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GLOBAL WELL-POSEDNESS OF THE VISCOUS BOUSSINESQ EQUATIONS

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Abstract. We prove the global well-posedness of the viscous incompressible Boussinesq equations in two spatial dimensions for general initial data in H^m with $m \geq 3$. It is known that when both the velocity and the density equations have finite positive viscosity, the Boussinesq system does not develop finite time singularities. We consider here the challenging case when viscosity enters only in the velocity equation, but there is no viscosity in the density equation. Using sharp and delicate energy estimates, we prove global existence and strong regularity of this viscous Boussinesq system for general initial data in H^m with $m \geq 3$.

1. **Introduction.** The question of global existence/finite time blowup of smooth solutions for the three-dimensional incompressible Euler or Navier-Stokes equations has been one of the most outstanding open problems in applied analysis. The answer to this question will undoubtedly play a key role in understanding core problems in hydrodynamics such as the onset of turbulence. This challenging problem has attracted significant attention but it has eluded resolution. The main difficulty is to understand the effect of vortex stretching, which is absent in the two-dimensional incompressible Euler or Navier-Stokes equations. As part of the effort to understand the vortex stretching effect for 3D flows, various simplified model equations have been proposed in the literature. Amongst these models, the Boussinesq system is one of the most commonly used because it shares a similar vortex stretching effect as that in the 3D incompressible flow.

In this paper, we consider the global existence of the viscous Boussinesq equations

$$u_t + u \cdot \nabla u = -\nabla p + \begin{pmatrix} 0 \\ \rho \end{pmatrix} + \nu \triangle u, \tag{1.1}$$

$$\nabla \cdot u = 0, \tag{1.2}$$

$$\rho_t + u \cdot \nabla \rho = 0, \ \ x \in R^2, \tag{1.3}$$

where u is the velocity, p is pressure, and ρ describes the variation of density from a global average constant density which has been normalized to 1, ν is diffusion coefficient for the momentum equation. We assume that $u_0 \in H^m(R^2)$, $\rho_0 \in H^{m-1}(R^2)$ for $m \geq 3$. The Boussinesq equations have been used as a model in many geophysical applications, see e.g. [15].

The Boussinesq equations (1.1)-(1.3) have received significant attention in the mathematical fluid dynamics community because of its close connection to the

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3-D incompressible flow. Let $u = (u_1, u_2)$, $x = (x_1, x_2)$, and define $W = \nabla^{\perp}\rho = (\partial_{x_2}\rho, -\partial_{x_1}\rho)$. Recall that the vorticity is defined by $\omega = curl(u) = (\partial_{x_1}u_2 - \partial_{x_2}u_1)$. It is easy to derive the following evolution equations for ω and W:

$$\omega_t + u \cdot \nabla\omega = \rho_{x_1} + \nu\triangle\omega, \tag{1.4}$$

$$W_t + u \cdot \nabla W = (\nabla u)W\,. \tag{1.5}$$

We can see that the growth of vorticity depends on ρ_{x_1}, which is the second component of W. On the other hand, one can show that W has the same degree of regularity as ∇u. Thus equation (1.5) for W formally shares the same difficulty as the 3D vorticity equation with the same vortex stretching term. If we follow the energy estimates for 3D Euler equation [1, 13], we would find that energy estimates for W in high order Sobolev norms require an *a priori* control on $\int_0^T \|\nabla u\|_{L^\infty} dt$. This is quite difficult to achieve because the velocity equation is coupled to the density equation and the latter lacks any viscous dissipation. Prior analytical work in this area does not give any definite result on whether or not a singularity would develop in finite time. In a recent paper, Córdoba, Fefferman, and De La Llave [7] obtain the following *a priori* bound on the velocity field:

$$\int_0^T \|u\|_{L^\infty} dt < \infty. \tag{1.6}$$

Based on this *a priori* estimate on the velocity field, they can exclude the possibility of certain semi-uniform collapsing singularities, which they refer to as "squirt" singularities [7].

A closely related system to (1.1)–(1.3) is the one where diffusion is also present in the density equation, i.e.

$$u_t + u \cdot \nabla u = -\nabla p + \begin{pmatrix} 0 \\ \rho \end{pmatrix} + \nu\triangle u\,, \tag{1.7}$$

$$\nabla \cdot u = 0\,, \tag{1.8}$$

$$\rho_t + u \cdot \nabla\rho = \mu\triangle\rho, \ \ x \in R^2\,. \tag{1.9}$$

The Cauchy problem for the system (1.7)-(1.9) has been studied both analytically and numerically, see [3, 11, 17]. In the case of $\mu > 0$, it is known that singularity does not develop in finite time.

In the absence of viscous effects, i.e. $\nu = \mu = 0$, the two-dimensional Boussinesq convection has also been studied numerically and analytically. In fact, the inviscid 2-D Boussinesq equation can be used as a model for the 3-D axi-symmetric Euler equation with swirl away from the symmetric axis $r = 0$. There have been some numerical studies which indicate the possibility of finite time singularity formation of the 3D axi-symmetric Euler equation with swirl [10, 16]. However, other numerical investigations on similar data for the 2-D inviscid Boussinesq equations appear to indicate that there is no finite time singularity formation [8]. Thus the numerical studies are still inconclusive. Other analytical work on the inviscid Boussinesq equation can be found in [4, 5].

