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Improved Geometric Conditions for Non-Blowup of the 3D Incompressible Euler Equation

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This is a follow-up of our recent article Deng et al. (2004). In Deng et al. (2004), we derive some local geometric conditions on vortex filaments which can prevent finite time blowup of the 3D incompressible Euler equation. In this article, we derive improved geometric conditions which can be applied to the scenario when velocity blows up at the same time as vorticity and the rate of blowup of velocity is proportional to the square root of vorticity. This scenario is in some sense the worst possible blow-up scenario for velocity field due to Kelvin's circulation theorem. The improved conditions can be checked by numerical computations. This provides a sharper local geometric constraint on the finite time blowup of the 3D incompressible Euler equation.

Keywords 3D Euler equations; Finite time blowup; Geometric properties; Global existence.

Mathematics Subject Classification Primary 76B03; Secondary 35L60, 35M10.

1. Introduction

The question of global existence/blowup of smooth solutions for the 3D Euler equation

$$\begin{aligned} u_t + (u \cdot \nabla)u &= -\nabla p, \\ \nabla \cdot u &= 0, \\ u|_{t=0} &= u_0, \end{aligned} \tag{1.1}$$

has been one of the long standing open problems. This question has attracted many researchers and a number of partial results have been obtained, see e.g., Ebin et al. (1970), Beale et al. (1984), Caflisch (1993), Constantin et al. (1996), Tadmor (2001), and Babin et al. (2001). But the definite answer to this challenging question is still open.

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In Beale et al. (1984), it is shown that a smooth solution $u(x, t)$ of (1.1) blows up at $t = T$ if and only if $\int_0^t \|\omega(\cdot, s)\|_\infty ds \nearrow \infty$ as $t \nearrow T$, where $\omega \equiv \nabla \times u$ is the vorticity. Many numerical studies have been conducted to study the possible finite time blowup of the 3D incompressible Euler equation, see, for example, Kerr (1993, 1995, 1996, 1997, 1998), Pelz (2001), and Grauer et al. (1998). Most of these studies suggest a growth rate $(T - t)^{-1}$ for $\|\omega\|_\infty$, which is the critical case of the Beale-Kato-Majda's criterion, implying a finite time singularity at time T.

Since numerical computations can never reach the blowup time T, even if finite time singularities do exist, it is useful to derive other complementary conditions which may exclude a finite-time blowup. In Constantin et al. (1996), one such condition involving the regularity of the direction of vorticity field in an order one region containing the maximum vorticity is given. More recently, Deng et al. (2004) obtained a set of more localized conditions on the geometric properties of vortex filaments which prevent a finite-time blowup. More specifically, we consider a family of vortex line segments L_t, along which the maximum vorticity is comparable to the maximum vorticity $\|\omega\|_\infty$. Denote by $L(t)$ the arc length of L_t, ξ the unit vorticity vector, $\xi \equiv \omega/|\omega|$, n the unit normal vector, and κ the curvature of the vortex line. Further, we define

$$U_\xi(t) \equiv \max_{x,y \in L_t} |(u \cdot \xi)(x, t) - (u \cdot \xi)(y, t)|,$$

$U_n(t) \equiv \max_{L_t} |u \cdot n|$, and $M(t) \equiv \max(\|\nabla \cdot \xi\|_{L^\infty(L_t)}, \|\kappa\|_{L^\infty(L_t)})$. With these notations, the main result in Deng et al. (2004) can be summarized as follows.

Let $A, B \in (0, 1)$ with $B < 1 - A$, and C_0 be a positive constant. If

(1) $U_\xi(t) + U_n(t)M(t)L(t) \lesssim (T - t)^{-A}$,
(2) $M(t)L(t) \leq C_0$,
(3) $L(t) \gtrsim (T - t)^B$,

then there will be no blowup up to time T.

The above result to some extent improves the previous results obtained by Constantin et al. (1996) and Cordoba and Fefferman (2001). On the one hand, our result requires a very localized and weaker assumption on the regularity of the vorticity vector ξ. In Constantin et al. (1996), the L^∞ norm of $\nabla\xi$ is assumed to be L^2 integrable in time in an $O(1)$ region containing the maximum vorticity. In contrast, we only assume that the divergence of the vorticity vector is integrable along a local vortex line segment and $L(t)\|\kappa\|_{L^\infty(L_t)}$ is bounded. The length of the vortex line segment, $L(t)$, can shrink to zero as the time approaches to the alleged singularity time.

