \int_{\mathbb{R}^2} \nabla E_2(r, z, \tilde{r}, \tilde{z}) u(\tilde{r}, \tilde{z})\, d\tilde{r} d\tilde{z}, \quad \nabla\Phi_u(r, z) = \int_{\mathbb{H}} \nabla E_3(r, z, \tilde{r}, \tilde{z}) u(\tilde{r}, \tilde{z})\, d\tilde{r} d\tilde{z},$$

where

$$\begin{aligned} \nabla E_2(r, z, \tilde{r}, \tilde{z}) &= -\frac{1}{2\pi} \frac{(r - \tilde{r}, z - \tilde{z})}{(r - \tilde{r})^2 + (z - \tilde{z})^2}, \\ \nabla E_3(r, z, \tilde{r}, \tilde{z}) &= -\frac{\tilde{r}}{2\pi} \int_0^\pi \frac{(r - \tilde{r}\cos(\theta), z - \tilde{z})}{(r^2 - 2r\tilde{r}\cos(\theta) + \tilde{r}^2 + (z - \tilde{z})^2)^{\frac{3}{2}}}\, d\theta. \end{aligned} \tag{A.5}$$

From (A.5), one can get the decay property of the Poisson fields:

$$|\nabla E_2(r,z,\tilde r,\tilde z)| \lesssim \frac{1}{((r-\tilde r)^2+(z-\tilde z)^2)^{\frac12}}, \quad |\nabla E_3(r,z,\tilde r,\tilde z)| \lesssim \frac{\tilde r}{(r-\tilde r)^2+(z-\tilde z)^2},$$

when $(r-\tilde r)^2+(z-\tilde z)^2 \gg 1$. In the following estimates of integrals, we denote $d := \sqrt{(r-\tilde r)^2+(z-\tilde z^2)}$ and $\tilde d := \sqrt{(\tilde r - R/\mu)^2+\tilde z^2}$ for brevity. We always bear in mind that $\mu \to 0$, which is much smaller than any fixed constant. Now, we assume $(r,z) \in B_{(R/\mu,0)}(\mu^{-s})$. By the expansion of elliptic integrals, one can show that (for example, see [6] for a related discussion) for any $(\tilde r,\tilde z) \in B_{(r,z)}(2\mu^{-s})$,

$$\nabla E_3(r,z,\tilde r,\tilde z) = \sqrt{\frac{\tilde r}{r}}\nabla E_2(r,z,\tilde r,\tilde z) + \mathcal{O}\left(\sqrt{\frac{\tilde r}{r^3}}\log(d)\right) = \left(1+\mathcal{O}(\mu^{1-s})\right)\nabla E_2(r,z,\tilde r,\tilde z) + \mathcal{O}\left(\mu\log(d)\right).$$

Thus, by Hardy's inequality and Cauchy's inequality,

$$\begin{aligned}
&\left|\int_{B_{(r,z)}(2\mu^{-s})} \nabla E_3(r,z,\tilde r,\tilde z)u(\tilde r,\tilde z) - \nabla E_2(r,z,\tilde r,\tilde z)u(\tilde r,\tilde z)\; d\tilde r d\tilde z\right| \\
&\lesssim \left|\int_{B_{(r,z)}(2\mu^{-s})} \left(1-\sqrt{\frac{\tilde r}{r}}\right)\nabla E_2(r,z,\tilde r,\tilde z)u(\tilde r,\tilde z)\; d\tilde r d\tilde z\right| + \mu\int_{B_{(r,z)}(2\mu^{-s})} |\log(d)u(\tilde r,\tilde z)|\; d\tilde r d\tilde z \\
&\lesssim \mu^{1-s}\int_{B_{(r,z)}(2\mu^{-s})} \frac{|u(\tilde r,\tilde z)|}{d}\; d\tilde r d\tilde z + \mu\|\log(d)\|_{L^2(B_{(r,z)}(2\mu^{-s}))}\cdot\|u\|_{L^2(B_{(r,z)}(2\mu^{-s}))} \\
&\lesssim \mu^{1-s}L_2' + \mu^{1-s}\int_{B_{(r,z)}(2\mu^{-s})\setminus B_{(r,z)}(\zeta^*)} \frac{|u(\tilde r,\tilde z)|}{d}\; d\tilde r d\tilde z + \mu^{1-s}|\log(\mu)|\cdot\|u\|_{L^2(B_{(r,z)}(2\mu^{-s}))} \\
&\lesssim \mu^{1-s}L_2' + \mu^{1-s}\|d^{-1}\|_{L^2(\zeta^*\le d\le 2\mu^{-s})}\cdot\|u\|_{L^2(\zeta^*\le d\le 2\mu^{-s})} + \mu^{1-s}|\log(\mu)|\cdot\|u\|_{L^2(B_{(r,z)}(2\mu^{-s}))} \\
&\lesssim \mu^{1-s}L_2' + \mu^{1-s}|\log(\mu)|\left(L_2+L_\infty\right).
\end{aligned} \tag{A.6}$$