In this paper, we consider the global existence of the viscous Boussinesq system (1.1)-(1.3) *in the absence of viscous effect for the density equation.* In particular, we prove the global existence and strong regularity of the viscous Boussinesq system (1.1)-(1.3) in finite Sobolev space, H^m with $m \geq 3$, without any smallness restriction on the initial data. There are two key ingredients in our analysis. The first one is to use a sharp Sobolev embedding estimate in two spatial dimensions which bounds the maximum norm of ∇u by the L^2 norm of $\triangle u$ with a logarithmic correction, i.e.

$$\|\nabla u\|_{L^\infty} \le C_0(\|\triangle u\|_{L^2} + \|\nabla u\|_{L^2} + 1)\,(\log(\|\triangle\nabla u\|_{L^2} + \|\nabla u\|_{L^2} + e))^{\frac{1}{2}}\,. \quad (1.10)$$

It can be shown that the above estimate is sharp in the sense that the exponent in the logarithmic correction term cannot be lower than 1/2. In fact, the square root exponent in the logarithmic correction is crucial in obtaining our global well-posedness. This inequality was first derived by Brezis and Wainger [2]. For the sake of completeness, we also include a different proof of (1.10) in the appendix.

The second technique is a novel local-in-time analysis to reduce the seemingly high order nonlinearity due to the coupling between the velocity and density equations. We use this technique to estimate the growth of vorticity and vortex stretching due to the density stratification. The main challenge in obtaining the global well-posedness of the Boussinesq system (1.1)-(1.3) is due to the fact that there is no viscous effect in the density equation. As a result, there is no control on the smallest scales for the gradient of density. As we mentioned before, the growth of $\|\nabla\rho\|_{L^2}$ is controlled by the time integral of the maximum norm of the velocity gradient, i.e. $\int_0^T \|\nabla u\|_{L^\infty} dt$. By taking advantage of the viscous effect in the velocity equation and the *a priori* bound of density in the L^2 norm, we can obtain an *a priori* estimate on the L^2 norm of ∇u and the time integral of the second derivatives of u, $\int_0^T \|\triangle u\|_{L^2} dt$. These *a priori* estimates on the velocity field enable us to use the above embedding estimate (1.10) to get a sharp bound on $\int_0^T \|\nabla u\|_{L^\infty} dt$, which in turn controls the growth of $\|\nabla\rho\|_{L^2}$. However, direct application of (1.10) globally in time could lead to high order nonlinear terms. To overcome this difficulty, we perform a local-in-time analysis which allows us to reduce the order of nonlinearity. Moreover, we show that the local-in-time analysis can be boot-strapped to any finite time. By combining these two techniques, we can prove the global well-posedness of the viscous Boussinesq system (1.1)-(1.3).

We remark that after we submitted our work for publication, we learned that a similar result for the 2D Boussinesq equation has been obtained by Chae [6]. We should also mention the recent work of Lin, Liu, and Zhang [12] who establish the local existence and global existence (with small initial data) of classical solutions for an Oldroyd-system in which viscosity enters only in the fluid equation, but not in the elasticity equation.

The rest of the paper is organized as follows. We state the main result in Section 2 and present its proof. The analysis is structured into several steps. We present an independent proof of the key Sobolev embedding estimate (1.10) to the appendix.

2. **Main result and its proof.** The main result of this paper is the following global well-posedness theorem.

Theorem 2.1. *Assume that $u_0 \in H^m(R^2)$, $\rho_0 \in H^{m-1}(R^2)$, $m \ge 3$, with $\|u_0\|_{H^m}$, $\|\rho_0\|_{H^{m-1}} \le M_0$. Then for any $\nu > 0$, the solution of the viscous Boussinesq system (1.1)-(1.3) has a unique global smooth solution. Moreover, we have for any $T > 0$ independent of initial data,*

$$\|u\|_{H^m}(t) \le C(\nu, T, M_0)(\|u_0\|_{H^m} + \|\rho_0\|_{H^{m-1}}), \quad (2.11)$$

$$\|\rho\|_{H^{m-1}}(t) \le C(\nu, T, M_0)(\|u_0\|_{H^m} + \|\rho_0\|_{H^{m-1}}), \ \textit{for} \ \ 0 \le t \le T. \quad (2.12)$$

Proof.

The proof can be divided into six steps. We will use the same generic constant C to denote various constants that depend on ν, T, and M_0 only.

