On Global Well-Posedness of the Lagrangian Averaged Euler Equations

Thomas Y. Hou* Congming Li †

Abstract

We study the global well-posedness of the Lagrangian averaged Euler equations in three dimensions. We show that a necessary and sufficient condition for the global existence is that the BMO norm of the stream function is integrable in time. We also derive a sufficient condition in terms of the total variation of certain level set functions, which guarantees the global existence. Furthermore, we obtain the global existence of the Lagrangian averaged 2D Boussinesq equations and the Lagrangian averaged 2D quasi-geostrophic equations in finite Sobolev space in the absence of viscosity or dissipation.

1 Introduction

The question of global existence for the 3D incompressible Euler equations is a very challenging open question. The main difficulty is to understand the effect of vortex stretching, which is absent in the 2D Euler equations. As part of the effort to understand the vortex stretching effect for 3D flows, various simplified model equations have been proposed in the literature. Amongst these models, the 2D Boussinesq system and the quasi-geostrophic equations are two of the most commonly used because they share a similar vortex stretching effect as that in the 3D incompressible flow. An interesting recent development is the Lagrangian averaged Euler equations [14, 15]. This work was originally motivated by the development of a one-dimensional shallow water theory [3]. The averaged Euler models have been used to study the average behavior of the 3D Euler and Navier-Stokes equations and used as a turbulent closure model (see e.g. [5]). The theoretical and computational aspects of the Lagrangian averaged Euler equations

*Applied and Computational Mathematics, Caltech, Pasadena, CA 91125. Email: hou@acm.caltech.edu

†Department of Applied Mathematics, University of Colorado at Boulder, Boulder, CO 80309. Email: cli@colorado.edu

has been studied by several authors [4, 5, 15, 21, 20]. However, the global existence of the 3D Lagrangian averaged Euler equations is still open, although the Lagrangian averaged Navier-Stokes equations have been shown to have global existence [20].

In this paper, we consider the global existence of the 3D Lagrangian averaged Euler equations and the corresponding 2D Lagrangian averaged Boussinesq equations and the averaged 2D quasi-geostrophic equations in the absence of viscosity or dissipation. The 3D Lagrangian averaged Euler equations have been derived by Holm, Marsden and Ratiu in [14, 15] (see page 1458 of [20]) in the following form:

$$\partial_t u + (u_\alpha \cdot \nabla) u + (\nabla u_\alpha)^T \cdot u = -\nabla p \tag{1}$$

here the notations are different from those in [20]. Our u_α corresponds to the original u and our u corresponds to $(1 - \alpha^2 \triangle) u$ in [20].

We will adopt the vorticity formulation [20]:

$$\partial_t \omega + (u_\alpha \cdot \nabla)\omega = \nabla u_\alpha \cdot \omega, \tag{2}$$

where u, ω, the α-averaged velocity u_α, and the divergence free vector stream function ψ are related by:

$$-\triangle \psi = \omega, \quad u = \nabla \times \psi \tag{3}$$

$$u_\alpha = (1 - \alpha^2 \triangle)^{-1} u, \tag{4}$$

One of the important properties of the averaged Euler equations is the following identity (see (3.3) in page 1457 of [20], recall that u_α is called u in [20]):

$$\frac{1}{2}\frac{d}{dt}\int_{R^3} \left(|u_\alpha|^2 + \alpha^2 |\nabla u_\alpha|^2\right) dx = 0.$$

This conservation property gives *a priori* bound on the H^1 norm of u_α:

$$\|u_\alpha\|_{H^1} \le C_\alpha. \tag{5}$$

The above reformulation gives a clear physical interpretation of the Lagrangian averaged Euler equations. The vorticity is convected by the α-averaged velocity field. If one discretizes the averaged Lagrangian Euler equations by the point vortex method, i.e. to approximate the initial vorticity by a collection of point vortices (Dirac delta functions), then the resulting numerical approximation is a vortex blob method with α being the vortex blob size [22, 13].

With the above interpretation of the averaged Lagrangian Euler equations, we can clearly apply the same averaging principle to other fluid dynamics equations. For example, if we apply the same Lagrangian averaging principle to the density equation, we would obtain the following Lagrangian averaged 2D Boussinesq equations

$$\omega_t + u \cdot \nabla \omega = \rho_{x_1}, \tag{6}$$

$$\rho_t + u_\alpha \cdot \nabla \rho = 0\,, \tag{7}$$

where $u_\alpha = (1-\alpha^2\triangle)^{-1}u$ and u is related to the vorticity ω through the usual vorticity stream function formulation, see (3). We refer to [23] for the derivation and discussions of the physical applications for the Boussinesq equations. Note that we only replace the velocity by the averaged velocity in the density equation, but not in the vorticity equation.

Similarly, we can derive the Lagrangian-averaged 2D quasi-geostrophic equations as follows:

$$\theta_t + u_\alpha \cdot \nabla \theta = 0, \tag{8}$$

$$u = \nabla^\perp \psi, \quad (-\triangle)^{1/2}\psi = \theta, \tag{9}$$

$$u_\alpha = (1-\alpha^2\triangle)^{-1/2}u\,, \tag{10}$$

where $\nabla^\perp\psi = (\partial_{x_2}\psi, -\partial_{x_1}\psi)$ and $(-\triangle)^{1/2}$ is defined as

$$(-\triangle)^{1/2}\psi \equiv \int e^{2\pi i x\cdot\xi}(2\pi|\xi|)\hat{\psi}(\xi)\; d\xi,$$

with $\hat{\psi}(\xi)$ being the Fourier transform of ψ. We refer to [9] for derivation and discussions of the quasi-geostrophic equation. Note that we use a weaker averaged velocity field for the 2D quasi-geostrophic equation. The exponent $1/2$ in the averaging operator corresponds to the critical case in the corresponding dissipative quasi-geostrophic equations [8].