It remains to estimate the tails of the integrals. Observe that $d/2 \le \tilde d \le 2d$ for any $(\tilde r,\tilde z) \in \mathbb{H}\setminus B_{(r,z)}(2\mu^{-s})$. By the decay property of u, u_* and the fundamental solutions, we have

$$\left|\int_{\mathbb{H}\setminus B_{(r,z)}(2\mu^{-s})} \nabla E_3(r,z,\tilde r,\tilde z)u(\tilde r,\tilde z)\; d\tilde r d\tilde z\right| \lesssim L_\infty\int_{\mathbb{H}\setminus B_{(r,z)}(2\mu^{-s})} \frac{d+R/\mu}{d^{2+\frac32}}\; d\tilde r d\tilde z \lesssim L_\infty(\mu^{\frac{s}{2}}+\mu^{\frac32 s-1}). \tag{A.7}$$

and

$$\left|\int_{\mathbb{R}^2\setminus B_{(r,z)}(2\mu^{-s})} \nabla E_2(r,z,\tilde r,\tilde z)u_*(\tilde r,\tilde z)\; d\tilde r d\tilde z\right| \lesssim L_\infty\int_{\mathbb{R}^2\setminus B_{(r,z)}(2\mu^{-s})} \frac{1}{d^{\frac52}}\; d\tilde r d\tilde z \lesssim L_\infty\mu^{\frac{s}{2}}. \tag{A.8}$$

Combining (A.6) (A.7) (A.8) gives (i).

Proof of (ii). Denoting $d_0 := \sqrt{(r-R/\mu)+z^2} - \frac12\mu^{-s} \ge \frac12\mu^{-s}$, we decompose the domain of integration into three parts:

$$D_1 := B_{(r,z)}(d_0), \quad D_2 := B_{(R/\mu,0)}(\mu^{-s}/2), \quad D_3 := \mathbb{H}\setminus(D_1\cup D_2).$$

For the first part, by the decay rates of u and ∇E_3, we have the estimate

$$\begin{aligned}
&\left|\int_{D_1} \nabla E_3(r,z,\tilde r,\tilde z)u(\tilde r,\tilde z)\; d\tilde r d\tilde z\right| \le \\
&\left|\int_{B_{(r,z)}(C)} \nabla E_3(r,z,\tilde r,\tilde z)u(\tilde r,\tilde z)\; d\tilde r d\tilde z\right| + \left|\int_{D_1\setminus B_{(r,z)}(C)} \nabla E_3(r,z,\tilde r,\tilde z)u(\tilde r,\tilde z)\; d\tilde r d\tilde z\right|,
\end{aligned}$$

where $C>0$ is some fixed constant. Since $|u(\tilde r,\tilde z)|\lesssim L_\infty d_0^{-\frac32}$ on $B_{(r,z)(C)}$ and $d_0\geq \mu^{-s}/2$, we have

$$\left|\int_{B_{(r,z)}(C)} \nabla E_3(r,z,\tilde r,\tilde z)u(\tilde r,\tilde z)\, d\tilde r d\tilde z\right| \lesssim L_\infty \frac{d_0+R/\mu}{d^{\frac32}} \lesssim L_\infty\left(\mu^{\frac{s}{2}}+\mu^{\frac32 s-1}\right).$$

