

STABLE NEARLY SELF-SIMILAR BLOWUP OF THE 2D BOUSSINESQ AND 3D EULER EQUATIONS WITH SMOOTH DATA II: RIGOROUS NUMERICS

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ABSTRACT. This is Part II of our paper in which we prove finite time blowup of the 2D Boussinesq and 3D axisymmetric Euler equations with smooth initial data of finite energy and boundary. In Part I of our paper [14], we establish an analytic framework to prove nonlinear stability of an approximate self-similar blowup profile using a combination of weighted L^∞ and weighted $C^{1/2}$ energy estimates. We reduce proving nonlinear stability to verifying several inequalities for the constants in the energy estimate which depend on the approximate steady state and the weights in the energy functional only. In Part II of our paper, we construct approximate space-time solutions with rigorous error control, which are used to obtain sharp stability estimates of the linearized operator in Part I. We also obtain sharp estimates of the regular part of the velocity using numerical integration with computer assistance. These results enable us to verify that the constants in the energy estimate obtained in Part I [14] indeed satisfy the inequalities for nonlinear stability. The nonlinear stability further implies the finite time singularity of the axisymmetric 3D Euler equations with smooth initial data and boundary.

1. INTRODUCTION

The three dimensional incompressible Euler equations are one of the most fundamental nonlinear partial differential equations that govern the motion of the ideal inviscid fluid flow. It is closely related to the incompressible Navier-Stokes equations. Due to the presence of nonlinear vortex stretching, the global regularity of the 3D incompressible Euler equations with smooth initial data and finite energy has been one of the longstanding open questions in nonlinear partial differential equations. Let \mathbf{u} be the divergence free velocity field and we define $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ as the *vorticity vector*. The 3D Euler equations governing the vorticity $\boldsymbol{\omega}$ are given by

$$(1.1) \quad \boldsymbol{\omega}_t + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \mathbf{u},$$

where \mathbf{u} is related to $\boldsymbol{\omega}$ via the *Biot-Savart law*. The velocity gradient $\nabla \mathbf{u}$ formally has the same scaling as vorticity $\boldsymbol{\omega}$. Thus the vortex stretching term, $\boldsymbol{\omega} \cdot \nabla \mathbf{u}$, has a nonlocal quadratic nonlinearity in terms of vorticity. Although many experts tend to believe that the 3D Euler equations would form a finite time singularity from smooth initial data, the nonlocal nature of the vortex stretching term could lead to dynamic depletion of nonlinearity, thus preventing a finite time blowup, see e.g. [21, 24, 37]. The interested readers may consult the excellent surveys [20, 31, 35, 39, 44] and the references therein.

Our work is inspired by the computation of Luo-Hou [42, 43] in which they presented some convincing numerical evidence that the 3D axisymmetric Euler equations with smooth initial data and boundary develop a potential finite time singularity. In Part I of our paper [14], we establish an analytic framework and obtain the essential stability estimates to prove finite time singularity of the 2D Boussinesq and 3D axisymmetric Euler equations with smooth initial data and boundary. The main results of this paper are stated by the two informal theorems below. The more precise and stronger statement of Theorem 1 can be found in Theorem 3 in Section 2.

Theorem 1. *Let θ , \mathbf{u} and $\boldsymbol{\omega}$ be the density, velocity and vorticity in the 2D Boussinesq equations (2.3)-(2.5), respectively. There is a family of smooth initial data $(\theta_0, \boldsymbol{\omega}_0)$ with $\theta_0(x, y)$ being even and $\boldsymbol{\omega}_0(x, y)$ being odd in x , such that the solution of the 2D Boussinesq equations develops a singularity in finite time $T < +\infty$. The velocity field \mathbf{u}_0 has finite energy. The blowup solution $(\theta(t), \boldsymbol{\omega}(t))$ is nearly self-similar in the sense that $(\theta(t), \boldsymbol{\omega}(t))$ with suitable dynamic rescaling is*

close to an approximate blowup profile $(\bar{\theta}, \bar{\omega})$ up to the blowup time. Moreover, the blowup is stable for initial data (θ_0, ω_0) close to $(\bar{\theta}, \bar{\omega})$ in some weighted L^∞ and $C^{1/2}$ norm.

Theorem 2. *Consider the 3D axisymmetric Euler equations in the cylinder $r, z \in [0, 1] \times \mathbb{T}$. Let u^θ and ω^θ be the angular velocity and angular vorticity, respectively. The solution of the 3D Euler equations (2.1)-(2.2) develops a nearly self-similar blowup (in the sense described in Theorem 1) in finite time for some smooth initial data $\omega_0^\theta, u_0^\theta$ supported away from the symmetry axis $r = 0$. The initial velocity field has finite energy, u_0^θ and ω_0^θ are odd and periodic in z . The blowup is stable for initial data $(u_0^\theta, \omega_0^\theta)$ that are close to the approximate blowup profile $(\bar{u}^\theta, \bar{\omega}^\theta)$ after proper rescaling subject to some constraint on the initial support size.*

We first review some main ideas in our stability analysis of the linearized operator presented in Part I [14]. We use the 2D Boussinesq system as an example. Let $\bar{\omega}, \bar{\theta}$ be an approximate steady state of the dynamic rescaling formulation. We denote $W = (\omega, \theta_x, \theta_y)$ and decompose $W = \bar{W} + \widetilde{W}$ with $\bar{W} = (\bar{\omega}, \bar{\theta}_x, \bar{\theta}_y)$. We further denote by \mathcal{L} the linearized operator around \bar{W} that governs the perturbation \widetilde{W} in the dynamic rescaling formulation (see Section 2):

$$(1.2) \quad \widetilde{W}_t = \mathcal{L}(\widetilde{W}).$$

We decompose the linearized operator \mathcal{L} into a leading order operator \mathcal{L}_0 plus a finite rank perturbation operator \mathcal{K} , i.e $\mathcal{L} = \mathcal{L}_0 + \mathcal{K}$. The leading order operator \mathcal{L}_0 is constructed in such way that we can obtain sharp stability estimates using weighted estimates and sharp functional inequalities.

In Part I [14], we have performed the weighted energy estimates using a combination of weighted L^∞ and $C^{1/2}$ norm. In our analysis, we decompose $\widetilde{W} = \widetilde{W}_1 + \widetilde{W}_2$, where \widetilde{W}_1 is the main part of the perturbation, which is essentially governed by the leading order operator \mathcal{L}_0 with a weak coupling to \widetilde{W}_2 through nonlinear interaction. The perturbation \widetilde{W}_2 captures the contribution from the finite rank operator. The key is to show that the energy estimate of the main part \widetilde{W}_1 satisfies the inequalities stated in our stability Lemma 2.1 (see Section 2). For this purpose, we need to obtain relatively sharp energy estimates for the leading order operator \mathcal{L}_0 by subtracting a finite rank operator \mathcal{K} . Without subtracting the finite rank operator, we would not be able to obtain linear and nonlinear stability of the approximate self-similar profile.

The constants in the weighted energy estimates obtained in Part I [14] depend on the approximate self-similar profile that we constructed numerically in Section 7 of Part I [14] and the singular weights we use. In this paper and in the supplementary material [12], we will provide sharp and rigorous upper bounds for these constants by estimating the higher order derivatives and then using interpolation estimates from numerical analysis. We also obtain sharp estimates of the regular part of the velocity, which is more regular than the vorticity, by bounding various integrals using numerical integration with computer assistance. These sharp estimates of the constants enable us to prove that the inequalities in our stability lemma hold for our approximate self-similar profile. Thus we can complete the stability analysis of the approximate self-similar profile and complete our blowup analysis for the 2D Boussinesq and 3D Euler equations. See Section 2.2 for more discussion of the main steps in our blowup analysis.

We use the following toy model to illustrate the main ideas of our stability analysis by considering \mathcal{K} as a rank-one operator $\mathcal{K}(\widetilde{W}) = a(x)P(\widetilde{W})$ for some operator P satisfying (i) $P(\widetilde{W})$ is constant in space; (ii) $\|P(\widetilde{W})\| \leq c\|\widetilde{W}\|$. Given initial data \widetilde{W}_0 , we decompose (1.2) as follows

$$(1.3) \quad \begin{aligned} \partial_t \widetilde{W}_1(t) &= \mathcal{L}_0 \widetilde{W}_1, & \widetilde{W}_1(0) &= \widetilde{W}_0, \\ \partial_t \widetilde{W}_2(t) &= \mathcal{L} \widetilde{W}_2 + a(x)P(\widetilde{W}_1(t)), & \widetilde{W}_2(0) &= 0. \end{aligned}$$

It is easy to see that $\widetilde{W} = \widetilde{W}_1 + \widetilde{W}_2$ solves (1.2) with initial data \widetilde{W}_0 since $\mathcal{L} = \mathcal{L}_0 + a(x)P$. By construction, the leading operator \mathcal{L}_0 has the desired structure that enables us to obtain sharp stability estimates. The second part \widetilde{W}_2 is driven by the rank-one forcing term $a(x)P(\widetilde{W}_1(t))$.

Using Duhamel's principle and the fact that $P(\widetilde{W}_1(t))$ is constant in space, we yield

$$(1.4) \quad \widetilde{W}_2(t) = \int_0^t P(\widetilde{W}_1(s))e^{\mathcal{L}(t-s)}a(x)ds.$$

If \widetilde{W}_1 is linearly stable in some $L^\infty(\varphi)$ space, by checking the decay of $e^{\mathcal{L}(t)}a(x)$ in the energy space for large t , we can obtain the stability estimate of \widetilde{W}_2 . Note that $e^{\mathcal{L}(t)}a(x)$ is equivalent to solving the linear evolution equation $v_t = \mathcal{L}(v)$ with initial data $v_0 = a(x)$. We can solve this initial value problem by constructing a space-time solution with rigorous error control.

We remark that our stability analysis is performed mainly for \widetilde{W}_1 since \widetilde{W}_2 is driven by \widetilde{W}_1 . The approximation errors in constructing the space-time approximation to \widetilde{W}_2 can be controlled by the decay estimate of \widetilde{W}_1 . Moreover, the region where we need to modify the linearized operator by a finite rank operator is mainly located in a small sector near the boundary where we have the smallest amount of damping. The total rank is less than 50. In our construction of approximate solution to \widetilde{W}_2 , we need to solve the linear PDE (1.2) in space-time with a number of initial data, which can be implemented in full parallel.

There has been a lot of effort in studying 3D Euler singularities. The most exciting recent development is Elgindi's breakthrough result in which he proved finite time singularity of the axisymmetric Euler equation with no swirl for C^α initial vorticity [25] (see also [26]). In [13], we established finite time blowup of the 2D Boussinesq and the 3D axisymmetric Euler equations with $C^{1,\alpha}$ velocity, large swirl and boundary in a setting similar to the Hou-Luo scenario [42,43]. See also [9] for further developments. Earlier efforts include the Constantin-Lax-Majda (CLM) model [22], the De Gregorio (DG) model [23], the generalized CML (gCLM) model [51] and the Hou-Li model [36]. See also [5–8,16,22,27,29] for the De Gregorio model and for the gCLM model with various parameters. Inspired by their work on the vortex sheet singularity [4], Caffisch and Siegel have studied complex singularity for 3D Euler equation, see [3,55] and also [52] for the complex singularities for 2D Euler equation.

In [18], the authors proved the blowup of the Hou-Luo model proposed in [43]. In [17], Chen-Hou-Huang proved the asymptotically self-similar blowup of the Hou-Luo model by extending the method of analysis established for the finite time blowup of the De Gregorio model by the same authors in [16]. In [19,32–34,40], the authors proposed several simplified models to study the Hou-Luo blowup scenario [42,43] and established finite time blowup of these models. In [28,30], Elgindi and Jeong proved finite time blowup for the 2D Boussinesq and 3D axisymmetric Euler equations in a domain with a corner using $\dot{C}^{0,\alpha}$ data.

The rest of the paper is organized as follows. In Section 2, we review the analytic framework that we established in Part I [14] and state the key lemmas which we use to prove the finite time blowup of the 2D Boussinesq and 3D Euler equations with smooth initial data. In Section 3, we discuss the construction of the approximate space-time solution to the linearized operator \mathcal{L} . This is crucial to obtain sharp estimates of the perturbed operator $\mathcal{L} - \mathcal{K}$ in the stability analysis. In Section 4, we show how to estimate the L^∞ and Hölder norms of the regular part of the velocity. Some technical estimates and derivations are deferred to the Appendix.

2. REVIEW OF THE ANALYTIC FRAMEWORK FROM PART I [14]

In this section, we will review some main ingredients in our analytic framework to establish stability analysis that we presented in Part I [14]. We will mainly focus on the 2D Boussinesq equations since the difference between the 3D Euler and 2D Boussinesq equations is asymptotically small. As in our previous works [13,16,17], we will use the dynamic rescaling formulation for the 2D Boussinesq equations to study the linear stability for the linearized operator around the approximate steady state of the dynamic rescaling equations. Passing from linear stability to nonlinear stability is relatively easier by treating the nonlinear terms and the residual error as small perturbations to the linear damping terms.

Denote by ω^θ , u^θ and ϕ^θ the angular vorticity, angular velocity, and angular stream function, respectively. The 3D axisymmetric Euler equations are given below:

$$(2.1) \quad \partial_t(ru^\theta) + u^r(ru^\theta)_r + u^z(ru^\theta)_z = 0, \quad \partial_t\left(\frac{\omega^\theta}{r}\right) + u^r\left(\frac{\omega^\theta}{r}\right)_r + u^z\left(\frac{\omega^\theta}{r}\right)_z = \frac{1}{r^4}\partial_z((ru^\theta)^2),$$

where the radial velocity u^r and the axial velocity u^θ are given by the Biot-Savart law:

$$(2.2) \quad -(\partial_{rr} + \frac{1}{r}\partial_r + \partial_{zz})\phi^\theta + \frac{1}{r^2}\phi^\theta = \omega^\theta, \quad u^r = -\phi_z^\theta, \quad u^z = \phi_r^\theta + \frac{1}{r}\phi^\theta,$$

with the no-flow boundary condition $\phi^\theta(1, z) = 0$ on the solid boundary $r = 1$ and a periodic boundary condition in z . For 3D Euler blowup that occurs at the boundary $r = 1$, we know that the scaling properties of the axisymmetric Euler equations are asymptotically the same as those of the 2D Boussinesq equations [44]. Thus, we also study the 2D Boussinesq equations on the upper half space:

$$(2.3) \quad \omega_t + \mathbf{u} \cdot \nabla \omega = \theta_x,$$

$$(2.4) \quad \theta_t + \mathbf{u} \cdot \nabla \theta = 0,$$

where the velocity field $\mathbf{u} = (u, v)^T : \mathbb{R}_+^2 \times [0, T] \rightarrow \mathbb{R}_+^2$ is determined via the Biot-Savart law

$$(2.5) \quad -\Delta \phi = \omega, \quad u = -\phi_y, \quad v = \phi_x,$$

where ϕ is the stream function with the no-flow boundary condition $\phi(x, 0) = 0$ at $y = 0$. By making the change of variables $\tilde{\theta} \triangleq (ru^\theta)^2$, $\tilde{\omega} = \omega^\theta/r$, we can see that $\tilde{\theta}$ and $\tilde{\omega}$ satisfy the 2D Boussinesq equations up to the leading order for $r \geq r_0 > 0$.

2.1. Dynamic rescaling formulation. Following [13, 16, 17], we consider the dynamic rescaling formulation of the 2D Boussinesq equations. Let $\omega(x, t)$, $\theta(x, t)$, $\mathbf{u}(x, t)$ be the solutions of (2.3)-(2.5). Then it is easy to show that

$$(2.6) \quad \begin{aligned} \tilde{\omega}(x, \tau) &= C_\omega(\tau)\omega(C_l(\tau)x, t(\tau)), & \tilde{\theta}(x, \tau) &= C_\theta(\tau)\theta(C_l(\tau)x, t(\tau)), \\ \tilde{\mathbf{u}}(x, \tau) &= C_\omega(\tau)C_l(\tau)^{-1}\mathbf{u}(C_l(\tau)x, t(\tau)), \end{aligned}$$

are the solutions to the dynamic rescaling equations

$$(2.7) \quad \tilde{\omega}_\tau(x, \tau) + (c_l(\tau)\mathbf{x} + \tilde{\mathbf{u}}) \cdot \nabla \tilde{\omega} = c_\omega(\tau)\tilde{\omega} + \tilde{\theta}_x, \quad \tilde{\theta}_\tau(x, \tau) + (c_l(\tau)\mathbf{x} + \tilde{\mathbf{u}}) \cdot \nabla \tilde{\theta} = c_\theta\tilde{\theta},$$

where $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v})^T = \nabla^\perp(-\Delta)^{-1}\tilde{\omega}$, $\mathbf{x} = (x, y)^T$,

$$(2.8) \quad C_\omega(\tau) = \exp\left(\int_0^\tau c_\omega(s)d\tau\right), \quad C_l(\tau) = \exp\left(\int_0^\tau -c_l(s)ds\right), \quad C_\theta = \exp\left(\int_0^\tau c_\theta(s)d\tau\right),$$

$t(\tau) = \int_0^\tau C_\omega(\tau)d\tau$ and the rescaling parameters $c_l(\tau)$, $c_\theta(\tau)$, $c_\omega(\tau)$ satisfy [13]

$$(2.9) \quad c_\theta(\tau) = c_l(\tau) + 2c_\omega(\tau).$$

To simplify our presentation, we still use t to denote the rescaled time in (2.7) and simplify $\tilde{\omega}$, $\tilde{\theta}$ as ω , θ

$$(2.10) \quad \omega_t + (c_l x + \mathbf{u}) \cdot \nabla \omega = \theta_x + c_\omega \omega, \quad \theta_t + (c_l x + \mathbf{u}) \cdot \nabla \theta = c_\theta \theta.$$

Following [17], we impose the following normalization conditions on c_ω , c_l

$$(2.11) \quad c_l = 2\frac{\theta_{xx}(0)}{\omega_x(0)}, \quad c_\omega = \frac{1}{2}c_l + u_x(0), \quad c_\theta = c_l + 2c_\omega.$$

For smooth data, these two normalization conditions play the role of enforcing

$$(2.12) \quad \theta_{xx}(t, 0) = \theta_{xx}(0, 0), \quad \omega_x(t, 0) = \omega_x(0, 0)$$

for all time.

We remark that the dynamic rescaling formulation was introduced in [41, 46] to study the self-similar blowup of the nonlinear Schrödinger equations. This formulation is closely related to the modulation technique in the literature and has been developed by Merle, Raphael, Martel, Zaag, and others, see, e.g. [1, 2, 38, 45, 47–49]. Moreover, it is related to the method of modulation equations developed by Soffer and Weinstein [56–58]. Recently, this method has been applied

to study singularity formation in incompressible fluids [13, 25] and related models [6–8, 16]. The more precise statement of our Theorem 1 is stated as follows.

Theorem 3. *Let $(\bar{\theta}, \bar{\omega}, \bar{\mathbf{u}}, \bar{c}_l, \bar{c}_\omega)$ be the approximate self-similar profile constructed in Section 7 of Part I [14] and $E_* = 5 \cdot 10^{-6}$. For initial data $\theta_0(x, y)$ even in x and $\omega_0(x, y)$ odd in x of (2.10) satisfying $E(\omega_0 - \bar{\omega}, \theta_{0,x} - \bar{\theta}_x, \theta_{0,y} - \bar{\theta}_y) < E_*$, we have*

$$(2.13) \quad \|\omega - \bar{\omega}\|_{L^\infty}, \|\theta_x - \bar{\theta}_x\|_{L^\infty}, \|\theta_y - \bar{\theta}_y\|_\infty < 200E_*, \quad |u_x(t, 0) - \bar{u}_x(0)|, |\bar{c}_\omega - c_\omega| < 100E_*$$

for all time. In particular, we can choose smooth initial data $\omega_0, \theta_0 \in C_c^\infty$ in this class with finite energy $\|\mathbf{u}_0\|_{L^2} < +\infty$ such that the solution to the physical equations (2.3)-(2.5) with these initial data blows up in finite time T .

The energy E is quite complicated, and we refer to Section 2.3 in Part I [14] for its formula.

Nearly self-similar blowup and the blowup time. Based on the main Theorem 3, the vorticity in the physical space (2.3),(2.4) ω_{phy} has the following form

$$\omega_{phy}(x, t(\tau)) = C_\omega^{-1}(\tau)\omega_{ss}(C_l(\tau)^{-1}x, \tau), \quad \|\omega_{ss}(\tau) - \bar{\omega}\|_{L^\infty} \ll 1,$$

where ω_{ss} is the self-similar variable ($\tilde{\omega}$ in (2.6)). We can generalize the rescaling parameters C_ω, C_l, C_θ (2.8) to $C_\omega(\tau) = C_\omega(0) \exp(\int_0^\tau c_\omega(s) ds)$, $C_l(\tau) = C_l(0) \exp(\int_0^\tau -c_l(s) ds)$, $C_\theta = C_\omega^2 C_l^{-1}$, $t(\tau) = \int_0^\tau C_\omega(\tau) d\tau$. Using the estimates (2.13) and $c_l(\tau) \equiv \bar{c}_l$ (2.11),(2.12), we obtain

$$\begin{aligned} \|\omega_{phy}(\tau)\|_{L^\infty} &\approx C_\omega^{-1}(\tau)\|\bar{\omega}\|_{L^\infty}, \quad C_\omega(\tau) \approx C_\omega(0)e^{\bar{c}_\omega\tau}, \quad C_l(\tau) = C_l(0)e^{-\bar{c}_l\tau}, \\ T = t(\infty) &\approx C_\omega(0)|\bar{c}_\omega|^{-1} = \frac{\|\bar{\omega}\|_{L^\infty}}{\|\omega_{phy}(0)\|_{L^\infty}|\bar{c}_\omega|}, \quad T - t(\tau) \approx C_\omega(\tau)|\bar{c}_\omega|^{-1}, \\ C_l(\tau)^{-1} &\approx C(T - t(\tau))^{\bar{c}_l/\bar{c}_\omega}, \quad \omega_{phy}(\tau) \approx (T - t(\tau))^{-1}|\bar{c}_\omega|^{-1}\bar{\omega}(Cx(T - t(\tau))^{\bar{c}_l/\bar{c}_\omega}), \end{aligned}$$

with $\bar{c}_l/\bar{c}_\omega \approx -2.92 < 0$, for some $C > 0$ depending on $\bar{c}_l, \bar{c}_\omega, C_l(0), C_\omega(0)$. The notation \approx means that the relation holds approximately. The exact relation can be inferred from (2.13), (2.6), (2.8). The blowup time is approximately inversely proportional to $\|\omega_{phy}(0)\|_{L^\infty}$. Since we only prove that $\omega_{ss}(\tau)$ is sufficiently close to the approximate profile $\bar{\omega}$ and do not prove convergence of $\omega_{ss}(\tau)$ as $\tau \rightarrow \infty$, Theorem 3 does not imply *asymptotically* self-similar blowup.

2.2. The main steps in the proof of Theorem 3. We will follow the framework in [13, 16, 17] to establish finite time blowup by proving the nonlinear stability of an approximate steady state to (2.10). We divide the proof of Theorem 3 into proving the following lemmas. The energy norm below is defined in Section 5 in Part I [14] for energy estimates, and the requirement of smallness is incorporated in the conditions (2.17), e.g. the term $a_{ij,3}$, in Lemma 2.5.

The upper bar notation is reserved for the approximate steady state, e.g. $\bar{\omega}, \bar{\theta}$. Given the approximate steady state $\bar{\omega}, \bar{\theta}, \bar{c}_l, \bar{c}_\omega$, we denote by $\bar{\mathcal{F}}_i$ and $\bar{F}_\omega, \bar{F}_\theta$ the residual error

$$(2.14) \quad \begin{aligned} \bar{F}_\omega &= -(\bar{c}_l x + \bar{\mathbf{u}}) \cdot \nabla \bar{\omega} + \bar{\theta}_x + \bar{c}_\omega \bar{\omega}, \quad \bar{F}_\theta = -(\bar{c}_l x + \bar{\mathbf{u}}) \cdot \nabla \bar{\theta} + \bar{c}_\theta \bar{\theta}, \\ \bar{\mathcal{F}}_1 &\triangleq \bar{F}_\omega, \quad \bar{\mathcal{F}}_2 \triangleq \partial_x \bar{F}_\theta, \quad \bar{\mathcal{F}}_3 \triangleq \partial_y \bar{F}_\theta, \quad \bar{c}_\theta = \bar{c}_l + 2\bar{c}_\omega. \end{aligned}$$

We have the following nonlinear stability Lemma for L^∞ -based energy estimate, which is proved in Appendix A.1 of Part I [14].

Lemma 2.1. *Suppose that $f_i(x, z, t) : \mathbb{R}_{++}^2 \times \mathbb{R}_{++}^2 \times [0, T] \rightarrow \mathbb{R}$, $1 \leq i \leq n$, satisfies*

$$(2.15) \quad \partial_t f_i + v_i(x, z) \cdot \nabla_{x,z} f_i = -a_{ii}(x, z, t)f_i + B_i(x, z, t) + N_i(x, z, t) + \bar{\varepsilon}_i,$$

where $v_i(x, z, t)$ are some vector fields Lipschitz in x, z with $v_i|_{x_1=0} = 0, v_i|_{z_1=0} = 0$. For some $\mu_i > 0$, we define the energy

$$E(t) = \max_{1 \leq i \leq n} (\mu_i \|f_i\|_{L^\infty}).$$

Suppose that B_i, N_i and $\bar{\varepsilon}_i$ satisfy the following estimate

$$(2.16) \quad \mu_i (|B_i(x, z, t)| + |N_i(x, z, t)| + |\bar{\varepsilon}_i|) \leq \sum_{j \neq i} (|a_{ij}(x, z, t)|E(t) + |a_{ij,2}(x, z, t)|E^2(t) + |a_{ij,3}(x, z, t)|).$$

If there exists some $E_*, \varepsilon_0, M > 0$ such that

$$(2.17) \quad \begin{aligned} a_{ii}(x, z, t)E_* - \sum_{j \neq i} (|a_{ij}|E_* + |a_{ij,2}|E_*^2 + |a_{ij,3}(x, z, t)|) &> \varepsilon_0, \\ \sum_{j \neq i} (|a_{ij}|E_* + |a_{ij,2}|E_*^2 + |a_{ij,3}(x, z, t)|) &< M, \end{aligned}$$

for all x, z and $t \in [0, T]$. Then for $E(0) < E_*$, we have $E(t) < E_*$ for $t \in [0, T]$.

Lemma 2.2. *There exists a nontrivial approximate steady state $(\bar{\omega}, \bar{\theta}, \bar{c}_l, \bar{c}_\omega)$ to (2.10), (2.11) with $\bar{\omega}, \bar{\theta} \in C^{4,1}$ and residual errors $\bar{F}_i, i = 1, 2, 3$ (2.14) sufficiently small in some energy norm.*

The construction of an approximate self-similar profile with a small residual error stated in Lemma 2.2 is provided in Section 7 of Part I [14] and the properties of $(\bar{\omega}, \bar{\theta}, \bar{c}_l, \bar{c}_\omega)$ are described in Section 2.4 of Part I [14]. We will estimate the local part of the residual error in Appendix C.4. We linearize (2.10) around $(\bar{\omega}, \bar{\theta}, \bar{c}_l, \bar{c}_\omega)$ and perform energy estimate of the perturbation $W = (\omega, \theta_x, \theta_y)$ in Section 5 in Part I [14]. In our estimates, we need to control a number of nonlocal terms.

Lemma 2.3. *Let ω be odd in x_1 . Denote $\delta(f, x, z) = f(x) - f(z)$. There exists finite rank approximations $\hat{\mathbf{u}}, \widehat{\nabla \mathbf{u}}$ for $\mathbf{u}(\omega), \nabla \mathbf{u}(\omega)$ with rank less than 50 such that we have the following weighted L^∞ and directional Hölder estimate for $f = u, v, \partial_t u, \partial_t v, x, z \in \mathbb{R}_2^{++}, i = 1, 2, \gamma_i > 0$*

$$(2.18) \quad \begin{aligned} |\rho_f(f - \hat{f})(x)| &\leq C_{f,\infty}(x, \varphi, \psi_1, \gamma) \max(\|\omega\varphi\|_\infty, s_f \max_{j=1,2} \gamma_j [\omega\psi_1]_{C_{x_j}^{1/2}(\mathbb{R}_2^+)}) , \\ \frac{|\delta(\psi_f(f - \hat{f}), x, z)|}{|x - z|^{1/2}} &\leq C_{f,i}(x, z, \varphi, \psi_1, \gamma) \max(\|\omega\varphi\|_\infty, s_f \max_{j=1,2} \gamma_j [\omega\psi_1]_{C_{x_j}^{1/2}(\mathbb{R}_2^+)}) , \end{aligned}$$

with $x_{3-i} = z_{3-i}$, where $s_f = 0$ for $f = u, v$, $s_f = 1$ for $f = \partial_t u, \partial_t v$, the functions $C(x), C(x, z)$ depend on γ , the weights, and the approximations, the singular weights $\varphi = \varphi_1, \varphi_{g,1}, \varphi_{elli}, \psi_{\partial u} = \psi_1, \psi_u$ are defined in (A.2), the weight ρ_{10} for \mathbf{u} and the weight for ρ_{ij} for $\nabla \mathbf{u}$ with $i + j = 2$ are given in (A.2). In the estimate of $f = u, v$, we do not need the Hölder semi-norm and we set $s_f = 0$. Moreover, $C(x), C(x, z)$ are bounded in any compact domain of \mathbb{R}_2^{++} . We have an additional estimate for $\rho_4(u - \hat{u})$ similar to the above with ρ_4 (A.2) singular along $x_1 = 0$.

Furthermore, we have the following estimate using the localized norm. There exist $D_1, D_2, \dots, D_n \subset \mathbb{R}_2^{++}$ and $D_S \in \mathbb{R}_2^+$ depending on x in the L^∞ estimate and x, z in the $C_{x_i}^{1/2}$ estimate, such that

$$\begin{aligned} |\rho_f(f - \hat{f})(x)| &\leq \sum_j C_{f,\infty,j}(x, \varphi, \psi_1, \gamma) \|\omega\varphi\|_{L^\infty(D_j)} + C_{f,\infty,S}(x, \varphi, \psi_1, \gamma) \max_{l=1,2} (\gamma_l [\omega\psi_1]_{C_{x_l}^{1/2}(D_S)}), \\ \frac{|\delta(\psi_f(f - \hat{f}), x, z)|}{|x - z|^{1/2}} &\leq \sum_j C_{f,i,j}(x, z, \varphi, \psi_1, \gamma) \|\omega\varphi\|_{L^\infty(D_j)} + C_{f,i,S}(x, z, \varphi, \psi_1, \gamma) \max_{l=1,2} (\gamma_l [\omega\psi_1]_{C_{x_l}^{1/2}(D_S)}), \end{aligned}$$

for $x_{3-i} = z_{3-i}, \varphi = \varphi_{elli}$ and the same notation as above, where $C_{f,\infty,S}, C_{f,i,S} = 0$ for $f = u, v$. Similarly, we have an estimate for $\rho_4(u - \hat{u})$ using localized norm with $C_{f,\infty,S} = 0$ similar to the above.

Since the weights $\rho_{10} \sim |x|^{-3}, \psi_1 \sim |x|^{-2}, \psi_u$ are singular near $x = 0$, without subtracting the approximation \hat{f} from f , $\rho_f f$ is not bounded near $x = 0$. We design the finite rank approximations $\hat{\mathbf{u}}, \widehat{\nabla \mathbf{u}}$ in Section 4.3 in Part I [14].

Based on these finite rank approximations, we can decompose the perturbations.

Lemma 2.4. *There exists $m < 50$ approximate solutions \hat{F}_i to the linearized equations $\partial_t W = \mathcal{L}W$ of (2.10) around $(\bar{\omega}, \bar{\theta}, \bar{c}_l, \bar{c}_\omega)$ in Lemma 2.2 from given initial data $\hat{F}_i(0)$ with residual error \mathcal{R} small in the energy norm. Further we can decompose the perturbation $W = W_1 + \widehat{W}_2$ with the following properties. (a) \widehat{W}_2 is constructed based on \hat{F}_i , see Section 4.2.4 of Part I [14]; (b) W_1 satisfies the equations with the leading order linearized operator $(\mathcal{L} - \mathcal{K})W_1$ up to the small residual error \mathcal{R} for some finite rank operator \mathcal{K} , and W_1 depends on \widehat{W}_2 weakly at the linear*

level via \mathcal{R} . The functionals $a_i(W_1)$, $a_{nl,i}(W)$ in the construction of \widehat{W}_2 and \mathcal{K} (see Section 4.2.4 of Part I [14]) are related to the finite rank approximations in Lemma 2.3.

Moreover, there exists an energy $E_4(t)$ for W_1, W (see Section 5.6.3. of Part I [14]) that controls the weighted L^∞ and $C^{1/2}$ seminorm of W_1 such that under the bootstrap assumption $E_4(t) < E_{*0}$ with $E_{*0} > 0$, we can establish nonlinear energy estimates for $E_4(t)$ using the estimates in Lemma 2.3.

If the bounds in Lemma 2.3 are tight, and the residual error in the constructions of $(\bar{\omega}, \bar{\theta})$, \widehat{F}_i are small enough, we can use Lemma 2.1 to obtain nonlinear stability.

Lemma 2.5. *For $E_* = 5 \cdot 10^{-6}$, the coefficients in the nonlinear energy estimates of $E_4(t)$ satisfy the conditions (2.17), and the statements in Theorem 3 hold true.*

The main purpose of Part II of our paper is the following. Firstly, we construct the approximate $\widehat{F}_i(t)$ in Lemma 2.4 numerically, and estimate its piecewise derivatives and the local residual error in Section 3. Secondly, in Section 4, we obtain sharp estimates of the constants in Lemma 2.3, which only depend on the weights. Thirdly, we estimate piecewise bounds of the approximate steady state in Appendix C, the singular weights in Appendix A, some explicit functions related to the approximate solutions in Appendix D. We remark that all of these estimates and constants depend on the given weights, some operators and functions, e.g. the approximate steady state and the specific initial conditions. With these estimates and constants, we obtain the concrete values of the inequalities in (2.17) and Lemma 2.5, which are given in Appendix D in Part I [14]. We further verify the inequalities for the stability conditions in Lemma 2.5.

Let us make a few comments on the above lemmas. Firstly, our energy estimate is based on weighted L^∞ functional spaces, which is crucial for extracting the damping terms for the energy estimate. See Section 2.7 of Part I [14] for the motivations. Given $\omega \in C^{1/2}$, we have $\mathbf{u} \in C^{3/2}$, $\nabla \mathbf{u} \in C^{1/2}$. To establish the nonlinear stability conditions (2.17) in Lemma 2.5, we need sharp constants in the estimates in Lemma 2.3. We use some techniques from optimal transport to obtain sharp $C^{1/2}$ estimate of $\nabla \mathbf{u}$ in Section 3 of Part I [14]. This corresponds to the limiting case in the $C_{x_i}^{1/2}$ estimate in Lemma 2.3 for a fixed x with $|x - z| \rightarrow 0$ and captures the most singular part in the estimates in Lemma 2.3. The constants in the sharp $C^{1/2}$ estimate established in Part I [14] are given by several integrals. In Section 5 of the supplementary material II [12], we estimate these integrals.

Other parts of the estimates in Lemma 2.3 are more regular since we work with the regular part of the velocity integral with a desingularized kernel. Given $\omega \in C^{1/2}$, we can reduce the estimates of these more regular terms to estimate some explicit L^1 integrals. We can obtain sharp estimates of these more regular integrals using some numerical quadrature with computer assistance. See Section 4.

By designing \mathcal{K} to approximate the nonlocal terms, we can obtain much better linear stability estimates for $\mathcal{L} - \mathcal{K}$. After we have shown that the stability conditions (2.17) are satisfied, we have nonlinear stability estimate $E_4(t) < E_*$ for all $t > 0$ using Lemma 2.1, which implies the bounds in Theorem 3. The remaining steps of obtaining finite time blowup from smooth initial data and finite energy follow [16] and a rescaling argument. We remark that the variable \widehat{W}_2 in Lemma 2.4 (see full definition in Section 4.2.4 of Part I [14]) plays an auxiliary role, and we do not perform energy estimate on \widehat{W}_2 directly.

Note that all the nonlocal terms in the linearized equations are not small. Without the sharp $C^{1/2}$ estimate, with the choice of energy E_4 , the stability conditions in (2.17) and Lemma 2.5 fail in the weighted Hölder estimate. Without the finite rank approximations for the nonlocal terms in Lemma 2.3, 2.4, the stability conditions for weighted L^∞ estimate also fail.

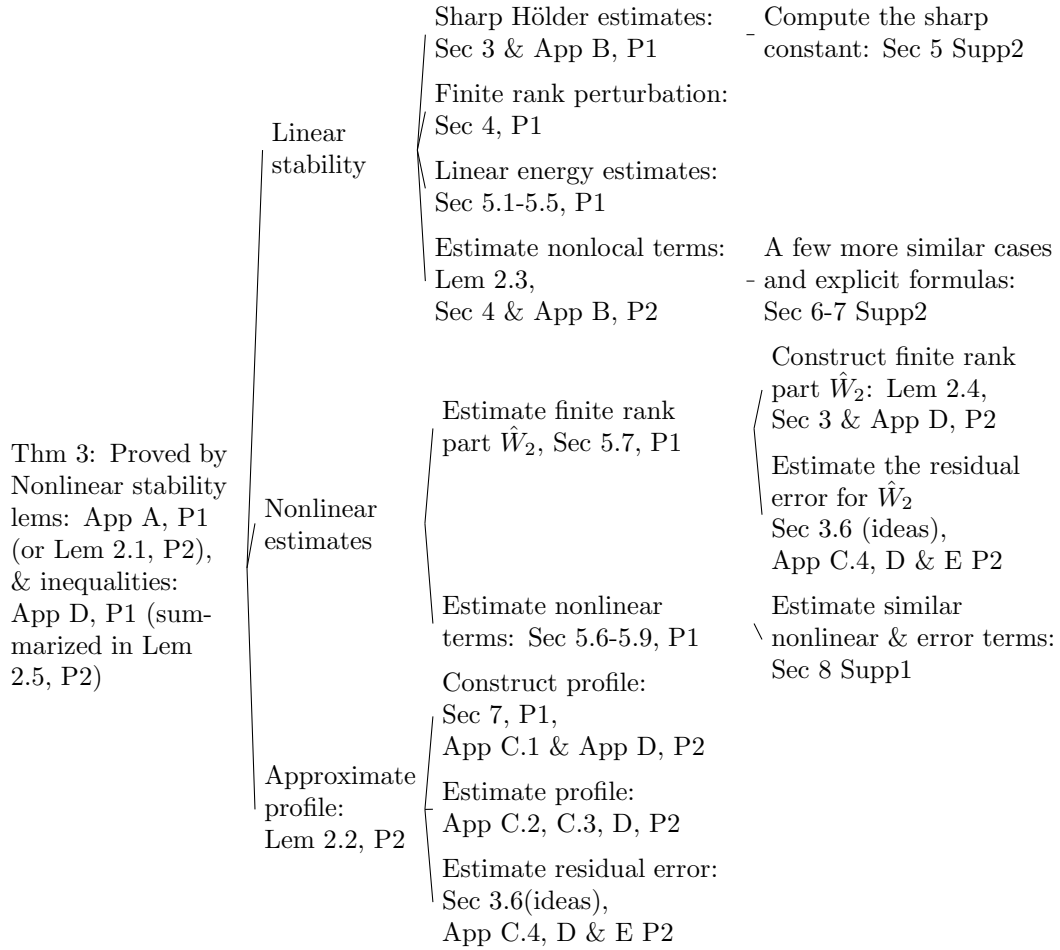
Rigorous numerics. We need to track two types of errors for rigorous numerics. The first type is the discretization error, e.g. the error terms in the Trapezoidal rule and in the interpolating polynomials. The second type is the round-off error in the computation. We use numerical analysis to estimate *all* the discretization errors, and use *only* the basic interval arithmetic [50,53], see e.g. (A.4), (A.5), in the INTLAB package [54] from MatLab to track the round-off error.

In our nonlinear estimates, we use a singular weight φ like $|x|^{-3}$ near $x = 0$ to measure the residual error $\bar{\mathcal{F}}_i$. To obtain a small weighted residual error $|\varphi\bar{\mathcal{F}}_i|$ near $x = 0$, we choose the mesh y_i (C.2) representing the approximate profile to be exact floating point numbers to reduce the round-off error near $x = 0$.

The codes for the computations are implemented in MatLab and can be found in [10]. The estimates of the constants in Lemma 2.3, integrals in Section 4, and the constructions and estimates of the approximate space-time solutions in Lemma 2.4 and in Section 3 are performed in parallel using the Caltech High Performance Computing¹. Other computer-assisted estimates and the main part of the verifications are done in Mac Pro (Rack,2019) with 2.5GHz 28-core Intel Xeon W processor and 768GB (6x128GB) of DDR4 ECC memory.

2.3. Dependency tree. The following tree schematizes various intermediate steps and related sections that lead to the main stability result Theorem 3, which implies the blowup result Theorem 1 for the 2D Boussinesq equations. The blowup for the Euler equations in Theorem 2 is proved by a perturbative argument in Section 6 in Part I [14].

Below, *Thm, Lem, App, Sec, P1, P2, Supp1, Supp2* are short for Theorem, Lemma, Appendix, Section, Paper I [14], Paper II (the current paper), the Supplementary material for Paper I [15] and Paper II [12], respectively. We present a few more detailed derivations in the Supplementary materials [15], [12], which expand and generalize discussions in the main papers and are less essential. Moreover, we present several explicit formulas we used in our computer-assisted estimates for the quantities derived in the main papers.



In Section 8 in Supp2, we generalize the standard interpolation estimate in numerical analysis to derive higher order interpolation estimates, which are used to estimate the residual error

¹See more details for Caltech HPC Resources <https://www.hpc.caltech.edu/resources>

effectively. See the discussion in Section 3.6. In Appendix A, we derive piecewise bounds for various weights, which are used in the weighted estimates of the nonlocal terms, the residual error, and in the linear, nonlinear estimates for stability.

In Appendix F, we collect the main notations used in this paper.

3. CONSTRUCT AND ESTIMATE THE APPROXIMATE SOLUTION TO THE LINEARIZED EQUATIONS

As we described in Section 2 of Part I [14] (see also the Introduction), we need to construct the approximate solutions to $e^{\mathcal{L}t}F_0$ for several initial data $\bar{F}_i, \bar{F}_{\chi,i}$. In this section, we discuss how to construct these space-time solutions numerically with some vanishing properties at the origin with rigorous error control.

The linearized equations associated with \mathcal{L} read

$$\begin{aligned} \partial_t \omega &= -(\bar{c}_l x + \bar{u}) \cdot \nabla \omega + \eta + \bar{c}_\omega \omega - \mathbf{u} \cdot \nabla \bar{\omega} + c_\omega \bar{\omega} = \mathcal{L}_1(\omega, \eta, \xi), \\ (3.1) \quad \partial_t \eta &= -(\bar{c}_l x + \bar{u}) \cdot \nabla \eta + (2\bar{c}_\omega - \bar{u}_x) \eta - \bar{v}_x \xi - \mathbf{u}_x \cdot \nabla \bar{\theta} - \mathbf{u} \cdot \nabla \bar{\theta}_x + 2c_\omega \bar{\theta}_x = \mathcal{L}_2(\omega, \eta, \xi), \\ \partial_t \xi &= -(\bar{c}_l x + \bar{u}) \cdot \nabla \xi + (2\bar{c}_\omega + \bar{u}_x) \xi - \bar{u}_y \eta - \mathbf{u}_y \cdot \nabla \bar{\theta} - \mathbf{u} \cdot \nabla \bar{\theta}_y + 2c_\omega \bar{\theta}_y = \mathcal{L}_3(\omega, \eta, \xi), \end{aligned}$$

with normalization condition

$$(3.2) \quad c_\omega = u_x(0), \quad c_l \equiv 0.$$

Although η, ξ represent θ_x, θ_y in the Boussinesq equations, we will consider initial data $(\omega_0, \eta_0, \xi_0)$ with $\partial_y \eta_0 \neq \partial_x \xi_0$. Thus, we do not have the relation $\partial_y \eta = \partial_x \xi$ and will treat η, ξ as two independent variables. The solutions $\omega(x), \eta(x)$ are odd in x_1 , $\xi(x)$ is even in x_1 with $\xi(0, y) = 0$. We consider initial data $(\omega_0, \eta_0, \xi_0) = O(|x|^2)$ near $x = 0$. Using a direct calculation, we can show that these vanishing conditions are preserved in time

$$(3.3) \quad \omega(t, x), \quad \eta(t, x), \quad \xi(t, x) = O(|x|^2).$$

We introduce the bilinear operator $B_{op,i}(\mathbf{u}, M), G$ for $(\mathbf{u}, M), G = (G_1, G_2, G_3)$

$$(3.4) \quad \begin{aligned} \mathcal{B}_{op,1} &= -\mathbf{u} \cdot \nabla G_1 + M_{11}(0)G_1, & \mathcal{B}_{op,2} &= -\mathbf{u} \cdot \nabla G_2 + 2M_{11}(0)G_2 - M_{11}G_2 - M_{21}G_3, \\ \mathcal{B}_{op,3} &= -\mathbf{u} \cdot \nabla G_3 + 2M_{11}(0)G_3 - M_{12}G_2 - M_{22}G_3. \end{aligned}$$

If $M = \nabla \mathbf{u}, M_{11} = u_x, M_{12} = u_y, M_{21} = v_x, M_{22} = v_y$, then we drop M to simplify the notation

$$(3.5) \quad \begin{aligned} \mathcal{B}_{op,1}(\mathbf{u}, G) &= -\mathbf{u} \cdot \nabla G_1 + u_x(0)G_1, & \mathcal{B}_{op,2} &= -\mathbf{u} \cdot \nabla G_2 + 2u_x(0)G_2 - u_x G_2 - v_x G_3, \\ \mathcal{B}_{op,3} &= -\mathbf{u} \cdot \nabla G_3 + 2u_x(0)G_3 - u_y G_2 - v_y G_3. \end{aligned}$$

The main result in this section is the following. Given n initial data $\bar{G}_i = (\bar{G}_{i,1}, \bar{G}_{i,2}, \bar{G}_{i,3})$ and n functions $c_i(t) (i = 1, 2, \dots, n)$ which are Lipschitz and bounded in t , we construct approximate space-time solution $\hat{W}_i = (\hat{W}_{i,1}, \hat{W}_{i,2}, \hat{W}_{i,3}), \hat{G} = (\hat{G}_1, \hat{G}_2, \hat{G}_3)$ and the approximate stream functions $\hat{\phi}_i^N, \hat{\phi}^N$ and the error $\hat{\varepsilon}$ associated with $\hat{W}_{i,1}, \hat{G}_1$

$$(3.6) \quad \hat{G} = \sum_{i \leq n} \int c_i(t-s) \hat{W}_i(s) ds, \quad \hat{\phi}^N = \sum_{i \leq n} \int c_i(t-s) \hat{\phi}_i^N(s) ds, \quad \hat{\varepsilon} = \sum_{i \leq n} \int c_i(t-s) (\hat{W}_{i,1} + \Delta \hat{\phi}_i^N)(s) ds,$$

with residual error

$$(3.7) \quad \mathcal{R} = \sum_{i \leq n} c_i(t) (\hat{W}_i(0) - \bar{W}_i) + \int_0^t c(t-s) (\partial_t - \mathcal{L}) \hat{W}_i(s) ds,$$

vanishing $O(|x|^3)$ near $x = 0$. Moreover, we can decompose $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3)$ as follows

$$(3.8) \quad \mathcal{R}_j(t) = \mathcal{R}_{loc,0,j}(t) + \mathcal{R}_{nloc,j}(t), \quad \mathcal{R}_{loc,0,j} = \sum_{i \leq n} \int_0^t c_i(t-s) \mathcal{R}_{num,i,j}(s) ds, \quad \mathcal{R}_{num,i,j} = O(|x|^3),$$

$$\mathcal{R}_{nloc,j} = P_j - D_j^2 P_j(0) \chi_{j,2}, \quad P_j = -\mathcal{B}_{op,j}(\mathbf{u}(\bar{\varepsilon}), \hat{G}) - \mathcal{B}_{op,j}(\mathbf{u}(\hat{\varepsilon}), (\bar{\omega}, \bar{\theta}_x, \bar{\theta}_y)),$$

where $\chi_{j,2}$ is given in (D.5), and $\bar{\varepsilon} = \bar{\omega} - (-\Delta) \bar{\phi}^N$ is the error of the approximate stream function for $(-\Delta)^{-1} \bar{\omega}$, $\mathcal{R}_{num,j}(t, x)$ depends on $\hat{W}_i, \hat{\phi}_i$ in x locally. We have absorbed the initial error

in \mathcal{R}_{num} . We derive the above decompositions and estimates of $\mathcal{R}_{loc,0,j}, \hat{G}, \hat{\phi}^N, \hat{\varepsilon}$, in Section 3.5-3.7. See (3.37), (3.35). The error in constructing the stream function $\hat{\phi}_i^N$ associated with $\hat{W}_{i,1}$ leads to nonlocal error, e.g. $\mathbf{u}(\hat{\varepsilon})$, in constructing the velocity. We combine the estimate of the nonlocal error in P_j and perturbation in Section 5.8 in Part I [14]. Furthermore, we track the piecewise bounds of the following quantities

$$(3.9) \quad \int_0^\infty |\partial_x^k \partial_y^l F(t)| dt, \quad F = \hat{W}_{i,j}, \quad F = \hat{\phi}_i^N, \quad F = \hat{\phi}_i^N - \partial_{xy} \phi_i^N(0)xy, \quad F = \hat{W}_{i,1} + \Delta \phi_i^N, \\ F = c_j \hat{W}_{i,j} - x \partial_x \hat{W}_{i,j} + y \partial_y \hat{W}_{i,j} - D_j^2 \hat{W}_{i,j}(0) f_{\chi,j}, \quad D^2 = (\partial_{xy}, \partial_{xy}, \partial_{xx}), \quad c = (1, 1, 3),$$

for $j = 1, 2, 3, i = 1, 2, \dots, n$, where $f_{\chi,j}$ is defined in (D.6). We track the C^2 bound of $\hat{W}_{i,j}$ and C^4 bounds for others following (3.37), and use these bounds to control \widehat{W}_2 in Lemma 2.4 and use them in the nonlinear energy estimates in Section 5 in Part I [14].

