MULTISCALE ANALYSIS FOR CONVECTION DOMINATED TRANSPORT EQUATIONS

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Dedicated to Prof. Ta-Tsien Li's 70th birthday with admiration

Abstract. In this paper, we perform a systematic multiscale analysis for convection dominated transport equations with a weak diffusion and a highly oscillatory velocity field. The paper primarily focuses on upscaling linear transport equations. But we also discuss briefly how to upscale two-phase miscible flows, in which case the concentration equation is coupled to the pressure equation in a nonlinear fashion. For the problem we consider here, the local Peclet number is of order $O(\epsilon^{-m+1})$ with $m \in [2, \infty]$ being any integer, where ϵ characterizes the small scale in the heterogeneous media. Due to the presence of the nonlocal memory effect, upscaling a convection dominated transport equation is known to be very difficult. One of the key ideas in deriving a well-posed homogenized equation for the convection dominated transport equation is to introduce a projection operator which projects the fluctuation onto a suitable subspace. This projection operator corresponds to averaging along the streamlines of the flow. In the case of linear convection dominated transport equations, we prove the well-posedness of the homogenized equations and establish rigorous error estimates for our multiscale expansion.

1. **Introduction.** Upscaling a convection dominated transport equation is known to be very difficult due to the presence of nonlocal memory effects. Even for a linear hyperbolic equation with a shear velocity field, the upscaled equation involves a nonlocal history dependent diffusion term which is not amenable to computation (see e.g. [23, 13, 8, 20]). In the case when the convection and diffusion are of the same order, the situation becomes relatively easier, but interesting phenomena can still occur in the limit of very large Peclet number, see e.g. [21, 10, 5]. Recently, Hou-Westhead-Yang [14] have introduced a novel multiscale analysis for the two-phase immiscible flows in heterogeneous porous media. In particular they derived the homogenized equations by projecting the fluctuation of saturation onto a suitable subspace. Further, they demonstrated by extensive numerical experiments that the

upscaling method can accurately capture the multiscale solution of the two-phase flow.

In this paper, we further improve the multiscale analysis of Hou-Westhead-Yang [14] and develop a systematic multiscale analysis to upscale convection dominated transport equations. We primarily focus on upscaling linear transport equations. But we also discuss briefly how to upscale two-phase miscible flows, in which case the concentration equation is coupled to the pressure equation in a nonlinear fashion. The problem we consider contains a strong convection term and a weak diffusion term whose viscosity coefficient is proportional to $O(\epsilon^m)$ with $m \in [2, \infty]$ being any integer, where ϵ characterizes the small scale in the porous media. The local Peclet number is of order $O(\epsilon^{-m+1})$. Because of the absence of diffusion in the leading order approximation, the upscaling of a convection-dominated transport equation is very difficult due to the presence of the nonlocal memory effect. One of the key ideas in the multiscale analysis is to introduce a projection operator, which projects the fluctuation of the concentration onto a suitable subspace. This projection operator corresponds to averaging along streamlines of the flow. This streamline averaging eliminates the "memory effects" at the small scales. The use of this projection operation allows us to separate the $O(1)$ fluctuation of the solution from its average.

One of the main contributions of this paper is that we prove the well-posedness of the homogenized equations in the case when the convection dominated transport equations are linear. We also prove that the multiscale expansion converges with an optimal rate of convergence. It is not an easy task to establish the well-posedness of the homogenized equations. This is due to the fact that the homogenized equations involve the streamline projection operator. This averaging operator along the streamlines is not a strictly positive definite operator. In fact, it can be expressed as a degenerated semi-definite elliptic operator with a zero eigenvalue. This degeneracy of the projection operator makes the well-posedness analysis very difficult. This is one of the main reasons why there has been no existence theory for the homogenized equations so far. By using a streamline-arclength coordinate, we successfully identify some of the essential features of the homogenized equations. This enables us to establish the existence and the regularity of the homogenized solution. Building upon these results, we further establish rigorous error estimates for the leading order multiscale expansion of the multiscale solution.

In addition to proving the well-posedness of the homogenized equations and establishing rigorous error estimates, we also improve the multiscale analysis of [14] in several ways. First, our multiscale analysis can be applied to the case when the initial condition of the concentration has order one oscillations. Secondly, we find that there is no need to include the fast time variable, $\tau = t/\epsilon$, in the first order and the high order correction terms of the multiscale expansion. This simplifies the multiscale analysis and clarifies the role for which the first order correction plays. Thirdly, we present a new rigorous proof of the equivalence of the subspace projection operator and the streamline averaging projection operator.

Our ultimate goal of our study is to develop a systematic multiscale analysis to upscale two-phase miscible flows. In this case, the multiscale velocity is determined from the elliptic equation for pressure through the Darcy law. The strongly heterogeneous porous medium is the source of multiscale oscillations for the Darcy velocity. Since the concentration is now coupled to the pressure equation, the convection dominated transport equation for concentration becomes nonlinear. In order to apply

the multiscale analysis we develop for the convection dominated transport equation, we need to derive an accurate approximation of the multiscale velocity field from the elliptic equation for pressure. To separate the nonlinear coupling of the elliptic equation from the convection equation, we can use an implicit pressure-explicit concentration method (similar to the so-called IMPES framework for immiscible two-phase flows) to solve this dynamically coupled system. We briefly outline a strategy in this paper. More detailed study will be provided in a subsequent paper.

The rest of the paper is organized as follows. In Section 2, we derive the homogenized equations for linear convection dominated transport equations. The existence and regularity of the homogenized equations are analyzed in Section 3, and error analysis is performed in Section 4. Finally, we discuss how to apply the homogenization result we obtain for the linear transport equation to upscale two-phase miscible flows with heterogeneous porous media in Section 5.

2. **Homogenization of linear convection dominated transport equations.**
In this section, we will perform a systematic multiscale analysis for linear convection dominated transport equations. Specifically, we will derive the homogenized equations for the following linear transport equation with a weak diffusion:

$$\frac{\partial c^\epsilon}{\partial t} + \mathbf{u}(\mathbf{x}, \frac{\mathbf{x}}{\epsilon}) \cdot \nabla c^\epsilon = \epsilon^m \nabla \cdot \left(D(\mathbf{x}, \frac{\mathbf{x}}{\epsilon}) \nabla c^\epsilon \right), \qquad (1)$$

$$c^\epsilon|_{t=0} = c^0(\mathbf{x}, \frac{\mathbf{x}}{\epsilon}), \qquad (2)$$

where $m \in [2, \infty]$ is an integer, $\mathbf{u}(\mathbf{x}, \mathbf{y})$, $D(\mathbf{x}, \mathbf{y})$, and $c^0(\mathbf{x}, \mathbf{y})$ are assumed to be periodic in $\mathbf{y}$ and sufficiently smooth as functions of $\mathbf{x}$ and $\mathbf{y}$. Here ϵ characterizes the small scale in the media, c^ϵ describes the concentration of the flow, $\mathbf{u}$ the multiscale velocity field, D is the non-dimensionalized viscous coefficient satisfying $0 < d_0 \leq D \leq d_1 < \infty$ for some finite d_0 and d_1. Moreover, $\mathbf{u}$ and the initial concentration, c_0, satisfy the following properties:

$$\nabla_y \cdot \mathbf{u} = 0, \qquad (3)$$

$$\mathbf{u} \cdot \nabla_y c^0 = 0. \qquad (4)$$

In this paper, we consider only the initial value problem of (1) in both two and three space dimensions. Moreover, we assume that the initial condition, $c^0(\mathbf{x}, \mathbf{y})$, has compact support as a function of $\mathbf{x}$. Under this assumption, it is easy to show using linear PDE theory that the solution, $c^\epsilon(\mathbf{x}, t)$, will decay rapidly to zero as $|\mathbf{x}| \to \infty$.

Our study is motivated by our desire to develop a systematic multiscale analysis for two-phase miscible flows in strongly heterogeneous porous media when the local Peclet number is very large. Developing a systematic multiscale analysis for the transport equation (1) for concentration is an essential step in this effort. Homogenization of convection dominated transport equations is known to be extremely difficult. For the problem we consider here, the local Peclet number, which is defined as $P_e = \frac{Ul}{\nu}$, is of order $O(\epsilon^{-m+1})$. Here U is the characteristic length scale of the flow velocity, which is of order one, l is the local space length scale, which is of order $O(\epsilon)$, and ν is the viscous coefficient, which is of order $O(\epsilon^m)$.

