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NONEXISTENCE OF LOCALLY SELF-SIMILAR BLOW-UP FOR THE 3D INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract. We study locally self-similar solutions of the three dimensional incompressible Navier-Stokes equations. The locally self-similar solutions we consider here are different from the global self-similar solutions. The self-similar scaling is only valid in an inner core region that shrinks to a point dynamically as the time, t, approaches a possible singularity time, T. The solution outside the inner core region is assumed to be regular, but it does not satisfy self-similar scaling. Under the assumption that the dynamically rescaled velocity profile converges to a limiting profile as $t \to T$ in L^p for some $p \in (3, \infty)$, we prove that such a locally self-similar blow-up is not possible. We also obtain a simple but useful non-blowup criterion for the 3D Euler equations.

1. **Introduction.** In this paper, we study locally self-similar solutions of the 3D Navier-Stokes equations

$$\begin{cases} u_t + (u \cdot \nabla)u = -\nabla p + \nu \Delta u, \\ \nabla \cdot u = 0, \\ u|_{t=0} = u_0(x), \end{cases} \tag{1}$$

where u is velocity, p is pressure, and ν is viscosity. The locally self-similar solutions we consider are very different from the global self-similar solutions considered by Leray [15]. The self-similar scaling is only valid in an inner core region that shrinks to a point dynamically as the time, t, approaches a possible singularity time, T. Typically the inner core region can be taken as a ball of radius proportional to some fractional power of $(T-t)$. The solution outside the inner core region is regular and does not satisfy self-similar scaling. A more refined notion of "asymptotically self-similar singularity" has been considered by Giga and Kohn in [10]. We remark that the nonexistence of global self-similar solutions has been proved by Necas, Ruzicka and Sverak [17] and by Tsai [21].

We prove our main result by using a dynamic rescaling technique. Assume that the solution of the 3D Navier-Stokes is smooth for $0 < t < T$ and may develop a possible locally self-similar singularity at $x = 0$ at time T. We introduce the

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following dynamic rescaling to the solution:

$$u(x,t) = \frac{U(y,t)}{\sqrt{T-t}}, \quad p(x,t) = \frac{P(y,t)}{T-t}, \quad y = \frac{x}{\sqrt{T-t}}, \quad 0 \le t < T. \tag{2}$$

This defines the rescaled velocity and pressure profiles, U and P. We assume that $U \in L^p$ for some $p \in (3,\infty)$. This is a reasonable assumption because a locally self-similar velocity field would typically satisfy the following scaling-invariant blow-up rate

$$|u(x,t)| \le \frac{C_*}{\sqrt{|x|^2 + (T-t)}}. \tag{3}$$

It is easy to see that the growth rate (3) would give a corresponding upper bound on the rescaled velocity field $|U(y,t)| \le C_*/\sqrt{|y|^2+1}$, which implies that $U \in L^p$ for $p \in (3,\infty]$ for all $0 \le t \le T$.

In this paper, we prove that if the rescaled velocity profile U converges to a limiting profile as $t \to T$ in L^p for some $p \in (3,\infty)$, then such a locally self-similar blow-up is not possible. One of the main observations of this paper is that if the rescaled velocity profile U converges to a limiting profile in L^p for some $p \in (3,\infty)$ as $t \to T$, then we can prove that

$$\lim_{t\to T} \|U(t)\|_{L^p} = 0. \tag{4}$$

The application of a classical result due to Leray [15] would imply that the solution is regular at $t = T$. In fact, we need something much weaker than (4). As long as one can show that $\lim_{n\to\infty} \|U(t_n)\|_{L^p} \to 0$ for a sequence of $t_n \to T$, this would be sufficient to show that u is regular at $t = T$.

A challenging open problem is to prove the nonexistence of a locally self-similar blow-up of the 3D Navier-Stokes equations by assuming only the boundedness of $\|U\|_{L^p}$ for some $p \in (3,\infty)$. The resolution of this open problem would rule out the possibility of a finite time blow-up solution that satisfies the scaling-invariant blow-up rate (3). To rule out such a locally self-similar blow-up is still very difficult at the technical level. We remark that recently Chen-Strain-Tsai-Yau [4] have made important progress along this direction for axisymmetric Navier-Stokes equations. They prove that if u is smooth for $0 \le t < T$ and satisfies $|u(x,t)| \le C_*/\sqrt{r^2 + (T-t)}$ with $r = \sqrt{x_1^2 + x_2^2}$, then u is regular at $t = T$. In their analysis, the fact that ru^θ (u^θ is the angular velocity component) satisfies a conservative convection diffusion equation plays an essential role.

