SIAM J. NUMER. ANAL.
Vol. 29, No. 3, pp. 615–639, June 1992


# CONVERGENCE OF A FINITE DIFFERENCE SCHEME FOR THE NAVIER–STOKES EQUATIONS USING VORTICITY BOUNDARY CONDITIONS*

THOMAS Y. HOU†‡ AND BRIAN T. R. WETTON†§

**Abstract.** A rigorous convergence result is presented for a finite difference scheme for the Navier–Stokes equations which uses vorticity boundary conditions. The approximating scheme is based on the vorticity-stream function formulation of the Navier–Stokes equations. The no-slip boundary condition is satisfied approximately by using a boundary condition of vorticity creation type. Convergence with second-order accuracy in vorticity and velocity is established for general domains in two space dimensions. Generalization to three space dimensions is also considered.

**Key words.** vorticity boundary conditions, energy estimates

**AMS(MOS) subject classifications.** primary 65M25; secondary 76D05

**1. Introduction.** The purpose of this paper is to analyze certain finite difference approximations for the incompressible Navier–Stokes equations in their vorticity formulation in domains with boundaries. We are especially interested in understanding stability and convergence properties of vorticity boundary conditions for these finite difference methods. The main result of this paper is a convergence proof of a finite difference method using a boundary condition of vorticity creation type. This result applies to general domains in two space dimensions as well as certain domains in three dimensions.

There are several ways of handling vorticity boundary conditions. Chorin [5] proposed a Lagrangean vortex blob scheme in which vorticity is created on the boundary in one step of the algorithm to approximately satisfy boundary conditions on the velocity. This step is followed by a random walk diffusion step and a convection step during which the boundary conditions on the velocity are violated. The vorticity creation boundary conditions correct this error. Several distinct types of vorticity boundary conditions which can be implemented in Eulerian difference schemes have also been proposed. Quartapelle and Valz-Gris [21], [17] use integral constraints to get enough information to determine the vorticity at the boundary. These integral constraints must be satisfied by the vorticity in order that the boundary conditions on the velocity be satisfied. Using a similar approach, Anderson [1] and Cottet [8] get boundary conditions for the vorticity in terms of an integral relationship. The last method is called the vorticity-stream function method discussed in the review article [16]. In this method, the vorticity on the boundary is determined by the values of the stream function at the interior points with coefficients determined by matching of Taylor series expansions. The above methods have been implemented numerically by the cited authors and all reflect the nonlocal nature of vorticity boundary conditions.

Chorin's creation scheme is perhaps the most interesting because of its ability to model high Reynold's number flows. A convergence analysis of a space-continuous model of Chorin's viscosity-splitting method is given in [7], [4]. Analysis using a

* Received by the editors July 16, 1990; accepted for publication (in revised form) June 23, 1991. This research was supported in part by Air Force Office of Scientific Research grant AFOSR-90-0090.

† Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, New York 10012.

‡ The research of this author was supported in part by National Science Foundation grant DMS-9003202 and by a Sloan foundation fellowship.

§ Present address, Department of Mathematics, University of British Columbia, Vancouver, British Columbia, Canada V6T 1Y4.

slightly different formulation is also given by [8]. However, a full analysis of this method is difficult because of its Lagrangean nature and the use of the random walk to simulate diffusion. The vorticity-stream function formulation seems the most tractable to analysis, and it is this type of scheme that is considered in this paper. Recently, Anderson [1], [2] has demonstrated that the different types of vorticity boundary conditions described in the preceding paragraph are related, at least in some restricted situations. Because of these relationships, the authors feel that understanding the stability and sources of error for one of the methods should shed some light on the others.

The main part of this paper is devoted to proving stability and convergence of a finite difference approximation for the Navier–Stokes equations which uses a vorticity boundary condition. This vorticity boundary condition can be interpreted as a finite difference version of the vorticity creation boundary condition. The analysis is based on discrete energy estimates. It is worth emphasizing that the energy estimates are performed on the velocity rather than vorticity since energy ($L^2$) estimates on the vorticity are difficult to obtain due to the nonlocal nature of the vorticity boundary conditions. This velocity error approach was first proposed by Naughton [15]. One of the main difficulties in the discrete energy estimates is the control of the boundary terms resulting from summation by parts. Typically these terms are large in magnitude and may not have a definite sign. There are three sources where the boundary terms can enter the analysis. The first one is from the linear diffusion term. In this case, the boundary terms can be handled in a way similar to Meth's [13]. However, Meth's energy estimate is not enough to imply convergence even for the Stokes equations, since the boundary condition is not accurate enough. The second source is from the linearized convection terms. This source gives rise to boundary terms which are very subtle to estimate. The smallness of the linearized velocity near the boundary is used in order to bound the boundary terms resulting from the repeated use of summation by parts. The last source is from the nonlinear convection terms, which are more difficult to handle.

We overcome the difficulties arising from the boundary conditions and the nonlinear stability by generalizing an argument of Strang [18] to our initial-boundary value problem. The idea is to construct a smooth function which satisfies the difference equations and the boundary conditions up to high-order accuracy. Then, when we compare the approximate solution with this smooth function, the error between the two can be made arbitrarily small. Consequently, the nonlinear terms can be estimated by the smallness of the error. This approach was used by Michelson [14] to analyze methods for hyperbolic initial-boundary value problems. The existence of such a function is nontrivial and requires certain regularity and compatibility conditions. Combining this argument with discrete energy estimates, we establish the convergence of the method. As a happy side effect, a sharper convergence rate is obtained in the maximum norm. The convergence proof is extended to general domains in two dimensions by using conformal mappings.

The order of accuracy of the finite difference method is an interesting question. The vorticity boundary condition is only of first-order accuracy, and the scheme is second-order accurate in the interior when using centered differencing. Thus, energy estimates would imply, at best, second-order accuracy in the velocity in a discrete $l^2$ norm. However, numerical experiments indicate that the vorticity actually converges in the maximum norm (in the interior) with second-order accuracy [13]. This seems hard to believe at first, since we may expect that the first-order accurate boundary condition would pollute the second-order accuracy in the interior. In the case of linear hyperbolic equations, it was shown by Gustafsson [10], using a quite involved argument, that lower-order boundary conditions do not affect the overall accuracy. In our situation, this property is explained using Strang's argument: the boundary term enters the error expansion only at second-order, and so the method indeed converges in the maximum norm with second-order accuracy.

Many numerical calculations on flows of interest have been conducted using methods with vorticity boundary conditions, especially those based on Chorin's algorithm [5], [6]. Recent numerical studies of Bell–Colella–Glaz for the projection method showed that high-order finite difference methods can capture fairly complicated flow features [3]. This leads us to believe that the finite difference algorithm we consider for analysis in this paper may also be useful for "real" computations, especially because vorticity-based methods have the advantage of eliminating the pressure term in a natural way. In our future work, we will perform extensive numerical studies for this method by using various vorticity boundary conditions. Some practical issues, such as local mesh refinement, upwind differencing, and time discretization will also be considered.

The rest of this paper is organized into two major sections and an appendix. Section 2 contains a discussion of the equations, the numerical algorithm, and the details of the convergence proof in two dimensions. In §3, the convergence result in three dimensions is presented. The discussion of the approximate solutions that satisfy the discrete equations to high-order accuracy in the manner of Strang is deferred to the appendix.

**2. Two-dimensional scheme.** In two space dimensions, the Navier–Stokes equations can be written in terms of the vorticity $\omega$ and stream function $\psi$ as follows:

$$\omega_t + u\omega_x + v\omega_y = \nu\Delta\omega, \tag{1}$$

$$\Delta\psi = -\omega, \tag{2}$$

$$u = \psi_y, \tag{3}$$

$$v = -\psi_x, \tag{4}$$

where $\nu$ is the viscosity and $u$ and $v$ are the $x$- and $y$-components of the velocity, respectively. These equations are considered in the infinite channel shown in Fig. 1. On the boundaries $y = 0$ and $y = 1$ we assume a no-flow condition $v = 0$ and specify slip velocity $u$. These boundary conditions can be written in terms of the stream function as

$$\psi(x, 0; t) = c_0, \qquad \psi(x, 1; t) = c_1,$$

where $c_0$ and $c_1$ are constants, and

$$\frac{\partial\psi}{\partial y}(x, 0; t) = u_0(x, t), \qquad \frac{\partial\psi}{\partial y}(x, 1; t) = u_1(x, t),$$

where $u_0$ and $u_1$ are the specified slip velocities. At first it seems that we have too many boundary conditions for (2) and none for (1). However, if (2) is substituted into (1), the resulting equation for the stream function has appropriate boundary conditions. Finding boundary conditions for the vorticity is a matter of converting one of the boundary conditions for the stream function to a boundary condition for the vorticity.

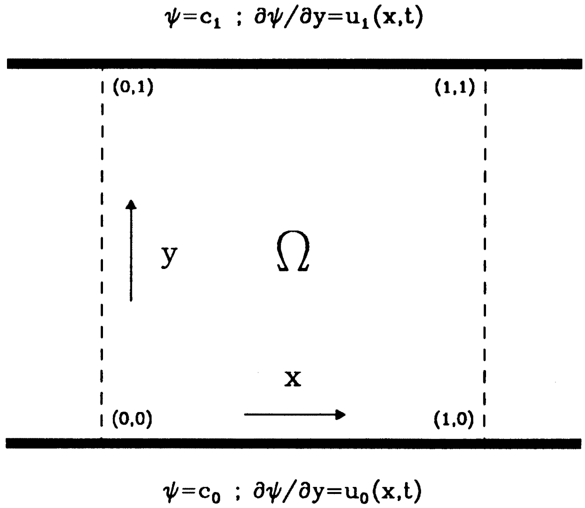


FIG. 1. *Two-dimensional channel.*

A semidiscrete finite difference approximation of (1)–(4) is given in the subsection below, followed by the convergence analysis. The final topic of this section deals with the extension of these results to more general domains and more general boundary conditions.

**2.1. The discrete equations.** To present the analysis more easily, some simplifying assumptions are made on the domain and the boundary conditions. First, we consider flows in the finite domain $\Omega$, shown with dashed lines in Fig. 1, and assume the flow is periodic in the $x$- direction. We also assume that $\psi = 0$ and $\partial\psi/\partial y = 0$ on the boundaries $y = 0$ and $y = 1$. More general boundary conditions will be considered later.