Step 1 H^1 **Estimate.**

First, we will obtain H^1 estimate. It is easy to see that

$$\|\rho(t)\|_{L^2} \le \|\rho_0\|_{L^2}, \;\; \|\rho(t)\|_{L^\infty} \le \|\rho_0\|_{L^\infty}. \tag{2.13}$$

Then straight forward energy estimate on velocity equation (1.1) gives

$$\frac{1}{2}\frac{d}{dt}\int_{R^2}|u|^2dx + \nu\int_{R^2}|\nabla u|^2dx \le \|\rho\|_{L^2}\|u\|_{L^2},$$

which implies

$$\|u\|_{L^2}(t) \le C, \quad \int_0^t \|\nabla u\|_{L^2}^2 dt \le C, \quad t \le T. \tag{2.14}$$

It is well-known that u can be recovered from the vorticity w via the Biot-Savart law:

$$u = K * \omega, \;\; K(x) = \frac{1}{2\pi|x|^2}(-x_2, x_1).$$

In particular, ∇K is a singular integral operator of order 0, satisfying $\|\nabla u\|_{L^2} = \|\omega\|_{L^2}$ and $\|\triangle u\|_{L^2} = \|\nabla\omega\|_{L^2}$.

Recall that the vorticity equation is given by

$$\omega_t + u\cdot\nabla\omega = \rho_{x_1} + \nu\triangle\omega. \tag{2.15}$$

Multiplying both sides of this equation by ω and integrating over R^2, we get

$$\frac{1}{2}\frac{d}{dt}\int_{R^2}|\omega|^2dx + \int_{R^2}(u\cdot\nabla\omega)\omega dx = \int_{R^2}\rho_{x_1}\omega dx + \nu\int_{R^2}\omega\triangle\omega dx. \tag{2.16}$$

Note that using integration by parts, we have

$$\int_{R^2}(u\cdot\nabla\omega)\omega dx = \frac{1}{2}\int_{R^2}(u\cdot\nabla)\omega^2dx = -\frac{1}{2}\int_{R^2}(\nabla\cdot u)\omega^2dx = 0,$$

since $\nabla\cdot u = 0$. Applying integration by parts to the last two terms of (2.16), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int|\omega|^2dx &= -\int\rho\omega_{x_1}dx - \nu\int|\nabla\omega|^2dx \\ &\le \|\rho\|_{L^2}\|\omega_{x_1}\|_{L^2} - \nu\int|\nabla\omega|^2dx \\ &\le C\|\rho\|_{L^2}^2 + \frac{\nu}{2}\|\omega_{x_1}\|_{L^2}^2 - \nu\int|\nabla\omega|^2dx \\ &\le C - \frac{\nu}{2}\int|\nabla\omega|^2dx\ .\end{aligned}$$

Therefore, we conclude that

$$\|\omega(\cdot,t)\|_{L^2} \le C\ , \quad \int_0^T\|\nabla\omega\|_{L^2}dx \le C, \quad t \le T, \tag{2.17}$$

which in turn implies

$$\|\nabla u\|_{L^2} = \|\omega\|_{L^2} \le C, \quad t \le T, \tag{2.18}$$

$$\int_0^T \|\triangle u\|_{L^2}^2 ds = \int_0^T \|\nabla\omega\|_{L^2}^2 ds \le C(T). \tag{2.19}$$

Step 2 H^2-Estimate.

Next, we will obtain H^2 estimate. Applying ∇ to vorticity equation (2.15) and dot multiplying the resulting equation with $\nabla\omega$, we get

$$\frac{1}{2}\frac{d}{dt}\int_{R^2} |\nabla\omega|^2 dx + \int_{R^2} \nabla\omega\cdot\nabla(u\cdot\nabla\omega)dx = \int \nabla\omega\cdot\nabla\rho_{x_1}dx + \nu\int \nabla\omega\cdot\nabla\Delta\omega dx. \tag{2.20}$$

Using integration by parts, we obtain

$$\left|\int \nabla\omega\cdot\nabla\rho_{x_1}dx\right| = \left|\int \nabla^2\omega\cdot\rho_{x_1}dx\right| \le C\|\nabla\rho\|_{L^2}^2 + \frac{\nu}{4}\|\triangle\omega\|_{L^2}^2. \tag{2.21}$$

For the last term on the right hand side of (2.20), we use the following simple formula:

$$\int \nabla\omega\cdot\nabla\triangle\omega dx = -\int |\triangle\omega|^2 dx. \tag{2.22}$$

For the convection term, which is the second term on the left hand side of (2.20), we have

$$\begin{aligned}\int \nabla\omega\cdot\nabla(u\cdot\nabla\omega)dx &= \int \nabla\omega\cdot(u\cdot\nabla(\nabla\omega))dx + \int \nabla\omega\cdot(\nabla u\cdot\nabla\omega)dx \\ &= \frac{1}{2}\int u\cdot\nabla|\nabla\omega|^2 dx + \int \nabla\omega\cdot(\nabla u\cdot\nabla\omega)dx \\ &= \int \nabla\omega\cdot(\nabla u\cdot\nabla\omega)dx,\end{aligned}$$

where we have used integration by parts and the divergence free condition of u in the last step. Thus, we have

$$\left|\int \nabla\omega\cdot\nabla(u\cdot\nabla\omega)dx\right| \le \|\nabla u\|_{L^2}\|\nabla\omega\|_{L^4}^2 = \|\omega\|_{L^2}\|\nabla\omega\|_{L^4}^2. \tag{2.23}$$