2.1. **Multiscale analysis of the transport equation.** Guided by our multiscale analysis, we look for a multiscale expansion of the concentration in the form

$$c^\epsilon(\mathbf{x}, t) = c_0(\mathbf{x}, \mathbf{x}/\epsilon, t) + \epsilon c_1(\mathbf{x}, \mathbf{x}/\epsilon, t) + \epsilon^2 c_2(\mathbf{x}, \mathbf{x}/\epsilon, t) + \epsilon^3 c_3(\mathbf{x}, \mathbf{x}/\epsilon, t) + O(\epsilon^4), \quad (5)$$

where c_j $(j = 0, 1, 2, 3)$ are periodic functions of $\mathbf{y}$. Using the relationship $\mathbf{y} = \mathbf{x}/\epsilon$, we have

$$\nabla \;\; = \;\; \nabla_x + \frac{1}{\epsilon}\nabla_y. \tag{6}$$

Below we will derive the homogenized equations for (1). All the analysis presented in this section can be applied to both two and three space dimensions. We will consider three cases: (1) $m = 3$, (2) $m = 2$, (3) $m \geq 4$. We will primarily focus on the first case to illustrate the main idea, but we will also point out the difference of the other two cases when appropriate. Substituting our expansion (5) into the concentration equation (1) and gathering together terms with the same power of ϵ, we obtain the following hierarchy of equations:

Case 1. $m = 3$.

$$\epsilon^{-1} \;\; : \;\; \mathbf{u} \cdot \nabla_y c_0 = 0, \tag{7}$$

$$\epsilon^{0} \;\; : \;\; \frac{\partial c_0}{\partial t} + \mathbf{u} \cdot \nabla_x c_0 + \mathbf{u} \cdot \nabla_y c_1 = 0, \tag{8}$$

$$\epsilon^{1} \;\; : \;\; \frac{\partial c_1}{\partial t} + \mathbf{u} \cdot \nabla_x c_1 + \mathbf{u} \cdot \nabla_y c_2 - \nabla_y\Big(D(\mathbf{x},\mathbf{y})\nabla_y c_0\Big) = 0. \tag{9}$$

Case 2. $m = 2$.

$$\epsilon^{-1} \;\; : \;\; \mathbf{u} \cdot \nabla_y c_0 = 0, \tag{10}$$

$$\epsilon^{0} \;\; : \;\; \frac{\partial c_0}{\partial t} + \mathbf{u} \cdot \nabla_x c_0 + \mathbf{u} \cdot \nabla_y c_1 - \nabla_y\Big(D(\mathbf{x},\mathbf{y})\nabla_y c_0\Big) = 0, \tag{11}$$

$$\epsilon^{1} \;\; : \;\; \frac{\partial c_1}{\partial t} + \mathbf{u} \cdot \nabla_x c_1 + \mathbf{u} \cdot \nabla_y c_2 - \nabla_y\Big(D(\mathbf{x},\mathbf{y})\nabla_y c_1\Big) \tag{12}$$

$$-\nabla_y\Big(D(\mathbf{x},\mathbf{y})\nabla_x c_0\Big) - \nabla_x\Big(D(\mathbf{x},\mathbf{y})\nabla_y c_0\Big) = 0.$$

Case 3. $m \geq 4$.

$$\epsilon^{-1} \;\; : \;\; \mathbf{u} \cdot \nabla_y c_0 = 0, \tag{13}$$

$$\epsilon^{0} \;\; : \;\; \frac{\partial c_0}{\partial t} + \mathbf{u} \cdot \nabla_x c_0 + \mathbf{u} \cdot \nabla_y c_1 = 0, \tag{14}$$

$$\epsilon^{1} \;\; : \;\; \frac{\partial c_1}{\partial t} + \mathbf{u} \cdot \nabla_x c_1 + \mathbf{u} \cdot \nabla_y c_2 = 0. \tag{15}$$

To better understand the structure of the homogenized equations for c_0, we introduce a subspace projection operator. First, we make the assumption that all functions of the fast variable $\mathbf{y}$ are periodic with period Y and square integrable. We denote this space in the usual way by L_Y^2, which is a Hilbert space with the scalar inner product

$$(f, g)_0 = (f, g)_{L_Y^2} := \int_Y f(\mathbf{y})g(\mathbf{y})d\mathbf{y}, \tag{16}$$

and the corresponding norm is $\|f\|_0 = \sqrt{(f, f)_0}$. We also introduce the related Sobolev space H_Y^m which consists of the set of all functions f in L_Y^2 possessing weak derivatives $\partial_y^\alpha f$ in L_Y^2 for all $|\alpha| \leq m$.

Next, we define a null space $\mathcal{N}$ as follows:

$$\mathcal{N} = \{f \in H_Y^1, \mathbf{u} \cdot \nabla_y f = 0, \forall \mathbf{y} \in Y\} \subset L_Y^2. \tag{17}$$

This functional space will play an important role in our multiscale analysis. Using (7), it is clear that the leading term $c_0 \in \mathcal{N}$. We also introduce a range space $\mathcal{W}$ as follows:

$$\mathcal{W} = \{\mathbf{u} \cdot \nabla_y v : v \in H^1_Y\}. \tag{18}$$

In [14], the authors have shown that $\mathcal{N}$ and $\mathcal{W}$ form an orthogonal decomposition of L^2_Y, i.e.

$$L^2_Y = \mathcal{N} \oplus \overline{\mathcal{W}}. \tag{19}$$

Let $\mathcal{P}_\mathcal{N}$ be the projection: $H^1_Y \to \mathcal{N}$. Define the projection $\mathcal{Q}_\mathcal{N}$: $L^2_Y \to \mathcal{W}$ as

$$\|v - \mathcal{Q}_\mathcal{N}(v)\| = \min_{\theta \in H^1_Y} \|v - \mathbf{u} \cdot \nabla_y \theta\|. \tag{20}$$

As pointed out in [14], the projection operator $\mathcal{P}_\mathcal{N}(g)$ can be obtained by using $\mathcal{P}_\mathcal{N}(g) = g - \mathcal{Q}_\mathcal{N}(g)$. The operator $\mathcal{Q}_\mathcal{N}$ can be computed by $\mathcal{Q}_\mathcal{N}(g) = \mathbf{u} \cdot \nabla_y \theta$, where θ is the solution of

$$\nabla_y \cdot (E \nabla_y \theta) = \mathbf{u} \cdot \nabla_y g, \quad \mathbf{y} \in Y, \tag{21}$$

with periodic boundary condition and the matrix is defined by $E = \mathbf{u}^T \mathbf{u}$ whose (i,j) entry is given by $u_i u_j$, where $\mathbf{u} = (u_1, u_2, u_3)$.

Now, we discuss how to solve for c_0 and c_1. We first consider the case of $m = 3$. Note that c_1 can not be determined by solving (7)-(8) alone. To solve for c_1, we need to project c_1 into $\mathcal{N}$ and $\mathcal{W}$:

$$c_1 = \widetilde{c_1} + w, \tag{22}$$

where $\widetilde{c_1} \in \mathcal{N}$ and $w \in \mathcal{W}$. Substituting (22) into (7)-(8), we obtain a coupled system for c_0 and w as follows:

$$\mathbf{u} \cdot \nabla_y c_0 = 0, \tag{23}$$

$$\frac{\partial c_0}{\partial t} + \mathbf{u} \cdot \nabla_x c_0 + \mathbf{u} \cdot \nabla_y w = 0, \tag{24}$$

where $w \in \mathcal{W}$. The initial condition is give by $c_0|_{t=0} = c^\epsilon|_{t=0} = c^0(\mathbf{x}, \mathbf{y})$.

Remark 1. Note that there are two equations for c_0 given by (23)-(24), but there is no evolution equation for w. The equation for w can be derived by imposing the algebraic constraint (23) for c_0. The role of w is to enforce $\mathbf{u} \cdot \nabla_y c_0 = 0$, which is similar to the role that the pressure plays in the incompressible Navier-Stokes equations.

To determine $\widetilde{c_1}$, we use (9). By applying the projection operator $\mathcal{P}_\mathcal{N}$ to (9), we obtain

$$\frac{\partial \widetilde{c_1}}{\partial t} + \mathcal{P}_\mathcal{N}(\mathbf{u}) \cdot \nabla_x \widetilde{c_1} + \mathcal{P}_\mathcal{N}\left(\mathbf{u} \cdot \nabla_x w - \nabla_y\left(D(\mathbf{x}, \mathbf{y})\nabla_y c_0\right)\right) = 0, \tag{25}$$

where $\widetilde{c_1} \in \mathcal{N}$ with zero initial condition, i.e. $\widetilde{c_1}|_{t=0} = 0$. Note that in (25) both c_0 and w are the known solutions which we have obtained by solving (23)-(24). By applying the projection operator $\mathcal{P}_\mathcal{N}$ to (9), we have decoupled $\widetilde{c_1}$ from c_2. This gives a closed equation to determine $\widetilde{c_1}$ uniquely. Once we have solved for c_0 and w from (23)-(24), we can solve for $\widetilde{c_1}$ from (25). Clearly, if c_0 and w are smooth, so is $\widetilde{c_1}$. Thus, equations (23)-(25) completely determine c_0 and c_1.