An $N \times N$ grid (with spacing $h = 1/N$) is laid on the periodic channel $\Omega$, and we then consider continuous time approximations $\tilde{\psi}_{i,j}(t)$ to $\psi(ih, jh; t)$. Approximations $\tilde{\omega}_{i,j}$, $\tilde{u}_{i,j}$, and $\tilde{v}_{i,j}$ are defined similarly. The difference operators that will be used in the paper are given below:

$$D_0^x f_{i,j} = (f_{i+1,j} - f_{i-1,j})/2h \quad \text{(centered)},$$
$$D_-^x f_{i,j} = (f_{i,j} - f_{i-1,j})/h \quad \text{(backward)},$$
$$D_+^x f_{i,j} = (f_{i+1,j} - f_{i,j})/h \quad \text{(forward)}.$$

The operators $D_0^y$, $D_-^y$, and $D_+^y$ are defined similarly. The centered difference approximation to the Laplacian is denoted by $\Delta_h$ which can be written as follows:

$$\Delta_h = D_-^x D_+^x + D_-^y D_+^y \,.$$

In terms of these difference operators, we approximate the Navier–Stokes equations by:

$$\frac{d\tilde{\omega}_{i,j}}{dt} = -\tilde{u}_{i,j} D_0^x \tilde{\omega}_{i,j} - \tilde{v}_{i,j} D_0^y \tilde{\omega}_{i,j} + \nu \Delta_h \tilde{\omega}_{i,j} \tag{5}$$

$$\Delta_h \tilde{\psi}_{i,j} = -\tilde{\omega}_{i,j}, \tag{6}$$

$$\tilde{u}_{i,j} = D_0^y \tilde{\psi}_{i,j}, \tag{7}$$

$$\tilde{v}_{i,j} = -D_0^x \tilde{\psi}_{i,j}. \tag{8}$$

The above approximation can be implemented as follows: given the discrete vorticity in the interior grid points, we compute the stream function in the interior grid points by (6) using the no-flow boundary condition $\psi = 0$. We then update the velocity field by (7)–(8). To update vorticity in the interior grid points, we use the vorticity equation (5), but this requires the values of the vorticity on the boundary. We use a vorticity-stream function type of boundary condition in which the vorticity on the boundary is related to the values of the stream function in the interior. To do this, we follow Orszag and Israeli [16] and Meth [13], and write $\psi_{i,1}$ in terms of an expansion around the grid point $(ih, 0)$:

$$\psi(ih, h) = \psi(ih, 0) + h\psi_y(ih, 0) + \frac{h^2}{2}\psi_{yy}(ih, 0) + O(h^3). \tag{9}$$

Since $\psi(ih, 0) = 0$, $\psi_y(ih, 0) = 0$, and $\psi_{yy}(ih, 0) = \Delta\psi(ih, 0) = -\omega(ih, 0)$, we obtain

$$\omega_{i,0} = -\frac{2}{h^2}\psi_{i,1} + O(h), \tag{10}$$

which is used as a boundary condition for the discrete vorticity:

$$\tilde{\omega}_{i,0} = -\frac{2}{h^2}\tilde{\psi}_{i,1}. \tag{11}$$

The boundary condition (11) can be interpreted as a finite difference equivalent of Chorin's creation scheme. Following Meth [13], we proceed as follows: (i) calculate the slip velocity introduced by the interior grid points, which is approximately equal to $(\tilde{\psi}_{i,1} - \tilde{\psi}_{i,0})/h = \tilde{\psi}_{i,1}/h$; (ii) create vorticity at the boundary of the amount $-(2/h)\tilde{\psi}_{i,1}/h$ to cancel the slip velocity. The factor $2/h$ is due to the fact that $\tilde{\psi}_{i,1}/h$ approximates the velocity at the point $(ih, h/2)$. Thus, using this finite difference version of Chorin's argument, we also obtain the boundary condition (11). Because of this fact, we refer to (11) as a vorticity boundary condition of creation type.

To summarize, the boundary conditions for $\psi$ and $\omega$ are given by

$$\tilde{\psi}_{i,j} = \tilde{\psi}_{i+N,j}, \qquad \tilde{\omega}_{i,j} = \tilde{\omega}_{i+N,j} \quad \text{(periodicity)}, \tag{12}$$

$$\tilde{\psi}_{i,0} = 0, \qquad \tilde{\psi}_{i,N} = 0 \quad \text{(no-flow)}, \tag{13}$$

and

$$\tilde{\omega}_{i,0} = -\frac{2}{h^2}\tilde{\psi}_{i,1}, \tag{14}$$

$$\tilde{\omega}_{i,N} = -\frac{2}{h^2}\tilde{\psi}_{i,N-1}. \tag{15}$$

The main convergence result is given below. It is assumed that the exact solution of the Navier–Stokes equations is sufficiently smooth.

THEOREM 2.1 (Convergence). *The solutions of* (5)–(8) *with boundary conditions* (12)–(13) *and vorticity boundary conditions* (14)–(15) *converge uniformly to the exact solutions of the Navier–Stokes equations with second-order accuracy:*

$$\|\omega - \tilde{\omega}\|_\infty \le C(T)h^2$$

*and*

$$\|u-\tilde{u}\|_\infty \le C(T)h^2, \qquad \|v-\tilde{v}\|_\infty \le C(T)h^2$$

*for all $t$ with $0 \le t \le T$. The $\|\cdot\|_\infty$ norm is the discrete maximum norm over the interior grid points defined in* (22) *below. The vorticity at the boundary, calculated using* (14)–(15), *converges with first-order accuracy in maximum norm.*

To analyze the finite difference approximation for the Navier–Stokes equations, several difficulties need to be overcome. These include the nonlinear convection terms, large boundary terms resulting from summation by parts, low-order accuracy of the boundary condition, and the discretization of a general domain. In the following subsections, we will show how these difficulties can be eliminated. The main ingredients are careful energy estimates and a generalization of Strang's argument [18] to the initial-boundary value problem.

**2.2. Convergence analysis.** An important element in the convergence analysis is the construction of approximate solutions $\hat{\psi}$ that satisfy the discrete equations to a high-order of accuracy.

LEMMA 2.2 (Consistency). *There exists a smooth function $\hat{\psi}$ that is an order $O(h^2)$ perturbation of $\psi$:*

$$\hat{\psi}(x,y,t;h) = \psi(x,y,t) + \sum_{p=2}^{q-1} h^p \psi^{(p)}(x,y,t), \tag{16}$$

*where the functions $\psi^{(p)}$ and their derivatives can be bounded in terms of $\psi$ and its derivatives. It satisfies the no-flow and the periodicity boundary conditions exactly and equals the exact stream function initially. Further, it satisfies*

$$\hat{\omega}_{i,0} = -\frac{2}{h^2}\hat{\psi}_{i,1} + O(h^{q-1}), \tag{17}$$

$$\hat{\omega}_{i,N} = -\frac{2}{h^2}\hat{\psi}_{i,N-1} + O(h^{q-1}), \tag{18}$$

$$\frac{d\hat{\omega}_{i,j}}{dt} = -D_0^y\hat{\psi}_{i,j}D_0^x\hat{\omega}_{i,j} + D_0^x\hat{\psi}_{i,j}D_0^y\hat{\omega}_{i,j} + \nu\Delta_h\hat{\omega}_{i,j} + O(h^q) \tag{19}$$

*to any desired degree of accuracy ($q$) provided the original solution $\psi$ is smooth enough. In the above equations, $\hat{\omega}_{i,j}$ is defined to be $-\Delta_h\hat{\psi}_{i,j}$.*

Notice that on the boundary, $\hat{\omega}_{i,0}$ is given by

$$\hat{\omega}_{i,0} = -\frac{1}{h^2}[\hat{\psi}_{i,1} - 2\hat{\psi}_{i,0} + \hat{\psi}_{i,-1}], \tag{20}$$

which is not well defined since the grid point $(ih,-h)$ is not in the domain. Therefore, $\hat{\omega}_{i,0}$ is defined to be the smooth continuation of $-\Delta_h\hat{\psi}$ onto the boundary, i.e., the Taylor expansion of (20) to sufficiently high order (depends on $q$) at the boundary:

$$\hat{\omega}_{i,0} = -\hat{\psi}_{yy}(ih,0) - \frac{h^2}{12}\hat{\psi}_{yyyy}(ih,0) + \cdots \text{ (finitely many terms)}. \tag{21}$$

This is needed to maintain the high-order accuracy in the interior. Since the proof of the consistency lemma is quite technical, we defer it to the appendix.

We define the following error terms:

$$\epsilon_{i,j} = \tilde{\psi}_{i,j} - \hat{\psi}_{i,j},$$
$$e_{i,j} = \tilde{\omega}_{i,j} - \hat{\omega}_{i,j},$$

and the following discrete norms on the grid:

$$\|f\|_2^2 = h^2 \sum_{i=0}^{N-1} \sum_{j=1}^{N-1} (f_{i,j})^2,$$

$$\|f\|_\infty = \max_i \max_{1\le j\le N-1} |f_{i,j}|, \tag{22}$$

$$\|f\|_{1,2}^2 = h^2 \sum_{i=1}^{N} \sum_{j=1}^{N} \left[(D_-^x f_{i,j})^2 + (D_-^y f_{i,j})^2\right],$$

$$\|f\|_{2,2}^2 = h^2 \sum_{i=0}^{N-1} \sum_{j=1}^{N-1} (\Delta_h f_{i,j})^2 + \frac{2}{h^2} \sum_{i=0}^{N-1} \left[(f_{i,1})^2 + (f_{i,N-1})^2\right].$$

Some useful relationships between these norms are derived below. Since $h^2(f_{i,j})^2 \le \|f\|_2^2$, $|f_{i,j}| \le \frac{1}{h}\|f\|_2$, and so

$$\|f\|_\infty \le \frac{1}{h}\|f\|_2. \tag{23}$$

Also, for all $f$:

$$\|D_0^y f\|_2 \le \|f\|_{1,2}. \tag{24}$$

Discrete functions $f$ with $f_{i,0} = 0$ and $f_{i,N} = 0$ for all $i$ satisfy a discrete version of the Poincaré inequality

$$\|f\|_2 \le \|f\|_{1,2} \tag{25}$$

and the estimate

$$\sum_{i=0}^{N-1} \left[(f_{i,1})^2 + (f_{i,N-1})^2\right] \le \|f\|_{1,2}^2. \tag{26}$$

The discrete Poincaré inequality (25) follows from $f_{i,j} = h\sum_{1\le k\le j} D_-^y f_{i,k}$. The last formula (26) can be verified as follows: Since $f_{i,0} = 0$ and $f_{i,N} = 0$, $f_{i,1} = hD_-^y f_{i,1}$ and $f_{i,N-1} = -hD_-^y f_{i,N}$. Therefore,

$$\sum_{i=0}^{N-1} \left[(f_{i,1})^2 + (f_{i,N-1})^2\right] = h^2 \sum_{i=1}^{N-1} \left[(D_-^y f_{i,1})^2 + (D_-^y f_{i,N})^2\right] \le \|f\|_{1,2}^2.$$

The expression, $O(h^p)$, will denote any quantity that can be bounded for $0 \le t \le T$ by $Kh^p$ for a constant K that depends only on $\psi$ and its derivatives (up to some sufficiently high order) up to time $T$. In addition, $c$ will denote an absolute constant, while $K$ and $B$ will denote $O(1)$ quantities.