In practice, we choose the initial data \bar{F}_i given in Appendix C.2.1 in Part I [14], and $c_i(t)$ some functionals of the perturbation W_1, \hat{W}_2 related to the finite rank perturbation.

Numerical methods. We solve (3.1) using the numerical method outlined in Section 7 of Part I [14] to obtain the solution $(\omega_k, \eta_k, \xi_k)$ at discrete time t_k . Since ξ is even with $\xi(0, y) = 0$, we write $\xi = x\zeta$ for an odd function ζ . We use the adaptive mesh discussed in Appendix C.1 to discretize the spatial domain. Then we represent ω, η, ζ using the piecewise 6-th order B-spline (C.6). See Appendix C.1. To solve the stream function $-\Delta\phi = \omega$ numerically, we use the B-spline based finite element method and obtain the numerical approximation ϕ^N for $(-\Delta)^{-1}\omega$. Then we can construct the velocity $\mathbf{u}^N = \nabla^\perp \phi^N$.

The gradients of several initial conditions \bar{F}_i are relatively large and the linearized equations (3.1) involve $\nabla \hat{W}$. To obtain a better approximation of the solution, we represent ω, η, ζ using a mesh $Y \times Y$ with Y refining the mesh y (C.2) in Appendix C.1 by a factor of three:

$$Y_{3i+j} = y_i + (y_{i+1} - y_i)j/3, \quad 0 \leq j \leq 3.$$

Since solving the Poisson equation is the main computational cost in each time step, we still represent ϕ^N using the coarse mesh $y \times y$ and solve it from source term with grid points value $\omega(y_i, y_j)$.

In the temporal variable, we use a third order Runge-Kutta method to update the PDE. To reduce the round-off error near $x = 0$, where we require a very small error in solving the linear PDE, we use a multi-level representation. We refer more details to Section 7 in Part I [14]. To keep the residual error smooth near $x = 0$, we apply a weak numerical filter near $x = 0$ every three steps. We do not add the semi-analytic part in constructing $(\omega_k, \eta_k, \xi_k)$ for efficiency consideration and that the far-field behavior of the solutions is changing over time.

After we obtain the numerical solution $(\omega_k, \eta_k, \xi_k, \phi_{k,1}^N)$ at discrete time, we will perform two rank-one corrections and interpolate the solution in time using a cubic polynomial to obtain the approximate space time solution \hat{W} , and estimate residual error in the energy space *a-posteriori*.

3.1. A posteriori error estimates: decomposition of errors. Since we cannot solve the Poisson equation exactly, we decompose the stream function $\bar{\phi}, \phi$ as follows

$$(3.10) \quad \bar{\phi} = (-\Delta)^{-1} \bar{\omega} = \bar{\phi}^N + \bar{\phi}^e, \quad \phi = (-\Delta)^{-1} \omega = \phi^N + \phi^e,$$

where $\bar{\phi}^N, \phi^N$ constructed using finite element method are the numeric approximation of the stream function, and the short hands N, e denote *numeric, error*, respectively. We use similar notations below for other nonlocal terms since we cannot construct them exactly. We will construct $\bar{\phi}^N, \phi^N$ numerically and treat $\bar{\phi}^e, \phi^e$ as error. The reader should not confuse ϕ^N with the N -th power of ϕ . We will never use power of ϕ throughout the paper. Similarly, we denote by $\mathbf{u}^N, \mathbf{u}^e$ the velocities corresponding to ϕ^N, ϕ^e . For example, we have

$$(3.11) \quad \mathbf{u}^N = \nabla^\perp \phi^N, \quad \mathbf{u}^e = \nabla^\perp \phi^e = \nabla^\perp (-\Delta)^{-1} (\omega - (-\Delta)\phi^N), \quad c_\omega^N = u_x^N(0), \quad c_\omega^e = u_x^e(0).$$

The above decomposition leads to the following decomposition of the operator \mathcal{L}

$$\begin{aligned}
(3.12) \quad & \mathcal{L}_1 = \mathcal{L}_1^N + \mathcal{L}_1^e + \mathcal{L}_1^{\bar{e}}, \quad \mathcal{L}_2 = \mathcal{L}_2^N + \mathcal{L}_2^e + \mathcal{L}_2^{\bar{e}}, \quad \mathcal{L}_3 = \mathcal{L}_3^N + \mathcal{L}_3^e + \mathcal{L}_3^{\bar{e}}, \\
& \mathcal{L}_1^N = \eta + \bar{c}_\omega^N \omega - (\bar{c}_1 x + \bar{\mathbf{u}}^N) \cdot \nabla \omega + c_\omega^N \bar{\omega} - \mathbf{u}^N \cdot \nabla \bar{\omega}, \\
& \mathcal{L}_1^e = c_\omega^e \bar{\omega} - \mathbf{u}^e \cdot \nabla \bar{\omega}, \quad \mathcal{L}_1^{\bar{e}} = \bar{c}_\omega^e \omega - \bar{\mathbf{u}}^e \cdot \nabla \omega, \\
& \mathcal{L}_2^N = -(\bar{c}_1 x + \bar{\mathbf{u}}^N) \cdot \nabla \eta + (2\bar{c}_\omega^N - \bar{u}_x^N) \eta - \bar{v}_x^N \xi - \mathbf{u}_x^N \cdot \nabla \bar{\theta} - \mathbf{u}^N \cdot \nabla \bar{\theta}_x + 2c_\omega^N \bar{\theta}_x, \\
& \mathcal{L}_2^e = -\mathbf{u}_x^e \cdot \nabla \bar{\theta} - \mathbf{u}^e \cdot \nabla \bar{\theta}_x + 2c_\omega^e \bar{\theta}_x, \quad \mathcal{L}_2^{\bar{e}} = -\bar{\mathbf{u}}^e \cdot \nabla \eta + (2\bar{c}_\omega^e - \bar{u}_x^e) \eta - \bar{v}_x^e \xi, \\
& \mathcal{L}_3^N = -(\bar{c}_1 x + \bar{\mathbf{u}}^N) \cdot \nabla \xi + (2\bar{c}_\omega^N - \bar{v}_y^N) \xi - \bar{u}_y^N \eta - \mathbf{u}_y^N \cdot \nabla \bar{\theta} - \mathbf{u}^N \cdot \nabla \bar{\theta}_y + 2c_\omega^N \bar{\theta}_y, \\
& \mathcal{L}_3^e = -\mathbf{u}_y^e \cdot \nabla \bar{\theta} - \mathbf{u}^e \cdot \nabla \bar{\theta}_y + 2c_\omega^e \bar{\theta}_y, \quad \mathcal{L}_3^{\bar{e}} = -\bar{\mathbf{u}}^e \cdot \nabla \xi + (2\bar{c}_\omega^e - \bar{v}_y^e) \xi - \bar{u}_y^e \eta,
\end{aligned}$$

where $\mathcal{L}_i^e, \mathcal{L}_i^{\bar{e}}$ denote the errors from $\psi^e, \bar{\psi}^e$, respectively. These operators depend on ω, η, ξ , and we drop the dependence in (3.12) to simplify the notations.

3.2. First correction and the construction of ϕ^N . According to the normalization condition and (3.3), the solution to (3.1) satisfies $\omega_x(0, t) = \eta_x(0, t) = 0$. To obtain an approximate solution with this condition, we make the first correction

$$(3.13) \quad \omega_k \rightarrow \omega_k - \omega_{k,x}(0, 0)\chi_{11}, \quad \eta_k \rightarrow \eta_k - \eta_{k,x}(0, 0)\chi_{21},$$

where χ_{ij} are cutoff functions defined in (3.17) with $\chi_{ij} = x + O(|x|^4)$ near 0. We do not modify ξ_k since ξ_k already vanishes quadratically near $(0, 0)$. We remark that the first correction does not change the second order derivatives of the solution near 0 and c_ω since

$$\partial_{xy}\chi_{11}(0) = \partial_{xy}\chi_{21}(0) = 0, \quad c_\omega(\chi_{11}) = -\partial_{xy}\phi_1(0) = 0.$$

where ϕ_1 is defined below

$$(3.14) \quad \phi_1 = -\frac{xy^2}{2}\kappa_*(x)\kappa_*(y),$$

where $\kappa_*(x)$ is the cutoff function chosen in (D.5) in Appendix D.2 satisfying $\kappa_*(x) = 1 + O(|x|^4)$ near $x = 0$, and ϕ_1 satisfies $-\Delta\phi_1 = x + O(|x|^4)$. For the numeric stream function $\phi_{k,1}^N$ constructed at the beginning of this Section 3, we correct it as follows

$$\phi_{k,1}^N \rightarrow \phi_{k,1}^N + \partial_x \Delta \phi_{k,1}^N(0)\phi_1 \triangleq \hat{\phi}_k^N.$$

Since $\partial_x \Delta \phi_1(0) = -1$, this allows us to obtain

$$(3.15) \quad \begin{aligned} \partial_x(-\Delta)\hat{\phi}_k^N(0) &= -\partial_x \Delta \phi_{k,1}^N(0) + \partial_x \Delta \phi_{k,1}^N(0) = 0, \\ \Delta \hat{\phi}_k^N &= O(|x|^2), \quad \omega_k - (-\Delta)\hat{\phi}_k^N = O(|x|^2). \end{aligned}$$

We further extend it to Lipschitz continuous solutions $\widehat{W}^{(1)} \triangleq (\hat{\omega}^{(1)}(t), \hat{\eta}^{(1)}(t), \hat{\xi}^{(1)}(t), \hat{\phi}^{N,(1)})$ in time using a cubic polynomial interpolation in t . See section 3.4 for more details. Here, we use $\hat{f}^{(1)}$ to denote the solution with the first correction.

3.3. The second correction. The error

$$(\partial_t - \mathcal{L}_i)(\hat{\omega}^{(1)}(t), \hat{\eta}^{(1)}(t), \hat{\xi}^{(1)}(t))$$

may not vanish to the order $O(|x|^3)$, which is a property that we require in the energy estimate. Then we add the second correction

$$(\hat{\omega}^{(1)}(t), \hat{\eta}^{(1)}, \hat{\xi}^{(1)}(t), \hat{\phi}^{N,(1)}) \rightarrow (\hat{\omega}^{(1)}(t) + a_1(t)\chi_{12}, \hat{\eta}^{(1)} + a_2(t)\chi_{22}, \hat{\xi}^{(1)}(t) + a_3(t)\chi_{32}, \hat{\phi}^{N,(1)} + a_1(t)\phi_2),$$

so that the error satisfies

$$(3.16) \quad \varepsilon_i^{(2)} \triangleq (\partial_t - \mathcal{L}_i)(\hat{\omega}^{(1)}(t) + a_1(t)\chi_{12}, \hat{\eta}^{(1)}(t) + a_2(t)\chi_{22}, \hat{\xi}^{(1)}(t) + a_3(t)\chi_{32}) = O(|x|^3)$$

near $x = 0$. We use the following functions for these two corrections

$$(3.17) \quad \begin{aligned} \chi_{11} &= -\Delta\phi_1, \quad \phi_1 = -\frac{xy^2}{2}\kappa_*(x)\kappa_*(y), \quad \chi_{21} = x\kappa_*(x)\kappa_*(y), \\ \chi_{12} &= -\Delta\phi_2, \quad \phi_2 = -\frac{xy^3}{6}\kappa_*(x)\kappa_*(y), \quad \chi_{22} = xy\kappa_*(x)\kappa_*(y), \quad \chi_{32} = \frac{x^2}{2}\kappa_*(x)\kappa_*(y), \end{aligned}$$

where $\kappa_*(x)$ is chosen in (D.5), $\chi_{\cdot,1}$ is used for the first correction, and $\chi_{\cdot,2}$ for the second correction. We do not have χ_{31} since we do need the first correction for ξ (3.13). Since $\kappa_*(x)$ satisfies $\kappa_*(x) = 1 + O(|x|^4)$ near $x = 0$, the behaviors of the above functions near $x = 0$ are given by

$$\chi_{11} = y + l.o.t., \quad \chi_{21} = x + l.o.t., \quad \chi_{12} = xy + l.o.t., \quad \chi_{22} = xy + l.o.t., \quad \chi_{32} = x^2/2 + l.o.t.$$

We choose $\chi_{1j} = -\Delta\phi_j$ for the correction of ω so that its associated velocity $\nabla^\perp(-\Delta)^{-1}\chi_{1j}$ can be obtained explicitly. We do not need such form for the correction of η, ξ since we do not compute the velocity of η, ξ .

For cutoff functions χ_1, χ_2, χ_3 with

$$(3.18) \quad c_\omega(\chi_1) = -\partial_{xy}(-\Delta)^{-1}\chi_1 = 0,$$

e.g. $\chi_i = \chi_{i2}$ chosen above, we have the following formulas of $\mathcal{L}_i(a_1(t)\chi_1, a_2(t)\chi_2, a_3(t)\chi_3)$ (3.1)

$$\begin{aligned} \mathcal{L}_1(a_1\chi_1, a_2\chi_2, a_3\chi_3) &= a_1(t) \left(-(\bar{c}_l x + \bar{\mathbf{u}}) \cdot \nabla \chi_1 + \bar{c}_\omega \chi_1 - \mathbf{u}(\chi_1) \cdot \nabla \bar{\omega} \right) + a_2(t) \chi_2, \\ \mathcal{L}_2(a_1\chi_1, a_2\chi_2, a_3\chi_3) &= a_2(t) \left(-(\bar{c}_l x + \bar{\mathbf{u}}) \cdot \nabla \chi_2 + (2\bar{c}_\omega - \bar{u}_x) \chi_2 \right) - a_3(t) \bar{v}_x \chi_3 - a_1(t) \left(\mathbf{u}(\chi_1) \cdot \nabla \bar{\theta} \right)_x, \\ \mathcal{L}_3(a_1\chi_1, a_2\chi_2, a_3\chi_3) &= a_3(t) \left(-(\bar{c}_l x + \bar{\mathbf{u}}) \cdot \nabla \chi_3 + (2\bar{c}_\omega + \bar{u}_x) \chi_3 \right) - a_2(t) \bar{u}_y \chi_2 - a_1(t) \left(\mathbf{u}(\chi_1) \cdot \nabla \bar{\theta} \right)_y, \end{aligned}$$

where $\mathbf{u}(\chi_1)$ is the velocity associated with χ_1 . We want to apply the above formulas to the second corrections $\chi_{i2}, i = 1, 2, 3$ in (3.17). We use the Hadamard product

$$(3.19) \quad (A \circ B)_i = A_i B_i,$$

and (3.12) to simplify the notation as follows

$$(3.20) \quad \begin{aligned} \mathcal{L}_i(a \circ \chi) &= Cor_{ij}(x; \chi) a_j(t), \quad Cor_{ij}(x; \chi) = Cor_{ij}^N(x; \chi) + Cor_{ij}^{\bar{e}}(x; \chi), \\ \mathcal{L}_i^N(a \circ \chi) &\triangleq Cor_{ij}^N(x; \chi) a_j(t), \quad \mathcal{L}_i^{\bar{e}}(a \circ \chi) \triangleq Cor_{ij}^{\bar{e}}(x; \chi) a_j(t). \end{aligned}$$

Note that $\mathcal{L}_i^e(a \circ \chi) = 0$ since we can obtain $\mathbf{u}(\chi_1)$ explicitly for $\chi_1 = \chi_{11}, \chi_{12}$ (3.17).

Next, we derive the equations for $a_i(t), i = 1, 2, 3$. Using (3.1) and the condition

$$\partial_{xy}\varepsilon_1^{(2)}(0) = \partial_{xy}\varepsilon_2^{(2)}(0) = \partial_{xx}\varepsilon_3^{(2)}(0) = 0,$$

from (3.16), we obtain the following ODEs for $a(t), b(t), c(t)$

$$(3.21) \quad \begin{aligned} \dot{a}_1(t) &= (-2\bar{c}_l + \bar{c}_\omega) a_1(t) + a_2(t) - F_1(t), \\ \dot{a}_2(t) &= (-2\bar{c}_l + 2\bar{c}_\omega - \bar{u}_x(0)) a_2(t) - F_2(t), \\ \dot{a}_3(t) &= (-2\bar{c}_l + 2\bar{c}_\omega - \bar{u}_x(0)) a_3(t) - F_3(t), \end{aligned}$$

where $F(t) = (F_1(t), F_2(t), F_3(t))^T$ is the error associated to the second order derivatives of $(\partial_t - \mathcal{L})\hat{W}^{(1)}$ near 0. More precisely, we have

$$(3.22) \quad \begin{aligned} F_1(t) &= \partial_{xy}(\partial_t - \mathcal{L}_1)\widehat{W}^{(1)}(0) = \frac{d}{dt}\hat{\omega}_{xy}^{(1)}(t, 0) - (-2\bar{c}_l + \bar{c}_\omega)\hat{\omega}_{xy}^{(1)}(t, 0) - \hat{\eta}_{xy}^{(1)}(t, 0) - c_\omega(t)\bar{\omega}_{xy}(0), \\ F_2(t) &= \partial_{xy}(\partial_t - \mathcal{L}_2)\widehat{W}^{(1)}(0) = \frac{d}{dt}\hat{\eta}_{xy}^{(1)}(t, 0) - (-2\bar{c}_l + 2\bar{c}_\omega - \bar{u}_x(0))\hat{\eta}_{xy}^{(1)}(t, 0) - c_\omega(t)\bar{\theta}_{xxy}(0), \\ F_3(t) &= \partial_x^2(\partial_t - \mathcal{L}_3)\widehat{W}^{(1)}(0) = \frac{d}{dt}\hat{\xi}_{xx}^{(1)}(t, 0) - (-2\bar{c}_l + 2\bar{c}_\omega - \bar{u}_x(0))\hat{\xi}_{xx}^{(1)}(t, 0) - c_\omega(t)\bar{\theta}_{xxy}(0). \end{aligned}$$

Denote

$$(3.23) \quad D^2 = (D_1^2, D_2^2, D_3^2) = (\partial_{xy}, \partial_{xy}, \partial_x^2)^T.$$

Then we can simplify (3.22) as

$$(3.24) \quad F_i = D_i^2(\partial_t - \mathcal{L}_i)\hat{W}^{(1)}(0) = D_i^2(\partial_t - \mathcal{L}_i^N - \mathcal{L}_i^e - \mathcal{L}_i^{\bar{e}})\hat{W}^{(1)}(0).$$

Denote by M the coefficients in (3.21)

$$(3.25) \quad M = \begin{pmatrix} -2\bar{c}_l + \bar{c}_\omega & 1 & 0 \\ 0 & -2\bar{c}_l + 2\bar{c}_\omega - \bar{u}_x(0) & 0 \\ 0 & 0 & -2\bar{c}_l + 2\bar{c}_\omega - \bar{u}_x(0) \end{pmatrix} \triangleq M^N + M^{\bar{e}},$$

where the last identity is based on the decomposition $\bar{c}_\omega = \bar{c}_\omega^N + \bar{c}_\omega^{\bar{e}}$, $\bar{u}_x(0) = \bar{u}_x^N(0) + \bar{u}_x^{\bar{e}}(0)$, and $M^{\bar{e}}$ only contains the contribution from $\bar{c}_\omega^{\bar{e}}, \bar{u}_x^{\bar{e}}(0)$. According to the normalization condition (3.2), we have $\bar{u}_x(0)^{\bar{e}} = \bar{c}_\omega^{\bar{e}}$. It follows

$$(3.26) \quad M^{\bar{e}} = \bar{c}_\omega^{\bar{e}} I_3.$$

We simplify the ODE for $a = (a_1, a_2, a_3)^T$ as

$$(3.27) \quad \dot{a}_i(t) = M_{ij} a_j(t) - F_i(t), \quad \dot{a}(t) = Ma - F = Ma - e_i D_i^2 (\partial_t - \mathcal{L}_i) \widehat{W}^{(1)}(0).$$

Recall $\chi_{\cdot 2} = (\chi_{12}, \chi_{22}, \chi_{32})$ from (3.17). In the i -th equation, the overall error for the approximate solution $\widehat{W}^{(1)} + a(t) \circ \chi_{\cdot 2}$ is

$$(3.28) \quad \begin{aligned} (\partial_t - \mathcal{L}_i)(\widehat{W}^{(1)} + a(t) \circ \chi_{\cdot 2}) &= (\partial_t - \mathcal{L}_i^N)(a(t) \circ \chi_{\cdot 2}) + \left((\partial_t - \mathcal{L}_i^N) \widehat{W}^{(1)} \right. \\ &\quad \left. - \mathcal{L}_i^{\bar{e}}(\widehat{W}^{(1)} + a(t) \circ \chi_{\cdot 2}) - \mathcal{L}_i^{\bar{e}}(\widehat{W}^{(1)} + a(t) \circ \chi_{\cdot 2}) \right) \triangleq J + I. \end{aligned}$$

Note that in the above notation, ∂_t acts on $a_i(t)\chi_{i,2}$. For J , using the ODE for $a(t)$ (3.27), (3.20), (3.24), and (3.25), we get

$$(3.29) \quad \begin{aligned} J &= (M_{ij} a_j - F_i) \chi_{i2} - Cor_{ij}^N(x; \chi_{\cdot 2}) a_j \\ &= (M_{ij}^N \chi_{i2} - Cor_{ij}^N(x; \chi_{\cdot 2})) a_j + M_{ij}^{\bar{e}} a_j \chi_{i2} - D_i^2 (\partial_t - \mathcal{L}_i^N - \mathcal{L}_i^{\bar{e}} - \mathcal{L}_i^{\bar{e}}) \widehat{W}^{(1)}(0) \chi_{i2} \triangleq J_1 + J_2 + J_3, \end{aligned}$$

where we have summation over $j = 1, 2, 3$. Since $\mathcal{L}^e(a(t) \circ \chi_{\cdot 2}) = 0$, using the above decomposition and combining I, J_2, J_3 , we yield

$$(3.29) \quad \begin{aligned} I + J_2 + J_3 &= \left((\partial_t - \mathcal{L}_i^N) \widehat{W}^{(1)} - D_i^2 (\partial_t - \mathcal{L}_i^N) \widehat{W}^{(1)}(0) \chi_{i2} \right) - \left(\mathcal{L}_i^{\bar{e}} \widehat{W}^{(1)} - D_i^2 \mathcal{L}_i^{\bar{e}} \widehat{W}^{(1)}(0) \chi_{i2} \right) \\ &\quad - \left(\mathcal{L}_i^{\bar{e}}(\widehat{W}^{(1)} + a(t) \circ \chi_{\cdot 2}) - D_i^2 \mathcal{L}_i^{\bar{e}}(\widehat{W}^{(1)} + a(t) \circ \chi_{\cdot 2})(0) \chi_{i2} - M_{ij}^{\bar{e}} a_j \chi_{i2} \right) \triangleq I_{i,N} + I_{i,e} + I_{i,\bar{e}}. \end{aligned}$$

Next, we check that $J_1, I_{i,N}, I_{i,e}, I_{i,\bar{e}}$ have a vanishing order $O(|x|^3)$. This is clear for $I_{i,N}, I_{i,e}$. Since we correct the second order derivatives and $\hat{\omega}^{(1)}, \hat{\eta}^{(1)}, \hat{\zeta}^{(1)}$ are odd with $\hat{\xi}^{(1)} = x\hat{\zeta}^{(1)}$, we get $\partial_x^i \partial_y^j I_{i,N}, \partial_x^i \partial_y^j I_{i,e} = 0, i + j \leq 2$ at the origin. For J_1 , we note that it is a linear combination of a_j with given coefficients $M_{ij}^N - Cor_{ij}^N$. Its cubic vanishing order follows from the definition. For example, when $i = j = 1$, we have

$$S = a_1(t) \cdot (Cor_{11}^{\bar{e}}(x) - M_{11}^{\bar{e}} \chi_{12}) = a_1(t) \left(-\bar{\mathbf{u}}^e \cdot \nabla \chi_{12} + \bar{c}_\omega^e \chi_{12} - \bar{c}_\omega^e \chi_{12} \right) = a_1(t) \left(-\bar{\mathbf{u}}^e \cdot \nabla \chi_{12} \right).$$

Since $\chi_{12} = xy + O(|x|^4)$ (3.17), $\bar{u}^e = \bar{u}_x^e(0)x + O(|x|^2), \bar{v}^e = -\bar{u}_x^e(0)y$ near 0, we have $S = O(|x|^3)$ near 0. The vanishing order of other terms in J_1 can be obtained similarly. Then for J_1 , we estimate the weighted norm for $Cor_{ij}^{\bar{e}}(x) - M_{ij}^{\bar{e}} \chi_{i2}$ and then apply the triangle inequality to further bound J_1 . Similarly, for a fixed i , we have the following vanishing order

$$\mathcal{L}_i^{\bar{e}}(a(t) \circ \chi_{\cdot 2}) - M_{ij}^{\bar{e}} a_j \chi_{i2} = Cor_{ij}^{\bar{e}} a_j(t) - M_{ij}^{\bar{e}} a_j \chi_{i2} = O(|x|^3), \quad D_i^2 \mathcal{L}_i^{\bar{e}}(a(t) \circ \chi_{\cdot 2})(0) = M_{ij}^{\bar{e}} a_j.$$

Thus, we can rewrite $I_{i,\bar{e}}$ as follows

$$(3.30) \quad I_{i,\bar{e}} = - \left(\mathcal{L}_i^{\bar{e}}(\widehat{W}^{(1)} + a(t) \circ \chi_{\cdot 2}) - D_i^2 \mathcal{L}_i^{\bar{e}}(\widehat{W}^{(1)} + a(t) \circ \chi_{\cdot 2})(0) \chi_{i2} \right),$$

which clearly has a cubic vanishing order. Note that $\widehat{W}^{(1)} + a(t) \circ \chi_{\cdot 2}$ is our final approximate solution for solving (3.1).

In summary, to estimate the error $(\partial_t - \mathcal{L})(\widehat{W}^{(1)} + a \circ \chi_{\cdot 2})$, we will estimate $J_1, I_{i,N}, I_{i,e}, I_{i,\bar{e}}$ separately. The term $I_{i,N}$ is the local error of solving (3.1) numerically, $I_{i,e}, I_{i,\bar{e}}$ are due to the error of solving the Poisson equations for ω and $\hat{\omega}^{(1)}$. Since we use a cubic polynomial interpolation to obtain the continuous function $\widehat{W}^{(1)}(t)$, the errors $I_{i,N}, I_{i,e}$ are piecewise cubic

polynomials in time, and we track the coefficients of these polynomials to verify that they are small. We discuss the estimate of nonlocal error in Section 3.7.

3.4. Cubic interpolation in time. Given the numerical solution with the first correction $\widehat{W}_n^{(1)} = (\widehat{\omega}_n^{(1)}, \widehat{\eta}_n^{(1)}, \widehat{\xi}_n^{(1)})$, we use a piecewise cubic interpolation to construct $\widehat{W}^{(1)}(t, x)$ over $(t, x) \in [0, T] \times \mathbb{R}_2^+$. We partition the whole time interval $[0, T]$ into small subintervals $[3mk, 3(m+1)k]$ with length $3k$. For $s \in [-3k/2, 3k/2]$ and $t_m = 3mk$, we construct

$$\begin{aligned} \widehat{W}^{(1)}(s + t_m + \frac{3k}{2}) &= \frac{1}{16}(-W_0 + 9W_1 + 9W_2 - W_3) + \frac{1}{24}(W_0 - 27W_1 + 27W_2 - W_3) \frac{s}{k} \\ &\quad + \frac{1}{4}(W_0 - W_1 - W_2 + W_3) \left(\frac{s}{k}\right)^2 + \frac{1}{6}(-W_0 + 3W_1 - 3W_2 + W_3) \left(\frac{s}{k}\right)^3 \\ &\triangleq \sum_{i \leq 3} C_i \cdot V \frac{1}{i!} \left(\frac{s}{k}\right)^i, \quad V = (W_0, W_1, W_2, W_3), \end{aligned}$$

where k is the time step, $W_i = \widehat{W}_{3m+i}^{(1)}$ for $t_m = 3mk$, and $C_i \in \mathbb{R}^4$ is the coefficient determined by the interpolation formula. A direct calculation yields

$$\begin{aligned} \partial_t \widehat{W}^{(1)} - \mathcal{L} \widehat{W}^{(1)} &= \sum_{1 \leq i \leq 3} \frac{C_i \cdot V}{k} \frac{1}{(i-1)!} \left(\frac{s}{k}\right)^{i-1} - \sum_{i \leq 3} \mathcal{L}(C_i \cdot V) \frac{1}{i!} \left(\frac{s}{k}\right)^i \\ &= \sum_{i \leq 2} \left(\frac{C_{i+1} \cdot V}{k} - \mathcal{L}(C_i \cdot V) \right) \frac{1}{i!} \left(\frac{s}{k}\right)^i - \mathcal{L}(C_4 \cdot V) \frac{s^3}{6k^3}. \end{aligned}$$

To estimate $\partial_t \widehat{W}^{(1)} - \mathcal{L} \widehat{W}^{(1)}$, we will use the triangle inequality and estimate $\frac{C_{i+1} \cdot V}{k} - \mathcal{L}(C_i \cdot V)$, $\mathcal{L}(C_4 \cdot V)$ rigorously using the methods in Section 3.6, 3.7.

Applying the triangle inequality and integrating the error over $s \in [-\frac{3k}{2}, \frac{3k}{2}]$ yield

$$\begin{aligned} (3.31) \quad &\int_{|s| \leq 3k/2} |\partial_t \widehat{W}^{(1)} - \mathcal{L} \widehat{W}^{(1)}| ds \leq \sum_{i \leq 2} \left| \frac{C_{i+1} \cdot V}{k} - \mathcal{L}(C_i \cdot V) \right| \int_{|s| \leq 3k/2} \frac{1}{i!} \left|\frac{s}{k}\right|^i \\ &+ |\mathcal{L}(C_4 \cdot V)| \int_{|s| \leq \frac{3k}{2}} \frac{1}{6} \left|\frac{s}{k}\right|^3 = k \left(\sum_{i \leq 2} \left| \frac{C_{i+1} \cdot V}{k} - \mathcal{L}(C_i \cdot V) \right| C_I(i) + |\mathcal{L}(C_4 \cdot V)| C_I(3) \right), \end{aligned}$$

where

$$C_I = \left[3, \frac{9}{4}, \frac{9}{8}, \frac{27}{64}\right].$$

3.4.1. Decomposing the time interval for parallel computing. To verify that the posteriori error is small, we need to estimate the error rigorously at each time step, which takes a significant amount of time. Consider a partition of the time interval $0 = T_0 < T_1 < \dots < T_n = T$, where T is the final time of the computation. To reduce the computational time, we first solve the equations on $[0, T]$ without any rigorous verification and save the solution $(\omega_k, \eta_k, \xi_k, \phi_{k,1}^N)$ at $t_k = T_i$. Since we do not need to perform verification at this step, the running time for each time step is short. Then we solve the equations on a smaller time interval $[T_i, T_{i+1}]$, $i = 0, 1, 2, \dots, n-1$ using $W(T_i)$ as the initial data and then perform the verification in each time interval in parallel. At the end of each time interval $[T_i, T_{i+1}]$, we use the pre-computed data $W(T_{i+1})$, which is the same as the initial data for next time interval $[T_{i+1}, T_{i+2}]$ for verification. This guarantees that we use the same discrete solution $(\omega_k, \eta_k, \xi_k, \phi_{k,1}^N)$ for verification in $[T_i, T_{i+1}]$ and $[T_{i+1}, T_{i+2}]$.

3.5. Compactly supported in time. To construct an approximate solution, we do not need to solve the linearized equations (3.1) for all time. In fact, since the solution decays in certain norm as t increases, we stop the computation at time T if $\widehat{W}^{(1)} - D^2 \widehat{W}^{(1)} \circ \chi$ is small in the energy norm. Then we extend $\widehat{W}^{(1)}(t, \cdot)$ trivially for $t > T$

$$\widehat{W}^{(1)}(t, \cdot) = 0, \quad t > T.$$

As a result, the error satisfies

$$\mathcal{R}_i = (\partial_t - \mathcal{L}_i) \widehat{W}^{(1)} = (\partial_t - \mathcal{L}_i) \widehat{W}^{(1)} \mathbf{1}_{t \leq T} - \delta_T(t) \widehat{W}_i^{(1)}(T).$$

Let $F = (F_1, F_2, F_3)$, $F_i = D_i^2(\partial_t - \mathcal{L}_i)\widehat{W}^{(1)}\Big|_{x=0}$ for $t \leq T$, where $D^2 = (D_{xy}, D_{xy}, D_x^2)$. Then similarly, we get

$$F_{ext} \triangleq D^2(\partial_t - \mathcal{L})\widehat{W}^{(1)}\Big|_{x=0} \mathbf{1}_{t \leq T} - D^2\widehat{W}^{(1)}(T, 0)\delta_T = F(t)\mathbf{1}_{t \leq T} - F_{end}(T)\delta_T, \quad F_{end}(T) \triangleq D^2\widehat{W}^{(1)}(T, 0).$$

We will test the above formulas with some Lipschitz function in time and the above formulas are well defined. Recall that the coefficients of the second correction a satisfy (3.27). Although \widehat{W} only has finite support in time, to achieve the vanishing order (3.16) for all time, we need to solve the ODE exactly for all time. If we stop solving the ODE at time T , we cannot achieve (3.16) at time T . Moreover, we cannot solve the ODE using a numerical method, e.g. the Runge-Kutta method, since it leads to an error. Instead, we solve the ODE exactly by diagonalizing the system. We introduce the following notations

$$(3.32) \quad \begin{aligned} \lambda_1 &= -2\bar{c}_l + \bar{c}_\omega, & \lambda_2 &= \lambda_3 = -2\bar{c}_l + 2\bar{c}_\omega - \bar{u}_x(0), & \lambda_1 - \lambda_2 &= -\bar{c}_l/2, \\ \tilde{a}_1 &= a_1 + \frac{a_2}{\lambda_1 - \lambda_2}, & \tilde{F}_1 &= F_1 + \frac{F_2}{\lambda_1 - \lambda_2}, & \tilde{a}_i &= a_i, \quad \tilde{F}_i = F_i, \quad i = 2, 3, \end{aligned}$$

and similar notations for \tilde{F}_{ext} , where we have used (2.11) to get $\lambda_1 - \lambda_2 = -\bar{c}_l/2$. The coefficients satisfy $\lambda_1 \approx -7$, $\lambda_2 = \lambda_3 \approx -5.5$. We diagonalize (3.21) as follows

$$\frac{d}{dt}\tilde{a}_i = \lambda_i\tilde{a}_i - \tilde{F}_{ext,i}.$$

Using Duhamel's formula and the definition of $\tilde{F}_{ext,i}$, we yield

$$(3.33) \quad \begin{aligned} \tilde{a}_j(t) &= e^{\lambda_j t}\tilde{a}_j(0) - \int_0^t e^{\lambda_j(t-s)}\tilde{F}_{ext,j}(s)ds \\ &= e^{\lambda_j t}\tilde{a}_j(0) - \int_0^{t \wedge T} e^{\lambda_j(t-s)}\tilde{F}_j(s)ds + \tilde{F}_{end}(T)e^{\lambda_j(t-T)}\mathbf{1}_{t \geq T} \triangleq S_1 + S_2 + S_3. \end{aligned}$$

For rank one perturbation, the full solution \widehat{W} with two corrections in (3.6),(3.7) is given by

$$(3.34) \quad \widehat{W} = \widehat{W}^{(1)} + a \circ \chi_{.2}, \quad \hat{\phi}^N = \hat{\phi}^{N,(1)} + a_1(t)\phi_2,$$

where $\chi_{.2}, \phi_2$ are defined in (3.17). With the above extension and the decomposition of error (3.28)-(3.29), the residual error for rank-one perturbation (3.7) with $n = 1$ is given by

$$(3.35) \quad \begin{aligned} \mathcal{R} &= c(t)(\widehat{W}_0^{(1)} + a_0 \circ \chi_{.2} - \bar{W}_0) + \int_0^t c(t-s)(\partial_t - \mathcal{L})(\widehat{W}^{(1)} + a \circ \chi_{.2})ds = \mathcal{R}_{loc,0,.} + \mathcal{R}_{nloc} \\ \mathcal{R}_{loc,0,.} &= c(t)(\widehat{W}_0^{(1)} + a_0 \circ \chi_{.2} - \bar{W}_0) - (\widehat{W}^{(1)}(T) - D^2\widehat{W}^{(1)}(T) \circ \chi_{.2})c(t-T)\mathbf{1}_{t \geq T} \\ &+ \int_0^{t \wedge T} c(t-s) \sum_{i \leq 3} e_i I_{i,N}(s)ds + \int_0^t c(t-s) \sum_{i \leq 3} e_i J_{1,i}(s)ds = \int_0^t c(t-s)\mathcal{R}_{num}(s)ds, \\ \mathcal{R}_{nloc} &= \int_0^{t \wedge T} c(t-s) \sum_{i \leq 3} e_i I_{i,e}(s)ds + \int_0^t c(t-s) \sum_{i \leq 3} e_i I_{i,\bar{e}}(s)ds, \end{aligned}$$

where $I_{i,N}, I_{i,e}, I_{i,\bar{e}}$ are given in (3.29), $J_{1,i}$ means J_1 (3.29) in the i -th equation, and \mathcal{R}_{num} is

$$(3.36) \quad \mathcal{R}_{num}(s) \triangleq \delta_0 \cdot (\widehat{W}_0^{(1)} + a_0 \circ \chi_{.2} - \bar{W}_0) - \delta_T \cdot (\widehat{W}^{(1)}(T) - D^2\widehat{W}^{(1)}(T) \circ \chi_{.2}) + e_j(\mathbf{1}_{t \leq T} I_{j,N} + J_{1,j}).$$

We only integrate the integrals for $e_i I_{i,N}, e_i I_{i,e}$ up to $\min(t, T)$ since these two integrands (3.29) do not involve $a_i(t)$ and have compact support $[0, T]$ in time. We obtain the local part $\mathcal{R}_{loc,0,.}$ in (3.8) for $n = 1$. The first term is the initial interpolation error for \bar{W}_0 , and we choose $a_0 \in \mathbb{R}^3$ to achieve vanishing order $\widehat{W}_0^{(1)} + a_0 \circ \chi - \bar{W}_0 = O(|x|^3)$. We use $\mathcal{R}_{loc,0,.}, \mathcal{R}_{nloc}$ to denote the error that depends on the solution locally and nonlocally. We use the bootstrap assumption to obtain uniform control of $c(t)$ in t . See Section 5.7 in Part I [14]. The error estimate of the local part $\mathcal{R}_{loc,0,.}$ follows Section 3.6. Moreover, we extract the essentially local part from \mathcal{R}_{nloc} and can estimate it with $\mathcal{R}_{loc,0,j}$ together (3.38). We decompose the nonlocal part \mathcal{R}_{nloc} in Section

3.7. To control the terms involving a_i , e.g. $J_{1,i}$ above (3.29), we can estimate the weighted norm of the functions $Cor_{ij}^N(x) - M_{ij}^N \chi_{i2}$ and then only need to estimate the integral of \tilde{a}_j .

Denote $x \wedge y \triangleq \min(x, y)$. Since the factor $\lambda_j < 0$, using the formula of \tilde{a}_j (3.33), we obtain

$$\begin{aligned} \int_0^\infty |S_1| dt &= \frac{1}{|\lambda_j|} |\tilde{a}_j(0)|, \quad \int_0^\infty |S_3(t)| dt = \int_T^\infty |\tilde{F}_{end,j}(T)| e^{\lambda_j(t-T)} dt = \frac{1}{|\lambda_j|} |\tilde{F}_{end,j}(T)|, \\ \int_0^\infty |S_2(t)| dt &\leq \int_0^\infty \left(\int_0^{t \wedge T} e^{\lambda_j(t-s)} |\tilde{F}_j(s)| ds \right) dt = \int_0^T |\tilde{F}_j(s)| \left(\int_s^\infty e^{\lambda_j(t-s)} dt \right) ds = \frac{1}{|\lambda_j|} \int_0^T |\tilde{F}_j(s)| ds. \end{aligned}$$

It follows

$$\int_0^\infty |\tilde{a}_j(t)| dt \leq \frac{1}{|\lambda_j|} \left(|\tilde{a}_j(0)| + \int_0^T |\tilde{F}_j(s)| ds + |\tilde{F}_{end,j}(T)| \right).$$

In the estimate of integral of \tilde{F}_j , (3.22), (3.32), we use $\bar{c}_\omega = \bar{c}_\omega^N + \bar{c}_\omega^e$, $c_\omega = c_\omega^N + c_\omega^e$ (3.11) and track the terms involving $\bar{c}_\omega^N, c_\omega^N$ in I_{e1} and error separately,

$$\begin{aligned} F_{j,e1} &= \int_0^T |F_j^N(t)| dt, \quad F_{j,e2} = \int_0^T |\bar{c}_\omega^e D_i^2 \widehat{W}^{(1)}(t, 0)| dt, \quad F_{j,e3} = \int_0^T |c_\omega^e(t, 0) D_i^2 \bar{W}(0)| dt, \\ F_j^N &= D_j^2 (\partial_t - \mathcal{L}_i^N) \widehat{W}^{(1)}(0), \quad D^2 = (\partial_{xy}, \partial_{xy}, \partial_{xx}). \end{aligned}$$

From (3.11), we get $2\bar{c}_\omega^e - \bar{u}_x^e(0) = \bar{u}_x^e(0)$ and only 1 unit of error I_{e2} in $F_j(t)$, $j = 2, 3$. We track \tilde{F}_j (3.32) similarly. Since $\widehat{W}^{(1)}, F, \tilde{F}, F^N, \tilde{F}^N$ (3.22) are cubic in time, we can estimate the above integrals following (3.31). Note that $|\tilde{F}_{end,j}(T)|$ does not involve the nonlocal error. Using the linear relation between a_j, \tilde{a}_j , we can estimate a_j .

Using the above estimates, we can represent the rank-one solution and estimate it as follows

$$\begin{aligned} \hat{G}(t, x) &= \int_0^t c(t-s) \hat{W}(s) ds, \quad \hat{W} = \hat{W}^{(1)} + a \circ \chi_{.2} \\ (3.37) \quad |\partial_x^i \partial_y^j G_i(t, x)| &\leq \sup_{t>0} |c(t)| \left(\int_0^T |\partial_x^i \partial_y^j \hat{W}_l^{(1)}(t)| dt + |\partial_x^i \partial_y^j \chi_{l2}| \int_0^\infty |a_l(t)| dt \right). \end{aligned}$$

Similarly, we can bound other quantities for \hat{G} and complete the estimates in (3.6).

We generalize the above formula and estimate directly to the finite rank perturbation operator using linearity. For different initial data \bar{W}_0 related to the finite rank perturbation, we choose a different stopping time $T(\widehat{W}_0^{(1)})$ to save computation cost. In practice, we construct the numerical solution up to time $T(\widehat{W}_0^{(1)}) \leq T = 12$. At that time, the solution $\hat{W}^{(1)}(T)$ is very small, which can be treated as a small perturbation. See figures in Section 4.3 in Part I [14].

Remark 3.1. Using linearity and the triangle inequality, we can assemble the estimates for \mathcal{R} (3.7) from the estimates of each mode \hat{W}_i in (3.6), (3.7). In practice, this means that we can implement the above estimate for each individual mode completely in parallel.

Finite support of the c_ω term in time. In Section 5 of Part I [14], we need to use $c_\omega(f)$, where $c_\omega(f) = u_x(f)(0) = -\partial_{xy}(-\Delta)^{-1} f(0)$. Since we choose the cutoff function χ_{12} for the second correction of $\hat{\omega}$ with properties (3.17), (3.18), we get

$$c_\omega(\hat{W}_1^{(1)} + a_1(t)\chi_{12}) = c_\omega(\hat{W}_1^{(1)}),$$

and it is supported in $[0, T]$.

3.6. Ideas of estimating the norm of the error. In this section, we discuss how to estimate the error derived in the previous section, e.g. $I_{i,N}$ (3.29), a-posteriori. The general idea is to first evaluate f on some grid points and estimate the higher order derivatives of f in a domain D . Then we can construct an approximation \hat{f} of f by interpolating the values of f at different points. The approximation error $f - \hat{f}$ can be bounded by $C_k \|f\|_{C^k} h^k$, where h measures the size of the domain. If the mesh h is sufficiently small, the error term is small. See a simple second order error estimate in (C.12).

To develop an efficient method for rigorous estimates, we have the following considerations. Firstly, we should evaluate as a small number of points as possible so that the method is efficient. Secondly, most functions f in the verification are complicated, e.g. $I_{i,N}$ (3.29), and it is difficult to obtain the sharp bound of the higher derivatives. Instead, we first estimate the piecewise derivatives of some simple functions, e.g. piecewise polynomials $(\hat{\omega}, \hat{\eta})$ or semi-analytic solutions following Appendix C, D. Then we use the triangle inequality and the Leibniz rule to estimate the products of these simple functions, and their linear combinations. Yet, in general, this approach overestimates the derivatives significantly. To compensate the overestimates, we use higher order interpolations and estimates with error bounds $Ch^k, k = 3, 4, 5$, which provide the small factor h^k . We develop three estimates based on different interpolations: the Newton interpolation, the Lagrangian interpolation, and the Hermite interpolation in Section 8 in the supplementary material II [12] (attached to this paper). The 1D interpolating polynomials are standard, and we generalize them to construct 2D interpolating polynomials.

We want to estimate the constant C in the error bound Ch^k as sharp as possible to reduce the computational cost and improve the efficiency. In fact, when $k = 4$, if we can obtain an interpolation method and reduce the constant C to $\frac{C}{16}$, to achieve the same level of error, we can increase h to $2h$. In this verification step, since the domain is 2D, it means that we can evaluate only $\frac{1}{4}$ of the grid point values of f , which can reduce the computational cost by 75%.

Using the above method, we can obtain a sharp estimate of the derivatives of f . Using the method in Section 8 in the supplementary material II [12] and Taylor expansion, we can further estimate the weighted norm of f with a singular weight near 0. We discuss the estimate of the nonlocal error in Section 3.7. Using these L^∞ estimates of f and its derivatives, we can further develop Hölder estimate for f . See Section E.1. We remark that the numerical solutions are regular, e.g. the approximate steady state and the solutions to the linearized equations are $C^{4,1}$. We use these methods to estimate piecewise $L^\infty(\varphi_{evo,i})$ norm of the local residual error $\mathcal{R}_{num,i}$ (3.36) and the $C_{x_i}^{1/2}$ partial Hölder seminorm of $\mathcal{R}_{num,i}\psi_i$, where $\varphi_{evo,i}, \psi_i$ are defined in (A.3).

We remark that the weights $\varphi_{evo,i}$ and $\varphi_i, i = 2, 3$ in the L^∞ energy estimate (see Section 5 in [14]) for η, ξ are similar but with different coefficient $p_{5,..}, p_{6,..}$. Since φ_i and $\varphi_{evo,i}$ are equivalent, after we obtain the piecewise weighted $L^\infty(\varphi_{evo,i})$ estimate of the error, we can obtain piecewise weighted $L^\infty(\varphi_i)$ estimate by estimating the ratio $\varphi_i/\varphi_{evo,i}$. Similarly, we can obtain weighted $L^\infty(\varphi_{g,i})$ estimate of the error, where $\varphi_{g,i}$ is another weight in the energy estimate in Section 5 in [14].