Similarly, we can decompose c_2 into $\mathcal{N}$ and $\mathcal{W}$:

$$c_2 = \widetilde{c_2} + w_2, \tag{26}$$

where $\widetilde{c}_2 \in \mathcal{N}$ and $w_2 \in \mathcal{W}$. Substituting (26) into (9), we obtain an equation for w_2 as follows:

$$\frac{\partial c_1}{\partial t} + \mathbf{u} \cdot \nabla_x c_1 + \mathbf{u} \cdot \nabla_y w_2 - \nabla_y \Big(D(\mathbf{x}, \mathbf{y}) \nabla_y c_0 \Big) = 0. \tag{27}$$

Define

$$q = \frac{\partial c_1}{\partial t} + \mathbf{u} \cdot \nabla_x c_1 - \nabla_y \Big(D(\mathbf{x}, \mathbf{y}) \nabla_y c_0 \Big). \tag{28}$$

Since c_0 and c_1 have been determined from (23)-(25), q is a known quantity. Further, we project q into $\mathcal{N}$ and $\mathcal{W}$:

$$q = q_N + q_w, \tag{29}$$

where $q_N \in \mathcal{N}$ and $q_w \in \mathcal{W}$. The construction of $\widetilde{c}_1$ and equation (25) imply that $q_N = 0$. Moreover, since $q_w \in \mathcal{W}$, we can express

$$q_w = \mathbf{u} \cdot \nabla_y \theta_q, \tag{30}$$

for some $\theta_q \in H_Y^1$. Substituting (26) into (27) and using (30), we obtain

$$\mathbf{u} \cdot \nabla_y \theta_q + \mathbf{u} \cdot \nabla_y w_2 = 0, \tag{31}$$

which implies that

$$w_2 = -\theta_q, \tag{32}$$

is a solution of (27). Further, we can determine w_2 uniquely by projecting θ_q into $\mathcal{W}$.

To determine $\widetilde{c}_2$, we need to use the dynamic equation for c_2 and apply the projection operator $\mathcal{P}_N$ to the c_2-equation as we did before for c_1. This procedure can be continued to as high order as we wish.

As we will show in the well-posedness analysis of the homogenized equations in Section 3, c_0, c_1 and w_2 as a function of $\mathbf{x}$ will decay rapidly to zero as $|\mathbf{x}| \to \infty$.

The above analysis can be extended to the case of $m \geq 4$ in a straightforward manner by simply setting $D = 0$ in the above analysis. As for the case of $m = 2$, we can proceed in the same way. As before, we decompose

$$c_1 = \widetilde{c}_1 + w, \tag{33}$$

where $\widetilde{c}_1 \in \mathcal{N}$ and $w \in \mathcal{W}$. Substituting (33) into (10)-(11), we obtain a coupled system for c_0 and w as follows:

$$\mathbf{u} \cdot \nabla_y c_0 = 0, \tag{34}$$

$$\frac{\partial c_0}{\partial t} + \mathbf{u} \cdot \nabla_x c_0 + \mathbf{u} \cdot \nabla_y w - \nabla_y \Big(D(\mathbf{x}, \mathbf{y}) \nabla_y c_0 \Big) = 0, \tag{35}$$

where $w \in \mathcal{W}$. The initial condition is give by $c_0|_{t=0} = c^\epsilon|_{t=0} = c^0(\mathbf{x}, \mathbf{y})$. Equations (34)-(35) completely determine c_0 and w.

Next we apply the projection operator $\mathcal{P}_N$ to equation (12) to obtain

$$\frac{\partial \widetilde{c}_1}{\partial t} + \mathcal{P}_N(\mathbf{u}) \cdot \nabla_x \widetilde{c}_1 + \mathcal{P}_N \Big(\mathbf{u} \cdot \nabla_x w - \nabla_y \Big(D(\mathbf{x}, \mathbf{y}) \nabla_y (\widetilde{c}_1 + w) \Big) \Big) \tag{36}$$

$$- \mathcal{P}_N \Big(\nabla_y \Big(D(\mathbf{x}, \mathbf{y}) \nabla_x c_0 \Big) + \nabla_x \Big(D(\mathbf{x}, \mathbf{y}) \nabla_y c_0 \Big) \Big) = 0,$$

where $\widetilde{c}_1 \in \mathcal{N}$ with zero initial condition, i.e. $\widetilde{c}_1|_{t=0} = 0$. This completely determines the first order correction c_1.

We would like to point out that w and w_2 in general may not be equal to zero at $t = 0$. Thus, the multiscale expansion may only match the exact initial condition up to $O(\epsilon)$.

2.2. **Average and fluctuation equations for c_0.** In this subsection, we derive a coupled system that governs the evolution of the average and the fluctuation of the leading order concentration, c_0. For this purpose, we first introduce the concept of an average quantity. Given a function $g(\mathbf{x}, \mathbf{y})$, we define the average of g as follows:

$$\overline{g} = \frac{1}{|Y|} \int_Y g(\mathbf{x}, \mathbf{y}) d\mathbf{y}. \tag{37}$$

The fluctuating part g' of function g is denoted in a natural way by

$$g'(\mathbf{x}, \mathbf{y}) = g(\mathbf{x}, \mathbf{y}) - \overline{g}(\mathbf{x}). \tag{38}$$

The average (37) can be thought of as a smoothing or spatial "filtering" of the small scales (c.f. Beckie et. al [3]). It is clear that the fluctuation has zero average, i.e. $\overline{g'} = 0$.

We now decompose c_0 and $\mathbf{u}$ into the sum of their average and fluctuation:

$$c_0(\mathbf{x}, \mathbf{y}, t) = \overline{c_0}(\mathbf{x}, t) + c_0'(\mathbf{x}, \mathbf{y}, t), \tag{39}$$

$$\mathbf{u}(\mathbf{x}, \mathbf{y}) = \overline{\mathbf{u}}(\mathbf{x}) + \mathbf{u}'(\mathbf{x}, \mathbf{y}). \tag{40}$$

In the cases of $m \geq 3$, we apply the average operator to (24) and use (3) to obtain

$$\frac{\partial \overline{c_0}}{\partial t} \quad + \quad \overline{\mathbf{u}} \cdot \nabla_x \overline{c_0} + \overline{\mathbf{u}' \cdot \nabla_x c_0'} = 0, \tag{41}$$

$$\frac{\partial c_0'}{\partial t} \quad + \quad \overline{\mathbf{u}} \cdot \nabla_x c_0' + \mathbf{u}' \cdot \nabla_x \overline{c_0} \tag{42}$$

$$+ \quad \mathbf{u}' \cdot \nabla_x c_0' - \overline{\mathbf{u}' \cdot \nabla_x c_0'} + \mathbf{u} \cdot \nabla_y w = 0.$$

In the case of $m = 2$, the average equation remains the same, but the fluctuation equation is slightly modified. The new coupled system becomes

$$\frac{\partial \overline{c_0}}{\partial t} \quad + \quad \overline{\mathbf{u}} \cdot \nabla_x \overline{c_0} + \overline{\mathbf{u}' \cdot \nabla_x c_0'} = 0, \tag{43}$$

$$\frac{\partial c_0'}{\partial t} \quad + \quad \overline{\mathbf{u}} \cdot \nabla_x c_0' + \mathbf{u}' \cdot \nabla_x \overline{c_0} + \mathbf{u}' \cdot \nabla_x c_0' - \overline{\mathbf{u}' \cdot \nabla_x c_0'} \tag{44}$$

$$+ \quad \mathbf{u} \cdot \nabla_y w - \nabla_y \Big(D(\mathbf{x}, \mathbf{y}) \nabla_y c_0' \Big) = 0.$$

We remark that the term, $\overline{\mathbf{u}' \cdot \nabla_x c_0'}$, in (41) and (43) plays a role similar to the Reynolds stress term in turbulence modeling. This is the term which introduces the nonlocal memory effect into the average equation.

2.3. **Equivalence of the subspace and the streamline projections.** Before we end this section, we give another interpretation of the projection operator, $\mathcal{P}_{\mathcal{N}}$. As pointed out in [14], the projection operator $\mathcal{P}_{\mathcal{N}}$ can be interpreted as a streamline averaging operator. To see this, we define an average projection $\mathcal{P}$ along the streamline. Let $\Theta(\mathbf{x}, t; \tau, \mathbf{y})$ be a flow map defined by

$$\frac{d\Theta}{d\tau} = \mathbf{u}(\mathbf{x}, \Theta, t), \quad \Theta(0) = \mathbf{y}. \tag{45}$$

The streamline projection of a given function $g(\mathbf{x}, \mathbf{y}, t)$ is defined as follows:

$$\mathcal{P}g(\mathbf{x}, \mathbf{y}, t) = \lim_{T \to \infty} \frac{1}{T} \int_0^T g(\mathbf{x}, \Theta(\tau), t) d\tau. \tag{46}$$

Moreover, as pointed out in [14], the streamline averaging projection operator $\mathcal{P}$ defined above is the same as the projection operator $\mathcal{P}_{\mathcal{N}}$ defined earlier. This is stated in the following lemma:

Lemma 2.1. *(Lemma 3.13 in* [14]*) Assume that* $g(\mathbf{x}, \mathbf{y}, t) \in H_Y^1$ *as a function of* $\mathbf{y}$. *Then we have* $\mathcal{P}_\mathcal{N}(g) = \mathcal{P}(g)$.

For the sake of completeness, we give a new constructive proof of this lemma.