Using the following Gagaliardo-Nirenberg (see [9] and [14]) inequality in 2-D:

$$\|v\|_{L^4} \le C\|v\|_{L^2}^{\frac{1}{2}}\|\nabla v\|_{L^2}^{\frac{1}{2}}, \tag{2.24}$$

we obtain

$$\begin{aligned}\left|\int \nabla\omega\cdot\nabla(u\cdot\nabla\omega)dx\right| &\le C\|\omega\|_{L^2}\|\nabla\omega\|_{L^2}\|\triangle\omega\|_{L^2} \\ &\le C\|\nabla\omega\|_{L^2}\|\triangle\omega\|_{L^2} \\ &\le C\|\nabla\omega\|_{L^2}^2 + \frac{\nu}{4}\|\triangle\omega\|_{L^2}^2,\end{aligned} \tag{2.25}$$

where we have used the fact $\|\omega\|_{L^2} \le C$ from our estimate (2.17) in Step 1.

Putting all the estimates (2.20)–(2.25) together, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla w\|_{L^2}^2 \le C\|\nabla w\|_{L^2}^2 + C\|\nabla\rho\|_{L^2}^2 - \frac{\nu}{2}\|\triangle w\|_{L^2}^2. \tag{2.26}$$

Next, we need to derive an estimate on $\|\nabla\rho\|_{L^2}$. Using an argument similar to that used in obtaining (2.20), we get

$$\frac{1}{2}\frac{d}{dt}\int_{R^2} |\nabla\rho|^2 dx + \int \nabla\rho\cdot\nabla(u\cdot\nabla\rho)dx = 0.$$

As before, we treat the convection term as follows,

$$\begin{aligned}\int \nabla\rho\cdot\nabla(u\cdot\nabla\rho)dx &= \int \nabla\rho\cdot(u\cdot\nabla(\nabla\rho))dx + \int\nabla\rho\cdot(\nabla u\cdot\nabla\rho)dx\\ &= -\frac{1}{2}\int(\nabla\cdot u)|\nabla\rho|^2dx + \int\nabla\rho\cdot(\nabla u\cdot\nabla\rho)dx\\ &= \int\nabla\rho\cdot(\nabla u\cdot\nabla\rho)dx.\end{aligned}$$

Thus, we obtain

$$\left|\int\nabla\rho\cdot\nabla(u\cdot\nabla\rho)dx\right| \le \left|\int\nabla\rho\cdot(\nabla u\cdot\nabla\rho)dx\right| \le \|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^2}^2,$$

which gives

$$\frac{1}{2}\frac{d}{dt}\|\nabla\rho\|_{L^2}^2 \le \|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^2}^2. \tag{2.27}$$

Step 3 A Key Embedding Estimate.

In order to control the growth of $\|\nabla\rho\|_{L^2}^2$, we need to use the following key embedding estimate:

$$\|f\|_{L^\infty} \le C(\|\nabla f\|_{L^2} + \|f\|_{L^2} + 1)\left(\log\left(\|\triangle f\|_{L^2}^2 + \|f\|_{L^2}^2 + e\right)\right)^{\frac{1}{2}} . \tag{2.28}$$

The proof of the above embedding estimate will be deferred to the appendix. Applying (2.28) with $f = \nabla u$, we immediately obtain

$$\|\nabla u\|_{L^\infty} \le C_0(\|\nabla^2 u\|_{L^2} + \|\nabla u\|_{L^2} + 1)(\log(\|\nabla\triangle u\|_{L^2}^2 + \|\nabla u\|_{L^2} + e))^{\frac{1}{2}}. \tag{2.29}$$

Step 4: **Local-In-Time Estimate.**

We now show how to use the key embedding estimate (2.29) to perform the local-in-time analysis. This local-in-time analysis is a crucial step in our well-posedness analysis.

Using (2.27) and (2.29), we have

$$\begin{aligned}&\|\nabla\rho(t)\|_{L^2}^2 \le \|\nabla\rho_0\|_{L^2}^2 e^{2\int_0^t \|\nabla u\|_{L^\infty}(s)ds}\\ &\le Ce^{2C_0\int_0^t\left(\|\triangle u\|_{L^2}+\|\nabla u\|_{L^2}\right)\left(\log(\|\nabla\triangle u\|_{L^2}^2+\|\nabla u\|_{L^2}^2+e)\right)^{\frac{1}{2}}ds}\\ &\le Ce^{C_0\int_0^t(2(\|\triangle u\|_{L^2}^2+\|\nabla u\|_{L^2}^2)+(\log(\|\nabla\triangle u\|_{L^2}^2+\|\nabla u\|_{L^2}^2+e)))ds}\\ &\le Ce^{C_0\int_0^t\log(\|\nabla\triangle u\|_{L^2}^2+\|\nabla u\|_{L^2}^2+e)ds},\end{aligned}$$

where we have used $\|\nabla u\|_{L^2}^2 \le C$ and $\int_0^t \|\triangle u\|_{L^2}^2 ds \le C$ from (2.18) and (2.19) in Step 1.