Estimate the local part of the residual error. Using the above methods, we can estimate the local part of the residual error $\bar{\mathcal{F}}_i$ for the approximate steady state and discuss the estimate in Appendix C.4. We further extract the local part of \mathcal{R}_{nloc} (3.35), which has the form (3.8) obtained in Section 3.7, and combine it with $\mathcal{R}_{loc,0,j}$ to get the essentially local residual error

$$(3.38) \quad \begin{aligned} \mathcal{R}_{loc,i} &= \mathcal{R}_{loc,0,i} + \mathcal{R}_{dif,i} + M, \quad \mathcal{R}_{dif,i} \triangleq D_i^2 \mathcal{B}_{op,i}(\mathbf{u}(\bar{\varepsilon}), \hat{G})(0) \cdot (\chi_{i2} - f_{\chi,i}), \\ M &\triangleq \mathcal{B}_{op,i}(\mathbf{u}(\hat{\varepsilon}), \bar{W}) - D_i^2 \mathcal{B}_{op,i}(\mathbf{u}(\hat{\varepsilon}), \bar{W})(0) \chi_{i2} - \mathcal{B}_{op,i}(\mathbf{u}_A(\hat{\varepsilon}_1), (\nabla \mathbf{u})_A(\hat{\varepsilon}_1), \bar{W}). \end{aligned}$$

where χ_{i2} is defined in (3.17). By definition (3.39) and following derivation of (3.24), we get

$$D_i^2 \mathcal{B}_{op,i}(\mathbf{u}(\bar{\varepsilon}), \hat{G})(0) = u_x(\bar{\varepsilon})(0) V_i, \quad V = (\hat{G}_{1,xy}(0), \hat{G}_{2,xy}(0), \hat{G}_{3,xx}(0)).$$

To estimate each term, we follow Section 3.6 and Appendix C.4. We perform the decomposition (C.18) $\mathbf{u}(\hat{\varepsilon}) = \mathbf{u}_A(\hat{\varepsilon}_1) + \hat{\mathbf{u}}(\hat{\varepsilon}_1) + \mathbf{u}(\hat{\varepsilon}_2)$ and similar decomposition for $\nabla \mathbf{u}(\hat{\varepsilon})$, with $(\bar{\varepsilon}, \chi_{\bar{\varepsilon}})$ in (C.18) replaced by $(\hat{\varepsilon}, \chi_{\hat{\varepsilon}})$, where $\chi_{\hat{\varepsilon}}$ is defined in (D.6). Using linearity of $\mathcal{B}_{op,i}$, we get

$$\begin{aligned} \mathcal{B}_{op,i}(\mathbf{u}(\hat{\varepsilon}), \bar{W}) - \mathcal{B}_{op,i}(\mathbf{u}_A(\hat{\varepsilon}_1), (\nabla \mathbf{u})_A(\hat{\varepsilon}_1), \bar{W}) &= II_i(\hat{\varepsilon}_1) + II_i(\hat{\varepsilon}_2), \\ II_i(\hat{\varepsilon}_1) &= \mathcal{B}_{op,i}(\hat{\mathbf{u}}(\hat{\varepsilon}_1), \widehat{\nabla \mathbf{u}}(\hat{\varepsilon}_1), \bar{W}), \quad II_i(\hat{\varepsilon}_2) = \mathcal{B}_{op,i}(\mathbf{u}(\hat{\varepsilon}_2), \nabla \mathbf{u}(\hat{\varepsilon}_2), \bar{W}). \end{aligned}$$

We have $\mathbf{u}_A = O(|x|^3)$, $(\nabla \mathbf{u})_A = O(|x|^2)$ near 0, which implies $\mathcal{B}_{op,i}(\mathbf{u}_A(\hat{\varepsilon}_1), (\nabla \mathbf{u})_A(\hat{\varepsilon}_1), \bar{W}) = O(|x|^3)$ (3.4) and

$$M = II_i(\hat{\varepsilon}_1) + II_i(\hat{\varepsilon}_2) - D_i^2 \mathcal{B}_{op,i}(\mathbf{u}(\hat{\varepsilon}), \bar{W})(0) \chi_{i2} = II_i(\hat{\varepsilon}_1) + II_i(\hat{\varepsilon}_2) - D_i^2 (II_i(\hat{\varepsilon}_1) + II_i(\hat{\varepsilon}_2))(0) \chi_{i2}.$$

The term $I_{i,N}$ in $\mathcal{R}_{loc,0,i}$ (3.29),(3.35) is similar to II_i^N , and M has a similar form as $II_i(\bar{\varepsilon}_1) + II_i(\bar{\varepsilon}_2)$ in Appendix C.4. We have done the above decomposition for $\mathbf{u}(\bar{\varepsilon})$ in (C.18), (C.19), and refer therein for more details. Then the estimate of $\mathcal{R}_{loc,j}$ is similar to that in Appendix C.4. See Section 5.8 in [14] for more discussion of the above forms.

Error for the initial data and at stopping time. The error $\widehat{W}^{(1)}(T) - D^2\widehat{W}^{(1)}(t) \circ \chi_{.2}$ at the stopping time has compact support and its estimate follows the methods in Section 3.6. To bound the initial interpolation error $err_{in} \triangleq \widehat{W}_0^{(1)} + a_0 \circ \chi_{.2} - \bar{W}_0$ (3.35) in a large domain, we follow similar methods. The error involves $\bar{\omega}, \bar{\theta}$ which are supported globally. To bound err_{in} in the middle and far-field, since $\bar{W}_0^{(1)} + a_0 \circ \chi_{i,2} = 0$, combining all the initial data from the finite rank perturbation (see Appendix C.2.1 of Part I [14]), we need to estimate

$$\begin{aligned} I_1 &= c_\omega(\omega_1)\bar{\omega} - \hat{\mathbf{u}}(\omega_1) \cdot \nabla \bar{\omega}, & I_2 &= 2c_\omega(\omega_1)\bar{\theta}_x - \hat{\mathbf{u}} \cdot \nabla \bar{\theta}_x - \hat{\mathbf{u}}_x \cdot \nabla \bar{\theta}, \\ I_3 &= 2c_\omega(\omega_1)\bar{\theta}_y - \hat{\mathbf{u}} \cdot \nabla \bar{\theta}_y - \hat{\mathbf{u}}_y \cdot \nabla \bar{\theta}, \end{aligned}$$

for large $|x|$. The approximation terms near 0 defined in Section 4.2.1 of Part I [14] are supported near 0 and decay to zero as $|x| \rightarrow \infty$. In the far-field, $\hat{\mathbf{u}}(\omega_1)$ is only a rank-one term. We estimate the above terms using (C.21), (C.22) with $a = c_\omega(\omega_1)$ and the estimates in Section C.4.

3.7. Posteriori error estimates of the velocity. In this section, we show that the nonlocal error in (3.35) has the desired forms in (3.8). Then we combine the estimate of such terms with the nonlinear energy estimate in Section 5.8 in [14]. Using (3.5) and the definition of $\mathcal{L}^{\bar{\varepsilon}}, \mathcal{L}^e$ (3.11), (3.12), we have

$$(3.39) \quad \mathcal{L}_j^{\bar{\varepsilon}}(G) = \mathcal{B}_{op,j}(\mathbf{u}(\bar{\varepsilon}), G), \quad \mathcal{L}_j^e(G) = \mathcal{B}_{op,j}(\mathbf{u}(G + (-\Delta)\phi_G^N), \bar{W}),$$

where ϕ_G^N is the numerical stream function associated with G .

Given $c_i(t)$ Lipschitz in t and $\bar{W}_i(0), i = 1, 2, \dots, n$, we construct $\hat{W}_i(t)$ following previous sections and \hat{G} using (3.6). Using the derivations in (3.35), (3.29), (3.30) and the above relation, the contribution from the error type $I_{j,\bar{\varepsilon}}$ term to the error (3.7) in the j th equation is the following

$$\mathcal{R}_j^{\bar{\varepsilon}} \triangleq \int c(t-s)I_{j,\bar{\varepsilon}}(s)ds = -(\mathcal{R}_{j0}^{\bar{\varepsilon}} - D_j^2\mathcal{R}_{j0}^{\bar{\varepsilon}}(0)\chi_{j2}), \quad \mathcal{R}_{j0}^{\bar{\varepsilon}} \triangleq \sum_{i \leq n} \int_0^t c_i(s)\mathcal{B}_{op,j}(\mathbf{u}(\bar{\varepsilon}), \hat{W}_i(t-s))dt.$$

Since $\mathcal{B}_{op,j}$ is bilinear and $c_i(t)$ is spatial-independent and Lipschitz in t , we get

$$(3.40) \quad \mathcal{R}_{j0}^{\bar{\varepsilon}} = \mathcal{B}_{op,j}(\mathbf{u}(\bar{\varepsilon}), \left(\sum_{i \leq n} \int_0^t c_i(t-s)\hat{W}_i(s)ds \right)) = \mathcal{B}_{op,j}(\mathbf{u}(\bar{\varepsilon}), \hat{G}(t)).$$

Denote by $\hat{G}^{(l)}, \hat{W}_j^{(l)}$ the approximate solution with extension in t in Section 3.4, and the first correction $l = 1$ in Section 3.2 or two corrections $l = 2$ in Section 3.2, 3.3. Let $\hat{\phi}_i^{(l)}$ be the stream function associated with $\hat{W}_i^{(l)}$ constructed numerically with the first correction for $l = 1$ and both corrections for $l = 2$. In particular, the full solution is given by $\hat{G} = \hat{G}^{(2)}, \hat{W}_i = \hat{W}_i^{(2)}, \hat{\phi}_i^N = \hat{\phi}_i^{N,(2)}$ (3.34). We construct the stream function $\hat{\phi}^{N,(l)}$ associated with $\hat{G}^{(l)}$ and error $\hat{\varepsilon}$ as follows

$$\hat{\phi}^{N,(l)} \triangleq \sum_{i \leq n} \int_0^t c_i(s)\hat{\phi}_i^{(l)}(t-s)ds, \quad \hat{\varepsilon} = \hat{W}^{(1)} + \Delta\hat{\phi}^{N,(1)} = \sum_{i \leq n} \int_0^t c_i(s)(\hat{W}_i^{(1)} + \Delta\hat{\phi}_i^{N,(1)})(t-s)ds.$$

Since we can obtain $\mathbf{u}(a(t)\chi_{12})$ exactly for the second correction (see Section 3.3), we have

$$(3.41) \quad \hat{\varepsilon}(t) = \hat{W}^{(1)} - (-\Delta)\hat{\phi}^{N,(1)} = \hat{W}^{(2)} - (-\Delta)\hat{\phi}^{N,(2)} = \hat{W} - (-\Delta)\hat{\phi}^N.$$

In practice, we estimate $\hat{\varepsilon}$ using the first identity since it does not involve $a_i(t)$ and the integrand $\hat{W}_i^{(1)} + \Delta\hat{\phi}_i^{(1)}$ is piecewise cubic in time. We decompose $\hat{\varepsilon}$ as follows

$$(3.42) \quad \hat{\varepsilon}_2 = \hat{\varepsilon}_{xy}(0)\Delta\left(\frac{x^3y}{2}\chi_{\hat{\varepsilon}}\right), \quad \hat{\varepsilon} = (\hat{\varepsilon} - \hat{\varepsilon}_2) + \hat{\varepsilon}_2 \triangleq \hat{\varepsilon}_1 + \hat{\varepsilon}_2,$$

where χ_ε is defined in (D.6). Since $\hat{\varepsilon}$ only vanishes $O(|x|^2)$ near 0, we perform the above decomposition so that $\hat{\varepsilon}_1 = O(|x|^3)$ near 0. See Appendix C.4 and Section 5.8 in [14] for motivations of (3.42). We estimate $\hat{\varepsilon}_1, \hat{\varepsilon}_{xy}(0), \hat{\phi}^N$ following (3.37). We establish (3.6).

Similarly, using linearity, (3.39), and (3.41), we can rewrite the residual error in (3.35) from $I_{j,\varepsilon}$ term in (3.29) related to $\mathcal{L}_j^\varepsilon(\cdot)$ as follows

$$\begin{aligned} \mathcal{R}_j^\varepsilon &\triangleq \int c(t-s)I_{j,\varepsilon}(s)ds = -(\mathcal{R}_{j0}^\varepsilon - D_j^2\mathcal{R}_{j0}^\varepsilon(0)\chi_{j2}), \\ \mathcal{R}_{j0}^\varepsilon &= \sum_{i \leq n} \int_0^t c_i(t-s)\mathcal{L}_j^\varepsilon(\hat{W}_i^{(1)})(s)ds = \sum_{i \leq n} \int_0^t c_i(t-s)\mathcal{B}_{op,j}(\mathbf{u}(\hat{W}_i^{(1)} + \Delta\hat{\phi}_i^{N,(1)})(s), \bar{W})ds \\ &= \mathcal{B}_{op,j}(\hat{\varepsilon}(t), \bar{W}), \end{aligned}$$

which along with (3.39)-(3.40) for $\mathcal{R}_j^\varepsilon$ establish the formula for \mathcal{R}_{nloc} in (3.8).

4. ESTIMATE THE NORM OF THE REGULAR PART OF THE VELOCITY

In this section, we derive the constants in the upper bound in Lemma 2.3. We have constructed the finite rank approximation \hat{f} for f in Lemma 2.3 in Section 4.3 in Part I [14]. The estimate of the most singular part, e.g. $u_{x,a,b}(\omega\psi)$, in the $C^{1/2}$ estimate in Lemma 2.3 can be obtained using the sharp Hölder estimates in Section 3 of Part I [14], where $u_{x,a,b}$ is defined via a localized kernel. In this section, we estimate other terms in Lemma 2.3, e.g. $I = \psi u_x(\omega) - u_{x,a,b}(\omega\psi) - \psi\hat{u}_x(\omega)$, involving the velocity with desingularized kernels, which are more regular.

In Section 4.1, we outline the strategies in the estimate and decompose the integrals from the nonlocal terms into several parts based on their regularities. In Section 4.2, we perform the L^∞ estimates in Lemma 2.3 and derive the constants. In Section 4.3, we perform the Hölder estimate of different parts. In Section 4.6, we combine the Hölder estimate of different parts, which provide the constants in Lemma 2.3. In particular, we reduce the L^∞ estimates and the $C^{1/2}$ estimates in Lemma 2.3 to bounding some explicit L^1 integrals depending on the weights, which can be estimated by a numerical quadrature with rigorously error control. We estimate these integrals with computer assistance. See discussions in Section 2.2.

We will apply the second estimates in Lemma 2.3 for the nonlocal error, e.g. $\mathbf{u}(\bar{\varepsilon})$ and $\bar{\varepsilon}$ is the error of solving the Poisson equations. Since we can estimate piecewise bounds of $\bar{\varepsilon}$ following Section 3.6, instead of using global norm, we improve the estimate using the localized norms, which are much smaller than the global norm. See Section 4.7.

The kernels associated with $\mathbf{u}, \nabla\mathbf{u}$ are given by

$$(4.1) \quad \begin{aligned} K_1 &\triangleq \frac{y_1 y_2}{|y|^4}, \quad K_2 \triangleq \frac{1}{2} \frac{y_1^2 - y_2^2}{|y|^4}, \quad K_u \triangleq \frac{y_2}{2|y|^2}, \quad K_v \triangleq -\frac{y_1}{2|y|^2}, \\ K_{u_x} &= -K_1, \quad K_{u_y} = K_{v_x} = K_2. \end{aligned}$$

Here, we have dropped the constant $\frac{1}{\pi}$, e.g. $u_x(\omega) = -\partial_{xy}(-\Delta)^{-1} = \frac{1}{\pi}K_{u_x} * \omega$. One needs to multiply $\frac{1}{\pi}$ back to obtain the final estimate.

Difficulties in the computations. In addition to the difficulties discussed in Section 5.1 of Part I [14], e.g. singularities caused by the weights and kernels, the singular integral introduces several technical difficulties in our estimates. To address these difficulties, we need to consider different scenarios and decompose the domain of the integrals carefully in our computer assisted estimates. Given $\omega\varphi \in L^\infty$, the velocity \mathbf{u} and the commutator $\psi \cdot (\nabla\mathbf{u})(\omega) - (\nabla\mathbf{u})(\omega\psi)$ are only log-Lipschitz. The logarithm singularity introduces several difficulties. For example, if \mathbf{u} is Lipschitz, a natural approach to estimate its Hölder norm in terms of $\|\omega\varphi\|_\infty$ is to estimate the piecewise bound of \mathbf{u} and $\partial\mathbf{u}$, which are local in \mathbf{u} , and then use the method in Section E.1. However, since \mathbf{u} is only log-Lipschitz, we need to perform a decomposition of \mathbf{u} into the regular part and the singular part carefully. For different parts, we will apply different estimates. See Section 4.1.11 for the ideas. For $\nabla\mathbf{u}$, the estimates are more involved since it is more singular.

4.1. Several strategies. We outline several strategies to estimate the nonlocal terms.

4.1.1. Integral with approximation. In our computation of $\mathbf{u}_A = \mathbf{u} - \widehat{\mathbf{u}}$, $(\nabla \mathbf{u})_A = \nabla \mathbf{u} - \widehat{\nabla \mathbf{u}}$, where the approximation terms $\widehat{\mathbf{u}}$, $\widehat{\nabla \mathbf{u}}$ are defined in Section 4.3 of Part I [14], the rescaling argument still applies. Note that we do not have $\partial \mathbf{u}_A = (\partial \mathbf{u})_A$ since we design approximations for \mathbf{u} , $\nabla \mathbf{u}$ separately. We consider one approximation term $c(x) \int \mathbf{1}_{y \notin S} K(x_a, y) \omega(y) dy$ for $\int K(x, y) \omega(y) dy$ to illustrate the ideas, where S is the singular region associated with x_a . Suppose that K is $-d$ -homogeneous. We want to estimate

$$I = \rho(x) \int_{\mathbb{R}^2} (K(x, y) - c(x)K(x_a, y)\mathbf{1}_{y \notin S})W(y)dy,$$

where W is the odd extension of ω from \mathbb{R}_+^2 to \mathbb{R}^2 (see (4.23)). Denote

$$(4.2) \quad f_\lambda(x) \triangleq f(\lambda x).$$

We choose $\lambda \asymp |x|$ and denote $x = \lambda \hat{x}$, $y = \lambda \hat{y}$, $x_a = \lambda \hat{x}_a$. Since $K(\lambda z) = \lambda^{-d}K(z)$, we have

$$(4.3) \quad \begin{aligned} I &= \rho_\lambda(\hat{x}) \int_{\mathbb{R}^2} \left(K(\lambda \hat{x}, \lambda \hat{y}) - c(\lambda \hat{x})\mathbf{1}_{\lambda \hat{y} \notin S} K(\lambda \hat{x}_a, \lambda \hat{y}) \right) W(\lambda \hat{y}) \lambda^2 d\hat{y} \\ &= \lambda^{2-d} \rho_\lambda(\hat{x}) \int_{\mathbb{R}^2} \left(K(\hat{x}, \hat{y}) - c(\lambda \hat{x})\mathbf{1}_{\hat{y} \notin S/\lambda} K(\hat{x}_a, \hat{y}) \right) W_\lambda(\hat{y}) d\hat{y}. \end{aligned}$$

The singular region becomes S/λ and close to $x_a/\lambda = \hat{x}_a$. For example, if $S = \{y : \max_i |y_i - x_{a,i}| \leq a\}$, we have $S/\lambda = \{y : \max_i |y_i - \hat{x}_{a,i}| \leq a/\lambda\}$. For the above integral, we will symmetrize the kernel and then estimate it using the norms $\|W\varphi\|_\infty$ and $[\omega\psi]_{C_{x_i}^{1/2}}$, $i = 1, 2$ (4.9).

The bulk and approximation. To take advantage of the scaling symmetry and overcome the singularity, in our computation for x away from the origin and not too large, we choose several dyadic rescaling parameters $\lambda = 2^i$, $i \in I$, e.g. $I = \{-4, -3, \dots, 10\}$. Then for any x with $\max(x_1, x_2) \in [2^i x_c, 2^{i+1} x_c]$, we can choose $\lambda = 2^i$ so that the rescaled $\hat{x} = \frac{x}{\lambda}$ satisfies

$$(4.4) \quad \hat{x} \in \begin{cases} [x_c, 2x_c] \times [0, 2x_c] \triangleq \Omega_1, & \text{if } x_2 \leq x_1, \\ [0, x_c] \times [x_c, 2x_c] \triangleq \Omega_2, & \text{if } x_2 > x_1. \end{cases}$$

We also choose x_i ($(x_i, 0)$ is the singularity) and the size of the singular region t_i for the approximation term defined in Section 4.3.2 of Part I [14] such that x_i/λ is on the grid point of the mesh and the boundary of the singular region $\{y : |x_i - y_1| \vee |y_2| \geq t_i/\lambda\}$, which aligns with one of the edges of a mesh cell. For example, this can be done by choosing the following y mesh in the near-field to discretize the y -integral, x_i , and t_i

$$y_{1,i} = ih, \quad y_{2,i} = ih, \quad x_i = 2^{n_i} h, \quad t_i = 2^{m_i} h.$$

Then when we discretize the rescaled integral in y , e.g. (4.3), the singular region is the union of several mesh cells. For large y , it is away from the singularity \hat{x} . Then we can use an adaptive mesh in y_1, y_2 to discretize the integral.

We remark that in (4.3), if $x_a \neq 0$ and $x_a/\lambda \asymp x_a/|x|$ is too large or too small, since $c(x)$ is supported near x_a , $c(\lambda \hat{x}) = c(x)$ will be 0. This means that when we compute $\mathbf{u}_A(x)$, $(\nabla \mathbf{u})_A$, if the coefficient of an approximation term with center x_i and parameter t_i is nonzero, e.g., $c(x) \neq 0$, then λ is comparable to x_i when we rescale the integral by λ . Thus $\hat{x}_i = x_i/\lambda$ is on the grid. We also choose t_i such that t_i/λ is a multiple of mesh size h for λ comparable to x_i .

Remark 4.1. Using the scaling symmetry and rescaling the integral by dyadic scales, we can compute the integral for $x \in [0, D]^2 \setminus [0, d]^2$ with roughly $O(\log(D/d))$ computational cost.

The near-field and the far-field. Recall the notations from Section 4.3 in Part I [14]

$$(4.5) \quad \begin{aligned} C_{u0} &= x, \quad C_{v0} = -y, \quad C_{u_x0} = 1, \quad C_{u_y0} = C_{v_x0} = 0, \\ C_{u_x} &= -(x^2 - y^2), \quad C_{v_x} = 2xy, \quad C_{u_y} = 2xy, \quad C_u = -\left(\frac{1}{3}x^3 - xy^2\right), \quad C_v = x^2y - \frac{1}{3}y^3, \\ K_{u_x0} &= -\frac{4y_1y_2}{|y|^4}, \quad K_{00} = \frac{24y_1y_2(y_1^2 - y_2^2)}{|y|^8}, \quad \mathcal{K}_{00}(\omega) \triangleq \frac{1}{\pi} \int_{\mathbb{R}_{++}^2} K_{00}(y)\omega(y)dy. \end{aligned}$$

If x is sufficiently small, i.e. $\max(x_1, x_2) < \min_{i \in I} 2^i x_c$, we choose $\lambda = \max(x_1, x_2)/x_c$ so that the rescaled $\hat{x} = \frac{x}{\lambda}$ is on the line $x_1 = x_c$ or $x_2 = x_c$. Assuming $\varphi(x) \geq |x|^{-\beta_1} |x|^{-\beta_2}$, $\rho \sim |x|^{-\alpha}$ near $x = 0$, and K is $-d$ -homogeneous, then we get

$$(4.6) \quad \begin{aligned} |\rho(x) \int_{\mathbb{R}_2^{++}} K(x, y) \omega(y) dy| &\leq \|\omega_\lambda \varphi_\lambda\|_{L^\infty} \rho_\lambda(\hat{x}) \int_{\mathbb{R}_2^{++}} |K(\hat{x}, \hat{y})| \varphi_\lambda(\hat{y})^{-1} \lambda^{2-d} d\hat{y} \\ &\leq \|\omega_\lambda \varphi_\lambda\|_{L^\infty} \lambda^{\beta_1 + \beta_2 + 2 - d} \rho_\lambda(\hat{x}) \int_{\mathbb{R}_2^{++}} |K(\hat{x}, \hat{y})| |\hat{y}|^{\beta_1} \hat{y}_1^{\beta_2} d\hat{y}. \end{aligned}$$

As $x \rightarrow 0$, $\lambda \rightarrow 0$. The factor $\lambda^{\beta_1 + \beta_2 + 2 - d}$ absorbs the large factor $\lambda^{-\alpha}$ in $\rho_\lambda(\hat{x})$. In our estimate of $\mathbf{u}_A, \nabla \mathbf{u}_A$, we have $\beta_1 + \beta_2 = 2.9$ for $\varphi_1, \varphi_{g,1}$, 2.5 for φ_{elli} (A.2), $(\alpha, d) = (2, 2)$ for $(\psi_1, \nabla \mathbf{u}_A)$ (A.1), $(\alpha, d) = (3, 1)$ for $(\rho_{10}, \mathbf{u}_A)$ (A.2). We have $\beta_1 + \beta_2 + 2 - d - 2 > 0$.

In general, the above integral may not be integrable due to the growing weight $|y|^{\beta_1} y_1^{\beta_2}$. For $\mathbf{u}_A, \nabla \mathbf{u}_A$ with small x , it takes the form (see Section 4.3 of Part I [14])

$$(4.7) \quad f(x) - C_{f0}(x)u_x(0) - C_f(x)\mathcal{K}_{00} = \int_{\mathbb{R}_2^{++}} (K_f^{sym} + C_{f0}(x) \frac{4}{\pi} \frac{y_1 y_2}{|y|^4} - C_f(x)K_{00}(y)) \omega(y) dy,$$

where C_{f0}, C_f and K_{00} are defined in (4.5), and $f = u, v, u_x, v_x, u_y, v_y$. In particular, the associated kernel has a much faster decay rate $|y|^{-6}$, which will be shown in Appendix B.1.1. Thus, the integral is integrable.

Since $\lambda = \max(x_1, x_2)/x_c$ is very small, $\rho_\lambda(\hat{x})$ can be well approximated by the most singular power $c\lambda^{-\alpha}|x|^{-\alpha}$ for some $c > 0$, which can be estimated effectively after factorizing out $\lambda^{-\alpha}$.

Similarly, if x is sufficiently large, i.e. $\max(x_1, x_2) > \max_{i \in I} 2^{i+1} x_c$, we choose $\lambda = \frac{\max(x_1, x_2)}{x_c}$ so that the rescaled $\hat{x} = x/\lambda$ is on the line $x_1 = x_c$ or $x_2 = x_c$. Since λ is sufficiently large, we can estimate the weight $\rho_\lambda, \varphi_\lambda$ based on their asymptotic behavior.

Integral near 0. We have an approximation $I = -C_{f0}(x)K_{ux0}(y) - C_2(x)K_{00}(y)$ (4.5) for $K_{f0}^{sym}(x, y)$ with some smooth coefficients C_2 (C_2 may not be C_f). The term $C_{f0}(x)K_{ux0}(y)$ and K_f are both $-d$ homogeneous, $d = 1$ or 2 . Since K_{ux0}, K_{00} are singular near 0, after rescale the integral following (4.3), we decompose the symmetrized integral for y near 0 as follows

$$(4.8) \quad \begin{aligned} II &= \int_{\mathbb{R}_2^{++}} \left(K_f^{sym}(\hat{x}, \hat{y}) \lambda^{2-d} - \lambda^{2-d} C_f(\hat{x}) K_{ux0}(\hat{y}) - C_2(\lambda \hat{x}) K_{00}(\hat{y}) \lambda^{-2} \right) \omega(\lambda \hat{y}) d\hat{y} \\ &= \lambda^{2-d} \left(\int_{\mathbb{R}_2^{++}} \left(K_f^{sym}(\hat{x}, \hat{y}) - C_f(\hat{x}) \mathbf{1}_{|\hat{y}|_\infty \geq k_{01} h} K_{ux0}(\hat{y}) - \lambda^{-4+d} C_2(\lambda \hat{x}) \mathbf{1}_{|\hat{y}|_\infty \geq k_{02} h} K_{00}(\hat{y}) \right) \omega(\lambda \hat{y}) d\hat{y} \right. \\ &\quad \left. - \int_{\mathbb{R}_2^{++}} \left(C_f(\hat{x}) \mathbf{1}_{|\hat{y}|_\infty \leq k_{01} h} K_{ux0}(\hat{y}) - \lambda^{-4+d} C_2(\lambda \hat{x}) \mathbf{1}_{|\hat{y}|_\infty \leq k_{02} h} K_{00}(\hat{y}) \right) \omega(\lambda \hat{y}) d\hat{y} \right) \end{aligned}$$

for some small integers k_{0i} with $k_{0i} h < |\hat{x}|_\infty / 2$, e.g. $k_{01} = 4, k_{02} = 20$, where $|a|_\infty = \max(a_1, a_2)$ and h is chosen in (4.14). We will estimate the first integral with regular integrand near $\hat{y} = 0$ using the method in Section 4.1.3, and the last two integrals for $|\hat{y}|_\infty \leq k_{01} h, |\hat{y}|_\infty \leq k_{02} h$ analytically in Section 4.4.1. We perform the above decomposition since $K_{00}(\hat{y}), K_{ux0}(\hat{y})$ are too singular to estimate them numerically.

We apply the above decompositions to the integrals in both L^∞ and $C^{1/2}$ estimates. We also apply the above decompositions to the approximation terms and estimate integral of K_{ux0} separately near $y = 0$.

4.1.2. The scaling relations. We discuss several scaling relations, which will be useful in later computation. For a $-d$ -homogeneous kernel K , i.e., $K(\lambda x) = \lambda^{-d} K(x)$, we have

$$I(x) = \rho(x) \int K(x, y) \omega(y) dy = \rho_\lambda(\hat{x}) \int K(\hat{x}, \hat{y}) \omega_\lambda(\hat{y}) \lambda^{2-d} d\hat{y} \triangleq \lambda^{2-d} I_\lambda(\hat{x}),$$

where $x = \lambda \hat{x}, y = \lambda \hat{y}$. To compute the derivative of $I(x)$, using the chain rule, we have

$$\partial_{x_i} I(x) = \lambda^{2-d} \frac{d\hat{x}_i}{dx_i} \partial_{\hat{x}_i} I_\lambda(\hat{x}) = \lambda^{1-d} \partial_{\hat{x}_i} I_\lambda(\hat{x}).$$

For the L^∞ part, clearly, we get $|I(x)| = |I_\lambda(\hat{x})|$. To compute the Hölder norm, we use the following relation $|x - z| = \lambda|\hat{x} - \hat{z}|$ and

$$\frac{|I(x) - I(z)|}{|x - z|^{1/2}} = \lambda^{-1/2} \frac{|I_\lambda(\hat{x}) - I_\lambda(\hat{z})|}{|\hat{x} - \hat{z}|^{1/2}}.$$

In particular, for $i = 1, 2$, we have

$$(4.9) \quad \|\omega_\lambda \varphi_\lambda\|_\infty = \|\omega\varphi\|_\infty, \quad [\omega_\lambda \psi_\lambda]_{C_{x_i}^{1/2}} = \lambda^{1/2} [\omega\psi]_{C_{x_i}^{1/2}}, \quad [f]_{C_{x_i}^{1/2}} := \sup_{y, z: y_i = z_i} \frac{|f(y) - f(z)|}{|y - z|^{1/2}}.$$

Using these scaling relations, we can perform the estimate in a rescaled domain with any $\lambda > 0$.

4.1.3. Mesh and the Trapezoidal rule. After rescaling the integral with suitable scaling factor λ , we can restrict the rescaled singularity $\hat{x} \in [0, 2x_c]^2 \setminus [0, x_c]^2$ (see (4.3), (4.4)).

If a domain Q is away from the singularity \hat{x} of the kernel, applying (4.9), we get

$$(4.10) \quad \int_Q |K(\hat{x}, y)| \omega_\lambda(y) dy \leq \|\omega_\lambda \varphi_\lambda\|_\infty \int_Q |K(\hat{x}, y)| \varphi_\lambda^{-1}(y) dy = \|\omega\varphi\|_\infty \int_Q |K(\hat{x}, y)| \varphi_\lambda^{-1}(y) dy.$$

Then, it suffices to estimate the integral of an explicit function $|K(\hat{x}, y)| \varphi_\lambda^{-1}(y)$. If in addition, the region Q is small, e.g. Q is the grid $[y_i, y_{i+1}] \times [y_j, y_{j+1}]$ introduced below, we further apply

$$\int_Q |K(\hat{x}, y)| \omega_\lambda(y) dy \leq \|\omega\varphi\|_\infty \|\varphi_\lambda^{-1}\|_{L^\infty(Q)} \int_Q |K(\hat{x}, y)| dy.$$

Since the domain Q is small, the estimate is sharp. We use the following method to estimate $\int |K(\hat{x}_i, y)| dy$ for a suitable kernel K and \hat{x}_i on the grid points.

We consider the estimate of the L^1 norm of some function f in \mathbb{R}_2^{++} , e.g. $f = K(\hat{x}_i, y)$ mentioned above. To discretize the integral, we design uniform mesh in the domain $[0, b]^2$ covering Ω_1 and Ω_2 with mesh size h and adaptive mesh in the larger domain $[0, D]^2$

$$(4.11) \quad 0 = y_0 < y_1 < \dots < y_n = D, \quad y_i = ih, \quad i \leq b/h.$$

The finer mesh in the near field $[0, b]^2$ allows us to estimate the integral with higher accuracy. We choose sparser mesh in the far-field since y is away from the singularity \hat{x} and the kernel decays in y . We partition the integral as follows

$$(4.12) \quad \int_{\mathbb{R}_2^{++}} |f(y)| dy = \sum_{0 \leq i, j \leq n-1} \int_{[y_i, y_{i+1}] \times [y_j, y_{j+1}]} |f(y)| dy + \int_{y \notin D} |f(y)| dy.$$

We focus on how to estimate the first part for nonsingular f . In Section 4.4, we estimate the integral beyond $[0, D]^2$ using the decay of the integral. We will discuss how to estimate the integral near the singularity of the kernel in a later subsection.

Denote $Q = [a, b] \times [d, c]$, $h_1 = b - a$, $h_2 = d - c$. We use the Trapezoidal rule

$$\int_{[a, b] \times [c, d]} |f(y)| dy \leq T(|f|, Q) + Err(f),$$

where

$$T(f, Q) \triangleq \frac{(b-a)(d-c)}{4} (f(a, c) + f(a, d) + f(b, c) + f(b, d)).$$

The error estimate of the above Trapezoidal rule is not obvious due to the absolute sign. In fact, even if f is smooth, $|f|$ is only Lipschitz near the zeros of f . Since the set of zeros is hard to characterize and that $|f|$ can have low regularity, we do not pursue higher order quadrature rule. We have the following error estimate.

Lemma 4.2 (Trapezoidal rule for the L^1 integral). *For $f \in C^2(Q)$, we have*

$$\int_Q |f(y)| dy \leq T(|f|, Q) + \frac{|Q|}{12} (h_1^2 \|f_{xx}\|_{L^\infty(Q)} + h_2^2 \|f_{yy}\|_{L^\infty(Q)}).$$

Remark 4.3. The above estimate shows that the Trapezoidal rule remains second order accurate from the above. In particular, this error estimate is comparable to the case without taking the absolute value.

Proof. Define the linear interpolation of f in Q

$$L(f) = \sum_{i=1}^4 \lambda_i(x) f_i, \quad E(f) = f - L(f),$$

where $\lambda_i(x)$ is linear and satisfies $\sum \lambda_i(x) = 1$ and $\lambda_i(x) \geq 0$ for $x \in Q$. Using the triangle inequality, we obtain

$$\int_Q |f| dy \leq \int_Q |E(f)| dy + \int_Q \lambda_i(x) |f_i| dy = T(|f|, Q) + \int_Q |E(f)| dy.$$

We have the standard error bound for linear interpolation $E(f)$

$$(4.13) \quad |E(f)| \leq \frac{\|f_{xx}\|_{L^\infty(Q)}}{2} |(x-a)(x-b)| + \frac{\|f_{yy}\|_{L^\infty(Q)}}{2} |(y-c)(y-d)|,$$

which can be obtained by first applying interpolation in x and then in y . It can also be established using the error estimate for the 2D Lagrangian interpolation with $k = 2$ in Section 8 in the supplementary material II [12]. Integrating the above estimate in x, y and using $\frac{1}{2} \int_0^1 t(1-t) dt = \frac{1}{12}$ conclude the proof. \square

To estimate the integral $\int |K(x, y)|$ for all $\hat{x} \in \Omega_1, \Omega_2$ (4.4), we discretize $[0, 2a]^2$ using uniform mesh with mesh size $h_x = h/2$. We use the above method to estimate $\int |K(\hat{x}_i, y)| dy$ for x_i on the grid points. After we estimate the derivatives of the kernel, we use the following Lemma to estimate the integral for any x in a domain.

Lemma 4.4. *Suppose that $K(x, y) \in C^2(P \times Q)$, $P = [a_1, b_1] \times [a_2, b_2]$, $h_i = b_i - a_i$, $i = 1, 2$, and $Q = [a, b] \times [c, d]$. Let $L(K)(x, y) = \sum_{i,j=1,2} \lambda_{ij}(x) K((a_i, b_j), y)$ be the linear interpolation of $K(x, y)$ in x using $K((a_i, b_j), y)$, $i, j = 1, 2$. Then for any $x \in P$, we have*

$$\int_Q |K(x, y)| dy \leq \sum_{i,j=1,2} \lambda_{ij}(x) \int_Q |K((a_i, b_j), y)| dy + \left(\frac{h_1^2}{8} \|K_{xx}\|_{L^\infty(P \times Q)} + \frac{h_2^2}{8} \|K_{yy}\|_{L^\infty(P \times Q)} \right) |Q|.$$

The proof follows from (4.13), the triangle inequality and $\frac{1}{2} |t(1-t)| \leq \frac{1}{8}$ for $t \in [0, 1]$. We will apply the above Lemma and sum Q over all the near-field domains $Q_{ij} = [y_i, y_{i+1}] \times [y_j, y_{j+1}]$ (4.11). Since $\sum_{ij} \lambda_{ij}(x) = 1$, we can simplify the first term as follows

$$\sum_{i,j=1,2} \lambda_{ij}(x) \sum_{k,l \leq n} \int_{Q_{kl}} |K((a_i, b_j), y)| dy \leq \max_{1 \leq i,j \leq 2} \sum_{k,l \leq n} \int_{Q_{kl}} |K((a_i, b_j), y)| dy.$$

Therefore, it suffices to estimate the integral for x on the grid points and the piecewise derivative bounds of the kernel.

We apply Lemmas 4.2, 4.4 to estimate the weighted integral related to the velocity. The integrands take the form (4.28), (4.29), (4.24). To estimate the error in the above integrals, we need to obtain piecewise L^∞ estimate of the derivatives of the integrands in P, Q . We estimate the derivatives of the weights in Appendix A.1 and the kernel in Appendix B.

Parameters for the integrals. In our computation, we choose

$$(4.14) \quad h_x = 13 \cdot 2^{-12}, \quad h = 13 \cdot 2^{-11}, \quad x_c = 13 \cdot 2^{-5},$$

which can be represented exactly in a binary system, to reduce the round off error. The approximate values of the above parameters are $h_x \approx 0.0032$, $h \approx 0.0064$, $x_c \approx 0.4$. For $x \in [0, 2x_c]^2 \setminus [0, x_c]^2$ (4.4), we have

$$(4.15) \quad \max(x_1, x_2) \geq x_c = 64h = 128h_x.$$

In our decomposition of the integral, e.g. (4.24), (4.45), (4.49), we impose a constraint on the size of the singular region to satisfy $(k+1)h < x_c$ such that the region does not cover the origin.

4.1.4. *Decomposition, commutators and the Lipschitz norm.* The most difficult part of the computation is to estimate the Hölder norm of $\nabla \mathbf{u}$, and we discuss several strategies. In this computation, we cannot first estimate the local Lipschitz norm of ∇u and then obtain the local Hölder norm due to the difficulties discussed at the beginning of Section 4. We need to decompose the integral related to ∇u into several parts according to the distance between y and the singularity and use different estimates for different parts.

We focus on the integral related to u_x without subtracting any approximation term and assume that $x \in [0, 2x_c]^2 \setminus [0, x_c]^2$. The approximation term $\widehat{\nabla \mathbf{u}}$ is nonsingular and can be estimated using the method in Section 4.1.3. Let h be the mesh size in the discretization of the integral in y . Suppose that

$$(4.16) \quad x \in \mathbb{R}_2^{++}, \quad x_2 \leq x_1, \quad x \in B_{i_1, j_1}(h_x) \subset B_{ij}(h), \quad j \leq i,$$

where $h_x = h/2$ and $B_{lm}(r)$ is defined as

$$(4.17) \quad B_{lm}(r) = [lr, (l+1)r] \times [mr, (m+1)r].$$

Denote by $R(x, k)$ the rectangle covering x

$$(4.18) \quad R(x, k) \triangleq [(i-k)h, (i+1+k)h] \times [(j-k)h, (j+1+k)h]$$

for any $k > 0$. If $k \in Z^+$, the boundary of $R(x, k)$ is along with the mesh grid and is at least kh away from x . Denote by $R_s, R_{s,1}, R_{s,2}$ different symmetric rectangles with respect to x

$$(4.19) \quad \begin{aligned} R_s(x, k) &\triangleq [x_1 - kh, x_1 + kh] \times [x_2 - kh, x_2 + kh], \\ R_{s,1}(x, k) &\triangleq [x_1 - kh, x_1 + kh] \times [(j-k)h, (j+1+k)h], \\ R_{s,2}(x, k) &\triangleq [(i-k)h, (i+1+k)h] \times [x_2 - kh, x_2 + kh]. \end{aligned}$$

We have $R_s(x, k) \subset R_{s,i}(x, k) \subset R(x, k)$, $i = 1, 2$. We introduce the upper, lower parts of $R(x, k)$

$$(4.20) \quad R^+(x, k) \triangleq R(x, k) \cap \{y : y_2 \geq x_2\}, \quad R^-(x, k) \triangleq R(x, k) \cap \{y : y_2 \leq x_2\}.$$

We use similar notations for $R_s(x, k), R_{s,1}(x, k), R_{s,2}(x, k)$. We further introduce the intersection of the rectangle and four half planes with reflection

$$(4.21) \quad \begin{aligned} R(x, k, N) &= R(x, k) \cap \{y : y_2 \geq 0\}, & R(x, k, S) &= \mathcal{R}_2(R(x, k) \cap \{y : y_2 \leq 0\}), \\ R(x, k, E) &= R(x, k) \cap \{y : y_1 \geq 0\}, & R(x, k, W) &= \mathcal{R}_1(R(x, k) \cap \{y : y_1 \leq 0\}), \end{aligned}$$

where N, E, S, W are short for *north, east, south, west*, respectively and the reflection operators $\mathcal{R}_1, \mathcal{R}_2$ are given by

$$\mathcal{R}_1(y_1, y_2) = (-y_1, y_2), \quad \mathcal{R}_2(y_1, y_2) = (y_1, -y_2).$$

It is clear that $R(x, k, S) \subset \mathbb{R}_2^+, R(x, k, W) \subset \{y : y_1 \geq 0\}$. An illustration of these domains is given in Figure 1. If $x, y \in \mathbb{R}_2^{++}$, we have the equivalence

$$(4.22) \quad \begin{aligned} (y_1, -y_2) \notin R(x, k) &\iff (y_1, -y_2) \notin R(x, k) \cap \{y : y_2 \leq 0\} \iff y \notin R(x, k, S), \\ (y_1, -y_2) \in R(x, k) &\iff y \in R(x, k, S). \end{aligned}$$

The above notations will be very useful in our later decomposition of the symmetrized kernel.

Define the odd extension of ω in y from \mathbb{R}_2^+ to \mathbb{R}_2

$$(4.23) \quad W(y) = \omega(y) \text{ for } y_2 \geq 0, \quad W(y) = -\omega(y_1, -y_2) \text{ for } y_2 < 0.$$

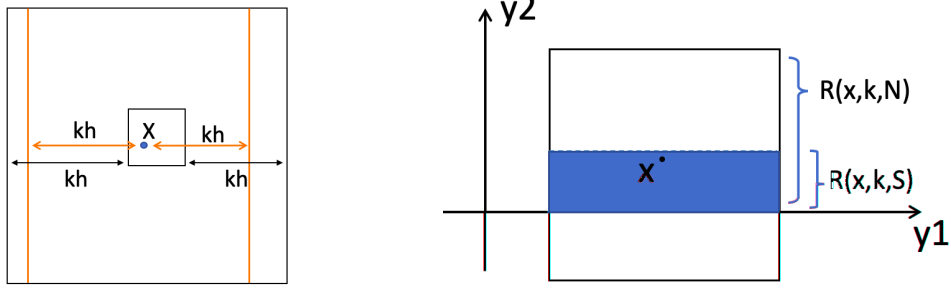


FIGURE 1. Left: The large box is $R(x, k)$ and the red box is $R_{s,1}(x, k)$. The small box containing x has size $h \times h$. Right: The upper box is $R(x, k, N)$, and the shaded box is $R(x, k, S)$, the reflection of the region below the y -axis.

W is odd in both y_1 and y_2 variables. Since we fix x (4.16) below, for simplicity, we drop x in the R notation. For $k > k_2, k, k_2 \in \mathbb{Z}^+$, we decompose the weighted $u_x(x)$ integral as follows²

$$\begin{aligned}
 \psi(x) \int K_1(x-y)W(y)dy &= \psi(x) \int_{R(k)^c} K_1(x-y)W(y)dy \\
 (4.24) \quad &+ \int_{R_{s,1}(k)} K_1(x-y)\psi(y)W(y)dy + \int_{R(k) \setminus R_{s,1}(k)} K_1(x-y)\psi(y)W(y)dy \\
 &+ \int_{R(k) \setminus R(k_2)} K_1(x-y)(\psi(x) - \psi(y))W(y)dy + \int_{R(k_2)} K_1(x-y)(\psi(x) - \psi(y))W(y)dy \\
 &\triangleq I_1(x, k) + I_2(x, k) + I_3(x, k) + I_4(x, k, k_2) + I_5(x, k_2),
 \end{aligned}$$

where

$$K_1(s) \equiv \frac{s_1 s_2}{|s|^4}.$$

We drop $-\frac{1}{\pi}$ in the integrand $-\frac{1}{\pi}K_1(s)$ for $u_x(x)$ (4.1) at this moment to simplify the notation. We will estimate different parts in Section 4.3.

Principle and Log-Lipschitz integrand. Our major motivation for the decomposition (4.24) and the integrands (4.28) with symmetrization is to obtain integrands J which is at least locally log-Lipschitz satisfying $J(x, y) \lesssim |x - y|^{-1}$ for y close to x , and the more singular one J_S . We will estimate the integral of J using the Trapezoidal rule in Section 4.1.3 and J_S analytically.

4.1.5. *Symmetrization.* After we obtain the decomposition, we use the odd symmetry of W in y_1, y_2 to symmetrize the integral and reduce the integral over \mathbb{R}_2 to the first quadrant \mathbb{R}_2^{++} . This enables us to exploit the cancellation in the integral and obtain a sharper estimate. In our computation, we symmetrize the integrals in $I_1(x, k)$ and $I_4(x, k, k_2)$, which are more regular. For a given kernel $K(x, y)$, we denote by K^{sym} the symmetrization of K

$$(4.25) \quad K^{sym}(x, y) \triangleq K(x, y) - K(x, -y_1, y_2) - K(x, y_1, -y_2) + K(x, -y).$$

We show how to symmetrize $I_1(x, k)$ as an example. Recall the notations in (4.21), (4.16). We assume $x_1 \geq x_2$. We choose $k < i$ so that $R(x, k) \subset \{y : y_1 > 0\}$ and $R(x, k, W) = \emptyset$. By definition (4.18), the domains $R(x, k), R(x, k, N), R^+(x, k)$ etc are the same for all $x \in B_{i_1, j_1}(h_x)$. Yet, $R(x, k)$ may cross the boundary $y_2 = 0$, i.e. $R(x, k, S) \neq \emptyset$. See the right figure in Figure 1 for a possible configuration. Using the equivalence (4.22) and the property that W is odd in

²Since we have no flow boundary condition for the velocity and stream function $-\Delta\phi = \omega, \phi(0, y) = 0$, the Poisson integral formula for $\mathbf{u} = \nabla^\perp \phi$ is equivalent to $\nabla^\perp(-\Delta_D)^{-1}W$ for Δ_{2D} defined in the whole space case.

y_1 and y_2 , for general $x \in \mathbb{R}_2^{++}$ (without $x_1 \geq x_2$), we can symmetrize $I_1(x, k)$ as follows

$$(4.26) \quad I_1(x, k) = \psi(x) \int_{\mathbb{R}_2^{++}} \left(K_1(x-y) \mathbf{1}_{y \in R(k)^c} - K_1(x_1 - y_1, x_2 + y_2) \mathbf{1}_{y \notin R(k, S)} \right. \\ \left. - K_1(x_1 + y_1, x_2 - y_2) \mathbf{1}_{y \notin R(k, W)} + K_1(x+y) \right) \omega(y) dy.$$

For $I_4(x)$ (4.24), we choose the weight $\psi(y)$ (A.1), (A.2) even in y_1, y_2 . Then the symmetrization of I_4 is

$$(4.27) \quad I_4(x, k, k_2) = \int_{\mathbb{R}_2^{++}} \left(K_1(x-y) \mathbf{1}_{y \in R(k) \setminus R(k_2)} - K_1(x_1 - y_1, x_2 + y_2) \mathbf{1}_{y \in R(k, S) \setminus R(k_2, S)} \right. \\ \left. - K_1(x_1 + y_1, x_2 - y_2) \mathbf{1}_{y \in R(k, W) \setminus R(k_2, W)} \right) (\psi(x) - \psi(y)) W(y) dy.$$

In (4.27), we do not have the term $K_1(x+y)$ since for $y \in \mathbb{R}_2^{++}$, $x+y \geq x_c > (k+1)h$ and $-y \notin R(k)$. See the discussion below (4.15). Thus after symmetrizing the kernel in I_4 , we do not have such a term.

