

MULTISCALE ANALYSIS IN LAGRANGIAN FORMULATION FOR THE 2-D INCOMPRESSIBLE EULER EQUATION

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Abstract. We perform a systematic multiscale analysis for the 2-D incompressible Euler equation with rapidly oscillating initial data using a Lagrangian approach. The Lagrangian formulation enables us to capture the propagation of the multiscale solution in a natural way. By making an appropriate multiscale expansion in the vorticity-stream function formulation, we derive a well-posed homogenized equation for the Euler equation. Based on the multiscale analysis in the Lagrangian formulation, we also derive the corresponding multiscale analysis in the Eulerian formulation. Moreover, our multiscale analysis reveals some interesting structure for the Reynolds stress term, which provides a theoretical base for establishing systematic multiscale modeling of 2-D incompressible flow.

1. Introduction. We develop a systematic multiscale analysis for the 2-D incompressible Euler equation with highly oscillating initial velocity field. The understanding of scale interactions for the incompressible Euler and Navier-Stokes equations has been a major challenge. For high Reynolds number flows, the degrees of freedom are so large that it is almost impossible to resolve all small scales by direct numerical simulations. Deriving an effective equation for the large scale solution is very useful for engineering applications. On the other hand, the nonlinear and nonlocal nature of the Euler equations makes it difficult to perform multiscale analysis. One of the main challenges is to understand how small scales propagate in time and whether the multiscale structure of the solution is preserved dynamically.

The homogenization of the incompressible Euler equation with highly oscillating initial data was first studied by McLaughlin-Papanicolaou-Pironneau (MPP for short) in 1985 [6]. To construct a multiscale expansion for the solution of the Euler

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equation, they made an important assumption that the oscillation is convected by the mean velocity field. Furthermore, by introducing the mean Lagrangian map θ and assuming that the multiscale solution is periodic in both the fast time variable $\tau = t/\epsilon$ and the fast spatial variable, $\mathbf{y} = \theta/\epsilon$, they obtained a cell problem for velocity and pressure. However, the well-posedness of the cell problem is not known. Additional assumptions were imposed on the solution of the cell problem in order to derive a variant of the $k - \epsilon$ model.

In this paper, we first study the homogenization of the Euler equation using a full Lagrangian formulation. The multiscale structure of the solution becomes apparent in the Lagrangian formulation. It is well known that the vorticity is conserved along the Lagrangian trajectory for the 2-D incompressible Euler equation. Thus vorticity preserves naturally the multiscale structure of its initial data via the Lagrangian formulation. Velocity can be constructed using the vorticity-stream function formulation. By using a Lagrangian description, we characterize the nonlinear convection of small scales exactly and turn a convection dominated transport problem into an elliptic problem for the stream function. Thus, traditional multiscale analysis for elliptic problems [1] can be used to obtain a multiscale expansion for the stream function. At the end, we derive a coupled multiscale system for the Lagrangian flow map and the stream function, which can be solved uniquely.

Next, we derive the homogenized equations using a semi-Lagrangian formulation. In this study, we use the Eulerian formulation to describe the large scale solution, and use the Lagrangian formulation to describe the propagation of the small scale solution. This semi-Lagrangian formulation reveals some interesting multiscale solution structure. This solution structure shows that there is an implicit relationship between the fast variable using the oscillatory Lagrangian map and the fast variable using the mean Lagrangian map. This relationship enables us to derive a multiscale analysis using only the mean Lagrangian map as the fast variable. This leads to a set of homogenized equations which are much easier to solve numerically.

Based on our multiscale analysis in the semi-Lagrangian formulation, we derive the corresponding homogenized equations for the Euler equation in the Eulerian formulation for velocity and pressure. In this formulation, we use the mean Lagrangian map to describe the propagation of the small scale velocity field, but use the Eulerian variable to describe the large scale velocity field. The homogenized equation for the large scale velocity field has a similar structure as the Large Eddy Simulation model with a Reynolds stress term [7]. The small scale solution is governed by a cell problem whose solution has a semi-analytical expression using the stream function-vorticity formulation. By exploring this semi-analytical solution structure of the cell problem, we reveal some intrinsic structure of the Reynolds stress term. Our study also shows that the effect of the Reynolds stress term in the 2-D Euler equation is dispersive in terms of the mean velocity field, which is very different from the diffusive effect of the Reynolds stress for the 3-D Euler equation [9]. The structure in the Reynolds stress term revealed by our multiscale analysis provides a useful guideline for us to develop a systematic Large Eddy Simulation model for incompressible flows.

We also discuss how to generalize the two-scale analysis to problems with infinitely many scales that are not separable. This is achieved by reorganizing the high frequency modes in the Fourier space into a form which has the two-scale structure. Then the two-scale analysis developed earlier can be applied to more practical fluid dynamic applications that have infinitely many scales. We have also generalized

the multiscale analysis for the Navier-Stokes equations and performed a careful numerical study to confirm the convergence of our multiscale analysis. We choose a random initial condition whose energy spectrum decays like $O(|k|^{-2})$. We compare the numerical simulation obtained by solving the homogenized equations with the well-resolved numerical simulation. We compute the solution for a relatively long time until the flow is sufficiently mixed and some large coherent structure emerges. At this stage, the energy spectrum reaches to an equilibrium distribution of order $O(|k|^{-3})$. Our numerical study demonstrates that the numerical simulation obtained by solving the homogenized equations captures very well the large scale feature as well as the small scale details of the well-resolved solution. Moreover, we obtain very good agreements with the well-resolved solution in terms of the Fourier spectrum, the mean kinetic energy, and the enstrophy.

The organization of the rest of the paper is as follows. In Section 2, we will present the formulation of the 2-D Euler equation with rapidly oscillating initial data. Section 3 is devoted to developing the two-scale analysis of the 2-D Euler equations in the Lagrangian and semi-Lagrangian formulation. In section 4, we derive the homogenized equations for velocity and pressure in the Eulerian formulation. Then, based upon our multiscale analysis in the Lagrangian formulation, we obtain some intrinsic structure of the Reynolds stress term for the 2-D Euler equation. In Section 5, we will briefly discuss how to generalize our method based on the two-scale analysis to flows with infinitely many scales. In Section 6, we will perform some numerical experiments to confirm our analytical results. Some technical proofs are deferred to the Appendices.

2. Formulation. We consider the 2-D incompressible Euler equation with highly oscillating initial velocity field:

$$\begin{aligned}
 (a) \quad & \partial_t \mathbf{u}^\epsilon + (\mathbf{u}^\epsilon \cdot \nabla) \mathbf{u}^\epsilon + \nabla p^\epsilon = \mathbf{0}, \\
 (b) \quad & \nabla \cdot \mathbf{u}^\epsilon = 0, \\
 (c) \quad & \mathbf{u}^\epsilon|_{t=0} = \mathbf{U}(\mathbf{x}) + \mathbf{W}(\mathbf{x}, \frac{\mathbf{x}}{\epsilon}),
 \end{aligned}
 \tag{2.1}$$

where $\mathbf{u}^\epsilon(t, \mathbf{x})$ and $p^\epsilon(t, \mathbf{x})$ are the velocity and the pressure field respectively. We assume that \mathbf{U} is a given smooth mean velocity field in R^2 , $\mathbf{W}(\mathbf{x}, \mathbf{y})$ with $\mathbf{y} = \mathbf{x}/\epsilon$ is the oscillatory component of the velocity field which is a smooth function of \mathbf{x} and \mathbf{y} , and is periodic in \mathbf{y} with period 1 in each dimension. Moreover, we assume that \mathbf{W} has zero mean

$$\langle \mathbf{W} \rangle \equiv \int_{[0,1]^2} \mathbf{W}(\mathbf{x}, \mathbf{y}) d\mathbf{y} = \mathbf{0}.
 \tag{2.2}$$

We do not consider the boundary effect in our study. The velocity field is assumed to be at rest at infinity. The question of interest is how to derive a homogenized equation for the averaged velocity field.

We remark that the assumption of scale separation and periodic structure is not realistic for fluid flows. On the other hand, the homogenization theory gives valuable insight into how small scale solution interacts with the large solution, and how small scale solution is propagated in time. In Section 5, we will generalize the two-scale analysis to the case when there are infinitely many scales that are not separable. This will be more applicable to fluid flows. Another important issue is viscous effect, which plays an important role in turbulent mixing. For the two-scale initial velocity considered here, if we include an order one viscosity, the small scale

velocity field will be damped out quickly for $t > 0$. Even if the viscosity is of the order $O(\epsilon)$, the small scale velocity component will be damped out after $O(\epsilon)$ time. Only for viscosity of order $O(\epsilon^2)$, will the small scale velocity component survive and remain of order $O(1)$. In this case, the viscosity would not enter the cell problem. Thus the formulation is essentially the same as the Euler equation. For this reason, we would just consider the inviscid Euler equation for now.

2.1. Some preliminary notations and results. Before we perform our multi-scale analysis for the 2-D incompressible Euler equation, we first introduce some notations. We denote by ∇^\perp the curl operator:

$$\nabla^\perp \psi = \begin{pmatrix} -\partial_2 \psi \\ \partial_1 \psi \end{pmatrix}, \quad \nabla^\perp \cdot \mathbf{v} = \partial_1 v_2 - \partial_2 v_1.$$

Then, we state some general results for the coordinate transform in Property 2.1.

Let $\boldsymbol{\alpha} = \boldsymbol{\theta}(\mathbf{x})$ be a coordinate transform from R^2 onto R^2 , $\mathbf{x} = \mathbf{X}(\boldsymbol{\alpha})$ be its inverse map. For each function $\phi = \phi(\mathbf{x})$, let $\widehat{\phi}(\boldsymbol{\alpha}) = \phi(\mathbf{X}(\boldsymbol{\alpha})) = \phi \circ \mathbf{X}$. Let $|\mathcal{B}|$ stand for the determinant of a matrix \mathcal{B} and $\mathbf{x}(\boldsymbol{\alpha}) = (x_1(\boldsymbol{\alpha}), x_2(\boldsymbol{\alpha}))$. We have

$$\begin{aligned} (a) \quad [D_\alpha \mathbf{X}]^{-1} &= \frac{1}{|D_\alpha \mathbf{X}|} (-\nabla_\alpha^\perp x_2, \nabla_\alpha^\perp x_1) = \frac{1}{|D_\alpha \mathbf{X}|} \begin{pmatrix} \partial_{\alpha_2} x_2 & -\partial_{\alpha_2} x_1 \\ -\partial_{\alpha_1} x_2 & \partial_{\alpha_1} x_1 \end{pmatrix}, \\ (b) \quad (\nabla_x^\perp \phi) \circ \mathbf{X} &= \frac{1}{|D_\alpha \mathbf{X}|} D_\alpha \mathbf{X} \nabla_\alpha^\perp \widehat{\phi}, \\ (c) \quad (\nabla_x^\perp \cdot \boldsymbol{\psi}) \circ \mathbf{X} &= \frac{1}{|D_\alpha \mathbf{X}|} (D_\alpha \mathbf{X} \nabla_\alpha^\perp) \cdot \widehat{\boldsymbol{\psi}} = \frac{1}{|D_\alpha \mathbf{X}|} \nabla_\alpha^\perp \cdot (D_\alpha \mathbf{X}^\top \widehat{\boldsymbol{\psi}}), \\ (d) \quad (\Delta_x \phi) \circ \mathbf{X} &= \frac{1}{|D_\alpha \mathbf{X}|} \nabla_\alpha^\perp \cdot \left(\frac{1}{|D_\alpha \mathbf{X}|} D_\alpha \mathbf{X}^\top D_\alpha \mathbf{X} \nabla_\alpha^\perp \widehat{\phi} \right). \end{aligned} \tag{2.3}$$

Derivation of Proposition 2.1 will be deferred to Appendix I. For two vectors \mathbf{a} and \mathbf{b} and matrix function $\mathcal{A} = (\mathbf{a}_1, \mathbf{a}_2)^\top$, we define

$$\mathbf{a} \otimes \mathbf{b} = \mathbf{a} \mathbf{b}^\top = \begin{pmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{pmatrix}, \quad \nabla \cdot \mathcal{A} = \begin{pmatrix} \nabla \cdot \mathbf{a}_1 \\ \nabla \cdot \mathbf{a}_2 \end{pmatrix}.$$

To distinguish the total and the partial derivative operators, we denote the total derivative operates by $\bar{\partial}_\eta, \bar{\nabla}_\beta, \bar{\nabla}_\beta^\perp$ and \bar{D}_β , etc, and the partial derivative operates by $\partial_\eta, \nabla_\beta, \nabla_\beta^\perp$ and D_β with respect to the time variable η and the space variable β , respectively. In case without confusion, we will omit the index or the superscript.

2.2. The vorticity-stream function formulation. Recall that vorticity is defined as $\omega^\epsilon = \nabla^\perp \cdot \mathbf{u}^\epsilon$ and the stream function ψ^ϵ satisfies the following elliptic equation:

$$-\Delta \psi^\epsilon = \omega^\epsilon. \tag{2.4}$$

The velocity field can be constructed from the stream function:

$$\mathbf{u}^\epsilon = -\nabla^\perp \psi^\epsilon. \tag{2.5}$$

The initial vorticity field is given by:

$$\omega_{\text{int}} = \frac{1}{\epsilon} \sigma_0(\mathbf{x}, \mathbf{y}) + \sigma_1(\mathbf{x}, \mathbf{y}) + \varrho(\mathbf{x}), \tag{2.6}$$

where

$$\begin{aligned} (a) \quad & \sigma_0(\mathbf{x}, \mathbf{y}) = \nabla_y^\perp \cdot \mathbf{W}(\mathbf{x}, \mathbf{y}), \quad \sigma_1(\mathbf{x}, \mathbf{y}) = \nabla_x^\perp \cdot \mathbf{W}(\mathbf{x}, \mathbf{y}), \\ (b) \quad & \varrho(\mathbf{x}) = \nabla_x^\perp \cdot \mathbf{U}(\mathbf{x}). \end{aligned} \tag{2.7}$$

The initial value problem for the vorticity field can now be written as follows:

$$\begin{aligned} (a) \quad & \partial_t \omega^\epsilon + \mathbf{u}^\epsilon \cdot \nabla \omega^\epsilon = 0, \\ (b) \quad & \omega^\epsilon|_{t=0} = \omega_{\text{int}}(\mathbf{x}, \frac{\mathbf{x}}{\epsilon}). \end{aligned} \tag{2.8}$$

Our analysis will be based upon the Lagrangian formulation for the 2-D incompressible Euler equation.

2.3. The Lagrangian formulation. We define the Lagrangian map, θ^ϵ , as follows:

$$\begin{aligned} (a) \quad & \frac{\partial \theta^\epsilon}{\partial t} + (\mathbf{u}^\epsilon \cdot \nabla) \theta^\epsilon = \mathbf{0}, \\ (b) \quad & \theta^\epsilon = \mathbf{x}. \end{aligned} \tag{2.9}$$

Let us denote by $\mathbf{x} = \mathbf{x}^\epsilon(t, \alpha)$ the inverse map of $\alpha = \theta^\epsilon(t, \mathbf{x})$, i.e. $\mathbf{x} \equiv \mathbf{x}^\epsilon(t, \theta^\epsilon(t, \mathbf{x}))$. It is easy to show that $\mathbf{x}^\epsilon(t, \alpha)$ is the flow map:

$$\begin{aligned} (a) \quad & \frac{d}{dt} \mathbf{x}^\epsilon(t, \alpha) = \mathbf{u}^\epsilon(t, \mathbf{x}^\epsilon(t, \alpha)), \\ (b) \quad & \mathbf{x}^\epsilon(0, \alpha) = \alpha. \end{aligned} \tag{2.10}$$

Using the Lagrangian map, we can characterize the evolution of the small scale vorticity field. For 2-D incompressible inviscid flows, the vorticity is conserved in time along the flow map. Thus we have

$$\omega^\epsilon(t, \mathbf{x}) = \omega_{\text{int}}(\theta^\epsilon(t, \mathbf{x}), \frac{\theta^\epsilon(t, \mathbf{x})}{\epsilon}), \tag{2.11}$$

and the equation for the stream function becomes:

$$-\Delta \psi^\epsilon = \omega_{\text{int}}(\theta^\epsilon(t, \mathbf{x}), \frac{\theta^\epsilon(t, \mathbf{x})}{\epsilon}). \tag{2.12}$$

Since the flow is incompressible, the flow map $\mathbf{x}^\epsilon(t, \alpha)$ is area-preserving, i.e. $|D_\alpha \mathbf{x}^\epsilon| = 1$. Moreover, we know from (2.3) that

$$[D_\alpha \mathbf{x}^\epsilon]^{-1} = (-\nabla_\alpha^\perp x_2^\epsilon, \nabla_\alpha^\perp x_1^\epsilon), \tag{2.13}$$

and

$$\Delta \psi^\epsilon = \nabla_\alpha^\perp \cdot ((D_\alpha \mathbf{x}^\epsilon)^\top D_\alpha \mathbf{x}^\epsilon \nabla_\alpha^\perp \psi^\epsilon). \tag{2.14}$$

The velocity field \mathbf{u}^ϵ can be expressed as

$$\mathbf{u}^\epsilon = -D_\alpha \mathbf{x}^\epsilon \nabla_\alpha^\perp \psi^\epsilon. \tag{2.15}$$

To summarize, we obtain the following coupled system for the stream function ψ^ϵ and the flow map \mathbf{x}^ϵ :

$$\begin{aligned} (a) \quad & -\nabla_\alpha^\perp \cdot (D_\alpha \mathbf{x}^{\epsilon \top} D_\alpha \mathbf{x}^\epsilon \nabla_\alpha^\perp \psi^\epsilon) = \omega_{\text{int}}(\alpha, \frac{\alpha}{\epsilon}), \\ (b) \quad & \partial_t \mathbf{x}^\epsilon + ((\nabla_\alpha^\perp \psi^\epsilon) \cdot \nabla) \mathbf{x}^\epsilon = \mathbf{0}, \quad t > 0, \quad \mathbf{x}^\epsilon(0, \alpha) = \alpha. \end{aligned} \tag{2.16}$$

Remark 2.1 : It is clear from (2.11) that the vorticity is essentially propagated along its flow map. Using the Lagrangian formulation, we can see clearly that the

microscopic periodic structure is preserved under the Lagrangian coordinate. This important feature provides us with a critical guideline in performing our multiscale analysis. By using the vorticity-stream function formulation in the Lagrangian formulation, we can treat the nonlinear convection exactly. We now turn the convection dominant transport problem, which is hyperbolic in nature, into an elliptic problem for the stream function and a quasi-linear convection equation for the flow map. The velocity field can be recovered from these two variables. The system (2.16) is a nonlinear coupling system of the elliptic equation and the transport equation. From this system, we can see that the multiscale periodic structure is convected by the full velocity field. The solution of this system is a periodic function of α/ϵ . This formulation plays an important role in our multiscale analysis.

3. Multiscale analysis in the Lagrangian formulation. In this section, we perform a systematic multiscale analysis for the 2-D incompressible Euler equation in the Lagrangian formulation. From the discussions in Section 2, we know that the small scale component of the vorticity field is propagated along the Lagrangian map, θ^ϵ . Thus it is natural to use the Lagrangian variable and the vorticity-stream function formulation to perform the multiscale analysis for the 2-D incompressible Euler equation.

3.1. Multiscale analysis along the exact flow map. In this subsection, we perform a multiscale analysis for the coupled stream function and flow map system (2.16). Throughout our analysis, we assume that the oscillatory component of the velocity field is of order one and is bounded independent of ϵ . The objective of our study is to obtain an averaged equation for the well-mixing long time solution of the Euler equation. For sufficiently well mixing flow, the order one oscillations in the velocity field would cancel out each other along the Lagrangian particle trajectories. Thus it is reasonable to expect that the flow map is one order smoother than the velocity field. The stream function is also one order smoother than the velocity field. Under this consideration, we look for multiscale expansions for the stream function and the flow map of the form:

$$\begin{aligned}
 (a) \quad \psi^\epsilon &= \psi^{(0)}(t, \alpha, \tau) + \epsilon \left(\bar{\psi}^{(1)}(t, \alpha, \tau) + \psi^{(1)}(t, \alpha, \tau, \mathbf{y}) \right) + \dots, \\
 (b) \quad \mathbf{x}^\epsilon &= \mathbf{x}^{(0)}(t, \alpha, \tau) + \epsilon \left(\bar{\mathbf{x}}^{(1)}(t, \alpha, \tau) + \mathbf{x}^{(1)}(t, \alpha, \tau, \mathbf{y}) \right) + \dots,
 \end{aligned}
 \tag{3.1}$$

where $\mathbf{y} = \alpha/\epsilon$ and $\tau = t/\epsilon$. We assume that $\psi^{(k)}$ and $\mathbf{x}^{(k)}$ are periodic functions with respect to \mathbf{y} and with zero-means for $k \geq 1$.