We also derive a simple but useful non-blow-up criterion for the 3D incompressible Euler equations. Let ω be the vorticity. Define $\Omega_t = \{x \mid \omega(x,t) \ne 0\}$ and $\xi(x,t) = \omega(x,t)/|\omega(x,t)|$ for $x \in \Omega_t$. We prove that if u is smooth for $0 \le t < T$ and satisfies the following growth estimate

$$\overline{\lim}_{t\to T} \left((T-t) \|\xi \cdot \nabla_x u \cdot \xi\|_{L^\infty(\Omega_t)} \right) < 1, \tag{5}$$

then the solution remains smooth at $t = T$. Note that in terms of the rescaled velocity, the non-blowup criterion (5) can be reformulated as

$$\overline{\lim}_{t\to T} \|\tilde{\xi} \cdot \nabla_y U \cdot \tilde{\xi}\|_{L^\infty(\tilde{\Omega}_t)} < 1, \tag{6}$$

where $\tilde{\xi} = \nabla_y \times U / |\nabla_y \times U|$ and $\tilde{\Omega}_t = \{y \mid \nabla_y \times U \ne 0\}$. The above non-blowup criterion also applies to the 3D Navier-Stokes equations for all viscosity $\nu \ge 0$.

We would like to emphasize that the behavior of the limiting velocity profile can be verified numerically if a locally self-similar blow-up is observed in a computation.

In those numerical studies where locally self-similar blow-up solutions were reported (see, e.g., [13, 2, 18, 11, 19, 14]), there seems to be a well-defined rescaled velocity profile as the time approaches the alleged singularity time. In particular, Kerr [13, 14] and Pelz [18, 11] provided some detailed description of the rescaled velocity and vorticity profiles close to the alleged singularity time. We have recently re-examined the locally self-similar blow-up solution of the 3D Euler equations obtained by Kerr [13, 14] for two antiparallel vortex tubes [12]. We found that $\|\xi \cdot \nabla_x u \cdot \xi\|_{L^\infty}$ is actually bounded by $C \log \|\omega\|_{L^\infty}$ as $t \to T$. Thus, even if $\|\omega\|_{L^\infty} = O((T-t)^{-1})$, as alleged in [13], the non-blow-up condition (5) is easily satisfied. In fact, we show that the maximum vorticity does not grow faster than doubly exponential in time [12]. Note that $\|\nabla_x u\|_{L^\infty}$ in general has the same blow-up rate as $\|\omega\|_{L^\infty}$ for a locally self-similar blow-up. The fact that $\|\xi \cdot \nabla_x u \cdot \xi\|_{L^\infty}$ can be bounded by $C \log \|\omega\|_{L^\infty}$ shows that there is tremendous cancellation in the vortex stretching term due to the anisotropic scaling of the solution near the region of maximum vorticity [14, 12]. The local geometric regularity of the vorticity vector ξ also plays an essential role in the dynamic depletion of vortex stretching [6, 7, 8].

We remark that Dr. Chae, motivated by the result presented in this paper, has recently obtained more general nonexistence results for asymptotically self-similar singularities in the Euler and Navier-Stokes equations [3]. For more discussions regarding other aspects of the Navier-Stokes equations, we refer the reader to [5, 20, 16].

In the remaining part of the paper, we will present and prove our main results.

2. The main results and their proofs.

Theorem 2.1. *Let $u_0 \in L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ for some $p \in (3, \infty)$. Assume that the solution u of the 3D incompressible Navier-Stokes equations is smooth for $0 < t < T$ and the rescaled velocity profile $U(y,t)$ converges to $\overline{U}$ in L^p as $t \to T$. Then the solution remains smooth at $t = T$.*

Theorem 2.2. *Assume that the solution u of the 3D incompressible Euler or Navier-Stokes equations is smooth for $0 \le t < T$ and satisfies the growth estimate (5); then the solution remains smooth at $t = T$.*

Proof. **of Theorem 2.1** We first introduce the following rescaling in time:

$$\tau = \frac{1}{2} \log \frac{T}{T-t}, \tag{7}$$

for $0 \le t < T$. Note that by this time rescaling, we have transformed the original Navier-Stokes equations from $[0,T)$ in t to $[0,\infty)$ in the new time variable τ. It is easy to derive the equivalent evolution equations for the rescaled velocity:

$$U_\tau + U + (y \cdot \nabla)U + 2(U \cdot \nabla)U = -2\nabla P + 2\nu \Delta U, \tag{8}$$

with initial condition $U|_{\tau=0} = \sqrt{T} u_0(y\sqrt{T})$, where U satisfies $\nabla \cdot U = 0$ for all times. The problem on the possible finite time blow-up of the Navier-Stokes equations is now converted to the problem on the large time behavior of the rescaled equations (8). Since u is the unique smooth solution for the original Navier-Stokes equations for $0 < t < T$, U is the unique smooth solution for the rescaled equation (8) for $0 < \tau < \infty$.