Though the symmetrized kernel is complicated, since these regions $R(l), R(l, \alpha), \alpha = N, E, l = k, k_2$ (4.18), (4.21) can be decomposed into the union of the mesh grids $[y_i, y_{i+1}] \times [y_j, y_{j+1}]$, in each grid, the indicator functions are constants. See also Remark 4.6. In each grid $y \in [y_i, y_{i+1}] \times [y_j, y_{j+1}]$, we can write the integrand in $I_1 + I_4$ as

$$(4.28) \quad J = K^{NC}(x, y) \cdot \psi(x) + K^C(x, y) \cdot (\psi(x) - \psi(y)), \\ \partial_{x_i} J = (K^{NC} + K^C) \partial_{x_i} \psi(x) + \partial_{x_i} K^{NC} \cdot \psi(x) + \partial_{x_i} K^C(x, y) \cdot (\psi(x) - \psi(y)),$$

where NC, C are short for *non-commutator*, *commutator*, respectively.

For y close to x , J is at least locally log-Lipschitz. See the Principle before Section 4.1.5 for motivation. For y away from x , e.g. $|y_1| \vee |y_2| \geq 4x_c$ in our computation, we have

$$(4.29) \quad J = K^{sym}(x, y) \psi(x).$$

In practice, we assemble the symmetrized integrand in $I_1 + I_4$ in \mathbb{R}_2^{++} together. Using (4.28), we only need to assemble K^{NC}, K^C . We first initialize the integrand with $(K^{NC}, K^C) = (K^{sym}, 0)$. To assemble the integrand in the singular regions, we perform two replacements. In the first replacement, we pretend that $R(k_2) = \emptyset$ and replace the integrand in $R(k) \cap \mathbb{R}_2^{++}$. Based on $x \in B_{ij}(h)$ (4.16), we determine the regions $R(x, k), R(x, k, S)$ (4.18), (4.21). Since $x_1 \geq x_2$, we get $R(x, k, W) = \emptyset$. See Figure 1. We partition $R(k) \cap \mathbb{R}_2^{++}$ as follows

$$(4.30) \quad R(k) \cap \mathbb{R}_2^{++} = R(k, N) = (R(k, N) \setminus R(k, S)) \cup R(k, S) \triangleq D_1 \cup D_2.$$

According to (4.26), (4.27) ($R(k_2) = \emptyset$), for $i = 1, 2$, we first replace (K^{NC}, K^C) in D_i by

$$(4.31) \quad (K^{NC}, K^C) = (K^{sym} - K_i^C, K_i^C), \quad K_1^C = K_1(x-y), \quad K_2^C = K_1(x-y) - K_1(x_1 - y_1, x_2 + y_2),$$

respectively, where K^C is from the integrand in (4.27). We have i singular terms in D_i in (4.27).

In the second replacement, we replace the integrand in the smaller singular region $R(k_2) \cap \mathbb{R}_2^{++} \subset R(k) \setminus \mathbb{R}_2^{++}$. Outside this region, we have obtained the symmetrized integrand using (4.31). Since we assume $x_1 \geq x_2$, we get $R(k, W) = \emptyset$ (see discussion below (4.25)) and $\mathbf{1}_{y \notin R(k, W)} \equiv 1, \mathbf{1}_{y \in R(k, W)} = 0$. Similar to $R(k) \cap \mathbb{R}_2^{++}$ (see Figure 1), we can decompose

$$R(k_2) \cap \mathbb{R}_2^{++} = (R(k_2, N) \setminus R(k_2, S)) \cap R(k_2, S) \triangleq D_3 \cup D_4.$$

In $D_4 = R(k_2, S) \subset R(k_2), R(k, S)$, from (4.26), (4.27), we completely remove the $K_1(x-y), K_1(x_1 - y_1, x_2 + y_2)$ terms in the integrand and have

$$(K^{NC}, K^C) = (K_1(x+y) - K_1(x_1 + y_1, x_2 - y_2), 0).$$

In D_3 , since $D_3 \subset R(k, N) = D_1 \cup D_2$ (4.30), there are two cases. In $D_3 \cap D_1, D_1 = R(k, N) \setminus R(k, S)$, we have three non-singular terms from (4.26) and 0 term from (4.27) and get

$$(K^{NC}, K^C) = (K_1(x+y) - K_1(x_1 + y_1, x_2 - y_2) - K_1(x_1 - y_1, x_2 + y_2), 0).$$

In $D_3 \cap D_2$, $D_2 = R(k, S)$, we have two terms from (4.26) and one term from (4.27). We get

$$(K^{NC}, K^C) = (K_1(x+y) - K_1(x_1+y_1, x_2-y_2), -K_1(x_1-y_1, x_2+y_2)).$$

For $x_1 < x_2$, we assemble the integrand similarly. Using (4.28), we obtain the integrand $\partial_{x_i} J$ for the Hölder estimate.

$C_y^{1/2}$ **estimate of u_y, v_x .** In the $C_y^{1/2}$ estimate of u_y, v_x with kernel K_2 (4.1), we symmetrize the integrand $K(x-y)(\psi(x) - \psi(y))$, see (4.68) in Section 4.3.9. In this case, the symmetrized integrand $W(y)T$ is similar to (4.26) with $\psi(x)$ replaced by $\psi(x) - \psi(y)$ and

$$T = (\psi(x) - \psi(y)) \left(K_2(x-y) \mathbf{1}_{y \in R(k)^c} - K_2(x_1-y_1, x_2+y_2) \mathbf{1}_{y \notin R(k, S)} - K_2(x_1+y_1, x_2-y_2) \mathbf{1}_{y \notin R(k, W)} + K_1(x+y) \right).$$

Due to the weight $(\psi(x) - \psi(y))$, we always have $K^{NC} = 0$. We initialize the T using (4.28) with $K^C = K_2^{sym}$ (4.25). In the singular region $R(x, k) \cap \mathbb{R}_2^{++}$, we only need to perform one replacement. Similar to (4.31), we use (4.30) and replace the integrand as follows

$$K^C = K_2^{sym} - K_2(x-y), y \in R(k, N) \setminus R(k, S), \quad K^C = K_2^{sym} - (K_2(x-y) - K_2(x_1-y_1, x_2+y_2)), y \in R(k, S).$$

We remove the most singular integrand in $R(k, N) \setminus R(k, S)$ and the most two singular integrands in $D_2 = R(k, S)$ to make T locally log-Lipschitz. See the Principle before Section 4.1.5.

L^∞ **estimate.** For L^∞ estimate, we do not multiply the integrand by the weight $\psi(x)$ or the commutator. We decompose the integral as (4.45), and symmetrize the nonsingular part in I_1 using (4.26) without the weight $\psi(x)$. Symmetrizing I_4 (4.45) is similar. We initialize the symmetrized integrand as K^{sym} (4.25), and then replace it in $R(k) \cap \mathbb{R}_2^{++}$. Without loss of generality, we assume $x_1 \geq x_2$ and have the decomposition (4.30). Similar to (4.31), we replace the integrand as follows

$$K^{sym} - K_1(x-y), y \in R(k, N) \setminus R(k, S), \quad K^{sym} - (K_1(x-y) - K_1(x_1-y_1, x_2+y_2)), y \in R(k, S).$$

That is, we remove one and two singular terms in $R(k, N) \setminus R(k, S)$, $R(k, S)$, respectively, making the integrand at least locally log-Lipschitz. See the Principle before Section 4.1.5.

4.1.6. *Integral in domains depending on x .* In the computation, we need to estimate several integrals in the domains $D(x)$ depending on x , e.g. I_3 in (4.24). Our fundamental idea is to cover $D(x)$ by some piecewise constant domains, which will be essentially treated as fixed domains. By refining the location of x , we can obtain tight covering.

We use the L^∞ estimate of I_3 to illustrate the ideas. A direct estimate yields

$$|I_3(x)| \leq \|W\varphi\|_\infty \int_{R(k) \setminus R_{s,1}(k)} |K_1(x-y)| \psi(y) \varphi^{-1}(y) dy.$$

We cannot apply the method in Section 4.1.3 to first estimate $I_3(x)$ for x on the grid points and then estimate $\partial^2 I_3(x)$ for the error since the kernel is singular and the error part associated with $\partial^2 I_3(x)$ is more singular (see Lemma 4.4).

Denote $f = \psi\varphi^{-1}$. We consider a change of variable $y = x + s$ to center our analysis around the singularity x . The domain for s is

$$(4.32) \quad \{y \in R(k) \setminus R_{s,1}(k)\} = \{s \in R(k) - x\} \cap \{|s_1| \geq kh\} \triangleq D(x, k).$$

It suffices to estimate

$$(4.33) \quad J = \int_{s \in D(x, k)} |K_1(-s)| f(x+s) dy, \quad f \geq 0,$$

for all $x \in B_{i_1, j_1}(h_x)$ (4.16). We want to further simplify the above domain so that it does not depend on x . Recall the location of x (4.16). To obtain a sharp estimate, we further partition the location of $x \in B_{i_1, j_1}(h_x)$ as follows

$$(4.34) \quad A_a = [i_1 h_x + a h_x/m, i_1 h_x + (a+1) h_x/m], \quad B_b \triangleq [j_1 h_x + b h_x/m, j_1 h_x + (b+1) h_x/m],$$

for some $m \in \mathbb{Z}^+$ and $0 \leq a, b \leq m-1$. Clearly, $A_a \times B_b$ is a partition of $B_{i_1, j_1}(h_x)$. Recall (4.16) and (4.18). We have

$$R(x, k) = [(i-k)h, (i+1+k)h] \times [(j-k)h, (j+1+k)h].$$

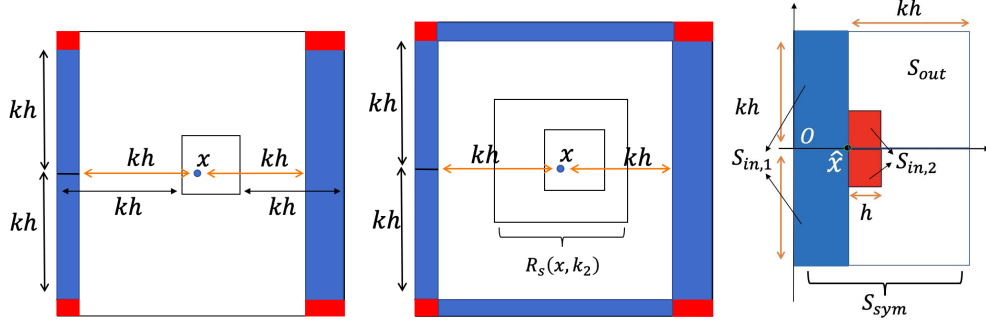


FIGURE 2. The largest box in the left and middle figure is $R(x, k)$. Left: The left and right blue regions are $X_{l,a} \times Y_{m,b}$, $X_{r,a} \times Y_{m,b}$. The four red regions correspond to $X_{\alpha,a} \times Y_{\beta,b}$, $\alpha = l, u, \beta = d, u$. Middle: Illustration of $R(x, k) \setminus R_s(x, k)$ and $R_s(x, k_2)$. $R(x, k) \setminus R_s(x, k)$ consists of the blue and the red regions. Right: different regions near the singularity for u/x_1 . Blue, red, and white regions represent $S_{in,1}$, $S_{in,2}$, S_{out} , respectively.

Now, for $x \in A_a \times B_b$, since $|s_1| \geq kh$, we have

$$(4.35) \quad \begin{aligned} s_1 = y_1 - x_1 &\in [(i-k)h - i_1 h_x - (a+1)h_x/m, -kh] \cup [kh, (i+1+k)h - i_1 h_x - ah_x/m] \\ &\triangleq X_{l,a} \cup X_{r,a}, \end{aligned}$$

where the subscripts l, r are short for left, right, respectively. Similarly, for s_2 , we have

$$(4.36) \quad \begin{aligned} s_2 = y_2 - x_2 &\in [(j-k)h - j_1 h_x - (b+1)h_x/m, (j+k+1)h - j_1 h_x - bh_x/m] \\ &\triangleq [(j-k)h - j_1 h_x - (b+1)h_x/m, -kh] \cup [-kh, kh] \cup [kh, (j+1+k)h - j_1 h_x - bh_x/m] \\ &\triangleq Y_{d,b} \cup Y_{m,b} \cup Y_{u,b} \end{aligned}$$

where the subscripts d, m, u are short for down, middle, upper, respectively. Note that the intervals X, Y do not depend on x . We have

$$(4.37) \quad D(x, k) \subset (X_{l,a} \cup X_{r,a}) \times (Y_{d,b} \cup Y_{m,b} \cup Y_{u,b}).$$

Now, we can decompose J (4.33) as follows

$$J \leq \sum_{\alpha=l,r,\beta=d,m,u} J_{\alpha,\beta}, \quad J_{\alpha,\beta} \triangleq \int_{X_{\alpha,a} \times Y_{\beta,b}} |K_1(-s)| f(s+x) dy, \quad \alpha = l, r, \beta = d, m, u.$$

See the left figure in Figure 2 for different domains in the above decomposition. From the definitions of X, Y , the total width of the left and the right domains $X_{\alpha,a} \times (Y_{d,b} \cup Y_{m,b} \cup Y_{u,b})$, $\alpha = l, u$ is

$$|X_{l,a}| + |X_{r,a}| = h + h_x/m.$$

For a fixed x , from the definition (4.18), the width of $R(k) \setminus R_{s,1}(k)$ is h . We choose a large m and further partition the location of x so that we do not overestimate the region too much.

For a small domain $Q = [a, b] \times [c, d]$, we can estimate the integral as follows

$$(4.38) \quad \int_Q |K_1(-s)| f(x+s) ds \leq \int_Q |K_1(-s)| ds \|f\|_{L^\infty(B_{i_1 j_1}(h_x) + Q)}.$$

Since Q is given, $K_1(s)$ is explicit and has scaling symmetries, we can estimate the integral of $|K_1(s)|$ easily. For example, if $Q = [ah, bh]^2$, we can use the scaling symmetries of $K_1(s)$ to obtain $\int_Q |K_1(-s)| = h^\beta \int_{[a,b]^2} |K_1(-s)|$ for some β . Moreover, for many kernels in our computations, e.g. $K(s) = \frac{s_1 s_2}{|s|^4}$, we have explicit formulas for the integral. See Section 5.1 in the supplementary material II [12].

We apply the above method to estimate the integral in $X_{\alpha,a} \times Y_{\beta,b}$, $\alpha = l, r, \beta = d, u$ (red region in Figure 2). Since $Y_{m,b} = [-kh, kh]$, for the integral in $X_{\alpha,a} \times Y_{m,b}$ (blue region), we further decompose it

$$(4.39) \quad J_{\alpha,m} = \sum_{-k \leq t \leq k-1} \int_{X_{\alpha,a} \times [th, (t+1)h]} |K_1(-s)| f(s+x) dy,$$

and then apply the above method to estimate it.

Next, we further simplify $\|f\|_{L^\infty(B_{i_1 j_1}(h_x) + Q)}$ in the above estimate. From (4.16), we get

$$ih \leq i_1 h_x < (i_1 + 1)h_x \leq (i + 1)h, \quad jh \leq j_1 h_x < (j_1 + 1)h_x \leq (j + 1)h.$$

For $X_{l,a}$ (4.35) with $0 \leq a \leq m - 1$, we have the lower bound for the endpoint

$$(i - k)h - i_1 h_x - (a + 1)h_x/m \geq (i - k)h - i_1 h_x - h_x \geq (i - k)h - ((i + 1)h - h_x) - h_x = -kh - h.$$

See the left figure in Figure 2. The width of blue region is less than h . Similarly, we can cover the intervals of X, Y (4.35), (4.36) uniformly for $0 \leq a, b \leq m - 1$ and obtain

$$\begin{aligned} X_{l,a} &\subset [(i - k)h - i_1 h_x - h_x, -kh] \subset [-(k + 1)h, -kh], \\ X_{r,a} &\subset [kh, (i + 1 + k)h - i_1 h_x] \subset [kh, (k + 1)h], \\ Y_{d,b} &\subset [-(k + 1)h, -kh], \quad Y_{u,b} \subset [kh, (k + 1)h]. \end{aligned}$$

Thus, we only need to estimate the L^∞ norm of f in

$$Q_{i_1 j_1}(h_x) + [ah, (\alpha + 1)h] \times [\beta h, (\beta + 1)h], \quad \alpha = -k - 1, k, \quad \beta = -(k + 1), -k, \dots, k.$$

These estimates are independent of the choice of m, a, b . Since the size of each domain is at most $2h \times 2h$, the above estimates based on (4.38) are sharp. We estimate the piecewise bound of the weights ψ, φ in Appendixes A.1, A.2, A.3.

Using the above decomposition and estimates, we obtain the estimate of J (4.33) for $x \in A_a \times B_b$ (4.34). Similarly, we can estimate J for any $0 \leq a, b \leq m - 1$. Taking the maximum of these m^2 estimates, we obtain the estimate of J and $I_3(x)$ for all $x \in B_{i_1 j_1}(h_x)$.

4.1.7. First generalization: integral in a ring. We generalize the above ideas to estimate the integrals in domain $D = R(x, k) \setminus R(x, k_2) = R(k) \setminus R(k_2)$

$$J = \int_{R(k) \setminus R(k_2)} |K(y - x)| |f(y)| dy = \int_{s \in D(x, k)} |K(s)| |f(x + s)| dy, \quad D(x, k) \triangleq R(k) \setminus R(k_2) - x$$

with $2 \leq k_2 = k - \frac{i}{2} < k$ for some integer $i \geq 1$ and some kernel $K(z)$. Note that the inner region $R(k_2)$ is different from (4.32). See I_4 in (4.24) for an example of this integral region. Suppose $x \in B_{ij}(h)$ (4.16). We partition location of x similar to (4.34) and introduce p_l, q_l

$$(4.40) \quad \begin{aligned} A_a &= [ih + ah/m, ih + (a + 1)h/m], B_b = [jh + bh/m, jh + (b + 1)h/m], \quad 0 \leq a, b \leq m - 1, \\ p_1 &= -k_2 - a/m, p_2 = k_2 + (m - a - 1)/m, p_3 = -k_2 - b/m, p_4 = k_2 + (m - b - 1)/m, \\ q_1 &= -k - (a + 1)/m, q_2 = k + (m - a)/m, q_3 = -k - (b + 1)/m, q_4 = k + (m - b)/m. \end{aligned}$$

For a fixed $x \in A_a \times B_b$, by comparing the boundaries of the following four rectangles, we get

$$D_{in} \triangleq [p_1 h, p_2 h] \times [p_3 h, p_4 h] \subset R(k_2) - x \subset R(k) - x \subset [q_1 h, q_2 h] \times [q_3 h, q_4 h] \triangleq D_{out}.$$

To obtain the above inclusions, for example, for $s = y - x, y \in R(k_2)$, we use

$$\min_{y \in R(k_2)} y_1 - x_1 = ih - k_2 h - x_1 \leq ih - k_2 h - (ih + ah/m) = -k_2 h - ah/m = p_1 h,$$

uniformly for $x \in A_a \times B_b \subset B_{ij}(h)$. For $R(k) - x \subset D_{out}$, we have $q_1 h \leq \min_{y \in R(k)} y_1 - x_1$. Other bounds for the inclusions are obtained similarly. We yield $D(x, k) \subset D_{ring}$, where

$$(4.41) \quad D_{ring} \triangleq D_{out} \setminus D_{in}$$

is fixed for $x \in A_a \times B_b$.

It suffices to estimate the integral J in D_{ring} . We partition $s \in D_{ring}$ using mesh

$$(4.42) \quad Z_1 = \{-k \leq i \leq k, i \in \mathbb{Z}\} \cup \{p_1, p_2, q_1, q_2\}, \quad Z_2 = \{-k \leq i \leq k, i \in \mathbb{Z}\} \cup \{p_3, p_4, q_3, q_4\},$$

and then order them in an increasing order $z_{l,1} < z_{l,2} < \dots < z_{l,2k+5} \in Z_l, l = 1, 2$. Note that we do not multiply $z_{l,c}$ by h here. We estimate the integral J_{cd} in each grid $Q_{c,d} = [z_{1,c}h, z_{1,c+1}h] \times [z_{2,d}h, z_{2,d+1}h]$ following (4.38) and using the norm $\|f\|_{L^\infty(x+Q_{c,d})}$. We turn off the integral in region $Q_{c,d}$ if $Q_{c,d} \subset D_{in}$ since it is not in D_{ring} (4.41).

Uniform covering. For fixed c, d , we want to cover $x + Q_{c,d}$ uniformly for $x \in A_a \times B_b$ and all $0 \leq a, b \leq m-1$ (the sub-partition of x) to bound $\|f\|_{L^\infty(x+Q_{c,d})}$. Since we add 4 extra points p_l, q_l in Z_1 and Z_2 , and order them in an increasing order, the region $Q_{c,d}$ can change for fixed c, d but with different a, b . We show that the $2k+4$ intervals $[z_{1,c}, z_{1,c+1}], 1 \leq c \leq 2k+4$ can be covered by $[\alpha_l, \beta_l]$ uniformly for a, b

$$(4.43) \quad \begin{aligned} & [\alpha_l, \beta_l], \quad \alpha_l \in Z_1^l, \beta_l \in Z_1^u, Z_1^l \triangleq \{-(k+1) \leq i \leq k, i \in \mathbb{Z}\} \cup \{-s_0 - 2, s_0\}, \\ & Z_1^u \triangleq \{-k \leq i \leq k+1, i \in \mathbb{Z}\} \cup \{-s_0, s_0 + 2\}, \quad s_0 = [k_2], \end{aligned}$$

with α_l, β_l increasing. From (4.40) and the definition of s_0 , we get

$$(4.44) \quad p_1 \in [-s_0 - 2, -s_0], \quad p_2 \in [s_0, s_0 + 2], \quad q_1 \in [-k - 1, -k], \quad q_2 \in [k, k + 1].$$

The uniform covering is based on the following observations. Suppose that $u_i \leq v_i, i = 1, 2, \dots, n$ (u_i, v_i may not be increasing). Let us denote by $\{U_i\}$ the re-ordering of $\{u_i\}$ in an increasing order and denote by $\{V_i\}$ the re-ordering of $\{v_i\}$ in an increasing order. Then we have $U_i \leq V_i$. In fact, for any $k \leq n$, from $u_i \leq v_i, V_k$ is larger than u_j with at least k different indexes j . Since U_k is the k -smallest value in $\{u_i\}_i$, we get $V_k \geq U_k$.

From (4.42), (4.44), since $q_2 = \max_c z_{1,c}, q_1 = \min_c z_{1,c}$, we get

$$\begin{aligned} \{z_{1,c}, c \leq 2k+4\} &= \{-k \leq i \leq k, i \in \mathbb{Z}\} \cup \{p_1, p_2, q_1\}, \quad -k-1 \leq q_1, \quad -s_0-2 \leq p_1, \quad s_0 \leq p_2, \\ \{z_{1,c+1}, c \leq 2k+4\} &= \{-k \leq i \leq k, i \in \mathbb{Z}\} \cup \{p_1, p_2, q_2\}, \quad p_1 \leq -s_0, \quad p_2 \leq s_0+2, \quad q_2 \leq k+1. \end{aligned}$$

We can bound each component in Z_1^l (4.43) by a component in the above list. Using the above observations, after reordering two sequences in an increasing order, which gives $\{\alpha_c\}, \{z_{1,c}\}_{c \leq 2k+4}$, we get $\alpha_c \leq z_{1,c}, c \leq 2k+4$ (4.43). Similarly, we obtain $z_{1,c+1} \leq \beta_c$, and yield $[z_{1,c}, z_{1,c+1}] \in [\alpha_c, \beta_c], c \leq 2k+4$.

Similarly, we obtain $[z_{2,d}, z_{2,d+1}] \subset [\alpha_d, \beta_d]$. Thus, we get $[z_{1,c}, z_{1,c+1}] \times [z_{2,d}, z_{2,d+1}] \in [\alpha_c, \alpha_{c+1}] \times [\beta_d, \beta_{d+1}]$ uniformly for the sub-partition of $x \in A_a \times B_b$ with $0 \leq a, b \leq m-1$, and can cover $x + Q_{cd}$ by $B_{i_1 j_1}(h_x) + [\alpha_c h, \alpha_{c+1} h] \times [\beta_d h, \beta_{d+1} h]$ (4.16) for $x \in B_{i_1 j_1}(h_x) \subset B_{ij}(h)$.

4.1.8. Second generalization: the boundary terms. We generalize the method to estimate some boundary terms. We estimate the x_1 -derivative of $I_3(x)$ (4.24) to illustrate the ideas. In $\partial_1 I_3$, we have an extra boundary term I_{32}

$$\partial_1 I_3(x) = \int_{R(k) \setminus R_{s,1}(k)} \partial_{x_1} K_1(x-y)(W\psi)(y) dy - \int_{(j-k)h}^{(j+1+k)h} K_1(x-y)(W\psi)(y) \Big|_{y_1=x_1-kh}^{x_1+kh} dy_2 \triangleq I_{31} + I_{32},$$

“ where we have used the domain for $R(x, k)$ (4.18).

For I_{31} , we apply the method in Section 4.1.6 to estimate it. Denote $\Gamma_k \triangleq [j-k]h, (j+1+k]h$. Using a change of variable $y = x + s$, we can rewrite I_{32} as follows

$$I_{32} = - \int_{s_2 \in \Gamma_k - x_2} \left(K_1(-kh, -s_2)(W\psi)(x_1+kh, x_2+s_2) - K_1(kh, -s_2)(W\psi)(x_1-kh, x_2+s_2) \right) ds_2.$$

We partition the location of x and assume $x \in A_a \times B_b \subset B_{i_1, j_1}(h_x)$ (4.34). From (4.36), we have

$$s_2 \in \Gamma_k - x_2 \subset Y_{d,b} \cup Y_{m,b} \cup Y_{u,b}.$$

Using the above decomposition and $|W\psi(x)| \leq \|W\varphi\|_\infty f(x), f = \psi\varphi^{-1}$, we obtain

$$|I_{32}| \leq \|W\varphi\|_\infty \sum_{\alpha=\pm, \beta=d, m, u} M_{\alpha, \beta}, \quad M_{\alpha, \beta} \triangleq \int_{Y_{\beta, b}} |K_1(-\alpha kh, -s_2)| \cdot |f(x_1 + \alpha kh, x_2 + s_2)| ds_2,$$

for $\alpha = \pm, \beta = u, m, d$. For $\beta = u, d$, the domain $Y_{\beta,b}$ is small $|Y_{\beta,b}| \leq h$. We apply the method in (4.38) to estimate $M_{\alpha,\beta}$. The only difference is that we need consider a 1D integral here

$$\int_Q |K_1(-\alpha kh, -s_2)| ds_2$$

for some interval Q , rather than a 2D integral in (4.38). For $M_{\alpha,m}$, we decompose the domain $Y_{m,b}$ into small intervals with length h similar to (4.39) and then apply the method in (4.38).

We combine these estimates to bound I_{32} for $x \in A_a \times B_b$. Then, we maximize the estimates over $0 \leq a, b \leq m-1$ to bound I_{32} for $x \in B_{i_1, j_1}(h_x)$.

4.1.9. *Third generalization.* In some of the computations, we need to estimate

$$J = \int_{R(k) \setminus R_s(k_2)} |K(x-y)| f(y) dy$$

for some $k_2 < k$ with $2k_2, k \in Z^+$, where $R_s(k)$ is defined in (4.19). Similarly, we use

$$R_s(k_2) \subset R_s(k) \subset R(k), \quad R(k) \setminus R_s(k_2) = R(k) \setminus R_s(k) \cup R_s(k) \setminus R_s(k_2),$$

and a change of variable $y = x + s$ to obtain

$$J = \left(\int_{s \in R(k) - x, |s_1| \vee |s_2| \geq kh} + \int_{k_2 h \leq |s_1| \vee |s_2| \leq kh} \right) K(-s) f(x+s) dy \triangleq J_1 + J_2.$$

Compared to $R(k) \setminus R_{s,1}(k)$, the domain $R(k) \setminus R_s(k)$ contains two more parts

$$X_{m,a} \triangleq [-kh, kh], \quad X_{m,a} \times Y_{u,b}, \quad X_{m,a} \times Y_{d,b},$$

i.e., the upper and lower blue regions in the right figure in Figure 2. The integral in these regions is estimated similar to that in $X_{\alpha,a} \times Y_{m,b}$ (4.37), and the estimate of J_1 is similar to J in (4.33).

For J_2 , the domain is simpler. Since $2k_2 \in Z^+$, we partition the domain into $h_x \times h_x$ grids

$$J_2 = \sum_{(c,d) \in S_k \setminus S_{k_2}} \int_{[ch_x, (c+1)h_x] \times [dh_x, (d+1)h_x]} |K(-s)| f(s+x) ds, \quad S_l \triangleq \{-k \leq c < k, -k \leq d < k\}.$$

For each integral, we estimate it using the method in (4.38). The remaining steps are the same as those of J in (4.33) studied previously.

Remark 4.5. In the estimates in Section 4.1.6-4.1.9, we use the important property that the weights are locally smooth to move them outside the integral. Moreover, we use the fact that the singular region depend on x monotonously to cover it effectively. Since the integral $\int_Q |K_1(s)| dy$ for different Q, a, b in the above estimates does not depend on x , we first compute these integrals once and store them, and then use them in later estimate of different x .

4.1.10. *Taylor expansion near the singularity.* We need to estimate the integral

$$J(x) \triangleq \int_D \partial_{x_i} \left(K(x-y) (\psi(x) - \psi(y)) W(y) \right) dy,$$

for $k_2 < k$ in some region D close to the singularity x . For example, $D = R(x, k_2) \setminus R(x, k_3), R(x, k_3) \setminus R_{s,1}(x, k_3)$ in $\partial_{x_i} I_{5,0}, \partial_{x_i} I_{5,1}$ (4.51). To obtain a sharp estimate, we perform Taylor expansion on $\psi(x)$. We focus on ∂_{x_1} . Denote $z = x - y, x_m = \frac{x+y}{2}$. A direct computation yields

$$I = \partial_{x_1} (K(x-y) \psi(x) - \psi(y)) = (\partial_1 K)(x-y) (\psi(x) - \psi(y)) + K(x-y) \partial_1 \psi(x).$$

Using Taylor expansion of ψ at x_m and following (B.26), we get

$$\begin{aligned} \psi(x) - \psi(y) &= (x-y) \cdot \nabla \psi(x_m) + \varepsilon_1, \quad \psi_x(x) = \psi_x(x_m) + \varepsilon_2 \\ |\varepsilon_1| &\leq \sum_{i+j=2} c_{ij} \|\partial_x^i \partial_y^j \psi\|_{L^\infty(Q(y))} |z_1|^i |z_2|^j, \quad |\varepsilon_2| \leq \frac{1}{2} (\|\partial_{xx} \psi\|_{L^\infty(Q(y))} |z_1| + \|\partial_{xx} \psi\|_{L^\infty(Q(y))} |z_2|), \end{aligned}$$

where $c_{20} = \frac{1}{4}$, $c_{11} = \frac{1}{2}$, $c_{02} = \frac{1}{4}$, and we have written $z_i = x_i - y_i$ and $Q(y)$ is one of the four quadrants $D \cap \{y : \text{sgn}(y_i - x_i) = \pm 1\}$ covering both x, y . Combining the term with the same derivative of ψ , we need to estimate the following integrals

$$\begin{aligned} & \left| \int_D \psi_x(x_m) (\partial_1 K(z) z_1 + K(z)) W(y) dy \right|, \quad \left| \int_D \psi_y(x_m) \partial_1 K(z) z_2 W(y) dy \right| \\ & \int_D |\partial_x^i \partial_y^j \psi|_{L^\infty(Q(y))} |\partial_1 K(z) z_1^i z_2^j W(y)| dy, \quad i + j = 2, \quad \int_D |\partial_x^{i+1} \partial_y^j \psi|_{L^\infty(Q(y))} |K(z) z_1^i z_2^j W(y)| dy, \quad i + j = 1, \end{aligned}$$

We partition the region of $z = x - y \in x - D$, e.g. $D = R(k_2) \setminus R(k_3)$ (4.51) into small mesh, and estimate the piecewise bounds of weights and each integral following Sections 4.1.6-4.1.9.

We estimate the integral of $|\partial_1^i \partial_2^j K(z) z_1^k z_2^l|$ in Section 5.1 in the supplementary material II [12].

4.1.11. *Hölder estimate of log-Lipschitz function.* In some computation, we need to perform $C^{1/2}$ estimate of some log-Lipschitz function. We consider an example to illustrate the ideas

$$F(x) = \int_{\max_i |x_i - y_i| \leq b} K(x, y) f(y) dy, \quad |K(x, y)| \leq C_1 |x - y|^{-1}, \quad |\partial K(x, y)| \leq C_2 |x - y|^{-2},$$

for some constant C_1, C_2 . Given $f \in L^\infty$, F is log-Lipschitz. To estimate $[f]_{C_x^{1/2}}$, we cannot first estimate the piecewise values of f and $\partial_x f$ and then combine them to obtain the $C_x^{1/2}$ estimate. Instead, given x, z , for a to be determined, we decompose F into the smooth part and the singular part

$$F_1(x) \triangleq \int_{a \leq \max_i |x_i - y_i| \leq b} K(x, y) f(y) dy, \quad F_2(x) \triangleq \int_{\max_i |x_i - y_i| \leq a} K(x, y) f(y) dy.$$

Using the assumptions of the kernel, we have

$$|\partial_{x_1} F_1(x)| \leq C_3 \log \frac{b}{a} \|f\|_\infty, \quad |F_2(x)| \leq C_4 |a| \cdot \|f\|_\infty,$$

where the constants C_3, C_4 depend on b, C_1, C_2 . Applying the above estimates, we obtain

$$\frac{|F(x) - F(z)|}{|x_1 - z_1|^{1/2}} \leq \frac{|F_1(x) - F_1(z)| + |F_2(x) - F_2(z)|}{|x_1 - z_1|^{1/2}} \leq \left(C_3 \log \frac{b}{a} \cdot |x_1 - z_1|^{1/2} + 2C_4 |a| |x_1 - z_1|^{-1/2} \right) \|f\|_\infty.$$

We optimize the estimates by choosing $a = C_5 |x_1 - z_1|$ for some constant C_5 depending on C_3, C_4 . Then we establish the estimate. The above simple estimates show that the choice of a depends on $|x - z|$. Thus, in our later Hölder estimates, we perform decomposition guided by the above estimates and optimize the choice of size of the singular region $[-a, a]^2$. On the other hand, since for different $|x - z|$ we need to choose different a , it increases the technicality of the computer-assisted estimates.

4.2. *L^∞ estimate.* Let \widehat{u}_x be the approximation term of u_x (see Section 4.3 of Part I [14]). We focus on the estimate of the piecewise L^∞ norm of $u_{x,A} = u_x - \widehat{u}_x$, which is a representative case. For simplicity, we assume the rescaling factor $\lambda = 1$. We assume that x satisfies (4.16) without loss of generality. We want to estimate $u_{x,A}$ for all $x \in B_{i_1 j_1}(h_x)$.

We can write $u_{x,A} = u_x - \widehat{u}_x$ as follows

$$u_{x,A} = \int (K(x - y) - \widehat{K}(x, y)) W(y) dy, \quad K_A \triangleq K(x - y) - \widehat{K}(x, y),$$

where $\widehat{K}(x, y)$ is the kernel for the approximation term and W is the odd extension of ω (see (4.23)). From Sections 4.3.2 and 4.3.3 of Part I [14], we remove the singular part in \widehat{K} , and then \widehat{K} is nonsingular. Given x with (4.16), similar to (4.24), for $k \geq k_2$, we perform the following decomposition

$$\begin{aligned} (4.45) \quad u_{x,A} &= \left(\int_{R(k)^c} + \int_{R(k) \setminus R_s(k_2)} + \int_{R_s(k_2)} \right) K(x - y) W(y) dy - \int \widehat{K}(x, y) W(y) dy \\ &\triangleq I_1 + I_2 + I_3 + I_4. \end{aligned}$$

where $R_s(k)$ is the symmetric singular region (4.19). See Section 4.2.3 for the choice of k .

Since $I_1 + I_4$ is nonsingular, we use the ideas in Section 4.1.5 to symmetrize the kernels in $I_1 + I_4$. Then we use the method in Section 4.1.3 to estimate it.

Remark 4.6. In our computation, the domain $[0, D]^2 \cap R(k)^c$ can be decomposed into the union of small grids $[y_i, y_{i+1}] \times [y_j, y_{j+1}]$ (4.11) since the boundary of $R(x, k)$ aligns with the mesh (4.18). In particular, in each grid, the indicator function is constant, and the integrand is smooth in y .

Next we consider I_2 . The domain of the integral is close to the singularity. If we use the method in Section 4.1.3 to estimate it, the error will be quite large since $\partial^2 K(x - y)$ is very singular. We want to estimate I_2 using $\|W\varphi\|_\infty$ and the singular part I_3 using $[W\psi]_{C^{1/2}}$. Since $K(z)$ is singular of order -2 , we expect an estimate

$$|I_2| + |I_3| \lesssim \log \frac{k}{k_2} \varphi^{-1}(x) \|W\varphi\|_{L^\infty[R(k)]} + \psi^{-1}(x) k_2^{1/2} [W\psi]_{C_x^{1/2}}.$$

Note that the weights φ, ψ have a different order of singularity for small x and a different rate of decay. Moreover, we need to control the right hand side using the energy, which assigns different weights to two norms (seminorms). Thus, to obtain a sharp estimate, we need to optimize the choice of k_2 .

Firstly, we consider $k_2 = 2, 2 + \frac{1}{2}, \dots, k$, we use the method in Section 4.1.9 to estimate I_2 . We also consider very small $k_2 < 2$. In this case, we further decompose I_2 as follows

$$I_2 = \left(\int_{R(k) \setminus R_s(2)} + \int_{R_s(2) \setminus R_s(k_2)} \right) K(x - y) W(y) dy \triangleq I_{21} + I_{22}.$$

For I_{21} , we apply the method in Section 4.1.9. For I_{22} , we use a change of variables $y = x + sh$

$$|I_{22}| = \left| \int_{k_2 \leq |s_1| \vee |s_2| \leq 2} K(-sh) W(x + sh) h^2 ds \right|.$$

Since the region is very small, $x + sh \in B_{i_1 j_1}(h_x) + [-2h, 2h]$, and $K_1(hs) = h^{-2} K_1(s)$, we get

$$|I_{22}| \leq \|W\varphi\|_\infty \|\varphi^{-1}\|_{L^\infty(B_{i_1 j_1}(h_x) + [-2h, 2h])} \int_{k_2 \leq |s_1| \vee |s_2| \leq 2} |K(s)| ds.$$

The integral can be computed explicitly and has the order $\log \frac{2}{k_2}$.

It remains to estimate the most singular part I_3 for different k_2 . Using a change of variables $y = x + sh$, the scaling symmetries, and the above derivations, we get

$$I_3 = \int_{[-k_2, k_2]^2} K(-s) W(x + sh) ds.$$

To use the Hölder norm of $W\psi$, we decompose it as follows (4.46)

$$I_3 = \int_{[-k_2, k_2]^2} K(-s) (W\psi)(x + sh) \left(\frac{1}{\psi(x + sh)} - \frac{1}{\psi(x)} \right) + K(-s) \frac{(W\psi)(x + sh)}{\psi(x)} ds \triangleq I_{31} + I_{32}.$$

For I_{32} , using the Hölder seminorm, the odd symmetry of $K(s) = c \frac{s_1 s_2}{|s|^4}$ in s_1 , and $|(W\psi)(x + sh) - (W\psi)(x - sh)| \leq \sqrt{2s_1 h}$, we get

$$|I_{32}| \leq \frac{h^{1/2}}{\psi(x)} [W\psi]_{C_x^{1/2}} \int_{[0, k_2] \times [-k_2, k_2]} |K(s)| \sqrt{2s_1} ds = \frac{2k_2^{1/2} h^{1/2}}{\psi(x)} [W\psi]_{C_x^{1/2}} \int_{[0, 1]^2} |K(s)| \sqrt{2s_1} ds,$$

where we used the scaling symmetry of K and a change of variables $s \rightarrow k_2 s$ in the last equality.

4.2.1. *The commutator.* For I_{31} , we apply the simple Taylor expansion to $f = \psi^{-1}$

$$(4.47) \quad |f(x + sh) - f(x)| \leq |f_x(x) h s_1 + f_y(x) h s_2| + h^2 \left(\frac{m_{20} s_1^2}{2} + m_{11} s_1 s_2 + \frac{m_{02} s_2^2}{2} \right),$$

where m_{ij} is the bound for the second derivatives of ψ^{-1}

$$m_{ij}(s) = \max_{B_{i_1 j_1}(h) + I(\text{sgn}(s_1)) \times I(\text{sgn}(s_2))} \|\partial_x^i \partial_y^j (\psi^{-1})\|_{L^\infty}, \quad I_+ = [0, k_2 h], \quad I_- = [-k_2 h, 0].$$

Note that m_{ij} is constant in each quadrant of $[-k_2, k_2]$. We plug in the expansion (4.47) to estimate I_{31} . We only discuss a typical term $m_{20}s_1^2h^2$

$$I_{31,20} \triangleq h^2 \int_{[-k_2, k_2]^2} |K(-s)(W\psi)(x+sh)|m_{20}(s)\frac{s_1^2}{2}ds.$$

If $k_2 \geq 2$, we can further partition $[-k_2, k_2]^2$ into $B_{2p,2q}(1/2) = [p, p+1/2] \times [q, q+1/2]$, $-k_2 \leq p, q \leq k_2 - 1/2$, where we use the notation (4.17). For each grid $B_{2p,2q}(1/2)$, the sign of s and $m_{20}(s)$ are fixed, and we have

$$\int_{B_{2p,2q}(\frac{1}{2})} |K(-s)|(W\psi)(x+sh)m_{20}(s)\frac{s_1^2}{2}ds \leq m_{20}\|W\varphi\|_\infty \int_{B_{2p,2q}(\frac{1}{2})} \frac{|K(s)|s_1^2}{2} \left(\frac{\psi}{\varphi}\right)(x+sh)ds.$$

The last integral can be estimated using the method in (4.38). Combining the estimate of integral in different regions $B_{2p,2q}(1/2)$, we obtain the estimate of $I_{31,20}$. Similarly, we can estimate the contributions of other terms in (4.47) to I_{31} .

For small $k_2 \leq 2$, we do not partition the domain. We denote $D(k_2) = B_{i_1, j_1}(h_x) + [-k_2h, k_2h]^2$. For $s \in [-k_2, k_2]$, we use $x+sh \subset D(k_2) \subset D(2)$ to get

$$|f(x+sh) - f(x)| \leq \|f_x\|_{L^\infty(D(k_2))}s_1h + \|f_y\|_{L^\infty(D(k_2))}s_2h. \quad |W\psi(x+sh)| \leq \|W\varphi\|_\infty \left\| \frac{\psi}{\varphi} \right\|_{L^\infty(D(2))}.$$

Plugging the above estimate into I_{31} , we get

$$I_{31} \leq \sum_{(i,j)=(1,0),(0,1)} h \|\partial_x^i \partial_y^j (\psi^{-1})\|_{L^\infty(D(k_2))} \|W\varphi\|_\infty \left\| \frac{\psi}{\varphi} \right\|_{L^\infty(D(2))} \int_{[-k_2, k_2]^2} |K(s)s_1^i s_2^j| ds.$$

Using the scaling symmetry, we can reduce the last integral to $k_2^{i+j} \int_{[-1,1]^2} |K(s)s_1^i s_2^j| ds$.

We apply the above estimates to a list of k_2 , and bound different norms using $\max(\|\omega\varphi\|_\infty, \max_i \gamma_i [\omega\psi_1]_{C_{x_i}^{1/2}(\mathbb{R}_2^+)})$. Then by optimizing the k_2 , we obtain the sharp estimate of $u_{x,A}$.

In (4.47), we do not bound $f(x+sh) - f(x)$ directly using the estimate (4.48) since s is large. Instead, we perform a higher order expansion.

Estimate of u_y, v_x . The estimates of u_y, v_x follow similar strategies and estimates. The only difference is the estimate of the most singular term similar to I_{32} (4.46) for u_y, v_x due to different symmetry property of the kernel. We estimate it using a combination of norms $\|\omega\varphi\|_\infty$, and semi-norms $[\omega\psi]_{C_{x_i}^{1/2}}$, and refer it to Section 6.1 in the supplementary material II [12].

4.2.2. Estimate of \mathbf{u}_A . The estimate of \mathbf{u}_A is much simpler since it is more regular. Let K and \hat{K} be the kernel of u, v and its approximation term, respectively. For $f = u$ or v , we perform a decomposition similar to (4.45)

$$(4.49) \quad f_A = \left(\int_{R(k)^c} + \int_{R(k) \setminus R_s(k)} + \int_{R_s(k)} \right) K(x-y)W(y)dy - \int \hat{K}(x,y)W(y)dy \triangleq I_1 + I_2 + I_3 + I_4.$$

The estimates of $I_1 + I_4$ follow the method for $u_{x,A}$. For I_2 , we use the method in Section 4.1.6. For I_3 , since K has a singularity of order $|x|^{-1}$, which is locally integrable, we use a change of variable $y = x + sh$ to obtain

$$I_3 = h \int_{[-k, k]^2} K(-s)W(x+sh)ds.$$

Then we partition $[-k, k]^2$ into small grids, and use the method in (4.38) to estimate the integral in each grid. Here, we get a factor h in the change of variables since $K(\lambda s) = \lambda^{-1}K(s)$.

4.2.3. Choice of parameters. Recall the choice of several parameters a, h, h_x from (4.14). We choose $3 \leq k \leq 10$. We choose k for the size of the singular region kh (4.45), (4.49) not so small such that the error $h^2\partial^2K$ in Lemma 4.2, which has the order $h^2|x-y|^{-\alpha-2}$ near the singularity, is smaller than the main term K , which has the order $|x-y|^{-\alpha}$, $\alpha = 1, 2$. Since we will estimate

$I_1 + I_4$, I_2 , I_3 in the decomposition separately using the triangle inequality, we do not choose k to be too large so that we can exploit the cancellation in $I_1 + I_4$.

4.3. Hölder estimates. We want to estimate $\frac{|f(x)-f(z)|}{|x-z|^{1/2}}$ for any $x, z \in \mathbb{R}_2^{++}$ with $x_1 = z_1$ or $x_2 = z_2$ and some function f , e.g. $f = u_{x,A}$. Without loss of generality, we assume $|z| > |x|$. Then in the $C_x^{1/2}$ estimate, we have $x_1 < z_1, x_2 = z_2$; in the $C_y^{1/2}$ estimate, we have $x_1 = z_1, x_2 < z_2$. Applying the rescaling argument in Section 4.1, we can restrict $\hat{x} = \frac{x}{\lambda}$ to $\hat{x} \in [0, 2x_c]^2 \setminus [0, x_c]^2$. For this reason, we assume $\lambda = 1$ for simplicity. We will only estimate the Hölder difference for comparable x, z : $|x| \asymp |z|$. If $|z| \gg |x|$, we simply apply the L^∞ estimate to $f(x), f(z)$ and use the triangle inequality.

We focus on the Hölder estimate of $u_{x,A}$, which is a representative and the most important nonlocal term to estimate in our energy estimate.

4.3.1. $C_x^{1/2}$ estimate. Recall I_i from the decomposition (4.24) and $K_1(s) = \frac{s_1 s_2}{|s|^4}$. We apply the same decomposition to $u_{x,A}(z)$. We assume that the approximation term \hat{u}_x (see Section 4.3 of Part I [14]) takes the following form

$$(4.50) \quad \hat{u}_x(x) = \int \hat{K}_1(x, y)W(y)dy, \quad I_6(x) \triangleq \psi(x)\hat{u}_x(x) = \psi(x) \int \hat{K}_1(x, y)W(y)dy,$$

with a nonsingular kernel \hat{K}_1 . We first discuss how to estimate the regular part I_1, I_3, I_4 in (4.24) and I_6 , which are Lipschitz. We will apply the sharp Hölder estimate in Lemmas 3.1-3.5 in Section 3 of Part I [14] to estimate the most singular part I_2 . The most technical part is to estimate I_5 , which is log-Lipschitz since the kernel $K_1(x-y)(\psi(x) - \psi(y))$ has a singularity of order -1 . We assemble the estimates of different parts to estimate $[u_{x,A}\psi]_{C_x^{1/2}}$ in Section 4.6.