The numerical computations by Kerr (1993, 1995, 1996, 1997, 1998), and Pelz (2001) have demonstrated that there is indeed a small region in which vorticity attains its global maximum and the vorticity vector has some partial regularity. However, the size of this region shrinks rapidly to zero in a rate proportional to some inverse power of maximum vorticity. While there is still a debate whether velocity field should blow up at the singularity time, recent numerical computations by Kerr (1997) suggest that the velocity field blowup at the singularity time with a growth rate proportional to the square root of the maximum vorticity. Kerr's computations also indicate that the maximum velocity of the flow is located on the boundary of the inner active region where maximum vorticity is attained. Moreover, he observes that the maximum velocity blows up like $(T - t)^{-1/2}$ and the length of the "relatively straight" inner vortex tube, denoted as $L(t)$ in the above notation, scales like $(T-t)^{1/2}$. If we consider the worst scenario in which $U_\xi(t)$ as defined above also blows up like $(T-t)^{-1/2}$, then we have $B = A = 1/2$, which is the critical case in the assumptions stated above and is not covered by our previous result.

In this article, we improve the previous result of Deng et al. (2004), and obtain sharper non-blowup conditions for the 3D incompressible Euler equation which include the critical case of $B = 1 - A$ (in particular, the case of $B = A = 1/2$ discussed above is included). This is made possible by improving our estimate on the relation between the vorticity growth and arc length stretching, and optimizing our dynamic estimate on the growth of maximum vorticity. Our improved non-blowup conditions are consistent with the state-of-the-art numerical computations, and can be applied to check whether finite time singularities would develop in an actual numerical computation. In order to apply our new result, we need to check whether the scaling constants in the variables of interest satisfy certain relationship. In the next section, we will state precisely this relationship and give some specific examples when the scaling parameters vary within certain regime.

Notations

Throughout this article, we will reserve some characters for several particular quantities according to the following rules of notations:

(i) ξ is always the direction of vorticity vectors, i.e., $\xi \equiv \omega/|\omega|$. Since we are considering small tubes where large vorticity concentrates, ξ is always well defined.
(ii) $\Omega(t)$ always denotes the maximum vorticity in the whole 3D space, that is, $\Omega(t) \equiv \|\omega(\cdot, t)\|_{L^\infty(\mathbb{R}^3)}$.
(iii) T will always denote the alleged time when a finite time blowup occurs.
(iv) $X(\alpha, \tau, t)$ denotes the particle path that passes α at time τ. That is, $X(\alpha, \tau, t)$ solves

$$\frac{\partial X(\alpha, \tau, t)}{\partial t} = u(X(\alpha, \tau, t), t),$$
$$X(\alpha, \tau, \tau) = \alpha.$$

For any set $A \subset \mathbb{R}^3$, we denote $X(A, \tau, t) = \bigcup_{\alpha \in A} X(\alpha, \tau, t)$. When $\tau = 0$, we use the conventional notation, $X(\alpha, t) \equiv X(\alpha, 0, t)$ (see e.g., Chorin and Marsden, 1993).

2. Main Result and Its Implications

In this section, we present our main result. As in Beale et al. (1984), we assume that the initial velocity field, u_0, is smooth and vanishes rapidly at infinity, for example $u_0 \in H^4(\mathbb{R}^3)$. We consider a family of vortex line segments L_t along which the maximum vorticity is comparable to $\Omega(t)$. Denote by $L(t)$ the arc length of L_t. Variables $U_\xi(t)$, $U_n(t)$, $M(t)$, κ, and n are defined as in Section 1. We should point out that, in general, we will not have $L_t = X(L_{t'}, t', t)$ for $t' < t$.

Now we can state our main result.