As for the second integral, by the decay of u, we obtain

$$\begin{aligned}\left|\int_{D_1\backslash B_{(r,z)}(C)} \nabla E_3(r,z,\tilde r,\tilde z)u(\tilde r,\tilde z)\, d\tilde r d\tilde z\right| &\lesssim \int_{D_1\backslash B_{(r,z)}(C)} \frac{\tilde r}{d^2}|u(\tilde r,\tilde z)|\, d\tilde r d\tilde z\\ &\lesssim L_\infty \int_{D_1\backslash B_{(r,z)}(C)} \frac{\tilde r}{\tilde d^{\frac32}}\cdot\frac{1}{d^2}\, d\tilde r d\tilde z\\ &\lesssim L_\infty \int_C^{d_0}\int_0^{2\pi} \frac{d_0+R/\mu+y\cos(\theta)}{(d_0+y\cos(\theta)+\mu^{-s})^{\frac32}}\cdot\frac1y\, d\theta dy.\end{aligned}$$

Fixing another constant $0<c<1$, when $\delta:=-\cos(\theta)>1-c$, we have the estimates:

$$\begin{aligned}\int_C^{d_0} \frac{d_0-\delta y}{(d_0-\delta y+\mu^{-s})^{\frac32}}\cdot\frac1y\, dy = \int_C^{\delta d_0}\frac{d_0-x}{(d_0-x+\mu^{-s})^{\frac32}}\cdot\frac1x\, dx &\leq \int_C^{d_0}\frac{1}{(d_0-x+1)^{\frac12}x}\, dx\\ &\lesssim \frac{\log(d_0)}{\sqrt{d_0}} \lesssim \mu^{\frac{s}{2}}|\log(\mu)|,\end{aligned}$$

and by Hölder's inequality,

$$\begin{aligned}\int_C^{d_0}\frac{R/\mu}{(d_0-\delta y+\mu^{-s})^{\frac32}}\cdot\frac1y\, dy &= R\mu^{\frac32 s-1}\int_C^{\delta d_0}\frac{d_0-x}{(\mu^s(d_0-x)+1)^{\frac32}}\cdot\frac1x\, dx\\ &\leq R\mu^{\frac32 s-1}\left(\int_C^{+\infty}\frac{1}{x^{\frac32}}\, dx\right)^{\frac23}\cdot\left(\int_0^{+\infty}\frac{1}{(\mu^s x+1)^{\frac92}}\, dx\right)^{\frac13} \lesssim \mu^{\frac76 s-1}.\end{aligned}$$

When $\delta:=-\cos(\theta)\leq 1-c$, the estimate is more straight forward:

$$\int_C^{d_0}\frac{d_0-\delta y+R/\mu}{(d_0-\delta y+\mu^{-s})^{\frac32}}\cdot\frac1y\, dy \lesssim \frac{d_0+R/\mu}{(d_0+\mu^{-s})^{\frac32}}\int_C^{d_0}\frac1y\, dy \leq \frac{d_0+R/\mu}{(d_0+\mu^{-s})^{\frac32}}\log(d_0) \lesssim (\mu^{\frac{s}{2}}+\mu^{\frac32 s-1})|\log(\mu)|.$$

In summary, we obtain

$$\left|\int_{D_1}\nabla E_3(r,z,\tilde r,\tilde z)u(\tilde r,\tilde z)\, d\tilde r d\tilde z\right| \lesssim L_\infty\left((\mu^{\frac{s}{2}}+\mu^{\frac32 s-1})|\log(\mu)|+\mu^{\frac76 s-1}\right). \tag{A.9}$$