4.3.2. Estimates of the regular terms I_1, I_3, I_4, I_6 . Recall I_1, I_3, I_4 from (4.24) and I_6 from (4.50). Since the integrands in I_1, I_3, I_4 are supported at least $k_2 h$ away from the singularity x , if W is in some suitable weighted L^∞ space, I_1, I_3, I_4 are piecewise smooth and their derivatives can be bounded by $\|W\varphi\|_{\infty(\mathbb{R}_2^+)} = \|\omega\varphi\|_\infty$. Their derivatives jump when $R(x, k), R(x, k_2)$ change, or equivalently, x moves from one grid to another. For $x \in B_{i_1, j_1}(h_x)$ (4.16), these rectangle domains are the same, and these functions are smooth. The approximation term I_6 (4.50) is locally smooth in x . To exploit the cancellation, we combine the estimates of I_1, I_4, I_6 together. We symmetrize the kernel in $I_1(x) + I_4(x) - I_6(x)$ following Section 4.1.5 and use the method in Section 4.1.3 to estimate the derivatives of $I_1(x) + I_4(x) - I_6(x)$. See also (4.28), (4.29) for the form of the symmetrized integrands in these integrals.

We estimate the piecewise Lipschitz norm of I_3 using the method in Sections 4.1.6, 4.1.8. We choose integer k, k_2 in the decomposition (4.24). Then in each grid $[y_i, y_{i+1}] \times [y_j, y_{j+1}]$, the indicator functions in $I_1 + I_4 - I_6$, e.g. $\mathbf{1}_{R(k)^c}, \mathbf{1}_{R(k) \setminus R(k_2)}$, are constant. See Remark 4.6. We will combine the estimates of different terms in Section 4.6, e.g. $I_1 + I_4 - I_6, I_3$ and part of I_5 defined later in (4.51), and obtain some Hölder continuous functions when x moves from one grid to another. We assemble the Hölder estimates in Section 4.6.

4.3.3. $C_x^{1/2}$ estimate of I_2 . We first estimate the second term I_2 in (4.24). Recall $R(x, k), R_{s_1}(x, k), R_s(x, k)$ from (4.18), (4.19) and the location of x (4.16). We have

$$x_2 - (j-k)h \leq (j+1)h - (j-k)h = (k+1)h, \quad (j+1+kh) - x_2 \leq (j+1+kh) - jh = (k+1)h.$$

Since $x_2 = z_2$, using Lemma 3.1 from Section 3 of Part I [14] with $(a, b_1, b_2) = (kh, x_2 - (j-k)h, (j+1+k)h - x_2)$ and $|b_1|, |b_2| \leq (k+1)h$, we obtain

$$\frac{1}{|x-z|^{1/2}}|I_2(x, k) - I_2(z, k)| \leq C_1 \left(\frac{(k+1)h}{|x-z|} \right) [W\psi]_{C_x^{1/2}} = C_1 \left(\frac{(k+1)h}{|x-z|} \right) [\omega\psi]_{C_x^{1/2}}.$$

We only apply the Hölder estimate to $|x-z| \leq \frac{kh}{2}$ (rescaled x, z) and the assumption $a \geq \frac{1}{2}|x_1 - z_1|$ in Lemma 3.1 in Part I [14] is satisfied. For $I_2(x, k)$ associated with other terms u, v, u_y, v_x , we can estimate it using similar ideas and Lemmas 3.1-3.5 in Part I [14]. The $C_y^{1/2}$ estimate of $I_2(x, k)$ is completely similar. See Section 4.3.8 for more details.

4.3.4. $C_x^{1/2}$ estimate of I_5 . For I_5 (4.24), $K_1(x-y)(\psi(x) - \psi(y))$ is singular of order -1 near $y = x$. Given $W \in L^\infty(\varphi)$, I_5 is log-Lipschitz. There are several approaches to estimate its Hölder norm, see e.g., Section 4.1.11. We use part of the $C_x^{1/2}$ seminorm of ω to get a better estimate. We choose $k_3 = k_2 - \frac{i}{2} \geq 2, i = 0, 1, 2, \dots, 2k_2 - 4$ and further decompose I_5 as follows

$$(4.51) \quad I_5(x, k_2) = \left(\int_{R(k_2) \setminus R(k_3)} + \int_{R(k_3) \setminus R_{s,1}(k_3)} + \int_{R_{s,1}(k_3)} \right) K_1(x-y)(\psi(x) - \psi(y))W(y)dy \\ \triangleq I_{5,0}(x, k_2, k_3) + I_{5,1}(x, k_3) + I_{5,2}(x, k_3).$$

The domain in $I_{5,0}$ depends on x . For x in a grid cell, it does not change with x . We estimate $\partial_{x_1} I_{5,0}$ using Taylor expansion in Section 4.1.10 and following the method in Section 4.1.7. We estimate the x -derivative of $I_{5,1}$ using the method in Sections 4.1.6, 4.1.8. We have

$$(4.52) \quad \partial_{x_1} I_{5,1} = \int_{R(k_3) \setminus R_{s,1}(k_3)} \partial_{x_1} \left(K_1(x-y)(\psi(x) - \psi(y)) \right) W(y) dy \\ - \int_{(j-k_3)h}^{(j+1+k_3)h} K_1(x-y)(\psi(x) - \psi(y))W(y) \Big|_{y_1=x_1-k_3h}^{x_1+k_3h} dy_2.$$

We estimate the first part following Section 4.1.10, and the second part following Section 4.1.8.

For $I_{5,2}$, we will estimate it using a method similar to that of I_2 . See the left figure in Figure 3 for the domains of the integrals in $I_{5,2}(x), I_{5,2}(z)$. The integrand satisfies

$$K_1(x-y)(\psi(x) - \psi(y))W(y) = \psi(x)K_1(x-y)(\psi^{-1}(y) - \psi^{-1}(x))(W\psi)(y) \\ \approx \psi(x)\partial_i(\psi^{-1}(x)) \cdot K_1(x-y)(y_i - x_i)(W\psi)(y).$$

Thus, $I_{5,2}(x)$ can be seen as a weighted version of I_2 (4.24) with a weight $\psi(x)\partial_i(\psi^{-1}(x))$, a more regular kernel $K_1(x-y)(y_i - x_i)$, and a smaller domain $R_{s,1}(k_3)$. Since the kernel is more regular and the domain is smaller, our estimate for $I_{5,2}$ is much smaller than that of I_2 .

Now, we justify this approach. Using a change of variables $y = x + s, s \in R_{s,1}(k_3) - x$ and the above identity, we yield

$$I_{5,2}(x, k_3) = \psi(x) \int_{R_{s,1}(k_3) - x} K_1(-s)(\psi^{-1}(x+s) - \psi^{-1}(x))(W\psi)(x+s)ds.$$

Using Newton's formula $f(1) = f(0) + f'(0) + \int_0^1 (1-t)f''(t)dt$ for $f(t) = \psi^{-1}(x+ts)$, we get

$$\psi^{-1}(x+s) - \psi^{-1}(x) = s \cdot \nabla \psi^{-1}(x) + \int_0^1 (1-t) \left(s \cdot (\nabla^2 \psi^{-1})(x+ts) \cdot s \right) dt \\ = \sum_{i=1,2} s_i \partial_i(\psi^{-1})(x) + \sum_{0 \leq i \leq 2} \binom{2}{i} s_1^i s_2^{2-i} \int_0^1 (1-t) \partial_1^i \partial_2^{2-i}(\psi^{-1})(x+ts) dt.$$

Denote

$$Q_{ij}(x) = \psi(x) \int_0^1 (1-t) \partial_1^i \partial_2^j(\psi^{-1})(x+ts) dt, \quad i+j=2, \quad D(x) = R_{s,1}(x, k_3) - x,$$

$$Q_{ij}(x) = \psi(x) \cdot \partial_1^i \partial_2^j(\psi^{-1})(x) = -\frac{\partial_1^i \partial_2^j \psi(x)}{\psi(x)}, \quad i+j=1, \quad P_{ij}(x) = \int_{D(x)} K_1(-s) s_1^i s_2^j (W\psi)(x+s) ds.$$

Using the above expansion and notations, we get

$$I_{5,2}(x, k_3) = \sum_{i+j=1} P_{ij} Q_{ij} + \sum_{i+j=2} \binom{2}{i} P_{ij} Q_{ij}.$$

Next, we use the above decomposition to estimate $I_{5,2}(x, k_3) - I_{5,2}(z, k_3)$. The leading order terms are $P_{ij} Q_{ij}$ with $i+j=1$. By definition of $R_{s,1}$ (4.19), we observe that if $x_2 = z_2$, we have

$$D(x) = R_{s,1}(x, k_3) - x = R_{s,1}(z, k_3) - z = D(z).$$

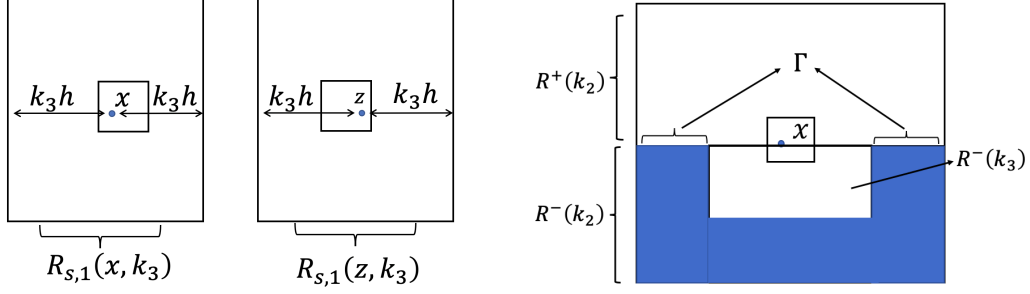


FIGURE 3. Left: $R_{s,1}(x, k_3)$ and $R_{s,1}(z, k_3)$ with $x_2 = z_2$. The small square is a mesh grid containing x or z . x, z can have different locations relative to the grids. Right: The large rectangle is $R(k_2)$, the upper part is $R^+(k_2)$, and the lower part is $R^-(k_2)$. The blue region is $R^-(k_2) \setminus R^-(k_3)$. Γ is part of its boundary.

Suppose that $x_1 < z_1$. We perform a decomposition

$$(4.53) \quad \begin{aligned} |P_{ij}(x)Q_{ij}(x) - P_{ij}(z)Q_{ij}(z)| &\leq J_1 + J_2, \\ J_1 &\triangleq |Q_{ij}(z)(P_{ij}(x) - P_{ij}(z))|, \quad J_2 \triangleq |P_{ij}(x)(Q_{ij}(x) - Q_{ij}(z))|. \end{aligned}$$

Using $D(x) = D(z)$, we bound J_1 as follows

$$\begin{aligned} |J_1| &\leq |Q_{ij}(z)| \left| \int_{D(x)} K_1(-s) s_1^i s_2^j ((W\psi)(x+s) - (W\psi)(z+s)) ds \right| \\ &\leq |Q_{ij}(z)| \cdot |x - z|^{1/2} \|\omega\psi\|_{C_x^{1/2}} \int_{s \in D(x)} |K_1(s) s_1^i s_2^j| ds. \end{aligned}$$

The term Q_{ij} only depends on the weight and is smoother than P_{ij} . We can estimate $Q_{ij}(x) - Q_{ij}(z)$ by bounding $\partial_1 Q_{ij}$ since Q_{ij} is locally smooth. For P_{ij} in J_2 , we use the method in (4.38) to bound it by $C\|\omega\varphi\|_\infty$ with some constant C . Then we obtain the estimate

$$|J_2| \leq C_2 |x - z| \cdot \|\omega\varphi\|_{L^\infty}$$

for some constant C_2 . Note that the second order term $P_{ij}Q_{ij}$, $i + j = 2$ is much smaller than the leading order terms. For $|x - z|$ not too small, we can estimate its contribution trivially

$$(4.54) \quad \frac{1}{|x - z|^{1/2}} |P_{ij}(x)Q_{ij}(x) - P_{ij}(z)Q_{ij}(z)| \leq \frac{1}{|x - z|^{1/2}} (|P_{ij}(x)Q_{ij}(x)| + |P_{ij}(z)Q_{ij}(z)|).$$

We optimize the above two estimates.

In summary, to obtain the above estimates, we estimate piecewise bounds for $|Q_{ij}(x)|$, $P_{ij}(x)$, $|\partial_k Q_{ij}(x)|$, and the integrals $\int_{D(x)} |K_1(s) s_1^i s_2^j| ds$, $i + j = 1, 2$.

The above estimate of $I_5(x, k_2)$ can be generalized to the $C_x^{1/2}$ estimate of u, v, v_x, u_y . Yet, it does not apply to the $C_y^{1/2}$ estimate of $\mathbf{u}, \nabla \mathbf{u}$ since it requires the estimate of $(W\psi)(x+s) - (W\psi)(z+s)$ for s in some rectangle $R = D(x) = D(z)$. However, since W is discontinuous across the boundary $y = 0$, $W\psi \notin C_y^{1/2}(R)$ if $x+s, z+s$ are not in the same half plane. If $x_1 < x_2$, then the rectangles $R(x, k_2), R(z, k_2)$ will not intersect the boundary and the previous estimate holds true. If $x_1 > x_2$, we consider two modifications for different kernels in the following subsections.

4.3.5. Ideas of the $C_y^{1/2}$ estimates of I_5 . The main idea in the following $C_y^{1/2}$ estimates is to use a combination of the estimates for the log-Lipschitz function in Section 4.1.11 and the estimate in Section 4.3.4. The latter provides better estimates, and we try to use this method *as much as possible*. Following the ideas in Section 4.1.11, we decompose $I_5(x)$ into the singular part and nonsingular part with different size k_3 of the singular region

$$I_5(x) = I_{5,S}(x, k_3) + I_{5,NS}(x, k_3).$$

Although we cannot apply the second method to the whole $I_5(x)$, we can apply it to the integrals in the upper part of the regions, e.g. $R^+(k_2), R^+(k_3)$ (4.20), since these integrals only involve $W\psi$ in \mathbb{R}_2^+ and we have $W\psi \in C^{1/2}$. Thus, we will further decompose some of the regions into the upper part and the lower part, and then apply the first method to the lower part, and the second method to the upper part.

4.3.6. $C_y^{1/2}$ estimate of the velocity with a kernel of the first type. The kernels

$$(4.55) \quad K = \frac{y_1 y_2}{|y|^4}, \quad \frac{y_2}{|y|^2}$$

associated with $u_x = -\partial_{xy}(-\Delta)^{-1}\omega, u = -\partial_y(-\Delta)^{-1}\omega$ vanish when $y_2 = 0$. We call them the first type kernel. Let K be a kernel of the first type. We use the following decomposition

$$(4.56) \quad I_5(x, k_2) = \left(\int_{R^+(k_2)} + \int_{R^-(k_2)} \right) K(x-y)(\psi(x) - \psi(y))W(y)dy \triangleq I_5^+(x, k_2) + I_5^-(x, k_2)$$

See the right figure in Figure 3 for $R^\pm(k_2)$. Since $R^+(x, k_2), R^+(z, k_2) \subset \mathbb{R}_2^+$, we can decompose

$$I_5^+ = I_{5,1}^+ + I_{5,2}^+$$

into the integral in the regions $I_{5,1}^+ : R^+(k_2) \setminus R_{s,2}^+(k_2)$ and $I_{5,2}^+ : R_{s,2}^+(k_2)$, and apply the same argument as that for $I_{5,1}(x, k_3), I_{5,2}(x, k_3)$ in Section 4.3.4 to obtain the desired estimates by restricting all the derivations in $R^+(x, k_2), R^+(z, k_2)$. Note that here, we do not further choose smaller window $R^+(x, k_3)$ to decompose $I_5^+(x, k_2)$, i.e. $k_3 = k_2$ and $I_{5,0} = 0$ in (4.51). For $I_{5,1}^+$, similar to (4.52), we get a boundary term from $\partial_2(R^+(k_2) \setminus R_{s,2}^+(k_2)) = [(i-k_2)h, (i+1+k_2)h] \times \{x_2 + k_2 h\}$. See (4.19), (4.18) for $R^+(k), R_{s,2}^+(k)$.

For the lower part $I_5^-(x, k_2)$, it is log-Lipschitz if $W \in L^\infty(\varphi)$. We cannot bound its derivative using $\|W\varphi\|_\infty$. We face the difficulty discussed at the beginning of Section 4.

Alternatively, we follow the ideas in Section 4.1.11. We decompose it into the smooth part and rough part. We introduce $0 < k_3 < k_2$ and consider the following decomposition

$$(4.57) \quad I_5^-(x, k_2) = \left(\int_{R^-(k_2) \setminus R^-(k_3)} + \int_{R^-(k_3) \setminus R_{s,2}^-(k_3)} + \int_{R_{s,2}^-(k_3)} \right) K(x-y)(\psi(x) - \psi(y))W(y)dy \\ \triangleq I_{5,0}^-(x, k_2) + I_{5,1}^-(x, k_3) + I_{5,2}^-(x, k_3).$$

See the right figure in Figure 3 for an illustration of different domains. Recall that $k_2 \in Z_+$. We choose $k_3 = k_2 - \frac{i}{2} \geq 2, i = 0, 1, 2, \dots, 2k_2 - 4$. Since the integrand in $I_{5,0}^-$ supports at least $k_3 h$ away from the singularity, $I_{5,0}^-(x, k_2)$ is piecewisely smooth. We can estimate $\partial_{x_2} I_{5,0}^-(x, k)$ following Sections 4.1.7, 4.1.10. The domain $R^-(k_2) \setminus R^-(k_3)$ is not piecewise constant since the upper part of its boundary, i.e.

$$\Gamma = \{(y_1, x_2) : y_1 \in [(i-k_2)h, (i+1+k_2)h] \setminus [(i-k_3)h, (i+1+k_3)h]\},$$

depends on x_2 . See Figure 3 for an illustration of Γ . Taking x_2 derivative on $I_{5,1}^-$, we get

$$(4.58) \quad |\partial_{x_2} I_{5,0}^-(x, k_2)| \leq \left| \int_{R^-(k_2) \setminus R^-(k_3)} \partial_{x_2} J(x, y)W(y)dy \right| + \left| \int_{y \in \Gamma} J(x, y)W(y)dy \right|, \\ J(x, y) = K(x-y)(\psi(x) - \psi(y))$$

Since $y \in \Gamma \subset \{y : y_2 = x_2\}$ and that $K(y_1, 0) \equiv 0$, the second term vanishes. The first term can be estimated using a change of variables $y = x + s$ and the method in Section 4.1.10, Section 4.1.7, since its support is at least $k_3 h$ away from the singularity.

For $I_{5,1}^-$, it is also piecewise Lipschitz, we estimate the x_2 derivative similar to $I_{5,1}$ in (4.52)

$$(4.59) \quad |\partial_{x_2} I_{5,1}^-| \leq \left| \int_{R^-(k_3) \setminus R_{s,2}^-(k_3)} \partial_{x_2} J(x, y)W(y)dy \right| + \left| \int_{(i-k_3)h}^{(i+1+k_3)h} J(x, y)W(y) \Big|_{y_2=x_2-k_3h} dy_1 \right|.$$

Different from $I_{5,1}$ in (4.52), the boundary term in the above estimate only involves the lower part $y_2 = x_2 - k_3 h$ since the domain in $I_{5,1}^-$ is $R^-(k_3) \setminus R_{s,2}^-(k_3)$.

For $I_{5,2}^-$, the kernel satisfies $K(x-y)(\psi(x) - \psi(y)) \sim |x-y|^{-1}$ for small $|x-y|$ and is locally integrable. We estimate its piecewise L^∞ bound using the method in Section 4.2.1 for the commutator.

The above decomposition can be applied to estimate

$$\frac{|I_5^-(x, k_2) - I_5^-(z, k_2)|}{|x-z|^{1/2}} \leq \min_{k_3=k_2-\frac{h}{2}} \frac{|(I_{5,0}^- + I_{5,1}^-)(x, k_3) - (I_{5,0}^- + I_{5,1}^-)(z, k_3)|}{|x-z|^{1/2}} + \frac{|I_{5,2}^-(x, k_3)| + |I_{5,2}^-(z, k_3)|}{|x-z|^{1/2}}$$

for $|x-z|$ not too small, e.g. $|x-z| \geq d_s = \frac{h}{10}$. When $|x-z|$ is sufficiently small, the second term in the above estimate can be very large.

According to the analysis in Section 4.1.11, for $|x-z|$ very small, we need to choose $k_3 h \sim |x-z|$ to get the sharp estimate. Thus, we consider one more decomposition for $a \leq 1$

$$(4.60) \quad \begin{aligned} I_5^-(x, k_2) &= \int_{R^-(k_2) \setminus R_s^-(a)} K(x-y)(\psi(x) - \psi(y))W(y)dy \\ &+ \int_{R_s^-(a)} K(x-y)(\psi(x) - \psi(y))W(y)dy \triangleq I_{5,3}^-(x, a) + I_{5,4}^-(x, a). \end{aligned}$$

The above decomposition is slightly different from (4.57). We choose $R_s^-(a)$ rather than $R^-(a)$, since we need to choose the singular region with size going to 0 as $|x-z| \rightarrow 0$. Yet, $R^-(a)$ (4.18) does not satisfy this requirement for $a \rightarrow 0$. We can estimate the derivative of $I_{5,3}^-(x, a)$ following Sections 4.1.6-4.1.8, and the L^∞ norm of $I_{5,4}^-(x, a)$ following Section 4.2.1. Again, in the computation of $\partial_{x_2} I_{5,3}^-(x, a)$, the boundary term vanishes due to $K(y_1, 0) \equiv 0$. In summary, we can obtain the following estimate

$$(4.61) \quad |\partial_{x_2} I_{5,3}^-(x, a)| \leq A(x) + B(x) \log(1/a), \quad |I_{5,4}^-(x, a)| \leq C(x)ah,$$

for any $a \leq 1$, where $A(x), B(x)$ can be estimated following the method in Appendix B.5.1, and the estimate of $C(x)$ follows the method in Section 4.2.1. Using the above estimates and the ideas in Section 4.1.11, we can estimate $d_y(I_5^-(\cdot, k_2), x, z)$ for small $|x-z|$ by optimizing a , where d_y is defined below

$$(4.62) \quad d_y(f, x, z) = |f(x) - f(z)||x-z|^{-1/2}.$$

We will assemble these estimates in Section 4.6.

4.3.7. $C_y^{1/2}$ estimate of the velocity with a kernel of the second type. For the kernels $K_2 = \frac{y_1^2 - y_2^2}{|y|^4}$ and $\frac{y_1}{|y|^2}$, they do not vanish on $y_2 = 0$ in general. We call them the second type kernel.

If we use the strategies in the previous subsection, the boundary term in the computation of $\partial_{x_2} I_{5,0}^-(x, k_3)$, $\partial_{x_2} I_{5,1}^-(x, k_3)$ or $\partial_{x_2} I_{5,3}^-(x, k_3)$ does not vanish on Γ and can be large. To avoid picking up a boundary term on Γ and apply the ideas in Section 4.3.5, we consider another estimate on $I_5(x, k_2)$. For $k_3 = k_2 - \frac{i}{2}$, $i = 0, 1, \dots, 2k_2 - 4$, we perform the following decomposition (4.63)

$$(4.63) \quad \begin{aligned} I_5(x, k_2) &= \left(\int_{R(k_2) \setminus R(k_3)} + \int_{R^-(k_3) \setminus R_{s,2}^-(k_3)} + \int_{R^+(k_3)} + \int_{R_{s,2}^-(k_3)} \right) K(x-y)(\psi(x) - \psi(y))W(y)dy \\ &\triangleq I_{5,0} + I_{5,1} + I_{5,2} + I_{5,3}. \end{aligned}$$

Following the ideas in Section 4.1.11, we estimate the derivative of the regular part and then the L^∞ norm of the singular part. Indeed, we can estimate the y -derivative of $I_{5,0}$ following Sections 4.1.10, 4.1.7, $I_{5,1}$ following the estimates of $I_{5,1}, I_{5,1}^-$ in (4.52), (4.59), and the L^∞ norm of $I_{5,3}$ following Section 4.2.1. The estimate of $I_{5,1}$ is similar to that of I_4 in Section 4.3.2. For $I_{5,2}$, since $R^+(k_3)$ is in \mathbb{R}_2^+ , we decompose

$$I_{5,2} = I_{5,2,1} + I_{5,2,2}$$

into the integral in the regions $I_{5,2,1} : R^+(k_3) \setminus R_{s,2}^+(k_3)$ and $I_{5,2,2} : R_{s,2}^+(k_3)$, and then estimate them following the method in the estimate of $I_{5,1}, I_{5,2}$ in Section 4.3.4.

After we estimate these quantities, we can estimate $d_y(I_5, x, z)$ (4.62) for $|x - z|$ not too small by optimizing k_3 . To estimate $d_y(I_5, x, z)$ (4.62) for sufficiently small $|x - z|$, following (4.60), we use the following decomposition

$$(4.64) \quad \begin{aligned} I_5(x, k_2) &= \int_{R(k_2) \setminus R_s(a)} K(x-y)(\psi(x) - \psi(y))W(y)dy + \int_{R_s^+(a)} K(x-y)(\psi(x) - \psi(y))W(y)dy \\ &+ \int_{R_s^-(a)} K(x-y)(\psi(x) - \psi(y))W(y)dy \triangleq I_{5,4} + I_{5,5} + I_{5,6}. \end{aligned}$$

Then we estimate the derivative of $I_{5,4}$ and the L^∞ norm of $I_{5,6}$ as follows

$$(4.65) \quad |\partial_{x_2} I_{5,4}| \leq A(x) + B(x) \log(1/a), \quad |I_{5,6}| \leq C(x)ah,$$

where the estimates of A, B are given in Appendix B.5.1, and the estimate of C follows the method in Section 4.2.1. The Hölder estimate of $I_{5,5}$ follows the method in the estimate of $I_{5,2}$ in Section 4.3.4. With these estimates, we can further bound $d_y(I_5, x, z)$

$$d_x(f, x, z) \triangleq \frac{|f(x) - f(z)|}{|x_1 - z_1|^{1/2}}, \quad d_y(f, x, z) \triangleq \frac{|f(x) - f(z)|}{|x_2 - z_2|^{1/2}}$$

for sufficiently small $|x - z|$ by optimizing a . See Section 4.6.

Remark 4.7. We do not apply the above computation with smaller window $[-ah, ah]^2$ in the $C_x^{1/2}$ estimate, since it leads to a worse estimate. See also the discussions in Section 4.3.5.

4.3.8. *Hölder estimate of u, v, u_y, v_x .* The ideas of the Hölder estimate for other terms are similar. For a kernel K associated with $\mathbf{u}, \nabla \mathbf{u}$, we perform another decomposition similar to (4.24)

$$(4.66) \quad \begin{aligned} \psi(x) \int K(x-y)W(y)dy &= \int \left(\psi(x)\mathbf{1}_{R(k)^c} + \mathbf{1}_{R_s(k)}\psi(y) + \mathbf{1}_{R(k) \setminus R_s(k)}\psi(y) \right. \\ &+ \left. \mathbf{1}_{R(k) \setminus R(k_2)}(\psi(x) - \psi(y)) + \mathbf{1}_{R(k_2)}(\psi(x) - \psi(y)) \right) K(x-y)W(y)dy \\ &\triangleq I_1(x, k) + I_2(x, k) + I_3(x, k) + I_4(x, k, k_2) + I_5(x, k_2), \end{aligned}$$

Here, we use $R_s(x, k)$ (4.19), which is symmetric with respect to both x_1 and x_2 , rather than $R_{s,1}(x, k)$, since the singular region in the sharp Hölder estimate of $[u_y]_{C_{x_i}^{1/2}}, [v_x]_{C_{x_i}^{1/2}}, [u_x]_{C_y^{1/2}}$ in Lemma 3.3-3.5 in Part I [14] needs to be symmetric in both x_1, x_2 . Denote by $I_{f_6}(x, k_2)$ the approximation term for $f = u_x, u_y, v_x, u, v$. It takes the form similar to (4.50).

We consider two cases of $\hat{x} \in [0, 2x_c]^2 \setminus [0, x_c]^2$ (4.4). In the first case, we consider $\hat{x} \in [x_c, 2x_c] \times [0, 2x_c] \triangleq D_{X1}$, where we have $\hat{x}_1 \geq c\hat{x}_2$ for some constant $c > 0$. In the second case, we consider $\hat{x} \in [0, x_c] \times [x_c, 2x_c] \triangleq D_{X2}$, where we have $\hat{x}_1 \leq c\hat{x}_2$. We distinguish these two cases since in the second case, the singular region does not touch the boundary, we can apply the method in Section 4.3.4.

$C_x^{1/2}$ **estimate of u_y, v_x .** In the $C_x^{1/2}$ estimate of u_y, v_x , we follow Section 4.3.2 to estimate the regular part $I_1 + I_4 - I_6$ and I_3 . We follow Section 4.3.3 and use Lemma 3.4 in Section 3 of Part I [14] to estimate I_2 . For I_5 , we follow Section 4.3.4.

$C_y^{1/2}$ **estimate of u_x .** We perform the decomposition (4.66) rather than (4.24). The estimates of $I_1 + I_4 - I_6, I_3$ follow Section 4.3.2. For I_2 , we use Lemma 3.3 in Section 3 of Part I [14]. We follow Section 4.3.6 to estimate I_5 if $\hat{x} \in D_{X1}$, and Section 4.3.4 if $\hat{x} \in D_{X2}$.

We remark that we use the decomposition (4.66) rather than (4.24) since in Lemma 3.3 in Section 3 of Part I [14], we need to assume that the singular region around x is symmetric in both x_1 and x_2 . The same reasoning applies to $C_{x_i}^{1/2}$ estimate of u_y, v_x .

$C_x^{1/2}$ and $C_y^{1/2}$ estimate of u, v . The Hölder estimates of u, v are substantially easier since u, v are more regular. We perform $C_x^{1/2}, C_y^{1/2}$ of $\rho \mathbf{u}_A$ for another weight $\rho = \psi_u$ (A.1). Below, we only use the weighted L^∞ norm $\|\omega\varphi\|_\infty$. We decompose the integral as follows

$$(4.67) \quad \rho(x) \int K(x-y)W(y)dy = \int \left(\mathbf{1}_{R(k)^c} \rho(x) + \mathbf{1}_{R(k)} \rho(x) \right) K(x-y)W(y)dy \\ \triangleq I_1(x, k) + I_2(x, k).$$

We choose k smaller than that in (4.24) for $\nabla \mathbf{u}$ since the kernel for \mathbf{u} is more regular. We follow Section 4.3.2 to estimate $I_1 - I_6$. For I_2 , we follow the ideas in Sections 4.1.11, 4.3.6, 4.3.7 to estimate the log-Lipschitz function. We choose a list of k_2 and associated region $S(k_2)$ and decompose I_2 as follows

$$I_2(x, k) \triangleq \left(\int_{R(k) \setminus R(k_2)} + \int_{R(k_2) \setminus S(k_2)} + \int_{S(k_2)} \right) \rho(x) K(x-y)W(y)dy \triangleq I_{20}(x, k_2) + I_{21}(x, k_2) + I_{22}(x, k_2).$$

For large $k_2 = k, k-1/2, \dots, 2$, we choose $S(k_2) = R_{s,i}(k_2)$ in the $C_{x_i}^{1/2}$ estimate, $i = 1, 2$. For $k_2 < 2$, we choose $S(k_2) = R_s(k_2)$. For $I_{20}(x, k_2), I_{21}(x, k_2)$, we estimate its derivatives following the estimate of $I_{50}, I_{5,1}$ (4.51), (4.52), respectively, or Section 4.1.7 when $k_2 \geq 2$, and the estimate of I_{54} when $k_2 < 2$ in Section 4.3.7. For $I_{22}(x, k_2)$, we estimate its L^∞ norm following the estimate of I_{53} when $k_2 \geq 2$, and the estimate of I_{56} when $k_2 < 2$ in Section 4.3.7. The estimate is simpler since the above kernel is much simpler than $K(x-y)(\psi(x) - \psi(y))$ in Section 4.3.7.

4.3.9. *Special case: $C_y^{1/2}$ estimate of u_y, v_x .* In this case, we apply Lemma 3.5 from Section 3 of Part I [14] to estimate the most singular part. Since in Lemma 3.5 from Section 3 of Part I, we do not localize the integral, we perform the following decomposition

$$(4.68) \quad \psi(x) \int K(x-y)W(y)dy = \int \left(\psi(y) + \mathbf{1}_{R(k_2)^c}(\psi(x) - \psi(y)) + \mathbf{1}_{R(k_2)}(\psi(x) - \psi(y)) \right) K(x-y)W(y)dy \\ \triangleq I_1(x, k) + I_2(x, k) + I_3(x, k).$$

For I_1 , we apply Lemma 3.5 from Part I [14]. We follow Section 4.3.7 to estimate I_3 if $\hat{x} \in D_{X1}$, and Section 4.3.4 if $\hat{x} \in D_{X2}$. We follow Section 4.3.2 to estimate $I_2 - I_6$, where I_6 is the approximation terms for u_y, v_x similar to (4.50). The symmetrized integrand is discussed in the paragraph “ $C^{1/2}$ estimate of u_y, v_x ” in Section 4.1.5. There are additional difficulties since the weight $\psi(y)$ and the symmetrized integrand $I(x, y) = K^C(x, y)(\psi(x) - \psi(y))$ for some kernel K^C (see similar derivations in (4.28), (4.29)) are singular near 0. Note that we do not have the K^{NC} term. See the paragraph $C^{1/2}$ estimate of u_y, v_x before Section 4.1.6.

The integral of $I(x, y)$ near 0 or in the far-field require some additional estimates, which we discuss below. Since y is away from the singularity x in these cases, the symmetrized integral is given by $I = K^{sym}(x, y)(\psi(x) - \psi(y))$. See (4.29) and Section 4.1.5 for related discussions.

Estimate the integral near 0. To estimate the $D_1 = \partial_{x_2}$ derivative, we use

$$|D_1 I| = |D_1 K^{sym}(\psi(x) - \psi(y)) + K^{sym} \cdot D_1 \psi(x)| \leq |D_1 K^{sym} \cdot \psi(x) + K^{sym} \cdot D_1 \psi(x)| + |D_1 K^{sym} \cdot \psi(y)|.$$

For y close to 0, since ψ is singular, $\psi(y)$ is much larger than $\psi(x)$, and $K^{sym}(x, y)$ is not singular. The main term in $D_1 I$ is given by $D_1 K^{sym} \psi(y)$. It follows

$$\int_Q |D_1 I \cdot W(y)| dy \leq \|W\varphi\|_\infty \left(\|\varphi^{-1}\|_{L^\infty(Q)} \int_Q |D_1 K^{sym} \psi(x) + K^{sym} \cdot D_1 \psi(x)| dy + \|\frac{\psi}{\varphi}\|_{L^\infty(Q)} \int_Q |D_1 K^{sym}| dy \right),$$

where Q is some grid near the origin. The integrands in both integrals do not involve the singular weight, and we can estimate them for each grid point x using the previous methods.

To estimate the X -discretization error, we need to estimate the integral of $\partial_{x_i}^2 \partial_{x_2} J$. Since $\psi(y)$ is independent of x , we get

$$I = K^{sym}(x, y) \left(\frac{\psi(x)}{\psi(y)} - 1 \right) \psi(y), \int_Q |\partial_{x_i}^2 \partial_{x_2} I \cdot W(y)| dy \leq \|W\varphi\|_\infty \|\frac{\psi}{\varphi}\|_{L^\infty(Q)} \int_Q \left| \partial_{x_i}^2 \partial_{x_2} K^{sym}(x, y) \left(\frac{\psi(x)}{\psi(y)} - 1 \right) \right| dy.$$

The last integrand is not singular in y near $y = 0$, and we estimate it using the previous method, e.g. Section 4.1.3.

For u_y, v_x , we have a rank-one approximation $K_{app}(x, y)$ from $C_{u_y}\chi_0 K_{00}$ (4.5) (see Section 4.3.2 from Part I [14]). The full integrand with approximation term and weight is given by

$$\begin{aligned} I_{app} &= K^{sym}(x, y)(\psi(x) - \psi(y)) - K_{app}(x, y)\psi(x) \\ &= (K^{sym}(x, y) - K_{app}(x, y))\psi(x) - K^{sym}(x, y)\psi(y) = I_{app,1} + I_{app,2}. \end{aligned}$$

For y away from the singularity x and 0, $I_{app,1}$ has the same form as the previous case, e.g. the $C_x^{1/2}$ estimate. We improve the error estimate $\partial_i^2 \partial_{x_2} I_{app}$ using the cancellation between the full symmetrized kernel $K(x, y)$ and K_{app} from Lemma B.2 and the estimate in (B.15) in Appendix B.1.1 and the property that $\psi(y)$ is much smaller than $\psi(x)$ for $|y|$ much larger than $|x|$.

Estimate in the far-field. For the tail part in this case, we have an improvement for small $|x|$ where $\chi_0(x) = 1$ due to the approximation term near 0

$$\hat{f} = C_{f_0}(x, y)u_x(0) + C_f(x, y)\mathcal{K}_{00} = C_f(x, y)\mathcal{K}_{00},$$

where $f = u_y, v_x$ and \mathcal{K}_{00} is defined in (4.5), and we have used $C_{f_0}(x, y) = 0$. Its associated integrand is given by

$$K_{app} \triangleq \pi^{-1}C_f(x, y)K_{00}(y),$$

where K_{00} is defined in (4.5). To estimate it, we use the following decomposition

$$D_1(I - \psi(x)K_{app}) = D_1((K^{sym} - K_{app}) \cdot \psi(x)) - D_1K^{sym} \cdot \psi(y) \triangleq P_1 + P_2.$$

We estimate P_1 using the method in Section 4.4. Due to the approximation, $K^{sym} - K_{app}$ has a much faster decay for large y beyond $[0, D]^2$. See (B.15) and Appendix B.1.1. For P_2 , we have

$$\int_{\Omega^c} |P_2| |W(y)| dy \leq \|W\varphi\|_\infty \int_{\Omega^c} |D_1K^{sym}| \frac{\psi}{\varphi}(y) dy,$$

where $\Omega = [0, D]^2$ with large D . The last integral is computed using the method in Section 4.4.

4.4. Estimate the integrals near 0 and in the far field. We use a combination of uniform mesh and adaptive mesh to compute the integral in a finite domain $[0, D]^2$, e.g. $D = 1000$. See Section 4.1.3. Since the kernel decays and the singularity is in the near-field, the integral beyond this domain is small, and we estimate it directly. In addition, for y near 0, we estimate the integrals (the last two integrals in (4.8)) from the approximations $u_x(0), K_{00}$ (4.7), which is singular of order $|y|^{-2}$ or $|y|^{-4}$. For simplicity, we consider $\lambda = 1$. The estimates can be generalized to other scaling parameter λ . To estimate $\int_D k(y)\omega(y)dy$ for D near 0 or D in the far-field, following (4.10), we only need to estimate $\int_D |k(y)|\varphi^{-1}(y)dy$. Since $|y|$ is either very small or very large, we can use the asymptotics of φ in these estimates.

4.4.1. Near-field estimate. Firstly, we estimate $\int_{[0, R_1]^2} |k(y)|\varphi^{-1}(y)dy$ for $k(y) = \frac{y_1 y_2}{|y|^4}, \frac{y_1 y_2 (y_1^2 - y_2^2)}{|y|^8}$ related to $u_x(0), K_{00}$ (4.7). We partition $[0, R_1]$ into

$$0 = z_0 < z_1 < \dots < z_n = R_1$$

with z_1 much smaller than R_1 . Denote $Q_{ij} = [z_{i-1}, z_i] \times [z_{j-1}, z_j]$. Clearly, we have

$$\int_{[0, R_1]^2} |k(y)|\varphi^{-1}(y)dy \leq \sum_{1 \leq i, j \leq n} I_{ij}, \quad I_{ij} \triangleq \int_{Q_{ij}} |k(y)|\varphi^{-1}(y)dy.$$

For $I_{ij}, (i, j) \neq (1, 1)$, we apply a trivial bound

$$(4.69) \quad I_{ij} \leq \|\varphi^{-1}\|_{L^\infty(Q_{ij})} \int_{Q_{ij}} |k(y)| dy \leq |Q_{ij}| \cdot \|k\|_{L^\infty(Q_{ij})} \|\varphi^{-1}\|_{L^\infty(Q_{ij})}.$$

For $k(y) = \frac{y_1 y_2}{|y|^4}, \frac{y_1 y_2 (y_1^2 - y_2^2)}{|y|^8}$, the estimate of $\|k\|_{L^\infty(Q_{ij})}$ is established in Appendix B. It remains to estimate the first term I_{11} . Denote $r = y_1$. Suppose that

$$\varphi(x) \geq q|x|^a (\cos \beta)^b, \quad b \leq 0.$$

See (A.2). If $k(y) = \frac{y_1 y_2}{|y|^4}$ and $a < 0$, we yield

$$\begin{aligned} I_{11} &\leq q^{-1} \int_0^{\sqrt{2}r} \int_0^{\pi/2} \frac{\sin \beta \cos \beta}{r^2} r^{-a} (\cos \beta)^{-b} r dr d\beta = q^{-1} \int_0^{\sqrt{2}r} r^{-a-1} dr \int_0^{\pi/2} \sin \beta (\cos \beta)^{-b+1} d\beta \\ &= q^{-1} \frac{(\sqrt{2}r)^{-a}}{-a} \int_0^1 t^{-b+1} dt = q^{-1} \frac{(\sqrt{2}r)^{-a}}{-a} \frac{1}{2-b}. \end{aligned}$$

If $k(y) = \frac{y_1 y_2 (y_1^2 - y_2^2)}{|y|^8}$, we yield $|k(y)| \leq \frac{1}{4} \frac{\sin 4\beta}{r^4}$. Since $b \leq 0$, if $a < -2$, we get $\varphi \geq qr^a$ and

$$\begin{aligned} I_{11} &\leq q^{-1} \int_0^{\sqrt{2}r} \int_0^{\pi/2} \frac{1}{4} \frac{|\sin 4\beta|}{s^4} s^{-a} s ds d\beta = \frac{1}{4q} \int_0^{\sqrt{2}r} s^{-a-3} ds \frac{1}{4} \int_0^{2\pi} |\sin \beta| d\beta \\ &= \frac{1}{4q} \frac{(\sqrt{2}r)^{-a-2}}{-2-a} \int_0^{\pi/2} \sin \beta d\beta = \frac{1}{4q} \frac{(\sqrt{2}r)^{-a-2}}{-2-a}. \end{aligned}$$

4.4.2. *Far-field estimate.* Denote $a \vee b = \max(a, b)$. To estimate the far field integral $I \triangleq \int_{y_1 \vee y_2 \geq R_0} |k(y)| \varphi^{-1}(y) dy$, we first pick sufficient large R , and then partition the domain

$$0 = z_0 < z_1 < \dots < z_m = R_0 < z_{m+1} < \dots < z_n = R_1 < +\infty.$$

Denote $Q_{ij} = [z_{i-1}, z_i] \times [z_{j-1}, z_j]$. Clearly, we have

$$I = \sum_{m+1 \leq \max(i,j) \leq n} I_{ij} + J, \quad I_{ij} \triangleq \int_{Q_{ij}} |k(y)| \varphi^{-1}(y) dy, \quad J = \int_{y_1 \vee y_2 \geq R_1} |k(y)| \varphi^{-1}(y) dy.$$

For I_{ij} , we apply the trivial estimate (4.69). Suppose that

$$\varphi \geq qr^a (\cos \beta)^b, \quad |k(y)| \leq |y|^{-p}, \quad b \in [-1, 0], \quad p + a > 2.$$

We get

$$J \leq \frac{1}{q} \int_{R_1}^{\infty} \int_0^{\pi/2} r^{-p-a} (\cos \beta)^{-b} r dr d\beta = \frac{1}{q} \frac{R_1^{-p-a+2}}{|p+a-2|} \int_0^{\pi/2} (\cos \beta)^{-b} d\beta.$$

Using Hölder's inequality and $b \in [-1, 0]$, we get

$$\int_0^{\pi/2} (\cos \beta)^{-b} d\beta \leq \left(\int_0^{\pi/2} \cos \beta d\beta \right)^{-b} \left(\int_0^{\pi/2} 1 d\beta \right)^{1+b} = (\pi/2)^{1+b}.$$

It follows

$$J \leq \frac{1}{q} \frac{R_1^{-p-a+2}}{|p+a-2|} (\pi/2)^{1+b}.$$

Application. We apply the above calculations to estimate the integral and its derivatives beyond the mesh $[0, D]^2$ (4.12). Since the domain is far away from the singularity, the integrand is the symmetrized kernel, e.g., (4.29). From Appendix B.1.1 and Lemma B.2 in Appendix B, for $\mathbf{u}_A, \nabla \mathbf{u}_A, \partial_i(\rho \mathbf{u}_A), \partial_i(\psi \nabla \mathbf{u}_A)$, the integrand in the far-field (y is large) satisfies

$$|K(x, y)| \leq C(x) \text{Den}^{-k},$$

with some $k \geq 2$ and coefficients $C(x)$, where Den is defined in (B.20).

In our computation, we rescale x to \hat{x} and restrict it to the near-field $[0, b]^2$ with $b < 2$. Note that $y \notin [0, D]^2$ and $|y| \geq D \gg b$. From (B.20), we get

$$\text{Den} \geq \min_{|z_1| \leq x_1, |z_2| \leq x_2} |y - z|^2 \geq \min_{|z_1| \leq x_1, |z_2| \leq x_2} (|y| - |z|)^2 \geq (|y| - |x|)^2 = |y|^2 \left(1 - \frac{|x|}{|y|}\right)^2.$$

Since $\frac{|x|}{|y|} \leq \sqrt{2}b/D$, we yield

$$\text{Den} \geq (1 - C_s)^2 |y|^2, \quad C_s = \sqrt{2}b/D.$$

It follows

$$\int_{y \notin [0, D]^2} |K(x, y)| \varphi^{-1}(y) dy \leq (1 - C_s)^{-2k} C(x) \int_{y \notin [0, D]^2} |y|^{-2k} \varphi^{-1}(y) dy.$$

Using the method in Section 4.4.2, we can estimate the above integral.

4.5. Estimate for very small or large x . The rescaling argument and the methods in the previous subsections apply to the estimate of $\mathbf{u}_A(x)$, $\nabla \mathbf{u}_A(x)$ for $x \in [0, x_M]^2 \setminus [0, x_m]^2$, $0 < x_m < x_M$. For very small or large x , we cannot use a finite number of dyadic scales $\lambda = 2^i$ to rescale x such that $x/\lambda \in [0, 2x_c]^2 \setminus [0, x_x]^2$. Instead, we choose $\lambda = \frac{\max(x_1, x_2)}{x_c}$. We want to estimate the rescaled integral with a $-d$ -homogeneous kernel K

$$p(x) \int K(x-y)W(y)dy = p_\lambda(x) \int K(\hat{x}-\hat{y})\lambda^{2-d}W_\lambda(\hat{y})dy,$$

uniformly for all small $\lambda \ll 1$ or large $\lambda \gg 1$, where p is some weight and p_λ is defined in (4.2). The rescaled singularity $\hat{x} = x/\lambda$ satisfies $\max_i \hat{x}_i = x_c$. We simplify \hat{x}, \hat{y} as x, y .

We can use the asymptotic of the weights to estimate the integral, see e.g. (4.6). The new difficulty is that the estimate involves the rescaled weight $p_\lambda(y)$. Since λ is not fixed and depends on x that tends to 0 or ∞ , we cannot evaluate $p_\lambda(y)$ and the integrand directly. In the following derivation, λ is comparable to $|x|$, which is either very small or very large.

For y away from the singular region, the integrand of the regular part is given by $J = K(x, y) \cdot p_\lambda(x)$ (4.29). We choose a radial weight p defined in Appendix A.1 $p(x) = \sum_{1 \leq i \leq n} q_i |x|^{a_i}$. See $\psi_1, \psi_u, \psi_{du}$ (A.1). We introduce the asymptotics of these weights

$$R_{lim} \triangleq \lim_{x \rightarrow A} \frac{D_1 p_\lambda(x)}{p_\lambda(x)}, \quad p_{lim} = q_i |x|^{a_i},$$

with $(A, i) = (0, 1)$ or $(A, i) = (\infty, n)$, where (q_n, a_n) denotes the last power in the weight. We use the following decomposition to compute $D_1 J$ with $D_1 = \partial_{x_i}$

$$\begin{aligned} |D_1 J| &= |D_1(K(x, y) \cdot p_\lambda(x))| = |D_1 K(x, y) \cdot p_\lambda(x) + K(x, y) \cdot D_1 p_\lambda(x)| \\ &= \left| p_\lambda(x) \left\{ D_1 K(x, y) + R_{lim} K(x, y) + \left(\frac{D_1 p_\lambda(x)}{p_\lambda(x)} - R_{lim} \right) K(x, y) \right\} \right|. \end{aligned}$$

Since we consider very small λ or very large λ , the error term $\frac{D_1 p_\lambda(x)}{p_\lambda(x)} - R_{lim}$ is small. Hence, we use a triangle inequality to bound $D_1 J$

$$|D_1 J| \leq p_\lambda(x) \left| D_1 K(x, y) + R_{lim} K(x, y) \right| + p_\lambda(x) \left| \left(\frac{D_1 p_\lambda(x)}{p_\lambda(x)} - R_{lim} \right) K(x, y) \right|.$$

The advantage of the above decomposition is that the main term $D_1 K(x, y) + R_{lim} K(x, y)$ does not depend on λ so that we can estimate it using previous methods.