In what follows, it is assumed that the order of accuracy of the constructed approximate solutions in Lemma 2.1 is sufficiently large ($q \ge 6$).

LEMMA 2.3 (Stability). *Given* $T > 0$, *suppose that* $\|\epsilon(t)\|_{1,2} \le h^{q-2}$ *for* $0 \le t \le T$ *and* $q \ge 6$. *Then there exists a positive number* $B$ *that depends on* $T$ *and the exact solution* $\psi$ *only such that for all* $0 \le t \le T$,

$$\frac{d}{dt}\|\epsilon(t)\|_{1,2} \le B(h^{q-3/2} + \|\epsilon(t)\|_{1,2}) . \tag{27}$$

*The function* $B(T)$ *can be chosen to be nondecreasing in time.*

The proof of Lemma 2.3 is lengthy and is left to the end of this section. We are now in a position to prove the main convergence result, Theorem 2.1.

*Proof of Theorem* 2.1. First it will be shown that Lemma 2.3 is equivalent to the following statement: Given $T > 0$ there is a $C(T)$, such that for all $t$ in the range $0 \le t \le T$,

$$\|\epsilon(t)\|_{1,2} \le C(T)h^{q-3/2} \tag{28}$$

for $h$ sufficiently small. The proof that Lemma 2.3 implies (28) is an easier version of the bootstrap argument in [9]. For a given $h$, define $T^*$ by

$$T^* = \inf\{t : \|\epsilon\|_{1,2} \ge h^{q-2}\}.$$

Since $\|\epsilon(0)\|_{1,2} = 0$, $T^* > 0$. If $T^* > T$, then the conditions of Lemma 2.3 are valid, and using Gronwall's inequality on (27), (28) is valid with $C(T) = e^{B(T)T}$. A simple contradiction argument shows that if $h \le \frac{1}{4}e^{-2B(T)T}$, then $T^*$ must be greater than $T$. Putting the definition of $\epsilon$ into (28), we obtain $\|\tilde{\psi} - \hat{\psi}\|_{1,2} \le C(T)h^{q-3/2}$. This result shows that the numerical solution converges with a high order of accuracy to the constructed approximate solution $\hat{\psi}$. In a manner similar to (23), the following can be shown:

$$\max_{i,j} |D_-^x \tilde{\psi}_{i,j} - D_-^x \hat{\psi}_{i,j}| \le C(T)h^{q-5/2},$$
$$\max_{i,j} |D_-^y \tilde{\psi}_{i,j} - D_-^y \hat{\psi}_{i,j}| \le C(T)h^{q-5/2} .$$

Since $-\tilde{\omega}_{i,j} = \Delta_h \tilde{\psi}_{i,j} = (D_+^x D_-^x + D_+^y D_-^y)\tilde{\psi}_{i,j}$, we have

$$\max_{i,j} | - \tilde{\omega}_{i,j} - \Delta_h \hat{\psi}| \le 2C(T)h^{q-7/2}, \tag{29}$$

where the maximum is taken over the interior only. Since $\hat{\psi}(x, y, t; h)$ is a second-order perturbation of the exact stream function (see (16)),

$$\begin{aligned} \Delta_h \hat{\psi}_{i,j} &= (\Delta\psi)_{i,j} + O(h^2) \\ &= -\omega_{i,j} + O(h^2) . \end{aligned}$$

This result inserted in (29) proves the second-order uniform convergence in the vorticity. The second-order convergence in the velocities and the first-order convergence in the boundary vorticity is obtained similarly. □

To get the discrete energy estimates needed to prove stability, we need to use the following summation by parts formulas.

LEMMA 2.4 (Summation by parts).

$$\sum_{j=1}^{N} f_j D_+ g_j = -\sum_{j=1}^{N} D_- f_j \cdot g_j + \frac{1}{h}(f_N g_{N+1} - f_0 g_1), \tag{30}$$

$$\text{(31)} \quad \sum_{j=1}^{N} f_j D_- g_j = -\sum_{j=1}^{N} D_+ f_j \cdot g_j + \frac{1}{h}(f_{N+1} g_N - f_1 g_0),$$

$$\text{(32)} \quad \sum_{j=1}^{N-1} f_j D_0 g_j = -\sum_{j=1}^{N-1} D_0 f_j \cdot g_j + \frac{1}{2h}(-f_0 g_1 + f_{N-1} g_N + f_N g_{N-1} - f_1 g_0).$$

*Proof of* (30). To verify this formula, we proceed as follows:

$$\begin{aligned}\sum_{j=1}^{N} f_j D_+ g_j &= \frac{1}{h} \sum_{j=1}^{N} f_j (g_{j+1} - g_j) \\ &= \frac{1}{h} \left( \sum_{j=1}^{N} f_{j-1} g_j + f_N g_{N+1} - f_0 g_1 - \sum_{J=1}^{N} f_j g_j \right) \\ &= -\sum_{j=1}^{N} D_- f_j \cdot g_j + \frac{1}{h}(f_N g_{N+1} - f_0 g_1) .\end{aligned}$$

The formulas (31) and (32) are proved similarly. □

In many of our applications, the functions $f$ and $g$ will have homogeneous boundary data ($f_0 = f_N = 0$) or will be periodic ($f_i = f_{i+N}$). In these cases, the boundary terms in the above formulas will vanish.

We now return to the proof of Lemma 2.3 (stability). It will be shown that the error $\epsilon$ in $\psi$ is stable in the $\| \cdot \|_{1,2}$ norm. This is the same as proving stability of the method in the velocity. The proof begins by taking the difference between (5) and (19):

$$\begin{aligned}
&\text{(33)} && \frac{de_{i,j}}{dt} = \\
&\text{(34)} && \quad - D_0^y \epsilon_{i,j} \cdot D_0^x e_{i,j} \\
&\text{(35)} && \quad + D_0^y \epsilon_{i,j} \cdot D_0^x \Delta_h \hat{\psi}_{i,j} \\
&\text{(36)} && \quad - D_0^y \hat{\psi}_{i,j} \cdot D_0^x e_{i,j} \\
&\text{(37)} && \quad + D_0^x \epsilon_{i,j} \cdot D_0^y e_{i,j} \\
&\text{(38)} && \quad - D_0^x \epsilon_{i,j} \cdot D_0^y \Delta_h \hat{\psi}_{i,j} \\
&\text{(39)} && \quad + D_0^x \hat{\psi}_{i,j} \cdot D_0^y e_{i,j} \\
&\text{(40)} && \quad + \nu \Delta_h e_{i,j} \\
&\text{(41)} && \quad + O(h^q).
\end{aligned}$$

The terms (35), (36), (38), and (39) are linearized convection terms whose stability needs to be analyzed. The linear diffusion term (40) leads to a decay in the error. Terms (34) and (37) are nonlinear terms whose stability is easy to analyze by using the high accuracy of the approximate solutions $\hat{\psi}$. Finally, term (41) is the truncation error (although errors from the boundary terms will be introduced when the others terms are integrated by parts to show stability). The above equation is multiplied by $h^2 \epsilon_{i,j}$ and summed over the interior. The resulting equation is evaluated term by term.

**Evolution term.**

*Term* (33). Note that $e_{i,j} = -\Delta_h \epsilon_{i,j}$ in the interior. Therefore this term can be written as

$$-h^2 \sum_{i=0}^{N-1} \sum_{j=1}^{N-1} \epsilon_{i,j} \left[ (D_+^x D_-^x + D_+^y D_-^y) \frac{d\epsilon_{i,j}}{dt} \right] .$$

After an application of Lemma 2.4 (summation by parts) in $x$ and $y$ directions, respectively, this term becomes

$$h^2 \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ (D_-^x \epsilon_{i,j}) \frac{d}{dt} (D_-^x \epsilon_{i,j}) + (D_-^y \epsilon_{i,j}) \frac{d}{dt} (D_-^y \epsilon_{i,j}) \right]$$

(the terms are periodic in $i$ and $\epsilon_{i,0} = 0$ and $\epsilon_{i,N} = 0$, so there are no boundary terms) or

$$\frac{1}{2} \frac{d}{dt} h^2 \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ (D_-^x \epsilon_{i,j})^2 + (D_-^y \epsilon_{i,j})^2 \right] ,$$

which is simply $\frac{1}{2}(d/dt)\|\epsilon\|_{1,2}^2$.