In this paper, we prove that a necessary and sufficient condition for the global existence is that the BMO norm of the stream function is integrable in time. This is an analogue of the well-known Beale-Kato-Majda condition [1] for the 3D Euler equations. For some recent results on the 3D Euler equations that explore the geometric properties of the Euler flow, we refer to [10, 11]. Moreover, using a level formulation, we derive a sufficient condition for the global existence. The non-blowup condition we obtain is expressed in terms of the total variation of a level set function, see (53) in Section 3 for the precise definition. Assume that the initial vorticity can be expressed in the form $\omega(0,x) = \omega_0(\phi_0,\psi_0)\nabla\phi_0 \times \nabla\psi_0$ for some smooth and bounded levelset functions

ϕ_0 and ψ_0. Let ϕ and ψ be the levelset functions satisfying

$$\phi_t + (u_\alpha \cdot \nabla)\phi = 0, \quad \phi(0,x) = \phi_0(x),$$
$$\psi_t + (u_\alpha \cdot \nabla)\psi = 0, \quad \psi(0,x) = \psi_0(x).$$

Then vorticity can be expressed in terms of these two levelset functions:

$$\omega = \omega_0(\phi, \psi)\nabla\phi \times \nabla\psi.$$

Moreover, if the total variation of either ϕ or ψ is intergral in time, then there is no finite time blow-up of the 3D averaged Euler equations. This result has a geometric interpretation. In particular, it excludes the possibility of a finite number of isolated singularities when vorticity is considered as a one-dimensional function by fixing the other two variables. If there is a finite time singularity, the one-dimensional restriction of vorticity must be highly oscillatory at the singularity time, and the singularities are dense in the singular region.

Application of the same argument to the corresponding 2D models gives much sharper existence results. In particular, we prove the global existence of the Lagrangian averaged 2D Boussinesq equations and the averaged 2D quasi-geostrophic equations in finite Sobolev spaces without any assumption on the solution itself.

The rest of the paper is organized as follows. In Section 2, we prove the necessary and sufficient condition for the 3D Lagrangian averaged Euler equations, and prove the global existence for the averaged 2D Boussinesq equations and the averaged 2D quasi-geostrophic equations. In Section 3 we present some result for the global existence of the 3D Lagrangian averaged Euler equations using a noval level set formulation.

2 Main Results and Proofs

In this section, we present three results. The first result is a necessary and sufficient condition for the global existence of the averaged Euler equations. The second result is the global existence of the averaged 2D Boussinesq equations. The third result is the global existence of the averaged 2D quasi-geostrophic equations. We begin by stating our first result for the 3D averaged Euler equations. Our result uses the BMO norm. Before we state our existence result, we remind the reader of the definition of the BMO norm which is defined as follows:

$$\|f\|_{BMO} = \sup_{x \in R^3} \sup_{r>0} \frac{1}{|B_r|} \int |f - \bar{f}| dx,$$

where $\bar{f} = \frac{1}{|B_r|} \int_{B_r} f(y)dy$, $B_r = \{y \in R^3, \quad |y - x| \le r\}$, and $|B_r|$ is the volume of B_r.

Theorem 1. *Assume that $\omega_0 \in H^m(R^3)$, $m \geq 0$. Then for any $\alpha > 0$, the solution of the Lagrangian averaged 3D Euler equations (2)-(4) has a unique global solution in $H^m(R^3)$ satisfying*

$$\|\omega(t)\|_{H^m} \leq C(T)\|\omega_0\|_{H^m}, \quad \text{for} \quad 0 \leq t \leq T,$$

if we have

$$\int_0^T \|\psi\|_{BMO} dt < \infty, \tag{11}$$

for any $T > 0$. Conversely, if the maximal time T of the existence of classical solutions is finite, then necessarily we have

$$\int_0^T \|\psi\|_{BMO} dt = \infty. \tag{12}$$

Proof. The proof relies on the following estimate obtained by Kozono and Taniuchi in [17]:

$$\|f\|_\infty \leq C\,(1 + \|f\|_{BMO}(1 + \log(\|f\|_{W^{s,p}} + e)) \tag{13}$$

for all $f \in W^{s,p}$ with $1 < p < \infty$ and $s > n/p$, n is the space dimension.

Another useful result is the following embedding estimate in the BMO norm:

$$\|Rf\|_{BMO} \leq C\|f\|_{BMO}, \tag{14}$$

for any Riesz type operator R (see [24] and appendix A of this paper).

It follows from (2)-(4) that

$$u_\alpha = (1 - \alpha^2 \triangle)^{-1} \nabla \times \psi. \tag{15}$$

This implies that

$$\nabla u_\alpha = \tilde{R}\psi, \tag{16}$$

where $\tilde{R} = \nabla(1 - \alpha^2\triangle)^{-1}\nabla\times$ is a Riesz type operator.

Now applying the embedding estimate (14) to (16), we obtain

$$\|\nabla u_\alpha\|_{BMO} \leq C\|\psi\|_{BMO}. \tag{17}$$

Using estimates (13) and (17), we get

$$\begin{aligned} \|\nabla u_\alpha\|_\infty &\leq C\,(1 + \|\nabla u_\alpha\|_{BMO} \log(\|\nabla u_\alpha\|_{W^{1,4}} + e)) \\ &\leq C\,(1 + \|\psi\|_{BMO} \log(\|\omega\|_{L^2} + e))\,, \end{aligned} \tag{18}$$

where we have used $u = \nabla \times (-\triangle)^{-1}\omega$ and the Sobolev embedding estimate

$$\|\nabla u_\alpha\|_{W^{1,p}} \leq C(\|u_\alpha\|_{H^3}) \leq C(\|u_\alpha\|_{H^1} + \|\omega\|_{L^2}),$$

for $p \in [2, 6]$ and the fact that $\|u_\alpha\|_{H^1}$ is bounded from (5).