For the second domain,

$$\begin{aligned}\left|\int_{D_2}\nabla E_3(r,z,\tilde r,\tilde z)u(\tilde r,\tilde z)\, d\tilde r d\tilde z\right| \lesssim \int_{D_2}\frac{R}{\mu}u(\tilde r,\tilde z)\mu^{2s}\, d\tilde r d\tilde z &\lesssim \mu^{2s-1}L_2+L_\infty\mu^{2s-1}\int_1^{\mu^{-s}}\frac{1}{\sqrt{x}}\, dx\\ &\lesssim L_2\mu^{2s-1}+L_\infty\mu^{\frac32 s-1}.\end{aligned} \tag{A.10}$$

Finally, for the third domain, observe that $\tilde d\leq 3d$ in this domain. Thus, we have the estimate

$$\left|\int_{D_3}\nabla E_3(r,z,\tilde r,\tilde z)u(\tilde r,\tilde z)\, d\tilde r d\tilde z\right| \lesssim L_\infty\int_{D_3}\frac{\tilde d+R/\mu}{\tilde d^{2+\frac32}}\, d\tilde r d\tilde z \lesssim L_\infty\int_{\mu^{-s}}^{+\infty}\frac{x+R/\mu}{x^{\frac52}}\, dx \lesssim L_\infty(\mu^{\frac12 s}+\mu^{\frac32 s-1}). \tag{A.11}$$

Finally, collecting (A.9)(A.10)(A.11), we obtain (ii).

□

Now we derive some useful estimates for the 2D Poisson field. For convenience, in the following we will switch between the Cartesian coordinate $\mathbf{y} = (\rho, \xi)$ and the polar coordinate (γ, θ) from time to time, where $\rho = \gamma \sin\theta,\ \xi = \gamma\cos\theta$. The specific choice of coordinates will be clear from the context.

Lemma 10 (Pointwise estimates of the 2D Poisson field)**.** *The following pointwise estimates of 2D Poisson fields hold:*
(i) For u and its 2D Poisson field $\Psi_u := -\frac{1}{2\pi}\log|\mathbf{y}| * u$, we have for any $\alpha > 0$,

$$\|\Psi_u\|^2_{L^\infty(\gamma\le 1)} + \left\|\frac{\Psi_u}{1+\log(\gamma)}\right\|^2_{L^\infty(\gamma>1)} \le C_\alpha \int_{\mathbb{R}^2} u^2(\mathbf{y})(1+\gamma)^{2+\alpha}\, d\mathbf{y}, \tag{A.12}$$

with constant $C_\alpha > 0$ depending on α. Moreover, if $\int u = 0$, we have the improved estimate

$$\|\Psi_u\|^2_{L^\infty} \le C_\alpha \int_{\mathbb{R}^2} u^2(\mathbf{y})(1+\gamma)^{2+\alpha}\, d\mathbf{y}. \tag{A.13}$$

(ii) If $u = u(\gamma)$ is a radial function on $\mathbb{R}^2$, we have for $0 \le \alpha \le 1$:

$$|\gamma\partial_\gamma \Psi_u(\gamma)|^2 = \left|\int_0^\gamma r u(r)\, dr\right|^2 \lesssim (1 + \mathbb{1}_{\{\gamma\ge 1\}}\log(\gamma))\frac{\gamma^2}{(1+\gamma)^{2\alpha}}\int_0^\gamma r u^2(r)(1+r)^{2\alpha}\, dr. \tag{A.14}$$

On the other hand, if u is without radial component, then for any $0 < \alpha < 2$, we have (in the polar coordinates)

$$\int_0^{2\pi} |\Psi_u(\gamma,\theta)|^2 + \gamma^2|\nabla\Psi_u(\gamma,\theta)|^2\, d\theta \lesssim \gamma^2(1+\gamma)^{-2\alpha}(1+\mathbb{1}_{\{\gamma\le 1\}}|\log(\gamma)|)\int_{\mathbb{R}^2} u^2(\mathbf{y})(1+\gamma)^{2\alpha}\, d\mathbf{y}. \tag{A.15}$$