First of all, we perform the expansion for the Jacobian matrix, $\mathcal{A}^\epsilon = (\bar{D}_\alpha \mathbf{x}^\epsilon)^\top \bar{D}_\alpha \mathbf{x}^\epsilon$, into a power series of ϵ :

$$\begin{aligned}
 (a) \quad \mathcal{A}^\epsilon &= \mathcal{A}^{(0)} + \epsilon \mathcal{A}^{(1)} + \dots, \\
 (b) \quad \mathcal{A}^{(0)} &= (D_\alpha \mathbf{x}^{(0)} + D_y \mathbf{x}^{(1)})^\top (D_\alpha \mathbf{x}^{(0)} + D_y \mathbf{x}^{(1)}), \\
 (c) \quad \mathcal{A}^{(1)} &= (D_\alpha \mathbf{x}^{(0)} + D_y \mathbf{x}^{(1)})^\top (D_\alpha (\bar{\mathbf{x}}^{(1)} + \mathbf{x}^{(1)}) + D_y \mathbf{x}^{(2)}) \\
 &\quad + (D_\alpha (\bar{\mathbf{x}}^{(1)} + \mathbf{x}^{(1)}) + D_y \mathbf{x}^{(2)})^\top (D_\alpha \mathbf{x}^{(0)} + D_y \mathbf{x}^{(1)}).
 \end{aligned}
 \tag{3.2}$$

Next, we will derive multiscale expansions for \mathbf{x}^ϵ and ψ^ϵ . By direct calculations, we can show that \mathbf{x}^ϵ and ψ^ϵ satisfy the following expansions:

$$\begin{aligned}
 & -\nabla_y^\perp \cdot \left(\mathcal{A}^{(0)} \nabla_y^\perp \psi^{(1)} \right) - \nabla_y^\perp \cdot \left(\mathcal{A}^{(0)} \nabla_\alpha^\perp \psi^{(0)} \right) \\
 & \quad - \epsilon \left[\nabla_y^\perp \cdot \left(\mathcal{A}^{(0)} \nabla_y^\perp \psi^{(2)} \right) + \nabla_y^\perp \cdot \left(\mathcal{A}^{(0)} \nabla_\alpha^\perp \left(\bar{\psi}^{(1)} + \psi^{(1)} \right) \right) \right] \\
 & \quad + \nabla_y^\perp \cdot \left(\mathcal{A}^{(1)} \nabla_y^\perp \psi^{(1)} \right) + \nabla_y^\perp \cdot \left(\mathcal{A}^{(1)} \nabla_\alpha^\perp \psi^{(0)} \right) \\
 & \quad + \nabla_\alpha^\perp \cdot \left(\mathcal{A}^{(0)} \nabla_y^\perp \psi^{(1)} \right) + \nabla_\alpha^\perp \cdot \left(\mathcal{A}^{(0)} \nabla_\alpha^\perp \psi^{(0)} \right) \Big] + \dots \\
 & = \sigma_0(\boldsymbol{\alpha}, \mathbf{y}) + \epsilon(\sigma_1(\boldsymbol{\alpha}, \mathbf{y}) + \varrho(\boldsymbol{\alpha})),
 \end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
 & \frac{1}{\epsilon} \partial_\tau \mathbf{x}^{(0)} + \partial_t \mathbf{x}^{(0)} + \partial_\tau \left(\bar{\mathbf{x}}^{(1)} + \mathbf{x}^{(1)} \right) \\
 & \quad + \epsilon \left[\partial_t \left(\bar{\mathbf{x}}^{(1)} + \mathbf{x}^{(1)} \right) + \partial_\tau \left(\bar{\mathbf{x}}^{(2)} + \mathbf{x}^{(2)} \right) \right] + \dots \\
 & = - \left(D_\alpha \mathbf{x}^{(0)} + D_y \mathbf{x}^{(1)} \right) \left(\nabla_\alpha^\perp \psi^{(0)} + \nabla_y^\perp \psi^{(1)} \right) \\
 & \quad - \epsilon \left(D_\alpha \mathbf{x}^{(0)} + D_y \mathbf{x}^{(1)} \right) \left[\nabla_\alpha^\perp \left(\bar{\psi}^{(1)} + \psi^{(1)} \right) + \nabla_y^\perp \psi^{(2)} \right] \\
 & \quad - \epsilon \left[D_\alpha \left(\bar{\mathbf{x}}^{(1)} + \mathbf{x}^{(1)} \right) + D_y \mathbf{x}^{(2)} \right] \left(\nabla_\alpha^\perp \psi^{(0)} + \nabla_y^\perp \psi^{(1)} \right) + \dots
 \end{aligned} \tag{3.4}$$

at each $(t, \boldsymbol{\alpha}, \tau, \mathbf{y}) = (t, \boldsymbol{\alpha}, t/\epsilon, \boldsymbol{\alpha}/\epsilon)$.

Based upon the multiscale expansions for \mathbf{x}^ϵ and ψ^ϵ , we can derive the homogenized equations for $\mathbf{x}^{(0)}$ and $\psi^{(0)}$ and the cell problem for $\mathbf{x}^{(1)}$ and $\psi^{(1)}$. The result is summarized in the following theorem.

Theorem 3.1. *Assume that the solution of the system (2.16) has the expansion of the form (3.1). Then we have*

$$\begin{aligned}
 (a) \quad & \mathbf{x}^{(0)} = \mathbf{x}^{(0)}(t, \boldsymbol{\alpha}), \quad \psi^{(0)} = \psi^{(0)}(t, \boldsymbol{\alpha}), \\
 (b) \quad & \bar{\mathbf{x}}^{(1)} = \bar{\mathbf{x}}^{(1)}(t, \boldsymbol{\alpha}), \quad \psi^{(1)} = \psi_0^{(1)}(t, \boldsymbol{\alpha}, \tau, \mathbf{y}) + (\nabla_\alpha \psi^{(0)})^\top (D_\alpha \mathbf{x}^{(0)})^{-1} \mathbf{x}^{(1)},
 \end{aligned} \tag{3.5}$$

and $\mathbf{x}^{(0)}$ and $\psi^{(0)}$ satisfy the following homogenized equations:

$$\begin{aligned}
 (a) \quad & \partial_t \mathbf{x}^{(0)} + (\nabla_\alpha^\perp \psi^{(0)} \cdot \nabla_\alpha) \mathbf{x}^{(0)} = \mathbf{0}, \\
 (b) \quad & \mathbf{x}^{(0)}|_{t=0} = \boldsymbol{\alpha},
 \end{aligned} \tag{3.6}$$

and

$$-\nabla_\alpha^\perp \cdot \left((D_\alpha \mathbf{x}^{(0)})^\top D_\alpha \mathbf{x}^{(0)} \nabla_\alpha^\perp \psi^{(0)} \right) - \nabla_\alpha^\perp \cdot \left\langle \mathcal{A}^{(0)} \nabla_y^\perp \psi_0^{(1)} \right\rangle = \varrho(\boldsymbol{\alpha}), \tag{3.7}$$

where ϱ is defined by (2.7)(b) and $\langle \mathcal{A}^{(0)} \nabla_y^\perp \psi_0^{(1)} \rangle$ is independent of τ . Moreover, the first order correctors, $\mathbf{x}^{(1)}$ and $\psi_0^{(1)}$, satisfy the following cell problem:

$$\begin{aligned}
 (a) \quad & \partial_\tau \mathbf{x}^{(1)} + ((\nabla_y^\perp \psi_0^{(1)}) \cdot \nabla_y) \mathbf{x}^{(1)} + ((\nabla_y^\perp \psi_0^{(1)}) \cdot \nabla_\alpha) \mathbf{x}^{(0)} = \mathbf{0}, \\
 (b) \quad & \mathbf{x}^{(1)}|_{\tau=t=0} = \mathbf{0},
 \end{aligned} \tag{3.8}$$

and

$$-\nabla_y^\perp \cdot \left(\mathcal{A}^{(0)} \nabla_y^\perp \psi_0^{(1)} \right) = \sigma_0(\boldsymbol{\alpha}, \mathbf{y}), \tag{3.9}$$

where σ_0 is defined by (2.7)(a) and $\mathcal{A}^{(0)}$ is given by (3.2)(b).

Once we obtain the multiscale expansion for \mathbf{x}^ϵ and ψ^ϵ , we can immediately derive the leading order term $\mathbf{u}^{(0)}$ of the velocity field \mathbf{u}^ϵ as follows:

$$\mathbf{u}^{(0)} = -D_\alpha \mathbf{x}^{(0)} \nabla_\alpha^\perp \psi^{(0)} - \left(D_\alpha \mathbf{x}^{(0)} + D_y \mathbf{x}^{(1)} \right) \nabla_y^\perp \psi_0^{(1)}. \tag{3.10}$$

Proof of theorem 3.1 will be deferred to Appendix II. Note that the terms $\bar{\mathbf{x}}^{(1)}$ and $\bar{\psi}^{(1)}$ do not affect the leading order term $\mathbf{u}^{(0)}$ of the velocity field. Thus we do not need to derive the corresponding evolution equations for these two quantities.

3.2. Semi-Lagrangian multiscale expansion for the velocity field. The expansion (3.1) is along the exact Lagrangian map. For engineering applications, it is more convenient to study the macroscopic behavior of the fluid flow in the Eulerian coordinate. In this subsection, we consider further expansion using the Eulerian coordinate for the large scale solution, but still use the Lagrangian coordinate to describe the propagation of the small scale solution. Let $\boldsymbol{\alpha} = \boldsymbol{\theta}(t, \mathbf{x})$ be the inverse map of $\mathbf{x} = \mathbf{x}^{(0)}(t, \boldsymbol{\alpha})$. i.e. $\mathbf{x} = \mathbf{x}^{(0)}(t, \boldsymbol{\theta}(t, \mathbf{x}))$. Applying $\boldsymbol{\theta}$ to the Lagrangian map

$$\mathbf{x} = \mathbf{x}^{(0)}(t, \boldsymbol{\theta}^\epsilon) + \epsilon \left(\bar{\mathbf{x}}^{(1)}(t, \boldsymbol{\theta}^\epsilon) + \mathbf{x}^{(1)}(t, \boldsymbol{\theta}^\epsilon, \frac{t}{\epsilon}, \frac{\boldsymbol{\theta}^\epsilon}{\epsilon}) \right) + \dots,$$

and using the identity, $\boldsymbol{\theta}^\epsilon = \boldsymbol{\theta}(t, \mathbf{x}^{(0)}(t, \boldsymbol{\theta}^\epsilon))$, we obtain

$$\boldsymbol{\theta}^\epsilon = \boldsymbol{\theta}(t, \mathbf{x}) - \epsilon D_x \boldsymbol{\theta}(t, \mathbf{x}) \left(\bar{\mathbf{x}}^{(1)}(t, \boldsymbol{\theta}^\epsilon) + \mathbf{x}^{(1)}(t, \boldsymbol{\theta}^\epsilon, \frac{t}{\epsilon}, \frac{\boldsymbol{\theta}^\epsilon}{\epsilon}) \right) + \dots. \tag{3.11}$$

Thus the multiscale expansion for the Lagrangian map is defined implicitly and has the following form

$$\boldsymbol{\theta}^\epsilon = \boldsymbol{\theta} + \epsilon \left(\bar{\boldsymbol{\theta}}^{(1)}(t, \mathbf{x}) + \boldsymbol{\theta}^{(1)}(t, \boldsymbol{\theta}, \frac{t}{\epsilon}, \frac{\boldsymbol{\theta}^\epsilon}{\epsilon}) \right) + \dots. \tag{3.12}$$

By comparing (3.11) with (3.12), we get

$$\boldsymbol{\theta}^{(1)} = -D_x \boldsymbol{\theta}(t, \mathbf{x}) \mathbf{x}^{(1)}(t, \boldsymbol{\theta}, \frac{t}{\epsilon}, \frac{\boldsymbol{\theta}^\epsilon}{\epsilon}) = -(D_\theta \mathbf{x}^{(0)}(t, \boldsymbol{\theta}))^{-1} \mathbf{x}^{(1)}(t, \boldsymbol{\theta}, \frac{t}{\epsilon}, \frac{\boldsymbol{\theta}^\epsilon}{\epsilon}). \tag{3.13}$$

Next, we look for the solution of the form

$$\begin{aligned} (a) \quad \psi^\epsilon &= \psi^{(0)}(t, \mathbf{x}) + \epsilon \left(\bar{\psi}^{(1)}(t, \mathbf{x}, \tau) + \psi^{(1)}(t, \boldsymbol{\theta}, \tau, \mathbf{y}) \right) + \dots, \\ (b) \quad \boldsymbol{\theta}^\epsilon &= \boldsymbol{\theta}(t, \mathbf{x}) + \epsilon \left(\bar{\boldsymbol{\theta}}^{(1)}(t, \mathbf{x}) + \boldsymbol{\theta}^{(1)}(t, \boldsymbol{\theta}, \tau, \mathbf{y}) \right) + \dots, \end{aligned} \tag{3.14}$$

where $\mathbf{y} = \boldsymbol{\theta}^\epsilon/\epsilon$ and $\tau = t/\epsilon$, and $\psi^{(k)}$ and $\boldsymbol{\theta}^{(k)}$ are periodic functions in \mathbf{y} with zero-means for $k \geq 1$.

3.2.1. A basic expansion formula. First, we would like to obtain a multiscale expansion for the Jacobian matrix $\bar{D}_x \boldsymbol{\theta}^\epsilon$. From (3.14), we get

$$\bar{D}_x \boldsymbol{\theta}^\epsilon = D_x \boldsymbol{\theta} + D_y \boldsymbol{\theta}^{(1)} \bar{D}_x \boldsymbol{\theta}^\epsilon + \mathcal{O}(\epsilon),$$

which implies

$$\left(\mathcal{I} - D_y \boldsymbol{\theta}^{(1)} \right) \bar{D}_x \boldsymbol{\theta}^\epsilon = D_x \boldsymbol{\theta} + \mathcal{O}(\epsilon).$$

Thus we obtain

$$\bar{D}_x \boldsymbol{\theta}^\epsilon = \mathcal{B}^{(0)} + \mathcal{O}(\epsilon), \quad \mathcal{B}^{(0)} = (\mathcal{I} - D_y \boldsymbol{\theta}^{(1)})^{-1} D_x \boldsymbol{\theta}. \tag{3.15}$$

For the inverse Jacobian, if we expand

$$(\bar{D}_x \boldsymbol{\theta}^\epsilon)^{-1} = \mathcal{S}^{(0)} + \epsilon \mathcal{S}^{(1)} + \mathcal{O}(\epsilon^2), \tag{3.16}$$

then we get

$$\mathcal{S}^{(0)} = (\mathcal{B}^{(0)})^{-1} = D_x \boldsymbol{\theta}^{-1} (\mathcal{I} - D_y \boldsymbol{\theta}^{(1)}). \tag{3.17}$$

Based on these expansions, we can derive the homogenized equations for the first two terms of (3.14).

Theorem 3.2. *Assume that $\boldsymbol{\theta}^\epsilon$ and ψ^ϵ have the expansion of the form (3.14). Then the homogenized equations for $\boldsymbol{\theta}$ and $\psi^{(0)}$ are given by*

$$\begin{aligned} (a) \quad & \partial_t \boldsymbol{\theta} - D_x \boldsymbol{\theta} \nabla_x^\perp \psi^{(0)} = \mathbf{0}, \\ (b) \quad & \boldsymbol{\theta}|_{t=0} = \mathbf{x}, \end{aligned} \tag{3.18}$$

and

$$-\Delta_x \psi^{(0)} = \varrho(\boldsymbol{\theta}) - \nabla_x^\perp \cdot (D_x \boldsymbol{\theta}^\top \langle \mathcal{D}^\top \mathbf{W}(\boldsymbol{\theta}, \cdot) \rangle), \tag{3.19}$$

where ϱ is defined by (2.7)(b) and \mathcal{D} is given by

$$\mathcal{D} = \begin{pmatrix} \partial_{y_2} \theta_2^{(1)} & -\partial_{y_2} \theta_1^{(1)} \\ -\partial_{y_1} \theta_2^{(1)} & \partial_{y_1} \theta_1^{(1)} \end{pmatrix}. \tag{3.20}$$

The cell problem for $\boldsymbol{\theta}^{(1)}$ and $\psi^{(1)}$ is given by

$$\begin{aligned} (a) \quad & \partial_\tau \boldsymbol{\theta}^{(1)} - (\mathcal{I} - D_y \boldsymbol{\theta}^{(1)}) \nabla_y^\perp \psi^{(1)} = \mathbf{0}, \\ (b) \quad & \boldsymbol{\theta}^{(1)}|_{\tau=t=0} = \mathbf{0} \end{aligned} \tag{3.21}$$

and

$$-\nabla_y^\perp \cdot \left((\mathcal{S}^{(0)})^\top \mathcal{S}^{(0)} \nabla_y^\perp \psi^{(1)} \right) = \sigma_0(\boldsymbol{\theta}, \mathbf{y}), \tag{3.22}$$

where σ_0 is defined by (2.7)(a) and $\mathcal{S}^{(0)}$ is defined by (3.17).

Moreover, the leading order velocity field, $\mathbf{u}^{(0)}$, is given by

$$\mathbf{u}^{(0)} = \mathbf{u} + \mathbf{w}, \tag{3.23}$$

where \mathbf{u} is the averaged component of $\mathbf{u}^{(0)}$, \mathbf{w} is the oscillating component of $\mathbf{u}^{(0)}$ with zero mean, satisfying

$$\mathbf{u} = -\nabla_x^\perp \psi^{(0)}, \quad \mathbf{w} = -D_x \boldsymbol{\theta}^{-1} (\mathcal{I} - D_y \boldsymbol{\theta}^{(1)}) \nabla_y^\perp \psi^{(1)}. \tag{3.24}$$

Proof of theorem 3.2 will be deferred to Appendix III.

3.2.2. A simplified formula for $\boldsymbol{\theta}^{(1)}$ and $\psi^{(1)}$. To simplify the computation of the cell problem, we introduce a change of variables from \mathbf{y} to \mathbf{z} as follows:

$$\mathbf{z} = \mathbf{y} - \boldsymbol{\theta}^{(1)}(t, \boldsymbol{\theta}, \tau, \mathbf{y}) \equiv \mathcal{G}(\mathbf{y}), \quad \mathbf{y} = \mathcal{G}^{-1}(\mathbf{z}), \tag{3.25}$$

where t , $\boldsymbol{\theta}$, and τ are considered as parameters. In Appendix IV, we will show that this change of variable is one to one, and is invertible.

$$\mathbf{z} + \mathbf{1} = \mathbf{y} + \mathbf{1} - \boldsymbol{\theta}^{(1)}(t, \boldsymbol{\theta}, \tau, \mathbf{y} + \mathbf{1}),$$

where $\mathbf{1} = (1, 1)^\top$, i.e., $\mathcal{G}^{-1}(\mathbf{z} + \mathbf{1}) = \mathbf{y} + \mathbf{1}$. Thus for each 1-periodic function $g = g(\mathbf{y})$, we have

$$\widehat{g}(\mathbf{z} + \mathbf{1}) = g(\mathbf{y} + \mathbf{1}) = g(\mathbf{y}) = \widehat{g}(\mathbf{z}).$$

This implies that \widehat{g} is also a 1-periodic function in \mathbf{z} . Using this new coordinate, we have the following simplified formula for $\boldsymbol{\theta}^{(1)}$ and $\psi^{(1)}$:

$$-\nabla_z^\perp \cdot \left(D_x \boldsymbol{\theta}^{-\top} D_x \boldsymbol{\theta}^{-1} \nabla_z^\perp \widehat{\psi}^{(1)} \right) = \sigma_0(\boldsymbol{\theta}, \mathbf{z} + \widehat{\boldsymbol{\theta}}^{(1)}), \tag{3.26}$$

where σ_0 is defined by (2.7)(a), and

$$\partial_\tau \widehat{\boldsymbol{\theta}}^{(1)} - (\mathcal{I} + D_z \widehat{\boldsymbol{\theta}}^{(1)}) \nabla_z^\perp \widehat{\psi}^{(1)} = \mathbf{0}. \tag{3.27}$$

The derivation of (3.26)-(3.27) will be given in Appendix IV. The existence of the nonlinear and nonlocal mapping \mathcal{G} from \mathbf{y} to \mathbf{z} is significant. To have a better understanding of this mapping, let us first consider the geometric meaning of the transformation of (3.25). From the expansion (3.14)(b), it is clear that

$$\mathbf{z} = \frac{\boldsymbol{\theta}}{\epsilon} + \bar{\boldsymbol{\theta}}^{(1)} + O(\epsilon), \tag{3.28}$$

since $\mathbf{y} = \boldsymbol{\theta}^\epsilon/\epsilon$. This means that the multiscale structure in the Lagrangian variable $\mathbf{y} = \boldsymbol{\theta}^\epsilon/\epsilon$ can be also expressed in terms of the new semi-Lagrangian variable $\mathbf{z} = (\boldsymbol{\theta} + \epsilon \bar{\boldsymbol{\theta}}^{(1)})/\epsilon$. The main difference between the expansion (3.26)-(3.27) in terms of the \mathbf{z} variable and the expansion in theorem 3.2 in terms of the \mathbf{y} variable is that the left-hand side of elliptic equation (3.26) is a constant coefficient elliptic operator with respect to the \mathbf{z} variable. This will significantly simplify the numerical calculation of $\widehat{\psi}^{(1)}$.