Next, we perform an energy estimate for the vorticity equation. Multiplying both sides of the vorticity equation (2) by ω and integrating over R^3, we get

$$\frac{1}{2}\frac{d}{dt}\int_{R^3}|\omega|^2 dx + \int_{R^3}(u_\alpha \cdot \nabla\omega)\cdot\omega dx = \int_{R^3}(\nabla u_\alpha \omega)\cdot\omega dx. \tag{19}$$

Note that using integration by parts, we have

$$\int_{R^3}(u_\alpha \cdot \nabla\omega)\cdot\omega dx = \frac{1}{2}\int_{R^3}(u_\alpha\cdot\nabla)|\omega|^2 dx = -\frac{1}{2}\int_{R^3}(\nabla\cdot u_\alpha)|\omega|^2 dx = 0, \tag{20}$$

since $\nabla \cdot u_\alpha = 0$.

On the other hand, we obtain by using estimate (18)

$$\begin{aligned} |\int_{R^3}(\nabla u_\alpha \omega)\cdot\omega dx| &\leq \|\nabla u_\alpha\|_\infty \int_{R^3}|\omega|^2 dx \\ &\leq C\left(1 + \|\psi\|_{BMO}\log(\|\omega\|_{L^2} + e)\right)\|\omega\|_{L^2}^2. \end{aligned} \tag{21}$$

Putting together estimates (19)-(21), we get

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 \leq C\left(1 + \|\psi\|_{BMO}\log(\|\omega\|_{L^2} + e)\right)\|\omega\|_{L^2}^2. \tag{22}$$

The Gronwall inequality then implies that

$$\|\omega(t)\|_{L^2} \leq C(T), \quad \text{for} \quad 0 \leq t \leq T, \tag{23}$$

since $\int_0^T \|\psi\|_{BMO} dt < \infty$ by our assumption (11). Moreover,

$$\|\nabla u_\alpha(t)\|_\infty \leq C||u_\alpha||_{H^3} \leq C(||u_\alpha||_{H^1} + ||\omega||_{L^2}) \leq C(T). \tag{24}$$

Using (2) and (24), we can easily show that

$$\|\omega(t)\|_\infty \leq \|\omega_0\|_\infty \exp\left(\int_0^T \|\nabla u_\alpha\|_\infty dt\right) \leq C(T), \quad \text{for} \quad 0 \leq t \leq T. \tag{25}$$

Now it is a standard exercise to obtain energy estimates in high order Sobolev norms [18]

$$\frac{d}{dt}\|\omega\|_{H^m} \leq C_m(\|\nabla u_\alpha\|_\infty + \|\omega\|_\infty)\|\omega\|_{H^m}. \tag{26}$$

Since $\|\nabla u_\alpha(t)\|_\infty$ and $\|\omega(t)\|_\infty$ are bounded for $0 \le t \le T$, we obtain the desired estimate for $\|\omega\|_{H^m}$ up to time T.

Now, if the maximal time T of the existence of classical solutions is finite, then we must have

$$\int_0^T \|\psi\|_{BMO} dt = \infty,$$

since if $\int_0^T \|\psi\|_{BMO} dt < \infty$, the above argument would imply that $\|\omega(t)\|_{H^m} \le C(T)\|\omega_0\|_{H^m}$ for $t \le T$, which is a contradiction. This completes the proof.

Next we prove the global existence of the averaged Boussinesq equations (6)-(7).

Theorem 2. *Assume that $\omega_0 \in H^m(R^2)$ and $\rho_0 \in H^{m+1}(R^2)$ for $m \ge 0$. Then for any $\alpha > 0$, the Lagrangian averaged 2D Boussinesq equations (6)-(7) has a unique global solution in $H^m(R^2)$ satisfying*

$$\|\omega(t)\|_{H^m} + \|\rho(t)\|_{H^{m+1}} \le C(T)(\|\omega_0\|_{H^m} + \|\rho_0\|_{H^{m+1}}), \quad 0 \le t \le T,$$

for any $T > 0$.

Proof. First of all, a standard energy estimate shows that $\|u\|_{L^2}$ is bounded since $\|\rho\|_{L^2}$ is conserved in time and bounded.

Let $W = \nabla^\perp \rho$. Then W satisfies the following evolution equation:

$$W_t + (u_\alpha \cdot \nabla)W = \nabla u_\alpha \cdot W. \tag{27}$$

For any odd integer $p > 2$, we multiply (6) by ω^{p-1} and (27) by $|W|^{p-2}W$ respectively and integrate over R^2. Upon using integration by parts for the convection terms and exploring the incompressibility of the velocity fields, u and u_α, we obtain

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\int_{R^2} (|\omega|^p + |W|^p)\, dx &\le (1 + \|\nabla u_\alpha\|_\infty)\int_{R^2} |W|^p dx + \int_{R^2} |\omega|^p dx \\ &\le (1 + \|\nabla u_\alpha\|_\infty)\left(\int_{R^2} |\omega|^p dx + \int_{R^2} |W|^p dx\right), \end{aligned} \tag{28}$$

where we have used the Yang's inequality to obtain

$$\begin{aligned} \int_{R^2} \omega^{p-1}|\rho_{x_1}| dx &\le \frac{p-1}{p}\int_{R^2} |\omega|^p dx + \frac{1}{p}\int_{R^2} |\rho_{x_1}|^p dx \\ &\le \int_{R^2} |\omega|^p dx + \int_{R^2} |W|^p dx. \end{aligned}$$

Using estimates (13), we get

$$\begin{aligned}\|\nabla u_\alpha\|_\infty &\le C\,(1+\|\nabla u_\alpha\|_{BMO}\log(\|\nabla u_\alpha\|_{W^{1,p}}+e)) \qquad (29)\\ &\le C\,(1+\|\nabla u_\alpha\|_{BMO}\log(\|\omega\|_{L^p}+e))\,,\end{aligned}$$

where we have used $u=\nabla^\perp(-\triangle)^{-1}\omega$ and the Sobolev embedding estimate

$$\|\nabla u_\alpha\|_{W^{1,p}} \le C(\|u_\alpha\|_{H^1}+\|\omega\|_{L^p}),$$

for $p\ge 2$ and the fact that $\|u_\alpha\|_{H^1}\le C\|u\|_{L^2}$ is bounded.