In order to complete our global energy estimates, we need to use a novel local-in-time analysis to control the seemingly high order nonlinear term resulting from the logarithmic correction in the above estimate. [1]

[1] After we have completed our manuscript, we also found a more direct and simpler analysis which overcomes the difficulty mentioned above. But we feel that this local-in-time analysis is of interest in itself and decide to present the original proof here.

Define $D \equiv Ce^{C_0 \int_0^{t-\epsilon} \log(\|\nabla\triangle u\|_{L^2}^2 + e)ds}$. The parameter $\epsilon > 0$ will be determined later. Then we can bound

$$\begin{aligned}
\|\nabla\rho(t)\|_{L^2}^2 &\leq De^{C_0 \int_{t-\epsilon}^t \log(\|\nabla\triangle u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + e)ds} \\
&= De^{C_0\epsilon\left(\frac{1}{\epsilon}\int_{t-\epsilon}^t \log(\|\nabla\triangle u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + e)ds\right)} \\
&\leq De^{C_0\epsilon \log\left(\frac{1}{\epsilon}\int_{t-\epsilon}^t (\|\nabla\triangle u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + e)ds\right)} \\
&\leq D\left(\frac{1}{\epsilon}\int_{t-\epsilon}^t (\|\nabla\triangle u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + e)ds\right)^{\epsilon C_0}
\end{aligned}$$

where we have used Jensen's inequality

$$\frac{1}{\epsilon}\int_{t-\epsilon}^t \log u(s)ds \leq \log(\frac{1}{\epsilon}\int_{t-\epsilon}^t u(s)ds),$$

since $\log(u)$ is a concave function.

Furthermore, we observe that $(\frac{1}{\epsilon})^\epsilon \leq C$ for all $\epsilon > 0$. We get

$$\|\nabla\rho\|_{L^2}^2(t) \leq CD\left(\int_0^t \|\nabla\triangle u\|_{L^2}^2 ds + \|\nabla u\|_{L^2}^2 + e\right)^{C_0\epsilon}.$$

Recall that C_0 is a constant in the embedding estimate (2.29), which is independent of M_0, ν and T. Choose ϵ such that $C_0\epsilon = \frac{1}{2}$, we get

$$\|\nabla\rho\|_{L^2}^2(t) \leq CD\left(\int_0^t \|\nabla\triangle u\|_{L^2}^2 ds + \|\nabla u\|_{L^2}^2 + e\right)^{\frac{1}{2}}. \tag{2.30}$$

Using a similar argument, we can show that for $t \leq T$

$$\begin{aligned}
D &\leq C\left(\frac{1}{t-\epsilon}\int_0^{t-\epsilon} \left(\|\nabla\triangle u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + e\right) ds\right)^{C_0(t-\epsilon)} \\
&\leq C\left(\int_0^{t-\epsilon} \left(\|\nabla\triangle u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + e\right) ds\right)^{C_0(t-\epsilon)}.
\end{aligned} \tag{2.31}$$

Substituting (2.30) into (2.26) and integrating in time from 0 to t, we have

$$\begin{aligned}
&\frac{1}{2}\|\nabla\omega(t)\|_{L^2}^2 - \frac{1}{2}\|\nabla\omega_0\|_{L^2}^2 \\
&\leq C\int_0^t \|\nabla\omega\|_{L^2}^2 ds + D\int_0^t \left(\int_0^s \|\nabla\triangle u\|_{L^2}^2 ds + C\right)^{\frac{1}{2}} ds - \frac{\nu}{2}\int_0^t \|\triangle\omega\|_{L^2}^2 ds \\
&\leq C\int_0^t \|\nabla\omega\|_{L^2}^2 ds + DCT\left(\int_0^t \|\triangle\omega\|_{L^2}^2 ds + C\right)^{\frac{1}{2}} - \frac{\nu}{2}\int_0^t \|\triangle\omega\|_{L^2}^2 ds \\
&\leq C\int_0^t \|\nabla\omega\|_{L^2}^2 ds + CD^2 + \frac{\nu}{4}\left(\int_0^t \|\triangle\omega\|_{L^2}^2 ds\right) - \frac{\nu}{2}\int_0^t \|\triangle\omega\|_{L^2}^2 ds\,,
\end{aligned}$$

where we have used the estimate $\|\nabla u\|_{L^2}^2 \leq C$ from (2.18) and $\|\nabla\triangle u\|_{L^2} \leq C\|\triangle\omega\|_{L^2}$. Therefore, we obtain

$$\frac{1}{2}\|\nabla\omega(t)\|_{L^2}^2 - \frac{1}{2}\|\nabla\omega_0\|_{L^2}^2 \leq CD^2 + C\int_0^t \|\nabla\omega\|_{L^2}^2 ds - \frac{\nu}{4}\int_0^t \|\triangle\omega\|_{L^2}^2 ds\,. \tag{2.32}$$

Note that $\epsilon = \frac{1}{2C_0}$ is a fixed constant independent of u_0, ρ_0, ν and time T.

Step 5: **Global Estimate.**

Next, we show how to boot-strap the local-in-time analysis to any finite time T. We argue as follows.