Theorem 1. *Assume that there is a family of vortex line segments* L_t *and* $T_0 \in [0, T)$, *such that* $X(L_{t_1}, t_1, t_2) \supseteq L_{t_2}$ *for all* $T_0 < t_1 < t_2 < T$. *Also assume that* $\Omega(t)$ *is* *monotonically increasing and* $\|\omega(t)\|_{L^\infty(L_t)} \geq c_0\Omega(t)$ *for some* $c_0 > 0$ *when* $t \in [T_0, T)$. *Furthermore, we assume that, there is* $A \in (0, 1)$ *and positive constants* C_U, C_0, c_L, *such that*

1. $[U_\xi(t) + U_n(t)M(t)L(t)] \leq C_U(T-t)^{-A}$,
2. $M(t)L(t) \leq C_0$,
3. $L(t) \geq c_L(T-t)^{1-A}$,

then there will be no blowup in the 3*D incompressible Euler flowup to time* T, *as long as the following condition is satisfied*:

$$R^3K < y_1(R^{A-1}(1-A)^{1-A}/(2-A)^{2-A}), \tag{2.2}$$

where $R = e^{C_0}/c_0$ *and* $K \equiv \frac{C_U c_0}{c_L(1-A)}$, *and* $y_1(m)$ *denotes the smallest positive* y *such that*

$$\frac{y}{(y+1)^{2-A}} = m.$$

Remark 1. The same result holds if we replace the first assumption in the above theorem by $U_\xi(t) + U_n(t) \lesssim (T-t)^{-A}$ for some $A \in (0, 1)$, since this assumption combined with the second assumption (2) will give us the first assumption (1) in the theorem.

In the remaining of this section, we discuss how we can apply the above theorem to actual numerical computations. In most numerical studies of 3D Euler singularities, it has been observed that the maximum vorticity blows up like $(T-t)^{-1}$. On the other hand, Kelvin's circulation theorem suggests that maximum velocity be bounded from above by the square root of maximum vorticity. Thus $A = 1/2$ is in some sense the worst blowup scenario for velocity filed if we consider the $(T-t)^{-1}$ blowup for vorticity field as generic. If we follow the vortex filament along which the maximum vorticity is attained, then we have $c_0 = 1$. Thus it is of practical interest to study Condition (2.2) for the case of $A = 1/2$ and $c_0 = 1$ and investigate the parameter range for C_0, C_U, c_L in which no finite time blowup would occur.

In the case of $A = 1/2$ and $c_0 = 1$, Theorem 1 implies that if

$$e^{3C_0}K < y_1\left(e^{-C_0/2}\frac{2}{3^{3/2}}\right),$$

then there will be no finite time singularity up to time T. We can rewrite the condition as

$$K < K_{\max}(C_0) \equiv e^{-3C_0} \cdot y_1(2e^{-C_0/2}/3^{3/2}),$$

where y_1 is the smallest positive number such that

$$\frac{y}{(y+1)^{3/2}} = \frac{2}{3^{3/2}}e^{-C_0/2}.$$

One can easily obtain $K_{\max}$ by solving (either numerically or analytically) the cubic equation

$$\frac{2}{3^{3/2}}e^{-C_0/2}(y+1)^3 - y^2 = 0,$$

Table 1

C_0	0.05	0.1	0.15	0.2	0.25	0.3
$K_{\max}(C_0)$	1.1770	0.8682	0.6644	0.5180	0.4088	0.3253

$K_{\max}$ as a function of C_0.

for each $C_0 > 0$. In the Table 1, we approximate $K_{\max}(C_0)$ numerically for some values of C_0.

Next we apply Theorem 1 to Kerr's computations. In a sequence of articles (Kerr, 1993, 1995, 1996, 1997, 1998), Kerr observed that when t is close enough to the alleged blowup time T, the region bounded by the contour of $0.6\|\omega\|_\infty$, i.e., the active region (Kerr, 1997), looks like two vortex sheets with thickness $\sim (T-t)$ meeting at an angle (Kerr, 1996). This active region has length scale $(T-t)^{1/2}$ in the vorticity direction. The maximum vorticity resides in the small tube-like region, with scaling $(T-t)^{1/2} \times (T-t) \times (T-t)$, which is the intersection of the two sheets. Inside the active region, vortex lines are "relatively straight" (Kerr, 1997). Thus, condition (3) in the theorem is satisfied, and we have $L(t) \geq c_L(T-t)^{1/2}$ for some $c_L > 0$. Since this observation is made according to the rescaled picture of vortex lines, it is likely that both the curvature κ and $\nabla \cdot \xi$ in this region are bounded by $(T-t)^{-1/2}$. In this case, condition (2) is satisfied. It is also observed that the maximum velocity of the flow is located on the boundary of the active region, that is $(T-t)^{1/2}$ away from the maximum vorticity, and grows like $(T-t)^{-1/2}$. If we take the worst scenario that $U_\xi(t)$ also blows up like $(T-t)^{-1/2}$, then we will have $A = 1/2$ in our theorem and conditions (1)–(3) in the theorem are all satisfied. Since the vortex lines are "relatively straight", we can expect C_0 to be quite small. If we take $C_0 \leq 0.1$ to be a reasonable guess, then there will be no finite time blowup if C_U and c_L satisfy the constraint