On the other hand, we obtain by using the John-Nirenberg type estimate (definition of BMO) in 2D

$$\|\nabla u_\alpha\|_{BMO}\le C\|\nabla u_\alpha\|_{H^1}\le C\|u\|_{L^2}\le C, \qquad (30)$$

where we have used the fact that $\|u\|_{L^2}$ is bounded. Therefore, we obtain by combining (28), (29) and (30) that

$$\frac{d}{dt}\left(\|\omega\|_{L^p}^p+\|W\|_{L^p}^p\right) \le \left(\|\omega\|_{L^p}^p+\|W\|_{L^p}^p\right)\left(1+\log(\|\omega\|_{L^p}^p+\|W\|_{L^p}^p+e)\right). \qquad (31)$$

The Gronwall inequality then implies that

$$\|\omega(t)\|_{L^p}+\|W(t)\|_{L^p}\le C(T), \quad \text{for} \quad 0\le t\le T. \qquad (32)$$

Using (29), (30) and (32), we get

$$\|\nabla u_\alpha(t)\|_\infty \le C(T), \quad \text{for} \quad 0\le t\le T. \qquad (33)$$

It follows from (27) and (33) that

$$\|W(t)\|_\infty\le C(T), \quad \text{for} \quad 0\le t\le T, \qquad (34)$$

which in turns implies

$$\|\omega(t)\|_\infty\le C(T), \quad \text{for} \quad 0\le t\le T. \qquad (35)$$

Now it is a standard exercise to show [18] that

$$\frac{d}{dt}\left(\|\omega\|_{H^m}+\|W\|_{H^m}\right)\le C(T)(\|\nabla u_\alpha\|_\infty+\|W\|_\infty)(\|\omega\|_{H^m}+\|W\|_{H^m}).$$

The theorem now follows from (34)-(35) and the Gronwall inequality. This completes the proof of the theorem.

Next we prove the global existence of the averaged 2D quasi-geostrophic equations (8)-(10).

Theorem 3. *Assume that* $\theta_0 \in H^{m+1}(R^2)$ *for* $m \geq 0$*. Then for any* $\alpha > 0$*, the solution of the Lagrangian averaged 2D quasi-geostrophic equations (8)-(10) has a unique global solution in* $H^{m+1}(R^2)$ *satisfying*

$$\|\theta(t)\|_{H^{m+1}} \leq C(T)\|\theta_0\|_{H^{m+1}}), \quad 0 \leq t \leq T,$$

for any $T > 0$.

Proof.

Again, we can perform a standard energy estimate to show that $\|\theta\|_{L^p}$ is bounded by $\|\theta_0\|_{L^p}$ (including $p = \infty$ which can be obtained via the so-called maximum principle).

Let $\omega = \nabla^{\perp}\theta$. Then ω satisfies the following evolution equation:

$$\omega_t + (u_\alpha \cdot \nabla)\omega = \nabla u_\alpha \cdot \omega. \tag{36}$$

Thus, ω shares the similar vortex stretching term as the 3D Euler equation. Now using an argument similar to our energy estimate for (27), we can obtain

$$\frac{1}{p}\frac{d}{dt}\int_{R^2} |\omega|^p dx \leq \|\nabla u_\alpha\|_\infty \int_{R^2} |\omega|^p dx. \tag{37}$$

Note that

$$\nabla u_\alpha = \nabla(1 - \alpha^2\triangle)^{-1/2}(-\triangle)^{-1/2}\nabla^{\perp}\theta \equiv R\theta,$$

for some Riesz type operator R. Using the following embedding estimates (see the Appendix)

$$\|\nabla u_\alpha\|_{BMO} \leq C\|\theta\|_{BMO},$$

and

$$\|\nabla u_\alpha\|_{W^{1,p}} \leq C\|\theta\|_{W^{1,p}} \leq C(\|\theta\|_{L^p} + \|\nabla\theta\|_{L^p}), \quad \text{for } 1 < p < \infty,$$

we obtain using (13) that

$$\begin{aligned} \|\nabla u_\alpha\|_\infty &\leq C\left(1 + \|\nabla u_\alpha\|_{BMO} \log(\|\nabla u_\alpha\|_{W^{1,p}} + e)\right) \\ &\leq C\left(1 + \|\theta\|_{BMO} \log(\|\theta\|_{L^p} + \|\omega\|_{L^p} + e)\right) \\ &\leq C\left(1 + \|\theta\|_\infty \log(\|\omega\|_{L^p} + e)\right) \\ &\leq C\left(1 + \log(\|\omega\|_{L^p} + e)\right). \end{aligned} \tag{38}$$

Substituting (38) into (37) gives

$$\frac{d}{dt}\|\omega\|_{L^p}^p \leq C\left(1 + \log(\|\omega\|_{L^p} + e)\right)\|\omega\|_{L^p}^p. \tag{39}$$

The Gronwall inequality then implies that

$$\|\omega(t)\|_{L^p} \leq C(T)\|\omega_0\|_{L^p}, \quad 0 \leq t \leq T, \tag{40}$$

which, together with (38), gives

$$\|\nabla u_\alpha\|_\infty \le C(T), \quad 0 \le t \le T. \tag{41}$$

Now it follows from (41) and (36) that

$$\|\omega\|_\infty \le C(T), \quad 0 \le t \le T. \tag{42}$$

Now it is a standard exercise to show [18] that

$$\frac{d}{dt}\|\omega\|_{H^m} \le C(T)(\|\omega\|_\infty + \|\nabla u_\alpha\|_\infty)\|\omega\|_{H^m}.$$

The theorem now follows from (41)-(42) and the Gronwall inequality. This completes the proof of the theorem.