3.3. Expansion along mean flow. In this subsection, we will further explore the advantage of the simplified cell problem using the new coordinate variable \mathbf{z} to obtain a multiscale expansion in the velocity field. This will further simplify the homogenized equations for the 2-D Euler equation and make them more suitable for numerical computations. Note that using the mapping \mathcal{G} and (3.28), we can express $\widehat{\boldsymbol{\theta}}^{(1)}$ as follows:

$$\widehat{\boldsymbol{\theta}}^{(1)} = \boldsymbol{\theta}^{(1)}(t, \boldsymbol{\theta}, \tau, \frac{\boldsymbol{\theta}}{\epsilon} + \bar{\boldsymbol{\theta}}^{(1)}) + \mathbf{O}(\epsilon). \tag{3.29}$$

Based upon this observation, we look for multiscale expansion for $\boldsymbol{\theta}^\epsilon$ and ψ^ϵ of the form:

$$\begin{aligned} (a) \quad \boldsymbol{\theta}^\epsilon &= \bar{\boldsymbol{\theta}}(t, \mathbf{x}, \tau) + \epsilon \widetilde{\boldsymbol{\theta}}(t, \bar{\boldsymbol{\theta}}, \tau, \mathbf{z}), \\ (b) \quad \psi^\epsilon &= \bar{\psi}(t, \mathbf{x}, \tau) + \epsilon \widetilde{\psi}(t, \bar{\boldsymbol{\theta}}, \tau, \mathbf{z}), \end{aligned} \tag{3.30}$$

where $\bar{\boldsymbol{\theta}}$ and $\bar{\psi}$ are averages of $\boldsymbol{\theta}^\epsilon$ and ψ^ϵ with respect to \mathbf{z} over one period, and $\widetilde{\boldsymbol{\theta}}$ and $\widetilde{\psi}$ are 1-periodic functions with zero mean in \mathbf{z} . Note that the above expansion has no higher order corrections. The reason for this expansion is because we would like to account for all the high order corrections in $\widetilde{\boldsymbol{\theta}}$ and $\widetilde{\psi}$. This is important for problems with many scales which we will discuss in Section 5. In the case when many scales are present, we cannot assume that the small scale ϵ is very small. Thus it is important to account for the effect of the high order terms. By including the high order corrections in our multiscale expansion, we can also account for some dispersive or diffusive effects that are present in the form of high order corrections. This is important for our understanding of the asymptotic structure of the Reynolds stress term which we will discuss in the next section.

To distinguish this new expansion from our previous multiscale expansions, we will refer to $\bar{\theta}$ and $\bar{\psi}$ as the total averages of θ^ϵ and ψ^ϵ , and $\tilde{\theta}$ and $\tilde{\psi}$ total fluctuations. Throughout this section, we take

$$\mathbf{z} = \frac{\bar{\theta}}{\epsilon}, \quad \tau = \frac{t}{\epsilon}, \tag{3.31}$$

as fast variables. Unlike expansion (3.14) in Section 3.2, expansion (3.30) is along the mean Lagrangian map $\bar{\theta}$. Let $\bar{\partial}_t = \partial_t + \epsilon^{-1}\partial_\tau$ denote the total derivative in time. We now state the averaged equations for $(\bar{\theta}, \bar{\psi})$ and the cell problem for $(\tilde{\theta}, \tilde{\psi})$ in Proposition 3.1.

Proposition 3.1. *Assume the solutions of the system (2.9) and (2.16) have the form (3.30). Then the velocity field is defined by*

$$\begin{aligned} (a) \quad & \mathbf{u}^\epsilon = \bar{\mathbf{u}} + \tilde{\mathbf{u}}, \\ (b) \quad & \bar{\mathbf{u}} = -\nabla_x^\perp \bar{\psi}, \quad \tilde{\mathbf{u}} = -\mathcal{H}\nabla_z^\perp \tilde{\psi} - \epsilon\nabla_x^\perp \tilde{\psi}, \end{aligned} \tag{3.32}$$

where \mathcal{H} is given by

$$\mathcal{H} = \begin{pmatrix} \partial_{x_2}\bar{\theta}_2, & -\partial_{x_2}\bar{\theta}_1 \\ -\partial_{x_1}\bar{\theta}_2, & \partial_{x_1}\bar{\theta}_1 \end{pmatrix}. \tag{3.33}$$

Moreover, the homogenized solution, $(\bar{\theta}, \bar{\psi})$, satisfies the averaged equations:

$$\begin{aligned} (a) \quad & \bar{\partial}_t \bar{\theta} - D_x \bar{\theta} \nabla_x^\perp \bar{\psi} - \epsilon \nabla_x \cdot \langle \tilde{\theta} \otimes (\mathcal{H}\nabla_z^\perp \tilde{\psi} + \epsilon \nabla_x^\perp \tilde{\psi}) \rangle = \mathbf{0}, \\ (b) \quad & \bar{\theta}|_{t=\tau=0} = \mathbf{x}, \end{aligned} \tag{3.34}$$

and

$$-\Delta_x \bar{\psi} = \langle \omega_{\text{int}}(\bar{\theta} + \epsilon \tilde{\theta}, \mathbf{z} + \tilde{\theta}) \rangle. \tag{3.35}$$

The oscillating component, $(\tilde{\theta}, \tilde{\psi})$, satisfies the following cell problem:

$$\begin{aligned} (a) \quad & \partial_\tau \tilde{\theta} - (D_x \bar{\theta}^\top \nabla_z) \cdot (\tilde{\theta} \otimes (\mathcal{H}\nabla_z^\perp \tilde{\psi})) - D_x \bar{\theta} \mathcal{H} \nabla_z^\perp \tilde{\psi} \\ & + \epsilon \partial_t \tilde{\theta} - \epsilon \nabla_x \cdot (\tilde{\theta} \otimes (\nabla_x^\perp \tilde{\psi} + \mathcal{H}\nabla_z^\perp \tilde{\psi} + \epsilon \nabla_x^\perp \tilde{\psi})) \\ & - \epsilon (D_x \bar{\theta}^\top \nabla_z) \cdot (\tilde{\theta} \otimes \nabla_x^\perp \tilde{\psi}) - \epsilon D_x \bar{\theta} \nabla_x^\perp \tilde{\psi} \\ & + \epsilon (\mathcal{I} + D_z \tilde{\theta}) \nabla_x \cdot \langle \tilde{\theta} \otimes (\mathcal{H}\nabla_z^\perp \tilde{\psi} + \epsilon \nabla_x^\perp \tilde{\psi}) \rangle = \mathbf{0}, \\ (b) \quad & \tilde{\theta}|_{\tau=t=0} = \mathbf{0}, \end{aligned} \tag{3.36}$$

and

$$\begin{aligned} & -\nabla_z^\perp \cdot (\mathcal{H}^\top \mathcal{H} \nabla_z^\perp \tilde{\psi}) - \epsilon \nabla_z^\perp \cdot (\mathcal{H}^\top \nabla_x^\perp \tilde{\psi}) - \epsilon \nabla_x^\perp \cdot (\mathcal{H} \nabla_z^\perp \tilde{\psi}) - \epsilon^2 \Delta_x \tilde{\psi} \\ & = \epsilon \omega_{\text{int}}(\bar{\theta} + \epsilon \tilde{\theta}, \mathbf{z} + \tilde{\theta}) - \epsilon \langle \omega_{\text{int}}(\bar{\theta} + \epsilon \tilde{\theta}, \mathbf{z} + \tilde{\theta}) \rangle. \end{aligned} \tag{3.37}$$

The proof of Proposition 3.1 will be deferred to Appendix V.

The averaged equations and the cell problem given in Proposition 3.1 can be further simplified if we expand the variables in powers of ϵ . Let

$$\begin{aligned} (a) \quad & \bar{\theta} = \theta(t, \mathbf{x}) + \epsilon \bar{\theta}^{(1)}(t, \mathbf{x}) + \epsilon^2 \bar{\theta}^{(2)}(t, \mathbf{x}, \tau) + \dots, \\ (b) \quad & \bar{\psi} = \psi^{(0)}(t, \mathbf{x}) + \epsilon \bar{\psi}^{(1)}(t, \mathbf{x}, \tau) + \dots, \end{aligned} \tag{3.38}$$

and

$$\begin{aligned} (a) \quad & \tilde{\boldsymbol{\theta}} = \boldsymbol{\theta}^{(1)}(t, \boldsymbol{\theta}, \tau, \mathbf{z}) + \epsilon \boldsymbol{\theta}^{(2)}(t, \boldsymbol{\theta}, \tau, \mathbf{z}) + \dots, \\ (b) \quad & \tilde{\psi} = \psi^{(1)}(t, \boldsymbol{\theta}, \tau, \mathbf{z}) + \epsilon \psi^{(2)}(t, \boldsymbol{\theta}, \tau, \mathbf{z}) + \dots \end{aligned} \tag{3.39}$$

and $\mathcal{H} = \mathcal{H}^{(0)} + \epsilon \mathcal{H}^{(1)} + \dots$, where

$$\mathcal{H}^{(0)} = (-\nabla_x^\perp \theta_2, \nabla_x^\perp \theta_1), \quad \mathcal{H}^{(k)} = (-\nabla_x^\perp \bar{\theta}_2^{(k)}, \nabla_x^\perp \bar{\theta}_1^{(k)}). \tag{3.40}$$

We have the following theorem.

Theorem 3.3. *Assume the solutions of systems (2.9) and (2.16) have the expansion of the form (3.38) and (3.39). The components of the initial vorticity, σ_0 and ϱ , are given by (2.7). Then the leading order averaged components, $\boldsymbol{\theta}$ and $\psi^{(0)}$, satisfy the homogenized equations:*

$$\begin{aligned} (a) \quad & \partial_t \boldsymbol{\theta} - D_x \boldsymbol{\theta} \nabla_x^\perp \psi^{(0)} = \mathbf{0}, \\ (b) \quad & \boldsymbol{\theta}|_{t=0} = \mathbf{x} \end{aligned} \tag{3.41}$$

and

$$-\Delta_x \psi^{(0)} = \varrho(\boldsymbol{\theta}) + (D_x \boldsymbol{\theta} \nabla_x^\perp) \cdot \left\langle (\mathcal{I} + D_z \boldsymbol{\theta}^{(1)})^\top \mathbf{W}(\boldsymbol{\theta}, \mathbf{z} + \boldsymbol{\theta}^{(1)}) \right\rangle. \tag{3.42}$$

The leading order oscillating components, $\boldsymbol{\theta}^{(1)}$ and $\psi^{(1)}$, satisfy the following cell problem:

$$\begin{aligned} (a) \quad & \partial_\tau \boldsymbol{\theta}^{(1)} - (\mathcal{I} + D_z \boldsymbol{\theta}^{(1)}) \nabla_z^\perp \psi^{(1)} = \mathbf{0}, \\ (b) \quad & \boldsymbol{\theta}^{(1)}|_{\tau=t=0} = \mathbf{0} \end{aligned} \tag{3.43}$$

and

$$-\nabla_z^\perp \cdot \left((\mathcal{H}^{(0)})^\top \mathcal{H}^{(0)} \nabla_z^\perp \psi^{(1)} \right) = \sigma_0(\boldsymbol{\theta}, \mathbf{z} + \boldsymbol{\theta}^{(1)}), \tag{3.44}$$

where $\mathcal{H}^{(0)}$ is given by (3.40). The leading order velocity field, $\mathbf{u}^{(0)}$, is given by

$$\begin{aligned} (a) \quad & \mathbf{u}^{(0)} = \mathbf{u} + \mathbf{w}, \\ (b) \quad & \mathbf{u} = -\nabla_x^\perp \psi^{(0)}, \quad \mathbf{w} = -\mathcal{H}^{(0)} \nabla_z^\perp \psi^{(1)}. \end{aligned} \tag{3.45}$$

The proof of Theorem 3.3 will be deferred to Appendix V. We would like to comment that among the three versions of the homogenized equations we have derived so far, the homogenized equations given by Theorem 3.3 are the most suitable for numerical computations. We can also see that without our understanding of the multiscale solution structure in the full Lagrangian coordinate, and the semi-Lagrangian multiscale analysis using the mapping \mathcal{G} , we would not have been able to derive the homogenized equations in Theorem 3.3. In Section 6, we will present some careful numerical study of the 2-D incompressible flow with multiscale solutions using the homogenized equations provided by Theorem 3.3. The agreements with the well-resolved numerical simulation are excellent.

4. The Eulerian formulation and structure of the Reynolds stress. In Section 3, we have derived the homogenized equation for the 2-D incompressible Euler equation using the vorticity-stream function formulation in the Lagrangian coordinate. In practical computations, it is easier to use the Eulerian formulation. It is also more convenient to interpret the physical meaning of the effect of small scales on the large scales using the Eulerian formulation. In this section, we will

perform a multiscale analysis in the Eulerian formulation building upon the insight we gain from the multiscale analysis in the Lagrangian framework. We also analyze the asymptotic structure of the Reynolds stress. This asymptotic structure provides a useful guideline for obtaining a systematic large eddy simulation model for incompressible flows.

4.1. Homogenized equations in the Eulerian formulation. In this subsection, we will derive the homogenized equations for the Euler equation in the Eulerian formulation. We take advantage of our detailed multiscale analysis performed in Section 3. Since the expansion along the mean Lagrangian map is the easiest for computational implementation, we look for multiscale expansions of the velocity field and the pressure in the following form:

$$\begin{aligned}
 (a) \quad \mathbf{u}^\epsilon &= \bar{\mathbf{u}}(t, \mathbf{x}, \tau) + \tilde{\mathbf{u}}(t, \bar{\boldsymbol{\theta}}, \tau, \mathbf{z}), \\
 (b) \quad p^\epsilon &= \bar{p}(t, \mathbf{x}, \tau) + \tilde{p}(t, \bar{\boldsymbol{\theta}}, \tau, \mathbf{z}), \\
 (c) \quad \boldsymbol{\theta}^\epsilon &= \bar{\boldsymbol{\theta}}(t, \mathbf{x}, \tau) + \epsilon \tilde{\boldsymbol{\theta}}(t, \bar{\boldsymbol{\theta}}, \tau, \mathbf{z}),
 \end{aligned}
 \tag{4.1}$$

where $\bar{\mathbf{u}}$ and $\tilde{\mathbf{u}}$ are defined by (3.32), and $\tilde{\mathbf{u}}$, \tilde{p} and $\tilde{\boldsymbol{\theta}}$ have zero mean in \mathbf{z} . Substituting the above expansion into the Euler equation, we obtain

$$\begin{aligned}
 (a) \quad & \bar{\partial}_t(\bar{\mathbf{u}} + \tilde{\mathbf{u}}) + ((\bar{\mathbf{u}} + \tilde{\mathbf{u}}) \cdot \nabla_x)(\bar{\mathbf{u}} + \tilde{\mathbf{u}}) + \nabla_x(\bar{p} + \tilde{p}) + \frac{1}{\epsilon} D_x \bar{\boldsymbol{\theta}}^\top \nabla_z \tilde{p} \\
 & + \frac{1}{\epsilon} D_z \tilde{\mathbf{u}}(\bar{\partial}_t \bar{\boldsymbol{\theta}} + D_x \bar{\boldsymbol{\theta}}(\bar{\mathbf{u}} + \tilde{\mathbf{u}})) = \mathbf{0}, \\
 (b) \quad & \nabla_x \cdot (\bar{\mathbf{u}} + \tilde{\mathbf{u}}) + \frac{1}{\epsilon} (D_x \bar{\boldsymbol{\theta}}^\top \nabla_z) \cdot \tilde{\mathbf{u}} = 0.
 \end{aligned}
 \tag{4.2}$$

By averaging the above equations with respect to \mathbf{z} , we get the following homogenized equations.

Theorem 4.1. *The total averaged velocity field $\bar{\mathbf{u}}$ satisfies the following homogenized equation:*

$$\begin{aligned}
 (a) \quad & \bar{\partial}_t \bar{\mathbf{u}} + (\bar{\mathbf{u}} \cdot \nabla_x) \bar{\mathbf{u}} + \nabla_x \bar{p} + \nabla_x \cdot \langle \tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}} \rangle = \mathbf{0}, \\
 (b) \quad & \nabla_x \cdot \bar{\mathbf{u}} = 0, \\
 (c) \quad & \bar{\mathbf{u}}|_{t=0} = \mathbf{U},
 \end{aligned}
 \tag{4.3}$$

and the homogenized equation for the Lagrangian map is given by

$$\begin{aligned}
 (a) \quad & \bar{\partial}_t \bar{\boldsymbol{\theta}} + (\bar{\mathbf{u}} \cdot \nabla_x) \bar{\boldsymbol{\theta}} + \epsilon \nabla_x \cdot \langle \tilde{\boldsymbol{\theta}} \otimes \tilde{\mathbf{u}} \rangle = \mathbf{0}, \\
 (b) \quad & \bar{\boldsymbol{\theta}}|_{t=0} = \mathbf{x},
 \end{aligned}
 \tag{4.4}$$

where $\bar{\partial}_t$ denotes the total derivative with respect to t , $\tilde{\boldsymbol{\theta}}$ and $\tilde{\mathbf{u}}$ are defined in Proposition 3.1.

The last term, $\langle \tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}} \rangle$, on the left hand side of (4.3)(a) is referred to as the Reynolds stress term.

Remark 4.1 By using asymptotic expansions, we can obtain the governing equation for the leading order term of $\tilde{\mathbf{u}}$. Specifically, let

$$\tilde{\mathbf{u}} = \mathbf{w}(t, \boldsymbol{\theta}, \tau, \mathbf{z}) + \mathbf{O}(\epsilon), \quad \tilde{p} = q(t, \boldsymbol{\theta}, \tau, \mathbf{z}) + O(\epsilon),
 \tag{4.5}$$

where $\tau = t/\epsilon$ and $\mathbf{z} = \boldsymbol{\theta}/\epsilon$. Then \mathbf{w} satisfies the following equations:

$$\begin{aligned} (a) \quad & \partial_\tau \mathbf{w} + (D_x \boldsymbol{\theta} \mathbf{w} \cdot \nabla_z) \mathbf{w} + D_x \boldsymbol{\theta}^\top \nabla_z q = \mathbf{0}, \\ (b) \quad & (D_x \boldsymbol{\theta}^\top \nabla_z) \cdot \mathbf{w} = 0, \\ (c) \quad & \mathbf{w}|_{\tau=t=0} = \mathbf{W}. \end{aligned} \tag{4.6}$$

Similarly, we can obtain higher order correction terms for $\tilde{\mathbf{u}}$. We remark that the averaged velocity $\bar{\mathbf{u}}$ in general depends on both the fast time variable τ and the slow time variable t since $\bar{\mathbf{u}}$ contains the average of the higher order correction terms. However, as we see from Proposition 3.1, the leading order term of $\bar{\mathbf{u}}$ is independent of τ .

Remark 4.2 For practical applications, it is important to consider the viscous effect. The multiscale analysis we develop for the Euler equations can be also extended to the Navier-Stokes equations. Specifically, we consider the incompressible Navier-Stokes equations:

$$\begin{aligned} (a) \quad & \partial_t \mathbf{u}^\epsilon + (\mathbf{u}^\epsilon \cdot \nabla_x) \mathbf{u}^\epsilon + \nabla_x p^\epsilon - \epsilon \nu \Delta \mathbf{u}^\epsilon = \mathbf{0}, \\ (b) \quad & \nabla_x \cdot \mathbf{u}^\epsilon = 0, \\ (c) \quad & \mathbf{u}^\epsilon|_{t=0} = \mathbf{U}(\mathbf{x}) + \mathbf{W}(\mathbf{x}, \mathbf{x}/\epsilon), \end{aligned} \tag{4.7}$$

where ν is viscosity. We still decompose the velocity and pressure in the same way as in (4.1). Then the total averaged velocity field $\bar{\mathbf{u}}$ satisfies the following the homogenized equations

$$\begin{aligned} (a) \quad & \bar{\partial}_t \bar{\mathbf{u}} + (\bar{\mathbf{u}} \cdot \nabla_x) \bar{\mathbf{u}} + \nabla_x \bar{p} + \nabla_x \cdot \langle \tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}} \rangle - \epsilon \nu \Delta_x \bar{\mathbf{u}} = \mathbf{0}, \\ (b) \quad & \nabla_x \cdot \bar{\mathbf{u}} = 0, \\ (c) \quad & \bar{\mathbf{u}}|_{t=0} = \mathbf{U}, \end{aligned} \tag{4.8}$$

where $\bar{\mathbf{u}} = \bar{\mathbf{u}}(t, \mathbf{x}, t/\epsilon)$, $\bar{p} = \bar{p}(t, \mathbf{x}, t/\epsilon)$ and $\bar{\partial}_t$ denotes the total derivative with respect to t . Let $\tilde{\mathbf{u}}$ have the form (4.5). The leading order term \mathbf{w} satisfies the following equations:

$$\begin{aligned} (a) \quad & \partial_\tau \mathbf{w} + (D_x \boldsymbol{\theta} \mathbf{w} \cdot \nabla_z) \mathbf{w} + D_x \boldsymbol{\theta}^\top \nabla_z q - \nu \nabla_z \cdot (D_x \boldsymbol{\theta} D_x \boldsymbol{\theta}^\top \nabla_z \mathbf{w}) = \mathbf{0}, \\ (b) \quad & (D_x \boldsymbol{\theta}^\top \nabla_z) \cdot \mathbf{w} = 0, \\ (c) \quad & \mathbf{w}|_{\tau=t=0} = \mathbf{W}. \end{aligned} \tag{4.9}$$

These two equations plus the homogenized equation (4.4) for $\bar{\boldsymbol{\theta}}$ give the leading order homogenized equations for the Navier-Stokes equations.

4.2. Structure of the Reynolds stress term. It is very important to understand the structure of the Reynolds stress term for engineering applications. A lot of work has been done in the fluid dynamics community to derive effective models for the Reynolds stress term. In this section, we will derive a new formulation for the Reynolds stress term based upon our multiscale analysis performed in Section 3.