: 1. For $t \le \epsilon$, we have $D = 0$, (2.32) implies that

$$\frac{1}{2}\|\nabla\omega(t)\|_{L^2}^2 - \frac{1}{2}\|\nabla\omega_0\|_{L^2} \le C\int_0^t \|\nabla\omega\|_{L^2}^2 ds \ .$$

The Gronwall inequality gives (for any $T > 0$)

$$\|\nabla\omega(t)\|_{L^2}^2 \le C(T,\nu,M_0),\ \ t \le \epsilon \ . \tag{2.33}$$

This in turn implies that

$$\int_0^t \|\triangle w\|_{L^2}^2 ds \le C(T,\nu,M_0),\ \ 0 \le t \le \epsilon \ . \tag{2.34}$$

: 2. For $\epsilon \le t \le 2\epsilon$, we have from (2.18), (2.31) and (2.34) that

$$\begin{aligned} D &= Ce^{C_0\int_0^{t-\epsilon}\log\left(\|\nabla\triangle u\|_{L^2}^2+\|\nabla u\|_{L^2}^2+e\right)ds} \\ &\le C\left(\int_0^\epsilon \left(\|\nabla\triangle u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + e\right)ds\right)^{C_0\epsilon} \\ &\le C\left(\int_0^\epsilon \left(\|\triangle\omega\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + e\right)ds\right)^{C_0\epsilon} \\ &\le C(T,\nu,M_0) \ . \end{aligned}$$

It follows from (2.32) that

$$\frac{1}{2}\|\nabla\omega(t)\|_{L^2}^2 - \frac{1}{2}\|\nabla\omega_0\|_{L^2}^2 \le CD^2 + C\int_0^t \|\nabla\omega\|_{L^2}^2 ds \ , \quad \epsilon \le t \le 2\epsilon \ .$$

The Gronwall inequality implies

$$\|\nabla\omega(t)\|_{L^2}^2 \le C(T,\nu,M_0),\ \ 0 \le t \le 2\epsilon. \tag{2.35}$$

This in turn gives

$$\int_0^t \|\triangle\omega\|_{L^2}^2 ds \le C(T,\nu,M_0),\ \ 0 \le t \le 2\epsilon \ . \tag{2.36}$$

Now estimate (2.36) would imply

$$D \le C(T,\nu,M_0) \text{ for } 0 \le t \le 3\epsilon.$$

Repeating the above local argument on (2.32) proves the boundedness of $\|\nabla\omega\|_{L^2}^2$, and $\int_0^t \|\triangle\omega\|_{L^2}^2 ds$ for $0 \le t \le 3\epsilon$. This procedure can be repeated indefinitely for any finite time. This proves the global estimate of the viscous Boussinesq equation with the estimate

$$\|\nabla\omega(t)\|_{L^2} \le C(T,\nu,M_0),\ \ 0 \le t \le T \ , \tag{2.37}$$

and

$$\int_0^t \|\triangle\omega\|_{L^2}^2 ds \le C(T,\nu,M_0),\ \ 0 \le t \le T \ . \tag{2.38}$$

From (2.37) and (2.38), we obtain for any $T > 0$,

$$\|\triangle u(t)\|_{L^2}^2 + \int_0^t \|\nabla\triangle u\|_{L^2}^2 ds \le C(T,\nu,M_0),\ \ 0 \le t \le T \ , \tag{2.39}$$

using the boundedness of the singular operator ∇K from L^2 to L^2. Finally, it follows from (2.30) that

$$\|\nabla\rho(t)\|_{L^2}^2 \leq CD\left(\int_0^t \|\nabla\triangle u\|_{L^2}^2 ds + \|\nabla u\|_{L^2}^2 + e)ds\right)^{\frac{1}{2}} \leq C(T,\nu,M_0)\ . \quad (2.40)$$

Step 6 Global Estimates in High Order Sobolev Norms.

First, we note that from (2.18), (2.39), and the Sobolev embedding, we have

$$\begin{aligned}\int_0^T \|\nabla u\|_{L^\infty}(s)ds &\leq C\int_0^T (\|\nabla u\|_{L^2} + \|\nabla\triangle u\|_{L^2})\, ds \\ &\leq C(T,\nu,M_0)\ . \qquad (2.41)\end{aligned}$$

Next, we obtain *a priori* bound for $\|\nabla\rho\|_{L^\infty}$. Let p be a positive odd integer, and denote $\nabla_j = \partial_{x_j}$. Multiplying the density equation by $\nabla_j\left((\nabla_j\rho)^p\right)$, we obtain

$$\int \nabla_j(\nabla_j\rho)^p\rho_t + \int (u\cdot\nabla\rho)(\nabla_j(\nabla_j\rho)^p)dx = 0\ .$$

Integration by parts gives

$$\int (\nabla_j\rho)^p(\nabla_j\rho)_t + \int (\nabla_j u\cdot\nabla\rho)(\nabla_j\rho)^p dx + \int (u\cdot\nabla(\nabla_j\rho))(\nabla_j\rho)^p dx = 0\ .$$

Note that

$$\int (\nabla_j\rho)^p((u\cdot\nabla)\nabla_j\rho)dx = \frac{1}{p+1}\int u\cdot\nabla(\nabla_j\rho)^{p+1}dx = -\frac{1}{p+1}\int (\nabla\cdot u)(\nabla_j\rho)^{p+1}dx = 0\ .$$