**Nonlinear terms.**

*Term* (37). It is supposed as in Lemma 2.3 that $\|\epsilon(t)\|_{1,2} \le h^{q-2}$ for $0 \le t \le T$. This allows us to bound this nonlinear term essentially by brute force. Using (25), $\|\epsilon\|_2 \le h^{q-2}$, and then, using (23),

$$\|\epsilon\|_\infty \le h^{q-3}. \tag{42}$$

The term (37) under consideration is

$$h^2 \sum_{i=0}^{N-1} \sum_{j=1}^{N-1} \epsilon_{i,j} D_0^x \epsilon_{i,j} D_0^y e_{i,j}.$$

This term is divided into one interior part and two boundary parts which will be analyzed separately:

$$h^2 \sum_{i=0}^{N-1} \sum_{j=2}^{N-2} \epsilon_{i,j} D_0^x \epsilon_{i,j} D_0^y e_{i,j} \tag{43}$$

$$+ h^2 \sum_{i=0}^{N-1} \epsilon_{i,1} D_0^x \epsilon_{i,1} \frac{1}{2h} (e_{i,2} - e_{i,0}) \tag{44}$$

$$+ h^2 \sum_{i=0}^{N-1} \epsilon_{i,N-1} D_0^x \epsilon_{i,N-1} \frac{1}{2h} (e_{i,N} - e_{i,N-2}). \tag{45}$$

Recall from the analysis of the evolution term that $e_{i,j} = -\Delta_h \epsilon_{i,j}$ in the interior. The term (43) contains only values of $e$ in the interior. Therefore, using the a priori bound (42), the term (43) can be bounded in absolute value by

$$h^{q-3} \cdot h^2 \sum_{i=0}^{N-1} \sum_{j=2}^{N-2} |D_0^x \epsilon_{i,j}| \cdot |D_0^y \Delta_h \epsilon_{i,j}|,$$

which, using the Cauchy–Schwarz inequality, can be bounded by

$$h^{q-3} \cdot \|D_0^x \epsilon\|_2 \|D_0^y \Delta_h \epsilon\|_2.$$

Since $D_0^y \Delta_h \epsilon_{i,j}$ can be written as a sum of backward differences of $\epsilon$ divided by $h^2$, $\|D_0^y \Delta_h \epsilon\|_2 \leq c\|\epsilon\|_{1,2}/h^2$. Therefore, the above term and hence (43) is bounded by

$$ch^{q-5}\|D_0^y \epsilon\|_2 \cdot \|D_-^x \epsilon\|_2 \leq c\|\epsilon\|_{1,2}^2, \tag{46}$$

where (24) and the fact that $q \geq 6$ is used. Since (44) and (45) are handled similarly, we will only consider the first. The term $e_{i,2}$ is in the interior and can be handled as above, so we will neglect it in what follows. In order to proceed, we need to relate $e_{i,0}$ to the error $\epsilon$. This is done as follows: taking the difference between equations (17) and (14) we have

$$e_{i,0} = -\frac{2}{h^2}\epsilon_{i,1} + O(h^{q-1}). \tag{47}$$

Using this fact and (42), term (44) can be bounded by

$$h^{q-3} \cdot h^2 \sum_{i=0}^{N-1} |D_0^x \epsilon_{i,1}| \cdot \left| \frac{1}{h^2} D_-^y \epsilon_{i,1} + O(h^{q-2}) \right|,$$

where the fact that $hD_-^y \epsilon_{i,1} = \epsilon_{i,1}$ is used ($\epsilon_{i,0} = 0$). Again, using the fact that $q \geq 6$ and (24), the above expression is bounded by

$$c\|\epsilon\|_{1,2}^2 + O(h^{2q-5})\|\epsilon\|_{1,2}.$$

To summarize, term (37) can be bounded by

$$c\|\epsilon\|_{1,2}^2 + O(h^{2q-5})\|\epsilon\|_{1,2}.$$

A similar estimate for term (34) can be obtained.

**Linearized convection terms.**

*Term* (35). Since the expression $D_0^x \Delta_h \hat{\psi}_{i,j}$ is bounded (in terms of the derivatives of $\hat{\psi}$), (35) can be bounded by

$$K\|\epsilon\|_2 \cdot \|D_0^y \epsilon\|_2 \leq K\|\epsilon\|_{1,2}^2, \tag{48}$$

where (25) and (24) have been used. A similar estimate for (38) can be obtained.

*Term* (39). This is the most troublesome of the terms. It is written out below:

$$h^2 \sum_{i=0}^{N-1} \sum_{j=1}^{N-1} \epsilon_{i,j} \cdot D_0^x \hat{\psi}_{i,j} \cdot D_0^y e_{i,j}. \tag{49}$$

After summation by parts using the formula (32) this becomes

$$-h^2 \sum_{i=0}^{N-1} \sum_{j=1}^{N-1} D_0^y (\epsilon D_0^x \hat{\psi})_{i,j} \cdot e_{i,j} \tag{50}$$

$$-\frac{h}{2} \sum_{i=0}^{N-1} \epsilon_{i,1} D_0^x \hat{\psi}_{i,1} e_{i,0} + \frac{h}{2} \sum_{i=0}^{N-1} \epsilon_{i,N-1} D_0^x \hat{\psi}_{i,N-1} e_{i,N}. \tag{51}$$

The other boundary terms drop out because $\epsilon$ has homogeneous boundary values. Term (51) will be analyzed first, and we will return to (50) later. When (47) is used, the first sum in (51) becomes

$$\frac{h}{2}\sum_{i=0}^{N-1}\epsilon_{i,1}D_0^x\hat{\psi}_{i,1}\left(\frac{2}{h^2}\epsilon_{i,1}+O(h^{q-1})\right). \tag{52}$$

To proceed, we need the following fact:

$$D_0^x\hat{\psi}_{i,1}=O(h). \tag{53}$$

To prove this, the thing to notice is that $D_0^x\hat{\psi}_{i,1} = v_{i,1} + O(h^2)$, where $v$ is the exact velocity (see the appendix for the details on the approximate solutions). Since $v(ih,0) = 0$ and $v$ is smooth, $v(ih,h)$ is $O(h)$. Using (53) and the Cauchy–Schwarz inequality, (52) can be bounded by

$$O(1)\sum_{i=0}^{N-1}(\epsilon_{i,1})^2+O(h^{q+1})N^{1/2}\left[\sum_{i=0}^{N-1}(\epsilon_{i,1})^2\right]^{1/2}$$

Using (26) on both factors above, this can be bounded by

$$K\|\epsilon\|_{1,2}^2+O(h^{q+1/2})\|\epsilon\|_{1,2}.$$

The other term in (51) is handled similarly. Now let us return to (50) and consider only the trickiest part:

$$h^2\sum_{i=0}^{N-1}\sum_{j=1}^{N-1}D_0^y(\epsilon D_0^x\hat{\psi})_{i,j}\cdot D_+^yD_-^y\epsilon_{i,j}.$$

Using the following formula

$$D_0^y(fg)_{i,j}=D_0^yf_{i,j}\cdot g_{i,j+1}+D_0^yg_{i,j}\cdot f_{i,j-1},$$

this becomes

$$h^2\sum_{i=0}^{N-1}\sum_{j=1}^{N-1}\left[(D_0^y\epsilon_{i,j})D_0^x\hat{\psi}_{i,j+1}+(D_0^yD_0^x\hat{\psi}_{i,j})\epsilon_{i,j-1}\right]D_+^yD_-^y\epsilon_{i,j}\ . \tag{54}$$

After an application of the formula

$$D_0^y\epsilon_{i,j}\cdot D_+^yD_-^y\epsilon_{i,j}=\tfrac{1}{2}D_+^y(D_-^y\epsilon_{i,j})^2,$$

the first summand (the second will be examined below) in (54) becomes

$$\frac{h^2}{2}\sum_{i=0}^{N-1}\sum_{j=1}^{N-1}D_0^x\hat{\psi}_{i,j+1}D_+^y(D_-^y\epsilon_{i,j})^2. \tag{55}$$

Using Lemma 3, (55) becomes

$$\frac{h^2}{2}\sum_{i=0}^{N}\sum_{j=1}^{N-1}D_+^yD_0^x\hat{\psi}_{i,j}(D_-^y\epsilon_{i,j})^2-\frac{h}{2}\sum_{i=0}^{N-1}D_0^x\hat{\psi}_{i,1}(D_-^y\epsilon_{i,1})^2. \tag{56}$$

The other boundary term is zero since $\hat{\psi}_{i,N} = 0$, and so $D_0^x \hat{\psi}_{i,N} = 0$. Both of the terms in (56) can be bounded by $K\|\epsilon\|_{1,2}^2$, although the second term requires (53).

We now return to the second factor of (54), which can be written as

$$h^2 \sum_{j=2}^{N-1} \sum_{i=0}^{N-1} D_-^y (D_0^y D_0^x \hat{\psi}_{i,j} \cdot \epsilon_{i,j-1}) D_-^y \epsilon_{i,j}$$

$$+ h^2 \sum_{i=0}^{N-1} D_0^y D_0^x \hat{\psi}_{i,N-1} \cdot \frac{\epsilon_{i,N-2}}{h} \cdot D_-^y \epsilon_{i,N} ,$$

when Lemma 2.4 (summation by parts) is used (the $j = 1$ term is zero). Using the fact that $\frac{1}{h}\epsilon_{i,N-2} = -D_-^y \epsilon_{i,N} - D_-^y \epsilon_{i,N-1}$, both of the sums in the above expression can be bounded by $K\|\epsilon\|_{1,2}^2$.

Summarizing the results of this case, (39) can be bounded by

$$K\|\epsilon\|_{1,2}^2 + O(h^{q+1/2})\|\epsilon\|_{1,2}.$$

Term (36) is handled in a similar way (easier because there are no boundary terms).