$$\frac{C_U}{c_L} \leq 0.4341.$$

In Deng et al. (2004), we argue that U_ξ is smaller than the maximum velocity field due to the local cancellation in the velocity kernel. Thus, if C_0 is small, it is reasonable to expect that C_U is also small. Currently, there are no numerical measurements of C_U and c_L available. Whether the scaling constants, c_L, C_U, etc, satisfy Condition (2.2) in Theorem 1 is still unknown. In a subsequent work, we plan to perform careful numerical studies to obtain accurate measurements for these scaling constants.

3. Proofs of the Main Result

In this section, we prove Theorem 1. Before we present the proof, we first state some known results from Deng et al. (2004) that will be useful for the proof of our main theorem.

3.1. *Preparatory Results*

In this subsection we recall some useful results from Deng et al. (2004). For the sake of completeness, we will give the key points in the proofs of these results. The detailed proofs of these results can be found in Deng et al. (2004).

First, we have the following lemma, which relates—through the incompressibility condition—the vortex line geometry to the magnitude of vorticity.

Lemma 1. *Let* $\xi(x,t) \equiv \frac{\omega(x,t)}{|\omega(x,t)|}$ *be the direction of the vorticity vector. Assume at a fixed time* $t > 0$ *the vorticity* $\omega(x,t)$ *is* C^1 *in* x*. We denote*

$$N = \{x \in \mathbb{R}^3 : \omega(x,t) \neq 0\}.$$

Then at this time t*, for any* $x \in N$*, there holds*

$$\frac{\partial|\omega|}{\partial s}(x,t) = -(\nabla \cdot \xi(x,t))|\omega|(x,t),$$

where s *is the arc length variable along the vortex line passing* x*. Furthermore, for any* y *that is on the same vortex line segment as* x*, we have*

$$|\omega(y,t)| = |\omega(x,t)| \cdot e^{\int_x^y (-\nabla\cdot\xi)(s,t)ds},$$

as long as the vortex line segment connecting x *and* y *lies in* N*, where the integration is along the vortex line.*

Proof. The proof is straightforward. First, note that since $\nabla \cdot \omega \equiv \nabla \cdot (\nabla \times u) \equiv 0$, we have

$$\begin{aligned} 0 = \nabla \cdot \omega &= \nabla \cdot (|\omega|\xi) \\ &= (\xi \cdot \nabla)|\omega| + (\nabla \cdot \xi)|\omega| \\ &= \frac{\partial}{\partial s}|\omega| + (\nabla \cdot \xi)|\omega|. \end{aligned}$$

Where s is the arc length parameter. Now it is easy to obtain the lemma by solving this ordinary differential equation along the vortex filament. □

Now we consider one vortex line segment l_t with length $l(t)$ transported by the flow. If we denote

$$m(t) \equiv \max(\|\nabla \cdot \xi\|_{L^\infty(l_t)}, \|\kappa\|_{L^\infty(l_t)}), \tag{3.3}$$

then by Lemma 1, for any two points $x, y \in l_t$, we have

$$e^{-m(t)l(t)} \leq \frac{|\omega(x,t)|}{|\omega(y,t)|} \leq e^{m(t)l(t)}, \tag{3.4}$$

if the vorticity does not vanish on any point in l_t.

Next we have the following lemma, which relates vorticity growth to vortex line stretching. For any starting time t_1 and some time $t > t_1$, consider the evolution of a vortex line. Let s and β be the arc length parameters of this vortex line at time t and t_1, respectively. We can write, for this very vortex line, $s = s(\beta, t)$. Note that $s(\beta, t_1) = \beta$.