Let $\phi(y) = (\phi_1, \phi_2, \phi_3)$ be a smooth, compactly supported, divergence free vector field in $\mathbb{R}^3$, and let $\psi(\tau)$ be a smooth, compactly supported test function in $(0,1)$

satisfying $\int_0^1 \psi(\tau)d\tau = 1$. Multiplying (8) by $\psi(\tau - n)\phi(y)$ and integrating over $\mathbb{R}^3 \times [n, n+1]$ for some $n > 0$, we obtain after integration by parts

$$\int_n^{n+1} \int_{\mathbb{R}^3} \left(-\psi_\tau \phi \cdot U + \psi\phi \cdot U - \psi \nabla \cdot (\phi \otimes y) \cdot U - 2\psi \nabla\phi \cdot (U \otimes U)\right) dy d\tau$$
$$= 2\nu \int_n^{n+1} \int_{\mathbb{R}^3} \psi \Delta\phi \cdot U dy d\tau, \tag{9}$$

where ψ is evaluated at $\tau - n$.

By the assumption of the theorem, we have

$$\lim_{\tau\to\infty} \|U(\tau) - \overline{U}\|_{L^p} = 0, \tag{10}$$

for some $p > 3$. Thus $\|U(\tau)\|_{L^p}$ is bounded for τ sufficiently large, and $\|\overline{U}\|_{L^p}$ is also bounded. Let $U(\tau) = \overline{U} + R(\tau)$. By (10), we have $\lim_{\tau\to\infty} \|R(\tau)\|_{L^p} = 0$. Substituting $U(\tau) = \overline{U} + R(\tau)$ into (9), we will show that all the terms involving R will go to zero as $n \to \infty$. It is sufficient to prove this for the nonlinear term:

$$\int_n^{n+1} \int_{\mathbb{R}^3} \psi \nabla\phi \cdot (R \otimes R) dy d\tau.$$

Let $q = p/(p-2) > 1$. Then we have $2/p + 1/q = 1$. Using the Hölder inequality, we obtain

$$\begin{aligned} \left|\int_n^{n+1} \int_{\mathbb{R}^3} \psi \nabla\phi \cdot (R \otimes R) dy d\tau\right| &\leq C \sup_{n\leq\tau\leq n+1} \int_{\mathbb{R}^3} |\nabla\phi||R|^2 dy \\ &\leq C\|\nabla\phi\|_{L^q} \sup_{n\leq\tau\leq n+1} \|R(\tau)\|_{L^p}^2 \to 0, \quad \text{as } n \to \infty. \end{aligned}$$

Other terms can be proved similarly. Therefore, by letting $n \to \infty$, we get

$$-\left(\int_0^1 \psi_\tau(\tau)d\tau\right) \int_{\mathbb{R}^3} \phi(y)\overline{U}(y)dy$$
$$+\left(\int_0^1 \psi(\tau)d\tau\right) \left(\int_{\mathbb{R}^3} \left(\phi \cdot \overline{U} - \nabla \cdot (\phi \otimes y) \cdot \overline{U} - 2\nabla\phi \cdot (\overline{U} \otimes \overline{U})\right) dy\right)$$
$$= 2\left(\int_0^1 \psi(\tau)d\tau\right) \left(\int_{\mathbb{R}^3} \Delta\phi \cdot \overline{U} dy\right). \tag{11}$$

Since ψ has compact support in $[0, 1]$, we conclude that

$$\int_0^1 \psi_\tau(\tau)d\tau = 0.$$

Moreover, we have $\int_0^1 \psi(\tau)d\tau = 1$ by assumption on ψ. Thus, we obtain

$$\int_{\mathbb{R}^3} \left(\phi \cdot \overline{U} - \nabla \cdot (\phi \otimes y) \cdot \overline{U} - 2\nabla\phi \cdot (\overline{U} \otimes \overline{U}) - 2\nu\Delta\phi \cdot \overline{U}\right) dy = 0. \tag{12}$$