Let

$$\sigma = \frac{1}{\epsilon} \nabla_z^\perp \cdot \mathbf{W}(\mathbf{x}, \mathbf{z}) + \nabla_x^\perp \cdot \mathbf{W}(\mathbf{x}, \mathbf{z}), \quad \varrho = \nabla_x^\perp \cdot \mathbf{U}(\mathbf{x}).$$

It is clear that σ is a function of the initial fluctuation \mathbf{W} and ϱ is a function of the initial mean vorticity field. Then we can write the vorticity as

$$\omega^\epsilon = \nabla^\perp \cdot \mathbf{u}^\epsilon = \sigma(\bar{\boldsymbol{\theta}} + \epsilon \tilde{\boldsymbol{\theta}}, \mathbf{z} + \tilde{\boldsymbol{\theta}}) + \varrho(\bar{\boldsymbol{\theta}} + \epsilon \tilde{\boldsymbol{\theta}}). \tag{4.10}$$

Since

$$\varrho(\bar{\boldsymbol{\theta}} + \epsilon \tilde{\boldsymbol{\theta}}) = \varrho(\bar{\boldsymbol{\theta}}) + \epsilon \nabla_{\bar{\boldsymbol{\theta}}} \varrho(\bar{\boldsymbol{\theta}}) \cdot \tilde{\boldsymbol{\theta}} + \frac{\epsilon^2}{2} D_{\bar{\boldsymbol{\theta}} \bar{\boldsymbol{\theta}}}^2 \varrho(\bar{\boldsymbol{\theta}})(\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\theta}}) + O(\epsilon^3),$$

we can write the mean vorticity, $\bar{\omega}$, as follows:

$$\bar{\omega} = \nabla^\perp \cdot \bar{\mathbf{u}} = \varrho(\bar{\boldsymbol{\theta}}) + \langle \sigma(\bar{\boldsymbol{\theta}} + \epsilon \tilde{\boldsymbol{\theta}}, \mathbf{z} + \tilde{\boldsymbol{\theta}}) \rangle + O(\epsilon^2). \tag{4.11}$$

Next, we decompose the oscillatory component of vorticity, $\tilde{\omega}$, into two parts:

$$\tilde{\omega} = \tilde{\omega}_0 + \tilde{\omega}_1, \tag{4.12}$$

with

$$\begin{aligned} (a) \quad & \tilde{\omega}_0 = \epsilon \nabla_{\bar{\boldsymbol{\theta}}} \bar{\omega} \cdot \tilde{\boldsymbol{\theta}}, \\ (b) \quad & \tilde{\omega}_1 = \sigma(\bar{\boldsymbol{\theta}} + \epsilon \tilde{\boldsymbol{\theta}}, \mathbf{z} + \tilde{\boldsymbol{\theta}}) - \langle \sigma(\bar{\boldsymbol{\theta}} + \epsilon \tilde{\boldsymbol{\theta}}, \mathbf{z} + \tilde{\boldsymbol{\theta}}) \rangle \\ & \quad - \epsilon \nabla_{\bar{\boldsymbol{\theta}}} \langle \sigma(\bar{\boldsymbol{\theta}} + \epsilon \tilde{\boldsymbol{\theta}}, \mathbf{z} + \tilde{\boldsymbol{\theta}}) \rangle \cdot \tilde{\boldsymbol{\theta}} + O(\epsilon^2). \end{aligned} \tag{4.13}$$

Corresponding to $\tilde{\omega}_0$ and $\tilde{\omega}_1$, we decompose the fluctuation of velocity, $\tilde{\mathbf{u}}$, into two parts

$$\tilde{\mathbf{u}} = \tilde{\mathbf{u}}_0 + \tilde{\mathbf{u}}_1, \tag{4.14}$$

where

$$\tilde{\mathbf{u}}_j = -\mathcal{H} \nabla_z^\perp \tilde{\psi}_j - \epsilon \nabla_x^\perp \tilde{\psi}_j, \quad j = 0, 1. \tag{4.15}$$

$\tilde{\psi}_0$ and $\tilde{\psi}_1$ are defined by

$$\begin{aligned} & -\nabla_z^\perp \cdot \left(\mathcal{H}^\top \mathcal{H} \nabla_z^\perp \tilde{\psi}_0 \right) - \epsilon \nabla_z^\perp \cdot \left(\mathcal{H}^\top \nabla_x^\perp \tilde{\psi}_0 \right) - \epsilon \nabla_x^\perp \cdot \left(\mathcal{H} \nabla_z^\perp \tilde{\psi}_0 \right) - \epsilon^2 \Delta_x \tilde{\psi}_0 \\ & = \epsilon^2 (D_x \bar{\boldsymbol{\theta}}^{-1} \tilde{\boldsymbol{\theta}})^\top \nabla_x \bar{\omega} \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} & -\nabla_z^\perp \cdot \left(\mathcal{H}^\top \mathcal{H} \nabla_z^\perp \tilde{\psi}_1 \right) - \epsilon \nabla_z^\perp \cdot \left(\mathcal{H}^\top \nabla_x^\perp \tilde{\psi}_1 \right) - \epsilon \nabla_x^\perp \cdot \left(\mathcal{H} \nabla_z^\perp \tilde{\psi}_1 \right) - \epsilon^2 \Delta_x \tilde{\psi}_1 \\ & = \epsilon \tilde{\omega}_1. \end{aligned} \tag{4.17}$$

We look for the solution of $\tilde{\psi}_0$ of the form

$$\tilde{\psi}_0 = \tilde{\psi}_0^{(0)} + \epsilon \tilde{\psi}_0^{(1)} + \epsilon^2 \tilde{\psi}_0^{(2)} + \dots$$

It is clear that $\tilde{\psi}_0^{(0)} = \tilde{\psi}_0^{(1)} = 0$, and $\tilde{\psi}_0^{(2)}$ satisfies

$$-\nabla_z^\perp \cdot \left((\mathcal{H}^{(0)})^\top \mathcal{H}^{(0)} \nabla_z^\perp \tilde{\psi}_0^{(2)} \right) = (D_x \boldsymbol{\theta}^{-1} \boldsymbol{\theta}^{(1)})^\top \nabla_x \bar{\omega}, \tag{4.18}$$

where $\mathcal{H}^{(0)}$ and $\boldsymbol{\theta}^{(1)}$ are defined in Theorem 3.3. Let $\boldsymbol{\xi} = -D_x \boldsymbol{\theta}^{-1} \boldsymbol{\theta}^{(1)}$ and $v = \mathcal{K}(f)$ be the solution operator of the boundary value problem:

$$-\nabla_z^\perp \cdot \left((\mathcal{H}^{(0)})^\top \mathcal{H}^{(0)} \nabla_z^\perp v \right) = f.$$

We have

$$\tilde{\psi}_0^{(2)} = -\mathcal{K}(\boldsymbol{\xi})^\top \nabla_x \bar{\omega}, \tag{4.19}$$

which implies

$$\tilde{\mathbf{u}}_0 = \epsilon^2 \mathcal{H}^{(0)} \nabla_z^\perp [\mathcal{K}(\boldsymbol{\xi})^\top \nabla_x \bar{\omega}] + \mathbf{O}(\epsilon^3) = \epsilon^2 \mathcal{H}^{(0)} \mathcal{A} \nabla_x \bar{\omega} + \mathbf{O}(\epsilon^3), \tag{4.20}$$

with

$$\mathcal{A} = (\nabla_z^\perp \mathcal{K}(\xi_1), \nabla_z^\perp \mathcal{K}(\xi_2)). \tag{4.21}$$

With the above information for the oscillatory component of the velocity field, we can now characterize the Reynolds stress term in the following theorem.

Theorem 4.2. *Let $\bar{\omega} = \nabla_x^\perp \cdot \bar{\mathbf{u}}$ be the mean vorticity. The Reynolds stress term has the following asymptotic structure:*

$$\begin{aligned} \nabla_x \cdot \langle \tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}} \rangle &= \nabla_x \cdot (\tilde{\mathbf{u}}_1 \otimes \tilde{\mathbf{u}}_1) - \epsilon^2 [\nabla_x \cdot \langle \tilde{\mathbf{u}}_1 \otimes (\mathcal{H}^{(0)} \mathcal{A} \nabla_x \bar{\omega}) \rangle \\ &\quad + \nabla_x \cdot \langle (\mathcal{H}^{(0)} \mathcal{A} \nabla_x \bar{\omega}) \otimes \tilde{\mathbf{u}}_1 \rangle] + \mathbf{O}(\epsilon^3), \end{aligned} \tag{4.22}$$

where $\mathcal{H}^{(0)}$ and \mathcal{A} are given by (3.40) and (4.21) respectively, and $\tilde{\mathbf{u}}_1$ is given by (4.15) and (4.17).

Note that

$$\nabla_x \bar{\omega} = \nabla_x \nabla_x^\perp \cdot \bar{\mathbf{u}} = \begin{pmatrix} \partial_{x_1}(\partial_{x_1} \bar{u}_2 - \partial_{x_2} \bar{u}_1) \\ \partial_{x_2}(\partial_{x_1} \bar{u}_2 - \partial_{x_2} \bar{u}_1) \end{pmatrix} = \begin{pmatrix} \Delta \bar{u}_2 \\ -\Delta \bar{u}_1 \end{pmatrix},$$

where $\nabla_x \cdot \bar{\mathbf{u}} = 0$ is used. Using this property, we can further express the Reynolds stress term in term of the mean vorticity field as follows.

Corollary 4.1. *Let*

$$\mathcal{E} = (-\nabla_z^\perp \mathcal{K}(\xi_2), \nabla_z^\perp \mathcal{K}(\xi_1)) = \begin{pmatrix} \partial_{z_2} \mathcal{K}(\xi_2) & -\partial_{z_2} \mathcal{K}(\xi_1) \\ -\partial_{z_1} \mathcal{K}(\xi_2) & \partial_{z_1} \mathcal{K}(\xi_1) \end{pmatrix}, \tag{4.23}$$

where $\boldsymbol{\xi} = -D_x \boldsymbol{\theta}^{-1} \boldsymbol{\theta}^{(1)}$. The Reynolds stress term has the following asymptotic structure:

$$\begin{aligned} \nabla_x \cdot \langle \tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}} \rangle &= \nabla_x \cdot (\tilde{\mathbf{u}}_1 \otimes \tilde{\mathbf{u}}_1) - \epsilon^2 [\nabla_x \cdot \langle \tilde{\mathbf{u}}_1 \otimes (\mathcal{H}^{(0)} \mathcal{E} \Delta_x \bar{\mathbf{u}}) \rangle \\ &\quad + \nabla_x \cdot \langle (\mathcal{H}^{(0)} \mathcal{E} \Delta_x \bar{\mathbf{u}}) \otimes \tilde{\mathbf{u}}_1 \rangle] + \mathbf{O}(\epsilon^3), \end{aligned} \tag{4.24}$$

where $\mathcal{H}^{(0)}$ is given by (3.40), and $\tilde{\mathbf{u}}_1$ is given by (4.15) and (4.17).

By using asymptotic expansions, one can obtain the leading order term of $\tilde{\mathbf{u}}_1$. We look for the solution of $\tilde{\psi}_1$ of the form

$$\tilde{\psi}_1 = \tilde{\psi}_1^{(0)} + \epsilon \tilde{\psi}_1^{(1)} + \dots$$

We have that $\tilde{\psi}_1^{(0)}$ satisfies

$$-\nabla_z^\perp \cdot \left((\mathcal{H}^{(0)})^\top \mathcal{H}^{(0)} \nabla_z^\perp \tilde{\psi}_1^{(0)} \right) = \sigma_0(\boldsymbol{\theta}, \mathbf{z} + \boldsymbol{\theta}^{(1)}), \tag{4.25}$$

where $\sigma_0 = \nabla_z^\perp \cdot \mathbf{W}(\mathbf{x}, \mathbf{z})$. This gives

$$\tilde{\mathbf{u}}_1 = -\mathcal{H}^{(0)} \nabla_z^\perp \tilde{\psi}_1^{(0)} + \mathbf{O}(\epsilon). \tag{4.26}$$

The structure of the Reynolds stress term provides important information for understanding some macroscopic properties of 2-D turbulence. From (4.22) and (4.24), we see that the oscillatory component of the velocity field, $\tilde{\mathbf{u}}$, is divided into two parts: $\tilde{\mathbf{u}}_0$ and $\tilde{\mathbf{u}}_1$. The term $\tilde{\mathbf{u}}_0$ comes from the interaction between the mean vorticity field and the oscillating component of the stream function. The net effect of $\tilde{\mathbf{u}}_0$ on the Reynolds stress is a dispersion of the mean velocity field. On the other hand, $\tilde{\mathbf{u}}_1$ mainly comes from the oscillating component of the initial velocity field, which may be considered as a high-frequency disturbance of some exterior forcing at the initial time. The net effect of this term on the Reynolds stress is an exterior forcing term to the homogenized equation. Therefore, the effect of the Reynolds stress term can be divided into two parts: a forcing term (corresponding to the first

term on RHS of (4.24)) and a dispersive term (corresponding to the second and third terms on RHS of (4.24)).

Another observation is that the term $\mathcal{H}^{(0)}$ plays an important role in the Reynolds stress term. Recall that $\mathcal{H}^{(0)} = D_x \boldsymbol{\theta}^{-1}$. When there is a strong shear in the flow, $\boldsymbol{\theta}$ could form a shear layer, and $\mathcal{H}^{(0)}$ becomes very large in the region of strong shear. The fluid is stretched in the region of strong shear. The locally stretched region has the effect of 'thinning' of the fluid. Equations (4.22) and (4.24) show that there exists a strong disturbance and dispersion in the region of large 'thinning' and $\mathcal{H}^{(0)}$ accounts for the effect of 'thinning'. We plan to perform further study of (4.22) and (4.24) to reveal more interesting properties of the homogenized equations.

5. Generalization to flows with infinitely many scales. In this section, we discuss how to apply the two-scale analysis we developed in the previous sections to problems with infinitely non-separable scales. This is important for practical computations. Specifically, we discuss how to transform a general initial condition with many non-separable scales into the form of a two-scale function so that we can use the two-scale analysis developed earlier to study more general multiscale problems.

Let $\mathbf{v}(\mathbf{x})$ be a general multiscale initial velocity field without periodic structure or scale separation. Our basic idea is to divide scales into two parts: the large scales and the small scales. First, we assume $\mathbf{v}(\mathbf{x})$ is a periodic function on a unit square $[0, 1]^2$. We expand \mathbf{v} into its Fourier series:

$$\mathbf{v}(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^2} \hat{\mathbf{v}}(\mathbf{k}) \exp\{2\pi i \mathbf{k} \cdot \mathbf{x}\}, \quad i = \sqrt{-1}, \quad \mathbf{k} = (k_1, k_2), \quad (5.1)$$

where $\hat{\mathbf{v}}(\mathbf{k})$ are the Fourier coefficients. Choose $0 < \epsilon = 1/E < 1$ as a reference wave-length, where E is an integer. Let $\Lambda_E = \{\mathbf{k}; |k_j| \leq \frac{E}{2}, 1 \leq j \leq 2\}$, $\Lambda'_E = \mathbb{Z}^2 \setminus \Lambda_E$. We decompose the initial data into two parts as follows

$$\mathbf{v} = \mathbf{v}^{(l)}(\mathbf{x}) + \mathbf{v}^{(s)}(\mathbf{x}), \quad (5.2)$$

where

$$\mathbf{v}^{(l)}(\mathbf{x}) = \sum_{\mathbf{k} \in \Lambda_E} \hat{\mathbf{v}}(\mathbf{k}) \exp\{2\pi i \mathbf{k} \cdot \mathbf{x}\}, \quad \mathbf{v}^{(s)}(\mathbf{x}) = \sum_{\mathbf{k} \in \Lambda'_E} \hat{\mathbf{v}}(\mathbf{k}) \exp\{2\pi i \mathbf{k} \cdot \mathbf{x}\}. \quad (5.3)$$

Clearly, the component $\mathbf{v}^{(l)}(\mathbf{x})$ corresponds to the large scale velocity field, and $\mathbf{v}^{(s)}(\mathbf{x})$ corresponds to the small scale velocity field. Here the superscripts s and l stand for small-scales and large-scales respectively. For each \mathbf{k} , we rewrite \mathbf{k} as

$$\mathbf{k} = E\mathbf{k}^{(s)} + \mathbf{k}^{(l)},$$

where $\mathbf{k}^{(s)}$ and $\mathbf{k}^{(l)}$, are integers with $\mathbf{k}^{(l)} \in \Lambda_E$. Based on this decomposition, we further decompose $\mathbf{v}^{(s)}(\mathbf{x})$ as follows:

$$\begin{aligned}
 \mathbf{v}^{(s)} &= \sum_{\mathbf{k} \in \Lambda'_E} \hat{\mathbf{v}}(\mathbf{k}) \exp\{2\pi i \mathbf{k} \cdot \mathbf{x}\} \\
 &= \sum_{E\mathbf{k}^{(s)} + \mathbf{k}^{(l)} \in \Lambda'_E} \hat{\mathbf{v}}(E\mathbf{k}^{(s)} + \mathbf{k}^{(l)}) \exp\{2\pi i (E\mathbf{k}^{(s)} + \mathbf{k}^{(l)}) \cdot \mathbf{x}\} \\
 &= \sum_{\mathbf{k} \neq \mathbf{0}} \left(\sum_{\mathbf{k}' \in \Lambda_E} \hat{\mathbf{v}}(E\mathbf{k} + \mathbf{k}') \exp\{2\pi i \mathbf{k}' \cdot \mathbf{x}\} \right) \exp\{2\pi i \mathbf{k} \cdot (E\mathbf{x})\} \quad (5.4) \\
 &\equiv \sum_{\mathbf{k} \neq \mathbf{0}} \hat{\mathbf{v}}^{(s)}(\mathbf{k}, \mathbf{x}) \exp\{2\pi i \mathbf{k} \cdot \frac{\mathbf{x}}{\epsilon}\} \\
 &= \mathbf{v}^{(s)}(\mathbf{x}, \frac{\mathbf{x}}{\epsilon}),
 \end{aligned}$$

where coefficient $\hat{\mathbf{v}}^{(s)}(\mathbf{k}, \mathbf{x})$ contains Fourier modes lower than $E/2$ only. Thus, we can decompose a periodic function formally into a two-scale function with periodic structure:

$$\mathbf{v} = \mathbf{v}^{(l)}(\mathbf{x}) + \mathbf{v}^{(s)}(\mathbf{x}, \frac{\mathbf{x}}{\epsilon}). \quad (5.5)$$

For general non-periodic function \mathbf{v} , by using the method of partition of unity, i.e., for a family of smooth cut-off functions $\{\phi_j\}_{j=1}^J$ such that

$$(i) \ \phi_j \in C_0^1([0, 1]^2), \quad (ii) \ 0 \leq \phi_j \leq 1, \quad (iii) \ \sum_{j=1}^J \phi_j = 1,$$

we can decompose \mathbf{v} as

$$\mathbf{v} = \sum_{j=1}^J \phi_j \mathbf{v} \equiv \sum_{j=1}^J \mathbf{v}_j.$$

We can then treat \mathbf{v}_j as a periodic function and use the same method mentioned-above to decompose the function \mathbf{v}_j into large and small scales. Thus, we can describe the initial velocity field \mathbf{v} in the following generic form:

$$\mathbf{v}(\mathbf{x}) = \mathbf{v}^{(l)}(\mathbf{x}) + \mathbf{v}^{(s)}(\mathbf{x}, \frac{\mathbf{x}}{\epsilon}), \quad (5.6)$$

where $\mathbf{v}^{(s)}(\mathbf{x}, \mathbf{y})$ is periodic function of the period 1 in \mathbf{y} . We can use a coarse grid with size H to resolve low frequency component of wavelength larger than ϵ and use a fine grid with size h to resolve high frequency component of wavelength smaller than ϵ . Since the initial data can be expressed in the form of a two-scale function, the homogenization results derived in section 3 and 4 can be applied to the more general initial data, which are more relevant to fluid dynamic applications. In the next section, we will use this technique and the multiscale analysis developed in the previous sections to study the decay of 2-D homogeneous turbulence. In this study, the initial velocity field is random and contains many non-separable scales. We find that many interesting features revealed by the direct numerical simulation are well captured by our multiscale method.

6. Numerical Experiments. We test the multiscale model for the 2-D incompressible Navier-Stokes equations in a doubly periodic box of size $2\pi \times 2\pi$. We choose viscosity $\nu = 5.0 \times 10^{-5}$, and we solve the homogenized equations given in Section 4.1. In particular, the averaged velocity $\bar{\mathbf{u}}$ and the leading oscillatory velocity \mathbf{w} satisfy the following homogenized equations,

$$\begin{aligned} (a) \quad & \bar{\partial}_t \bar{\mathbf{u}} + (\bar{\mathbf{u}} \cdot \nabla_x) \bar{\mathbf{u}} + \nabla_x \bar{p} + \nabla_x \cdot \langle \tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}} \rangle - \nu \Delta_x \bar{\mathbf{u}} = \mathbf{0}, \\ (b) \quad & \nabla_x \cdot \bar{\mathbf{u}} = 0, \\ (c) \quad & \bar{\mathbf{u}}|_{t=0} = \mathbf{U}, \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} (a) \quad & \partial_\tau \mathbf{w} + (D_x \boldsymbol{\theta} \mathbf{w} \cdot \nabla_z) \mathbf{w} + D_x \boldsymbol{\theta}^\top \nabla_z q - \nu' \nabla_z \cdot (D_x \boldsymbol{\theta} D_x \boldsymbol{\theta}^\top \nabla_z \mathbf{w}) = \mathbf{0}, \\ (b) \quad & (D_x \boldsymbol{\theta}^\top \nabla_z) \cdot \mathbf{w} = 0, \\ (c) \quad & \mathbf{w}|_{\tau=t=0} = \mathbf{W}(\mathbf{x}, \mathbf{x}/\epsilon), \end{aligned} \tag{6.2}$$

where $\nu' = \nu/\epsilon$ is the cell viscosity, and $\boldsymbol{\theta}$ is governed by

$$\begin{aligned} (a) \quad & \bar{\partial}_t \boldsymbol{\theta} + (\bar{\mathbf{u}} \cdot \nabla_x) \boldsymbol{\theta} = \mathbf{0}, \\ (b) \quad & \boldsymbol{\theta}|_{t=0} = \mathbf{x}. \end{aligned} \tag{6.3}$$

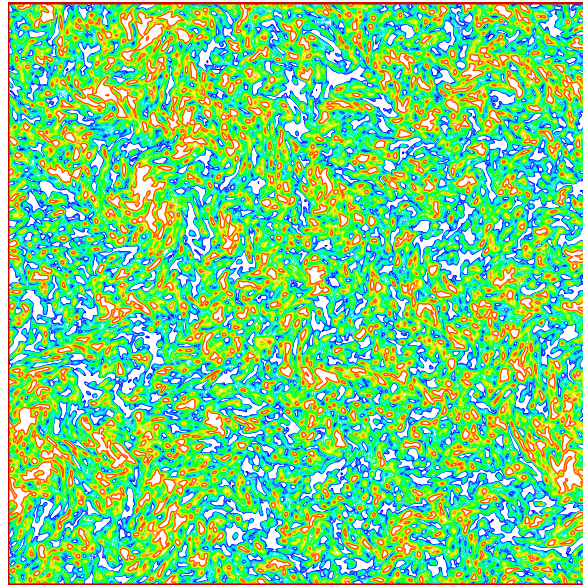
The pseudo-spectral method is used to solve both the large-scale homogenized equations and the small-scale cell problem. We use the second-order Runge-Kutta method to discretize the equation in time. In our computations, we solve the homogenized equation using a coarse grid and large time step, Δt . For each coarse grid point within the time interval from t_n to t_{n+1} with $t_n = n\Delta t$, we solve the cell problem with the periodic boundary condition in the \mathbf{z} variable from $\tau = t_n/\epsilon$ to $\tau = t_{n+1}/\epsilon$ using a subgrid $\Delta \mathbf{z}$ and a subgrid time step $\Delta \tau$. We then average the cell solution from $\tau = t_n/\epsilon$ to $\tau = t_{n+1}/\epsilon$ to evaluate the Reynolds stress term and update the mean velocity at t_{n+1} . In solving the viscous cell problem, we use a splitting method. We first solve the Euler equation using a streamline vorticity formulation as described in Theorem 3.3, and then add the viscous correction. To eliminate the aliasing error in the pseudo-spectral method, we use a 15th order Fourier smoothing function to damp the high frequency modes.