Lemma 2. *For any point α at time t_1 such that $\omega(\alpha, t_1) \neq 0$, let $X(\alpha, t_1, t)$ be the position of the same particle at time $t \geq t_1$. Then we have*

$$\frac{\partial s}{\partial \beta}(X(\alpha, t_1, t), t) = \frac{|\omega(X(\alpha, t_1, t), t)|}{|\omega(\alpha, t_1)|}. \tag{3.5}$$

Proof. First, by translation we can set $t_1 = 0$, and use the notation $X(\alpha, t) \equiv X(\alpha, 0, t)$. It is well known that

$$\omega(X(\alpha, t), t) = \nabla_\alpha X(\alpha, t) \cdot \omega(\alpha, 0).$$

Therefore we have

$$\begin{aligned} |\omega(X(\alpha, t), t)| &= \frac{\partial X(\alpha, t)}{\partial s} \cdot \nabla_\alpha X(\alpha, t) \cdot \frac{\partial \alpha}{\partial \beta} |\omega(\alpha, 0)| \\ &= \frac{\partial X(\alpha, t)}{\partial s} \cdot \frac{\partial X(\alpha, t)}{\partial \beta} |\omega(\alpha, 0)| \\ &= \frac{\partial s}{\partial \beta} |\omega(\alpha, 0)|, \end{aligned}$$

where β is the arc length parameter at time $t = 0$, and we have used the fact that

$$\frac{\partial X(\alpha, t)}{\partial s} \cdot \frac{\partial X(\alpha, t)}{\partial s} = \xi \cdot \xi = 1. \qquad \square$$

In Deng et al. (2004), we prove that

$$\frac{D(s_\beta)}{Dt} = (u \cdot \xi)_\beta - \kappa (u \cdot n) s_\beta, \tag{3.6}$$

where $\kappa = |\frac{\partial \xi}{\partial s}| = |(\xi \cdot \nabla)\xi|$ is the curvature of the vortex line, and $n = \kappa^{-1} \frac{\partial \xi}{\partial s}$ is the normal direction. Then, by integrating (3.6) along one Lagrangian vortex line segment $l_\tau = X(l_{t_1}, t_1, \tau)$, and then from time t_1 to t, we obtain

$$l(t) \leq l(t_1) + \int_{t_1}^{t} [U_\xi(\tau) + m(\tau) l(\tau) U_n(\tau)] d\tau. \tag{3.7}$$

Next we give a sharper version of Lemma 3 in Deng et al. (2004) to relate (3.7) to the growth of the vorticity.

Lemma 3. *For any t_1, let l_t be a vortex line segment that is carried by the flow, i.e., $l_t = X(l_{t_1}, t_1, t)$, $t > t_1$. Denote its length by $l(t)$, and let $m(t)$ be defined as in* (3.3). *If we further denote $\Omega_l(t) \equiv \max_{x \in l_t} |\omega(x, t)|$, then*

$$e^{-m(t) l(t)} \frac{\Omega_l(t)}{\Omega_l(t_1)} \leq \frac{l(t)}{l(t_1)} \leq e^{m(t_1) l(t_1)} \frac{\Omega_l(t)}{\Omega_l(t_1)}. \tag{3.8}$$

Proof. Let β denote the arc length parameter at time t_1, and s denote the arc length parameter at time t. Let the two ends points of l_{t_1} be β_1 and β_2. Then we have

$$\begin{aligned} l(t) &= \int_{\beta_1}^{\beta_2} s_\beta \, d\beta \\ &= \int_{\beta_1}^{\beta_2} \left| \frac{\omega(X(\alpha, t_1, t), t)}{\omega(\alpha, t_1)} \right| d\beta \\ &\le \int_{\beta_1}^{\beta_2} \frac{\Omega_l(t)}{e^{-m(t_1)l(t_1)} \Omega_l(t_1)} d\beta \\ &= e^{m(t_1)l(t_1)} \frac{\Omega_l(t)}{\Omega_l(t_1)} l(t_1). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} l(t) &= \int_{\beta_1}^{\beta_2} s_\beta \, d\beta \\ &= \int_{\beta_1}^{\beta_2} \left| \frac{\omega(X(\alpha, t_1, t), t)}{\omega(\alpha, t_1)} \right| d\beta \\ &\ge \int_{\beta_1}^{\beta_2} \frac{e^{-m(t)l(t)} \Omega_l(t)}{\Omega_l(t_1)} d\beta \\ &= e^{-m(t)l(t)} \frac{\Omega_l(t)}{\Omega_l(t_1)} l(t_1). \end{aligned}$$