**Diffusion term.**

*Term* (40). One part of (40) is given below:

$$\nu h^2 \sum_{i=0}^{N-1} \sum_{j=1}^{N-1} \epsilon_{i,j} \cdot D_+^x D_-^x e_{i,j}.$$

Using the periodicity in the $x$-direction, this is easily summed by parts twice to give

$$\nu h^2 \sum_{i=0}^{N-1} \sum_{j=1}^{N-1} D_+^x D_-^x \epsilon_{i,j} \cdot e_{i,j}. \tag{57}$$

The other part of this term is

$$\nu h^2 \sum_{i=0}^{N-1} \sum_{j=1}^{N-1} \epsilon_{i,j} \cdot D_+^y D_-^y e_{i,j}.$$

Using the fact that $\epsilon_{i,0} = 0$ and $\epsilon_{i,N} = 0$, this can be summed by parts once to get:

$$- \nu h^2 \sum_{i=0}^{N-1} \sum_{j=1}^{N} D_-^y \epsilon_{i,j} \cdot D_-^y e_{i,j}. \tag{58}$$

After a second summation by parts (the boundary terms are not zero in this case) this becomes:

$$\nu h^2 \sum_{i=0}^{N-1} \sum_{j=1}^{N-1} D_+^y D_-^y \epsilon_{i,j} \cdot e_{i,j} \tag{59}$$

$$+ \nu h \sum_{i=0}^{N-1} D_-^y \epsilon_{i,1} e_{i,0} \tag{60}$$

$$- \nu h \sum_{i=0}^{N-1} D_-^y \epsilon_{i,N} e_{i,N}. \tag{61}$$

The last two terms above will be handled similarly, so only (60) will be considered:

$$\nu h \sum_{i=0}^{N-1} D_-^y \epsilon_{i,1} e_{i,0}.$$

Using (47) and $\epsilon_{i,0} = 0$, it can be written as

$$(62) \quad -\nu \sum_{i=0}^{N-1} \epsilon_{i,1} \left( \frac{2}{h^2} \epsilon_{i,1} + O(h^{q-1}) \right) = -\nu \frac{2}{h^2} \sum_{i=0}^{N-1} (\epsilon_{i,1})^2 - O(h^{q-1}) \sum_{i=0}^{N-1} \epsilon_{i,1}.$$

The first term in the right-hand side of the equation above and the equivalent term from (61) can be combined with (57) and (59) to give $-\|\epsilon\|_{2,2}^2$. The second term on the right-hand side of (62) can be bounded in absolute value by

$$(63) \qquad O(h^{q-1}) N^{1/2} \left[ \sum_{i=0}^{N-1} (\epsilon_{i,1})^2 \right]^{1/2} \leq O(h^{q-3/2}) \|\epsilon\|_{1,2},$$

where (26) is used.

**Final term and summary.**

*Term* (41). This term can be bounded in absolute value by $O(h^q)\|\epsilon\|_2$, which is bounded by $O(h^q)\|\epsilon\|_{1,2}$ (use (25)).

When the results from all of the terms above are combined (the lowest order coefficient for $\|\epsilon\|_{1,2}$ is $h^{q-3/2}$), we have the following:

$$\begin{aligned} \frac{d}{dt} \|\epsilon\|_{1,2}^2 &\leq -\nu \|\epsilon\|_{2,2}^2 + K \|\epsilon\|_{1,2}^2 + O(h^{q-3/2}) \|\epsilon\|_{1,2} \\ &\leq 2B \|\epsilon\|_{1,2} (h^{q-3/2} + \|\epsilon\|_{1,2}) \ , \end{aligned}$$

or

$$\frac{d}{dt} \|\epsilon\|_{1,2} \leq B(h^{q-3/2} + \|\epsilon\|_{1,2}).$$

It is clear that $B(T)$ can be chosen to be nondecreasing, so the proof of Lemma 2.3 (stability) is complete. □

**Dependence on the Reynold's number.** In the above proof of Lemma 2.3 (stability), the constant $B(T)$ does not depend explicitly on $\nu$, and so the same is true for the factor $C(T)$ in the statement of Theorem 2.1 (convergence). It seems then that the convergence properties of the scheme are independent of the Reynold's number. However, this statement is misleading because of the following argument: the constant $C(T)$ does depend on the the exact solution, which can be expected to develop large gradients near the boundary as the Reynold's number increases.

**2.3. Other domains and boundary values.** So far, the authors have restricted their attention to boundary conditions of the form $\psi = 0$ and $\partial\psi/\partial n = 0$. The general boundary conditions considered in the section below are $\psi$ and $\partial\psi/\partial n$ given but not necessarily zero. This would allow the specification of general velocities on the boundary. A method of implementing these boundary conditions is given below. In order to prove convergence as before, the specified velocity at the boundary must

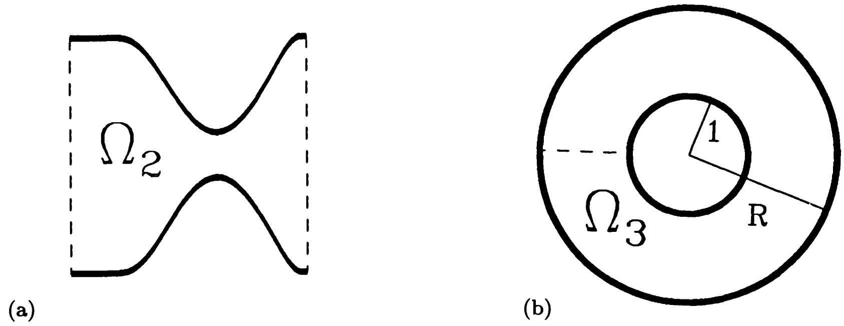


FIG. 2. *General domains* (a) *irregular periodic channel;* (b) *Annulus.*

be tangential to the boundary (this translates to $\psi =$ constant on the boundary). In practice, complicated matching boundary conditions are often specified on an artificial surface when the domain under consideration is unbounded (see, for instance, [1]). The convergence of such schemes is not addressed here.

The authors also show how to perform calculations in any domain that can be conformally mapped into the original channel domain shown in Fig. 1. Two such domains, an irregular periodic channel, and an annulus are shown in Fig. 2. It will be shown that the convergence proofs of §2.2 apply to the algorithm proposed for these domains.

**2.3.1. Nonhomogeneous boundary data.** Consider flow in the periodic channel, where

$$\psi(x,0;t) = f(x) \quad \text{and} \quad \frac{\partial \psi}{\partial y}(x,0;t) = g(x),$$

and where $f$ and $g$ are given functions which could also depend on time. Similar conditions could be applied on the line $y = 1$. The computational boundary conditions for $\tilde{\psi}$ are the exact conditions, $\tilde{\psi}_{i,0} = f_i$, so that the error $\epsilon = \psi - \tilde{\psi}$ is zero at the boundary as before. Taking a Taylor series for $\psi_{i,1}$, based at the point $(ih, 0)$, and proceeding as in (9)–(10), we obtain a computational boundary condition of

$$-\tilde{\omega}_{i,0} = D_+D_-f_i + \frac{2}{h^2}\tilde{\psi}_{i,1} - \frac{2}{h}g_i - \frac{2}{h^2}f_i, \tag{64}$$

in which the exact solution satisfies to first order. To prove convergence of the method using these boundary conditions, we proceed by showing consistency and stability as before. An approximate solution $\hat{\psi}$ can be constructed that satisfies this condition to high order as in Lemma 2.2 (consistency). To get the proof of Lemma 2.3 (stability), (53) must be satisfied. This means that the normal velocity at the boundary must be zero, which means $f =$ constant. This restriction can be interpreted as follows: boundary conditions (14)–(15) and their generalizations to nonhomogeneous boundary data are appropriate only on a slip surface, where the vorticity created is only diffused, not convected, away from the boundary. Provided that the restriction $f =$ constant is satisfied, Lemma 2.3 (stability) can be proved for the nonhomogeneous boundary conditions (64). Theorem 2.1 (convergence) follows from the stability lemma as in §2.2. Actually, by taking into account the diffusion terms that reduce the error, we can also prove convergence for nonhomogeneous flow boundary conditions ($f$ not necessarily constant). In this case, the constant $C$ in the statement of Theorem 2.1 will have an explicit dependence on the viscosity $\nu$.

**2.3.2. Other domains.** Suppose that $\alpha(x,y)$, $\beta(x,y)$ is a mapping of a domain $\Omega_*$ onto the periodic channel $\Omega$ of Fig. 1 that is conformal, i.e.,

$$\alpha_x = \beta_y, \quad \alpha_y = -\beta_x, \quad \Delta\alpha = \Delta\beta = 0. \tag{65}$$

For instance, if $R = e^{2\pi}$, then the conformal mapping from the annulus $\Omega_3$ of Fig. 2 to the periodic channel is

$$\alpha(x,y) = -\frac{\theta}{2\pi} + \frac{1}{2},$$
$$\beta(x,y) = \frac{\log r}{2\pi},$$

where $(r,\theta)$ are the polar coordinates of $(x,y)$. The above transformation comes by suitably scaling and rotating the analytic transformation $\log z$. Let $J$ denote the Jacobian of the general conformal transformation, which is always positive and given by

$$J = \left| \frac{(\partial\alpha, \partial\beta)}{(\partial x, \partial y)} \right| = \alpha_x^2 + \alpha_y^2 = \beta_x^2 + \beta_y^2$$

by using (65). We restrict our attention to domains for which the conformal transformation onto the periodic channel has a Jacobian that is not singular ($J$ bounded and bounded away from zero). Under this restriction, we cannot directly consider unbounded domains. However, for computational purposes, the restriction to finite domains is not a serious one.

A simple calculation using (65) shows that $\Delta = J\tilde{\Delta}$, where

$$\tilde{\Delta} = \frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}.$$

Suppose that we want to solve the Navier–Stokes equations in the new domain $\Omega_*$ with boundary conditions $\psi = 0$ and $\partial\psi/\partial n = 0$ (more general conditions are handled as above). When these equations are transformed into the $(\alpha,\beta)$ plane using the chain rule and (65), they become

$$\omega_t = J(-U\omega_\alpha - V\omega_\beta + \tilde{\Delta}\omega), \tag{66}$$

$$J\tilde{\Delta}\psi = -\omega, \tag{67}$$

$$U = \psi_\beta, \tag{68}$$

$$V = -\psi_\alpha, \tag{69}$$

with boundary conditions

$$\psi = 0, \qquad \psi_\beta = 0 \tag{70}$$

on the lines $\beta = 0$ and $\beta = 1$ (conformal maps preserve Neumann boundary conditions). The equations (66)–(69) are defined in the periodic channel $\Omega$ and can be discretized like (5)–(8) on a regular grid in $\alpha$ and $\beta$:

$$\frac{d\tilde{\omega}_{i,j}}{dt} = J_{i,j}\left(-\tilde{U}_{i,j} \cdot D_0^\alpha \tilde{\omega}_{i,j} - \tilde{V}_{i,j} \cdot D_0^\beta \tilde{\omega}_{i,j} + \tilde{\Delta}_h \tilde{\omega}_{i,j}\right), \tag{71}$$

$$J_{i,j}\tilde{\Delta}_h\tilde{\psi}_{i,j} = -\tilde{\omega}_{i,j}, \tag{72}$$

$$\tilde{U}_{i,j} = D_0^\beta\tilde{\psi}_{i,j}, \tag{73}$$

$$\tilde{V}_{i,j} = -D_0^\alpha\tilde{\psi}_{i,j}, \tag{74}$$

where $\tilde{\Delta}_h = D_+^\alpha D_-^\alpha + D_+^\beta D_-^\beta$. The computational boundary conditions are

$$\tilde{\psi}_{i,0} = 0, \tag{75}$$

$$\tilde{\omega}_{i,0} = -\frac{2J_{i,0}}{h^2}\tilde{\psi}_{i,1}, \tag{76}$$

where (76) is derived like (14) and is also first-order accurate. Boundary conditions at $\beta = 1$ are derived similarly. The following convergence result can be obtained, provided the exact solution of the Navier–Stokes equations is sufficiently smooth.