3 The Level Set Formulation for the 3D Euler Equations

In this section, we will present a level set formulation for the 3D Euler equations and show how they can be used to obtain a sufficient condition to guarantee the global existence of the averaged Euler equations.

We consider the 3D Euler equations in the vorticity form:

$$\partial_t \omega + (u \cdot \nabla)\omega = \nabla u \cdot \omega, \quad \omega(0, x) = \omega_0(x), \tag{43}$$

where $\omega = \nabla \times u$ and u is divergence free.

Let $X(t, \alpha)$ be the Lagrangian flow map, satisfying

$$\frac{d}{dt}X(t, \alpha) = u(t, X(t, \alpha)), \quad X(0, \alpha) = \alpha. \tag{44}$$

Since u is divergence free, we know that the determinant of the Jacobian matrix $\frac{\partial X}{\partial \alpha}$ is identically equal to one. It is well-known that vorticity along the Lagrangian trajectory has the following analytical expression [6]

$$\omega(t, X(t, \alpha)) = \frac{\partial X}{\partial \alpha}\omega_0(\alpha). \tag{45}$$

Let $\theta(t, x)$ be the inverse map of $X(t, \alpha)$, i.e. $X(t, \theta(t, x)) \equiv x$. Then it is easy to show that θ satisfies the following evolution equation:

$$\theta_t + (u \cdot \nabla)\theta = 0, \quad \theta(0, x) = x. \tag{46}$$

Let $\theta = (\theta_1, \theta_2, \theta_3)$ and $\omega_0 = (\omega_0^{(1)}, \omega_0^{(2)}, \omega_0^{(3)})$. Using (45) and the fact that $X_\alpha \theta_x = I$ and $|\theta_x| = 1$, we can show that

$$\omega(t,x) = \omega_0^{(1)}(\theta)\nabla\theta_2 \times \nabla\theta_3 + \omega_0^{(2)}(\theta)\nabla\theta_3 \times \nabla\theta_1 + \omega_0^{(3)}(\theta)\nabla\theta_1 \times \nabla\theta_2. \quad (47)$$

Note that $\theta_j (j = 1, 2, 3)$ are level set functions convected by the flow velocity u. In general, one can show that if the initial vorticity $\omega(0, x) = \omega_0(\phi_0, \psi_0)\nabla\phi_0 \times \nabla\psi_0$, and the level set functions ϕ and ψ satisfy

$$\phi_t + (u \cdot \nabla)\phi = 0, \quad \phi(0, x) = \phi_0(x), \quad (48)$$

$$\psi_t + (u \cdot \nabla)\psi = 0, \quad \psi(0, x) = \psi_0(x), \quad (49)$$

then the vorticity at later time can be expressed in terms of these two level set functions and their gradients:

$$\omega(t, x) = \omega_0(\phi, \psi)\nabla\phi \times \nabla\psi. \quad (50)$$

This level set formulation has been considered by Deng, Hou, and Yu in their study of the 3D Euler equations [12]. The special case when $\omega_0 = 1$ is also known as the Clebsch representation [7]. In this case, the velocity field has the form

$$u = \nabla p + \phi\nabla\psi,$$

for some potential function p.

It is easy to see that the above level set formulation of vorticity for the 3D Euler equations also applies to the 3D Lagrangian averaged Euler equations. The only change is that the level set functions now satisfy

$$\phi_t + (u_\alpha \cdot \nabla)\phi = 0, \quad \phi(0, x) = \phi_0(x), \quad (51)$$

$$\psi_t + (u_\alpha \cdot \nabla)\psi = 0, \quad \psi(0, x) = \psi_0(x). \quad (52)$$

Now we state a sufficient condition for the global existence of the Lagrangian averaged Euler equations in terms of the property of the level set functions defined by (51)-(52). Before we state our result, we first introduce a definition of the total variation of a level set function, ϕ, as follows:

$$\|\phi\|_{TVx_1} = \sup_{x_2, x_3} \int_{-\infty}^{\infty} |\frac{\partial}{\partial x_1}\phi(x_1, x_2, x_3)| dx_1. \quad (53)$$

We can define $\|\phi\|_{TVx_2}$ and $\|\phi\|_{TVx_3}$ similarly and let $\|\phi\|_{TV} = \sum_{i=1}^{3} \|\phi\|_{TVx_i}$.

Theorem 4. *Assume that the initial vorticity has the form $\omega(0,x) = \omega_0(\phi_0,\psi_0)\nabla\phi_0 \times \nabla\psi_0$ with ω_0, ϕ_0 and ψ_0 being smooth and bounded. Moreover, we assume that ϕ and ψ satisfy (51)-(52) such that either $\int_0^T \|\phi\|_{TV} dt < \infty$ or $\int_0^T \|\psi\|_{TV} dt < \infty$ for any $T > 0$. Then the averaged 3D Euler equations have a unique smooth global solution satisfying*

$$\|\omega(t)\|_{H^m} \leq C(T)\|\omega(0)\|_{H^m}), \quad 0 \leq t \leq T,$$

for any $T > 0$.