In this section we present numerical results on the decay of 2-D homogeneous turbulence, and compare the homogenization model with well resolved direct numerical simulation (DNS). The DNS uses a 1024×1024 fine grid. The simulation starts with a random initial condition, whose stream function in the Fourier space is given by,

$$|\widehat{\psi}(k)| = \frac{k}{k^{7/2} + \delta}, \quad k = |\mathbf{k}|, \tag{6.4}$$

with random phases. This choice of the initial velocity field is similar to the earlier work of Henshaw-Kreiss-Reyna [10]. In the computation, we choose $\delta = 10^{-14}$. The energy spectrum, $E(k)$, is given by $|\widehat{\mathbf{u}}_k|^2 k$ (see page 55 of [3]). The corresponding energy spectrum is $E(k) = O(k^{-2})$. The initial vorticity distribution is plotted in Figure 1. From $t = 0$ to $t = 5$, coherent vortices emerge from the random initial condition, which is denoted as 'vortex generation period'. At later stages, the flow is dominated by the mutual interactions of coherent vortices. The number of vortices decreases and the averaged vortex radius and circulation increase.

We use the technique presented in Section 5 to prepare the initial velocity in the form of a two-scale initial condition so that we can apply the homogenized equations (6.1)-(6.2) to solve the multiscale problem. The dimension of the coarse

FIGURE 1. Vorticity contour at $t = 0$.

grid (slow variables) is 64×64 and that of small scale (fast variables) is 32×32 . The cell problem is solved for every other coarse grid cell with total number of 32×32 cells. The corresponding ϵ in our two-scale analysis is $\epsilon = 1/32$. The time step for the homogenized equation is $\Delta t = 0.05$ and the subgrid time step for the cell problem is $\Delta \tau = 0.05$. In Figure 2, we compare the vorticity distribution obtained from solving the homogenized equations with that obtained from the DNS at $t = 10.0$ and 20.0 . The plots show the stretching of vortex filament by the mean flow. In addition, the size of the vortices grows due to the merger of vorticity of the same sign, which is one of the mechanisms that contributes to the inverse energy cascade. The vorticity distribution obtained from solving the homogenized equations is in excellent agreement with that obtained from the DNS, suggesting that the multiscale model captures the vortex interactions at both large scales and small scales.

Further, we compare the mean velocity field by filtering the DNS calculation and the multiscale computation with a low pass Gaussian filter $\exp(-sk^2)$, where s is the filter scale and k is the module of Fourier modes. We use $s = 0.005$ in our computations. The mean horizontal velocity, u_1 , is plotted in Figure 3. The agreement is quite good up to $t = 20$. The computations based on the homogenized equations can be continued up to $t = 25$, which is roughly of order $O(1/\epsilon)$. Beyond this time, the Lagrangian flow map θ will develop small scale features dynamically. We will have to reinitialize the Lagrangian flow map to take into account these small scales features generated dynamically. In a subsequent paper, we will introduce a different approach which takes into account the dynamically generated small scales due to the nonlinear interaction of the large scale solution. This will remove the difficulty of reinitializing the Lagrangian flow map dynamically.

Next, we study the decay of mean kinetic energy and mean enstrophy. It is known that the temporal evolution of mean kinetic energy and mean enstrophy is

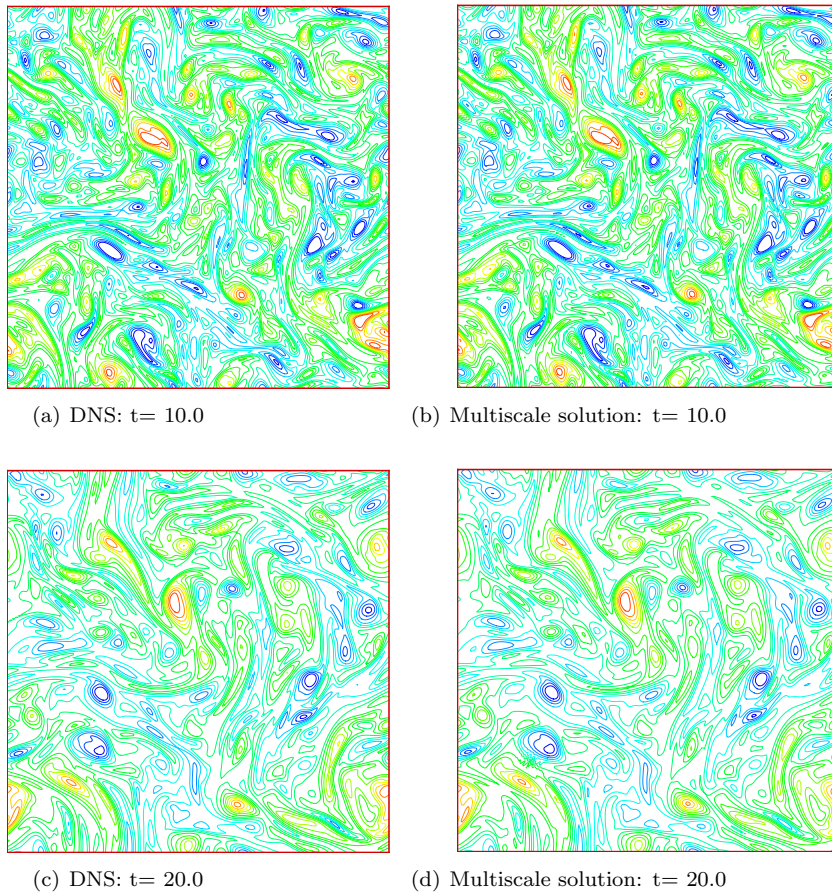


FIGURE 2. Contour plot of vorticity. Contour levels: max=1.4, min=-0.9.

governed by the following equations [11, 12]

$$\frac{d}{dt} \langle \frac{1}{2} |\mathbf{u}|^2 \rangle = -\nu \langle \omega^2 \rangle, \tag{6.5}$$

$$\frac{d}{dt} \langle \frac{1}{2} \omega^2 \rangle = -\nu \langle \nabla \omega^2 \rangle. \tag{6.6}$$

From equation (6.6), we can see that the enstrophy is bounded by its initial value. Therefore in the limit of $\nu \rightarrow 0$, $d\langle |\mathbf{u}|^2 \rangle / dt \rightarrow 0$, i.e. the total kinetic energy is conserved. On the other hand, there exists a cascade of enstrophy from large scale to small scale [11, 12], which causes the total enstrophy to decay in time. Specifically, vorticity gradients are amplified with the formation of thin filaments. These filaments are stretched until they reach the very small dissipation scales, so that the enstrophy and all positive-order vorticity moments decay. In Figure 4, we show the temporal evolution of the total kinetic energy and the total enstrophy using three different approaches, which are (i) the DNS, (ii) the simulation using the homogenized equations, and (iii) the simulation of the homogenized equations ignoring the Reynolds stress term. With very small viscosity, the energy decay

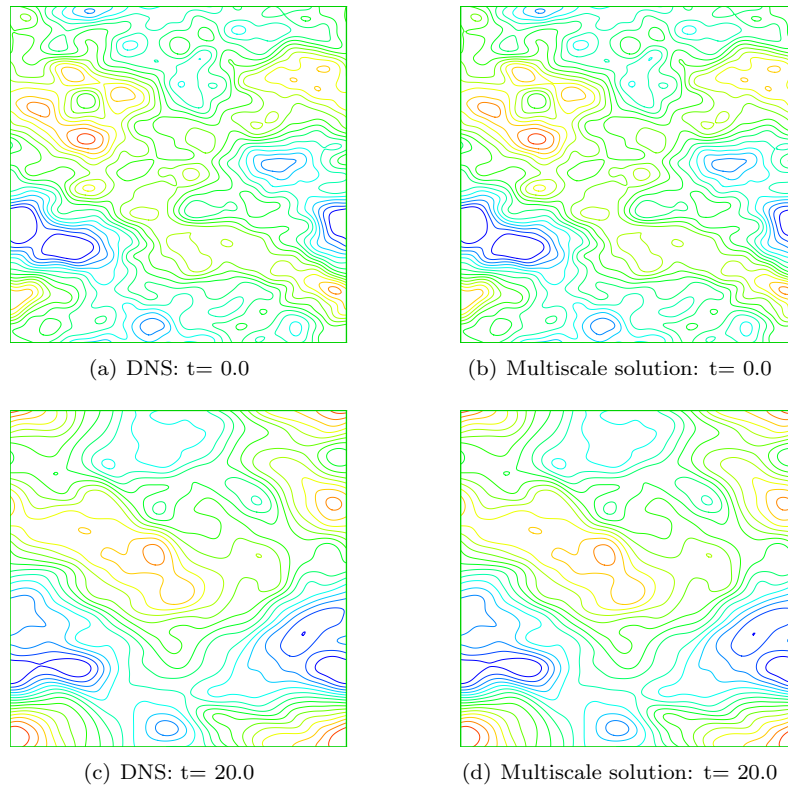


FIGURE 3. Filtered velocity field u_1 with Gaussian filter $\exp(-sk^2)$, $s=0.005$. Contour levels: $\max=0.45$, $\min=-0.45$.

from all 3 simulations is negligible. On the other hand, the enstrophy decays continuously, with maximum decay rate during the initial vortex formation period. The decay rate becomes smaller during the vortex merger stage. We can see that the simulation with coarse grid using approach (iii) leads to much slower enstrophy decay because it could not capture the enstrophy cascade from large scales to the dissipation scale. On the other hand, the enstrophy decay rate of the simulation using the homogenized equations is very close to that of the DNS, suggesting that the dissipation mechanism is well resolved within each cell.

We also compare the spectrum between the DNS and the simulation using the homogenized equations at $t = 20.0$ in Figure 5. We reconstruct the fine grid velocity field by

$$\mathbf{u}^\epsilon(t, \mathbf{x}) \approx \mathbf{u}(t, \mathbf{x}) + \mathbf{w}(t, \boldsymbol{\theta}, \tau, \frac{\boldsymbol{\theta}(t, \mathbf{x})}{\epsilon}), \quad (6.7)$$

where the fine grid phase function is obtained using the spectral interpolation. The spectrum obtained from the homogenized equations decays at a rate asymptotic to k^{-3} while the initial spectrum decays at a rate of order k^{-2} . The agreement between the multiscale computation and the DNS is very good at low wave numbers. At high wave numbers, the DNS spectrum decays faster than the spectrum obtained from the homogenized equations. The difference is partly due to the fact that we neglect the higher order terms in the homogenized equations and the cell problem.

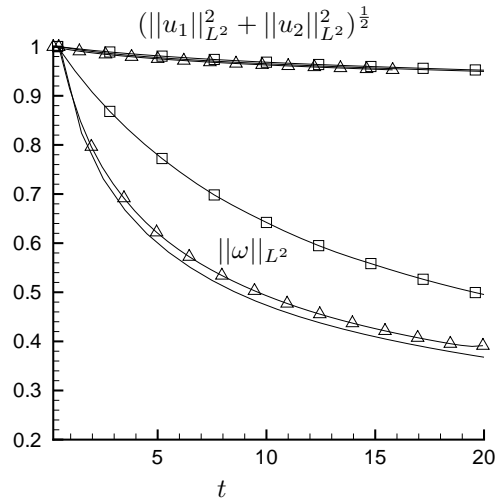


FIGURE 4. Temporal evolution of kinetic energy $(\|u_1\|_{L^2}^2 + \|u_2\|_{L^2}^2)^{\frac{1}{2}}$ and enstrophy $\|\omega\|_{L^2}$. (—), DNS; (Δ), homogenization; (\square), without Reynolds stress terms.

Another reason is that the leading order viscous term in the cell problem may not capture all the viscous effect in the DNS computation. Nonetheless, the model accurately captures the dynamics of the large scale, as well as the averaged effect from the small scales.

As we mentioned before, we do not compute every cell problem for each coarse grid node, but for every other coarse grid node with total number of 32×32 cells. We further reduce the number of cell problems by computing only every other four coarse grid node with total number of 16×16 cells. As we can see from Figure 5, there is only a small discrepancy at the high wave numbers between the computation using 32×32 cells and that using 16×16 cells. However, the spectra of these two computations are almost identical at low wave numbers.

Finally, we remark that when the initial velocity field has an energy spectrum that is close to the equilibrium state, i.e. $E(k) \approx O(k^{-3})$, then there is no need to apply the viscous correction to the cell problem, i.e. we can set the cell viscosity term in (6.2) to zero. The cell problem based on the Euler equations is sufficient to capture the leading order effect of the Navier-Stokes equations at high Reynolds numbers. In Figure 6, we present the numerical results corresponding to an initial velocity field with energy spectrum $E(k) \approx O(k^{-3})$. The homogenized equations we use in our computations are based on the Euler equations without the viscous correction. We observe that the computation based on the homogenized equations gives very good agreement with the well-resolved DNS solution. We observe some discrepancy of the two computations at high frequency regime. This could be due to the lack of viscous effect in the cell problem. However, we found that this discrepancy in high frequency does not seem to affect the accuracy in the low to moderate frequencies.

Appendix I. Derivation of Proposition 2.1. Before derivations of conclusions in Property 2.1 and other theorems, we first give two equalities (0.2) and (0.3). Note that

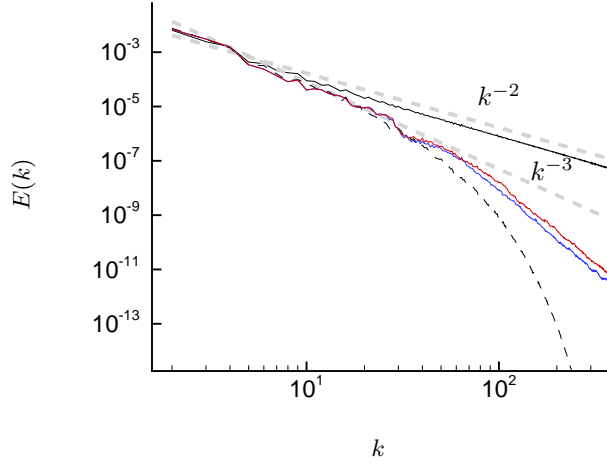


FIGURE 5. Energy spectrum $E(k)$. (black) DNS.(—), $t = 0.0$; (----) $t = 20.0$; (blue), homogenization at $t = 20.0$, with 32×32 cells; (red), homogenization at $t = 20.0$, with 16×16 cells.

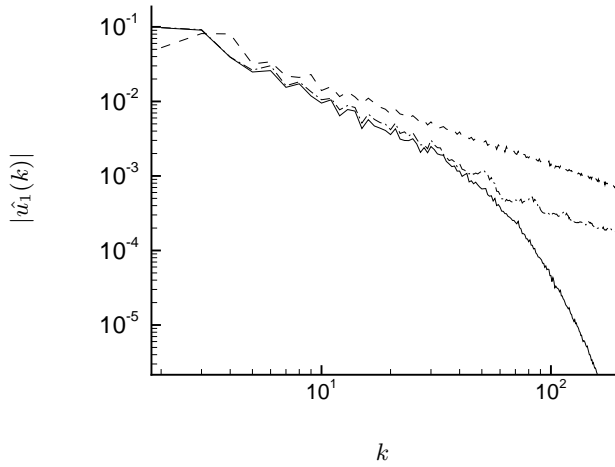


FIGURE 6. Spectrum of velocity component u_1 , with initial energy spectrum $E(k) \approx O(k^{-3})$. (----), $t = 0.0$; (—), DNS, $t = 20.0$; (— · —), homogenization, $t = 20.0$.

for g and f ,

$$\nabla f \cdot \nabla^\perp g = \nabla^\perp \cdot (g \nabla f) - g \nabla^\perp \cdot (\nabla f) = \nabla^\perp \cdot (g \nabla f), \tag{0.1}$$

where $\nabla^\perp \cdot (\nabla f) = 0$ is used in the last step. From (0.1) we have that for each g and \mathbf{f} ,

$$D\mathbf{f} \nabla^\perp g = \begin{pmatrix} \nabla f_1 \cdot \nabla^\perp g \\ \nabla f_2 \cdot \nabla^\perp g \end{pmatrix} = \begin{pmatrix} \nabla^\perp \cdot (g \nabla f_1) \\ \nabla^\perp \cdot (g \nabla f_2) \end{pmatrix} \tag{0.2}$$

and that for \mathbf{f} and $\boldsymbol{\psi}$,

$$\begin{aligned} (D\mathbf{f}\nabla^\perp) \cdot \boldsymbol{\psi} &= \nabla f_1 \cdot \nabla^\perp \psi_1 + \nabla f_2 \cdot \nabla^\perp \psi_2 \\ &= \nabla^\perp \cdot (\psi_1 \nabla f_1 + \psi_2 \nabla f_2) = \nabla^\perp \cdot (D\mathbf{f}^\top \boldsymbol{\psi}). \end{aligned} \tag{0.3}$$

Let $\boldsymbol{\alpha} = \boldsymbol{\theta}(\mathbf{x})$ be a coordinate transform from R^2 onto R^2 , $\mathbf{x} = \mathbf{X}(\boldsymbol{\alpha})$ be its inverse map. For each function $\phi = \phi(\mathbf{x})$, let $\widehat{\phi}(\boldsymbol{\alpha}) = \phi(\mathbf{X}(\boldsymbol{\alpha})) = \phi \circ \mathbf{X}$. Let us derive Property 2.1.

Firstly, (2.3)(a), i.e.,

$$D_\alpha \mathbf{X}^{-1} = \frac{1}{|D_\alpha \mathbf{X}|} \begin{pmatrix} \partial_{\alpha_2} x_2 & -\partial_{\alpha_2} x_1 \\ -\partial_{\alpha_1} x_2 & \partial_{\alpha_1} x_1 \end{pmatrix} = \frac{1}{|D_\alpha \mathbf{X}|} (-\nabla_\alpha^\perp x_2, \nabla_\alpha^\perp x_1) \tag{0.4}$$

is a well-known fact.

Secondly, we prove (2.3)(b). It is clear that

$$(\nabla_x^\perp \phi) \circ \mathbf{X} = (\nabla_x^\perp \widehat{\phi}) = \begin{pmatrix} \partial_{x_2} \alpha_2 & -\partial_{x_2} \alpha_1 \\ -\partial_{x_1} \alpha_2 & \partial_{x_1} \alpha_1 \end{pmatrix} \begin{pmatrix} -\partial_{\alpha_2} \widehat{\phi} \\ \partial_{\alpha_1} \widehat{\phi} \end{pmatrix} = |D_x \boldsymbol{\alpha}| D_x \boldsymbol{\alpha}^{-1} \nabla_\alpha^\perp \widehat{\phi},$$

where (0.4) is used for $D_x \boldsymbol{\alpha}$. On the other hand, $D_x \boldsymbol{\alpha}^{-1} = D_\alpha \mathbf{X}$ and $|D_x \boldsymbol{\alpha}| = |D_\alpha \mathbf{X}|^{-1}$. Substituting these facts into the above equality leads to

$$(\nabla_x^\perp \phi) \circ \mathbf{X} = |D_\alpha \mathbf{X}|^{-1} D_\alpha \mathbf{X} \nabla_\alpha^\perp \widehat{\phi}. \tag{0.5}$$

This is (2.3)(b).

Thirdly, let us see (2.3)(c). It follows from (0.3) and (0.5) that

$$(\nabla_x^\perp \cdot \boldsymbol{\psi}) \circ \mathbf{X} = |D_\alpha \mathbf{X}|^{-1} (D_\alpha \mathbf{X} \nabla_\alpha^\perp) \cdot \widehat{\boldsymbol{\psi}} = |D_\alpha \mathbf{X}|^{-1} \nabla_\alpha^\perp \cdot (D_\alpha \mathbf{X}^\top \widehat{\boldsymbol{\psi}}). \tag{0.6}$$

(2.3)(c) is derived.