THEOREM 2.5. *The solutions of* (71)–(74) *with boundary conditions* (75) *and vorticity boundary conditions* (76) *converge uniformly to the exact solutions of the Navier–Stokes equations with second-order accuracy:*

$$\|\omega - \tilde{\omega}\|_\infty \le C(T)h^2.$$

*Second-order convergence in the velocities can also be shown.*

*Proof.* An approximate solution can be constructed that satisfies the discrete boundary conditions and interior equations to a high order of accuracy as in Lemma 2.2 (consistency). Equations (71)–(74), along with the specified boundary conditions, have the same structure as (5)–(8) and (14)–(15) but with smooth positive weights $J_{i,j}$. It is the preservation of this structure under conformal mappings that allows Lemma 2.3 (stability) to be proved for these equations almost exactly as before. Theorem 2.5 follows from the stability lemma as before. □

**3. Three-dimensional results.** The vector form of the Navier–Stokes equations in three dimensions written in vorticity, stream function formulation are

$$\omega_t = -u\cdot\nabla\omega + \omega\cdot\nabla u + \nu\Delta\omega, \tag{77}$$

$$\Delta\psi = -\omega, \tag{78}$$

$$u = \nabla\times\psi, \tag{79}$$

where $\omega$, $\psi$, and $u$ are now vector valued. The three terms on the right-hand side of (77) are called the convective, stretching, and dissipative terms, respectively. Components of the vectors will be written with superscripts. A solution is considered in the domain shown in Fig. 3, which is a generalization of the periodic channel considered in the two-dimensional case in which the flow is assumed to be periodic in the $x-$ and $z$-directions. Boundary conditions of $\psi = 0$ and $\partial\psi/\partial y = 0$ are imposed on the upper and lower walls, which guarantees no-flow ($v = 0$) and no-slip ($u, w = 0$) boundary conditions on these surfaces. More general boundary conditions can be handled as in §2.3.

**3.1. Discretization.** Our three-dimensional scheme is a straightforward generalization of (5)–(8) in the two-dimensional case. Discrete variables $\tilde{\psi}^\alpha_{i,j,k}$ are defined at mesh points $(ih, jh, kh)$ and the discrete vorticity is related to the discrete stream function through the relation

$$\tilde{\omega}_{i,j,k} = -\Delta_h\tilde{\psi}_{i,j,k}. \tag{80}$$

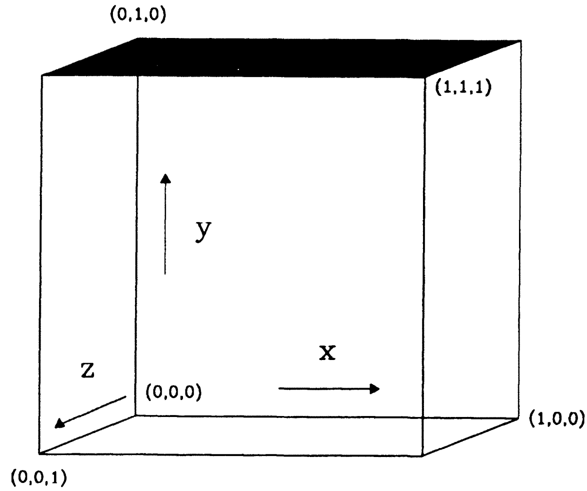


FIG. 3. *Three-dimensional channel.*

The velocities $\tilde{u}_{i,j,k}$ are the centered difference approximations of $\nabla \times \psi$, i.e.,

$$\tilde{u}^1_{i,j,k} = D^y_0 \tilde{\psi}^3_{i,j,k} - D^z_0 \tilde{\psi}^2_{i,j,k}, \text{ etc.} \tag{81}$$

The discrete equations can now be written as

$$\frac{d\tilde{\omega}^\alpha_{i,j,k}}{dt} = -\tilde{u}^\beta_{i,j,k} \cdot D^\beta_0 \tilde{\omega}^\alpha_{i,j,k} + \tilde{\omega}^\beta_{i,j,k} \cdot D^\beta_0 \tilde{u}^\alpha_{i,j,k} + \nu \Delta_h \tilde{\omega}^\alpha_{i,j,k}, \tag{82}$$

where $D^1_-$ is equivalent to $D^x_-$, etc., and summation on $\beta$ is assumed.

The question of boundary conditions remains. Periodic conditions in the $x$- and $z$-directions are assumed. On the surface $y = 0$, the following conditions are defined:

$$\tilde{\psi}^\alpha_{i,0,k} = 0, \tag{83}$$

$$\tilde{\omega}^\alpha_{i,0,k} = -\frac{2}{h^2} \tilde{\psi}^\alpha_{i,1,k}. \tag{84}$$

These boundary conditions are the discrete versions of $\psi = 0$ and $\partial\psi/\partial y = 0$ (in vorticity form), respectively. Equation (84) is derived exactly like (10) in the two-dimensional case. Because of the addition of the vortex stretching term in the three-dimensional equations, values for the discrete velocities on the boundaries are needed. The following boundary conditions are also used:

$$\tilde{u}^\alpha_{i,0,k} = 0. \tag{85}$$

Boundary conditions similar to (83)–(85) are defined on the surface $y = 1$.

**3.2. Convergence analysis.** Just as in the two-dimensional case, approximate solutions $\hat{\psi}$ are introduced which satisfy the equations (82) and boundary conditions (84) to a high order of accuracy as in Lemma 2.2 (consistency). These solutions are constructed in an analogous way to the two-dimensional ones (see the appendix). The discrete norms in §2.2 are generalized to the three-dimensional setting:

$$\|f\|_2^2 = h^3 \sum_{\alpha=1}^{3} \sum_{i,j,k} (f^\alpha_{i,j,k})^2,$$

$$\|f\|_{1,2}^2 = h^3 \sum_{\alpha,\beta=1}^{3} \sum_{i,j,k} (D_-^\beta f_{i,j,k}^\alpha)^2.$$

The error terms $\epsilon_{i,j,k}^\alpha$ and $e_{i,j,k}^\alpha$ are defined as in the two-dimensional case and the velocity error is written as

$$E_{i,j,k}^1 = \tilde{u}_{i,j,k}^1 - (D_0^y \hat{\psi}_{i,j,k}^3 - D_0^z \hat{\psi}_{i,j,k}^2), \text{ etc.} \tag{86}$$

In the discussion of boundary values in the appendix it is shown that

$$E_{i,0,k}^\alpha = O(h^q). \tag{87}$$

The following formulas are equivalent to (26) and (23), respectively, from the two-dimensional analysis:

$$h \sum_{i,k} (\epsilon_{i,1,k}^\alpha)^2 \le \|\epsilon\|_{1,2}^2, \tag{88}$$

$$\|f\|_\infty \le h^{-3/2} \|f\|_2. \tag{89}$$

The following convergence result can be obtained, provided the exact solution of the Navier–Stokes equations is sufficiently smooth.

THEOREM 3.1. *The solutions of* (80)–(82) *with boundary conditions* (83) *and* (85) *and vorticity boundary conditions* (84) *converge uniformly to the exact solutions of the Navier–Stokes equations with second-order accuracy:*

$$\|\omega - \tilde{\omega}\|_\infty \le C(T) h^2.$$

*Second-order convergence in the velocities and first-order convergence in the boundary vorticity can also be shown.*

*Proof.* To prove Lemma 2.3 (stability) in the three-dimensional case, we proceed as before. The approximate solutions $\hat{\psi}$ satisfy the discrete equations (82) up to $O(h^q)$ as in Lemma 2.2 (consistency). Subtract this result from (82) and write out the difference as in (33)–(41). Multiply the result by $h^3 \epsilon_{i,j,k}^\alpha$ and sum over the interior and $\alpha$. The terms coming from the vorticity convection and dissipation are estimated as in the two-dimensional case. There are new terms coming from the vortex stretching which must be examined. The nonlinear terms are bounded by brute force as before. Representative linear terms are given below:

$$h^3 \sum_{i,k} \sum_{j=1}^{N-1} \epsilon_{i,j,k}^1 e_{i,j,k}^2 D_0^y \hat{u}_{i,j,k}^1 \tag{90}$$

and

$$h^3 \sum_{i,k} \sum_{j=1}^{N-1} \epsilon_{i,j,k}^1 \hat{\omega}_{i,j,k}^2 D_0^y E_{i,j,k}^1. \tag{91}$$

By noting that $e_{i,j,k}^2 = -\Delta_h \epsilon_{i,j,k}^2$ in the interior, the first term (90) can be bounded by $K\|\epsilon\|_{1,2}^2$ after summation by parts. The second term (91) requires a little more work. After summation by parts in the $y$ direction, this term becomes

$$-h^3 \sum_{i,k} \sum_{j=1}^{N-1} D_0^y(\epsilon^1_{i,j,k}\hat{\omega}^2_{i,j,k})E^1_{i,j,k}$$
$$-\frac{h^2}{2}\sum_{i,k}\epsilon^1_{i,1,k}\hat{\omega}^2_{i,1,k}E^1_{i,0,k} + \frac{h^2}{2}\sum_{i,k}\epsilon^1_{i,N-1,k}\hat{\omega}^2_{i,N-1,k}E^1_{i,N,k}.$$

The first expression above can be bounded by $K\|\epsilon\|^2_{1,2}$ since the error terms $E$ are just divided differences of the error terms $\epsilon$ in the interior. The other expressions above can be bounded by $O(h^{q+1/2})\|\epsilon\|_{1,2}$ using (87) and (88). Therefore, the analysis that led to Lemma 2.3 (stability) can be duplicated for the three-dimensional case. The convergence result, Theorem 3.1, follows from the stability lemma as in the two-dimensional case. □

**4. Summary.** The authors have presented a rigorous convergence analysis for a finite difference approximation of the Navier–Stokes equations in two and three dimensions. The difference method considered is of the vorticity-stream function type and uses vorticity boundary conditions of creation type. The main ingredients *in the proof are the use of approximate solutions that satisfy the discrete equations* and boundary conditions to high-order accuracy and careful discrete energy estimates. Both of these techniques are most naturally used on a fixed, simple grid (hence our use of conformal mappings for general two-dimensional domains). An application of these techniques to a Lagrangean method such as Chorin's algorithm still seems difficult at this stage. However, we are hopeful that our analysis can be generalized to certain deterministic vortex methods, including vortex-in-cell methods. The general philosophy that comes from our analysis is that the linear stability of a proposed numerical scheme for the initial-boundary value problem is the crucial question: the nonlinear stability is free once the approximate solutions are constructed and the accuracy is determined by the accuracy of the interior scheme.