Thus ends the proof. Note that in the above derivation we have used (3.4). □

By combining (3.7) and (3.8), we obtain

$$\Omega_l(t) \le e^{m(t)l(t)} \Omega_l(t_1) \left[1 + \frac{1}{l(t_1)} \int_{t_1}^{t} (U_\xi(\tau) + m(\tau) U_n(\tau) l(\tau)) d\tau \right], \tag{3.9}$$

where $\Omega_l(t) \equiv \max_{x \in l_t} |\omega(x, t)|$. If we use (3.8) one more time, we get

$$\frac{\Omega_l(t)}{\Omega_l(t_1)} \le e^{m(t)l(t)} \left[1 + e^{m(t_1)l(t_1)} \frac{\Omega_l(t)}{\Omega_l(t_1)} \frac{1}{l(t)} \int_{t_1}^{t} (U_\xi(\tau) + m(\tau) U_n(\tau) l(\tau)) d\tau \right]. \tag{3.10}$$

This is the key estimate in the proof of our main theorem.

3.2. *Proof of the Main Theorem*

This subsection is devoted to the proof of Theorem 1. The proof relies heavily on (3.10) which bounds the ratio of maximum vorticity at two different times by local properties of the vorticity and velocity fields. If vorticity blew up at a finite time T, then for any constant $r > 1$, we could divide the time interval $[0, T)$ into an infinite number of subintervals, $[t_k, t_{k+1})$, in which the maximum vorticity increases geometrically, i.e., $\Omega(t_{k+1}) = r\Omega(t_k)$. By using our assumptions in Theorem 1, we have

$$\int_{t_1}^{T} \Omega(t)dt \le \sum_{k=1}^{\infty} \Omega(t_{k+1})(t_{k+1} - t_k) \le \Omega(t_1) \sum_{k=1}^{\infty} r^k (T - t_k).$$

The key of our proof is to show the existence of one particular $r > 1$ such that the corresponding t_k converges to T so fast that $\limsup_{k\to\infty} \frac{r^{k+1}(T-t_{k+1})}{r^k(T-t_k)} = \limsup_{k\to\infty} \frac{r(T-t_{k+1})}{(T-t_k)} < 1$, which makes the summation finite, and thus get a contradiction.

Proof of Theorem 1. We prove Theorem 1 by contradiction. First, by translating the initial time we can assume that the assumptions in Theorem 1 hold in $[0, T)$. Let $r > (e^{C_0}/c_0)$ be some constant that will be fixed later, where C_0 is the constant in the theorem such that $M(t)L(t) \le C_0$ for all $t \in [0, T)$, and c_0 is the constant such that $\Omega_L(t) \ge c_0 \Omega(t)$. Throughout the proof we denote $\Omega_L(t) \equiv \|\omega(\cdot, t)\|_{L^\infty(L_t)}$.

If there were a finite time blowup at time T, we would have

$$\int_0^T \Omega(t)dt = \infty,$$

or equivalently for any $t_1 \in [0, T)$,

$$\int_{t_1}^T \Omega(t)dt = \infty.$$

Then, necessarily, we have $\Omega(t) \nearrow \infty$ as $t \nearrow T$. Now we can take a time sequence $t_1, t_2, \ldots, t_n, \ldots$ such that

$$\Omega(t_{k+1}) = r\Omega(t_k), \tag{3.11}$$

where $r > R/c_0$ is a constant to be fixed later. Since $\Omega(t)$ is monotone, and T is the smallest time such that $\int_0^T \Omega(t)dt = \infty$, it is obvious that $t_n \nearrow T$ as $n \to \infty$.

Now we choose $l_{t_2} = L_{t_2}$. By our assumptions on L_t, there is $l_{t_1} \subset L_{t_1}$ such that $X(l_{t_1}, t_1, t_2) = l_{t_2}$. This is a crucial step to our theorem and we illustrate it in the following Figure 1.