Finally, it follows from (0.5) and (0.6) that

$$(\Delta_x \phi) \circ \mathbf{X} = \nabla_x^\perp \cdot (\nabla_x^\perp \widehat{\phi}) = |D_\alpha \mathbf{X}|^{-1} \nabla_\alpha^\perp \cdot (|D_\alpha \mathbf{X}|^{-1} D_\alpha \mathbf{X}^\top D_\alpha \mathbf{X} \nabla_\alpha^\perp \widehat{\phi}). \tag{0.7}$$

This is (2.3)(d).

Appendix II. Proof of Theorem 3.1. To prove Theorem 3.1, we first prove the following two properties.

Proposition A. $\mathbf{x}^{(0)}$ is independent of τ , i.e., $\mathbf{x}^{(0)}(t, \boldsymbol{\alpha}, \tau) = \mathbf{x}^{(0)}(t, \boldsymbol{\alpha})$.

Proof. From (2.10), we get

$$\frac{1}{\epsilon} \partial_\tau \mathbf{x}^{(0)} + \partial_t \mathbf{x}^{(0)} + \partial_\tau \bar{\mathbf{x}}^{(1)} + \mathbf{x}^{(1)} + \dots = \mathbf{u}^\epsilon.$$

Since we assume that \mathbf{u}^ϵ is bounded independent of ϵ , we conclude that $\frac{1}{\epsilon} \partial_\tau \mathbf{x}^{(0)} = \mathbf{0}$.

This implies $\mathbf{x}^{(0)} = \mathbf{x}^{(0)}(t, \boldsymbol{\alpha})$.

Proposition B. $\psi^{(0)}$ and $\psi_0^{(1)}$ satisfy the following equations:

$$-\nabla_\alpha^\perp \cdot \kappa^{-1} (D_\alpha \mathbf{x}^{(0)})^\top D_\alpha \mathbf{x}^{(0)} \nabla_\alpha^\perp \psi^{(0)} - \nabla_\alpha^\perp \cdot \langle \mathcal{A}^{(0)} \nabla_y^\perp \psi_0^{(1)} \rangle = \varrho(\boldsymbol{\alpha}) \tag{0.8}$$

and

$$-\nabla_y^\perp \cdot \mathcal{A}^{(0)} \nabla_y^\perp \psi_0^{(1)} = \sigma_0(\boldsymbol{\alpha}, \mathbf{y}). \tag{0.9}$$

where $\kappa = |D_\alpha \mathbf{x}^{(0)}|$, ϱ and σ_0 are defined by (2.7).

Proof. Let $\boldsymbol{\eta} = (D_\alpha \mathbf{x}^{(0)})^{-1} \mathbf{x}^{(1)}$, $\mathcal{H}^\epsilon = D_\alpha \mathbf{x}^\epsilon$, and $\mathcal{B}^\epsilon = (D_\alpha \mathbf{x}^\epsilon)^{-1}$. Expand \mathcal{H}^ϵ into a power series of ϵ :

$$\mathcal{H}^\epsilon = \mathcal{H}^{(0)} + \epsilon \mathcal{H}^{(1)} + \dots, \tag{0.10}$$

with

$$\mathcal{H}^{(0)} = D_{\alpha}\mathbf{x}^{(0)} + D_y\mathbf{x}^{(1)}, \quad \mathcal{H}^{(1)} = D_{\alpha}(\bar{\mathbf{x}}^{(1)} + \mathbf{x}^{(1)}) + D_y\mathbf{x}^{(2)}, \dots \tag{0.11}$$

which in turn gives the expansions for \mathcal{A}^ϵ and \mathcal{B}^ϵ as follows:

$$\mathcal{A}^\epsilon = (\mathcal{H}^\epsilon)^\top \mathcal{H}^\epsilon = (\mathcal{H}^{(0)})^\top \mathcal{H}^{(0)} + \epsilon[(\mathcal{H}^{(0)})^\top \mathcal{H}^{(1)} + (\mathcal{H}^{(1)})^\top \mathcal{H}^{(0)}] + \dots \tag{0.12}$$

and

$$\mathcal{B}^\epsilon = (-\nabla_{\alpha}^{\perp} x_2^\epsilon, \nabla_{\alpha}^{\perp} x_1^\epsilon) = \mathcal{B}^{(0)} + \epsilon \mathcal{B}^{(1)} + \mathcal{O}(\epsilon^2) \tag{0.13}$$

with

$$\mathcal{B}^{(0)} = (-\nabla_{\alpha}^{\perp} x_2^{(0)} - \nabla_y^{\perp} x_2^{(1)}, \nabla_{\alpha}^{\perp} x_1^{(0)} + \nabla_y^{\perp} x_1^{(1)}), \dots \tag{0.14}$$

Using the expansion for \mathcal{H}^ϵ , we get by direct calculations that

$$\begin{aligned} & \mathcal{H}^\epsilon \nabla_{\alpha}^{\perp} \psi^{(0)} + \epsilon \mathcal{H}^\epsilon \left(\nabla_{\alpha}^{\perp} + \frac{1}{\epsilon} \nabla_y^{\perp} \right) \cdot \boldsymbol{\eta} \cdot \nabla_{\alpha} \psi^{(0)} \\ &= \kappa^{-1} D_{\alpha} \mathbf{x}^{(0)} \nabla_{\alpha}^{\perp} \psi^{(0)} + \epsilon \mathcal{H}^\epsilon \nabla_{\alpha}^{\perp} \cdot \boldsymbol{\eta} \cdot \nabla_{\alpha} \psi^{(0)} \\ & \quad - \epsilon \kappa^{-1} \mathcal{H}^\epsilon \mathcal{B}^{(1)} D_{\alpha} \mathbf{x}^{(0)} \nabla_{\alpha}^{\perp} \psi^{(0)} + \mathbf{O}(\epsilon^2). \end{aligned} \tag{0.15}$$

Applying first the matrix $(\mathcal{H}^\epsilon)^\top$ and then the operator $\nabla_{\alpha}^{\perp} + \frac{1}{\epsilon} \nabla_y^{\perp} \cdot$ to the both sides of (0.15), we obtain

$$\begin{aligned} & \nabla_{\alpha}^{\perp} + \frac{1}{\epsilon} \nabla_y^{\perp} \cdot \mathcal{A}^\epsilon \left(\nabla_{\alpha}^{\perp} + \frac{1}{\epsilon} \nabla_y^{\perp} \right) \psi^{(0)} + \epsilon \boldsymbol{\eta} \cdot \nabla_{\alpha} \psi^{(0)} \\ &= \nabla_y^{\perp} \cdot \left[(\mathcal{H}^{(1)})^\top \left(\kappa^{-1} D_{\alpha} \mathbf{x}^{(0)} \nabla_{\alpha}^{\perp} \psi^{(0)} + \mathcal{H}^{(0)} \nabla_{\alpha}^{\perp} (\boldsymbol{\eta} \cdot \nabla_{\alpha} \psi^{(0)}) \right. \right. \\ & \quad \left. \left. - \kappa^{-1} \mathcal{H}^{(0)} \mathcal{B}^{(1)} D_{\alpha} \mathbf{x}^{(0)} \nabla_{\alpha}^{\perp} \psi^{(0)} \right) \right] \\ & \quad + \nabla_{\alpha}^{\perp} \cdot (\mathcal{H}^{(0)})^\top \left(\kappa^{-1} D_{\alpha} \mathbf{x}^{(0)} \nabla_{\alpha}^{\perp} \psi^{(0)} \right) + \mathbf{O}(\epsilon). \end{aligned} \tag{0.16}$$

Substitute (0.16) into the equation

$$\begin{aligned} & - \nabla_{\alpha}^{\perp} + \frac{1}{\epsilon} \nabla_y^{\perp} \cdot \mathcal{A}^\epsilon \left(\nabla_{\alpha}^{\perp} + \frac{1}{\epsilon} \nabla_y^{\perp} \right) \psi^{(0)} + \epsilon (\bar{\psi}^{(1)} + \psi_0^{(1)} + \boldsymbol{\eta} \cdot \nabla_{\alpha} \psi^{(0)}) \\ & \quad + \epsilon^2 (\bar{\psi}^{(2)} + \psi^{(2)}) + \dots \left] = \omega_{\text{int}}(\boldsymbol{\alpha}, \mathbf{y}). \end{aligned} \tag{0.17}$$

After some algebra and using the definition (2.6) of ω_{int} , the equation (0.17) is reduced to

$$\begin{aligned} & - \nabla_y^{\perp} \cdot \mathcal{A}^{(0)} \nabla_y^{\perp} \psi_0^{(1)} - \epsilon \nabla_y^{\perp} \cdot \left[\mathcal{A}^{(0)} \nabla_y^{\perp} \psi^{(2)} + \mathcal{A}^{(1)} \nabla_y^{\perp} \psi_0^{(1)} \right. \\ & \quad + \mathcal{A}^{(0)} \nabla_{\alpha}^{\perp} (\bar{\psi}^{(1)} + \psi_0^{(1)}) + (\mathcal{H}^{(1)})^\top \left(\kappa^{-1} D_{\alpha} \mathbf{x}^{(0)} \nabla_{\alpha}^{\perp} \psi^{(0)} \right. \\ & \quad \left. \left. + \mathcal{H}^{(0)} \nabla_{\alpha}^{\perp} (\boldsymbol{\eta} \cdot \nabla_{\alpha} \psi^{(0)}) - \kappa^{-1} \mathcal{H}^{(0)} \mathcal{B}^{(1)} D_{\alpha} \mathbf{x}^{(0)} \nabla_{\alpha}^{\perp} \psi^{(0)} \right) \right] \\ & \quad - \epsilon \left[\nabla_{\alpha}^{\perp} \cdot \mathcal{A}^{(0)} \nabla_y^{\perp} \psi_0^{(1)} + \nabla_{\alpha}^{\perp} \cdot (\mathcal{H}^{(0)})^\top \left(\kappa^{-1} D_{\alpha} \mathbf{x}^{(0)} \nabla_{\alpha}^{\perp} \psi^{(0)} \right) \right] \\ & = \sigma_0(\boldsymbol{\alpha}, \mathbf{y}) + \epsilon [\sigma_1(\boldsymbol{\alpha}, \mathbf{y}) + \varrho(\boldsymbol{\alpha})] + \mathcal{O}(\epsilon^2). \end{aligned} \tag{0.18}$$

By matching the terms of $O(1)$ in (0.18), we obtain (0.9). Then by averaging (0.18) with respect to \mathbf{y} and using $\langle \mathcal{H}^{(0)} \rangle = D_{\alpha} \mathbf{x}^{(0)}$, we derive (0.8). This completes the proof of Proposition B.

Proof of Theorem 3.1 First, $\mathbf{x}^{(0)} = \mathbf{x}^{(0)}(t, \boldsymbol{\alpha})$ is a direct consequence of Proposition A and (3.9) follows from Proposition B. We now prove the rest of Theorem 3.1. We divide the proof into three steps:

Step 1. Derivation of (3.8). First, by matching the terms of $O(1)$ in (3.4), we have that

$$\partial_t \mathbf{x}^{(0)} + \partial_{\tau} (\bar{\mathbf{x}}^{(1)} + \mathbf{x}^{(1)}) + D_{\alpha} \mathbf{x}^{(0)} + D_y \mathbf{x}^{(1)} \left(\nabla_{\alpha}^{\perp} \psi^{(0)} + \nabla_y^{\perp} \psi^{(1)} \right) = \mathbf{0}. \tag{0.19}$$

On the other hand, (0.15) shows that

$$D_{\alpha\mathbf{x}}^{(0)} + D_{y\mathbf{x}}^{(1)} [\nabla_{\alpha}^{\perp}\psi^{(0)} + \nabla_y^{\perp}(\boldsymbol{\eta} \cdot \nabla_{\alpha}\psi^{(0)})] = \kappa^{-1}D_{\alpha\mathbf{x}}^{(0)}\nabla_{\alpha}^{\perp}\psi^{(0)}. \tag{0.20}$$

Recall that $\psi^{(1)} = \psi_0^{(1)} + \boldsymbol{\eta} \cdot \nabla_{\alpha}\psi^{(0)}$. Using this relation and (0.20), we get

$$\begin{aligned} & D_{\alpha\mathbf{x}}^{(0)} + D_{y\mathbf{x}}^{(1)} \nabla_{\alpha}^{\perp}\psi^{(0)} + \nabla_y^{\perp}\psi^{(1)} \\ &= \kappa^{-1}D_{\alpha\mathbf{x}}^{(0)}\nabla_{\alpha}^{\perp}\psi^{(0)} + D_{\alpha\mathbf{x}}^{(0)} + D_{y\mathbf{x}}^{(1)} \nabla_y^{\perp}\psi_0^{(1)}. \end{aligned} \tag{0.21}$$

Substituting (0.21) into (0.19) gives

$$\begin{aligned} & \partial_t\mathbf{x}^{(0)} + \partial_{\tau} \bar{\mathbf{x}}^{(1)} + \mathbf{x}^{(1)} + \kappa^{-1}D_{\alpha\mathbf{x}}^{(0)}\nabla_{\alpha}^{\perp}\psi^{(0)} \\ &+ D_{\alpha\mathbf{x}}^{(0)} + D_{y\mathbf{x}}^{(1)} \nabla_y^{\perp}\psi_0^{(1)} = \mathbf{0}. \end{aligned} \tag{0.22}$$

Averaging (0.22) with respect to \mathbf{y} leads to

$$\begin{aligned} (a) \quad & \partial_t\mathbf{x}^{(0)} + \partial_{\tau}\bar{\mathbf{x}}^{(1)} + \kappa^{-1}D_{\alpha\mathbf{x}}^{(0)}\nabla_{\alpha}^{\perp}\psi^{(0)} = \mathbf{0}, \\ (b) \quad & \partial_{\tau}\mathbf{x}^{(1)} + D_{\alpha\mathbf{x}}^{(0)} + D_{y\mathbf{x}}^{(1)} \nabla_y^{\perp}\psi_0^{(1)} = \mathbf{0}. \end{aligned} \tag{0.23}$$

(0.23)(b) is equivalent to (3.8).

Step 2. Derivation of (3.7). By averaging (0.19) with respect to \mathbf{y} , we have

$$\partial_t\mathbf{x}^{(0)} + \partial_{\tau}\bar{\mathbf{x}}^{(1)} + D_{\alpha\mathbf{x}}^{(0)}\nabla_{\alpha}^{\perp}\psi^{(0)} = \mathbf{0}. \tag{0.24}$$

Comparing (0.24) with (0.23)(a), we obtain

$$\kappa^{-1}D_{\alpha\mathbf{x}}^{(0)}\nabla_{\alpha}^{\perp}\psi^{(0)} = D_{\alpha\mathbf{x}}^{(0)}\nabla_{\alpha}^{\perp}\psi^{(0)}. \tag{0.25}$$

Substituting the above equality into (0.8) gives (3.7).

Step 3. Derivation of (3.5) and (3.6).

From (0.24), we see that (3.5) and (3.6) follow from the fact that $D_{\alpha\mathbf{x}}^{(0)}\nabla_{\alpha}^{\perp}\psi^{(0)}$ is independent of τ . Let

$$\begin{aligned} (a) \quad & \mathbf{u} = -D_{\alpha\mathbf{x}}^{(0)}\nabla_{\alpha}^{\perp}\psi^{(0)}, \\ (b) \quad & \mathbf{w} = - D_{\alpha\mathbf{x}}^{(0)} + D_{y\mathbf{x}}^{(1)} \nabla_y^{\perp}\psi_0^{(1)}. \end{aligned} \tag{0.26}$$

It follows from (0.8), (0.9) and (0.23)(b) that

$$\begin{aligned} (a) \quad & \nabla_{\alpha}^{\perp} \cdot (D_{\alpha\mathbf{x}}^{(0)})^{\top}\mathbf{u} + \langle (\mathcal{H}^{(0)})^{\top}\mathbf{w} \rangle - \mathbf{U} = 0, \\ (b) \quad & \nabla_y^{\perp} \cdot (\mathcal{H}^{(0)})^{\top}\mathbf{w} - \mathbf{W} = 0, \\ (c) \quad & \partial_{\tau}\mathbf{x}^{(1)} = \mathbf{w}. \end{aligned} \tag{0.27}$$

Equations (0.27)(a) and (0.27)(b) imply

$$\begin{aligned} (a) \quad & (D_{\alpha\mathbf{x}}^{(0)})^{\top}\mathbf{u} + \langle (\mathcal{H}^{(0)})^{\top}\mathbf{w} \rangle = \mathbf{U} + \nabla_{\alpha}p, \\ (b) \quad & (\mathcal{H}^{(0)})^{\top}\mathbf{w} - \langle (\mathcal{H}^{(0)})^{\top}\mathbf{w} \rangle = \mathbf{W} + \nabla_yq, \end{aligned} \tag{0.28}$$

for some functions p and q . Summing (0.28)(a) and (0.28)(b) gives

$$(\mathcal{H}^{(0)})^{\top}\mathbf{w} = \mathbf{W} + \mathbf{U} + \nabla_{\alpha}p + \nabla_yq - (D_{\alpha\mathbf{x}}^{(0)})^{\top}\mathbf{u}. \tag{0.29}$$

Differentiating (0.29) with respect to τ implies

$$(\partial_{\tau}\mathcal{H}^{(0)})^{\top}\mathbf{w} + (\mathcal{H}^{(0)})^{\top}\partial_{\tau}\mathbf{w} = \nabla_{\alpha}\partial_{\tau}p + \nabla_y\partial_{\tau}q - (D_{\alpha\mathbf{x}}^{(0)})^{\top}\partial_{\tau}\mathbf{u}. \tag{0.30}$$

Note that $\partial_\tau \mathcal{H}^{(0)} = D_y \partial_\tau \mathbf{x}^{(1)} = D_y \mathbf{w}, D_y \mathbf{w}^\top (\mathbf{w} + \mathbf{u}) = \frac{1}{2} \nabla_y |\mathbf{w} + \mathbf{u}|^2$. Moreover, using (0.11) we get

$$\begin{aligned} (D_\alpha \mathbf{x}^{(0)})^\top \partial_\tau \mathbf{u} &= (\mathcal{H}^{(0)})^\top \partial_\tau \mathbf{u} - (D_y \mathbf{x}^{(1)})^\top \partial_\tau \mathbf{u} \\ &= (\mathcal{H}^{(0)})^\top \partial_\tau \mathbf{u} - \partial_\tau ((D_y \mathbf{x}^{(1)})^\top \mathbf{u}) + (D_y (\partial_\tau \mathbf{x}^{(1)}))^\top \mathbf{u} \\ &= (\mathcal{H}^{(0)})^\top \partial_\tau \mathbf{u} - \partial_\tau \left(\sum_{k=1}^2 \nabla_y x_k^{(1)} u_k \right) + (D_y \mathbf{w})^\top \mathbf{u}, \end{aligned}$$

where we have used (0.27)(c) in the last step. Substituting the above equation into (0.30) would give

$$\begin{aligned} &\partial_\tau \mathbf{w} + \partial_\tau \mathbf{u} \\ &= (\mathcal{H}^{(0)})^{-\top} \left[\nabla_\alpha \partial_\tau p + \nabla_y \left(\partial_\tau q + \sum_{k=1}^2 x_k^{(1)} u_k \right) - \frac{1}{2} |\mathbf{w} + \mathbf{u}|^2 \right]. \end{aligned} \tag{0.31}$$

By averaging (0.31) with respect to \mathbf{y} , we get the equation for (\mathbf{u}, p) with respect to (τ, α) :

$$\begin{aligned} (a) \quad &\partial_\tau \mathbf{u} + (D_\alpha \mathbf{x}^{(0)})^{-\top} \nabla_\alpha (\partial_\tau p) = \mathbf{0}, \\ (b) \quad &\nabla_\alpha \cdot ((D_\alpha \mathbf{x}^{(0)})^{-1} \mathbf{u}) = 0. \end{aligned} \tag{0.32}$$

Further, (0.32)(b) implies $\nabla_\alpha \cdot ((D_\alpha \mathbf{x}^{(0)})^{-1} \partial_\tau \mathbf{u}) = 0$. Note that

$$\int_{\mathbb{R}^2} \partial_\tau \mathbf{u} \cdot \left[(D_\alpha \mathbf{x}^{(0)})^{-\top} \nabla_\alpha (\partial_\tau p) \right] d\alpha = - \int_{\mathbb{R}^2} \left[\nabla_\alpha \cdot (D_\alpha \mathbf{x}^{(0)})^{-1} \partial_\tau \mathbf{u} \right] \partial_\tau p d\alpha = 0.$$

Multiplying (0.32)(a) with $\partial_\tau \mathbf{u}$ and using the above orthogonality condition, we obtain that

$$\int_{\mathbb{R}^2} (\partial_\tau \mathbf{u} \cdot \partial_\tau \mathbf{u}) d\alpha = 0.$$

This implies $\partial_\tau \mathbf{u} = \mathbf{0}$, i.e., $\mathbf{u} = \mathbf{u}(t, \alpha)$. From the definition of \mathbf{u} in (0.26), we conclude that $\psi^{(0)}$ is also independent of τ , i.e., $\psi^{(0)} = \psi^{(0)}(t, \alpha)$. Now it follows from (0.23)(a) that $\partial_\tau \bar{\mathbf{x}}^{(1)}$ is independent of τ . This shows that $\bar{\mathbf{x}}^{(1)}$ is a linear function with respect to τ , i.e., $\bar{\mathbf{x}}^{(1)} = \mathbf{a}(t, \alpha) + \mathbf{b}(t, \alpha)\tau$. Thus we have

$$\begin{aligned} \mathbf{x}^\epsilon &= \mathbf{x}^{(0)} + \epsilon \mathbf{x}^{(1)} + \epsilon \bar{\mathbf{x}}^{(1)} + \dots \\ &= \mathbf{x}^{(0)} + \epsilon \mathbf{x}^{(1)} + \epsilon \mathbf{a}(t, \alpha) + t \mathbf{b}(t, \alpha) + \dots \end{aligned}$$

since $\tau = t/\epsilon$. The last term in the expansion is now a function of t and α , and is of order $O(1)$. Therefore, we can include it to the $\mathbf{x}^{(0)}$ term. The remaining $\bar{\mathbf{x}}^{(1)}$ is a function of t and α . Hence we have $\bar{\mathbf{x}}^{(1)} = \bar{\mathbf{x}}^{(1)}(t, \alpha)$. This proves (3.5). Finally, (3.6) follows from (0.24) since we have proved that $\partial_\tau \bar{\mathbf{x}}^{(1)} = \mathbf{0}$.