Remark. As we mentioned before, the above result has a clear geometric interpretation. It implies that if the one-dimensional restriction of the levelset function ϕ or ψ has a total variation which is integrable in time, then there is no finite time blow-up. This excludes the possibility of a finite number of isolated singularities when vorticity is considered as a one-dimensional function by fixing the other two variables. In particular, if there is a finite time singularity, the one-dimensional restriction of vorticity must be highly oscillatory at the singularity time, and the singularities are dense in the singular region.

Proof.

Recall that

$$u_\alpha = (1 - \alpha^2\triangle)^{-1}\nabla \times (-\triangle)^{-1}\omega.$$

Thus we have

$$\nabla u_\alpha = (1 - \alpha^2\triangle)^{-1}R\omega,$$

where $R = \nabla\nabla \times (-\triangle)^{-1}$ is a Riesz operator.

First, we consider the special case when $\omega_0 \equiv 1$. In this case, we have $\omega = \nabla\phi \times \nabla\psi$ for all times. Without loss of generality, we may assume that $\int_0^T \|\psi\|_{TV} dt < \infty$. We can further rewrite $\omega = \nabla \times (\phi\nabla\psi)$.

Let $B(y)$ be the integral kernel of the operator $(1 - \alpha^2\triangle)^{-1}R$ in R^3. We set $x = 0$ and omit the reference to time. Then we can express

$$\begin{aligned} |\nabla u_\alpha(0)| &= |\int_{R^3} B(y)\omega(y)dy| \\ &= |\int_{R^3} \nabla B(y) \times (\phi(y)\nabla\psi(y))dy|. \end{aligned}$$

One can show that

$$|\nabla B(y)| \leq \frac{C_\alpha}{|y|^2(1 + |y|^{\frac{1}{4}})}. \tag{54}$$

Let B_ϵ denote the ball centered at the origin with radius $\epsilon < 1$. Note that the level set functions ϕ and ψ are bounded for all times. Let $p > 3$, q be the conjugate of p satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Further, we denote $r = \frac{3-2q}{q}$. If y_i is one of the three components of y in R^3, we denote by y' the remaining two dimensional vector excluding y_i. Then we have

$$\begin{aligned}
|\nabla u_\alpha(0)| &= |\int_{B_\epsilon} + \int_{|y|\geq\epsilon} \nabla B(y) \times (\phi(y)\nabla\psi(y))dy| \\
&\leq \|\phi\|_\infty \left(\epsilon^{\frac{3-2q}{q}} \|\nabla\psi\|_{L^p} + \sum_{i=1}^{3} \int_{|y'|^2+|y_i|^2\geq\epsilon^2} \frac{dy'}{(|y_i|^2+|y'|^2)(1+|y|^{\frac{1}{4}})} \int |\frac{\partial\psi}{\partial y_i}| dy_i \right) \\
&\leq \|\phi\|_\infty \left(\epsilon^r \|\nabla\psi\|_{L^p} + \|\psi\|_{TV} \int_{R^2} \frac{dy'}{(\epsilon^2+|y'|^2)(1+|y|^{\frac{1}{4}})} + \right. \\
&\quad \left. + \|\psi\|_{TV} \int_{|y'|\geq\epsilon} \frac{dy'}{|y'|^2(1+|y'|^{\frac{1}{4}})} \right) \\
&\leq \|\phi\|_\infty \left(\epsilon^r \|\nabla\psi\|_{L^p} + \|\psi\|_{TV} \log\frac{1}{\epsilon} \right),
\end{aligned}$$

where we have used the Hölder inequality in the estimate for the inner part. Note that $\|\phi\|_\infty \leq \|\phi_0\|_\infty$. By setting $\epsilon^r(e + \|\nabla\psi\|_{L^p}) = 1$, we obtain

$$|\nabla u_\alpha(0)| \leq C\left(1 + \|\psi\|_{TV} \log(\|\nabla\psi\|_{L^p} + e)\right). \tag{55}$$

Differentiating (52) with respect to x, we obtain

$$(\nabla\psi)_t + (u_\alpha \cdot \nabla)(\nabla\psi) + \nabla u_\alpha \nabla\psi = 0. \tag{56}$$

Performing the energy estimate to (56), we get

$$\begin{aligned}
\frac{\partial}{\partial t}\|\nabla\psi\|_{L^p} &\leq \|\nabla u_\alpha\|_\infty \|\nabla\psi\|_{L^p} \\
&\leq C\left(1 + \|\psi\|_{TV}\log(\|\nabla\psi\|_{L^p} + e)\right)\|\nabla\psi\|_{L^p}.
\end{aligned}$$

The Gronwall inequality then implies

$$\|\nabla\psi\|_{L^p} \leq C(T), \tag{57}$$

provided that $\int_0^T \|\psi\|_{TV} dt < \infty$. Substituting (57) back to (55), we conclude that

$$\int_0^T \|\nabla u_\alpha\|_\infty dt \leq C \int_0^T \|\psi\|_{TV} dt \leq C(T). \tag{58}$$

The bound on $\int_0^T \|\nabla u_\alpha\|_\infty dt$ immediately gives the maximum bound on $\nabla\psi$ from (56). Similarly we obtain the maximum bound for $\nabla\phi$. Combining the maximum estimates for $\nabla\psi$ and $\nabla\phi$, we obtain the maximum bound for vorticity ω. Then it is a standard argument to prove the energy estimate for ω in H^m norm using

$$\frac{\partial}{\partial t}\|\omega\|_{H^m} \le C(\|\nabla u_\alpha\|_\infty + \|\omega\|_\infty)\|\omega\|_{H^m}.$$