In the above graph, segments AB and $C'D'$ are L_{t_1} and L_{t_2}. Since by our assumptions, L_t is shrinking with time, the flow image of AB, denoted by $A'B'$ will be much longer than $C'D'$. Note that the segments $A'C'$ and $D'B'$ do not have a good bound for $\nabla\xi$. Now our choice of l_{t_1} and l_{t_2} is the following. We take l_{t_2} to be $C'D'$. Then l_{t_1} has to be the preimage of l_{t_2} and is denoted by CD.

If we further denote

$$\Omega_l(t_i) \equiv \|\omega(\cdot, t_i)\|_{L^\infty(l_{t_i})} \qquad i = 1, 2,$$

we have

$$\Omega_l(t_2) = \Omega_L(t_2), \qquad \Omega_l(t_1) \ge e^{-C_0}\Omega_L(t_1).$$

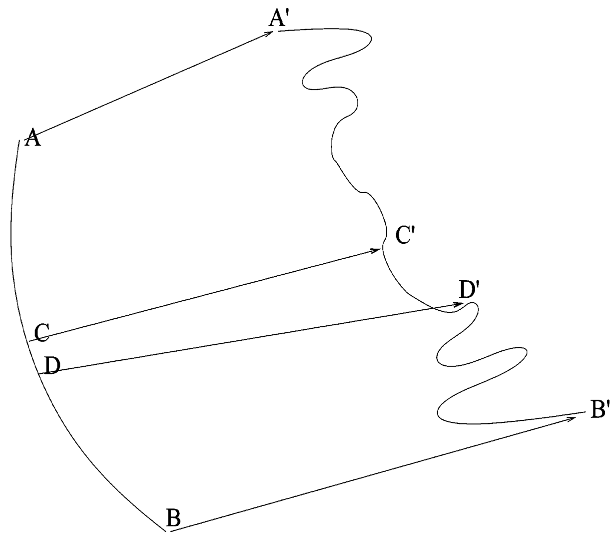

Figure 1. Illustration of L_t.

Now by taking $t = t_2$ in (3.10), we would have

$$\frac{\Omega_L(t_2)}{\Omega_L(t_1)} \le e^{C_0}\left[1 + e^{2C_0}\frac{\Omega_L(t_2)}{\Omega_L(t_1)}\frac{1}{L(t_2)}\int_{t_1}^{t_2}(U_\xi(\tau) + M(\tau)U_n(\tau)L(\tau))d\tau\right].$$

By the assumptions of Theorem 1, we have $\Omega_L(t) \ge c_0\Omega(t)$. Thus, we get

$$\begin{aligned} r = \frac{\Omega(t_2)}{\Omega(t_1)} &\le \frac{e^{C_0}}{c_0}\left[1 + e^{2C_0}\frac{\Omega(t_2)}{\Omega(t_1)}\frac{1}{L(t_2)c_0}\int_{t_1}^{t_2}(U_\xi(\tau) + M(\tau)U_n(\tau)L(\tau))d\tau\right] \\ &\le R + R^3Kr\left[\left(\frac{T-t_1}{T-t_2}\right)^{1-A} - 1\right], \end{aligned}$$

where

$$R = \frac{e^{C_0}}{c_0}, \qquad K = \frac{C_U c_0}{c_L(1-A)}.$$

Now we have

$$\left(\frac{T-t_1}{T-t_2}\right)^{1-A} \ge \frac{(R^3K+1)r - R}{R^3Kr},$$

which gives

$$\frac{T-t_2}{T-t_1} \le \left(\frac{R^3Kr}{(R^3K+1)r - R}\right)^{1/(1-A)}.$$

The same argument applies to each pair (t_k, t_{k+1}). This gives

$$\frac{T - t_{k+1}}{T - t_k} \le \left(\frac{R^3 K r}{(R^3 K + 1)r - R}\right)^{1/(1-A)}.$$

We will get a contradiction if we can find $r > R$ such that

$$\sum_{k=1}^{\infty} \Omega(t_k)(t_{k+1} - t_k) < \infty.$$

Since

$$\int_{t_1}^{T} \Omega(t)dt \le \sum_{k=1}^{\infty} \Omega(t_{k+1})(t_{k+1} - t_k) \le \Omega(t_1) \sum_{k=1}^{\infty} r^k (T - t_k),$$

a sufficient condition would be

$$\limsup_{k\to\infty} \frac{r(T - t_{k+1})}{T - t_k} < 1.$$

This is satisfied if there exists $r > R$ such that

$$\frac{r^{2-A} R^3 K}{(R^3 K + 1)r - R} < 1. \tag{3.12}$$

In the appendix (see Lemma 4), we will show that the existence of such r is equivalent to (2.2):

$$R^3 K < y_1(R^{A-1}(1 - A)^{1-A}/(2 - A)^{2-A}),$$

with $y_1(m)$ as defined below (2.2). This completes the proof of Theorem 1.