It is also convenient to write equivalently in the parabolic variables ($\zeta = \nu\gamma$) in our setting:

$$\begin{aligned} &|\zeta\partial_\zeta\Psi_u(\zeta)|^2 \lesssim (1+\mathbb{1}_{\{\zeta\ge\nu\}}\log(\zeta/\nu))\frac{\zeta^2}{(\nu+\zeta)^{2\alpha}}\int_{\mathbb{R}^2} u^2(\nu+\zeta)^{2\alpha}\, d\mathbf{x} \quad \textit{(radial)}\ 0\le\alpha\le 1,\\ &\int_0^{2\pi} |\Psi_u(\zeta,\theta)|^2 + \zeta^2|\nabla\Psi_u(\zeta,\theta)|^2\, d\theta \lesssim \zeta^2(\nu+\zeta)^{-2\alpha}(1+\mathbb{1}_{\{\zeta\le\nu\}}|\log(\zeta/\nu)|)\int_{\mathbb{R}^2} u^2(\mathbf{x})(\nu+\zeta)^{2\alpha}\, d\mathbf{x},\\ &\qquad\qquad\qquad\qquad\qquad\qquad \textit{(without radial component)}\ 0<\alpha<2. \end{aligned} \tag{A.16}$$

(iii) For any $1 \le p < 2$, we have the following estimate based on the L^∞-norm of u:

$$\|\nabla\Psi_u\|_{L^\infty} \le C(p)\left(\|u\|_{L^\infty} + \|u\|_{L^p}\right). \tag{A.17}$$

Proof. See *Lemma 7.2* in [31] for a proof of (i) and (iii). As for (ii), (A.14) follows directly from the explicit expression of $\partial_\zeta\Psi_u$ when u is radial and Cauchy's inequality. To prove (A.15), we expend both u and Ψ_u into trigonalmetric series:

$$\begin{aligned} u(\gamma,\theta) &= \sum_{j=1}^{+\infty} u^{+,j}(\gamma)\sin(j\theta) + \sum_{j=1}^{+\infty} u^{-,j}(\gamma)\cos(j\theta),\\ \Psi_u(\gamma,\theta) &= \sum_{j=1}^{+\infty} \Psi_u^{+,j}(\gamma)\sin(j\theta) + \sum_{j=1}^{+\infty} \Psi_u^{-,j}(\gamma)\cos(j\theta). \end{aligned}$$

By $-\Delta\Psi_u = u$, we have

$$-\left(\partial_\gamma^2 + \frac{1}{\gamma}\partial_\gamma - \frac{j^2}{\gamma^2}\right)\Psi_u^{\pm,j}(\gamma) = u^{\pm,j}(\gamma) \quad j\ge 1,$$

which admits explicit solutions

$$\Psi_u^{\pm,j}(\gamma) = \frac{\gamma^j}{2j}\int_\gamma^{+\infty} u^{\pm,j}(y)y^{1-j}\,dy + \frac{\gamma^{-j}}{2j}\int_0^\gamma u^{\pm,j}(y)y^{1+j}\,dy,$$
$$\partial_\gamma\Psi_u^{\pm,j}(\gamma) = \frac{\gamma^{j-1}}{2}\int_\gamma^{+\infty} u^{\pm,j}(y)y^{1-j}\,dy - \frac{\gamma^{-j-1}}{2}\int_0^\gamma u^{\pm,j}(y)y^{1+j}\,dy.$$

By Cauchy's inequality, we have

$$\begin{aligned}\left|\int_\gamma^{+\infty} u^{\pm,j}(y)y^{1-j}\,dy\right| &\lesssim \left(\int_\gamma^{+\infty} y^{1-2j}(1+y)^{-2\alpha}\,dy\right)^{\frac12}\left(\int_\gamma^{+\infty}(u^{\pm,j}(y))^2(1+y)^{2\alpha}y\,dy\right)^{\frac12}\\ &\lesssim \gamma^{1-j}(1+\gamma)^{-\alpha}(1+\mathbb{1}_{\{\gamma\le1\}}|\log\gamma|)^{\frac12}\left(\int_\gamma^{+\infty}(u^{\pm,j}(y))^2(1+y)^{2\alpha}y\,dy\right)^{\frac12},\end{aligned}$$