Since the estimate of derivative of u, v does not involve the commutator, see, e.g. (4.67), we can apply the above method to compute the integral of $D_1 u$ for small x or large x .

For y near the singular region, from (4.28), the symmetrized integrand is given by

$$J = K^C(p_\lambda(x) - p_\lambda(y)) + K^{NC} p_\lambda(x),$$

where we use p for the weight. Firstly, we have

$$|D_1 J| = |D_1 K^C(p_\lambda(x) - p_\lambda(y)) + D_1 K^{NC} p_\lambda(x) + (K^C + K^{NC}) D_1 p_\lambda(x)|$$

Denote $K = K^C + K^{NC}$. We use the following method to bound $D_1 J$

$$\begin{aligned} |D_1 J| &\leq p_\lambda(x) \left| D_1 K^C \cdot \left(1 - \frac{p_\lambda(y)}{p_\lambda(x)} \right) + D_1 K^{NC} + K \cdot \frac{D_1 p_\lambda}{p_\lambda} \right| \\ &\leq p_\lambda(x) \left\{ \left| D_1 K^C \cdot \left(1 - \frac{p_{lim}(y)}{p_{lim}(x)} \right) + D_1 K^{NC} + K \cdot \frac{D_1 p_{lim}}{p_{lim}} \right| \right. \\ &\quad \left. + \left| D_1 K^C \left(\frac{p_\lambda(y)}{p_\lambda(x)} - \frac{p_{lim}(y)}{p_{lim}(x)} \right) \right| + K \left| \frac{D_1 p_{lim}}{p_{lim}} - \frac{D_1 p_\lambda}{p_\lambda} \right| \right\}. \end{aligned}$$

The second and the third term on the right hand side can be seen as an error term. The main term $\left| D_1 K^C \cdot \left(1 - \frac{p_{lim}(y)}{p_{lim}(x)} \right) + D_1 K^{NC} + K \cdot \frac{D_1 p_{lim}}{p_{lim}} \right|$ does not depend on λ , and the singularity x is in the near-field and away from 0. We can apply all the delicate decompositions developed in previous sections to estimate $D_1 J$.

In the Hölder estimates, we need various bounds for the weights p_λ . Using the asymptotics of $p(x)$, we can estimate the derivatives of p_λ for very small λ or very large λ uniformly. See Appendix A.1, A.2. Once we obtain the estimates of ψ_λ , and the weight φ_λ in the L^∞ norm $\|\omega_\lambda \varphi_\lambda\|_\infty$, we can use the methods in the previous subsections and the scaling relations in Section 4.1.2 to perform the Hölder estimates.

The L^∞ estimate follows similar ideas and is much easier. We refer more details to Section 7 in the supplementary material II [12].

We remark that since we have much larger damping coefficients in the energy estimates (see Section 5 in Part I [14]) near $x = 0$ and in the far-field, the estimates of the nonlocal terms in these regions, though technical, only have minor effects on the nonlinear stability estimates.

4.6. Assemble the Hölder estimates. In Section 4.3, we decompose the velocity in several parts and estimate them separately using the norms $\|\omega\varphi\|_\infty, [\omega\psi]_{C_{x_i}^{1/2}}$. In this section, we assemble these estimates and estimate

$$\delta(f, x, z) \triangleq \frac{|f(x) - f(z)|}{|x - z|^{1/2}},$$

for $f = \psi_u \mathbf{u}_A, \psi \nabla \mathbf{u}_A$ with weights in (A.1). To obtain better estimates, we combine some of the estimates.

In the proof of the first inequality in Lemma 2.3, we combine and bound different norms using $\max(\|\omega\varphi\|_\infty, \max_{j=1,2} \gamma_j [\omega\psi_1]_{C_{x_j}^{1/2}(\mathbb{R}_2^+)})$. We apply the second inequality to the error $\varepsilon = \omega - (-\Delta)\phi^N$ (3.10) and can evaluate the localized norm using piecewise bounds of the error. See Section 4.7.

To illustrate the ideas, we focus on the $C_x^{1/2}$ estimate, $x \in [x_c, 2x_c] \times [0, 2x_c]$, i.e. x_1 is large relative to x_2 , $z_1 \geq x_1$, and $x_2 = z_2$. For general pairs (x, z) , we can rescale (x, z) to $(\lambda x, \lambda z)$ such that $\lambda x \in [0, 2x_c]^2 \setminus [0, x_c]^2$. Using the scaling relations in (4.1.2), we can estimate the rescaled version of $\delta(f, x, z)$. See also the discussion at the beginning of Section 4.3.

We assume that $z_1 \in [x_c, 2(1+\nu)x_c]$ with $\nu < 1$. For $z_1 \geq 2(1+\nu)x_c$, we have $z_1 > (1+\nu)x_1$. Since z_1, x_1 are large relative to z_2, x_2 , respectively, we have

$$|x - z| = |z_1 - x_1| \asymp |z_1| \gtrsim |x|, |z|.$$

Then, we can use the L^∞ estimate and triangle inequality to estimate $\delta(f, x, z)$. Note that we can estimate the piecewise L^∞ norm of $|x|^{-1/2} \rho(x) \mathbf{u}_A(x)$ and $|x|^{-1/2} \psi \nabla \mathbf{u}_A$ following Section 4.2, where ρ, ψ are the weights in the Hölder estimate of $\rho \mathbf{u}_A, \psi \nabla \mathbf{u}_A$. See Section 7.4 in the supplementary material II [12] for more details.

We focus on $f = \psi u_{x,A}$. We partition the domain $D_\nu = [x_c, 2(1+\nu)x_c] \times [0, 2x_c]$ into $h_x \times h_x$ grids $D_{ij}, 1 \leq i \leq 2(1+\nu)x_c/h_x, 1 \leq j \leq 2x_c/h_x$. We apply the decomposition (4.67) with the same parameters k, k_2 to x in different grids D_{ij} . For $x \in D_{ij}$, using the method in Section 4.3, we obtain the estimate

$$(4.70) \quad \begin{aligned} f(x) &= I_1(x) + I_2(x) + I_3(x) + I_4(x) + I_5(x) - I_6(x), \quad I_5 = I_{5,0} + I_{5,1} + I_{5,2}, \\ |\partial_x(I_1 + I_4 + I_{5,0} - I_6)| &\leq a_{ij,1} \|\omega\varphi\|_\infty, \quad |\partial_x I_3| \leq a_{ij,2} \|\omega\varphi\|_\infty, \quad |\partial_x I_{5,1}| \leq a_{ij,3} \|\omega\varphi\|_\infty, \end{aligned}$$

for some constants $a_{ij,l}, b_{ij} \geq 0$, where $I_{5,1}, I_{5,2}$ are defined and estimated in Section 4.3.4.

For $x, z \in D_\nu$ with $x_2 = z_2, z_1 \leq z_1$, we have $x \in D_{i_1, j}, z \in D_{i_2, j}$ for some $i_1 \leq i_2$. We apply the method in Section 4.3.3 to estimate $\delta(I_2, x, z)$ and the method in Section 4.3.4 to estimate J_1 related to $\delta(I_{5,2}, x, z)$ (4.53). These estimates contribute to the bound $C_{hol}[\omega\psi]_{C_x^{1/2}}$ for some $C_{hol} > 0$, which can be computed.

Regularity of the combination. While $I_1 + I_4 + I_{5,0} - I_6, I_3, I_{5,1}$ are only piecewise smooth and can be discontinuous when x crosses the grids D_{ij} , the sum $I_{lp} = I_1 + I_4 + I_{5,0} - I_6 + I_3 + I_{5,1}$

is continuous and Lipschitz in x_1 for fixed x_2 . In fact, by definition (4.24), (4.51), we get

$$\begin{aligned} I_{lip} &= \int_{R(k) \setminus R_{s,1}(k_3)} K_1(x-y)(\psi(x) - \psi(y))W(y)dy \\ &\quad + \psi(x) \int_{R(k)^c} K_1(x-y)W(y)dy + \int_{R(k) \setminus R_{s,1}(k)} K_1(x-y)\psi(y)W(y)dy - I_6 \\ &= \psi(x) \int_{R_{s,1}^c(k_3)} K_1(x-y)W(y)dy - \int_{R_{s,1}(k) \setminus R_{s,1}(k_3)} K_1(x-y)W(y)\psi(y)dy - I_6. \end{aligned}$$

For fixed x_2 , since I_6 (4.50) for the approximation term is smooth in x and the domain $R_{s,1}(l)$ (4.19) depends on x_1 continuously, we obtain that I_{lip} is continuous in x_1 when x crosses the grids D_{ij} . Since $I_{lip}(x)$ is smooth for $x \in D_{ij}$, we get that I_{lip} is continuous and Lipschitz in x_1 with piecewise Lipschitz norm bounded by $a_{ij,1} + a_{ij,2} + a_{ij,3}$.

Similarly, for the case in Section 4.3.4, we have $I_{lip} = I_1 + I_3 + I_4 + I_{5,0} + I_{5,1} - I_6$ (4.24), (4.51) is Lipschitz in x_i in the $C_i^{1/2}$ estimate for fixed x_{3-i} , $i = 1, 2$.

For the case in Section 4.3.6, $I_{lip} = I_1 + I_3 + I_4 + I_{5,0}^- + I_{5,1}^- + I_{5,1}^+ - I_6$ (4.24), (4.56), (4.57) and $I_{lip} = I_1 + I_3 + I_4 + I_{5,3}^- + I_{5,1}^+ - I_6$ (4.24), (4.56), (4.60) are Lipschitz, where $I_{5,1}^+$ associated with I_5^+ (4.56) is defined similar to $I_{5,1}$ in (4.51).

For the case in Section 4.3.7, $I_{lip} = I_1 + I_3 + I_4 + I_{5,0} + I_{5,1} + I_{5,2,1} - I_6$ (4.24), (4.63), and $I_{lip} = I_1 + I_3 + I_4 + I_{5,4} - I_6$ are Lipschitz, where $I_{5,2,1}$ associated with $I_{5,2}$ (4.63) is defined similar to $I_{5,1}$ in (4.51).

In summary, the sum of the terms in $f(x)$ (4.70) with piecewise derivative estimates is Lipschitz. Using the triangle inequality, we obtain the piecewise Lipschitz bound for I_{lip} . The remaining parts in $f(x)$ (4.70) are continuous and are estimated by the piecewise L^∞ bounds, e.g. $I_{5,2}^-$ (4.57), $I_{5,4}^-$ (4.60), $I_{5,3}$ (4.63), and the improved Hölder estimates, e.g. $I_{5,2}$ (4.51)

By averaging the piecewise derivative bounds and using the estimates in Appendix E.2, for $x \in D_{i_1,j}$, $z \in D_{i_2,j}$, we can obtain

$$|I_{lip}(x) - I_{lip}(z)| \leq C_{lip}|x_1 - z_1| \cdot \|\omega\varphi\|_\infty$$

for constant C_{lip} depending only on $\{a_{kl,j}\}_{k,l \geq 1, j \leq 3}$ and the mesh h_x explicitly. Hence, for the remaining terms in f not estimated using the seminorm $[\omega\psi]_{C_x^{1/2}}$, e.g. $I_1 + I_4 - I_6 + I_3 + I_{5,0} + I_{5,1}$ and J_2 related to $I_{5,2}$ (4.53), each term is continuous and they satisfy ³

$$f_R(x) = \sum_{1 \leq l \leq N} f_l(x), \quad |f_l(x) - f_l(z)| \leq \min(p_l|x_1 - z_1|, q_l) \cdot \|\omega\varphi\|_\infty$$

for some N , where we can choose $q_l = \infty$ if we do not have L^∞ estimate for $f_l(x)$. Similar consideration applies to p_l . In our problem, there are only a few terms and $N < 10$. In the $C_{x_i}^{1/2}$ Hölder estimate of $P_{ij}Q_{ij}(x)$ (continuous in x_i) in $I_{5,2}$ (4.53), we optimize two estimates (see the estimates between (4.53) and (4.54)), which is a nontrivial example of the above summand.

Now, for $x \in D_{i_1,j}$, $z \in D_{i_2,j}$, we have

$$(4.71) \quad \frac{|f_R(x) - f_R(z)|}{|z_1 - x_1|^{1/2}} \leq \sum_{1 \leq l \leq N} \min(p_l \delta^{1/2}, q_l \delta^{-1/2}) \|\omega\varphi\|_\infty,$$

$$\delta = z_1 - x_1 \in [\max(i_2 - i_1 - 1, 0)h_x, (i_2 - i_1 + 1)h_x],$$

The upper bound can be obtained explicitly by partitioning the range of $z_1 - x_1$ into finite many subintervals M_l according to the threshold $\delta_l = q_l/p_l$. In each M_l , the bound reduces to

$$P\delta^{1/2} + Q\delta^{-1/2}$$

for some constants P, Q . It is convex in $\delta^{1/2}$ and can be optimized easily and explicitly in any interval $[\delta_l, \delta_u]$, $\delta_l > 0$.

³In the previous version of this paper [11], some term $f_l(x)$ is not continuous when x crosses the grids. We have corrected this minor issue by reorganizing different terms so that each $f_l(x)$ is continuous. See the above paragraph *Regularity of the combination*. Related computer-assisted estimates have been updated and the full nonlinear stability estimates remain valid.

Remark 4.8. We combine the estimates of different parts in (4.70) using (4.71) to obtain a sharp estimate. If one estimate different parts separately, the distance $\delta = z_1 - x_1$ for the optimizer may not be achieved for the same value, which leads to an overestimate. We remark that for small distance $|z_1 - x_1|$, such an overestimate can be significant since the ratio between the endpoints $|i_2 - i_1 + 1|/\max(i_2 - i_1 - 1, 0)$ varies a lot.

In some estimates, e.g. the $C_y^{1/2}$ estimate of u_x in Section 4.3.6, we need to decompose I_5 using different size of small singular region k_3 . In such a case, we have a list of estimates associated to different k_3 for the part f_R not estimated by $[\omega\psi]_{C_x^{1/2}}$ or $[\omega\psi]_{C_y^{1/2}}$:

$$\frac{|f_R(x) - f_R(z)|}{|z_1 - x_1|^{1/2}} \leq \sum_{1 \leq l \leq N} \min(p_{l,k_3} \delta^{1/2}, q_{l,k_3} \delta^{-1/2}) \|\omega\varphi\|_\infty.$$

For $|x_1 - z_1|$ bounded away from 0, e.g. $|x_1 - z_1| \geq \frac{1}{10}h_x$, we can still partition the range of $|x_1 - z_1|$ and optimizing the above estimates first over δ and then k_3 .

4.6.1. *Hölder estimate for small distance.* In some Hölder estimates, e.g. the $C_y^{1/2}$ estimate in Sections 4.3.6, 4.3.7, when $|x - z|$ is very small, e.g. $|x - z| \leq ch_x$ with $c < 1$, we need to choose a singular region with size a to be arbitrary small. See also Section 4.1.11 for the estimates of a log-Lipschitz function. In these estimates, we can decompose $f_R(x)$ that is not estimated using the Hölder norm of $\omega\psi$ as follows

$$f_R(x) = f_1(x, a, b) + f_2(x, a),$$

for $a < b$ and b is fixed. We can estimate the derivative of f_1 , and the L^∞ norm for f_2

$$|\partial_x f_1(x, a, b)| \leq (A_i + B_i \log \frac{b}{a}) \|\omega\varphi\|_\infty, \quad |f_2| \leq \frac{C_i a}{2} \|\omega\varphi\|_\infty$$

in each grid D_{ij} for any $a \leq b$, see e.g., (4.61) and (4.70). We drop j since we consider x, z with $x_2 = z_2$. For $t = |x - z| \leq h_x$, we get

$$(4.72) \quad \frac{|f(x) - f(z)|}{|x - z|^{1/2}} \leq (A + B \log \frac{b}{a}) \sqrt{t} + \frac{Ca}{\sqrt{t}} \triangleq F(a, t)$$

where $A = \max(A_i, A_{i+1}), B = \max(B_i, B_{i+1}), C = \max(C_i, C_{i+1})$. For each $t \leq ch_x$, we can optimize the above estimate over $a \leq b$ explicitly. Then we maximize the estimate over $t \leq ch_x$ to obtain uniform estimate for small $|x - z| \leq ch_x$. We refer the derivations to Appendix B.5.2.

4.7. **Improved estimate for the nonlocal error.** In Section 3.7, we discuss the estimates of the nonlocal error $\mathbf{u}(\bar{\varepsilon})$ based on the functional inequalities established in this section. Since the weight is singular $\varphi \sim |x|^{-2}|x_1|^{-1/2}, \varphi = \varphi_{elli}$ (A.2) near the origin, $\bar{\varepsilon}_1\varphi$ is much larger near $x = 0$. Due to the anisotropic mesh for large x and small y , or small x and large y , and the round off error, $\bar{\varepsilon}_1$ is not very small in these far-field regions. On the other hand, these regions are small since either $|(x, y)|$ is very small or the ratio $x/y, y/x$ is very small, and the error is very small in the bulk, e.g. $x = O(1)$. See Figure 4 for the rigorous weighted bound of the error in the adaptive mesh. The weighted error of $\bar{\varepsilon}_1$ is larger near 0, while the error for $\hat{\varepsilon}_1$ is larger in the far-field. If we simply use the global norm $\|\omega\varphi\|_\infty, \omega = \bar{\varepsilon}, \hat{\varepsilon}$, and then apply the previous estimates to bound $\mathbf{u}(\bar{\varepsilon})$, we overestimate the nonlocal error significantly. For $x = O(1)$, where we have the smallest damping for the energy estimate, due to the decay of kernel and the smallness of these regions, the integral $\int K(x, y)\bar{\varepsilon}(y)dy$ near $y = 0$ or in the far-field is very small.

Note that we can obtain the piecewise derivative bounds for the error $\bar{\varepsilon}_1, \hat{\varepsilon}_1$ and we partition the domain of the integral into different regions (4.45). Instead of using the global norm to bound the integral, we use the localized norms $\|W\varphi_{elli}\|_{l^\infty(D)}, [W\psi_1]_{C_{x_i}^{1/2}(D)}$ (A.2), (A.1) to exploit the smallness of the error in most part of the domains and improve the error estimate.

Recall the regions of rescaled \hat{x} (4.4) and the mesh y_i partitioning the domain (4.11). We fix a scale λ and assume $\hat{c}[x_c, 2x_c] \times [0, 2x_c]$. By definition, the singular region $R(\hat{x}, k)$ (4.18)

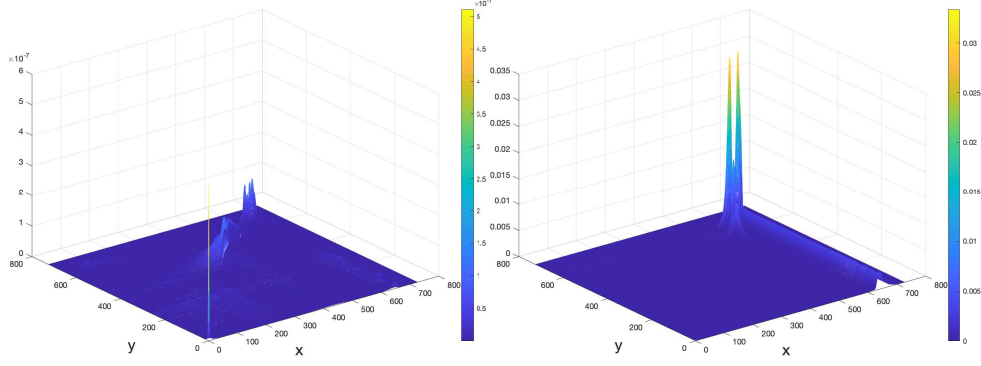


FIGURE 4. Piecewise $L^\infty(\varphi_{elli})$ bound of the error $\bar{\varepsilon}_1, \hat{\varepsilon}_1$ in solving the Poisson equations. Left: error for the approximate steady state. Right: error for the approximate space-time solution \hat{W}_2

satisfies

$$-R(\hat{x}, k) \cap \mathbb{R}_2^+, R(\hat{x}, k) \cap \mathbb{R}_2^+ \subset [x_c - kh, 2x_c + kh] \times [0, 2x_c + kh] \triangleq S_{kh}.$$

Thus, in the estimates of I_2, I_3, I_4 in (4.45), instead of using the global norm $\|W\varphi\|_{L^\infty}$, we use $\|\omega_\lambda \varphi_\lambda\|_{L^\infty(S_{kh})} = \|\omega\varphi\|_{L^\infty(\lambda S_{kh})}$. For the error $\omega = \bar{\varepsilon}, \hat{\varepsilon}$, we can bound $\|\omega\varphi\|_{L^\infty(\lambda S_{kh})}$ by using the piecewise estimates of $\bar{\varepsilon}, \hat{\varepsilon}$ and covering the region λS_{kh} . Similarly, we use the localized bound $[\omega_\lambda \psi_\lambda]_{C_{x_i}^{1/2}(S_{kh})} = \lambda^{1/2} [\omega\psi]_{C_{x_i}^{1/2}(\lambda S_{kh})}$ for the Hölder seminorm in the estimate of I_2, I_3, I_4 , and similar localized norms for I_5 .

For the regular part I_1 , we partition $[0, D]^2, \mathbb{R}_2^{++}$ into disjoint domains: near-field $D_{n,i}$ the bulk D_B and the far-field $D_{f,i}$, e.g.

$$D_{n,1} = [8h, 16h], D_B = [0, 2]^2 \setminus D_{n,1}, D_{f,1} = [0, D]^2 \setminus [0, 2]^2, D_{f,2} = \mathbb{R}_2^{++} \setminus [0, D]^2,$$

where h is the mesh size in (4.11). Then we use the norm $\|\omega_\lambda \varphi_\lambda\|_{L^\infty(D)} = \|\omega\varphi\|_{L^\infty(\lambda D)}$ for the estimate of the integral in region D .

In (4.8), we estimate the integral of $K_{00}(y)$ (4.5) for $|\hat{y}|_\infty \leq k_{02}h$ and $|\hat{y}|_\infty \geq k_{02}h$ separately. Since the kernel is very singular near 0, the L^1 estimate of the integral in $|\hat{y}|_\infty \leq k_{02}h$ in Section 4.4.1 is not very small. Since we can evaluate $\omega = \bar{\varepsilon}, \hat{\varepsilon}$, we change the rescaling from \hat{y} back to y by using $y = \lambda\hat{y}$ in (4.8)

$$J = \int_{|\hat{y}|_\infty \leq k_{02}h} K_{00}(\hat{y})\omega(\lambda\hat{y})d\hat{y} = \lambda^2 \int_{|y|_\infty \leq \lambda k_{02}h} K_{00}(y)\omega(y)dy,$$

where we get λ^2 since K_{00} is -4 homogeneous. For a list of dyadic scales $\lambda = 2^k$, we estimate the integral using Simpson's rule with very small mesh. This allows us to exploit the cancellation in the integral. For $|y|$ very close to 0, we use Taylor expansion. See Section 6.4.1 in supplementary material II [12] (attached to this paper) for more details.

In the estimate of the integral for very small x or large x in Section 4.5 (see more details in Section 7 in the Supplementary Material II [12]), we estimate the rescaled integral for $\lambda \leq \lambda_1$ and $\lambda \geq \lambda_n$ with small λ_1 and large λ_n uniformly. In the case of $\lambda \leq \lambda_1$, we bound $\|\omega_\lambda \varphi_\lambda\|_{L^\infty([a,b] \times [c,d])} \leq \|\omega\varphi\|_{L^\infty(\lambda_1[0,b] \times [0,d])}$. Other norms in different cases are estimated similarly.

We do not track the bound $\|\omega_\lambda \varphi_\lambda\|_{L^\infty(Q_{ij})}$ in each small grid Q_{ij} for computational efficiency.

APPENDIX A. WEIGHTS AND PARAMETERS

A.1. Estimate of the weights. Recall the weights for the Hölder estimate of ω, η, ξ and \mathbf{u}

$$(A.1) \quad \begin{aligned} \psi_1 &= |x|^{-2} + 0.5|x|^{-1} + 0.2|x|^{-1/6}, & \psi_{du} &= \psi_1, & \psi_u &= |x|^{5/2} + 0.2|x|^{-7/6}, \\ \psi_2 &= p_{2,1}|x|^{-5/2} + p_{2,2}|x|^{-1} + p_{2,3}|x|^{-1/2} + p_{2,4}|x|^{1/6}, \\ \psi_3 &= \psi_2, & \vec{p}_{2,\cdot} &= (0.46, 0.245, 0.3, 0.112), \end{aligned}$$

and the following weights for ω, ρ_i for \mathbf{u} and the error

$$(A.2) \quad \begin{aligned} \varphi_1 &= x^{-1/2}(|x|^{-2.4} + 0.6|x|^{-1/2}) + 0.3|x|^{-1/6}, & \varphi_{g1} &= \varphi_1 + |x|^{1/16}, \\ \varphi_{elli} &= |x_1|^{-1/2}(|x|^{-2} + 0.6|x|^{-1/2}) + 0.3|x|^{-1/6}, & \rho_{10} &= |x|^{-3} + |x|^{-7/6}, & \rho_{20} &= \psi_1. \\ \rho_3 &= |x|^{-1} + |x|^{-1/6}, & \rho_4 &= x^{-1/2}(|x|^{-2.5} + 0.6|x|^{-1/2}) + 0.3|x|^{-1/6}. \end{aligned}$$

To estimate the weighted L^∞ norm of the residual error in Section 3, we use $\psi_i, \varphi_{evo,i}$

$$(A.3) \quad \begin{aligned} \varphi_{evo,1} &= \varphi_1, & \varphi_{evo,2} &= x^{-1/2}(\tilde{p}_{5,1}|x|^{-5/2} + \tilde{p}_{5,2}|x|^{-3/2} + \tilde{p}_{5,3}|x|^{-1/6}) + \tilde{p}_{5,4}|x|^{-1/4} + \tilde{p}_{5,5}|x|^{1/7}, \\ \varphi_{evo,3} &= x^{-1/2}(\tilde{p}_{6,1}|x|^{-5/2} + \tilde{p}_{6,2}|x|^{-3/2} + \tilde{p}_{6,3}|x|^{-1/6}) + \tilde{p}_{6,4}|x|^{-1/4} + \tilde{p}_{6,5}|x|^{1/7}, \\ \tilde{p}_{5,\cdot} &= (0.42, 0.135, 0.216, 0.182, 0.0349) \cdot \mu_0, & \mu_0 &= 0.917, \\ \tilde{p}_{6,\cdot} &= (2.5 \cdot \tilde{p}_{5,1}, 2.9 \cdot \tilde{p}_{5,2}, 3.115 \cdot \tilde{p}_{5,3}, 1.82 \cdot \tilde{p}_{5,4}, 2.72 \cdot \tilde{p}_{5,5}), \end{aligned}$$

where φ_1 is defined in (A.2).

In our energy estimates and the estimates of the nonlocal terms, we need various estimates of the weights and their derivatives. From Appendix C.1 of Part I [14] and (A.2), (A.1), we have two types of weights. The first one is the radial weight

$$\rho(x, y) = \sum_i p_i r^{a_i}, \quad r = (x^2 + y^2)^{1/2},$$

where a_i is increasing and $p_i \geq 0$. We use these weights for the Hölder estimates. See e.g. (A.1).

The second type of weights is the following

$$\rho(x, y) = \rho_1(r)|x|^{-\alpha} + \rho_2(r),$$

where ρ_1, ρ_2 are the radial weights.

We use f_l, f_u to denote the lower and upper bound of f . We have the following simple inequalities

$$(A.4) \quad \begin{aligned} (f-g)_l &= f_l - g_u, & (f-g)_u &= f_u - g_l, & (f+g)_\gamma &= f_\gamma + g_\gamma, \\ (fg)_l &= \min(f_l g_l, f_u g_l, f_l g_u, f_u g_u), & (fg)_u &= \max(f_l g_l, f_u g_l, f_l g_u, f_u g_u). \end{aligned}$$

where $\gamma = l, u$. If $g \geq 0$, we can simplify the formula for the product

$$(A.5) \quad (fg)_l = \min(f_l g_l, f_l g_u), \quad (fg)_u = \max(f_u g_l, f_u g_u).$$

Given the piecewise bounds of $\partial^j f, \partial^j g, j \leq k$, we can estimate $\partial^k(fg)$ using the Leibniz rule

$$(A.6) \quad |\partial_x^i \partial_y^j (fg)| \leq \sum_{k \leq i, l \leq j} \binom{i}{k} \binom{j}{l} |\partial_x^k \partial_y^l f| \cdot |\partial_x^{i-k} \partial_y^{j-l} g|.$$

A.2. Radial weights. The advantage of radial weights ρ is that we can estimate them easily. Since $\rho(x, y)$ is in x, y , we restrict the estimate of piecewise bounds to the case of $x \geq 0, y \geq 0$. The bound in general domain $D = [a, b] \times [c, d]$ can be obtained by decomposing D into four quadrants and then using the symmetry and combining the bounds from different quadrants.

A.2.1. *Bounds for the derivatives.* We can easily derive the derivatives and their upper and lower bound as follows. Firstly, we have

$$(A.7) \quad (\partial_x^i \partial_y^j \rho(x, y))_\gamma = \sum_{1 \leq k \leq n} p_k (\partial_x^i \partial_y^j r^{a_k})_\gamma,$$

where $\gamma = l, u$. Using induction, for any α, i, j , we can obtain

$$\partial_x^i \partial_y^j r^\alpha = \sum_{k \leq i+j, l \leq \min(j, 1)} C_{i,j,k,l}(\alpha) x^k y^l r^{\alpha-i-j-k-l} = \sum_{k \leq i+j, l \leq \min(j, 1)} (C_{i,j,k,l}^+(\alpha) - C_{i,j,k,l}^-(\alpha)) x^k y^l r^{\alpha-i-j-k-l},$$

with $C_{i,j,k,l}^\pm(\alpha) \triangleq \max(0, C_{i,j,k,l}(\alpha))$. The bounds for $C_{i,j,k,l}^\pm(\alpha) x^k y^l r^{\alpha-i-j-k-l}$ are simple:

$$(A.8) \quad (C_{i,j,k,l}^\pm(\alpha) x^k y^l r^{\alpha-i-j-k-l})_\gamma = C_{i,j,k,l}^\pm(\alpha) x_\gamma^k y_\gamma^l r_\gamma^{\alpha-i-j-k-l}.$$

In particular, we use the derivatives bound for $i + j \leq 4$ and we have

$$\begin{aligned} \partial_x r^a &= a x r^{a-2}, \quad \partial_x^2 r^a = a r^{a-2} + a(a-2)x^2 r^{a-4}, \quad \partial_{xy} r^a = a(a-2) x y r^{a-4}, \\ \partial_x^3 r^a &= a(a-2)(a-4)x^3 r^{a-6} + 3a(a-2)x r^{a-4}, \quad \partial_x^2 \partial_y r^a = a(a-2) y r^{a-4} + a(a-2)(a-4)x^2 y r^{a-6}, \\ \partial_x^4 r^a &= 3a(a-2)r^{a-4} + 6a(a-2)(a-4)x^2 r^{a-6} + a(a-2)(a-4)(a-6)x^4 r^{a-8}, \\ \partial_x^3 \partial_y r^a &= a(a-2)(a-4) x y r^{a-6} + 2a(a-2)(a-4) x y r^{a-6} + a(a-2)(a-4)(a-6)x^3 y r^{a-8}, \\ \partial_x^2 \partial_y^2 r^a &= a(a-2)(a-3)r^{a-4} + a(a-2)(a-4)(a-6)x^2 r^{a-6} - x^4 a(a-2)(a-4)(a-6)r^{a-8}. \end{aligned}$$

Using (A.4), the above identities, and linearity, we can obtain the upper and lower bounds for $\partial_x^i \partial_y^j \rho$. Since $\rho(x, y)$ is symmetric in x, y , we have $\partial_1^i \partial_2^j \rho(x, y) = (\partial_1^j \partial_2^i \rho)(y, x)$ and can obtain piecewise bounds of $\partial_1^i \partial_2^j \rho$ from that of $\partial_1^j \partial_2^i \rho$.

For the estimate in Section 4.5, we need to use the estimates of $\partial_x^i \partial_y^j \rho(\lambda x)$ for very small $\lambda \leq \lambda_*$ or very large $\lambda \geq \lambda_*$ uniformly. Obviously, the bounds are mainly determined by the leading order power of $p(\lambda x)$, i.e. $p_1 |\lambda r|^{a_1}$ for small λ and $p_n |\lambda r|^{a_n}$ for large λ . We would like to estimate $(\partial_x^i \partial_y^j \rho(\lambda x))_\gamma \lambda^{-\beta}$ for $\lambda \leq \lambda_*, \beta = a_1$ and $\lambda \geq \lambda_*, \beta = a_n, \gamma = l, u$. Using the above derivations (A.7), we have

$$\lambda^{-\beta} (\partial_x^i \partial_y^j \rho(x, y))_\gamma = \sum_{1 \leq k \leq n} p_k (\partial_x^i \partial_y^j \lambda^{a_k - \beta} r^{a_k})_\gamma, \quad \gamma = l, u,$$

and we only need to derive the upper and the lower bounds for $C_{i,j,k,l}^\pm(a_m) x^k y^l r^{\alpha-i-j-k-l} \lambda^{a_m - \beta}$ uniformly for $\lambda \leq \lambda_*, \beta = a_1$ or $\lambda \geq \lambda_*, \beta = a_n$. Since a_i is increasing, in the first case, we have

$$\lambda^{a_1 - a_1} = 1, \quad a_m - a_1 > 0, \quad (\lambda^{a_m - a_1})_l = 0, \quad (\lambda^{a_m - a_1})_u = \lambda_*^{a_m - a_1}, \quad m > 1.$$

In the second case, we get

$$\lambda^{a_n - a_n} = 1, \quad a_m - a_n < 0, \quad (\lambda^{a_m - a_n})_l = 0, \quad (\lambda^{a_m - a_n})_u = \lambda_*^{a_m - a_n}, \quad m > 1.$$

In both cases, if $a_m = \beta$, we get a trivial bound 1 for $\lambda^{a_m - \beta}$; if $a_m \neq \beta$, we get $0 \leq \lambda^{a_m - \beta} \leq \lambda_*^{a_m - \beta}$. Using these bounds for $\lambda^{a_m - \beta}$, (A.8), (A.4), (A.5), we obtain the bounds for $\lambda^{-\beta} \partial_x^i \partial_y^j \psi(\lambda x)$ uniformly for small $\lambda, \beta = a_1$ and large $\lambda, \beta = a_n$.

We also need to bound $M = \lambda^{-\beta} \rho_\lambda(x) \left| \frac{\rho_\lambda(y)}{\rho_\lambda(x)} - \frac{\rho_{lim}(y)}{\rho_{lim}(x)} \right|$ used in Section 4.5, uniformly for $\lambda \leq \lambda_*, \beta = a_1, \rho_{lim}(y) = p_1 |y|^{a_1}$ or $\lambda \geq \lambda_*, \beta = a_n, \rho_{lim}(y) = p_n |y|^{a_n}$. Using the formula of ρ and a direct computation yield

$$\frac{\rho_{lim}(y)}{\rho_{lim}(x)} = \frac{|y|^\beta}{|x|^\beta}, \quad M \leq \sum_{i \leq n} p_i \lambda^{a_i - \beta} \left| |y|^{a_i} - |x|^{a_i} \frac{|y|^\beta}{|x|^\beta} \right| \leq \sum_{i \leq n} p_i \lambda_*^{a_i - \beta} |y|^\beta \left| |y|^{a_i - \beta} - |x|^{a_i - \beta} \right|.$$

We remark that the leading power $\lambda_*^{a_i - \beta}$ for $a_i = \beta$ is cancelled due to $|y|^0 = |x|^0 = 1$ in the above estimate and we gain the small factor $\lambda_*^{a_i - \beta}$ for $a_i \neq \beta$.

A.2.2. *Leading order behavior of $\partial\rho/\rho$.* In our verification, we need to bound $\partial\rho(\lambda x)/\rho(\lambda x)$ as $\lambda \rightarrow 0$ or $\lambda \rightarrow \infty$ uniformly. A direct calculation yields

$$\frac{\partial_{x_i}\rho}{\rho} = \frac{x_i}{|x|^2} \frac{\sum_i p_i a_i r^{a_i}}{\sum_i p_i r^{a_i}} \triangleq \frac{x_i}{|x|^2} S(x), \quad S(x) \triangleq \frac{\sum_i p_i a_i r^{a_i}}{\sum_i p_i r^{a_i}}.$$

For x close to 0, we introduce $b = a - a_1$. Clearly, we get $b_i \geq 0$ and

$$S(x) = a_1 + \frac{\sum_i p_i b_i r^{a_i}}{\sum_i p_i r^{a_i}} = a_1 + \frac{\sum_i p_i b_i r^{b_i}}{\sum_i p_i r^{b_i}} \triangleq a_1 + \frac{A(r)}{B(r)}.$$

Using $b_i \geq 0$ and the Cauchy-Schwarz inequalities, we yield

$$A'B - AB' = r^{-1} \left(\left(\sum p_i b_i^2 r^{b_i} \right) \left(\sum p_i r^{b_i} \right) - \left(\sum p_i b_i r^{b_i} \right)^2 \right) = r^{-1} \frac{1}{2} \sum_{ij} p_i p_j (b_i - b_j)^2 r^{b_i + b_j} \geq 0,$$

and thus A/B is increasing. For $\lambda \leq \lambda_*$, $r \in [r_l, r_u]$, we get the uniform bound for $S(\lambda x)$

$$a_1 \leq S(\lambda x) \leq a_1 + \frac{A(\lambda_* r_u)}{B(\lambda_* r_u)}.$$

For $\lambda = 1$, we simply obtain

$$a_1 + \frac{A(r_l)}{B(r_l)} \leq S(x) \leq a_1 + \frac{A(r_u)}{B(r_u)}.$$

Similarly, for $\lambda \geq \lambda_*$, $r \in [r_l, r_u]$, we get

$$a_n + \frac{A(\lambda_* r_l)}{B(\lambda_* r_l)} \leq S(\lambda x) \leq a_n, \quad \frac{A(r)}{B(r)} = \frac{\sum_i p_i b_i r^{b_i}}{\sum_i p_i r^{b_i}},$$

where $b = a - a_n \leq 0$. Here, we have used that $A(r)/B(r)$ is increasing. Though b_i is negative, we still have $(A/B)' = \frac{A'B - AB'}{B^2} > 0$. From the above estimates, we yield

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \lambda \frac{\partial_{x_i}\rho}{\rho}(\lambda x) &= \frac{x_i}{|x|^2} a_1 \triangleq R_0(x), & \left| \frac{\partial_{x_i}\rho}{\rho}(\lambda x) - R_0(\lambda x) \right| &\leq \lambda^{-1} \frac{x_i}{|x|^2} \frac{|A(\lambda_* x)|}{|B(\lambda_* x)|}, \quad \lambda \leq \lambda_*, \\ \lim_{\lambda \rightarrow \infty} \lambda \frac{\partial_{x_i}\rho}{\rho}(\lambda x) &= \frac{x_i}{|x|^2} a_n \triangleq R_\infty(x), & \left| \frac{\partial_{x_i}\rho}{\rho}(\lambda x) - R_\infty(\lambda x) \right| &\leq \lambda^{-1} \frac{x_i}{|x|^2} \frac{|A(\lambda_* x)|}{|B(\lambda_* x)|}, \quad \lambda \geq \lambda_*. \end{aligned}$$

A.2.3. *Bounds for the derivatives of $1/\rho$.* The bounds for $d_x^i d_y^j \rho^{-1}$ is more complicated since ρ^{-1} is not linear in the summand $p_i r^{a_i}$. We need such estimates in the estimate of the velocity. Firstly, using the bounds in Section A.2.1 and (A.5), we can obtain the upper and the lower bounds for R_{ij}

$$R_{ij} = \frac{\partial_x^i \partial_y^j \rho}{\rho}.$$

For $i + j = 1$ and $k = 2, 3$, we use the estimate in Section A.2.1 to obtain the bounds for

$$R_{10} = \frac{x}{|x|^2} S(x), \quad R_{0,1} = \frac{y}{|x|^2} S(x), \quad (R_{ij})^k.$$

In our estimate, we need $\partial_x^i \partial_y^j \rho^{-1}$ for $i + j \leq 3$. A direct calculation yields

$$\begin{aligned} \partial_x \rho^{-1} &= -\frac{\rho_x}{\rho^2} = -\frac{R_{10}}{\rho}, & \partial_{xx} \rho^{-1} &= -\frac{\rho_{xx}}{\rho^2} + 2\frac{\rho_x^2}{\rho^3} = \rho^{-1}(-R_{20} + 2R_{10}^2), \\ \partial_{xy} \rho^{-1} &= -\frac{\rho_{xy}}{\rho} + \frac{2\rho_x \rho_y}{\rho^3} = \rho^{-1}(-R_{11} + 2R_{10}R_{01}), \\ \partial_{xxx} \rho^{-1} &= -\frac{\rho_{xxx}}{\rho^2} + \frac{6\rho_{xx}\rho_x}{\rho^3} - \frac{6\rho_x^3}{\rho^4} = \rho^{-1}(-R_{30} + 6R_{20}R_{10} - 6R_{10}^3), \\ \partial_{xxy} \rho^{-1} &= -\frac{\rho_{xxy}}{\rho^2} + \frac{2\rho_{xx}\rho_y}{\rho^3} + \frac{4\rho_x\rho_{xy}}{\rho^3} - 6\frac{\rho_x^2\rho_y}{\rho^4} = \rho^{-1}(-R_{21} + 2R_{20}R_{01} + 4R_{10}R_{11} - 6R_{10}^2R_{01}). \end{aligned}$$

Next, we estimate $\partial_x^i \partial_y^j (\partial_{x_i} \rho / \rho)$ for $i \leq 2, j = 0$ or $i = 0, j \leq 2$. Denote $f = \partial_{x_1} \rho$. Using a direct computation, for $D_2 = \partial_x^{i_2} \partial_y^{j_2}$ with $i_2 + j_2 = 1$, we yield

$$D_2 \frac{f}{\rho} = \frac{D_2 f}{\rho} - \frac{f D_2 \rho}{\rho^2} = \rho^{-1} (D_2 f - f R_{i_2, j_2}).$$

For $(i_2, j_2) = (2, 0), (0, 2)$, denote $i_3 = i_2/2, j_3 = j_2/2, D_3 = \partial_x^{i_3} \partial_y^{j_3}$. We yield

$$\begin{aligned} D_3^2 \frac{f}{\rho} &= \frac{D_3^2 f}{\rho} - \frac{2D_3 f \cdot D_3 \rho}{\rho^2} + f D_3^2 \left(\frac{1}{\rho} \right) = \frac{D_3^2 f}{\rho} - \frac{2D_3 f \cdot D_3 \rho}{\rho^2} + f \left(-\frac{D_3^2 \rho}{\rho^2} + \frac{2(D_3 \rho)^2}{\rho^3} \right) \\ &= \rho^{-1} (D_3^2 f - 2D_3 f R_{i_3, j_3} - f R_{i_2, j_2} + 2f R_{i_3, j_3}^2), \end{aligned}$$

where we have used $D_3^2 \frac{1}{\rho} = D_3 \left(-\frac{D_3 \rho}{\rho^2} \right) = -\frac{D_3^2 \rho}{\rho^2} + \frac{2(D_3 \rho)^2}{\rho^3}$.

Since we have estimated $\partial_x^i \partial_y^j \rho$ and R_{ij} , we can bound these derivatives of $D_1 \rho / \rho$ using (A.4).

We also need to obtain the uniform estimates of $\lambda^\beta \partial_x^i \partial_y^j (\rho^{-1}(\lambda x))$ for $\lambda \leq \lambda_*, \beta = a_1$ and $\lambda \geq \lambda_*, \beta = a_n$. Denote $\rho_\lambda(x) = \rho(\lambda x)$. For example, for $D_1 = \partial_{x_1}$, we have

$$\lambda^\beta D_1 (\rho_\lambda^{-1}(x)) = -\lambda^{1+\beta} \frac{(D_1 \rho)(\lambda x)}{\rho_\lambda^2(x)} = -\lambda^{1+\beta} \rho_\lambda^{-1}(x) \lambda^{-1} \frac{x_i}{|x|^2} S(\lambda x) = -\lambda^\beta \rho_\lambda^{-1}(x) \frac{x_i}{|x|^2} S(\lambda x),$$

which can be estimated using the estimates in Sections A.2.1, A.2.2. The power λ^β and the leading power $\lambda^{-\beta}$ in $\rho_\lambda^{-1}(x)$ cancel each other. The estimates of $\lambda^\beta \partial_x^i \partial_y^j (\rho^{-1}(\lambda x))$ with $i+j \geq 2$ and $\partial_x^i \partial_y^j \frac{\partial_{x_i}(\rho_\lambda)}{\rho_\lambda}$ are similar, and follow from the above derivations for $\partial_x^i \partial_y^j \rho^{-1}, \partial_x^i \partial_y^j (\partial \rho / \rho)$, the piecewise estimates for $\partial_x^i \partial_y^j \rho(\lambda x)$ in Section A.2.1 and $\frac{\partial \rho}{\rho}(\lambda x)$ in Section A.2.2, which are uniform in small $\lambda \leq \lambda_*$ or large $\lambda \geq \lambda_*$. We remark that in all of these estimates for $\rho_\lambda(x)$, taking derivatives in x does not change the asymptotic power in λ .

A.2.4. *Improved estimates for ρ^{-1} near $x = 0$.* For the special case $a_1 = -2$, we can write

$$\rho(x) = r^{-2} \sum_i p_i r^{a_i+2} = r^{-2} \tilde{\rho}(x), \quad \rho^{-1} = (x^2 + y^2) \tilde{\rho}(x)^{-1}$$

To obtain a better estimate of ρ^{-1} , we use the fact that $x^2 + y^2$ is a polynomial. Firstly, we can obtain the bounds for $\partial_x^i \partial_y^j \tilde{\rho}^{-1}$. The bound for $S_0 = x^2 + y^2$ is trivial, e.g.,

$$(\partial_x S_0)_\gamma = 2x_\gamma, (\partial_y S_0)_\gamma = 2y_\gamma, \quad \gamma = u, l, \quad \partial_{xy} S_0 = 0, \quad \partial_{xx} S_0 = \partial_{yy} S_0 = 2.$$

Then using (A.4)-(A.5), we can bound ρ^{-1} .

A.3. **The mixed weight.** For the second type of weights $W = \rho_1(r)|x|^{-1/2} + \rho_2(r)$, we can compute its derivatives and its upper and lower bounds using linearity and the Leibniz rule (A.6). We consider $x, y \geq 0$. For example, we have

$$W_l = \rho_{1,l} x_u^{-1/2} + \rho_{2,l}, \quad (W^{-1})_u = (W_l)^{-1}, \quad W_x = \partial_x \rho_1 x^{-1/2} - \frac{1}{2} \rho_1 x^{-3/2} + \partial_x \rho_2.$$

To obtain the upper bound for $\partial_x^i \partial_y^j W$, we use the Leibniz rule (A.6):

$$|\partial_x^i \partial_y^j W| \leq \sum_{k \leq i} \binom{i}{k} |\partial_x^{i-k} \partial_y^j \rho_1| \frac{(2k-1)!!}{2^k} x^{-1/2-k} + |\partial_x^i \partial_y^j \rho_2|.$$

We need to bound $\rho(r)/W(x, y)$ in the estimate of the integrals. Suppose that the leading and the last powers of ρ is a_1, a_n . The leading and the last terms of W are given by $p_i r^{b_i} \cos(\beta)^{-\alpha_i}, \alpha_i \geq 0$.

$$W \geq p_1 r^{b_1}, \quad W \geq p_n r^{b_n}.$$

We estimate

$$\frac{\rho}{W} \leq C_1 r^{a_1 - b_1}, \quad \frac{\rho}{W} \leq C_2 r^{a_n - b_n},$$

for all $x, y \in \mathbb{R}_2^+$. We apply the above estimates for x near 0 or x sufficiently large.

Using $W(\lambda x) \geq \rho_1(\lambda x) \lambda^{-1/2} |x_1|^{-1/2}$, $W(\lambda x) \geq \rho_2(\lambda x)$, the uniform estimates of $\rho_i(\lambda x)$ in λ in Section A.2.1, we can obtain the lower bound of $W(\lambda x)$ and the upper bound of $\frac{\rho(\lambda x)}{W(\lambda x)}$ uniformly in λ .

APPENDIX B. ESTIMATE THE DERIVATIVES OF THE VELOCITY KERNEL AND INTEGRANDS

In this appendix, we estimate the derivatives of the kernel $-\frac{1}{2\pi}\log|x|$ associated to the velocity $\mathbf{u} = \nabla^\perp(-\Delta)^{-1}\omega$ and its symmetrization (4.25). These estimates are used to estimate the error terms in Lemmas 4.2, 4.4. We will perform an additional estimate for u with weight $\varphi(x)$ singular along $x_1 = 0$ in Section B.4. Some additional derivations related to the estimate of the velocity are given in Appendix B.5.

B.1. Estimate the symmetrized kernel. In this section, we estimate the symmetrized kernel. We develop several symmetrized estimates for harmonic functions. Before we introduce the estimates, we have a simple 1D estimate, which is useful for later estimates.