Proof. First, we observe that

$$
\begin{aligned}
\nabla_y \cdot (E\nabla_y \theta) &= (\mathbf{u} \cdot \nabla_y)(\mathbf{u} \cdot \nabla_y \theta) + (\nabla_y \cdot \mathbf{u})(\mathbf{u} \cdot \nabla_y \theta) \\
&= (\mathbf{u} \cdot \nabla_y)(\mathbf{u} \cdot \nabla_y \theta),
\end{aligned}
\tag{47}
$$

since $\nabla_y \cdot \mathbf{u} = 0$. Moreover, we have from (45) that

$$
(\mathbf{u} \cdot \nabla_y)\theta(\mathbf{x}, \Theta, t) = \frac{d\theta}{d\tau}(\mathbf{x}, \Theta, t), \quad (\mathbf{u} \cdot \nabla_y)(\mathbf{u} \cdot \nabla_y \theta)(\mathbf{x}, \Theta, t) = \frac{d}{d\tau}\left(\frac{d\theta}{d\tau}\right). \tag{48}
$$

Using (47) and (48), we can reformulate equation (21) along $\mathbf{y} = \Theta$ as follows:

$$
\frac{d}{d\tau}\left(\frac{d\theta}{d\tau}\right) = \frac{dg}{d\tau}, \tag{49}
$$

which implies that

$$
\frac{d\theta}{d\tau} = g + c, \tag{50}
$$

where c is a constant along the characteristics $\Theta(\mathbf{y}, \tau)$ for a given initial condition $\mathbf{y}$. By integrating (50) along the characteristics, we can show that $c = -\mathcal{P}(g)$, where $\mathcal{P}(g)$ is defined in (46). By integrating (50) along the characteristics from 0 to τ, we obtain

$$
\theta(\mathbf{x}, \Theta(\tau), t) = \theta(\mathbf{x}, \mathbf{y}, t) + \int_0^\tau g(\mathbf{x}, \Theta(\tau'), t)d\tau' - \tau\mathcal{P}(g). \tag{51}
$$

We can see that if g and $\mathbf{u}$ are smooth, so is θ. Moreover, we have from (46) that

$$
\lim_{T \to \infty}\left(\frac{\theta(\mathbf{x}, \Theta(T), t) - \theta(\mathbf{x}, \mathbf{y}, t)}{T}\right) = \lim_{T \to \infty}\frac{1}{T}\int_0^T g(\mathbf{x}, \Theta(\tau), t)d\tau - \mathcal{P}(g) = 0. \tag{52}
$$

Using (50), we get

$$
Q_\mathcal{N}(g) = \mathbf{u} \cdot \nabla_y \theta = \frac{d\theta}{d\tau} = g - \mathcal{P}(g). \tag{53}
$$

Therefore, we obtain

$$
P_\mathcal{N}(g) = g - Q_\mathcal{N}(g) = \mathcal{P}(g). \tag{54}
$$

This proves the lemma. □

Remark 2. We remark that Bourgeat, Jurak and Piatnitski have performed multiscale analysis and error estimates for transport equations in [5] in the case when local Peclet number P_e is of order one, which corresponds to the case of $m = 1$ in (1). With $O(\epsilon)$ diffusion coefficient, the leading order equation for c_0 becomes

$$
\mathbf{u} \cdot \nabla_y c_0 = \frac{1}{P_e}\nabla_y \cdot \left(D(\mathbf{x}, \mathbf{y})\nabla_y c_0\right). \tag{55}
$$

This leading order constraint equation is diffusion dominated if P_e is $O(1)$. This is very different from the corresponding constraint equation (7) that we consider for $m \geq 2$, which is hyperbolic in nature. Equation (55) and the divergence free condition $\nabla_y \cdot \mathbf{u} = 0$ immediately imply that c_0 is independent of $\mathbf{y}$, which shows that c^ϵ has no small scale oscillation to the leading order in the case they considered

in [5]. This is very different from the case that we consider here with $m \geq 2$ since c_0 has order one oscillation.

3. **The well-posedness of the homogenized equations for c_0 and w.** In this section, we will prove the existence of the leading order term c_0 as well as w. As we have shown in the previous section, the leading order term $c_0(\mathbf{x}, \mathbf{y}, t)$ satisfies the following equation:

$$\frac{\partial c_0}{\partial t} + \mathbf{u} \cdot \nabla_x c_0 + \mathbf{u} \cdot \nabla_y w - \nu \nabla_y (D(\mathbf{y}) \nabla_y c_0) = 0, \quad on \ \Omega \times Y, \qquad (56)$$

$$\mathbf{u} \cdot \nabla_y c_0 = 0, \quad on \ \Omega \times Y, \qquad (57)$$

with initial condition $c_0|_{t=0} = c^0(\mathbf{x}, \mathbf{y})$, where c_0, w and $\mathbf{u}$ are periodic in $\mathbf{y}$ and $w \in \mathcal{W}$. Here $\Omega = \mathbb{R}^n$, n is the space dimension. We introduce a parameter ν here to unify different cases that we consider. In the case of $m \geq 3$, we have $\nu = 0$. In the case of $m = 2$, we have $\nu = 1$.

The main purpose of this section is to prove the existence and regularity of solution c_0 and w of problem (56) - (57).

Theorem 3.1. *Assume that $\Omega = \mathbb{R}^2$ and that the initial condition, the velocity and the diffusion coefficient are smooth on $\Omega \times Y$. Further, we assume that $|\mathbf{u}| \geq \alpha_0 > 0$ on $\Omega \times Y$. Then there exists a unique smooth solution c_0 and w of problem (56) - (57), satisfying*

$$\|c_0\|_{H^1(\Omega \times Y)}(t) \leq M \|c_0\|_{H^1(\Omega \times Y)}(0). \qquad (58)$$

Moreover, if the initial condition for c_0 and the coefficients are sufficiently smooth, then both c_0 and w are also sufficiently smooth.

Proof. It is difficult to analyze the well-posedness of the homogenized equation for c_0 directly. To better understand the structure of the homogenized equation for c_0, we introduce the streamline function ψ for each given $\mathbf{x}$, which is defined as follows:

$$\nabla_y \psi \ = \ (-u_2, u_1)^\tau. \qquad (59)$$

Note that $\mathbf{u}(\mathbf{x}, \mathbf{y}) = \overline{\mathbf{u}}(\mathbf{x}) + \mathbf{u}'(\mathbf{x}, \mathbf{y})$. Thus ψ can be written in the following form:

$$\psi = -y_1 \overline{u}_2(\mathbf{x}) + y_2 \overline{u}_1(\mathbf{x}) + \psi'(\mathbf{x}, \mathbf{y}) \qquad (60)$$

where ψ' is a Y-periodic function with mean zero, satisfying

$$\nabla_y \psi' = (-u_2', u_1')^\tau. \qquad (61)$$

Alternatively, we have $\Delta_y \psi' = \dfrac{\partial u_1'}{\partial y_2} - \dfrac{\partial u_2'}{\partial y_1}$. Note that the stream function ψ is a function of $(\mathbf{x}, \mathbf{y})$.

Next, we introduce a weighted arclength variable along a streamline. We assume that $|\mathbf{u}| \neq 0$, $\mathbf{y} \in Y$. The weighted arclength variable s satisfies

$$\nabla_y s = \frac{1}{|\mathbf{u}|^2} (u_1, u_2)^\tau. \qquad (62)$$

The change of variables from (y_1, y_2) to (ψ, s) is nonsingular since

$$\det \left(\frac{\partial(\psi, s)}{\partial(y_1, y_2)} \right) = (-\psi_{y_2}, \psi_{y_1}) \cdot (s_{y_1}, s_{y_2}) = (-\mathbf{u}) \cdot \nabla_y s = -1, \qquad (63)$$

by (62). Along a streamline $\psi = C$, we have

$$\frac{ds}{dl} = \frac{1}{|\mathbf{u}|}, \qquad (64)$$

where l is the arclength variable along the streamline. Thus $s(\mathbf{y}) = \int_{\Gamma(\mathbf{y}_0, \mathbf{y})} \frac{1}{|\mathbf{u}|} dl$, where $\Gamma(\mathbf{y}_0, \mathbf{y})$ is the part of the streamline from $\mathbf{y}_0$ to $\mathbf{y}$.

For each fixed $\mathbf{x}$ and t, we make a change of variable from the $\mathbf{y}$ variable to the $\{\psi, s\}$ coordinates. Since $|\mathbf{u}| \neq 0$, $\mathbf{y} \in Y$, this change of variables is nonsingular. By the chain rule, we get

$$\nabla_y c_0 = \nabla_y \psi \frac{\partial c_0}{\partial \psi} + \nabla_y s \frac{\partial c_0}{\partial s}, \tag{65}$$

which implies that

$$\mathbf{u} \cdot \nabla_y c_0 = \mathbf{u} \cdot \nabla_y \psi \frac{\partial c_0}{\partial \psi} + \mathbf{u} \cdot \nabla_y s \frac{\partial c_0}{\partial s}. \tag{66}$$

Note that $\mathbf{u} \cdot \nabla_y \psi = 0$, $\mathbf{u} \cdot \nabla_y s = 1$, and $\mathbf{u} \cdot \nabla_y c_0 = 0$. Therefore, we obtain

$$\mathbf{u} \cdot \nabla_y c_0 = \frac{\partial c_0}{\partial s} = 0. \tag{67}$$

We conclude from (67) that c_0 does not depend on s, i.e. $c_0 = c_0(\mathbf{x}, \psi)$. Thus we have

$$\nabla_y c_0 = \nabla_y \psi \frac{\partial c_0}{\partial \psi}. \tag{68}$$

Further, for any smooth function $k(\mathbf{y})$ we have

$$\nabla_y \left(k(\mathbf{x}, \mathbf{y}) \nabla_y c_0 \right) = |\mathbf{u}|^2 k \frac{\partial^2 c_0}{\partial \psi^2} + \left(|\mathbf{u}|^2 \frac{\partial k}{\partial \psi} + k \Delta_y \psi \right) \frac{\partial c_0}{\partial \psi}. \tag{69}$$