Therefore, we obtain

$$\frac{1}{p+1}\frac{d}{dt}\int (\nabla_j\rho)^{p+1}dx \leq \|\nabla u\|_{L^\infty}\int |\nabla\rho|^{p+1}dx,\ j=1,2,$$

which implies

$$\frac{d}{dt}\int |\nabla\rho|^{p+1}dx \leq (p+1)\|\nabla u\|_{L^\infty}\int |\nabla\rho|^{p+1}dx\ .$$

Thus, we get

$$\|\nabla\rho(t)\|_{L^{p+1}}^{p+1} \leq \|\nabla\rho_0\|_{L^{p+1}}^{p+1} e^{(p+1)\int_0^t \|\nabla u\|_{L^\infty}ds}$$

which gives

$$\|\nabla\rho(t)\|_{L^{p+1}} \leq \|\nabla\rho_0\|_{L^{p+1}} e^{\int_0^t \|\nabla u\|_{L^\infty}(s)ds} \leq C(T,\nu,M_0)\ .$$

Therefore, we conclude

$$\|\nabla\rho(t)\|_{L^\infty} \leq lim_{p\to\infty}\|\nabla\rho(t)|_{p+1} \leq C(T,\nu,M_0)\ . \quad (2.42)$$

Now, we are ready to perform H^m-estimate for ρ and u. We need to use the following Calculus Inequality (see e.g. Lemma 3.4 on page 98 of [13])

$$\sum_{0\leq|\alpha|\leq m} \|D^\alpha(uv) - uD^\alpha v\|_{L^2} \leq C\left(\|\nabla u\|_{L^\infty}\|v\|_{H^{m-1}} + \|v\|_{L^\infty}\|u\|_{H^m}\right)\ . \quad (2.43)$$

Let $0\leq|\alpha|\leq m$ be a multi-index. Apply D^α to the density equation (1.3), multiply $D^\alpha\rho$, and integrate over space. The result is

$$\frac{1}{2}\frac{d}{dt}\int_{R^2} |D^\alpha\rho|^2 dx + \int_{R^2} D^\alpha\rho D^\alpha(u\cdot\nabla\rho)dx = 0\ .$$

Applying the Calculus Inequality (2.43) with $v = \nabla\rho$ for $0 \le |\alpha| \le (m-1)$, we get

$$\begin{aligned} \left|\int D^\alpha \rho D^\alpha (u\cdot\nabla\rho)dx\right| &= \left|\int D^\alpha\rho(u\cdot\nabla)D^\alpha\rho dx + \int D^\alpha\rho\left(D^\alpha(u\cdot\nabla\rho) - u\cdot\nabla D^\alpha\rho\right)dx\right| \\ &\le \left|\frac{1}{2}\int u\cdot\nabla(D^\alpha\rho)^2 dx\right| + \|D^\alpha\rho\|_{L^2}\|D^\alpha(u\cdot\nabla\rho) - u\cdot D^\alpha\nabla\rho\|_{L^2} \\ &\le \|D^\alpha\rho\|_{L^2} C\left(\|\nabla u\|_{L^\infty}\|\rho\|_{H^{m-1}} + \|u\|_{H^{m-1}}\|\nabla\rho\|_{L^\infty}\right) . \end{aligned}$$

Summing over α for $0 \le |\alpha| \le (m-1)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\rho\|_{H^{m-1}}^2 \le C\left(\|\nabla u\|_{L^\infty}\|\rho\|_{H^{m-1}}^2 + \|\nabla\rho\|_{L^\infty}\|\rho\|_{H^{m-1}}\|u\|_{H^{m-1}}\right) ,$$

which is equivalent to

$$\frac{d}{dt}\|\rho\|_{H^{m-1}}^2 \le C(\|\nabla u\|_{L^\infty} + 1)\|\rho\|_{H^{m-1}}^2 + C\|u\|_{H^{m-1}}^2 , \tag{2.44}$$

where we have used $\|\nabla\rho\|_{L^\infty} \le C$. Next we apply the same estimate to the velocity equation (1.1). We have

$$\begin{aligned} \frac{d}{dt}\|u\|_{H^m}^2 &\le C\|\nabla u\|_{L^\infty}\|u\|_{H^m}^2 + \frac{\nu}{2}\|\nabla u\|_{H^m}^2 + C\|\rho\|_{H^{m-1}}^2 - \nu\|\nabla u\|_{H^m}^2 \\ &= C\|\nabla u\|_{L^\infty}\|u\|_{H^m}^2 + C\|\rho\|_{H^{m-1}}^2 - \frac{\nu}{2}\|\nabla u\|_{H^m}^2 . \end{aligned} \tag{2.45}$$

Adding (2.44) and (2.45) gives,

$$\frac{d}{dt}\left(\|u\|_{H^m}^2 + \|\rho\|_{H^{m-1}}^2\right) \le C(\|\nabla u\|_{L^\infty} + 1)(\|u\|_{H^m}^2 + \|\rho\|_{H^{m-1}}^2) - \frac{\nu}{2}\|\nabla u\|_{H^m}^2 \tag{2.46}$$