This completes the proof of Theorem 3.1.

Appendix III. Proof of Theorem 3.2 We divide the proof into three steps:

Step 1. Derivation of (3.23) and (3.24). From the expansion (3.14) and (3.16), i.e., $(\bar{D}_x \theta^\epsilon)^{-1} = \mathcal{S}^{(0)} + \mathcal{O}(\epsilon)$, we see that

$$\begin{aligned} \mathbf{u}^\epsilon &= -\bar{\nabla}_x^\perp \psi^\epsilon = - \nabla_x^\perp + \frac{1}{\epsilon} ((D_x \theta^\epsilon)^\top \nabla_y)^\perp \psi^\epsilon = - \nabla_x^\perp + \frac{1}{\epsilon} (D_x \theta^\epsilon)^{-1} \nabla_y^\perp \psi^\epsilon \\ &= - \nabla_x^\perp + \frac{1}{\epsilon} (\mathcal{S}^{(0)} + \mathcal{O}(\epsilon)) \nabla_y^\perp \psi^\epsilon = -\nabla_x^\perp \psi^{(0)} - \mathcal{S}^{(0)} \nabla_y^\perp \psi^{(1)} + \mathbf{O}(\epsilon). \end{aligned}$$

To leading order term, by using $\mathcal{S}^{(0)} = D_x \theta^{-1} (\mathcal{I} - D_y \theta^{(1)})$ in (3.17), we have

$$\mathbf{u}^{(0)} = -\nabla_x^\perp \psi^{(0)} - D_x \theta^{-1} (\mathcal{I} - D_y \theta^{(1)}) \nabla_y^\perp \psi^{(1)}. \tag{0.33}$$

From (0.2), we know that the mean of $D_y \theta^{(1)} \nabla_y^\perp \psi^{(1)}$ is zero such that the mean of $D_x \theta^{-1} (\mathcal{I} - D_y \theta^{(1)}) \nabla_y^\perp \psi^{(1)}$ is zero. Hence (3.23) and (3.24) hold.

Step 2. Derivation of (3.18) and (3.21). Denote the total derivatives, by \bar{D}_x and $\bar{\partial}_t$, and the partial derivatives, by D_x and ∂_t , with respect to \mathbf{x} and t respectively. By using $\bar{\partial}_t \boldsymbol{\theta}^\epsilon + \bar{D}_x \boldsymbol{\theta}^\epsilon \mathbf{u}^\epsilon = \mathbf{0}$, we have

$$\begin{aligned} \bar{\partial}_t \boldsymbol{\theta}^\epsilon + (\mathbf{u}^\epsilon \cdot \bar{\nabla}_x) \boldsymbol{\theta}^\epsilon &= \partial_t + \frac{1}{\epsilon} \partial_\tau \boldsymbol{\theta}^\epsilon + D_x \boldsymbol{\theta}^\epsilon \mathbf{u}^\epsilon + \frac{1}{\epsilon} D_y \boldsymbol{\theta}^\epsilon (\bar{\partial}_t \boldsymbol{\theta}^\epsilon + \bar{D}_x \boldsymbol{\theta}^\epsilon \mathbf{u}^\epsilon) \\ &= \partial_t + \frac{1}{\epsilon} \partial_\tau \boldsymbol{\theta}^\epsilon + D_x \boldsymbol{\theta}^\epsilon \mathbf{u}^\epsilon = \partial_t \boldsymbol{\theta} + (\mathbf{u}^{(0)} \cdot \nabla_x) \boldsymbol{\theta} + \partial_\tau \boldsymbol{\theta}^{(1)} + \mathbf{O}(\epsilon) = \mathbf{0}, \end{aligned} \quad (0.34)$$

By matching the terms of $O(1)$, we have

$$\partial_t \boldsymbol{\theta} + (\mathbf{u}^{(0)} \cdot \nabla_x) \boldsymbol{\theta} + \partial_\tau \boldsymbol{\theta}^{(1)} = \mathbf{0}, \quad (0.35)$$

or equivalently, by using $(\mathbf{u}^{(0)} \cdot \nabla_x) \boldsymbol{\theta} = D_x \boldsymbol{\theta} \mathbf{u}^{(0)}$ and (0.33),

$$\partial_t \boldsymbol{\theta} - D_x \boldsymbol{\theta} \nabla_x^\perp \psi^{(0)} + \partial_\tau \boldsymbol{\theta}^{(1)} - (\mathcal{I} - D_y \boldsymbol{\theta}^{(1)}) \nabla_y^\perp \psi^{(1)} = \mathbf{0}. \quad (0.36)$$

Averaging (0.36) with respect to \mathbf{y} , we obtain

$$\begin{aligned} (a) \quad \partial_t \boldsymbol{\theta} - D_x \boldsymbol{\theta} \nabla_x^\perp \psi^{(0)} &= \mathbf{0}, \quad \boldsymbol{\theta}|_{t=0} = \mathbf{x}, \\ (b) \quad \partial_\tau \boldsymbol{\theta}^{(1)} - (\mathcal{I} - D_y \boldsymbol{\theta}^{(1)}) \nabla_y^\perp \psi^{(1)} &= \mathbf{0}, \quad \boldsymbol{\theta}^{(1)}|_{\tau=t=0} = \mathbf{0}, \end{aligned} \quad (0.37)$$

which gives (3.18) and (3.21).

Step 3. Derivation of (3.19) and (3.22). For each $g = g(\mathbf{x}, \boldsymbol{\theta}^\epsilon / \epsilon)$, it follows from (0.5), (0.6), $|D_x \boldsymbol{\theta}^\epsilon| = 1$ and $D_{\boldsymbol{\theta}^\epsilon} \mathbf{x} = (D_x \boldsymbol{\theta}^\epsilon)^{-1}$ that

$$\begin{aligned} \Delta g &= \frac{1}{\epsilon^2} \nabla_y^\perp \cdot (\bar{D}_x \boldsymbol{\theta}^\epsilon)^{-\top} (\bar{D}_x \boldsymbol{\theta}^\epsilon)^{-1} \nabla_y^\perp g + \frac{1}{\epsilon} \nabla_y^\perp \cdot (\bar{D}_x \boldsymbol{\theta}^\epsilon)^{-\top} \nabla_x^\perp g \\ &\quad + \frac{1}{\epsilon} \nabla_x^\perp \cdot (\bar{D}_x \boldsymbol{\theta}^\epsilon)^{-1} \nabla_y^\perp g + \Delta_x g. \end{aligned} \quad (0.38)$$

Hence, we have that

$$\begin{aligned} -\Delta \psi^\epsilon &= -\frac{1}{\epsilon} \nabla_y^\perp \cdot (\mathcal{S}^{(0)})^\top \mathcal{S}^{(0)} \nabla_y^\perp \psi^{(1)} \\ &\quad - \frac{1}{\epsilon} \nabla_y^\perp \cdot (\mathcal{S}^{(0)})^\top \nabla_x^\perp \psi^{(0)} - \nabla_y^\perp \cdot (\mathcal{S}^{(0)})^\top \mathcal{S}^{(0)} \nabla_y^\perp \psi^{(2)} \\ &\quad - \nabla_y^\perp \cdot ((\mathcal{S}^{(1)})^\top \mathcal{S}^{(0)} + (\mathcal{S}^{(0)})^\top \mathcal{S}^{(1)}) \nabla_y^\perp \psi^{(1)} \\ &\quad - \nabla_y^\perp \cdot (\mathcal{S}^{(0)})^\top \nabla_x^\perp (\bar{\psi}^{(1)} + \psi^{(1)}) - \nabla_y^\perp \cdot (\mathcal{S}^{(1)})^\top \nabla_x^\perp \psi^{(0)} \\ &\quad - \nabla_x^\perp \cdot \mathcal{S}^{(0)} \nabla_y^\perp \psi^{(1)} - \Delta_x \psi^{(0)} + O(\epsilon) \\ &= \omega_{\text{int}}(\boldsymbol{\theta}^\epsilon, \frac{\boldsymbol{\theta}^\epsilon}{\epsilon}) = \omega_{\text{int}}(\boldsymbol{\theta} + \epsilon(\bar{\boldsymbol{\theta}}^{(1)} + \boldsymbol{\theta}^{(1)}) + \mathbf{O}(\epsilon^2), \mathbf{y}). \end{aligned} \quad (0.39)$$

From (3.17) that $\mathcal{S}^{(0)} = D_x \boldsymbol{\theta}^{-1} (\mathcal{I} - D_y \boldsymbol{\theta}^{(1)})$ and (0.3), we have

$$\begin{aligned} \nabla_y^\perp \cdot (\mathcal{S}^{(0)})^\top \nabla_x^\perp \psi^{(0)} &= -\nabla_y^\perp \cdot (D_y \boldsymbol{\theta}^{(1)})^\top D_x \boldsymbol{\theta}^{-\top} \nabla_x^\perp \psi^{(0)} \\ &= -(D_y \boldsymbol{\theta}^{(1)} \nabla_y^\perp) \cdot D_x \boldsymbol{\theta}^{-\top} \nabla_x^\perp \psi^{(0)} = 0. \end{aligned} \quad (0.40)$$

On the other hand, we have that

$$\omega_{\text{int}}(\boldsymbol{\theta}^\epsilon, \boldsymbol{\theta}^\epsilon / \epsilon) = \varrho(\boldsymbol{\theta}^\epsilon) + \nabla_{\boldsymbol{\theta}^\epsilon}^\perp \cdot \mathbf{W}(\boldsymbol{\theta}^\epsilon, \boldsymbol{\theta}^\epsilon / \epsilon) \quad (0.41)$$

and, by using (0.6) and $\bar{D}_x \boldsymbol{\theta}^\epsilon = \mathcal{B}^{(0)} + \mathcal{O}(\epsilon)$,

$$\begin{aligned} \nabla_{\bar{\boldsymbol{\theta}}^\epsilon}^\perp \cdot \mathbf{W}(\boldsymbol{\theta}^\epsilon, \frac{\boldsymbol{\theta}^\epsilon}{\epsilon}) &= \bar{\nabla}_x^\perp \cdot ((\bar{D}_x \boldsymbol{\theta}^\epsilon)^\top \mathbf{W}(\boldsymbol{\theta}^\epsilon, \frac{\boldsymbol{\theta}^\epsilon}{\epsilon})) \\ &= \nabla_x^\perp + \frac{1}{\epsilon} (\bar{D}_x \boldsymbol{\theta}^\epsilon)^{-1} \nabla_y^\perp \cdot ((\bar{D}_x \boldsymbol{\theta}^\epsilon)^\top \mathbf{W}(\boldsymbol{\theta}^\epsilon, \frac{\boldsymbol{\theta}^\epsilon}{\epsilon})) \\ &= \frac{1}{\epsilon} \nabla_y^\perp \cdot \mathbf{W}(\boldsymbol{\theta}^\epsilon, \mathbf{y}) + \nabla_x^\perp \cdot ((\bar{D}_x \boldsymbol{\theta}^\epsilon)^\top \mathbf{W}(\boldsymbol{\theta}^\epsilon, \mathbf{y})) \\ &= \frac{1}{\epsilon} \nabla_y^\perp \cdot \mathbf{W}(\boldsymbol{\theta}, \mathbf{y}) + \nabla_x^\perp \cdot ((\mathcal{B}^{(0)})^\top \mathbf{W}(\boldsymbol{\theta}, \mathbf{y})) \\ &\quad + \nabla_y^\perp \cdot D_\theta \mathbf{W}(\boldsymbol{\theta}, \mathbf{y}) (\bar{\boldsymbol{\theta}}^{(1)} + \boldsymbol{\theta}^{(1)}) + \mathcal{O}(\epsilon). \end{aligned} \tag{0.42}$$

Substituting (0.40), (0.41) and (0.42) into (0.39) and matching the terms of $\mathcal{O}(\frac{1}{\epsilon})$, we get

$$-\nabla_y^\perp \cdot (\mathcal{S}^{(0)})^\top \mathcal{S}^{(0)} \nabla_y^\perp \psi^{(1)} = \nabla_y^\perp \cdot \mathbf{W}(\boldsymbol{\theta}^\epsilon, \mathbf{y}) = \sigma_0(\boldsymbol{\theta}, \mathbf{y}). \tag{0.43}$$

(3.22) follows from (0.43).

Then by averaging (0.39) with respect to \mathbf{y} , we have

$$-\Delta_x \psi^{(0)} = \varrho(\boldsymbol{\theta}) + \nabla_x^\perp \cdot \langle (\mathcal{B}^{(0)})^\top \mathbf{W}(\boldsymbol{\theta}, \cdot) \rangle. \tag{0.44}$$

From $|\bar{D}_x \boldsymbol{\theta}^\epsilon| = |D_x \boldsymbol{\theta}| = 1$ and (3.16) that $\bar{D}_x \boldsymbol{\theta}^\epsilon = \mathcal{B}^{(0)} + \mathcal{O}(\epsilon)$ and $\mathcal{B}^{(0)} = (\mathcal{I} - D_y \boldsymbol{\theta}^{(1)})^{-1} D_x \boldsymbol{\theta}$, we know

$$|\mathcal{I} - D_y \boldsymbol{\theta}^{(1)}| = 1 + \mathcal{O}(\epsilon). \tag{0.45}$$

Since the inverse of a matrix is the product of its adjoint matrix and the reciprocal of its determinant, we have

$$(\mathcal{I} - D_y \boldsymbol{\theta}^{(1)})^{-1} = \mathcal{I} - \begin{pmatrix} \partial_{y_2} \theta_2^{(1)} & -\partial_{y_2} \theta_1^{(1)} \\ -\partial_{y_1} \theta_2^{(1)} & \partial_{y_1} \theta_1^{(1)} \end{pmatrix} + \mathcal{O}(\epsilon) = \mathcal{I} - \mathcal{D} + \mathcal{O}(\epsilon) \tag{0.46}$$

Thus we derive that $\langle \mathcal{B}^{(0)} \rangle = D_x \boldsymbol{\theta}$ and $\langle (\mathcal{B}^{(0)})^\top \mathbf{W}(\boldsymbol{\theta}, \cdot) \rangle = -D_x \boldsymbol{\theta}^\top \langle \mathcal{D}^\top \mathbf{W}(\boldsymbol{\theta}, \cdot) \rangle$. Substituting this into (0.44) leads to (3.19).

This completes the proof of Theorem 3.2.

Appendix IV. Derivation of Expansions (3.26)-(3.27)

Derivation of (3.26). Since

$$\mathbf{z} = \mathbf{y} - \boldsymbol{\theta}^{(1)}(t, \boldsymbol{\theta}, \tau, \mathbf{y}) = \mathcal{G}(\mathbf{y}), \tag{0.47}$$

where $(t, \boldsymbol{\theta}, \tau)$ are parameters, hence

$$D_y \mathbf{z} = \mathcal{I} - D_y \boldsymbol{\theta}^{(1)}. \tag{0.48}$$

(0.45) shows $|\mathcal{I} - D_y \boldsymbol{\theta}^{(1)}| = 1 + \mathcal{O}(\epsilon)$. This implies that the variable change is one-to-one. Let $\widehat{g}(\mathbf{z}) = g(\mathbf{y}) = g \circ \mathcal{G}^{-1}$. The inverse transform of (0.47) is

$$\mathbf{y} = \mathbf{z} + \widehat{\boldsymbol{\theta}}^{(1)}(t, \boldsymbol{\theta}, \tau, \mathbf{z}). \tag{0.49}$$

by using (0.5) and $\mathcal{S}^{(0)} = D_x \boldsymbol{\theta}_x^{-1} (\mathcal{I} - D_y \boldsymbol{\theta}^{(1)})$ in (3.17), we see that for each function g

$$\begin{aligned} \mathcal{S}^{(0)} \nabla_y^\perp g \circ \mathcal{G}^{-1} &= D_x \boldsymbol{\theta}^{-1} (\mathcal{I} - D_y \boldsymbol{\theta}^{(1)}) \nabla_y^\perp g \circ \mathcal{G}^{-1} \\ &= D_x \boldsymbol{\theta}^{-1} D_y \mathbf{z} \nabla_y^\perp g \circ \mathcal{G}^{-1} = D_x \boldsymbol{\theta}^{-1} \nabla_z^\perp \widehat{g} + \mathcal{O}(\epsilon). \end{aligned} \tag{0.50}$$

On the other hand, it follows from (0.3) and (0.6) that for each $\phi(\mathbf{y})$

$$\begin{aligned} \nabla_y^\perp \cdot (\mathcal{S}^{(0)})^\top \phi(\mathbf{y}) \circ \mathcal{G}^{-1} &= \nabla_y^\perp \cdot (\mathcal{I} - D_y \boldsymbol{\theta}^{(1)})^\top D_x \boldsymbol{\theta}^{-\top} \phi(\mathbf{y}) \circ \mathcal{G}^{-1} \\ &= (\mathcal{I} - D_y \boldsymbol{\theta}^{(1)}) \nabla_y^\perp \cdot D_x \boldsymbol{\theta}^{-\top} \phi(\mathbf{y}) \circ \mathcal{G}^{-1} \\ &= D_y \mathbf{z} \nabla_y^\perp \cdot D_x \boldsymbol{\theta}^{-\top} \phi(\mathbf{y}) \circ \mathcal{G}^{-1} = \nabla_z^\perp \cdot D_x \boldsymbol{\theta}^{-\top} \widehat{\phi}(\mathbf{z}) + \mathcal{O}(\epsilon). \end{aligned} \tag{0.51}$$

Applying (0.50) and (0.51) to (0.43)(a), we get

$$-\nabla_z^\perp \cdot D_x \boldsymbol{\theta}^{-\top} D_x \boldsymbol{\theta}^{-1} \nabla_z^\perp \widehat{\boldsymbol{\psi}}^{(1)} = \sigma_0(\boldsymbol{\theta}, \mathbf{z} + \widehat{\boldsymbol{\theta}}^{(1)}) + O(\epsilon). \quad (0.52)$$

Matching the terms of $O(1)$ in the above equation leads to (3.26).

Derivation of (3.27). We consider equations for $\widehat{\boldsymbol{\theta}}^{(1)}$. From (0.49), we see $\widehat{\boldsymbol{\theta}}^{(1)} = \boldsymbol{\theta}^{(1)}(t, \boldsymbol{\theta}, \tau, \mathbf{y}) = \boldsymbol{\theta}^{(1)}(t, \boldsymbol{\theta}, \tau, \mathbf{z} + \widehat{\boldsymbol{\theta}}^{(1)})$. Hence, it is clear that

$$\partial_\tau \widehat{\boldsymbol{\theta}}^{(1)} = (\partial_\tau \boldsymbol{\theta}^{(1)}) \circ \mathcal{G}^{-1} + ((D_y \boldsymbol{\theta}^{(1)}) \circ \mathcal{G}^{-1}) \partial_\tau \widehat{\boldsymbol{\theta}}^{(1)}$$

Thus we get from (0.5) and (0.37)(b), i.e., $\partial_\tau \boldsymbol{\theta}^{(1)} = (\mathcal{I} - D_y \boldsymbol{\theta}^{(1)}) \nabla_y^\perp \psi^{(1)}$, that

$$\begin{aligned} ((\mathcal{I} - D_y \boldsymbol{\theta}^{(1)}) \circ \mathcal{G}^{-1}) \partial_\tau \widehat{\boldsymbol{\theta}}^{(1)} &= \partial_\tau \boldsymbol{\theta}^{(1)} \circ \mathcal{G}^{-1} = ((\mathcal{I} - D_y \boldsymbol{\theta}^{(1)}) \nabla_y^\perp \psi^{(1)}) \circ \mathcal{G}^{-1} \\ &= (D_y \mathbf{z} \nabla_y^\perp \psi^{(1)}) \circ \mathcal{G}^{-1} = \nabla_z^\perp \widehat{\boldsymbol{\psi}}^{(1)} + \mathbf{O}(\epsilon). \end{aligned} \quad (0.53)$$

It follows from (0.48) and (0.49) that

$$(\mathcal{I} - D_y \boldsymbol{\theta}^{(1)})^{-1} \circ \mathcal{G}^{-1} = (D_y \mathbf{z})^{-1} = D_z \mathbf{y} \circ \mathcal{G}^{-1} = \mathcal{I} + D_z \widehat{\boldsymbol{\theta}}^{(1)}.$$

Substituting this into (0.53) leads to

$$\partial_\tau \widehat{\boldsymbol{\theta}}^{(1)} = ((\mathcal{I} - D_y \boldsymbol{\theta}^{(1)})^{-1} \circ \mathcal{G}^{-1}) \nabla_z^\perp \widehat{\boldsymbol{\psi}}^{(1)} + \mathbf{O}(\epsilon) = (\mathcal{I} + D_y \widehat{\boldsymbol{\theta}}^{(1)}) \nabla_z^\perp \widehat{\boldsymbol{\psi}}^{(1)} + \mathbf{O}(\epsilon).$$

Matching the terms of $\mathbf{O}(1)$, we obtain (3.27).