**Appendix. Proof of the consistency lemma.** In this appendix, the approximate solutions $\hat{\psi}(x, y, t; h)$ that satisfy Lemma 2.2 (consistency) will be constructed. In order to do this, we proceed in a manner similar to Strang [18] and consider an expansion

$$\hat{\psi}(x, y, t; h) = \sum_{p=0}^{q-1} h^p \psi^{(p)}(x, y, t). \tag{92}$$

The desired properties of $\hat{\psi}$ can be obtained by choosing the correct functions $\psi^{(p)}$ (they will depend only on the exact solution $\psi$). The procedure for determining $\psi^{(p)}$ will be outlined below.

**Boundary conditions.** Since the condition $\hat{\psi}_{i,0} = 0$ must be satisfied exactly, it is assumed that $\psi^{(p)}(x, 0) = 0$ for all $x$ and $p$. Similarly, $\psi^{(p)}(x, 1) = 0$. Also, all functions $\psi^{(p)}$ are assumed to be periodic in $x$. The normal boundary conditions for the functions $\psi^{(p)}$ must be chosen so that the condition in equation (17) is satisfied, keeping in mind that $\hat{\omega}_{i,0}$ is given by (21) (this is needed to maintain the interior accuracy). It will be assumed that the functions $\psi^{(p)}$ are as smooth as necessary (this will be verified in a later section). First, consider the Taylor expansion of $\hat{\omega}(x, 0)$ in

(21):

$$\hat{\omega}(x,0) = -D^y_- D^y_+ \hat{\psi}(x,0) := -\sum_{n=0}^{N} \frac{2h^{2n}}{(2n+2)!} \frac{\partial^{2n+2}\hat{\psi}}{\partial y^{2n+2}}(x,0) + O(h^{2N+2}), \tag{93}$$

where $N$ is such that $2N \geq q$. A Taylor expansion for $-\frac{2}{h^2}\hat{\psi}(x,h)$ (using the fact that $\hat{\psi}(x,0) = 0$) is given by

$$-\frac{2}{h^2}\hat{\psi}(x,0) = \sum_{n=1}^{2N+2} \frac{2h^{n-2}}{n!} \frac{\partial^n \hat{\psi}}{\partial y^n}(x,0) + O(h^{2N+1}). \tag{94}$$

Subtracting the terms (93) and (94) above, we find that the even powers of $h$ drop out and the following remains:

$$\sum_{n=0}^{N} \frac{2h^{2n-1}}{(2n+1)!} \frac{\partial^{2n+1}\hat{\psi}}{\partial y^{2n+1}}(x,0) + O(h^{2N+1}). \tag{95}$$

In order to satisfy (17), the above expression must be $O(h^{q-1})$, i.e.,

$$\sum_{n=0}^{N} \frac{h^{2n-1}}{(2n+1)!} \frac{\partial^{2n+1}\hat{\psi}}{\partial y^{2n+1}}(x,0) = O(h^{q-1}). \tag{96}$$

The sum (92) is substituted into the above equation and the coefficients of powers of $h$ are examined:

$$\begin{aligned}
\frac{1}{h} &: \frac{\partial \psi^{(0)}}{\partial y}(x,0), \\
1 &: \frac{\partial \psi^{(1)}}{\partial y}(x,0), \\
h &: \frac{\partial \psi^{(2)}}{\partial y}(x,0) + \frac{1}{6}\frac{\partial^3 \psi^{(0)}}{\partial y^3}(x,0), \\
&\vdots \\
h^r &: \frac{\partial \psi^{(r)}}{\partial y}(x,0) + \frac{1}{6}\frac{\partial^3 \psi^{(r-2)}}{\partial y^3}(x,0) + \frac{1}{5!}\frac{\partial^3 \psi^{(r-4)}}{\partial y^5}(x,0) + \cdots \\
&\vdots
\end{aligned}$$

If the first $q$ coefficients of powers of $h$ given above were zero, then the condition (96) would be satisfied, and so the relationship given in (17) would be valid. To ensure that the coefficients above are zero, we impose the following for all $x$:

$$\frac{\partial \psi^{(0)}}{\partial y}(x,0) = 0, \tag{97}$$

$$\frac{\partial \psi^{(1)}}{\partial y}(x,0) = 0, \tag{98}$$

$$\frac{\partial \psi^{(2)}}{\partial y}(x,0) = -\frac{1}{6}\frac{\partial^3 \psi^{(0)}}{\partial y^3}(x,0),$$

$$\vdots$$

$$\frac{\partial\psi^{(r)}}{\partial y}(x,0) = -\frac{1}{6}\frac{\partial^3\psi^{(r-2)}}{\partial y^3}(x,0) - \frac{1}{5!}\frac{\partial^5\psi^{(r-4)}}{\partial y^5}(x,0) - \cdots. \tag{99}$$

$$\vdots$$

Similar conditions can be obtained for the upper boundary. The formulas above give boundary conditions for the normal derivative of $\psi^{(r)}$ in terms of the derivatives of $\psi^{(p)}$ on the boundary for $p < r$.

**A note on the order of the boundary conditions.** At this point it is appropriate to discuss the question of the accuracy of the vorticity boundary condition. Recalling (20), we know that the vorticity at the boundary is given formally by

$$\hat{\omega}_{i,0} = -\frac{1}{h^2}(\hat{\psi}_{i,1} + \hat{\psi}_{i,-1})$$

and that the expansion of the vorticity on the boundary must satisfy

$$\hat{\omega}_{i,0} = -\frac{2}{h^2}\hat{\psi}_{i,1}$$

to high-order accuracy as in (17). Comparing the two equations above, we see that they are equivalent if $\hat{\psi}_{i,-1} = \hat{\psi}_{i,1}$ or

$$\frac{\hat{\psi}_{i,1} - \hat{\psi}_{i,-1}}{2h} = 0. \tag{100}$$

If we interpret (100) as a Taylor series approximation at the boundary, we see that it is a second-order approximation of the no-slip condition, $\partial\psi/\partial y = 0$. Since the no-slip condition is the physical boundary condition of the problem and it is satisfied to second-order accuracy, we can expect second-order convergence in terms of the stream function. This observation was first made by Naughton [15].

A subtle point which can be answered at this point is whether the boundary conditions for the velocity in the three-dimensional case (85) (which are satisfied exactly by the exact velocity) will also be satisfied to high accuracy by the approximate solutions $\hat{\psi}$. A priori there is no way to guarantee that the expansion for the stream function at the boundary will not introduce errors in the expressions for the boundary velocities. However, since the velocities are given by (81) and all three components of the stream function expansion will satisfy (100), we see that the expansions in the boundary conditions for the velocity and the vorticity are compatible. This discussion leads us to the result (87) used in §3.2.

**Interior equations.** In this section, we construct relationships between the functions $\psi^{(p)}$ so that (19) is satisfied. As in the previous section, the sum (92) is inserted into the expression (19), the finite differences are expanded in Taylor series, and the coefficients of the powers of $h$ are set equal to zero. The resulting expressions which must be satisfied for all $x$, $y$, and $t$ are:

$$1: \frac{\partial\Delta\psi^{(0)}}{\partial t} = -\psi_y^{(0)}\Delta\psi_x^{(0)} + \psi_x^{(0)}\Delta\psi_y^{(0)} + \nu\Delta\Delta\psi^{(0)}, \tag{101}$$

$$h: \frac{\partial\Delta\psi^{(1)}}{\partial t} = -\psi_y^{(0)}\Delta\psi_x^{(1)} + \psi_x^{(0)}\Delta\psi_y^{(1)} \tag{102}$$

$$-\psi_y^{(1)}\Delta\psi_x^{(0)} + \psi_x^{(1)}\Delta\psi_y^{(0)} + \nu\Delta\Delta\psi^{(0)},$$

$$h^2 : \frac{\partial\Delta\psi^{(2)}}{\partial t} = -\psi_y^{(0)}\Delta\psi_x^{(2)} - \psi_y^{(2)}\Delta\psi_x^{(0)}$$
$$+\psi_x^{(0)}\Delta\psi_y^{(2)} + \psi_x^{(2)}\Delta\psi_y^{(0)} + \nu\Delta\Delta\psi^{(2)}$$
$$-\psi_y^{(1)}\Delta\psi_x^{(1)} + \psi_x^{(1)}\Delta\psi_y^{(1)}$$
$$-\frac{1}{12}\left(\frac{\partial^5\psi^{(0)}}{\partial x^4\partial t} + \frac{\partial^5\psi^{(0)}}{\partial y^4\partial t}\right) + \frac{1}{6}\left(\frac{\partial^4\Delta\psi^{(0)}}{\partial x^4} + \frac{\partial^4\Delta\psi^{(0)}}{\partial y^4}\right),$$
$$+\cdots$$

$$\vdots$$

$$(103)\quad h^r : \frac{\partial\Delta\psi^{(r)}}{\partial t} = -u^{(r)}\Delta\psi_x^{(r)} - \psi_y^{(r)}X^{(r)} + v^{(r)}\Delta\psi_y^{(r)} + \psi_x^{(r)}Y(r)$$
$$+\nu\Delta\Delta\psi^{(r)} + f^{(r)},$$

$$\vdots$$

where $u^{(r)}$, $v^{(r)}$, $X^{(r)}$, $Y^{(r)}$, and $f^{(r)}$ are linear combinations of derivatives of $\psi^{(p)}$ with $p < r$. If the relationships above are satisfied, then the desired result (19) will be obtained. The relationships above and the boundary conditions in the previous section can be thought of as equations for $\psi^{(p)}$, which can be solved inductively: once $\psi^{(0)}$ is known, then the boundary values and terms in the equation for $\psi^{(1)}$ are known, and so we can solve for $\psi^{(1)}$, and so on.