Similarly, we have $\mathbf{u} \cdot \nabla_y w = \frac{\partial w}{\partial s}$. Using the relation (69) with $k(\mathbf{x}, \mathbf{y}) = D(\mathbf{x}, \mathbf{y})$, equations (56) and (57) become

$$\frac{\partial c_0}{\partial t} + \mathbf{u} \cdot \nabla_x c_0 - \nu |\mathbf{u}|^2 D \frac{\partial^2 c_0}{\partial \psi^2} - \nu \left(|\mathbf{u}|^2 \frac{\partial D}{\partial \psi} + D \Delta_y \psi \right) \frac{\partial c_0}{\partial \psi} + \frac{\partial w}{\partial s} = 0, \tag{70}$$

$$\frac{\partial c_0}{\partial s} = 0. \tag{71}$$

Differentiating (70) with respect to s and using (71) lead to the following equation:

$$\frac{\partial}{\partial s} \left[\mathbf{u} \cdot \nabla_x c_0 - \nu |\mathbf{u}|^2 D \frac{\partial^2 c_0}{\partial \psi^2} - \nu \left(|\mathbf{u}|^2 \frac{\partial D}{\partial \psi} + D \Delta_y \psi \right) \frac{\partial c_0}{\partial \psi} \right] + \frac{\partial}{\partial s} \left(\frac{\partial w}{\partial s} \right) = 0, \tag{72}$$

which implies

$$\frac{\partial w}{\partial s} + \mathbf{u} \cdot \nabla_x c_0 - \nu |\mathbf{u}|^2 D \frac{\partial^2 c_0}{\partial \psi^2} - \nu \left(|\mathbf{u}|^2 \frac{\partial D}{\partial \psi} + D \Delta_y \psi \right) \frac{\partial c_0}{\partial \psi} = g(\psi), \tag{73}$$

where $g(\psi)$ is a function of ψ.

Denote $\mathcal{P}_s$ as the average projection along the streamline with respect to the weighted arc-length coordinate s:

$$\mathcal{P}_s f(\psi) = \lim_{L \to \infty} \frac{1}{L} \int_0^L f(\psi, s) ds. \tag{74}$$

Since the velocity field $\mathbf{u}$ is independent of t, it is easy to see that $\mathcal{P}_s$ is equivalent to the streamline average projection operator defined in (46). From the definition (74), it is clear that if $f(\psi, s)$ is smooth then $\mathcal{P}_s f(\psi)$ is smooth. Moreover, we have $\mathcal{P}_s \left(\frac{\partial f}{\partial \psi} \right)(\psi) = \frac{\partial}{\partial \psi} \left(\mathcal{P}_s f \right)$. Thus $\mathcal{P}_s f$ is sufficiently smooth if $\mathbf{u}$ and f are sufficiently smooth.

Taking the average of (73) along the streamline for each ψ, we obtain

$$
\begin{aligned}
g(\psi) &= \mathcal{P}_s\Big[\mathbf{u}\cdot\nabla_x c_0 - \nu|\mathbf{u}|^2 D\frac{\partial^2 c_0}{\partial\psi^2} - \nu\Big(|\mathbf{u}|^2\frac{\partial D}{\partial\psi} + D\Delta_y\psi\Big)\frac{\partial c_0}{\partial\psi}\Big] \qquad (75) \\
&= \mathcal{P}_s(\mathbf{u})\cdot\nabla_x c_0 - \nu\mathcal{P}_s\Big(|\mathbf{u}|^2 D\Big)\frac{\partial^2 c_0}{\partial\psi^2} - \nu\mathcal{P}_s\Big(|\mathbf{u}|^2\frac{\partial D}{\partial\psi} + D\Delta_y\psi\Big)\frac{\partial c_0}{\partial\psi}.
\end{aligned}
$$

With $g(\psi)$ defined as above, we can easily show that w satisfies the property:

$$
\lim_{L\to\infty}\frac{1}{L}\int_0^L\frac{\partial w}{\partial s}ds = \lim_{L\to\infty}\frac{w(\psi,L) - w(\psi,0)}{L} = 0. \qquad (76)
$$

Since the change of variable is $\mathbf{x}$-dependent, we have $\nabla_x c_0 = \nabla_x c_0 + \nabla_x\psi\frac{\partial c_0}{\partial\psi}$. Using (70), (73), and (75), we derive an equivalent equation for $c_0(\mathbf{x},\psi,t)$ as follows:

$$
\frac{\partial c_0}{\partial t} + \widetilde{\mathbf{v}}\cdot\nabla_x c_0 + \widetilde{\mathbf{v}}_\psi\frac{\partial c_0}{\partial\psi} - \nu\widetilde{D}_\psi\frac{\partial^2 c_0}{\partial\psi^2} = 0, \qquad (77)
$$

where

$$
\widetilde{\mathbf{v}} = \mathcal{P}_s(\mathbf{u}), \qquad (78)
$$

$$
\widetilde{\mathbf{v}}_\psi = \mathcal{P}_s(\mathbf{u})\cdot\nabla_y\psi - \nu\mathcal{P}_s\Big(|\mathbf{u}|^2\frac{\partial D}{\partial\psi} + D\Delta_y\psi\Big), \qquad (79)
$$

$$
\widetilde{D}_\psi = \mathcal{P}_s\Big(|\mathbf{u}|^2 D\Big). \qquad (80)
$$

Since $D(\mathbf{x},\mathbf{y})$ and $\mathbf{u}(\mathbf{x},\mathbf{y})$ are smooth functions on Y, we conclude that $\widetilde{\mathbf{v}}$ and $\widetilde{\mathbf{v}}_\psi$ are smooth vector and scalar functions respectively.

It is worth emphasizing that the above analysis provides a new dynamic equation for c_0 by averaging along the streamline. Equation (77) is now self-contained and is decoupled from the w variable. In some sense, we have explicitly carried out the projection operator to the c_0-equation and obtained a well-posed convection diffusion equation for c_0. This is a critical step in our analysis.

Once we have obtained for c_0 from (77), we can solve for w by integrating (73) along each streamline. Note that w is uniquely determined by projecting a solution obtained this way to the space of $\mathcal{W}$ to satisfy the condition $w\in\mathcal{W}$. Moreover, since the change of variables from (y_1,y_2) to (ϕ,s) is nonsingular, we can solve for w for all (y_1,y_2) in Y by integrating (73) along streamlines.

Unfortunately, we can not analyze the well-posedness of the c_0 equation directly in the streamline coordinate since the problem is not necessarily periodic in the ψ variable. To overcome this difficulty, we transfer the above equation for c_0 back to $\{y_1,y_2\}$ coordinates. In doing this, we still maintain the essential structure of the c_0-equation in the streamline coordinate, but we now have periodic boundary conditions with respect to the $\mathbf{y}$ variable.

Using the transformation $y_1 = y_1(\psi,s)$, $y_2 = y_2(\psi,s)$ and the chain rule, we get

$$
\frac{\partial c_0}{\partial\psi} = \frac{\partial c_0}{\partial y_1}\frac{\partial y_1}{\partial\psi} + \frac{\partial c_0}{\partial y_2}\frac{\partial y_2}{\partial\psi}. \qquad (81)
$$

Moreover, using the relation (69) with $k(\mathbf{x},\mathbf{y}) = \dfrac{\widetilde{D}_\psi}{|\mathbf{u}|^2}$, we obtain

$$
\widetilde{D}_\psi\frac{\partial^2 c_0}{\partial\psi^2} = \nabla_y\Big(\frac{\widetilde{D}_\psi}{|\mathbf{u}|^2}\nabla_y c_0\Big) - \Big(\frac{\partial\widetilde{D}_\psi}{\partial\psi} + \frac{1}{|\mathbf{u}|^2}\widetilde{D}_\psi\frac{\partial|\mathbf{u}|^2}{\partial\psi} + \frac{\widetilde{D}_\psi}{|\mathbf{u}|^2}\Delta_y\psi\Big)\frac{\partial c_0}{\partial\psi}. \qquad (82)
$$

Using (81)-(82) and (77), we obtain the equation for c_0 in the $\{y_1, y_2\}$ coordinates as follows:

$$\frac{\partial c_0}{\partial t} + \widetilde{\mathbf{v}} \cdot \nabla_x c_0 + \widetilde{\boldsymbol{\eta}} \cdot \nabla_y c_0 - \nu \nabla_y \Big(\frac{\widetilde{D}_\psi}{|\mathbf{u}|^2} \nabla_y c_0 \Big) = 0, \tag{83}$$

with initial condition $c_0|_{t=0} = c^0(\mathbf{x}, \mathbf{y})$, where c_0 is Y - periodic in the $\mathbf{y}$ variable, and

$$\widetilde{\mathbf{v}} = \mathcal{P}_s(\mathbf{u}), \tag{84}$$

$$\widetilde{\boldsymbol{\eta}} = \Big[\widetilde{\mathbf{v}}_\psi - \nu \frac{\partial \widetilde{D}_\psi}{\partial \psi} - \nu \frac{1}{|\mathbf{u}|^2} \widetilde{D}_\psi \frac{\partial |\mathbf{u}|^2}{\partial \psi} - \nu \frac{\widetilde{D}_\psi}{|\mathbf{u}|^2} \Delta_y \psi \Big] \Big(\frac{\partial y_1}{\partial \psi}, \frac{\partial y_2}{\partial \psi} \Big)^\tau. \tag{85}$$