Thus, $\overline{U}$ is a weak solution of the steady state rescaled Navier-Stokes equations:

$$\overline{U} + (y \cdot \nabla)\overline{U} + 2(\overline{U} \cdot \nabla)\overline{U} = -2\nabla\overline{P} + 2\nu\Delta\overline{U}, \tag{13}$$

with $\nabla \cdot \overline{U} = 0$. Since $\overline{U} \in L^p$ for some $p \in (3, \infty)$, we can apply Theorem 1 of [21] to conclude that $\overline{U} \equiv 0$. As a result, we obtain the following *a priori* decay estimate for $\|U(\tau)\|_{L^p}$.

$$\lim_{\tau\to\infty} \|U(\tau)\|_{L^p} = 0. \tag{14}$$

Using the rescaling relation (2), we can obtain the following estimate in terms of the original velocity field:

$$\lim_{t\to T}(T-t)^{1/2-3/2p}\|u(t)\|_{L^p}=0. \tag{15}$$

This would imply that u must be regular at $t=T$. If this were not the case, then the classical result of Leray [15] (also see the excellent summary of Leray's results in [9]) would imply that

$$\|u(t)\|_{L^p}\geq\frac{C}{(T-t)^{1/2-3/2p}}, \tag{16}$$

for some positive constant C that depends on p but is independent of T and t. This contradicts estimate (15). In fact, we need something much weaker than (14) to obtain a contradiction with Leray's result. We just need a subsequence $\tau_n\to\infty$ such that $\lim_{n\to\infty}\|U(\tau_n)\|_{L^p}=0$. This already contradicts the blow-up rate estimate of Leray. This observation may be useful for future study. This completes the proof of Theorem 2.1. □

Proof. **of Theorem 2.2.** We prove the result for the Navier-Stokes equations for all $\nu\geq 0$. Let $\tilde{\Omega}_\tau=\{y\mid\nabla\times U\neq 0\}$. By the assumption of Theorem 2.2, we have

$$\overline{\lim}_{\tau\to\infty}\|\tilde{\xi}\cdot\nabla U\cdot\tilde{\xi}\|_{L_\infty(\tilde{\Omega}_\tau)}<1. \tag{17}$$

Thus, there exists $\tau_M>0$ large enough and $\epsilon>0$ small enough such that

$$\|\tilde{\xi}\cdot\nabla U\cdot\tilde{\xi}\|_{L_\infty(\tilde{\Omega}_\tau)}\leq 1-\epsilon, \tag{18}$$

for $\tau\geq\tau_M$. Define $W\equiv\nabla\times U$. By taking the curl of (8), we obtain an equation for the rescaled vorticity W as follows:

$$W_\tau+2W+(y\cdot\nabla)W+2(U\cdot\nabla)W=2\nabla U\cdot W+2\nu\Delta W\ . \tag{19}$$

For $y\in\tilde{\Omega}_\tau$, we derive by taking the inner product of W with (19) that

$$\frac{1}{2}(|W|^2)_\tau+2|W|^2+(y\cdot\nabla)|W|^2+(U\cdot\nabla)|W|^2=2(\tilde{\xi}\cdot\nabla U\cdot\tilde{\xi})|W|^2+\nu\Delta(|W|^2)-2\nu|\nabla W|^2\ , \tag{20}$$

where we have used $W\cdot\Delta W=\Delta(|W|^2/2)-|\nabla W|^2$, which can be verified directly. It follows from (18) and (20) that

$$\frac{d}{d\tau}\|W\|_{L^\infty}\leq -2\epsilon\|W\|_{L^\infty}, \tag{21}$$

for $\tau\geq\tau_M$ and for all $\nu\geq 0$. This implies that

$$\|W(\tau)\|_{L^\infty}\leq\|W(\tau_M)\|_{L^\infty}e^{-2\epsilon\tau},\quad \tau\geq\tau_M\ . \tag{22}$$

In terms of the original vorticity variable, we obtain

$$\|\omega(t)\|_{L^\infty}\leq\frac{\|\omega(t_M)\|_{L^\infty}(T-t_M)}{T^\epsilon(T-t)^{1-\epsilon}}, \tag{23}$$

for $t_M\leq t<T$, where $t_M=T(1-e^{-2\tau_M})<T$. Therefore, we have

$$\int_0^T\|\omega(t)\|_{L^\infty}dt=\int_0^{t_M}\|\omega(t)\|_{L^\infty}dt+\int_{t_M}^T\|\omega(t)\|_{L^\infty}dt<\infty, \tag{24}$$

since ω is smooth for $0\leq t\leq T_M$. Now the theorem follows from the Beale-Kato-Majda non-blowup criterion [1]. This completes the proof of Theorem 2.2. □

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