It remains to comment on the more general case when $\omega_0 \neq 1$. Note that

$$\begin{aligned} \omega_0(\phi,\psi)\nabla\phi\times\nabla\psi &= \nabla(\omega_0\phi)\times\nabla\psi - \phi(\omega_0)_\phi\nabla\phi\times\nabla\psi \\ &= \nabla\times(\omega_0\phi\nabla\psi) - \phi(\omega_0)_\phi\nabla\phi\times\nabla\psi. \end{aligned}$$

Define

$$h(\phi,\psi) = \int_0^\phi s(\omega_0)_\phi(s,\psi)ds.$$

Then we have

$$\begin{aligned} \nabla h(\phi,\psi) &= h_\phi\nabla\phi + h_\psi\nabla\psi \\ &= \phi(\omega_0)_\phi\nabla\phi + h_\psi\nabla\psi. \end{aligned}$$

This implies that

$$\begin{aligned} (\phi(\omega_0)_\phi\nabla\phi)\times\nabla\psi &= \nabla h\times\nabla\psi - h_\psi\nabla\psi\times\nabla\psi \\ &= \nabla\times(h\nabla\psi). \end{aligned}$$

Note that h is bounded since both ϕ and $(\omega_0)_\phi$ are bounded. Therefore, we can rewrite

$$\omega_0(\phi,\psi)\nabla\phi\times\nabla\psi = \nabla\times(\omega_0\phi\nabla\psi) - \nabla\times(h\nabla\psi),$$

with both $\omega_0\phi$ and h being bounded. Thus the previous argument for $\omega_0 = 1$ applies the case when $\omega_0 \neq 1$. This completes the proof of the theorem.

4 Appendix

In this appendix, we present some basic estimates about Riesz type operators that have been used many times in this paper. The L^p $(1 < p < \infty)$ estimates are based on the well-known Calderon-Zygmund [2] decomposition and the Marcinkiewicz [19] interpolations. The $L^{p,\Phi}$ estimates are due to Peetre [24]. We note that for $\Phi(r) = r^\lambda$, $L^{p,\Phi}$ is the Morrey space if $0 < \lambda < n$, the John-Nirenberg space (BMO) [16] if $\lambda = n$, and C^α if $n < \lambda < n + p$. These beautiful and elegant results are very enlightening and we collect only the main results that are closely related to the operators we have used in this paper.

The Riesz type operators are the translation invariant singular integral operators of the following form:

$$(Tf)(x) = \int_{R^n} K(x-y)f(y)dy = \int_{R^n} K(z)f(z+x)dz, \tag{59}$$

where the integral is taken in the sense of principle integration when needed. As usual, the integral is first defined for $f \in C_0^\infty(R^n)$ (functions that are smooth with compact support) and then extended to more general functions (L^p, $L^{p,\Phi}$, etc.) based on the so-called *a priori* estimates.

The operators we have used are the copoments of the vector valued operators: $\nabla(1-\alpha^2\triangle)^{-1}\nabla\times$, $\nabla(1-\alpha^2\triangle)^{-1/2}(-\triangle)^{-1/2}\nabla^\perp$ and $\nabla\nabla\times(-\triangle)^{-1}$. They share the following common properties:

(i): $\|Tf\|_{L^2} \le C\|f\|_{L^2}$,

(ii): $|K(x)| \le \frac{B}{|x|^n}$,

(iii): $\int_{|y|>2|x|} |K(y-x) - K(y)|dy \le A$.

Remark: The bound C in (i) can be obtained by computing the maximum value of the Fourier transform of the operators for our Riesz type operators listed above. In all three cases, C is equal to 1 or $\frac{1}{\alpha^2}$. Also, our operators can be regarded as the composition of the standard Riesz transform $R_i = \frac{\partial}{\partial x_i}(-\triangle)^{-1/2}$ with another operator whose kernel is given by $K(x) = \partial_{x_i}\partial_{x_j}G_\alpha(x)$ $(i,j = 1,2,3)$ where $G_\alpha(x) = C_\alpha\frac{\exp(-|x|/\alpha)}{|x|}$ for $x \in R^3$. Note that G_α is the Green's (potential) function for the oparetor $(1-\alpha^2\triangle)$ in R^3.

The above conditions and the Calderon-Zygmund decomposition [2] show that T is weak $(1,1)$ type operator (see also page 31 in [25]). We can then employ the Marcinkiewicz interpolations [19] to show that $\|Tf\|_{L^p} \le C_p\|f\|_{L^p}$ for $1 < p \le 2$. The translation invariant nature of T implies that the dual operator T^* also satisfies (i)-(iii). Thus the duality argument implies the following result.

Theorem 5. *If the operator T defined by (59) satisfies (i)-(iii), then*

$$\|Tf\|_{L^p} \le C_K\|f\|_{L^p} \text{ for } 1 < p < \infty, \tag{60}$$

where C depends on the constants A, B, C, p, and n only.

Next, we present the BMO estimate of Peetre for our operators. It is a special case of the $L^{P,\Phi}$ estimates [24].

Let $\Phi = \Phi(r)$, $r > 0$, be a positive and nondecreasing function that satisfies: $\Phi(2r) \le C\Phi(r)$. The space $L^{P,\Phi}$ consists of locally integrable functions f on R^n such that:

$$\|f\|_{L^{P,\Phi}} = \sup_{x_0\in R^n, r>0} \inf_\tau \|\frac{f(y)-\tau}{\Phi(r)}\|_{L^p(B_r(x_0))} < \infty. \tag{61}$$

If we identify functions that differ by a constant then the above norm gives us a Banach space.

To get the $L^{P,\Phi}$ estimate, we need more assumptions on T and Φ:

(iv): $|\nabla K(x)| \leq \frac{B}{|x|^{n+1}}$, $\lim_{|x|\to\infty} K(x) = 0$,

(v): $\int_{|x|=r} K(x) d\sigma = 0$ for any $r > 0$,

(vi): $\int_r^\infty s^{-2-\frac{n}{p}} (\Phi(s))^{\frac{1}{p}} ds \leq C r^{-1-\frac{n}{p}} (\Phi(r))^{\frac{1}{p}}$,

(vii) $\Phi(2r) \leq C\Phi(r)$.