Lemma B.1. *We have*

$$|f(x) + f(-x) - 2f(0)| \leq x^2 \|f_{xx}\|_{L^\infty[-x,x]}, \quad |f(x) + f(-x) - 2f(0) - x^2 f_{xx}(0)| \leq \frac{x^4}{12} \|\partial_x^4 f\|_{L^\infty[-x,x]}.$$

Proof. Denote $G(x) = f(x) + f(-x)$. Clearly, G is even and

$$(B.1) \quad G(0) = 2f(0), \quad G'(0) = 0, \quad \partial_x^2 G(0) = 2f_{xx}(0), \quad \partial_x^3 G(0) = 0.$$

Using the Taylor expansion, we obtain

$$G(x) = G(0) + G'(0)x + \frac{\partial_x^2 G(0)x^2}{2} + \frac{\partial_x^3 G(0)x^3}{6} + \frac{\partial_x^4 G(\xi)x^4}{24},$$

for some $\xi \in [0, x]$. Using (B.1), we get

$$|G(x) - G(0) - G''(x)\frac{x^2}{2}| \leq \|\partial_x^4 G\|_{L^\infty[0,x]}\frac{x^4}{24} \leq \|\partial_x^2 f\|_{L^\infty[-x,x]}\frac{x^4}{12}.$$

Plugging the identity (B.1) into the above estimate proves the second estimate in Lemma B.1. The first estimate is simpler. \square

The following lemma is useful for estimating the symmetrized kernel (4.25) and its derivatives.

Lemma B.2. *Suppose that $Q_x = [-x_1, x_1] \times [-x_2, x_2]$ and $f \in C^4(Q_x)$ is harmonic. Denote*

$$(B.2) \quad \begin{aligned} G_1(1, x) &\triangleq f(x_1, x_2) + f(-x_1, x_2) + f(x_1, -x_2) + f(-x_1, -x_2) - 4f(0, 0), \\ G_2(1, x) &\triangleq f(x_1, x_2) - f(-x_1, x_2) - f(x_1, -x_2) + f(-x_1, -x_2), \\ \hat{G}_1(x) &\triangleq 2x_1^2 f_{xx}(0, 0) + 2x_2^2 f_{yy}(0, 0), \quad \hat{G}_2(x) \triangleq 4x_1 x_2 f_{xy}(0, 0). \end{aligned}$$

We have

$$(B.3) \quad |G_1(1, x)| \leq 2|x|^2 \|f_{xx}\|_{L^\infty(Q_x)}, \quad |\partial_{x_i} G_1(1, x)| \leq 4|x_i| \cdot \|f_{xx}\|_{L^\infty(Q_x)},$$

$$(B.4) \quad |G_1(1, x) - \hat{G}_1(x)| \leq \frac{(x_1^4 + 6x_1^2 x_2^2 + x_2^4)}{6} \|\partial^4 f\|_{L^\infty(Q_x)} \leq \frac{|x|^4}{3} \|\partial^4 f\|_{L^\infty(Q_x)},$$

$$(B.5) \quad |G_1(1, x_1, 0) - \hat{G}_1(x_1, 0)| \leq \frac{1}{6} x_1^4 \|\partial^4 f\|_{L^\infty(Q_x)},$$

$$(B.6) \quad |\partial_{x_i}(G_1(1, x) - \hat{G}_1(x))| \leq \frac{2}{3}(3x_{3-i}^2 x_i + x_i^3) \|\partial^4 f\|_{L^\infty(Q_x)} \leq \frac{2\sqrt{2}}{3} |x|^3 \|\partial^4 f\|_{L^\infty(Q_x)},$$

where $\|\partial^4 f\|_{L^\infty} = \max_{0 \leq i \leq 4} \|\partial_x^i \partial_y^j f\|_{L^\infty(Q_x)}$. For G_2 , we have the following estimate

$$(B.7) \quad |G_2(1, x)| \leq 4x_1 x_2 \|f_{xy}\|_{L^\infty(Q_x)}, \quad |\partial_{x_i} G_2(1, x)| \leq 4|x_{3-i}| \cdot \|f_{xy}\|_{L^\infty(Q_x)},$$

$$(B.8) \quad |G_2(1, x) - \hat{G}_2(x)| \leq \frac{2x_1 x_2 |x|^2}{3} \|\partial^4 f\|_{L^\infty(Q_x)},$$

$$(B.9) \quad |\partial_{x_i}(G_2(1, x) - \hat{G}_2(x))| \leq \frac{2}{3}(3x_i^2 x_{3-i} + x_{3-i}^3) \|\partial^4 f\|_{L^\infty(Q_x)} \leq \frac{2\sqrt{2}}{3} |x|^3 \|\partial^4 f\|_{L^\infty(Q_x)}.$$

Note that $G_1(\cdot, x)$ is even in x_i , and $G_2(\cdot, x)$ is odd in x_i . The polynomials of x_i in the upper bounds (without absolute value) have the same symmetries. Similar properties hold for $\partial G_1, \partial G_2$. Moreover the above bound satisfies the differential relation. These properties are useful for tracking different bounds for G_1, G_2 .

Proof. Recall $Q_x = [-x_1, x_1] \times [-x_2, x_2]$. Denote

$$A_{ij}(x) = \|\partial_x^i \partial_y^j f\|_{L^\infty(Q_x)}.$$

Using Lemma B.1, for any $t \in [0, 1]$, we obtain

$$|f(tx_1, x_2) + f(tx_1, -x_2) - 2f(tx_1, 0)| \leq A_{02}x_2^2, \quad |f(x_1, 0) + f(-x_1, 0) - 2f(0, 0)| \leq A_{20}x_1^2.$$

Since f is harmonic function, we have $\partial_x^{i+2}\partial_y^j f = -\partial_x^i\partial_y^{j+2}f$ and obtain $A_{i+2,j} = A_{i,j+2}$. Taking $t = \pm 1$ in the above estimate and using the triangle inequality, we prove

$$|G(1, x)| \leq 2A_{20}x_1^2 + 2A_{02}x_2^2 = 2A_{20}(x_1^2 + x_2^2) = 2A_{20}|x|^2,$$

which is the first estimate in (B.3).

The second estimate in (B.3) is simple. We consider $i = 1$ without loss of generality. We get $|\partial_{x_1} G_1(1, x)| = |(\partial_1 f)(x_1, x_2) - (\partial_1 f)(-x_1, x_2) + (\partial_1 f)(x_1, -x_2) - (\partial_1 f)(-x_1, -x_2)| \leq 4x_1 A_{20}(x)$.

For (B.4), using Lemma B.1, we yield

$$(B.10) \quad \begin{aligned} |f(tx_1, x_2) + f(tx_1, -x_2) - 2f(tx_1, 0) - x_2^2(\partial_2^2 f)(tx_1, 0)| &\leq A_{04}(x)\frac{x_2^4}{12}, \\ |\partial_2^2 f(x_1, 0) + \partial_2^2 f(-x_1, 0) - 2\partial_2^2 f(0, 0)| &\leq x_1^2 A_{2,2}(x), \\ |f(x_1, 0) + f(-x_1, 0) - 2f(0) - x_1^2 \partial_1^2 f(0)| &\leq A_{40}\frac{x_1^4}{12}, \end{aligned}$$

for $t = \pm 1$. Combining the above estimates and using the triangle inequality and $A_{40} = A_{22} = A_{04}$, we prove the first estimate in (B.4). The second estimate follows from $2|x|^4 - x_1^4 - 6x_1^2x_2^2 - x_2^4 = (x_1^2 - x_2^2)^2 \geq 0$.

Estimate (B.5) follows from (B.4) by taking $x_2 = 0$.

For (B.6), we consider the estimate of ∂_{x_1} . The other case is similar. Using

$$\partial_1 f(x_1, s) - (\partial_1 f)(-x_1, s) = \int_0^{x_1} (\partial_1^2 f)(t, s) + (\partial_1^2 f)(-t, s) dt,$$

we obtain

$$\begin{aligned} \partial_1(G(1, x) - \hat{G}_1(x)) &= (\partial_1 f)(x_1, x_2) - (\partial_1 f)(-x_1, x_2) + (\partial_1 f)(x_1, -x_2) - (\partial_1 f)(-x_1, -x_2) - 4x_1 \partial_1^2 f(0) \\ &= \int_0^{x_1} \left((\partial_1^2 f)(z, x_2) + (\partial_1^2 f)(-z, x_2) + (\partial_1^2 f)(z, -x_2) + (\partial_1^2 f)(-z, -x_2) - 4\partial_1^2 f(0) \right) dz. \end{aligned}$$

Applying (B.3), we yield

$$|\partial_1(G(1, x) - \hat{G}_1(x))| \leq \int_0^{x_1} 2(z^2 + x_2^2) dz A_{4,0}(x) = \left(\frac{2}{3}x_1^3 + 2x_1x_2^2\right) A_{4,0}(x),$$

and complete the proof of the first estimate in (B.6). For the second estimate, we use the AM-GM inequality to yield

$$(B.11) \quad (3x_2^2x_1 + x_1^3)^2 = (3x_2^2 + x_1^2)^2 x_1^2 = \frac{1}{4}(3x_2^2 + x_1^2)^2 4x_1^2 \leq \frac{1}{4} \left(\frac{2(3x_2^2 + x_1^2) + 4x_1^2}{3} \right)^3 = 2|x|^6.$$

Taking a square root completes the estimate.

To estimate G_2 in (B.2), we rewrite it as follows

$$(B.12) \quad \begin{aligned} G_2(1, x) - c\hat{G}_2(x) &= \int_{-x_1}^{x_1} \int_{-x_2}^{x_2} \partial_{12} f(z_1, z_2) - c\partial_{12} f(0) dz \\ &= \int_0^{x_1} \int_0^{x_2} (\partial_{12} f)(z_1, z_2) + (\partial_{12} f)(-z_1, z_2) + (\partial_{12} f)(z_1, -z_2) \\ &\quad + (\partial_{12} f)(-z_1, -z_2) - 4c(\partial_{12} f)(0) dz, \end{aligned}$$

for $c = 0, 1$. The integrand has the same form as G_1 in (B.2). For $c = 0$, using the above decomposition, we prove

$$|G_2(1, x)| \leq 4x_1x_2A_{11}.$$

When $c = 1$, using (B.6), we yield

$$|G_2(1, x) - \hat{G}_2(x)| \leq A_{40} 2 \int_0^{x_1} \int_0^{x_2} |y|^2 dy = A_{40} \frac{2}{3} (x_1^3 x_2 + x_1 x_2^3) = A_{40} \frac{2}{3} x_1 x_2 |x|^2.$$

To estimate the derivatives, we focus on ∂_{x_1} . Using the above representation, we obtain

$$\begin{aligned} \partial_{x_1}(G_2(1, x) - c\hat{G}_2(x)) &= \int_0^{x_2} ((\partial_{12}f)(x_1, y_2) + (\partial_{12}f)(-x_1, y_2)) \\ &\quad + ((\partial_{12}f)(x_1, -y_2) + (\partial_{12}f)(-x_1, -y_2) - 4c(\partial_{12}f)(0)) dy. \end{aligned}$$

We apply the same estimates to the integrands with $c = 0, 1$ and yield

$$|\partial_{x_1} G_2(1, x)| \leq 4x_2 A_{11}, \quad |\partial_{x_1}(G_2(1, x) - \hat{G}_2(x))| \leq A_{31} 2 \int_0^{x_2} (x_1^2 + y_2^2) dy_2 = A_{31} (2x_1^2 x_2 + \frac{2}{3} x_2^3).$$

The second inequality in (B.9) follows from (B.11). The above estimates imply (B.7)-(B.9). \square

Recall the kernels associated with $\nabla \mathbf{u}, \mathbf{u}$ in (4.1). These kernels are the derivatives of the Green function $-\frac{1}{2\pi} \log|x|$ and are harmonic away from 0. We have the following estimates for their derivatives.

Lemma B.3. *Denote $r = (x^2 + y^2)^{\frac{1}{2}}$ and $f(x, y) = \log r$. For any $i, j \geq 0$ with $i + j \geq 1$, we have*

$$|\partial_x^i \partial_y^j f(x, y)| \leq (i + j - 1)! \cdot r^{-i-j}.$$

As a result, for $K_1(y) = -\frac{1}{2} \partial_{12} f(y), K_2(y) = -\frac{1}{2} \partial_1^2 f(y)$, we have

$$|K_i| \leq \frac{1}{2|y|^2}, \quad |\partial_{y_1}^j \partial_{y_2}^{2-j} K_i| \leq \frac{3}{|y|^4}, \quad |\partial_{y_1}^j \partial_{y_2}^{4-j} K_i| \leq \frac{60}{|y|^6}, \quad |\partial_{y_1}^j \partial_{y_2}^{6-j} K_i| \leq \frac{2520}{|y|^8}, \quad i = 1, 2.$$

Proof. Consider the polar coordinate $\beta = \arctan(y/x), r = (x^2 + y^2)^{1/2}$. We use induction on $n = i + j$ to prove

$$(B.13) \quad \partial_x^i \partial_y^j f = (n - 1)! \cos(n\beta - \beta_{ij}) r^{-n},$$

for some constant β_{ij} . We have the formula

$$(B.14) \quad \partial_x g = (\cos \beta \partial_r - \frac{\sin \beta}{r} \partial_\beta) g, \quad \partial_y g = (\sin \beta \partial_r + \frac{\cos \beta}{r} \partial_\beta) g.$$

Firstly, for $n = 1$, a direct calculation yields

$$\partial_x f = \frac{x}{r^2} = \frac{\cos \beta}{r}, \quad \partial_y f = \frac{y}{r^2} = \frac{\sin \beta}{r} = \frac{\cos(\beta - \pi/2)}{r}.$$

Suppose that (B.13) holds for any i, j with $i + j = n$ and $n \geq 1$. Now, since

$$\begin{aligned} \partial_x \partial_x^i \partial_y^j f &= (n - 1)! \partial_x (\cos(n\beta - \beta_{ij}) r^{-n}) \\ &= (n - 1)! (-n \cos \beta \cos(n\beta - \beta_{ij}) r^{-n-1} + n \sin \beta \sin(n\beta - \beta_{ij}) r^{-n-1}) \\ &= n! (-\cos(n\beta - \beta_{ij} + \beta) r^{-n-1}) = n! \cos((n + 1)\beta - \beta_{ij} - \pi) r^{-n-1}, \end{aligned}$$

using a similar computation and $\sin(x) = \cos(x - \pi/2)$, we can obtain that $\partial_y \partial_x^i \partial_y^j f$ has the form (B.13). Using induction, we prove (B.13). The desired estimate follows from (B.13). \square

Using the above two Lemmas, we can estimate the error in the discretization of the kernels $K(x, y)$ in both x and y directions.

B.1.1. Estimate the kernels in the far field. We apply Lemma B.2 to estimate the decay of F_1, F_2 (B.15)

$$F_0 \triangleq G(y - x) - G(y_1 - x_1, y_2 + x_2) - G(y_1 + x_1, y_2 - x_2) + G(y + x), \quad G(y) = -\log|y|/2,$$

$$F_1 \triangleq F_0 - 4x_1 x_2 \partial_{12} G(y), \quad F_2 \triangleq F_1 - \frac{2(x_1^2 - x_2^2)x_1 x_2}{3} \partial_1^3 \partial_2 G(y), \quad I_{ijkl}(P) \triangleq \partial_{x_1}^i \partial_{x_2}^j \partial_{y_1}^k \partial_{y_2}^l P(x, y).$$

Note that for stream function $\phi = (-\Delta)^{-1} \omega(y) = C \cdot G * W$, where W is the odd extension of ω from \mathbb{R}_2^+ to \mathbb{R}_2^{++} , since $G(z)$ is even in z_i , after symmetrization, we have

$$\tilde{\phi}(x) = \phi(x) - x_1 x_2 \phi_{12}(0) = C \int_{\mathbb{R}^2} G(y - x) W(y) dy = C \int F_1(x, y) W(y) dy,$$

where $\phi_{12}(0)$ is related to $C_{f0}K_{ux0}$ in (4.5). In the estimate of $\mathbf{u}, \nabla \mathbf{u}$ related to $\partial_{x_1}^i \partial_{x_2}^j \tilde{\phi}$, e.g. (1, 1) for $u_x = -\partial_{x_1 x_2} \phi$, for $y \in Q$ away from the singularity, we get the symmetrized integrand

$$\partial_{x_1}^i \partial_{x_2}^j \int_Q F_1(x, y) W(y) dy = \int_Q \partial_{x_1}^i \partial_{x_2}^j F_1(x, y) W(y) dy.$$

In the error estimate of the Trapezoidal rule Lemma 4.2, we estimate $\partial_{x_1}^i \partial_{x_2}^j \partial_{y_i}^2 F_1(x, y)$, which is $I_{ij20}(F_1)$ or $I_{ij02}(F_1)$ in (B.15). We apply the estimate of F_2 to $K_f - C_{f0} \tilde{K}_{ux0} - C_f K_{00}$ (4.5). Below, we show that $I_{ijkl}(F_i)$, $i = 1, 2$ has faster decay in $|y|$ than $\partial_{x_1}^i \partial_{x_2}^j \partial_{y_1}^k \partial_{y_2}^l G(y+x)$.

By definition, we get $i_1, j_1 \leq 1$. Next, we fix y and introduce

$$(B.16) \quad g_{pq}(z) \triangleq \partial_{y_1}^p \partial_{y_2}^q G(y+z), \quad M_{G,k} \triangleq \max_{a+b=k} \|(\partial_{y_1}^a \partial_{y_2}^b G)(y+\cdot)\|_{L^\infty(Q_x)}, \quad Q_x = [-x_1, x_1] \times [-x_2, x_2].$$

Since G is harmonic, we have

$$(B.17) \quad \begin{aligned} \partial_{x_i}^k G(y_1 + s_1 x_1, y_2 + s_2 x_2) &= s_i^k \partial_{y_i}^k G(x_1 + s_1 y_1, x_2 + s_2 y_2), \quad s_i \in \{\pm 1\}, \\ \partial_1^2 G(y) &= -\partial_2^2 G(y), \quad \partial_{x_1 x_2} g_{pq}(x)|_{x=0} = \partial_{y_1}^{p+1} \partial_{y_2}^{q+1} G(y), \quad \partial_{22} g_{rs}(0) = -\partial_{11} g_{rs}(0). \end{aligned}$$

Second approximation F_2 . Note that taking ∂_{y_i} in F_i does not change the sign of coefficient of G term in (B.15). Applying (B.12) with $c = 1$ and $f(z) = g_{rs}(z)$ in G_2 , we yield

$$\begin{aligned} I_{pqrs}(F_2) &= \partial_{x_1}^p \partial_{x_2}^q \int_0^{x_1} \int_0^{x_2} g_{r+1, s+1, all}(z) dz, \\ g_{\alpha\beta, all}(z) &= g_{\alpha\beta}(z) + g_{\alpha\beta}(-z) + g_{\alpha\beta}(z_1, -z_2) + g_{\alpha\beta}(-z_1, z_2) - 4g_{\alpha\beta}(0) - 2(z_1^2 - z_2^2) \partial_{11} g_{\alpha\beta}(0). \end{aligned}$$

If $\max(i, j) \leq 1$, using the above notation to $I_{ijkl}(F_2)$ and the estimate of $G_1 - \hat{G}_1$ in Lemma B.2 with $f = g_{kl}$, and then integrating the bounds in z_2 , we get

$$|I_{10kl}(F_2)| = \left| \int_0^{x_2} g_{k+1, l+1, all}(x_1, z_2) dz_2 \right| \leq M_{G, d_2} \int_0^{x_2} \frac{x_1^4 + 6x_1^2 z_2^2 + z_2^4}{6} dz_2 = \left(\frac{x_1^4 x_2}{6} + \frac{x_1^2 x_2^3}{3} + \frac{x_2^5}{30} \right) M_{G, d_2},$$

where $d_2 = k + l + 6$. Similarly, we get

$$|I_{01kl}(F_2)| \leq \left(\frac{x_1^5}{30} + \frac{x_1^3 x_2^2}{3} + \frac{x_1^4 x_2}{6} \right) A_{G, d_2}, \quad I_{11kl}(F_2) \leq \frac{x_1^4 + 6x_1^2 x_2^2 + x_2^4}{6} A_{G, d_2}.$$

If $\max(i, j) \geq 2$, $i + j \leq 3$, without loss of generality, we consider $i \geq 2$. We choose $(i_1, j_1, k_1, l_1) = (i - 2, j, k + 2, l)$. From (B.17), we get

$$\begin{aligned} \partial_{x_1}^2 (x_1 x_2 \partial_{12} G(y)) &= 0, \\ \partial_{x_1}^2 \partial_{y_1}^k \partial_{y_2}^l \left(\frac{2(x_1^2 - x_2^2) x_1 x_2}{3} \partial_1^3 \partial_2 G(y) \right) &= 4x_1 x_2 \partial_{y_1}^{k+1} \partial_{y_2}^{l+1} G(y) = 4x_1 x_2 \partial_{12} g_{k_1 l_1}(0). \end{aligned}$$

Using (B.17) again, we rewrite $\partial_{x_1}^i \partial_{y_1}^k G(x+y) = \partial_{x_1}^{i_1} \partial_{y_1}^{k_1} G(x+y)$ and get

$$(B.18) \quad I_{ijkl}(F_2) = \partial_{x_1}^{i_1} \partial_{x_2}^{j_1} (g_{k_1 l_1}(x) - g_{k_1 l_1}(x_1, -x_2) - g_{k_1 l_1}(-x_1, x_2) + g_{k_1 l_1}(-x) - 4x_1 x_2 \partial_{12} g_{k_1 l_1}(0)).$$

The same derivation applies to the case of $j \geq 2$, where we choose $(i_1, j_1, k_1, l_1) = (i, j - 2, k, l + 2)$. Since $i_1, j_1 \leq 1$, using the estimate of $G_2 - \hat{G}_2$ in Lemma B.2 with $f = g_{k_1 l_1}$, we get

$$\begin{aligned} |I_{20kl}(F_2)|, |I_{02kl}(F_2)| &\leq \frac{2x_1 x_2 |x|^2}{3} M_{G, d_2}, \quad (i_1, j_1) = (0, 0), \\ |I_{30kl}(F_2)|, |I_{12kl}(F_2)| &\leq \frac{2}{3} (3x_1^2 x_2 + x_2^3) M_{G, d_2}, \quad (i_1, j_1) = (1, 0), \\ |I_{21kl}(F_2)|, |I_{03kl}(F_2)| &\leq \frac{2}{3} (x_1^3 + 3x_1 x_2^2) M_{G, d_2}, \quad (i_1, j_1) = (0, 1), \quad d_2 = k_1 + l_1 + 4 = k + l + 6. \end{aligned}$$

Note that the form (B.18) can be seen as the $\partial_{x_1}^{i_1} \partial_{x_2}^{j_2} F_1$. If $4 \leq i + j \leq 5$, we still first perform (B.18) by choosing $(i_1, j_1, k_1, l_1) = (i - 2, j, k + 2, l)$ or $(i, j - 2, k, l + 2)$ and get

$$I_{ijkl}(F_2) = I_{i_1 j_1 k_1 l_1}(\tilde{F}_1),$$

where \tilde{F}_1 is similar to F_1 in (B.15) with G replaced by $g_{i-i_1, j-j_1} = \partial_{y_1}^{i-i_1} \partial_{y_2}^{j-j_1} G(y)$. Then we apply the estimate for the first approximation below with $i_1 + j_1 \leq 3$.

First approximation. The estimate of $I_{ijkl}(F_1)$ is similar. Denote

$$i_2 = i - 2\lfloor \frac{i}{2} \rfloor, \quad j_2 = j - 2\lfloor \frac{j}{2} \rfloor, \quad k_2 = k + 2\lfloor \frac{i}{2} \rfloor, \quad l_2 = l + 2\lfloor \frac{j}{2} \rfloor.$$

If $\max(i, j) \leq 1$, we get $(i, j, k, l) = (i_2, j_2, k_2, l_2)$. Applying the estimate $G_2 - \hat{G}_2$ in Lemma (B.2) with $f = g_{k_2 l_2}$, we get

$$\begin{aligned} I_{10kl}(F_1) &\leq \frac{2}{3} x_2 (x_2^2 + 3x_1^2) \|\partial^d G(y + \cdot)\|_{L^\infty(Q_x)} = \frac{2}{3} x_2 (x_2^2 + 3x_1^2) M_{G,d}, \\ I_{01kl}(F_1) &\leq \frac{2}{3} x_1 (x_1^2 + 3x_2^2) M_{G,d}, \quad I_{00kl}(F_1) \leq \frac{2x_1 x_2 |x|^2}{3} M_{G,d}, \quad d = k_2 + l_2 + 4 = k + l + 4. \end{aligned}$$

If $(i, j) = (1, 1)$, we apply the estimate of G_1 in Lemma (B.2) with $f = \partial_{x_1 x_2} g_{kl}(x)$ (k, l are number of derivatives on $G(y + z)$) to get

$$|I_{11kl}(F_1)| \leq 2|x|^2 M_{G,d}, \quad d = k_2 + l_2 + 4 = k + l + 4.$$

If $\max(i, j) \geq 2, i + j \leq 3$, $x_1 x_2 \partial_{12} G(0)$ vanishes in I_{ijkl} . We apply derivation similar to (B.18) without $4x_1 x_2 \partial_{12} g_{k_2 l_2}(0)$ and the estimate of G_2 in Lemma B.2 with $f = g_{k_2 l_2}$ to get

$$|I_{ijkl}(F_1)| \leq 4x_1^{1-i_2} x_2^{1-j_2} \|\partial^2 g_{k_2 l_2}\|_{L^\infty(Q_x)} \leq 4x_1^{1-i_2} x_2^{1-j_2} M_{G,d}, \quad d = k_2 + l_2 + 2 = k + l + 4.$$

To bound $M_{G,k}$, we apply Lemma B.3 to get

$$(B.19) \quad M_{G,k} = \max_{a+b=k} \|(\partial_{y_1}^a \partial_{y_2}^b G)(y + \cdot)\|_{L^\infty(Q_x)} \leq \frac{(k-1)!}{2 \cdot \text{Den}(x, y)^{k/2}}, \quad \text{Den}(x, y) = \min_{z \in Q_x} |y - z|^2.$$

It is not difficult to obtain that for $x, y \in \mathbb{R}_2^{++}$, we have

$$(B.20) \quad \text{Den}(x, y) = \sum_{i=1,2} \min_{|z_i| \leq x_i} |y_i - z_i|^2 = \sum_{i=1,2} (\max(y_i - x_i, 0))^2.$$

Using the above estimates, for $|y| \gg |x|$, we get $\text{Den} \sim |y|^2$ and the decay estimate for $I_{ijkl}(F_1)$ (B.15) with a rate $|y|^{-k-l-4}$ and $I_{ijkl}(F_2)$ with a rate $|y|^{-k-l-6}$.

B.2. Piecewise L^∞ estimate of derivatives of the Green function. In this section, we develop sharp L^∞ estimates of the derivatives of the Green function $G(x) = -\frac{1}{2\pi} \log|x|$ and their linear combinations in a small domain $[a, b] \times [c, d]$. They will be used in Lemmas 4.2, 4.4 to estimate the error, especially near the singularity of the kernel. We remark that the linear combinations of $\partial_1^i \partial_2^j G$ can be quite complicated. If we simply use the triangle inequality to estimate it, we can overestimate some terms with cancellation significantly, especially near the singularity of G . These sharp estimates are useful for reducing the estimate of the error term in Lemmas 4.2, 4.4 without choosing very small mesh, which can lead to large computational cost.

B.2.1. Coefficients of the derivatives of the Green function. To simplify the notation, we drop $\frac{1}{\pi}$ from G and denote $f_p = -\frac{1}{2} \log|x|$. Firstly, we derive the formulas of $\partial_1^i \partial_2^j f_p$. Due to homogeneity, for $k + l \geq 1$, we assume

$$(B.21) \quad \partial_{x_1}^k \partial_{x_2}^l f_p = \frac{\sum_{i+j=k+l} c_{ij} x_1^i x_2^j}{|x|^{2(k+l)}}.$$

Next, we derive the recursive formula for c_{ij} . Using induction, we can obtain

$$\begin{aligned} \partial_{x_1}^{k+1} \partial_{x_2}^l f_p &= \frac{\sum_{i+j=k+l} c_{ij} i x_1^{i-1} x_2^j}{|x|^{2(k+l)}} - \frac{2(k+l)x_1}{|x|^{2(k+l+1)}} \sum_{i+j=k+l} c_{ij} x_1^i x_2^j \\ &= \frac{1}{|x|^{2(k+l+1)}} \left(\sum_{i+j=k+l} c_{ij} i x_1^{i+1} x_2^j + c_{ij} i x_1^{i-1} x_2^{j+2} - 2(k+l) c_{ij} x_1^{i+1} x_2^j \right) \\ &= \frac{1}{|x|^{2(k+l+1)}} \left(\sum_{i+j=k+l} (c_{ij} i + c_{i+2, j-2} (i+2) - 2(k+l) c_{ij}) x_1^{i+1} x_2^j \right). \end{aligned}$$

Therefore, we obtain the recursive formula

$$c_{i+1,j} = ic_{ij} + (i+2)c_{i+2,j-2} - 2(k+l)c_{ij},$$

for all $i+j = k+l$, or equivalently,

$$c_{i,j} = (i-1)c_{i-1,j} - 2(k+l)c_{i-1,j} + (i+1)c_{i+1,j-2},$$

for all $i+j = k+l+1$. Similarly, for ∂_{x_2} , we yield

$$c_{i,j} = (j-1)c_{i,j-1} - 2(k+l)c_{i,j-1} + (j+1)c_{i-2,j+1},$$

for all $i+j = k+l+1$.

B.2.2. Estimates of rational functions. We use the above formulas to develop sharp estimates of the derivatives of f_p and their linear combinations in a small grid cell $[y_{1l}, y_{1u}] \times [y_{2l}, y_{2u}]$. For $k < k_2$ and $S \subset \{(i, j) : i+j = k\}$, we estimate

$$(B.22) \quad I_S \triangleq \frac{\sum_{(i,j) \in S} c_{ij} y_1^i y_2^j}{|y|^{k_2}}.$$

We assume that $I_S(x)$ is either odd in x_i or even in x_i for $i = 1, 2$. Clearly, this properties hold for $\partial_{x_1}^k \partial_{x_2}^l f_p$ (B.21). Denote $i_1 = \min_{i \in S} i$, $j_1 = \min_{j \in S} j$. We yield

$$I_S = \frac{y_1^{i_1} y_2^{j_1} \sum_{(i,j) \in S} c_{ij} y_1^{i-i_1} y_2^{j-j_1}}{|y|^{i_1+j_1} |y|^{k_2-i_1-j_1}}.$$

We further introduce

$$P \triangleq \sum_{(i,j) \in S} c_{ij}^+ y_1^{i-i_1} y_2^{j-j_1}, \quad Q \triangleq \sum_{(i,j) \in S} c_{ij}^- y_1^{i-i_1} y_2^{j-j_1}.$$

We claim that $i-i_1, j-j_1$ are even for all $(i, j) \in S$. Since I_S is either odd or even in $x_i, i = 1, 2$, the numerator $\sum c_{ij} x_1^i x_2^j$ in (B.22) have the same symmetries in x_1, x_2 . In particular, each monomial $c_{ij} x_1^i x_2^j$ in (B.22) also enjoys the same symmetries in x_1, x_2 as I_S . If $i-i_1$ is odd for some i , then $c_{ij} x_1^{i-i_1} x_2^{j-j_1}$ must be odd in x_1 . It implies $i-i_1 \geq 1$ for any $(i, j) \in S$ and contradicts the minimality of i_1 . The same argument applies to j_1 .

As a result, P and Q are monotone increasing in $|y_1|, |y_2| \geq 0$. For $|y_i|_l \leq |y_i| \leq |y_i|_u, i = 1, 2$, we can derive the upper and lower bounds for P, Q and yield

$$\begin{aligned} |I| &\leq \frac{\max(P_u - Q_l, Q_u - P_l)}{|y|_l^{k_2-i_1-j_1}} \max_{y \in \Omega} \frac{|y_1|^{i_1} |y_2|^{j_1}}{|y|^{i_1+j_1}} \\ &\leq \frac{\max(P_u - Q_l, Q_u - P_l)}{|y|_l^{k_2-i_1-j_1}} \left(\frac{|y_1|_u}{(|y_1|_u^2 + |y_2|_l^2)^{1/2}} \right)^{i_1} \left(\frac{|y_2|_u}{(|y_1|_l^2 + |y_2|_u^2)^{1/2}} \right)^{j_1}, \end{aligned}$$

where $|y|_l$ is the lower bound of $|y|$ and we have used the fact that $z_i/|z|$ is increasing in z_i for $z_i \geq 0$ to obtain its upper bound. Now, for $y_i \in [y_{il}, y_{iu}]$, we estimate $|y_i|_l, |y_i|_u$ as follows

$$(B.23) \quad \begin{aligned} |y_i| &\geq \max(0, |y_{il} + y_{iu}|/2 - (y_{iu} - y_{il})/2) \triangleq |y_i|_l, \\ |y_i| &\leq \max(|y_{il}|, |y_{iu}|) \triangleq |y_i|_u, \quad |y|_l \triangleq (|y_1|_l^2 + |y_2|_l^2)^{1/2}. \end{aligned}$$

Note that for $y_i \in [y_{il}, y_{iu}]$, y_i can change sign.

B.3. Improved estimate of the higher order derivatives of the integrands. In the Hölder estimate, we need to estimate the derivatives of the integrands (4.28), (4.29), (4.24), which take the form

$$K^C(x, y)(p(x) - p(y)) + K^{NC}p(x),$$

for some weight p and kernels K^C, K^{NC} . Using the estimates of the kernels in Appendix B.1, B.2 and the weights in Section A.1, the Leibniz rule (A.6), and the triangle inequality, we can estimate the derivative of the integrands. However, such an estimate can lead to significant overestimates near the singularity of the integrand. We use the estimates in Appendix B.2 to

handle the cancellations among different terms and obtain improved estimates for the integrand and its derivatives near the singularity:

$$(B.24) \quad T_{00}(x, y) \triangleq K(y - x)(p(x) - p(y)), \quad \partial_{x_i} T_{00}(x, y).$$

We choose weight $p(x)$ that is even in x and y . The basic idea is to perform a Taylor expansion on $p(x) - p(y)$ and obtain the factor $|x - y|$, which cancels one order of singularity from $K(x, y)$. We use the formulas in Appendix B.2 to collect the terms with the same singularity and exploit the cancellation.

B.3.1. Y -discretization. In the Y -discretization of the integral, we need to estimate the y -derivatives of the integrand (B.24). For $a, b = 1, 2$, denote

$$(B.25) \quad D_1 = \partial_a, \quad D_2 = \partial_b, \quad x_m = \frac{x + y}{2}.$$

Next, we compute $\partial_{y_b}^j \partial_{x_a}^i T_{00}$. The reader should be careful about the sign. Note that

$$\partial_{x_a}(K(y - x)) = -(\partial_a K)(y - x) = -D_1 K(y - x).$$

Using the Leibniz rule, we get

$$\begin{aligned} \partial_{y_b}^2 \partial_{x_a} T_{00} &= \partial_{y_b}^2 (-D_1 K(p(x) - p(y)) + K \cdot D_1 p(x)) = \partial_{y_b}^2 (D_1 K \cdot (p(y) - p(x)) + K \cdot D_1 p(x)) \\ &= D_2^2 D_1 K \cdot (p(y) - p(x)) + 2D_2 D_1 K \cdot D_2 p(y) + D_1 K \cdot D_2^2 p(y) + D_2^2 K \cdot D_1 p(x). \end{aligned}$$

We use Taylor expansion at $x = x_m$ and write

$$(B.26) \quad \begin{aligned} p(y) - p(x) &= (y - x) \cdot \nabla p(x_m) + p_{m,2,err}, \quad \partial_i p(z) = \partial_i p(x_m) + (\partial_i p(z) - \partial_i p(x_m)), \quad z = x, y, \\ |f(z) - f(x_m) - (z - x_m) \cdot \nabla f(x_m)| &\leq \frac{1}{2} \frac{d_1^2}{4} \|f_{xx}\|_{L^\infty(Q)} + \frac{d_1 d_2}{4} \|f_{xy}\|_{L^\infty(Q)} + \frac{1}{2} \frac{d_1^2}{4} \|f_{xx}\|_{L^\infty(Q)} \triangleq I_f. \end{aligned}$$

for $d = y - x, z = x, y$ and any f , where Q is the rectangle covering x, y . Then $p_{m,2,err}$ is bounded by $2I_p = O(|x - y|^2)$. Combining the terms involving ∇p , we get

$$(B.27) \quad \begin{aligned} \partial_{y_b}^2 \partial_{x_a} T_{00} &= \sum_{i=1,2} \left(D_2^2 D_1 K \cdot (y_i - x_i) + \mathbf{1}_{D_2=\partial_i} 2D_2 D_1 K + \mathbf{1}_{D_1=\partial_i} D_2^2 K \right) \cdot \partial_{x_i} p(x_m) + D_2^2 D_1 K \cdot p_{m,2,err} \\ &\quad + 2D_2 D_1 K \cdot (D_2 p(y) - D_2 p(x_m)) + D_2^2 K \cdot (D_1 p(x) - D_1 p(x_m)) + D_1 K \cdot D_2^2 p(y) \\ &\triangleq \left(\sum_{i=1,2} I_i \cdot \partial_{x_i} p(x_m) \right) + II_1 + II_2 + II_3 + II_4, \\ I_i &\triangleq D_2^2 D_1 K \cdot (y_i - x_i) + \mathbf{1}_{D_2=\partial_i} 2D_2 D_1 K + \mathbf{1}_{D_1=\partial_i} D_2^2 K, \end{aligned}$$

where $\partial_i^j \partial_2^k K$ is evaluated at $y - x$, and II_i denotes the last four terms in the second equation. The first term is the most singular term. We combine the most singular terms to exploit the cancellation and improve the estimates. We estimate the kernels

$$(B.28) \quad K_{mix}(D_1, D_2, i, s)(z_1, z_2) \triangleq D_2^2 D_1 K(z) z_i + \mathbf{1}_{D_2=\partial_i} 2D_2 D_1 K(z) + s \mathbf{1}_{D_1=\partial_i} D_2^2 K(z),$$

with $s = \pm 1$ and $D_1, D_2 \in \{\partial_1, \partial_2\}$. Then we can bound $\partial_{y_b}^2 \partial_{x_a} T_{00}$ using the triangle inequality. When $D_1 = D_2$, we have an improved estimate for II_2, II_3

$$(B.29) \quad II_2 + II_3 = D_2^2 K(D_2 p(y) - D_2 p(x_m)) + (D_2 p(y) + D_2 p(x) - 2D_2 p(x_m)).$$

We estimate $D_2 p(y) + D_2 p(x) - 2D_2 p(x_m)$ using (B.26) with $f = D_2 p$ and $z = x, y$.

B.3.2. The second singular term. For $x = (x_1, x_2)$ close to the y -axis or the x -axis, since we have symmetrized the integral (see (4.28) and Section 4.1.5), we have another singular term in the integrand

$$T_{01} \triangleq K(y_1 - x_1, y_2 + x_2)(p(x) - p(y)), \quad \text{or } T_{10} \triangleq K(y_1 + x_1, y_2 - x_2)(p(x) - p(y)).$$

We have the first term if $x_2 < x_1$ and x_2 close to 0, and the second term if $x_1 < x_2$ and x_1 close to 0. We label the former case with $side = 1$ and the latter $side = 2$. See the right figure

in Figure 1 for an illustration of the first case. The T_{01} term is supported in the blue region $R(x, k, S)$. Denote

$$(B.30) \quad (s_1, s_2) = (1, -1) \text{ if } side = 1, \quad (s_1, s_2) = (-1, 1) \text{ if } side = 2.$$

Case I. If $(D_1, side) = (\partial_1, 1)$ or $(\partial_2, 2)$, we obtain

$$\partial_{x_a} K(y_1 - s_1 x_1, y_2 - s_2 x_2) = -\partial_{y_a} K(y_1 - s_1 x_1, y_2 - s_2 x_2),$$

for $(a, s_1, s_2) = (1, 1, -1)$ or $(2, -1, 1)$. The computations for $\partial_{y_b}^2 \partial_{x_1} T_{01}, \partial_{y_b}^2 \partial_{x_2} T_{10}$ are the same as (B.27) with K and its derivatives evaluating at $z = (y_1 - s_1 x_1, y_2 - s_2 x_2)$.

We estimate II_i in (B.27) directly using the triangle inequality and the bounds for $\partial_1^i \partial_2^j K$ in Section B.1, B.2 and p in Section A.1. For I_i in (B.27) in the most singular term, if $i = side$, from definition (B.30), we get

$$s_i = 1, \quad s_{3-i} = -1, \quad z_i = y_i - s_i x_i = y_i - x_i, \quad z_{3-i} = y_{3-i} + x_{3-i}.$$

Therefore, it follows

$$I_i = D_2^2 D_1 K(z) \cdot (y_i - x_i) + \mathbf{1}_{D_2=\partial_i} 2D_2 D_1 K(z) + \mathbf{1}_{D_1=\partial_i} D_2^2 K(z) = K_{mix}(D_1, D_2, i, 1)(z),$$

where K_{mix} is defined in (B.28). If $i \neq side$, we have $z_i = y_i + x_i \geq |y_i - x_i|, z_{3-i} = y_{3-i} - x_{3-i}$. We simply bound the summand using the triangle inequality

$$|I_i| \leq |D_2^2 D_1 K(z)| \cdot |y_i - x_i| + \mathbf{1}_{D_2=\partial_i} 2|D_2 D_1 K(z)| + \mathbf{1}_{D_1=\partial_i} |D_2^2 K(z)|.$$

Case II. If $(D_1, side) = (\partial_1, 2)$ or $(\partial_2, 1)$, we obtain

$$\partial_{x_a} K(y_1 - s_1 x_1, y_2 - s_2 x_2) = (\partial_{y_a} K)(y_1 - s_1 x_1, y_2 - s_2 x_2),$$

for $(a, s_1, s_2) = (1, -1, 1)$ or $(2, 1, -1)$. Recall the definitions of D_1, D_2 (B.25). Using the above identity, we yield

$$\partial_{y_b}^2 \partial_{x_a} T = \partial_{y_b}^2 (D_1 K \cdot (p(x) - p(y)) + K \cdot D_1 p) = -(\partial_{y_b}^2 (D_1 K \cdot (p(y) - p(x)) - K \cdot D_1 p)),$$

for $T = T_{01}$ or T_{10} . Using an expansion similar to that in (B.27), (B.26), we get

$$(B.31) \quad \begin{aligned} -\partial_{y_b}^2 \partial_{x_a} T &= \sum_{i=1,2} (D_2^2 D_1 K \cdot (y_i - x_i) + \mathbf{1}_{D_2=\partial_i} 2D_2 D_1 K - \mathbf{1}_{D_1=\partial_i} D_2^2 K) \cdot \partial_{x_i} p(x_m) + D_2^2 D_1 K \cdot p_{m,2,err} \\ &\quad + 2D_2 D_1 K \cdot (D_2 p(y) - D_2 p(x_m)) - D_2^2 K \cdot (D_1 p(x) - D_1 p(x_m)) + D_1 K \cdot D_2^2 p(y) \\ &\triangleq \left(\sum_{i=1,2} I_i \cdot \partial_{x_i} p(x_m) \right) + II_1 + II_2 + II_3 + II_4, \\ I_i &\triangleq D_2^2 D_1 K \cdot (y_i - x_i) + \mathbf{1}_{D_2=\partial_i} 2D_2 D_1 K - \mathbf{1}_{D_1=\partial_i} D_2^2 K, \end{aligned}$$

where $\partial_1^i \partial_2^j K$ is evaluated at $z = (y_1 - s_1 x_1, y_2 - s_2 x_2)$. We bound II_i using triangle inequality, the estimate (B.29), and the bounds for K , its derivatives, and p in Sections B.1, B.2, and A.1.

For I_i , if $i = side$, from (B.30), we get $s_i = 1$ and $z_i = y_i - s_i x_i = y_i - x_i$. Hence, we get

$$I_i = D_2^2 D_1 K \cdot (y_i - x_i) + \mathbf{1}_{D_2=\partial_i} 2D_2 D_1 K - \mathbf{1}_{D_1=\partial_i} D_2^2 K = K_{mix}(D_1, D_2, i, -1)(z),$$

where K_{mix} is defined in (B.28).

If $i \neq side$ and $D_1 = D_2 = \partial_i$, we have $z_i = y_i - s_i x_i = y_i + x_i$ and get a cancellation between $D_2 D_1 K$ and $D_2^2 K$ and yield

$$|I_i| = |D_2^2 D_1 K \cdot (y_i - x_i) + D_2 D_1 K| \leq |D_2^2 D_1 K| \cdot |y_i - x_i| + |D_2 D_1 K|.$$

Otherwise, we simply bound each term in I_i using the triangle inequality.

B.3.3. *X-discretization.* For $K(s) = \frac{s_1 s_2}{|s|^4}, \frac{1}{2} \frac{s_2^2 - s_1^2}{|s|^4}$, we have $K(s) = K(-s)$. Denote

$$T = K(y - x)(p(x) - p(y)) = K(x - y)(p(x) - p(y)).$$

In this section, we compute $\partial_{x_b}^i \partial_{x_a}^j T$. Using the Taylor expansion at x

$$p(x) - p(y) = (x - y) \cdot \nabla p(x) + p_{x,2,err},$$

and calculations similar to those in Section B.3.1, we get

$$\begin{aligned} (B.32) \quad \partial_{x_b}^2 \partial_{x_a} T &= \partial_{x_b}^2 (D_1 K \cdot (p(x) - p(y)) + K D_1 p(x)) = D_2^2 D_1 K \cdot (p(x) - p(y)) + 2D_1 D_2 K \cdot D_2 p(x) \\ &\quad + D_1 K \cdot D_2^2 p(x) + D_2^2 K \cdot D_1 p(x) + 2D_2 K \cdot D_1 D_2 p(x) + K \cdot D_1 D_2^2 p(x) \\ &= \sum_{i=1,2} (D_2^2 D_1 K \cdot (x_i - y_i) + \mathbf{1}_{D_2=\partial_i} 2D_1 D_2 K + \mathbf{1}_{D_1=\partial_i} D_2^2 K) \partial_i p(x) + D_2^2 D_1 K \cdot p_{x,2,err} \\ &\quad + D_1 K \cdot D_2^2 p(x) + 2D_2 K \cdot D_1 D_2 p(x) + K \cdot D_1 D_2^2 p(x) \triangleq \left(\sum_{i=1,2} I_i \cdot \partial_i p(x) \right) + II, \\ I_i &\triangleq D_2^2 D_1 K \cdot (x_i - y_i) + \mathbf{1}_{D_2=\partial_i} 2D_1 D_2 K + \mathbf{1}_{D_1=\partial_i} D_2^2 K, \end{aligned}$$

where II consists of the last four terms in the third equation, K and its derivatives are evaluated at $x - y$. Since $D_1, D_2 = \partial_{x_i}$, we get

$$I_i = D_2^2 D_1 K \cdot (x_i - y_i) + \mathbf{1}_{D_2=\partial_i} 2D_1 D_2 K + \mathbf{1}_{D_1=\partial_i} D_2^2 K = K_{mix}(D_1, D_2, i, 1)(x - y),$$

where K_{mix} is defined in (B.28). We use the bound for K_{mix} , $\partial_1^i \partial_2^j K$ and p to estimate $D_2^2 D_1 T$.

B.3.4. *The second singular term.* Similar to Section B.3.2, we have the second singular term for x close to the x -axis or y -axis

$$T_{01} \triangleq K(x_1 - y_1, x_2 + y_2)(p(x) - p(y)), \quad T_{10} \triangleq K(x_1 + y_1, x_2 - y_2)(p(x) - p(y)).$$

We have the former if $x_2 < x_1$ and x_2 close to 0, and the latter if $x_1 < x_2$ and x_1 close to 0. Using the definition of *side*, s_1, s_2 from Section B.3.2 and (B.30), we get

$$\partial_{x_a} K(x_1 - y_1 s_1, x_2 - y_2 s_2) = (D_1 K)(x_1 - y_1 s_1, x_2 - y_2 s_2).$$

Then the computations of $D_2^2 D_1 T$ are the same as those in (B.32) with $\partial_1^i \partial_2^j K$ evaluated at $z = (x_1 - s_1 y_1, x_2 - s_2 y_2)$. We bound II in (B.32) directly using the triangle inequality and the bounds for $\partial_1^i \partial_2^j K$ and p . For I_i in (B.32), if $i = \text{side}$, from (B.30), we get s_i and $z_i = x_i - s_i y_i = x_i - y_i$. It follows

$$I_i = D_2^2 D_1 K \cdot z_i + \mathbf{1}_{D_2=\partial_i} 2D_1 D_2 K + \mathbf{1}_{D_1=\partial_i} D_2^2 K = K_{mix}(D_1, D_2, i, 1)(z).$$

If $i \neq \text{side}$, we have $z_i = x_i + y_i > |x_i - y_i|$. We bound each term in I_i separately by following the previous argument.