and

$$\begin{aligned}\left|\int_0^\gamma u^{\pm,j}(y)y^{1+j}\,dy\right| &\lesssim \left(\int_0^\gamma y^{1+2j}(1+y)^{-2\alpha}\,dy\right)^{\frac12}\left(\int_0^\gamma(u^{\pm,j}(y))^2(1+y)^{2\alpha}y\,dy\right)^{\frac12}\\ &\lesssim \gamma^{1+j}(1+\gamma)^{-\alpha}\left(\int_0^\gamma(u^{\pm,j}(y))^2(1+y)^{2\alpha}y\,dy\right)^{\frac12},\end{aligned}$$

where all constants of the inequalities above are independent of j. It follows that

$$|\Psi_u^{\pm,j}(\gamma)|^2 \lesssim \frac{1}{j^2}\gamma^2(1+\mathbb{1}_{\{\gamma\le1\}}|\log\gamma|)\int_0^{+\infty}(u^{\pm,j}(y))^2(1+y)^{2\alpha}y\,dy,$$
$$|\partial_\gamma\Psi_u^{\pm,j}(\gamma)|^2 \lesssim (1+\mathbb{1}_{\{\gamma\le1\}}|\log\gamma|)\int_0^{+\infty}(u^{\pm,j}(y))^2(1+y)^{2\alpha}y\,dy.$$

Finally, by Parseval's identity, we obtain

$$\begin{aligned}\int_0^{2\pi}|\Psi_u(\gamma,\theta)|^2+\gamma^2|\nabla\Psi_u(\gamma,\theta)|^2\,d\theta &\lesssim \sum_\pm\sum_{j\ge1} j^2|\Psi_u^{\pm,j}(\gamma)|^2 + \sum_\pm\sum_{j\ge1}\gamma^2|\partial_\gamma\Psi_u^{\pm,j}(\gamma)|^2\\ &\lesssim \gamma^2(1+\mathbb{1}_{\{\gamma\le1\}}|\log\gamma|)\int_0^{+\infty}\sum_\pm\sum_{j\ge1}(u^{\pm,j}(y))^2(1+y)^{2\alpha}y\,dy\\ &\lesssim \gamma^2(1+\mathbb{1}_{\{\gamma\le1\}}|\log\gamma|)\int_{\mathbb{R}^2}|u(\mathbf{y})|^2(1+|\mathbf{y}|)^{2\alpha}\,d\mathbf{y}.\end{aligned}$$

which is (A.15). □

Inequalities

Here is one Hardy-Poincaré type inequality:

Lemma 11. *Let $b>0$ be a small parameter. Then, there exists a universal constant $C>0$, such that for any function ε on $\mathbb{R}^2$, it holds that ($|\mathbf{y}|:=\gamma$, $\mathbf{y}\in\mathbb{R}^2$)*

$$\int_{\mathbb{R}^2}\left((1+\gamma)^2\varepsilon^2+b^2\gamma^2(1+\gamma)^4\varepsilon^2\right)e^{-\frac{b\gamma^2}{2}} \le C\int_{\mathbb{R}^2}\left(b\varepsilon^2+|\nabla\varepsilon|^2\right)(1+\gamma)^4e^{-\frac{b\gamma^2}{2}}. \tag{A.18}$$

If we change of variable $\gamma:=\zeta/\sqrt{b}$, an equivalent form of this inequality is written as

$$\int_{\mathbb{R}^2}\left((\sqrt{b}+\zeta)^2\varepsilon^2+\zeta^2(\sqrt{b}+\zeta)^4\varepsilon^2\right)e^{-\frac{\zeta^2}{2}} \le C\int_{\mathbb{R}^2}\left(\varepsilon^2+|\nabla\varepsilon|^2\right)(\sqrt{b}+\zeta)^4e^{-\frac{\zeta^2}{2}}. \tag{A.19}$$

Proof. It suffices to prove (A.18). Denote $\mathbf{y} = (y_1, y_2)$. Integrating by parts, we have for $i = 1, 2$,

$$2\int (y_i + y_i^3)\varepsilon\partial_i\varepsilon e^{-\frac{b\gamma^2}{2}} = -\int (1 + 3y_i^2)\varepsilon^2 e^{-\frac{b\gamma^2}{2}} + \int b(y_i^2 + 3y_i^4)\varepsilon^2 e^{-\frac{b\gamma^2}{2}}.$$