**Properties of the solutions.** Since the solution $\hat{\psi}$ should agree exactly with the exact solution $\psi$ at $t = 0$ for all $h$, we give the following initial data for the terms $\psi^{(p)}$:

$$\psi^{(0)}(x, y, 0) = \Psi(x, y),$$
$$\psi^{(p)}(x, y, 0) = 0 \text{ for } p > 0,$$

where $-\Delta\Psi$ is the initial vorticity distribution. The equation (101), boundary conditions (97), and initial conditions (above) for $\psi^{(0)}$ and $\psi$ are the same, so $\psi^{(0)} = \psi$. We assume that $\psi$ is as smooth as required. It is clear from (98) and (102) and the initial conditions above that $\psi^{(1)} \equiv 0$. It can be shown inductively that $\psi^{(p)} \equiv 0$ for all odd $p$. Now consider the equations (103) and boundary conditions (99) for $\psi^{(r)}$ for $r$ even. These equations are linear and simple energy estimates (like those for the heat equation) show that $\psi^{(r)}$ and its derivatives are bounded by the coefficients and nonhomogeneous terms in the equations for $\psi^{(r)}$ which can inductively be bounded by the exact solution $\psi$ and its derivatives. The solutions are smooth for $t > 0$, but the problem of showing smoothness of the solution up to the initial time is a delicate one, even for the heat equation [12]. Certain compatibility conditions on the initial data must be imposed. In the case of the Navier–Stokes equations and the equations for $\psi^{(r)}$, we can derive nonlocal compatibility conditions on the initial data in order for the solution to be smooth at $t = 0$. This is discussed below. Assuming that these conditions are met, the constructed expansion $\hat{\psi}$ has the desired properties, and the proof of Lemma 2.2 (consistency) is complete. □

**A remark on the smoothness of solutions of Navier–Stokes equations at $t = 0$.** Consider the Navier–Stokes equations written in vorticity-stream function formulation (1)–(4) with no-slip and no-flow boundary conditions in the periodic channel $\Omega$ with initial data $\psi(x,y,0) = \Psi(x,y)$. There is a hierarchy of compatibility conditions that $\Psi$ must satisfy if the solution $\psi$ is to be smooth at $t = 0$:

(1) Clearly, $\Psi(x,0) = 0$, $\Psi(x,1) = 0$, $\Psi_y(x,0) = 0$, and $\Psi_y(x,1) = 0$ to have a continuous solution $\psi$.

(2) Let $g^{(1)}(x,y) = \psi_t(x,y,0)$. By differentiating the no-flow and no-slip boundary conditions, obtain $g^{(1)} = g_y^{(1)} = 0$ on the boundary. From (1), we have

$$\Delta g^{(1)} = -\Psi_y \Delta \Psi_x + \Psi_x \Delta \Psi_y + \nu \Delta\Delta\Psi := \gamma^{(1)}. \tag{104}$$

It appears that $g^{(1)}$ is overdetermined (both Dirichlet and Neumann boundary conditions for $g^{(1)}$ are given), so $\gamma^{(1)}$ must be in the class of functions $\mathcal{C}$ for which (104) can be solved. In this case, the class $\mathcal{C}$ consists of all functions $\gamma$ such that

$$\int_\Omega \gamma u = 0 \tag{105}$$

for all harmonic functions $u$ in $\Omega$ [21]. Assuming that the solvability condition is met, $g^{(1)}$ can be determined in terms of $\Psi$.

(3) Let $g^{(2)}(x,y) = \psi_{tt}(x,y,0)$. By differentiating the no-flow and no-slip boundary conditions twice, obtain $g^{(2)} = g_y^{(2)} = 0$ on the boundary. Differentiating (1) with time, we have

$$\begin{aligned} \Delta g^{(2)} &= -\psi_{yt}\Delta\Psi_x - \Psi_y \Delta\psi_{xt} + \psi_{xt}\Delta\Psi_y + \Psi_x \Delta\psi_{yt} + \nu\Delta\Delta\psi_t \\ &= -g_y^{(1)}\Delta\Psi_x - \Psi_y\Delta g_x^{(1)} + g_x^{(1)}\Delta\Psi_y + \Psi_x\Delta g_y^{(1)} + \nu\Delta\Delta g^{(1)} \\ &:= \gamma^{(2)}. \end{aligned}$$

In order to satisfy both boundary conditions, $\gamma^{(2)}$ must also be in $\mathcal{C}$. This imposes a further restriction on $\Psi$.

There is also a hierarchy of compatibility conditions for $\Psi$ that come from the smoothness requirements for the functions $\psi^{(p)}$. These conditions are straightforward generalizations of the ones considered above.

In principle, we would expect that smooth solutions will exist up to $t = 0$ if these compatibility conditions are satisfied, but the proof of such a theorem is a delicate issue and not the purpose of this paper. See the paper by Temam [19] for a proof of the sufficiency of nonlocal compatibility conditions for smoothness in the case of the Navier–Stokes equations written in primitive variables, velocity, and pressure. Heywood and Rannacker [11] describe what behaviour the solutions have as $t$ goes to zero when these compatibility conditions are not met and show convergence of finite element approximations in this case. Naughton [15] shows convergence of a vorticity-stream function method in a linear one-dimensional model problem with incompatible initial data. For a general discussion of existence theorems for the Navier–Stokes equations, see the books by Temam [20] and Kreiss and Lorentz [12].

**Acknowledgments.** The authors wish to thank Professor Chris Anderson for several beneficial discussions. We also want to thank Professor Jonathan Goodman for bringing Meth's thesis to our attention.

## REFERENCES

[1] C. R. ANDERSON, *Vorticity boundary conditions for boundary vorticity generation for two-dimensional viscous incompressible flows*, J. Comput. Phys., 80 (1989), pp. 72–97.

[2] C. R. ANDERSON, *Observations on vorticity creation boundary conditions*, in Mathematical Aspects of Vortex Dynamics, R. E. Caflisch, ed., Society for Industrial and Applied Mathematics, Philadelphia, PA, 1989.

[3] J. B. BELL, P. COLELLA, AND H. M. GLAZ, *A second-order projection method for the incompressible Navier–Stokes equations*, J. Comput. Phys., 85 (1989), pp. 257–283.

[4] G. BENFATTO AND M. PULVIRENTI, *Convergence of the Chorin–Marsden product formula in the half-plane*, Comm. Math. Phys., 106 (1986), pp. 427–458.

[5] A. J. CHORIN, *Numerical study of slightly viscous flows*, J. Fluid Mech., 57 (1973), pp. 785–796.

[6] ———, *Vortex sheet approximation of boundary layers*, J. Comput. Phys., 27 (1978), pp. 428–442.

[7] A. J. CHORIN, T. J. R. HUGHES, M. F. MCCRACKEN, AND J. E. MARSDEN, *Product formulas and numerical algorithms*, Comm. Pure Appl. Math., 31 (1978), pp. 205–256.

[8] G.-H. COTTET, *Boundary conditions and deterministic vortex methods for the Navier–Stokes equations*, in Mathematical Aspects of Vortex Dynamics, R. E. Caflisch, ed., Society for Industrial and Applied Mathematics, Philadelphia, PA, 1989.

[9] J. GOODMAN, T. Y. HOU, AND J. LOWENGRUB, *Convergence of the point vortex method for the* 2-D *Euler equations*, Comm. Pure Appl. Math, 43 (1990), pp. 415–430.

[10] B. GUSTAFSSON, *The Convergence Rate for Difference Approximations to Mixed Initial Boundary Value Problems*, Math. Comp., 29 (1975), pp. 396–406.

[11] J. G. HEYWOOD AND R. RANNACHER, *Finite element approximation of the nonstationary Navier–Stokes problem.* I. *Regularity of solutions and second-order error estimates for spatial discretization*, SIAM J. Numer. Anal., 19 (1982), pp. 275–311.

[12] H.-O. KREISS AND J. LORENTZ, *Initial-Boundary Value Problems and the Navier–Stokes Equations*, Academic Press, San Diego, CA, 1989.

[13] K. METH, *A vortex and finite difference hybrid method to compute the flow of an incompressible, inviscid fluid past a semi-infinite plate*, Ph.D. thesis, New York University, New York, 1988.

[14] D. MICHELSON, *Convergence theorems for finite difference approximations for hyperbolic quasi-linear initial-boundary value problems*, Math. Comp., 49 (1987), pp. 445–459.

[15] M. J. NAUGHTON, *On Numerical Boundary Conditions for the Navier–Stokes Equations,* Ph.D. thesis, Department of Applied Mathematics, California Institute of Technology, Pasadena, CA, 1986.

[16] S. A. ORSZAG AND M. ISRAELI, *Numerical simulation of viscous incompressible flows*, Ann. Rev. of Fluid Mech., 6 (1974), pp. 281–318.

[17] L. QUARTAPELLE, *Vorticity conditioning in the computation of two-dimensional viscous flows*, J. Comput. Phys., 40 (1981), pp. 453–477.

[18] G. STRANG, *Accurate partial differential methods* II. *Non-linear problems*, Numer. Math., 6 (1964), pp. 37–46.

[19] R. TEMAM, *Behaviour at time* $t = 0$ *of the solutions of semi-linear evolution equations*, J. Differential Equations, 43 (1982), pp. 73–92.

[20] ———, *Navier Stokes Equations*, North-Holland, New York, Amsterdam, 1977.

[21] F. V. VALZ-GRIS AND L. QUARTAPELLE, *Projection conditions on the vorticity in viscous incompressible flows*, Internat. J. Numer. Math. Fluids, 1 (1981), pp. 129–144.