Note that equation (83) is a linear convection diffusion equation in $\mathbb{R}^2 \times Y$ with smooth coefficients. In particular, the equation is hyperbolic along the $\mathbf{x}$-direction. Let $\boldsymbol{\alpha}(\mathbf{x}, t)$ be the Lagrangian coordinate corresponding to the velocity field $\widetilde{\mathbf{v}}$. By changing the variable from $\mathbf{x}$ to $\boldsymbol{\alpha}(\mathbf{x}, t)$, we can further eliminate the convection term along the $\mathbf{x}$ direction. The equation can be considered as a function $\mathbf{y}$ alone, with α being a parameter. Since the initial condition, $c_0(\mathbf{x}, \mathbf{y})$, has compact support as a function of $\mathbf{x}$ and since the velocity field $\widetilde{\mathbf{v}}$ is bounded, the Lagrangian image of $\mathbf{x}$, $\boldsymbol{\alpha}(\mathbf{x}, t)$ for $|\mathbf{x}|$ sufficiently large would lie outside the support of the initial condition, $c_0(\mathbf{x}, \mathbf{y})$. Thus the corresponding initial condition formulated in the Lagrangian coordinate $\boldsymbol{\alpha}$ and as a function of $\mathbf{y}$ is zero if $|\mathbf{x}|$ is sufficiently large. As a consequence, the governing equation of c_0 as a function of $\mathbf{y}$ has only zero solution for $|\mathbf{x}|$ sufficiently large. This shows that the solution of c_0 as a function of $\mathbf{x}$ also has compact support for any finite time. Now we can use (73) and integrate along streamline to conclude that w also has compact support in $\mathbf{x}$.

Let $\alpha_1 = \max\limits_{Y \times \Omega} |\mathbf{u}|^2$ and $\alpha_0 = \min\limits_{Y \times \Omega} |\mathbf{u}|^2$. Since $|\mathbf{u}|^2 \neq 0$ on $Y \times \Omega$, we have $\alpha_1 \geq \alpha_0 > 0$. Further, we have $0 < d_0 \leq D \leq d_1$ by our assumption. Thus we get

$$d_0 \alpha_0 \leq \widetilde{D}_\psi = \mathcal{P}_s\Big(|\mathbf{u}|^2 D\Big) = \lim_{L \to \infty} \frac{1}{L} \int_0^L |\mathbf{u}|^2 D ds \leq d_1 \alpha_1, \tag{86}$$

and

$$0 < d_0^* \equiv \frac{d_0 \alpha_0}{\alpha_1} \leq \frac{\widetilde{D}_\psi}{|\mathbf{u}|^2} \leq \frac{d_1 \alpha_1}{\alpha_0}, \tag{87}$$

on $Y \times \Omega$.

Using equation (83), we can prove the the existence and regularity of solution c_0 in the Sobolev norm. We first estimate $\|c_0\|_{L^2(\Omega \times Y)}(t)$. Let Ω denote the entire $\mathbb{R}^2$ space. Multiplying c_0 to equation (83) and integrating over $\Omega \times Y$, we estimate each term on the both sides of the equation as follows:

$$\int_\Omega \int_Y \frac{\partial c_0}{\partial t} c_0 d\mathbf{y} d\mathbf{x} = \frac{1}{2} \frac{d}{dt} \|c_0\|_{L^2(\Omega \times Y)}^2, \tag{88}$$

$$\int_\Omega \int_Y \widetilde{v} \cdot (\nabla_x c_0) c_0 d\mathbf{y} d\mathbf{x} = -\frac{1}{2} \int_\Omega \int_Y \nabla_x \cdot \widetilde{\mathbf{v}} c_0^2 d\mathbf{y} d\mathbf{x}, \tag{89}$$

$$\int_\Omega \int_Y \widetilde{\boldsymbol{\eta}} \nabla_y c_0 c_0 d\mathbf{y} d\mathbf{x} = -\frac{1}{2} \int_\Omega \int_Y \nabla_y \cdot \widetilde{\boldsymbol{\eta}} c_0^2 d\mathbf{y} d\mathbf{x}, \tag{90}$$

$$\int_\Omega \int_Y \nabla_y \Big(\frac{\widetilde{D}_\psi}{|\mathbf{u}|^2} \nabla_y c_0 \Big) c_0 d\mathbf{y} d\mathbf{x} = -\int_\Omega \int_Y \frac{\widetilde{D}_\psi}{|\mathbf{u}|^2} \nabla_y c_0 \cdot \nabla_y c_0 d\mathbf{y} d\mathbf{x}, \tag{91}$$

where we have used the periodicity of c_0 and $\widetilde{\boldsymbol{\eta}}$ as a function of $\mathbf{y}$ and the fact that c_0 has compact support as a function of $\mathbf{x}$. From the Hölder inequality, it holds that

$$\frac{1}{2}\frac{d}{dt}\|c_0\|^2_{L^2(\Omega\times Y)} + \nu d_0^* \int_\Omega \int_Y |\nabla_y c_0|^2 d\mathbf{y}d\mathbf{x} \leq M\|c_0\|^2_{L^2(\Omega\times Y)}. \tag{92}$$

It follows from the Gronwall inequality that

$$\|c_0\|^2_{L^2(\Omega\times Y)}(t) \leq C_M\|c_0\|^2_{L^2(\Omega\times Y)}(0). \tag{93}$$

We now estimate $c_{0,x_1} = \frac{\partial c_0}{\partial x_1}$ and $c_{0,x_2} = \frac{\partial c_0}{\partial x_2}$. Differentiating (83) with respect to x_1, we get

$$\frac{\partial c_{0,x_1}}{\partial t} \quad + \quad \widetilde{\mathbf{v}}\cdot\nabla_x c_{0,x_1} + \frac{\partial\widetilde{\mathbf{v}}}{\partial x_1}\cdot\nabla_x c_0 + \widetilde{\boldsymbol{\eta}}\cdot\nabla_y c_{0,x_1} + \frac{\partial\widetilde{\boldsymbol{\eta}}}{\partial x_1}\cdot\nabla_y c_0$$
$$- \quad \nu\nabla_y\Big(\frac{\widetilde{D}_\psi}{|\mathbf{u}|^2}\nabla_y c_{0,x_1}\Big) - \nu\nabla_y\Big[\frac{\partial}{\partial x_1}\Big(\frac{\widetilde{D}_\psi}{|\mathbf{u}|^2}\Big)\nabla_y c_0\Big] = 0. \tag{94}$$

Multiply c_{0,x_1} to equation (94) and integrate over $\Omega\times Y$. Note that

$$\int_\Omega\int_Y \frac{\partial\widetilde{\mathbf{v}}}{\partial x_1}\cdot(\nabla_x c_0)c_{0,x_1}d\mathbf{y}d\mathbf{x} \leq M(\|c_{0,x_1}\|^2_{L^2(\Omega\times Y)} + \|c_{0,x_2}\|^2_{L^2(\Omega\times Y)}), \tag{95}$$

and

$$\int_\Omega\int_Y \frac{\partial\widetilde{\boldsymbol{\eta}}}{\partial x_1}\cdot(\nabla_y c_0)c_{0,x_1}d\mathbf{y}d\mathbf{x} \leq M(\|c_{0,x_1}\|^2_{L^2(\Omega\times Y)} + \|\nabla_y c_0\|^2_{L^2(\Omega\times Y)}). \tag{96}$$

We obtain

$$\frac{1}{2}\frac{d}{dt}\|c_{0,x_1}\|^2_{L^2(\Omega\times Y)} \quad + \quad \frac{1}{2}\nu d_0^*\|\nabla_y c_{0,x_1}\|^2_{L^2(\Omega\times Y)} \leq M\Big(\|c_{0,x_1}\|^2_{L^2(\Omega\times\mathbf{Y})}$$
$$+ \quad \|c_{0,x_2}\|^2_{L^2(\Omega\times Y)} + \|\nabla_y c_0\|^2_{L^2(\Omega\times Y)}\Big). \tag{97}$$

The estimates for c_{0,x_2} and $\nabla_y c_0$ can be carried out similarly. This gives

$$\frac{1}{2}\frac{d}{dt}\|c_{0,x_2}\|^2_{L^2(\Omega\times Y)} \quad + \quad \frac{1}{2}\nu d_0^*\|\nabla_y c_{0,x_2}\|^2_{L^2(\Omega\times Y)} \leq M\Big(\|c_{0,x_1}\|^2_{L^2(\Omega\times Y)}$$
$$+ \quad \|c_{0,x_2}\|^2_{L^2(\Omega\times Y)} + \|\nabla_y c_0\|^2_{L^2(\Omega\times Y)}\Big), \tag{98}$$

and

$$\frac{1}{2}\frac{d}{dt}\|\nabla_y c_0\|^2_{L^2(\Omega\times Y)} \quad + \quad \frac{1}{2}\nu d_0^*\|\nabla_y\nabla_y c_0\|^2_{L^2(\Omega\times Y)} \leq M\Big(\|c_{0,x_1}\|^2_{L^2(\Omega\times Y)}$$
$$+ \quad \|c_{0,x_2}\|^2_{L^2(\Omega\times Y)} + \|\nabla_y c_0\|^2_{L^2(\Omega\times Y)}\Big). \tag{99}$$

Combining (97)-(99) and using the Gronwall inequality, we obtain the desired estimate for $\nabla_x c_0$ and $\nabla_y c_0$:

$$\|\nabla_x c_0\|^2_{L^2(\Omega\times Y)}(t) \quad + \quad \|\nabla_y c_0\|^2_{L^2(\Omega\times Y)}(t)$$
$$\leq \quad C_M\Big(\|\nabla_x c_0\|^2_{L^2(\Omega\times Y)}(0) + \|\nabla_y c_0\|^2_{L^2(\Omega\times Y)}(0)\Big).$$

Furthermore, we can obtain the estimates of any order derivatives of c_0 if the initial condition and the coefficients are sufficiently smooth.