Appendix V. Proofs of Proposition 3.1 and Theorem 3.3 We will derive Proposition 3.1 and Theorem 3.3 respectively.

Derivation of Proposition 3.1 We divide the proof of Property 3.1 into two steps.

Step 1. Derivation of (3.32), (3.34) and (3.36). Since

$$\begin{aligned} \bar{\nabla}_x^\perp \psi^\epsilon &= \begin{pmatrix} -\partial_{x_2} \psi^\epsilon - \frac{1}{\epsilon} (\partial_{z_1} \psi^\epsilon \partial_{x_2} \bar{\theta}_1 + \partial_{z_2} \psi^\epsilon \partial_{x_2} \bar{\theta}_2) \\ \partial_{x_1} \psi^\epsilon + \frac{1}{\epsilon} (\partial_{z_1} \psi^\epsilon \partial_{x_1} \bar{\theta}_1 + \partial_{z_2} \psi^\epsilon \partial_{x_2} \bar{\theta}_1) \end{pmatrix} \\ &= \nabla_x^\perp \psi^\epsilon + \frac{1}{\epsilon} \begin{pmatrix} \partial_{x_2} \bar{\theta}_2 & -\partial_{x_2} \bar{\theta}_1 \\ -\partial_{x_2} \bar{\theta}_1 & \partial_{x_1} \bar{\theta}_1 \end{pmatrix} \nabla_z^\perp \psi^\epsilon = \nabla_x^\perp + \frac{1}{\epsilon} \mathcal{H} \nabla_z^\perp \psi^\epsilon \end{aligned} \quad (0.54)$$

with $\mathcal{H} = (-\nabla_x^\perp \bar{\theta}_2, \nabla_x^\perp \bar{\theta}_1)$, hence

$$\mathbf{u}^\epsilon = -\bar{\nabla}_x^\perp \psi^\epsilon = -\nabla_x^\perp + \frac{1}{\epsilon} \mathcal{H} \nabla_z^\perp \psi^\epsilon = -\nabla_x^\perp \bar{\psi} - \mathcal{H} \nabla_z^\perp \bar{\psi} - \epsilon \nabla_x^\perp \bar{\psi}. \quad (0.55)$$

This shows that the expression (3.32) of the velocity field holds. Then, substituting the expansion (3.30) into (2.9), we get

$$\bar{\partial}_t \bar{\boldsymbol{\theta}} + D_x \bar{\boldsymbol{\theta}} \mathbf{u}^\epsilon + \partial_\tau \bar{\boldsymbol{\theta}} + D_z \bar{\boldsymbol{\theta}} [\bar{\partial}_t \bar{\boldsymbol{\theta}} + D_x \bar{\boldsymbol{\theta}} \mathbf{u}^\epsilon] + \epsilon [\partial_t \bar{\boldsymbol{\theta}} + D_x \bar{\boldsymbol{\theta}} \mathbf{u}^\epsilon] = \mathbf{0}. \quad (0.56)$$

Noting that

$$\frac{1}{\epsilon} D_z \bar{\boldsymbol{\theta}} D_x \bar{\boldsymbol{\theta}} \mathbf{u}^\epsilon + D_x \bar{\boldsymbol{\theta}} \mathbf{u}^\epsilon = (\mathbf{u}^\epsilon \cdot \bar{\nabla}_x) \bar{\boldsymbol{\theta}} = \bar{\nabla}_x \cdot (\bar{\boldsymbol{\theta}} \otimes \mathbf{u}^\epsilon) = \nabla_x \cdot (\bar{\boldsymbol{\theta}} \otimes \mathbf{u}^\epsilon) + \frac{1}{\epsilon} (D_x \bar{\boldsymbol{\theta}}^\top \nabla_z) \cdot (\bar{\boldsymbol{\theta}} \otimes \mathbf{u}^\epsilon),$$

where $\bar{\nabla}_x \cdot \mathbf{u}^\epsilon = 0$ is used, we can rewrite (0.56) as

$$\bar{\partial}_t \bar{\boldsymbol{\theta}} + D_x \bar{\boldsymbol{\theta}} \mathbf{u}^\epsilon + \partial_\tau \bar{\boldsymbol{\theta}} + D_z \bar{\boldsymbol{\theta}} \bar{\partial}_t \bar{\boldsymbol{\theta}} + \epsilon \partial_t \bar{\boldsymbol{\theta}} + \epsilon \nabla_x \cdot (\bar{\boldsymbol{\theta}} \otimes \mathbf{u}^\epsilon) + (D_x \bar{\boldsymbol{\theta}}^\top \nabla_z) \cdot (\bar{\boldsymbol{\theta}} \otimes \mathbf{u}^\epsilon) = \mathbf{0}. \quad (0.57)$$

By averaging (0.57) over \mathbf{z} and using (0.55), we have

$$\bar{\partial}_t \bar{\boldsymbol{\theta}} - D_x \bar{\boldsymbol{\theta}} \nabla_x^\perp \bar{\psi} - \epsilon \nabla_x \cdot \left\langle \bar{\boldsymbol{\theta}} \otimes (\mathcal{H} \nabla_z^\perp \bar{\psi} + \epsilon \nabla_x^\perp \bar{\psi}) \right\rangle = \mathbf{0}. \quad (0.58)$$

This proves (3.34).

Thirdly, substituting (0.58) into (0.57), we have

$$\begin{aligned}
 & \partial_\tau \tilde{\boldsymbol{\theta}} - (D_x \tilde{\boldsymbol{\theta}}^\top \nabla_z) \cdot (\tilde{\boldsymbol{\theta}} \otimes (\mathcal{H} \nabla_z^\perp \tilde{\psi})) - D_x \tilde{\boldsymbol{\theta}} \mathcal{H} \nabla_z^\perp \tilde{\psi} \\
 & + D_z \tilde{\boldsymbol{\theta}} D_x \tilde{\boldsymbol{\theta}} \nabla_x^\perp \tilde{\psi} - (D_x \tilde{\boldsymbol{\theta}}^\top \nabla_z) \cdot (\tilde{\boldsymbol{\theta}} \otimes \nabla_x^\perp \tilde{\psi}) \\
 & + \epsilon \partial_t \tilde{\boldsymbol{\theta}} - \epsilon \nabla_x \cdot (\tilde{\boldsymbol{\theta}} \otimes (\nabla_x^\perp \tilde{\psi} + \mathcal{H} \nabla_z^\perp \tilde{\psi} + \epsilon \nabla_x^\perp \tilde{\psi})) \\
 & - \epsilon (D_x \tilde{\boldsymbol{\theta}}^\top \nabla_z) \cdot (\tilde{\boldsymbol{\theta}} \otimes \nabla_x^\perp \tilde{\psi}) - \epsilon D_x \tilde{\boldsymbol{\theta}} \nabla_x^\perp \tilde{\psi} \\
 & + \epsilon (\mathcal{I} + D_z \tilde{\boldsymbol{\theta}}) \nabla_x \cdot \left\langle \tilde{\boldsymbol{\theta}} \otimes (\mathcal{H} \nabla_z^\perp \tilde{\psi} + \epsilon \nabla_x^\perp \tilde{\psi}) \right\rangle = \mathbf{0}.
 \end{aligned} \tag{0.59}$$

By calculating, we see that

$$\begin{aligned}
 D_z \tilde{\boldsymbol{\theta}} D_x \tilde{\boldsymbol{\theta}} \nabla_x^\perp \tilde{\psi} &= \begin{pmatrix} \nabla_z \tilde{\theta}_1 \cdot \partial_{x_1} \tilde{\boldsymbol{\theta}} & \nabla_z \tilde{\theta}_1 \cdot \partial_{x_2} \tilde{\boldsymbol{\theta}} \\ \nabla_z \tilde{\theta}_2 \cdot \partial_{x_1} \tilde{\boldsymbol{\theta}} & \nabla_z \tilde{\theta}_2 \cdot \partial_{x_2} \tilde{\boldsymbol{\theta}} \end{pmatrix} \nabla_x^\perp \tilde{\psi} \\
 &= \begin{pmatrix} (\partial_{x_1} \tilde{\boldsymbol{\theta}} \cdot \nabla_z)(\tilde{\theta}_1(-\partial_{x_2} \tilde{\psi})) + (\partial_{x_2} \tilde{\boldsymbol{\theta}} \cdot \nabla_z)(\tilde{\theta}_1 \partial_{x_1} \tilde{\psi}) \\ (\partial_{x_1} \tilde{\boldsymbol{\theta}} \cdot \nabla_z)(\tilde{\theta}_2(-\partial_{x_2} \tilde{\psi})) + (\partial_{x_2} \tilde{\boldsymbol{\theta}} \cdot \nabla_z)(\tilde{\theta}_2 \partial_{x_1} \tilde{\psi}) \end{pmatrix} \\
 &= (D_x \tilde{\boldsymbol{\theta}}^\top \nabla_z) \cdot (\tilde{\boldsymbol{\theta}} \otimes \nabla_x^\perp \tilde{\psi})
 \end{aligned}$$

Substituting this into (0.59) leads to

$$\begin{aligned}
 & \partial_\tau \tilde{\boldsymbol{\theta}} - (D_x \tilde{\boldsymbol{\theta}}^\top \nabla_z) \cdot (\tilde{\boldsymbol{\theta}} \otimes (\mathcal{H} \nabla_z^\perp \tilde{\psi})) - D_x \tilde{\boldsymbol{\theta}} \mathcal{H} \nabla_z^\perp \tilde{\psi} \\
 & + \epsilon \partial_t \tilde{\boldsymbol{\theta}} - \epsilon \nabla_x \cdot (\tilde{\boldsymbol{\theta}} \otimes (\nabla_x^\perp \tilde{\psi} + \mathcal{H} \nabla_z^\perp \tilde{\psi} + \epsilon \nabla_x^\perp \tilde{\psi})) \\
 & - \epsilon (D_x \tilde{\boldsymbol{\theta}}^\top \nabla_z) \cdot (\tilde{\boldsymbol{\theta}} \otimes \nabla_x^\perp \tilde{\psi}) - \epsilon D_x \tilde{\boldsymbol{\theta}} \nabla_x^\perp \tilde{\psi} \\
 & + \epsilon (\mathcal{I} + D_z \tilde{\boldsymbol{\theta}}) \nabla_x \cdot \left\langle \tilde{\boldsymbol{\theta}} \otimes (\mathcal{H} \nabla_z^\perp \tilde{\psi} + \epsilon \nabla_x^\perp \tilde{\psi}) \right\rangle = \mathbf{0}.
 \end{aligned} \tag{0.60}$$

This is (3.36).

Step 2. Derivation of (3.35) and (3.37). Since $\bar{\nabla}_x^\perp \psi^\epsilon = \nabla_x^\perp + \frac{1}{\epsilon} \mathcal{H} \nabla_z^\perp \psi^\epsilon$, we get

$$\Delta \psi^\epsilon = \frac{1}{\epsilon^2} \nabla_z^\perp \cdot \mathcal{H}^\top \mathcal{H} \nabla_z^\perp \psi^\epsilon + \frac{1}{\epsilon} \nabla_z^\perp \cdot \mathcal{H}^\top \nabla_x^\perp \psi^\epsilon + \frac{1}{\epsilon} \nabla_x^\perp \cdot \mathcal{H} \nabla_z^\perp \psi^\epsilon + \Delta_x \psi^\epsilon,$$

which implies

$$\begin{aligned}
 & -\nabla_z^\perp \cdot \mathcal{H}^\top \mathcal{H} \nabla_z^\perp \tilde{\psi} - \epsilon \nabla_z^\perp \cdot \mathcal{H}^\top \nabla_x^\perp \tilde{\psi} - \epsilon \nabla_x^\perp \cdot \mathcal{H} \nabla_z^\perp \tilde{\psi} - \epsilon \Delta_x (\tilde{\psi} + \epsilon \tilde{\psi}) \\
 & = \epsilon \omega_{\text{int}}(\boldsymbol{\theta}^\epsilon, \frac{\boldsymbol{\theta}^\epsilon}{\epsilon}) = \epsilon \omega_{\text{int}}(\bar{\boldsymbol{\theta}} + \epsilon \tilde{\boldsymbol{\theta}}, \mathbf{z} + \tilde{\boldsymbol{\theta}})
 \end{aligned} \tag{0.61}$$

with $\mathbf{z} = \bar{\boldsymbol{\theta}}/\epsilon$. Averaging the equation (0.61) over \mathbf{z} , we have

$$-\Delta_x \bar{\psi} = \left\langle \omega_{\text{int}}(\bar{\boldsymbol{\theta}} + \epsilon \tilde{\boldsymbol{\theta}}, \mathbf{z} + \tilde{\boldsymbol{\theta}}) \right\rangle. \tag{0.62}$$

Substituting (0.62) into (0.61), we get

$$\begin{aligned}
 & -\nabla_z^\perp \cdot \mathcal{H}^\top \mathcal{H} \nabla_z^\perp \tilde{\psi} - \epsilon \nabla_z^\perp \cdot \mathcal{H}^\top \nabla_x^\perp \tilde{\psi} - \epsilon \nabla_x^\perp \cdot \mathcal{H} \nabla_z^\perp \tilde{\psi} - \epsilon^2 \Delta_x \tilde{\psi} \\
 & = \epsilon \omega_{\text{int}}(\bar{\boldsymbol{\theta}} + \epsilon \tilde{\boldsymbol{\theta}}, \mathbf{z} + \tilde{\boldsymbol{\theta}}) - \epsilon \langle \omega_{\text{int}}(\bar{\boldsymbol{\theta}} + \epsilon \tilde{\boldsymbol{\theta}}, \mathbf{z} + \tilde{\boldsymbol{\theta}}) \rangle.
 \end{aligned} \tag{0.63}$$

This prove (3.35) and (3.37) and completes the derivation of Property 3.1

Derivation of Theorem 3.3

Substituting $\bar{\boldsymbol{\theta}} = \boldsymbol{\theta} + \epsilon \bar{\boldsymbol{\theta}}^{(1)} + \dots$ and $\bar{\psi} = \psi^{(0)} + \epsilon \bar{\psi}^{(1)} + \dots$ into (3.34) (or (0.58)) and then matching the terms of $\mathbf{O}(1)$, we get (3.41).

Substituting $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta}^{(1)} + \epsilon \boldsymbol{\theta}^{(2)} + \dots$ and $\tilde{\psi} = \psi^{(1)} + \epsilon \psi^{(1)} + \dots$ into (3.36) (or (0.60)), and then matching the terms of $\mathbf{O}(1)$, we have

$$\partial_\tau \boldsymbol{\theta}^{(1)} - (D_x \boldsymbol{\theta}^\top \nabla_z) \cdot (\boldsymbol{\theta}^{(1)} \otimes (\mathcal{H}^{(0)} \nabla_z^\perp \psi^{(1)})) - D_x \boldsymbol{\theta} \mathcal{H}^{(0)} \nabla_z^\perp \psi^{(1)} = \mathbf{0}, \tag{0.64}$$

where $\mathcal{H}^{(0)} = (-\nabla_x^\perp \theta_2, \nabla_x^\perp \theta_1)$. Since $\mathcal{H}^{(0)} = D_x \bar{\boldsymbol{\theta}}^{-1}$ and

$$\begin{aligned} (D_x \boldsymbol{\theta}^\top \nabla_z) \cdot (\boldsymbol{\theta}^{(1)} \otimes (\mathcal{H}^{(0)} \nabla_z^\perp \psi^{(1)})) &= \begin{pmatrix} (D_x \boldsymbol{\theta}^\top \nabla_z) \cdot (\theta_1^{(1)} \mathcal{H}^{(0)} \nabla_z^\perp \psi^{(1)}) \\ (D_x \boldsymbol{\theta}^\top \nabla_z) \cdot (\theta_2^{(1)} \mathcal{H}^{(0)} \nabla_z^\perp \psi^{(1)}) \end{pmatrix} \\ &= \begin{pmatrix} \nabla_z \cdot (\theta_1^{(1)} D_x \boldsymbol{\theta} \mathcal{H}^{(0)} \nabla_z^\perp \psi^{(1)}) \\ \nabla_z \cdot (\theta_2^{(1)} D_x \boldsymbol{\theta} \mathcal{H}^{(0)} \nabla_z^\perp \psi^{(1)}) \end{pmatrix} = \begin{pmatrix} \nabla_z \cdot (\theta_1^{(1)} \nabla_z^\perp \psi^{(1)}) \\ \nabla_z \cdot (\theta_2^{(1)} \nabla_z^\perp \psi^{(1)}) \end{pmatrix} \\ &= \begin{pmatrix} \nabla_z \theta_1^{(1)} \cdot \nabla_z^\perp \psi^{(1)} \\ \nabla_z \theta_2^{(1)} \cdot \nabla_z^\perp \psi^{(1)} \end{pmatrix} = D_z \boldsymbol{\theta}^{(1)} \nabla_z^\perp \psi^{(1)} \end{aligned}$$

Substituting these results into (0.64), we get

$$\partial_\tau \boldsymbol{\theta}^{(1)} - (\mathcal{I} + D_z \boldsymbol{\theta}^{(1)}) \nabla_z^\perp \psi^{(1)} = \mathbf{0}. \tag{0.65}$$

This proves (3.43)

Then we consider the right-hand side terms of (0.62). We see

$$\begin{aligned} \nabla_{\boldsymbol{\theta}^\epsilon}^\perp \cdot \mathbf{W}(\boldsymbol{\theta}^\epsilon, \frac{\boldsymbol{\theta}^\epsilon}{\epsilon}) &= \bar{D}_x \boldsymbol{\theta}^\epsilon \bar{\nabla}_x^\perp \cdot \mathbf{W}(\boldsymbol{\theta}^\epsilon, \frac{\boldsymbol{\theta}^\epsilon}{\epsilon}) = \bar{\nabla}_x^\perp \cdot ((\bar{D}_x \boldsymbol{\theta}^\epsilon)^\top \mathbf{W}(\boldsymbol{\theta}^\epsilon, \frac{\boldsymbol{\theta}^\epsilon}{\epsilon})) \\ &= (\nabla_x^\perp + \frac{1}{\epsilon} \mathcal{H} \nabla_z^\perp) \cdot [D_x \bar{\boldsymbol{\theta}}^\top (\mathcal{I} + D_z \tilde{\boldsymbol{\theta}} + \epsilon D_{\tilde{\boldsymbol{\theta}}} \tilde{\boldsymbol{\theta}})^\top \mathbf{W}(\bar{\boldsymbol{\theta}} + \epsilon \tilde{\boldsymbol{\theta}}, \mathbf{z} + \tilde{\boldsymbol{\theta}})]. \end{aligned} \tag{0.66}$$

Averaging this equality over \mathbf{z} , we get

$$\begin{aligned} \nabla_{\boldsymbol{\theta}^\epsilon}^\perp \cdot \mathbf{W}(\boldsymbol{\theta}^\epsilon, \frac{\boldsymbol{\theta}^\epsilon}{\epsilon}) &= \nabla_x^\perp \cdot \left\langle D_x \bar{\boldsymbol{\theta}}^\top (\mathcal{I} + D_z \tilde{\boldsymbol{\theta}} + \epsilon D_{\tilde{\boldsymbol{\theta}}} \tilde{\boldsymbol{\theta}})^\top \mathbf{W}(\bar{\boldsymbol{\theta}} + \epsilon \tilde{\boldsymbol{\theta}}, \mathbf{z} + \tilde{\boldsymbol{\theta}}) \right\rangle \\ &= (D_x \boldsymbol{\theta} \nabla_x^\perp) \cdot \left\langle (\mathcal{I} + D_z \boldsymbol{\theta}^{(1)})^\top \mathbf{W}(\boldsymbol{\theta}, \mathbf{z} + \boldsymbol{\theta}^{(1)}) \right\rangle + O(\epsilon). \end{aligned} \tag{0.67}$$

Thus we have

$$\langle \omega_{\text{int}}(\bar{\boldsymbol{\theta}} + \epsilon \tilde{\boldsymbol{\theta}}, \mathbf{z} + \tilde{\boldsymbol{\theta}}) \rangle = \varrho(\boldsymbol{\theta}) + (D_x \boldsymbol{\theta} \nabla_x^\perp) \cdot \left\langle (\mathcal{I} + D_z \boldsymbol{\theta}^{(1)})^\top \mathbf{W}(\boldsymbol{\theta}, \mathbf{z} + \boldsymbol{\theta}^{(1)}) \right\rangle + O(\epsilon).$$

Substituting this into the equation (0.62) and then matching the terms of $O(1)$ leads to (3.42).

Finally, since

$$\epsilon [\omega_{\text{int}}(\bar{\boldsymbol{\theta}} + \epsilon \tilde{\boldsymbol{\theta}}, \mathbf{z} + \tilde{\boldsymbol{\theta}}) - \langle \omega_{\text{int}}(\bar{\boldsymbol{\theta}} + \epsilon \tilde{\boldsymbol{\theta}}, \mathbf{z} + \tilde{\boldsymbol{\theta}}) \rangle] = \sigma_0(\boldsymbol{\theta}, \mathbf{z} + \boldsymbol{\theta}^{(1)}) + O(\epsilon).$$

where $\sigma_0(\mathbf{x}, \mathbf{z}) = \nabla_z^\perp \cdot \mathbf{W}(\mathbf{x}, \mathbf{z})$, hence substituting this into the equation (0.63) and then matching the terms of $O(1)$ leads to (3.44)

This completes the proof of Theorem 3.3.

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