In particular, we have

$$\frac{d}{dt}\left(\|u\|_{H^m}^2 + \|\rho\|_{H^{m-1}}^2\right) \le C\left(\|\nabla u\|_{L^\infty} + 1\right)\left(\|u\|_{H^m}^2 + \|\rho\|_{H^{m-1}}^2\right) . \tag{2.47}$$

The Gronwall inequality implies

$$\begin{aligned} \|u(t)\|_{H^m}^2 + \|\rho(t)\|_{H^{m-1}}^2 &\le \left(\|u_0\|_{H^m}^2 + \|\rho_0\|_{H^{m-1}}^2\right) e^{\int_0^t C(\|\nabla u\|_{L^\infty}+1)ds} \\ &\le C(T,\nu,M_0) , \end{aligned} \tag{2.48}$$

for $0 \le t \le T$, where we have used (2.41). Substituting (2.48) back to (2.46), we get

$$\int_0^t \|\nabla u\|_{H^m}^2 ds \le C(T,\nu,M_0) , \quad 0 \le t \le T. \tag{2.49}$$

With the above global regularity estimates, we can easily prove the uniqueness of solutions to the Boussinesq system and the continuous dependence on initial data (see e.g. [13]). This completes the proof of the global well-posedness estimate for the viscous Boussinesq system (1.1)-(1.3).

3. **Appendix.** In this appendix, we prove the key embedding estimate (2.29) in Step 3.

$$\|f\|_{L^\infty} \le C_0(\|\nabla f\|_{L^2} + \|f\|_{L^2} + 1)\left(\log(\|\triangle f\|_{L^2} + \|f\|_{L^2} + e)\right)^{\frac{1}{2}} . \tag{3.50}$$

Proof

Denote by $B_\epsilon = \{x, |x| \le \epsilon\}$ the disk centered at the origin with radius ϵ. Let $w = \varphi f$, where φ is a smooth cut-off function with support in the unit disk B_1 and satisfying

$$\varphi(0) = 1,\ |\nabla\varphi| \le C,\ |\nabla^2\varphi| \le C,\ \mathrm{supp}\varphi \subset B_1.$$

By translation, it is sufficient to prove (2.29) for the origin, $x = 0$. Using the Cauchy formula, we have

$$\begin{aligned} |w(0)| &= \left| \frac{1}{2\pi} \int (\log |y| - \log \epsilon) \triangle w(y) dy \right| \\ &\leq \frac{1}{2\pi} \left| \int_{B_\epsilon} (\log |y| - \log \epsilon) \triangle w(y) dy \right| + \frac{1}{2\pi} \left| \int_{B_1 \setminus B_\epsilon} (\log |y| - \log \epsilon) \triangle w(y) dy \right| \\ &\leq \frac{1}{2\pi} \left(\int_{B_\epsilon} |\log (\frac{y}{\epsilon})|^2 dy \right)^{\frac{1}{2}} \|\triangle w\|_{L^2} + \frac{1}{2\pi} \int_{B_1 \setminus B_\epsilon} \frac{|\nabla w|}{|y|} dy \qquad (3.51) \end{aligned}$$

where we have used the Hölder inequality in the first estimate, and performed integration by parts for the second term. There is no boundary contribution from the integration by parts because the integrand vanishes at the boundary of $(B_1 \setminus B_\epsilon)$.

By rescaling the integration domain in the last term of (3.51) with $z = y/\epsilon$ and using the properties on the cut-off function φ, we obtain

$$\begin{aligned} |f(0)| &\leq \frac{1}{2\pi} \left(\int_{B_1} \epsilon^2 |\ln z|^2 dz \right)^{\frac{1}{2}} \|\triangle w\|_{L^2} + \frac{1}{2\pi} \left(\int_{B_1 \setminus B_\epsilon} \frac{dy}{|y|^2} \right)^{\frac{1}{2}} \|\nabla w\|_{L^2} \\ &\leq C\epsilon \left(\|\triangle f\|_{L^2} + \|f\|_{L^2} \right) + C \left(\log \frac{1}{\epsilon} \right)^{\frac{1}{2}} \left(\|\nabla f\|_{L^2} + \|f\|_{L^2} \right) , \end{aligned}$$

where we have used $\|\nabla f\|_{L^2} \leq C \left(\|f\|_{L^2} + \|\triangle f\|_{L^2} \right)$.

Choose ϵ such that $\epsilon \left(\|\triangle f\|_{L^2} + \|f\|_{L^2} + 1 \right) = 1$, i.e. $\epsilon = \frac{1}{\|\triangle f\|_{L^2} + \|f\|_{L^2} + 1}$. We obtain

$$|f(0)| \leq C \left(\|\nabla f\|_{L^2} + \|f\|_{L^2} + 1 \right) \left(\log \left(\|\triangle f\|_{L^2} + \|f\|_{L^2} \right) \right)^{\frac{1}{2}} .$$

The argument that we have presented for $x = 0$ obviously applies to any other point. This completes the proof of (3.50).

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