Note that the above analysis applies to both the case of $\nu = 1$ and $\nu = 0$. Thus we obtain the existence and regularity of c_0 for all the cases $m \geq 2$.

Recall that w can be constructed from c_0 by integrating (73) along the streamline and projecting it to the space of $\mathcal{W}$. The regularity of c_0 implies that w is also sufficiently smooth since the change of variables between (y_1, y_2) and (ψ, s) is non-singular and is sufficiently smooth. This proves Theorem 3.1. $\square$

Remark 3. In the three dimensional case, we can obtain a similar well-posedness result for problem (56) - (57) if we make the additional assumption that the velocity field can be expressed globally by a pair of stream functions $\{\psi_1, \psi_2\}$ as follows:

$$\mathbf{u} = \nabla_y \psi_1 \times \nabla_y \psi_2. \tag{100}$$

This representation of dual stream functions of velocity (100) has been used in the visualization of a relatively large class of 3D incompressible flows that contain no critical points, see e.g, [7, 11, 18, 22]. As in the two-dimensional case, we can define a weighted arclength variable s along a streamline as follows:

$$\nabla_y s = \frac{1}{|\mathbf{u}|^2} (u_1, u_2, u_3)^\tau. \tag{101}$$

Thus, we obtain a change of coordinates from the $\mathbf{y}$ coordinates to the (ψ_1, ψ_2, s) coordinates. The change of variables from $\mathbf{y}$ to (ψ_1, ψ_2, s) is nonsingular since we have

$$\det\left(\frac{\partial(\psi_1, \psi_2, s)}{\partial(y_1, y_2, y_3)} \right) = -\nabla_y \psi \times \nabla_y \psi_2 \cdot \nabla_y s = (-\mathbf{u}) \cdot \nabla_y s = -1, \tag{102}$$

by (101) and our assumption that $|\mathbf{u}| \neq 0$. By the chain rule, we get

$$\nabla_y c_0 = \nabla_y \psi_1 \frac{\partial c_0}{\partial \psi_1} + \nabla_y \psi_2 \frac{\partial c_0}{\partial \psi_2} + \nabla_y s \frac{\partial c_0}{\partial s}. \tag{103}$$

Observe that $\mathbf{u} \cdot \nabla_y \psi_1 = 0$, $\mathbf{u} \cdot \nabla_y \psi_2 = 0$, and $\mathbf{u} \cdot \nabla_y s = 1$. Thus, equation (103) implies that

$$\frac{\partial c_0}{\partial s} = \mathbf{u} \cdot \nabla_y c_0 = 0, \tag{104}$$

which is equation (71) in the two-dimensional case. Further, we can derive that

$$\frac{\partial c_0}{\partial t} + \mathbf{u} \cdot \nabla_x c_0 \quad - \quad \nu D\Big(|\nabla_y \psi_1|^2 \frac{\partial^2 c_0}{\partial \psi_1^2} + 2\nabla_y \psi_1 \cdot \nabla_y \psi_2 \frac{\partial^2 c_0}{\partial \psi_1 \partial \psi_2} + |\nabla_y \psi_2|^2 \frac{\partial^2 c_0}{\partial \psi_2^2} \Big)$$

$$- \quad \nu \xi_{\psi_1} \frac{\partial c_0}{\partial \psi_1} - \nu \xi_{\psi_2} \frac{\partial c_0}{\partial \psi_2} + \frac{\partial w}{\partial s} = 0, \tag{105}$$

where

$$\xi_{\psi_1} \quad = \quad D\Delta_y \psi_1 + \frac{\partial D}{\partial \psi_1} |\nabla_y \psi_1|^2 + \frac{\partial D}{\partial \psi_2} \nabla_y \psi_1 \cdot \nabla_y \psi_2,$$

$$\xi_{\psi_2} \quad = \quad D\Delta_y \psi_2 + \frac{\partial D}{\partial \psi_2} |\nabla_y \psi_2|^2 + \frac{\partial D}{\partial \psi_1} \nabla_y \psi_1 \cdot \nabla_y \psi_2,$$

which corresponds to equation (70) in the two-dimensional case. Following the similar steps as in the proof of the two dimensional case, we can prove the existence and regularity of c_0 and w in the three dimensional case.

We would like to mention that not every incompressible velocity field $\mathbf{u}$ that contains no critical points (i.e. $|\mathbf{u}| \neq 0$) can be globally represented by the dual stream function formulation. Additional assumption on the property of the stream lines may be required to ensure the existence of the dual stream function formulation.

The precise mathematical condition under which a 3D incompressible velocity field has the dual stream function formulation (101) still requires further investigation.

4. **The error estimate for the leading order approximation.** In the last section, we have proved the existences and regularities of the leading order approximation c_0 and w. In this section, we will perform error analysis to estimate the error of the leader order approximation.

The main result of this section is the following error estimate.

Theorem 4.1. *Assume that the conditions stated in Theorem 3.1 hold. Let c^ϵ be the exact solution of (1)-(2), and c_0 be the solutions of (23) - (24) respectively. Then we have the following estimate:*

$$\|c^\epsilon - c_0\|_{L^\infty((0,T];L^2(\Omega))} \le C_M \epsilon, \tag{106}$$

where C_M is a constant depending only on T and the regularity of $\mathbf{u}$ and D.

Proof. Define

$$c^{1,\epsilon}(\mathbf{x}, \frac{\mathbf{x}}{\epsilon}, t) = c_0(\mathbf{x}, \frac{\mathbf{x}}{\epsilon}, t) + \epsilon w(\mathbf{x}, \frac{\mathbf{x}}{\epsilon}, t), \tag{107}$$

where c_0 and w are solutions defined by (23)-(24). Substituting $c^{1,\epsilon}$ into equation (1), we obtain by using (23)-(24) that

$$\frac{\partial c^{1,\epsilon}}{\partial t} + \mathbf{u}(\mathbf{x}, \frac{\mathbf{x}}{\epsilon}) \cdot \nabla c^{1,\epsilon} \quad - \quad \epsilon^m \nabla\Big(D(\mathbf{x}, \frac{\mathbf{x}}{\epsilon})\nabla c^{1,\epsilon}\Big) = \epsilon F(c_0, w). \tag{108}$$

The definition of F depends on m. In the case of $m \ge 3$, F is defined by

$$F(c_0, w) = -\frac{\partial w}{\partial t} - \mathbf{u}(\mathbf{x}, \mathbf{x}) \cdot \nabla_x w$$
$$+ \epsilon^{m-3}\Big(\nabla_y + \epsilon\nabla_x\Big)\Big(D(\mathbf{x}, \mathbf{y})\big(\nabla_y + \epsilon\nabla_x\big)\Big)(c_0 + \epsilon w).$$

In the case of $m = 2$, F is defined by

$$F(c_0, w) = -\frac{\partial w}{\partial t} - \mathbf{u}(\mathbf{x}, \mathbf{y}) \cdot \nabla_x w + \Big(\nabla_y + \epsilon\nabla_x\Big)\Big(D(\mathbf{x}, \mathbf{y})\big(\nabla_y + \epsilon\nabla_x\big)\Big) w,$$
$$+ (\nabla_y(D\nabla_x) + \nabla_x(D\nabla_y) + \epsilon\nabla_x(D\nabla_x)) c_0.$$

Theorem 3.1 implies that

$$\|F(c_0, w)\|_{L^2((0,T];L^2(\Omega))} \le M_3. \tag{109}$$

Let $e^\epsilon = (c^{1,\epsilon} - c^\epsilon)$ be the error function. Subtracting (1) from (108) gives

$$\frac{\partial e^\epsilon}{\partial t} + \mathbf{u}(\mathbf{x}, \frac{\mathbf{x}}{\epsilon}) \cdot \nabla e^\epsilon = \epsilon^m \nabla\Big(D(\frac{\mathbf{x}}{\epsilon})\nabla e^\epsilon\Big) + \epsilon F(c_0, w), \tag{110}$$

with initial condition

$$e^\epsilon|_{t=0} = \epsilon w(\mathbf{x}, \frac{\mathbf{x}}{\epsilon}, 0). \tag{111}$$