By Cauchy's inequality,

$$\left|\int (y_i + y_i^3)\varepsilon\partial_i\varepsilon e^{-\frac{b\gamma^2}{2}}\right| \le C\int (1+\gamma)^4|\nabla\varepsilon|^2 e^{-\frac{b\gamma^2}{2}} + \frac{1}{10}\int (1 + 3y_i^2)\varepsilon^2 e^{-\frac{b\gamma^2}{2}},$$

for some constant $C > 0$. Thus, we have

$$\int_{\mathbb{R}^2} (1+\gamma^2)\varepsilon^2 e^{-\frac{b\gamma^2}{2}} \le C\int_{\mathbb{R}^2} \left(b\varepsilon^2 + |\nabla\varepsilon|^2\right)(1+\gamma)^4 e^{-\frac{b\gamma^2}{2}}$$

The other part is similar. Integrate by parts:

$$2\int b(y_i + y_i^5)\varepsilon\partial_i\varepsilon e^{-\frac{b\gamma^2}{2}} = \int b^2(y_i^2 + y_i^6)\varepsilon^2 e^{-\frac{b\gamma^2}{2}} - \int b(1 + 5y_i^4)\varepsilon^2 e^{-\frac{b\gamma^2}{2}}.$$

Then, apply the Cauchy's inequality

$$\left|\int b(y_i + y_i^5)\varepsilon\partial_i\varepsilon e^{-\frac{b\gamma^2}{2}}\right| \le C\int (1+\gamma)^4|\nabla\varepsilon|^2 e^{-\frac{b\gamma^2}{2}} + \frac{1}{10}\int b^2(y_i^2 + y_i^6)\varepsilon^2 e^{-\frac{b\gamma^2}{2}},$$

and it follows that

$$\int b^2\gamma^2(1+\gamma)^4\varepsilon^2 e^{-\frac{b\gamma^2}{2}} \le C\int_{\mathbb{R}^2} \left(b\varepsilon^2 + |\nabla\varepsilon|^2\right)(1+\gamma)^4 e^{-\frac{b\gamma^2}{2}}.$$

The proof is thus complete. □

We recall the Hardy-Littlewood-Sobolev (HLS) inequality in $\mathbb{R}^n$: for $0 < s < n$, $1 < p < q < \infty$ with $\frac{1}{q} = \frac{1}{p} - \frac{s}{n}$, we have

$$\left\|\frac{1}{|\mathbf{x}|^{n-s}} * f\right\|_{L^q} \le C\|f\|_{L^p},$$

where $C = C(p)$. Now combining the 2D HLS ($p > 2$) and Hölder inequality, we obtain:

$$\|\nabla\Psi_u\|_{L^p} \lesssim \left\|\frac{1}{|\mathbf{x}|} * u\right\|_{L^p} \lesssim \|u\|_{L^{\frac{2p}{2+p}}} \lesssim \|u/\sqrt{W}\|_{L^2}\|W\|_{L^{\frac{p}{2}}}^{\frac{1}{2}}, \tag{A.20}$$

for any weight function W. A useful corollary in our setting is the following:

$$\|\nabla\Psi_\varepsilon\|_{L^p} \lesssim \|\varepsilon\|_{L^{\frac{2p}{2+p}}} \lesssim \|\varepsilon\nu/\sqrt{U_\nu}\|_{L^2}\|U_\nu/\nu^2\|_{L^{\frac{p}{2}}}^{\frac{1}{2}} \lesssim \nu^{-\frac{2p-2}{p}}\|\varepsilon\nu/\sqrt{U_\nu}\|_{L^2}. \tag{A.21}$$

Acknowledgments: The research of P.Song and T.Y.Hou was in part supported by NSF Grant DMS-2205590 and the Choi Family Gift Fund. V.T. Nguyen is supported by the National Science and Technology Council of Taiwan.

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