Remark: Assumption (iv) implies (ii) and (iii). Assumptions (iv) and (v) imply (i) (see theorem 4 of page 306 of [25]).

Simple calculations show that our operators satisfies all the conditions on T. The function $\Phi(r) = r^\lambda$ with $0 \leq \lambda < n + p$ satisfies (vi).

Theorem 6. *(Peetre) Assume Φ and T satisfies (iv)-(vii), $1 < p < \infty$, then*

$$\|Tf\|_{L^{P,\Phi}} \leq C_p \|f\|_{L^{P,\Phi}} \text{ for } 1 < p < \infty. \tag{62}$$

As mentioned in the first paragraph, the special case that $\Phi(r) = r^n$ leads to the BMO estimate:

$$\|Tf\|_{BMO} \leq C\|f\|_{BMO}.$$

Acknowledgments. Hou's research was supported in part by the National Science Foundation under an FRG Grant DMS-0353838 and an ITR Grant ACI-0204932. Li's research was supported in part by the National Science Foundation under Grant DMS-0401174.

References

[1] Beale, J.T., Kato, T. and Majda, A. *Remarks on the breakdown of smooth solutions for the 3 D Euler equations.* Comm. Math. Phys., 94, 1984, pp. 61-64.

[2] Calderon, A. P. and Zygmund A. *On the existence of certain singular integrals.* Acta Math. 88, 1952, pp. 85-139.

[3] Camassa, R. and Holm, D. D., *An integrable shallow water equation with peaked solitons.* Phys. Rev. Lett. 71, 1993, pp. 1661-1664.

[4] Chen, S. Y., Holm, D. D., Margolin, L. G., and Zhang, R., *Direct numerical simulations of the Navier-Stokes alpha model.* Physica D., 133, 1999, pp. 66-83.

[5] Chen, S. Y., Foias, C., Holm, D. D., Olson, E. J., Titi, E. S., and Wynne, S., *The Camassa-Holm equations as a closure model for turbulent channel and pipe flows.* Phys. Fluids, 11, 1999, pp. 2343-2353.

[6] Chorin AJ, Marsden JE., *A Mathematical Introduction to Fluid Mechanics.* 3rd ed., Springer-Verlag, New York, 1993.

[7] Clebsch, A., *Uber die integration der hydrodynamicschen gleichungen,* J. Reine Angew. Math., 56, 1859, pp. 1-.

[8] Constantin, P., Cordoba, D., and Wu, J., *On the critical dissipative quasi-geostrophic equation,* Indiana University Mathematics Journal, 50, 2001, pp. 97-107.

[9] Constantin, P., Majda, A., and Tabak, E., *Formation of strong fronts in the 2-D quasi-geostrophic thermal active scalar.* Nonlinearity, 7, 1994, pp. 1495-1533.

[10] Constantin P, Fefferman C, Majda A. Geometric constraints on potentially singular solutions for the 3-D Euler equation. Commun PDE, 21, 1996, pp. 559-571.

[11] Deng, J., Hou, T. Y., and Yu, X., Geometric Properties and the non-Blow-up of the Three-Dimensional Euler Equation. Commun PDE, 30, 2005, pp. 225-243.

[12] Deng, J., Hou, T. Y., and Yu, X., private communications.

[13] Hald, O., *Convergence of vortex blob methods for Euler's equations. III.* SIAM J. Numer. Anal., 24, 1987, pp. 538-582.

[14] Holm, D. D., Marsden, J. E., and Ratiu, T. S., *Euler-Poincaré models of ideal fluids with nonlinear dispersion.* Phys. Rev. Lett., 349, 1998, pp. 4173-4177.

[15] Holm, D. D., Marsden, J. E., and Ratiu, T. S., *Euler-Poincaré equations and semidirect products with applications to continuum theories.* Adv. Math., 137, 1998, pp. 1-81.

[16] John, F. and Nirenberg L. *On functions of bounded mean oscillation.* Comm. Pure Appl. Math. 14, 1961, pp. 415-426.

[17] Kozono, H. and Taniuchi, Y. *Limiting case of the Sobolev inequality in BMO, with application to the Euler equations.* Comm. Math. Phys., 214, 2000, pp. 191-200.

[18] Majda, A. and Bertozzi, A. *Vorticity and Incompressible Flow,* Cambridge University Press, Cambridge, UK, 2002.

[19] Marcinkiewicz, J. *Syr L'interpolation d'operations.* Acad. Sci. Paris, 208, 1939, pp. 1272-1273.

[20] Marsden, J. and Shkoller, S. *Global well-posedness for the lagrangian averaged Navier-Stokes (LANS-α) equations on bounded domains,* Phil. Trans R. Soc Lond. A, 359, 2001, pp. 1449-1468.

[21] Marsden, J. E., Ratiu, T., and Shkoller, S., *The geometry and analysis of the averaged Euler equations and a new differmophism group.* Geom. Funct. Analysis, 10, 2000, pp. 582-599.

[22] Oliver, M. and Shkoller, S., The vortex blob method as a second-grade non-Newtonian fluid. Commun. PDEs, 26, 2001, pp. 91-110.

[23] Pedlosky, J., *Geophysical Fluid Dynamics,* Springer-Verlag, New York, 1987.

[24] Peetre, J., *On Convolution Operators Leaving $L^{p,\lambda}$ Spaces Invariant,* Ann. Mat. Pura Appl., 72, 1966, pp. 295-304.

[25] Stein, E. M. *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals* Princeton Mathematical Series, 1993