Multiplying e^ϵ to (110) and integrating over Ω, we get

$$\int_\Omega \frac{\partial e^\epsilon}{\partial t} e^\epsilon \, d\mathbf{x} + \int_\Omega (\mathbf{u}(\mathbf{x}, \frac{\mathbf{x}}{\epsilon}) \cdot \nabla e^\epsilon) e^\epsilon \, d\mathbf{x}$$
$$= \epsilon^m \int_\Omega \nabla\Big(D(\mathbf{x}, \frac{\mathbf{x}}{\epsilon})\nabla e^\epsilon\Big) e^\epsilon \, d\mathbf{x}$$
$$+ \epsilon \int_\Omega F(c_0, w)(\mathbf{x}, \frac{\mathbf{x}}{\epsilon}, t) e^\epsilon \, d\mathbf{x}. \tag{112}$$

Observe that

$$\nabla \cdot \mathbf{u}(\mathbf{x}, \frac{\mathbf{x}}{\epsilon}) = \nabla_x \cdot \mathbf{u}(\mathbf{x}, \mathbf{y}) + \frac{1}{\epsilon}\nabla_y \cdot \mathbf{u}(\mathbf{x}, \mathbf{y}) = \nabla_x \cdot \mathbf{u}(\mathbf{x}, \mathbf{y}), \quad \mathbf{y} = \frac{\mathbf{x}}{\epsilon},$$

since $\nabla_y \cdot \mathbf{u}(\mathbf{x}, \mathbf{y}) = 0$ by our assumption (3). Performing integration by parts, we obtain that

$$\frac{1}{2}\frac{d}{dt}\|e^\epsilon\|^2_{L^2(\Omega)} + \epsilon^m \int_\Omega D(\mathbf{x}, \frac{\mathbf{x}}{\epsilon})\nabla e^\epsilon \cdot \nabla e^\epsilon \ d\mathbf{x} \leq C\|e^\epsilon\|^2_{L^2(\Omega)} + \epsilon^2\|F(c_0, w)\|^2_{L^2(\Omega)} \quad (113)$$

where we have used the assumption that the forcing $\mathbf{u}$ is smooth as a function of $\mathbf{x}$. There is no boundary contribution in the integration because $c^\epsilon(\mathbf{x}, t)$ decays to zero rapidly as $|\mathbf{x}| \to \infty$ and $c^{1,\epsilon}(\mathbf{x}, \mathbf{y}, t)$ has compact support as a function $\mathbf{x}$ (since c_0 and w have compact support in $\mathbf{x}$). Integrating above equation over $[0, t]$, applying the Gronwall's inequality, it follows from (109) that

$$\|e^\epsilon\|^2_{L^2(\Omega)}(t) \quad + \quad 2d_0\epsilon^m \int_0^t \|\nabla e^\epsilon\|^2_{L^2(\Omega)}(t') \ dt' \leq M^*\epsilon^2, \tag{114}$$

for $0 < t \leq T$. This proves that

$$\|c^\epsilon - (c_0 + \epsilon w)\|^2_{L^2(\Omega)}(t) \leq C_M^*\epsilon^2, \tag{115}$$

for all $0 \leq t \leq T$. Since w is smooth, we also obtain

$$\|c^\epsilon - c_0\|^2_{L^2(\Omega)}(t) \leq C_M\epsilon^2, \tag{116}$$

for all $0 \leq t \leq T$. This completes the proof of Theorem 4.1.   □

5. **Remark on two-phase miscible flows in heterogeneous media.** In this section, we will address more complex two-phase miscible flows which involve a similar convection-dominated transport equations of concentration. We consider miscible displacement of a fluid with a dissolved solute in a heterogeneous confined aquifer. The evolution of the concentration is governed by the following convection diffusion equation (see, e.g. [1, 2, 12]):

$$\phi(\mathbf{x})\frac{\partial c}{\partial t} + \mathbf{u}(\mathbf{x}) \cdot \nabla c = \nabla \cdot \left(d(\mathbf{x})\nabla c\right), \qquad \mathbf{x} \in \Omega, \tag{117}$$

$$\mathbf{u} = -\frac{K}{\mu(c)}\nabla p, \qquad \mathbf{x} \in \Omega, \tag{118}$$

$$\nabla \cdot \mathbf{u} = f, \qquad \mathbf{x} \in \Omega, \tag{119}$$

subject to appropriate boundary and initial conditions. Here Ω is the physical domain of interest, $c(\mathbf{x}, t)$ is the concentration distribution function of solute, p is the pressure of the fluid, and u is the Darcy velocity of the fluid, ϕ is the porosity, K is the permeability, and d is the diffusion coefficient of the solute.

In practice we assume that the heterogeneous porous medium in the reservoir has a network of uniformly spaced heterogeneities with a block size l which is much small compared to the size L of the reservoir, namely, $0 < l \ll L$. If ϵ denotes the ratio of the small block size to the size of the whole region, then $0 < \epsilon \ll 1$. Let $\phi^*, d^*,$ and u^* be the characteristic porosity, diffusivity and Darcy velocity respectively. To obtain a dimensionless form of the model, we introduce the dimensionless space variable $\mathbf{x} \mapsto \mathbf{x}/L$ and the dimensionless characteristic time $t \mapsto t/\tau_c$, with $\tau_c = \phi^* L/u^*$. We assume that the local Peclet number, $P_e = \frac{u^0 l}{D^0}$, is of order $O(\epsilon^{-m+1})$

for some integer $m \geq 2$. In this case, we derive the following dimensionless governing system of equations:

$$\phi\Big(\frac{\mathbf{x}}{\epsilon}\Big)\frac{\partial c^\epsilon}{\partial t} + \mathbf{u}^\epsilon(\mathbf{x}, \frac{\mathbf{x}}{\epsilon}) \cdot \nabla c^\epsilon = \epsilon^m \nabla \cdot \Big(D(\mathbf{x}, \frac{\mathbf{x}}{\epsilon})\nabla c^\epsilon\Big), \qquad \mathbf{x} \in \Omega, \quad (120)$$

$$\mathbf{u}^\epsilon(\mathbf{x}, \frac{\mathbf{x}}{\epsilon}) = -\frac{K(\mathbf{x}, \frac{\mathbf{x}}{\epsilon})}{\mu(c^\epsilon)}\nabla p^\epsilon, \qquad \mathbf{x} \in \Omega, \quad (121)$$

$$\nabla \cdot \mathbf{u}^\epsilon(\mathbf{x}, \frac{\mathbf{x}}{\epsilon}) = f(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \quad (122)$$

subject to appropriate boundary condition and smooth initial condition, where $\epsilon = \frac{l}{L}$, and $D(\mathbf{x}, \frac{\mathbf{x}}{\epsilon})$ is the modified diffusion coefficient. The forcing function $f(\mathbf{x})$ is assumed to be smooth. We assume that $0 < \epsilon \ll 1$ and the dimensionless parameters ϕ, K and D are assumed to be Y-periodic functions where the rescaled periodicity cell is $Y = [0,1] \times [0,1]$. Let $\mathbf{y} = \mathbf{x}/\epsilon$, we further assume that the porosity, permeability and diffusion functions are all strictly positive functions such that $0 < \phi_0 \leq \phi(\mathbf{y}) \leq \phi_1$, $0 < d_0 \leq D(\mathbf{x}, \mathbf{y}) \leq d_1$, $0 < K_0 \leq K(\mathbf{x}, \mathbf{y}) \leq K_1$ for all $\mathbf{y} \in Y$, and $0 < \mu_0 \leq \mu(\cdot) \leq \mu_1$. All these functions as well as the initial condition of the concentration are sufficiently smooth.

In the miscible fluid flows above, equation (120) is similar to the linear transport equation (1) but the velocity is defined through Darcy's law (121). Equations (121)(122) lead to elliptic equation of pressure p but is coupling with (120) through parameter $\mu(c^\epsilon)$. In order to apply the multiscale analysis that we develop for the convection dominated transport equation, we need to obtain an accurate approximation to the multiscale velocity field. This can be done by applying the classical homogenization techniques to the elliptic equations (121)-(122) for the pressure. Once we obtain the multiscale velocity field, we can use it to upscale the convection dominated transport equation for concentration based on the homogenization method we developed in the previous sections.

To solve this nonlinear coupled elliptic/transport system, we can use the implicit pressure-explicit concentration method (IMPEC for short). First, assume that at time step t^n, we have already obtained the approximation to the leading order concentration, $C(t^n)$. To update the multiscale Darcy velocity U in the next time step t^{n+1}, we need to apply classical multiscale analysis to the elliptic equation for pressure and solve numerically the upscaled pressure equations and the corresponding cell problems. We would solve the average solution on a coarse-grid and capture the small scale solution using a subgrid. The multiscale finite element methods developed in [15, 9] and [6], and the multiscale finite volume method [16, 19] can be used to solve the upscaled pressure equation together with the cell problem. Once we have obtained the leading order multiscale velocity field U at t^{n+1}, we can apply the multiscale analysis to update the concentration at the next time step t^{n+1}. This procedure can be repeated in time. It provides an effective multiscale algorithm to upscale the two-phase miscible flow. In [14], the authors presented many numerical experiments for the immiscible flows in porous media based on a multiscale analysis similar to the one described in this paper. They showed that their upscaling method captures the average very well and demonstrated the efficiency of the upscaling method.

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