Appendix: A Technical Result

In this appendix we prove the following technical result.

Lemma 4. *There exists $r > R$ such that* (3.12) *is satisfied if and only if*

$$R^3 K < y_1(R^{A-1}(1 - A)^{1-A}/(2 - A)^{2-A}), \tag{3.13}$$

where $y_1(m)$ is defined as in Theorem 1.

Proof. Define

$$f(r) \equiv \frac{r^{2-A} R^3 K}{(R^3 K + 1)r - R},$$

and

$$r_c \equiv \frac{2 - A}{1 - A} \frac{R}{R^3 K + 1}.$$

Since

$$f'(r) = \frac{r^{1-A}R^3K}{((R^3K+1)r-R)^2}\big[(1-A)(R^3K+1)r-(2-A)R\big]$$

is negative for $r < r_c$ and positive for $r > r_c$, we conclude that r_c is the only minimizer of $f(r)$ in (R, ∞). Furthermore, since

$$f(R) = \frac{R^{2-A}R^3K}{(R^3K+1)R-R} = R^{1-A} > 1$$

due to $R = e^{C_0}/c_0 > 1$, condition (3.12) is equivalent to

1. $r_c > R$ and
2. $f(r_c) < 1$.

Next we investigate the above two conditions. The first condition $r_c > R$ is just

$$\frac{2-A}{1-A}\frac{R}{R^3K+1} > R,$$

which reduces to

$$R^3K < \frac{1}{1-A}. \tag{3.14}$$

As for the second condition, after some algebra, we can rewrite it as

$$\frac{R^3K}{(R^3K+1)^{2-A}} < R^{A-1}\frac{(1-A)^{1-A}}{(2-A)^{2-A}}. \tag{3.15}$$

Now let $y = R^3K$ and consider $g(y) \equiv \frac{y}{(y+1)^{2-A}}$. We study its behavior on $\mathbb{R}^+$. It is easy to see that $g(0) = g(+\infty) = 0$. By simple calculations, we have

$$\begin{aligned} g'(y) &= \frac{1}{(y+1)^{4-2A}}\big[(y+1)^{2-A} - (2-A)y(y+1)^{1-A}\big] \\ &= \frac{(y+1)^{1-A}}{(y+1)^{4-2A}}(1-(1-A)y). \end{aligned}$$

Thus it is clear that $g(y)$ is increasing in $\left(0, \frac{1}{1-A}\right)$, decreasing in $\left(\frac{1}{1-A}, +\infty\right)$, and reaches its only maximum at $y = \frac{1}{1-A}$. Since

$$g\left(\frac{1}{1-A}\right) = \frac{(1-A)^{1-A}}{(2-A)^{2-A}},$$

and $R^{1-A} > 1$, there exist two values y_1 and y_2, satisfying $y_2 > \frac{1}{1-A} > y_1 > 0$, such that

$$g(y_1) = g(y_2) = R^{A-1}\frac{(1-A)^{1-A}}{(2-A)^{2-A}},$$

and for all other $y > 0$, $g(y) \neq R^{A-1}\frac{(1-A)^{1-A}}{(2-A)^{2-A}}$.

Now it is easy to see that the two conditions (3.14) and (3.15) are equivalent to

1. $R^3K < \frac{1}{1-A}$, and
2. $R^3K < y_1$ or $R^3K > y_2$.

Since $y_1 < \frac{1}{1-A} < y_2$, conditions (1) and (2) above are equivalent to

$$R^3K < y_1,$$

where y_1 is the smallest $y > 0$ such that $f(y) \equiv \frac{y}{(y+1)^{2-A}} = R^{A-1}\frac{(1-A)^{1-A}}{(2-A)^{2-A}}$. This completes the proof of Lemma 4. □

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