B.4. Estimate of $u(x)$ for small x_1 . In the energy estimate, we need to estimate $(u(x) - \hat{u}(x))\varphi(x)$ with weight φ singular along the line $x_1 = 0$, e.g. φ_1 (A.2), where $\hat{u}(x)$ is a finite rank approximation of $u(x)$. We use the property that u vanishes on $x_1 = 0$ to establish such an estimate.

By definition and symmetrizing the kernel using the odd symmetry of ω , we have

$$u(x, y) = \frac{1}{2\pi} \int_{y_1 \geq 0} \left(\frac{x_2 - y_2}{|x - y|^2} - \frac{x_2 - y_2}{(x_1 + y_1)^2 + (x_2 - y_2)^2} \right) \omega(y) dy = \frac{1}{\pi} \int_{y_1 \geq 0} K(x, y) W(y) dy,$$

where

$$\begin{aligned} (B.33) \quad K &= \frac{1}{2} \left(\frac{x_2 - y_2}{|x - y|^2} - \frac{x_2 - y_2}{(x_1 + y_1)^2 + (x_2 - y_2)^2} \right) = x_1 \cdot \frac{2(x_2 - y_2)y_1}{|x - y|^2((x_1 + y_1)^2 + (x_2 - y_2)^2)} \\ &\triangleq x_1 K_{du}(x, y) = x_1 \tilde{K}_{du}(x_1, y_1, x_2 - y_2), \quad \tilde{K}_{du}(x, y, z) = \frac{2yz}{((x - y)^2 + z^2)((x + y)^2 + z^2)}. \end{aligned}$$

We define K_{app} as the symmetrized kernel in \mathbb{R}_2^{++} for \hat{u} similar to that in Section 4.2. Since W is odd in y_2 , we can symmetrize the integral in y_2 and obtain the full symmetrized integrand

$$x_1 K_{du}(x, y) - x_1 K_{du}(x_1, x_2, y_1, -y_2) = x_1 (\tilde{K}_{du}(x_1, y_1, x_2 - y_2) - \tilde{K}_{du}(x_1, y_1, x_2 + y_2)).$$

Since K is -1 homogeneous, using a rescaling argument, for $x = \lambda \hat{x}, y = \lambda \hat{y}$, we have

$$(B.34) \quad u = \frac{\lambda}{\pi} \int_{\hat{y}_1 \geq 0} \left(\mathbf{1}_{S^c}(\hat{y}) K(\hat{x}, \hat{y}) - K_{app, \lambda}(\hat{x}, \hat{y}) \right) \omega_\lambda(\hat{y}) + \mathbf{1}_S(\hat{y}) K(\hat{x}, \hat{y}) \omega_\lambda(\hat{y}) d\hat{y} \triangleq I + II,$$

for some rescaled kernel $K_{app, \lambda}(\hat{x}, \hat{y})$, where $S = R(\hat{x}, k)$ is the singular region (4.18) adapted to \hat{x} . For I , we further rewrite it and estimate it as follows

$$I = \frac{\lambda}{\pi} \hat{x}_1 \int_{\hat{y}_1 \geq 0, \hat{y} \notin S} \left(\mathbf{1}_{S^c}(\hat{y}) K_{du}(\hat{x}, \hat{y}) - \frac{1}{\hat{x}_1} K_{app, \lambda}(\hat{x}, \hat{y}) \right) \omega_\lambda(\hat{y}) d\hat{y}.$$

Since the integrand is not singular, we further symmetrize the integrand in y_2 and then use the method in Section 4.1.3 to discretize and estimate the integral to obtain its tight bound.

Derivative bounds. To estimate the error in the Trapezoidal rule in Lemma 4.2, we need to bound $\partial_{x_i}^2 K_{du}(x, y), \partial_{y_i}^2 K_{du}(x, y)$. Since $\frac{1}{x} C_{u0}(x, y), \frac{1}{x} C_u(x, y)$ (4.5) are smooth, from the construction in Section 4.3, the kernel $\frac{1}{x_1} K_{app}(x, y)$ and its rescaled version are regular in \hat{x} . We estimate its derivatives following Section 4.1. Since $K_{du}(x, y) = \frac{1}{x_1} K(x, y)$ (B.33), $K(x, y)$ is harmonic in y , and $|\partial_{x_2}^2 K(x, y)| = |\partial_{y_2}^2 K(x, y)|$, we get

$$\partial_{y_1}^2 K_{du}(x, y) = -\partial_{y_2}^2 K_{du}(x, y), \quad |\partial_{y_2}^2 K_{du}(x, y)| = |\partial_{x_2}^2 K_{du}(x, y)|.$$

Thus, we only need to bound $|\partial_{x_1}^2 K_{du}|$ and $|\partial_{y_1}^2 K_{du}|$, or $\partial_x^2 \tilde{K}_{du}$ and $\partial_y^2 \tilde{K}_{du}$ using the relation (B.33). We derive the formulas of $\partial_x^2 \tilde{K}_{du}$ and $\partial_y^2 \tilde{K}_{du}$ and then estimate them using methods similar to that in Appendix B.2. We have an improved estimate for $\partial_y \tilde{K}_{du}$ in $\{x\} \times [y_l, y_u] \times [z_l, z_u]$ near the singularity. A direct computation yields

$$\begin{aligned} \partial_y^2 \tilde{K}_{du}(x, y, z) &= 24yz \frac{(z^4 - (x^2 - y^2)^2)(x^2 + y^2 + z^2)}{T_-^3 T_+^3} + 64 \frac{x^2 y^3 z^3}{T_-^3 T_+^3} \\ &= \frac{yz}{T_-^2 T_+^2} \left(12 \left(\frac{1}{T_-} + \frac{1}{T_+} \right) (z^4 - (x - y)^2 (x + y)^2) + 64x^2 \frac{y^2 z^2}{T_- T_+} \right), \quad T_\pm = (x \pm y)^2 + z^2. \end{aligned}$$

where we have used $\frac{1}{T_-} + \frac{1}{T_+} = 2 \frac{x^2 + y^2 + z^2}{T_- T_+}$. We apply the estimate of K_{du} to $x, y \geq 0$. Since $|\partial_y^2 \tilde{K}_{du}|$ is even in z , without loss of generality, we consider $z \geq 0$. Then for P_2 , we have $z/T_-^{1/2}, y/T_+^{1/2}$ are increasing in z, y , respectively. To bound other terms, we simply use the monotonicity of the polynomials, (B.22), interval operation (A.4), (A.5), and follow Section B.2.1. For example, we use (B.23) to bound $(x - y)^2, (x + y)^2$ and

$$0 \leq \frac{y}{T_+^{1/2}} \leq \frac{y_u}{((x + y_u)^2 + z_l^2)^{1/2}}, \quad 0 \leq \frac{z}{T_-^{1/2}} \leq \frac{z_u}{(|x - y_l|^2 + z_u^2)^{1/2}}.$$

\hat{x}_1 not small. For II in (B.34), if $\hat{x}_1 \geq x_l = 2h > 0$ away from 0, we have $|K_{du}(\hat{x}, \hat{y})| \lesssim \frac{1}{x_l} \frac{1}{|\hat{x} - \hat{y}|}$, which is integrable near the singularity \hat{x} . We estimate II using

$$|II| \leq \frac{\lambda}{\pi} \hat{x}_1 \int_{\hat{y}_1 \geq 0, \hat{y} \in S} |K_{du}(\hat{x}, \hat{y})| \varphi_\lambda^{-1}(\hat{y}) d\hat{y} \|\omega\varphi\|_\infty, \quad S = R(\hat{x}, k).$$

We follow Section 4.1.6 by introducing $\hat{y} = \hat{x} + s, s \in S - \hat{x}$, decomposing $S - \hat{x}$ into the symmetric part D_{sym} and non-symmetric part D_{ns} and estimating the piecewise integral of $K_{du}(\hat{x}, \hat{y})$

$$D_{sym} = R_s(\hat{x}, k) - \hat{x}, \quad D_{ns} = (R(\hat{x}, k) \setminus R_s(\hat{x}, k)) - \hat{x},$$

$$|K_{du}(\hat{x}, \hat{y})| \mathbf{1}_{\hat{y}_1 \geq 0} = |F| \mathbf{1}_{\hat{x}_1 + s_1 \geq 0}, \quad F = \frac{(\hat{x}_1 + s_1) s_2}{|s|^2 ((s_1 + 2\hat{x}_1)^2 + s_2^2)},$$

and piecewise bounds of $\varphi_\lambda^{-1}(y)$, where we have used (B.33) to obtain the above formula. We observe that $|F|$ is even in s_2 and $F \geq 0$ for $s \in Q = [a, b] \times [c, d]$ with $c, d \geq 0$. We estimate

the piecewise integrals of F in Q in Section 6.2 in the supplementary material II [12]. Denote $X_1^+ \triangleq \{y : y_1 \geq 0\}$. If $\hat{x}_1 \geq kh$, we get $S \cap X_1^+ = R(\hat{x}, k)$ and the regions D_{sym}, D_{ns} are the same as those in Section 4.1.6. If $\hat{x}_1 \in [ih, (i+1)h), i < k$, the region S touches $\{y : y_1 = 0\}$ and we get

$$S \cap X_1^+ = [0, (i+k+1)h] \times [(j-k)h, (j+1+k)h], \quad \text{for } x_2 \in [jh, (j+1)h]$$

In this case, the symmetric and non-symmetric region becomes smaller. We do not have the left edge in the middle figure in Figure 2, part of the upper and the lower edge due to the restriction $\hat{y}_1 = s_1 + \hat{x}_1 \geq 0$. The estimate of the integrals for $s \in S \cap X_1^+ - \hat{x}_1$ follows similar argument.

Small \hat{x}_1 . The difficulty is to estimate II for small $\hat{x}_1 \leq 2h$. It is not difficult to obtain that

$$(B.35) \quad |II| \lesssim \frac{\lambda}{\pi} \|\omega_\lambda\|_{L^\infty(S)} \hat{x}_1 |\log(\hat{x}_1)|.$$

Thus we cannot bound II by $C\hat{x}_1$ for some constant C uniformly for small \hat{x}_1 . Denote by

$$(B.36) \quad \begin{aligned} S_{sym} &= [0, \hat{x}_1 + kh] \times [\hat{x}_2 - kh, \hat{x}_2 + kh], & S_{in,1} &= [0, \hat{x}_1] \times [\hat{x}_2 - kh, \hat{x}_2 + kh], \\ S_{in,2} &= [\hat{x}_1, \hat{x}_1 + h] \times [\hat{x}_2 - h, \hat{x}_2 + h], & S_{in} &= S_{in,1} \cup S_{in,2} \\ S_{out} &= [\hat{x}_1, \hat{x}_1 + hk] \times [\hat{x}_2 - kh, \hat{x}_2 + kh] \setminus S_{in,2}, & \hat{y} &= \hat{x} + \hat{x}_1 s. \end{aligned}$$

See the right figure in Figure 2 for an illustration of different regions. By definition, we have $S_{sym} = S_{out} \cup S_{in,1} \cup S_{in,2}$. Here S_{in} captures the most singular region. Then $\hat{y} \in S_{in}$ is equivalent to

$$(B.37) \quad \begin{aligned} s \in \hat{x}_1^{-1}(S_{in} - \hat{x}) &= \hat{x}_1^{-1}([- \hat{x}_1, 0] \times [-kh, kh] \cup [0, h] \times [-h, h]) \triangleq R_1(B_1) \cup R_2(B_2), \\ R_1(B_1) &= [-1, 0] \times \left[-\frac{1}{B_1}, \frac{1}{B_1}\right], & R_2(B_2) &= [0, \frac{1}{B_2}] \times \left[-\frac{1}{B_2}, \frac{1}{B_2}\right], & B_1 &= \frac{\hat{x}_1}{kh}, & B_2 &= \frac{\hat{x}_1}{h}. \end{aligned}$$

We further decompose II as follows

$$II = \frac{\lambda}{\pi} \hat{x}_1 \int_{y_1 \geq 0} (\mathbf{1}_{S \setminus S_{sym}}(\hat{y}) + \mathbf{1}_{S_{out}}(\hat{y}) + \mathbf{1}_{S_{in,1}}(\hat{y}) + \mathbf{1}_{S_{in,2}}(\hat{y})) K_{du}(\hat{x}, \hat{y}) \omega_\lambda(\hat{y}) d\hat{y} = \frac{\lambda \hat{x}_1}{\pi} (II_1 + II_2 + II_{in,1} + II_{in,2}).$$

The integrals II_1, II_2 capture the non-symmetric part and the symmetric part away from the singularity. We apply L^∞ estimate and the method in Sections 4.1.6, 4.1.9. For $II_{in,i}$, using a change of variables (B.36), (B.37), we derive

$$II_{in,i} = \int_{s \in R_i(B_i)} K_{du}(\hat{x}, \hat{x} + \hat{x}_1 s) \hat{x}_1^2 \omega_\lambda(\hat{x} + \hat{x}_1 s) ds.$$

Note that $\hat{y} - \hat{x} = \hat{x}_1 s$, $\hat{y}_1 + \hat{x}_1 = \hat{x}_1(2 + s_1)$, $\hat{y}_2 - \hat{x}_2 = \hat{x}_1 s_2$. By definition (B.33), we get

$$K_{du}(\hat{x}, \hat{x} + \hat{x}_1 s) \hat{x}_1^2 = -\frac{2\hat{x}_1 s_2 \cdot (\hat{x}_1 + \hat{x}_1 s_1)}{\hat{x}_1^2 |s|^2 \cdot \hat{x}_1^2 ((s_1 + 2)^2 + s_2^2)} \hat{x}_1^2 = -\frac{2(s_1 + 1)s_2}{|s|^2 ((s_1 + 2)^2 + s_2^2)} \triangleq -K_s(s),$$

$$II_{in,i} = -\int_{R_i(B_i)} K_s(s) \omega_\lambda(\hat{x} + \hat{x}_1 s) ds.$$

Since $K_s(s)$ is symmetric in s_2 , we derive

$$\begin{aligned} |II_{in,1}| &\leq \|\omega_\lambda\|_\infty \left(\max_{z \in [-\hat{x}_1, 0] \times [0, kh]} \varphi_\lambda^{-1}(\hat{x} + z) + \max_{z \in [-\hat{x}_1, 0] \times [-kh, 0]} \varphi_\lambda^{-1} \right) J_1(B_1), \\ |II_{in,2}| &\leq \|\omega_\lambda\|_\infty \left(\max_{z \in [0, h] \times [0, h]} \varphi_\lambda^{-1} + \max_{z \in [0, h] \times [-h, 0]} \varphi_\lambda^{-1} \right) J_2(B_2), \end{aligned}$$

where B_i is given in (B.37) and

$$J_1(B_1) = \left| \int_{[-1, 0] \times [0, 1/B_1]} K_s(s) ds \right| = \int_{[-1, 0] \times [0, 1/B_1]} K_s(s) ds, \quad J_2(B_2) = \int_{[0, 1/B_2]^2} K_s(s) ds.$$

The formula of J_i can be obtained using the analytic integral formula for K_s , and obviously J_i is decreasing in B . Note that $J_1(B)$ is bounded, but $J_2(B) \lesssim 1 + \log(B) \lesssim 1 + |\log \hat{x}_1|$, which relates to the estimate (B.35). We refer the formulas of J_i to Section 6.2 in the supplementary material II [12].

B.5. Additional derivations.

B.5.1. *Estimate of the log-Lipschitz integral.* In this section, we derive the coefficient in the estimate of $\partial_{x_2} I_{5,4}(x)$ (4.64), (4.65). For $I_{5,4}$, we further decompose it as follows

$$I_{5,4} = \left(\int_{R(k_2) \setminus R_s(k_2)} + \int_{R_s(k_2) \setminus R_s(b)} + \int_{R_s(b) \setminus R_s(a)} \right) K(x-y)(\psi(x) - \psi(y))W(y)dy \triangleq I_{5,4,1} + I_{5,4,2} + I_{5,4,3}.$$

In practice, we choose $b = 2$. The first two terms are nonsingular and their derivatives can be estimated using the method in Sections 4.1.6-4.1.9. In the estimate of $\partial_{x_i} I_{5,4}$, we only need to estimate the boundary term on $\partial R_s(a)$ since the boundary terms on $\partial R_s(k_2), \partial R_s(b)$ are canceled in $\partial_{x_i} I_{5,j}, j = 1, 2, 3$. For $I_{5,4,3}$, using the second order Taylor expansion to $\psi(x) - \psi(y)$ centered at x , we have

$$\begin{aligned} \partial_{x_2}(K(x-y)(\psi(x) - \psi(y))) &= (\partial_2 K)(x-y)(\psi(x) - \psi(y)) + K(x-y)\partial_2 \psi(x) \\ &= (\partial_2 K(x-y)(x_2 - y_2) + K(x-y))\partial_2 \psi(x) + \partial_2 K(x-y)(x_1 - y_1)\partial_1 \psi(x) + \mathcal{R}_K, \end{aligned}$$

where the remainder \mathcal{R}_K coming from the higher order term in the Taylor expansion satisfies

$$|\mathcal{R}_K| \leq \sum_{i+j=2} \|\partial_x^i \partial_y^j \psi\|_{L^\infty(Q)} |x_1 - y_1|^i |x_2 - y_2|^j c_{ij},$$

where $Q = B_{i_1 j_1}(h_x) + [-bh, bh]^2$ and $c_{20} = c_{02} = \frac{1}{2}, c_{11} = 1$. It follows

$$|\partial_{x_2} I_{5,4,3}| \leq \|\omega \varphi\|_\infty \sum_{0 \leq i \leq 1, 0 \leq j \leq i+1} Scoe_{ij}(x) \cdot f_{ij}(a, b),$$

where the coefficients $Scoe_{ij}(x)$ depend on the weight ψ, φ , and $f_{ij}(a, b)$ bounds the integral

$$(B.38) \quad \int_{[-b, b]^2 \setminus [-a, a]^2} |\partial_2 K(y) \cdot y_1^i y_2^j + \mathbf{1}_{(i,j)=(0,1)} K(y)| dy \leq f_{ij}(a, b).$$

For example, $Scoe_{01}$ comes from the following estimate for $I_{5,4,3}$

$$\begin{aligned} & \int_{R_s(b) \setminus R_s(a)} |(\partial_2 K(x-y)(x_2 - y_2) + K(x-y))\partial_2 \psi(x)| \omega(y) dy \\ & \leq \|\omega \varphi\|_\infty \|\varphi^{-1}\|_{L^\infty(Q)} \cdot |\partial_2 \psi(x)| \int_{[-b, b]^2 \setminus [-a, a]^2} |\partial_2 K(s)s_2 + K(s)| ds. \end{aligned}$$

The function $f_{ij}(a, b)$ satisfies the following estimates for some constants $B_{1j} > 0$

$$f_{1j}(a, b) \leq B_{1j} \log(b/a), \quad j = 1, 2.$$

We refer the derivations to Section 5.1.5 in the supplementary material II [12].

B.5.2. *Optimization in the Hölder estimate.* Consider

$$\max_{t \leq t_u} \min_{a \leq b} F(a, t), \quad F(a, t) = (A + B \log \frac{b}{a})\sqrt{t} + \frac{Ca}{\sqrt{t}},$$

in the upper bound in (4.72). For each $t \leq t_u$, we first optimize $F(a, t)$ over $a \leq b$. We assume that A, B, C, b, c, h, h_x are given. Denote

$$t_u = ch_x, \quad t_1 = \frac{Cb}{B}.$$

For a fixed t , since $\partial_a^2 F > 0, \partial_a F(0, t) < 0$ and $\partial_a F(a, t) = 0$ if $a = \frac{Bt}{C}$, we choose $a = \min(b, \frac{Bt}{C})$. For $t \leq \frac{Cb}{B} = t_1$, we get

$$\min_{a \leq b} F(a, t) \leq F\left(\frac{Bt}{C}, t\right) = (A + B \log \frac{bC}{B} + B)\sqrt{t} - B\sqrt{t} \log t.$$

The right hand side can be further estimated by studying the concave function on $s = t^{1/2} \leq s_u$

$$f(p, q, s) = (p - q \log s)s \leq f(p, q, \min(s_u, s_*)), \quad s_* = \exp\left(\frac{p-q}{q}\right)$$

with $p = A + B \log(\frac{bC}{B}) + B, q = 2B, s_u = \min(t_u^{1/2}, t_1^{1/2})$. We get the above inequality since $f(p, q, s)$ is increasing for $s \leq s_*$ and is decreasing for $s \geq s_*$.

If $\frac{Cb}{B} \leq t \leq t_u$, we choose $a = b$ and get

$$\min_{a \leq b} F(a, t) \leq F(b, t) = A\sqrt{t} + \frac{Cb}{\sqrt{t}},$$

which is convex in $t^{1/2}$. Thus its maximum is achieved at the endpoints.

APPENDIX C. REPRESENTATIONS AND ESTIMATES OF THE SOLUTIONS

In Section 7 of Part I [14], we represent the approximate steady state as follows

$$(C.1) \quad \begin{aligned} \bar{\omega} &= \bar{\omega}_1 + \bar{\omega}_2, \quad \bar{\theta} = \bar{\theta}_1 + \bar{\theta}_2, \quad \bar{\omega}_1 = \chi(r)r^{-\bar{\alpha}_1}g_1(\beta), \quad \bar{\theta}_1 = \chi(r)r^{1-2\bar{\alpha}_1}g_2(\beta), \\ \bar{\phi}^N &= \bar{\phi}_1^N + \bar{\phi}_2^N + \bar{\phi}_3^N + \bar{\phi}_{cor}^N, \quad \bar{\phi}_3^N = \bar{a}\chi_{\phi,2D}, \quad \chi_{\phi,2D} = -xy\chi_\phi(x)\chi_\phi(y), \\ \bar{\phi}_{cor}^N &= -c \cdot \frac{xy^2}{2}\kappa_*(x)\kappa_*(y) = c\phi_1, \quad c = \partial_x(\bar{\omega} + \Delta(\bar{\phi}_1^N + \bar{\phi}_2^N + \bar{\phi}_3^N)), \quad |\bar{\alpha}_1 + \frac{\bar{c}_\omega}{\bar{c}_l}| \ll 1, \quad \bar{\alpha}_1 \approx \frac{1}{3}, \end{aligned}$$

where $\bar{\omega}_2, \bar{\theta}_2, \bar{\phi}_2^N$ have compact supports and are represented as piecewise polynomials, $\bar{a} \in \mathbb{R}$ is some coefficient, κ_* is given in (D.5), ϕ_1 is the same as (3.14), χ_ϕ is given in (D.7). We choose a small correction $\bar{\phi}_{cor}^N$ similar to that in Section 3.2 so that $\bar{\omega} + \Delta\bar{\phi}^N = O(|x|^2)$ near 0. We use upper script N to distinguish the numerical approximation $\bar{\phi}^N$ for the exact stream function $\bar{\phi} = (-\Delta)^{-1}\bar{\omega}$. The exponent $\bar{\alpha}_1$ and angular profiles $g_i(\beta)$ are obtained by fitting the far-field asymptotics of an approximate steady state with $\bar{\omega}_1 = 0, \bar{\theta}_1 = 0$. Then we construct $(\bar{\omega}_1, \bar{\theta}_1)$ using the above formulas. Afterward, we refine the construction of the near-field part $(\bar{\omega}_2, \bar{\theta}_2)$ and exponents $(\bar{c}_\omega, \bar{c}_l)$ by fixing $(\bar{\omega}_1, \bar{\theta}_1, \bar{\alpha}_1)$. See more details on how to find the semi-analytic part in Section 7 of Part I [14]. We will discuss how to estimate the semi-analytic part in Section C.3. In the following sections, we discuss more details about the representations and establish rigorous estimate of the derivatives of $\bar{\omega}, \bar{\theta}$.

Note that we do not need an approximation term $\bar{\phi}_3$ for the stream function in solving the linearized equation in Section 3 since we can allow a larger residual error in Section 3.

C.1. Representations. In a large domain $[0, L]^2$, we use piecewise polynomials to represent the solution. Firstly, we choose a large L of order 10^{15} and then design the adaptive mesh $y_{-5} < \dots < y_0 = 0 < y_1 < \dots < y_{N-1} = L, N = 748$ to partition $[0, L]$.

Adaptive mesh. We design three parts of the mesh $y_i, i \in I_j \triangleq [a_j, b_j], a_0 = 0$ as follows

$$(C.2) \quad \begin{aligned} y_i &= \frac{i}{256}, i = -5, -4, \dots, 1, \dots, b_1, \quad y_{a_2+i} = y_{a_2} + F(ih_3), i = 1, \dots, b_2 - a_2, \\ y_{a_3+i} &= y_{a_3} \exp(ir_1), i = 1, \dots, b_3 - a_3, \quad r_0 = 1.025, r_1 = 1.15 \\ F(z) &= \frac{h_2}{h_3}z \exp(rz^2), \quad r = \log\left(\frac{r_0}{1+h_3}\right) \frac{1}{(1+h_3)^2 - 1}, \quad h_2 = \frac{1}{128}, h_3 = \frac{1}{b_2 - a_2}. \end{aligned}$$

Since we need to estimate the weighted L^∞ norm of the residual error with a singular weight of order $|x|^{-\beta}, \beta \approx 3$ near $x = 0$, we use uniformly dense mesh near 0 so that we have a very small residual error. We choose the parameters $\frac{1}{256}, h_2 = \frac{1}{128}$ since they can be represented exactly as floating point numbers. Thus, we can reduce the round-off error in the computation. In the far-field, we use a mesh that grows exponentially fast in space. Note that the error estimate $f - I(f)$ for the k -th order interpolation of f on $[y_i, y_{i+1}]$ reads

$$|f - I(f)| \leq C(y_{i+1} - y_i)^k |\partial_x^k f|.$$

For large x , we expect that $\partial_x^k f$ has a decay rate $|y|^{-k-\alpha}$ if $|f| \lesssim |y|^{-\alpha}$ for $\alpha > 0$. Thus, to get a uniformly small error in the far-field, we just require $\frac{y_{i+1} - y_i}{y_i} \leq \varepsilon$ with $\varepsilon < 1$. This allows us to choose an exponentially growing mesh in the far-field and cover a very large domain without using too many points. We use the second part of the mesh to glue the first part of the mesh,

which grows linearly, and the third part of the mesh. The functions $F(z)$ behaves linearly for z close to 0, and it grows exponentially fast with rate r_1 for z close to 1:

$$F(1+h_3)/F(1) = (1+h_3) \exp(r((1+h_3)^2-1)) = (1+h_3) \exp(\log(r_0/(1+h_3))) = r_0.$$

Parameters h_2, h_3 control the mesh size $y_{a_2+1} - y_{a_2} = F(h_3) = h_2 \exp(rh_3^2) \approx h_2$. One can design another $F(z)$ by gluing the first and the third part of the mesh. The above explicit and simple form of $F(z)$ serves our purpose. We further glue $y_i, i \in [b_j, a_{j+1}], j = 1, 2$ using the Lagrangian interpolation for $j = 1$. For $j = 2$, we interpolate the growth rate using $\exp(\log(r_0)l(i) + (1-l(i))\log(r_1))$ with $l(i)$ linear in $i \in [b_2, a_3]$. Note that we do not use the specific property of the profile to design the adaptive mesh (C.2).

In our numerical computation, we compute the derivatives of the solution using the B-spline basis, see e.g., (C.6), and do not use the Jacobian related to the adaptive mesh. In particular, we do not use derivatives of the map $f(i) = y_i$, and have more flexibility to design the mesh.

Let $n_1 = 720 < N$. We solve the dynamic rescaling equation (2.10)-(2.11) on first $n_1 \times n_1, (y_i, y_j), i, j \leq n_1 - 1$ grids. We construct

$$(C.3) \quad \bar{\omega}_2(x, y) = \sum_{0 \leq i \leq n_1+1, -2 \leq j \leq n_1+1} a_{ij} B_{1,i}(x) B_j(y),$$

where $a_{ij} \in \mathbb{R}$ is the coefficient, $B_i(x), B_j(y)$ are constructed from the 6-th order B-spline

$$(C.4) \quad B_i(x) = C_i B_{i0}(x), \quad B_{i0}(x) = \sum_{0 \leq j \leq k} k \frac{(s_{ij} - x)_+^{k-1}}{d_j}, \quad d_j = \prod_{0 \leq l \leq k, l \neq j} (s_{ij} - s_{il}),$$

with $k = 6$. The constant C_i will be chosen in (C.10), (C.11) so that the stiffness matrix associated to these B-spline basis has a better condition number. We choose s_{ij} as follows

$$s_{ij} = y_{i+j-3}, \quad 0 \leq j \leq k = 6.$$

Then the B-spline B_i is supported in $[y_{i-3}, y_{i+3}]$ and is centered around y_i . Since ω is odd in x , to impose this symmetry in the representation, we modify the first few basis

$$(C.5) \quad B_{1,i}(x) = B_i(x) - B_i(-x), \quad i \leq 2.$$

Then B_i is odd. We remark that $B_{1,0}(x) \equiv 0$.

B-Spline and the tensor structure. We also use the B-spline basis to represent the stream function (C.8) and solve the Poisson equation using the B-spline based finite element method. We use the B-spline basis since it is easy to design a high-order numerical scheme to solve the Poisson equation. Each basis function in (C.3), (C.7), (C.8) has the form $f(x)g(y)$, which allows us to evaluate and estimate the 2D function very effectively using the method in Appendix C.2.2.

Remark C.1. While the method described below to obtain the coefficients a_i is technical, since we perform a-posteriori estimates of the profiles and residual error using the given a_i , the method of deriving a_i is not involved in the a-posteriori estimates and the verification process.

Extrapolation. Near the boundary $y = 0$, we need 2 extra basis functions $a_{i,-j} B_{-j}(y), j = 1, 2$ that are not zeros in $y_1 \geq 0$. Without these 2 functions, the representation (C.3) does not approximate $\bar{\omega}$ with a 6-th order error. We use a 7-th order extrapolation [42, 43] to determine $a_{i,-j}$

$$a_{i,-j} = \sum_{0 \leq l \leq 6} C_{3-j, l+1} a_{i,l}, \quad C_{1,\cdot} = (28, -112, 210, -224, 140, -48, 7), \quad C_{2,\cdot} = (7, -21, 35, -35, 21, -7, 1).$$

We choose $C_{j,l}$ such that the 7-th difference of $a_{i,j}, -2 \leq j \leq 6$ is 0. Since $a_{i,-j}$ depends on $a_{i,l}$ linearly, we can combine $a_{i,-j} B_{i,-j}, j = 1, 2$ with $a_{i,l} B_{i,l}$ and modify (C.3) as follows

$$(C.6) \quad \bar{\omega}_2(x, y) = \sum_{0 \leq i, j \leq n_1+1} a_{ij} B_{1,i}(x) B_{2,j}(y),$$

$$B_{2,j}(y) = B_j(y) + C_{2,j+1} B_{-1}(y) + C_{1,j+1} B_{-2}(y), \quad 0 \leq j \leq 6, \quad B_{2,j}(y) = B_j(y), \quad j \geq 7.$$

The modified basis functions $B_{1,i}, B_{2,j}$ are still piecewise polynomials in $[y_l, y_{l+1}]$.

Far-field extension. In (C.3),(C.6), we use Bspline $B_{1,i}(x), B_j(y)$ up to $i, j \leq n_1 + 1$ rather than $n_1 - 1$ since the support of $B_{1,i}, B_j$ intersects $[0, y_{n-1}]^2$ for $i, j \leq n_1 - 1$. To determine the extra coefficients, we first extend the grid point values of $\omega_2(x, y)$ from (y_i, y_j) with $i, j \leq n_1 - 1$ to $i, j \leq n_1 + l_0 - 1$ by $\omega_2(y_{n_1+l}, y_j) = P(y_{n_1+l}; y_j), l = 0, 1, \dots, l_0 - 3$, where P is the Lagrangian interpolation polynomials on $(y_{n_1-1}, \omega(y_{n_1-1}, y_j)), (y_{n_1+l_0-3}, 0), (y_{n_1+l_0-2}, 0)$. We impose $\omega_2(y_{n_1+l}, y_j) = 0, l = l_0 - 3, l_0 - 2, l_0 - 1$. Similarly, we extend $\omega(y_i, y_{n+l})$. Note that ω_2 is odd and $B_{1,0} = 0$. We solve the coefficients $a_{kl}, 1 \leq k \leq M, 0 \leq l \leq M$ from

$$\omega_2(y_p, y_q) = \sum_{1 \leq i \leq M, 0 \leq j \leq M} a_{ij} B_{1,i}(x) B_{2,j}(y), \quad 1 \leq p \leq M, \quad 0 \leq q \leq M, \quad M = n_1 + l_0 - 1.$$

The value a_{0j} is not used since $B_{1,0} \equiv 0$. To simplify the notation, we keep it. We only keep $a_{ij}, i, j \leq n_1 + 1$ and obtain (C.6). In practice, we choose $l_0 = 8$ and the above construction provides a solution with tail decaying smoothly to 0 for $|y|_\infty \geq y_{n_1+l_0-1}$.

To solve the dynamic rescaling equations numerically (2.10)-(2.12) (see Section 7 Part I), we update the grid point value of ω_{n+1} at time t_{n+1} , and then use the above method to obtain a_{ij} .

For the density $\bar{\theta}_2$, the representation is similar

$$(C.7) \quad \bar{\theta}_2 = x \sum_{0 \leq i, j \leq n_1+1} a_{ij} B_{1,i}(x) B_{2,j}(y).$$

Here, we multiply x since $\bar{\theta}$ is even and vanishes $O(x^2)$ near $x = 0$.

For the stream function $\bar{\phi}_2^N$ (C.1), we choose $n_2 > n_1$ and represent it as follows

$$(C.8) \quad \bar{\phi}_2^N = \sum_{0 \leq i, j \leq n_2-1} a_{ij} \tilde{B}_{1,i}(x) \tilde{B}_{2,j}(y) \rho_p(y).$$

Instead of using the above extension to determine the extra coefficients, we perform an additional extrapolation for the basis in the far-field similar to (C.6)

$$\tilde{B}_{l,j}(z) = B_{l,j}(z), \quad j \leq n_2 - 8, \quad \tilde{B}_{l,j}(z) = B_j(z) + C_{2,n_2-j} B_{n_2}(z) + C_{1,n_2-j} B_{n_2+1}(z).$$

We multiply $\rho_p(y)$ given below to impose the Dirichlet boundary condition

$$(C.9) \quad \rho_p(y) = \arctan(1+y) - \arctan(1).$$

We can obtain the exact formulas of $\partial_x^i \rho_p$ using a symbolic computation. We use induction to obtain rigorous estimate of $\partial_x^i \rho_p$. See Section D.3.

We choose C_i in (C.4) of order $s_{i,j+1} - s_{i,j}$ as follows

$$(C.10) \quad C_i = y_1, i \leq 9, \quad C_i = (s_{i,4} - s_{i,2})/2, \quad i > 9,$$

so that the summand in (C.4) has order 1 for x in the support $[y_{i-3}, y_{i+3}]$. When we need to perform extrapolation for $a_n B_n, a_{n+1} B_{n+1}$ from $a_i B_i, i \leq n - 1$, e.g. (C.8), we modify the last few terms as follows

$$(C.11) \quad C_i = (y_n - y_{n-1})/100, \quad n - 9 \leq i.$$

We choose C_i to be constant for i close to 0 or i close to n_1 since we need to perform extrapolation, and the choice of the constant does not affect the extrapolation formula for a_{ij} .

Far-field angular profile. To represent the far-field angular profile of $\bar{\omega}_1, \bar{\theta}_1, \bar{\phi}_1^N$ (C.1), we design adaptive mesh $0 = \beta_0 < \beta_1 < \dots < \beta_m = \pi/2$, and use 8-th order Bspline to represent $\bar{\omega}, \bar{\zeta} = \frac{\bar{\theta}}{x_1}$

$$g(\pi/2 - \beta) = \sum_{i \geq 0} b_i B_{1,i}^{(8)}(\beta), \quad g_\phi(\pi/2 - \beta) = ((\pi/2)^2 - \beta^2) \sum_i b_i \tilde{B}_i^{(8)}(\beta), \quad \beta \in [0, \pi/2],$$

where $B_{1,i}^{(8)}$ is 8-th order Bspline (C.4) $k = 8$ with odd modification (C.5). Since $\bar{\omega}, \bar{\zeta}$ are odd in x , in the angular direction, this symmetry becomes odd in $\beta = \pi/2$. To impose it, we write g in terms of $\pi/2 - \beta$ and modify the first few B-spline B_i (C.4) following (C.5) so that $\tilde{B}_{1,i}$ is odd at $\beta = 0$. Then g is odd in $\beta = \pi/2$. The stream function $\bar{\phi}^N$ satisfies the boundary condition $\bar{\phi}^N(x, 0) = 0$. For the angular profile, we need $g_\phi(0) = 0$, and use the weight $\pi/2 - \beta$ to impose this condition. We further modify a few Bspline $B_{1,i}(\beta)$ supported near $\beta = \pi/2$ using 9-th

order extrapolation similar to (C.6) near $\beta = \pi/2$ and get $\tilde{B}_{1,i}(\beta)$. We choose the mesh β_i to be equi-spaced near $\beta = \pi/2$ and determine the coefficients for extrapolation similar to (C.6). We remark that to evaluate the derivative $\partial_{\beta}^k g$ at $\pi/2 - \beta$, we have the sign $(-1)^k$

$$(\partial_{\beta}^k g)(\pi/2 - \beta) = (-1)^k \partial_{\beta}^k g(\pi/2 - \beta) = (-1)^k \sum b_i \partial_{\beta}^k B_{1,i}^{(8)}(\beta).$$

We discuss how to obtain these angular profiles using the curve fitting in Section 7 in [14].

C.2. Estimate of the derivatives of piecewise polynomials. Our approximate steady state in a very large domain is represented as piecewise polynomials. We discuss how to estimate its derivatives. Suppose that we can evaluate a function f on finite many points. For example, f is an explicit function or a polynomial. To obtain a piecewise sharp bound of f on $I = [x_l, x_u]$, we use the following standard error estimate

$$(C.12) \quad \max_{x \in I} |f(x)| \leq \max(|f(x_l)|, |f(x_u)|) + \frac{h^2}{8} \|f_{xx}\|_{L^\infty(I)}, \quad h = x_u - x_l,$$

If we can obtain a rough bound for f_{xx} , as long as the interval I is small, i.e., h is small, the error part is small. Similarly, if we can obtain a rough bound for $\partial_x^{k+2} f$, using induction and the above estimate recursively,

$$\max_{x \in I} |\partial_x^i f(x)| \leq \max(|\partial_x^i f(x_l)|, |\partial_x^i f(x_u)|) + \frac{h^2}{8} \|\partial_x^{i+2} f\|_{L^\infty(I)},$$

for $i = k, k-1, \dots, 0$, we can obtain the sharp bound for $\partial_x^i f$ on I . We call the above method the second order method since the error term is second order in h .

C.2.1. Estimate a piecewise polynomial in 1D. Suppose that $p(x)$ is a piecewise polynomials on $x_0 < x_1 < \dots < x_n$ with degree d , e.g. Hermite spline. Denote $I_i = [x_i, x_{i+1}]$. Then $p(x)$ is a polynomial in each I_i with degree $\leq d$. Our goal is to estimate $\partial_x^k p(x)$ in I_i for all k by only finite many evaluations of $p(x)$ and its derivatives. Firstly, we have

$$\partial_x^k p(x) = 0, \quad k > d, \quad \partial_x^d p(x) = c_p,$$

for some constant c_p in I_i . Using induction from $k = d-1, d-2, \dots, 0$, we have

$$\max_{x \in I_i} |\partial_x^k p(x)| \leq \max(|\partial_x^k p(x_i)|, |\partial_x^k p(x_{i+1})|) + \frac{h_i^2}{8} \|\partial_x^{k+2} p\|_{L^\infty(I_i)}, \quad h_i = x_{i+1} - x_i.$$

Since we know $\partial_x^{d+1} p(x) = 0$ on I_i , using the above method, we can obtain the sharp piecewise bounds for all derivatives of $p(x)$ on I_i . Using the above approach, we can estimate the derivatives of the angular profile defined in Section 7.1 of Part I [14] rigorously.

C.2.2. Estimate a piecewise polynomial in 2D. Now, we generalize the above ideas to 2D so that we can estimate the approximate steady state (C.6). We assume that $p(x, y)$ is a piecewise polynomials in the mesh $Q_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ with degree d . That is, in Q_{ij} , $p(x, y)$ can be written as a linear combination of

$$x^k y^l, \quad \max(k, l) \leq d,$$

e.g. (C.6). For (C.6), we have $d = 5$. Similar to the 1D case, we have

$$\partial_x^k \partial_y^l p(x, y) = 0, \quad \max(k, l) > d.$$

Moreover, we know $\partial_x^{d-1} \partial_y^{d-1}$ is linear in x, y .

We use the following direct generalization of (C.12) to 2d

$$(C.13) \quad \max_{(x,y) \in Q} |f(x, y)| \leq \max_{\alpha, \beta=l, u} |f(x_\alpha, y_\beta)| + \frac{\|f_{xx}\|_{L^\infty(Q)}(x_u - x_l)^2}{8} + \frac{\|f_{yy}\|_{L^\infty(Q)}(y_u - y_l)^2}{8},$$

$$Q = [x_l, x_u] \times [y_l, y_u].$$

Denote

$$A_{kl} \triangleq \max_{Q_{ij}} \|\partial_x^k \partial_y^l p\|_{L^\infty(Q_{ij})}, B_{kl} \triangleq \max_{\alpha, \beta=l, u} |\partial_x^k \partial_y^l p(x_\alpha, y_\beta)|, \quad h_1 = x_{i+1} - x_i, \quad h_2 = y_{j+1} - y_j.$$

Since p is given, we can evaluate B_{kl} . Clearly, we have $A_{kl} = 0$ for $\max(k, l) > d$. For $k = d-1, d$, using (C.13) and induction in the order $l = d, d-1, d-2, \dots, 0$, we can obtain

$$A_{kl} \leq B_{kl} + \frac{1}{8}(h_1^2 A_{k+2,l} + h_2^2 A_{k,l+2}).$$

This allows us to bound A_{kl} for $k = d, d-1$ and all l . Similarly, we can bound A_{kl} for $l = d, d-1$ and all k .

For the remaining cases, we can use induction on $n = \max(k, l) = d-2, d-1, \dots, 0$ to estimate

$$A_{kl} \leq B_{kl} + \frac{1}{8}(h_1^2 A_{k+2,l} + h_2^2 A_{k,l+2}).$$

This allows us to estimate all derivatives of $p(x, y)$ in Q_{ij} .

C.2.3. Estimate a piecewise polynomial in 2D with weights. We consider how to estimate the derivatives of $f = \rho(y)p(x, y)$, where ρ is a given weight in y and $p(x, y)$ is the piecewise polynomials in 2D. For example, our construction of the stream function (C.8) has such a form. Firstly, we can estimate the derivatives of $p(x, y)$ using the method in Appendix C.2.2. For the weight ρ , we estimate its derivatives in Section D.3. Then, using the Leibniz rule (A.6) and the triangle inequality, we can estimate the derivatives f

$$|\partial_x^i \partial_y^j f| \leq \sum_{l \leq j} \binom{j}{l} |\partial_x^i \partial_y^l p(x, y)| |\partial_y^{j-l} \rho(y)|$$

for high enough derivatives.

Now, we plug the above bounds for $\partial_x^{i+2} \partial_y^j, \partial_x^i \partial_y^{j+2} f$ in (C.13) and evaluate $\partial_x^i \partial_y^j f$ on the grid points to obtain the sharp estimate of $\partial_x^i \partial_y^j f$.

C.3. Estimate of the far-field approximation. We estimate the derivatives of

$$g(x, y) = g(r, \beta) = A(r)B(\beta), \quad r = (x^2 + y^2)^{1/2}, \quad \beta = \arctan(y/x),$$

where (r, β) is the polar coordinate. The semi-analytic parts of $\bar{\omega}, \bar{\theta}$ have the above forms.

C.3.1. Formulas of the derivatives of g . Firstly, we use induction to establish

$$(C.14) \quad F_{i,j} \triangleq \partial_x^i \partial_y^j g(r, \beta) = \sum_{k+l \leq i+j} C_{i,j,k,l}(\beta) r^{-i-j+k} \partial_r^k A \partial_\beta^l B,$$

with $C_{i,j,k,l} = 0$, for $k < 0, l < 0$, or $k+l > i+j$. Let us motivate the above ansatz. Recall from (B.14) that

$$\partial_x = \cos \beta \partial_r - \frac{\sin \beta}{r} \partial_\beta, \quad \partial_y = \sin \beta \partial_r + \frac{\cos \beta}{r} \partial_\beta.$$

For each derivative ∂_x or ∂_y , we get the factor $\frac{1}{r}$ or a derivative ∂_r , which leads to the form $r^{-i-j+k} \partial_r^k A$. Moreover, we get a derivative ∂_β and some functions depending on β , which leads to the form $C_{i,j,k,l}(\beta) \partial_\beta^l B$.

For $D = \partial_x$ or ∂_y , a direct calculation yields

$$(C.15) \quad DF_{i,j} = \sum_{k+l \leq i+j} D(C_{i,j,k,l} r^{-i-j+k}) \cdot \partial_r^k A \partial_\beta^l B + C_{i,j,k,l} r^{-i-j+k} (D \partial_r^k A \cdot \partial_\beta^l B + \partial_r^k A \cdot D \partial_\beta^l B).$$

Using the formula of ∂_x, ∂_y , we get

$$\begin{aligned} \partial_x(C_{i,j,k,l}(\beta) r^{-i-j+k}) &= -\sin \beta \partial_\beta C_{i,j,k,l} r^{-i-j-1+k} + (k-i-j) \cos \beta C_{i,j,k,l} r^{-i-j-1+k}, \\ \partial_x \partial_r^k A &= \cos \beta \partial_r^{k+1} A, \quad \partial_x \partial_\beta^l B = -\frac{\sin \beta}{r} \partial_\beta^{l+1} B, \end{aligned}$$

Using $\partial_x F_{i,j} = F_{i+1,j}$ and comparing the above formulas and the ansatz (C.14), we yield

$$(C.16) \quad C_{i+1,j,k,l} = (k-i-j) \cos \beta C_{i,j,k,l} - \sin \beta \partial_\beta C_{i,j,k,l} + \cos \beta C_{i,j,k-1,l} - \sin \beta C_{i,j,k,l-1},$$

for $k \leq i + j$. Similarly, for $D = \partial_y$, plugging the following identities

$$\begin{aligned} \partial_y(C_{i,j,k,l}(\beta)r^{-i-j+k}) &= \cos\beta\partial_\beta C_{i,j,k,l}r^{-i-j-1+k} + (k-i-j)\sin(\beta)C_{i,j,k,l}r^{-i-j-1+k}, \\ \partial_y\partial_r^k A &= \sin\beta\partial_r^{k+1}A, \quad \partial_y\partial_\beta^l B = \frac{\cos\beta}{r}\partial_\beta^{l+1}B \end{aligned}$$

into (C.15) and then comparing (C.14) and (C.15), we yield

$$(C.17) \quad C_{i,j+1,k,l} = (k-i-j)\sin\beta C_{i,j,k,l} + \cos\beta\partial_\beta C_{i,j,k,l} + \sin\beta C_{i,j,k-1,l} + \cos\beta C_{i,j,k,l-1}.$$

The based case is given by

$$F_{0,0} = A(r)g(\beta), \quad C_{0,0,0,0} = 1.$$

Using induction and the above recursive formulas, we can derive $C_{i,j,k,l}(\beta)$ in (C.14).

C.3.2. Estimates of $F_{i,j}$. To estimate $F_{i,j}$, using (C.14) and triangle inequality, we only need to estimate $\partial_r^k A, \partial_\beta^l B(\beta)$, and $C_{i,j,k,l}(\beta)$. In our case, $B(\beta)$ is piecewise polynomials, whose estimates follow the method in Appendix (C.2.1). Function $A(r)$ is some explicit function, which will be constructed and estimated in Section D.1. To estimate $C_{i,j,k,l}(\beta)$ on $\beta \in [\beta_1, \beta_2]$, we use the second order estimate in (C.12) and the induction ideas in Section C.2.1. We can evaluate $C_{i,j,k,l}$ using its exact formula. It remains to bound $\partial_\beta^2 C_{i,j,k,l}$.

An important observation from (C.16), (C.15) is that $C_{i,j,k,l}$ is a polynomial on $\sin\beta$ and $\cos\beta$ with degree less than $i+j$, which can be proved easily using induction. In particular, we can write $C_{i,j,k,l}$ as follows

$$C_{i,j,k,l} = \sum_{0 \leq k \leq n} a_k \sin(k\beta) + b_k \cos(k\beta), \quad f \triangleq \partial_\beta^2 C_{i,j,k,l} = \sum_{1 \leq k \leq n} c_k \sin(k\beta) + d_k \cos(k\beta), \quad n = i+j$$

for some $a_k, b_k, c_k, d_k \in \mathbb{R}$. It is easy to see that $C_{i,j,k,l}$ is either odd or even in β depending on $j-l$, which implies $c_k \equiv 0$ or $d_k \equiv 0$. Using Cauchy-Schwarz's inequality, we get

$$\|f\|_\infty \leq \sum_{1 \leq k \leq n} (|c_k| + |d_k|) \leq \left(n \sum_{k \leq n} (c_k^2 + d_k^2) \right)^{1/2} = \left(\frac{n}{\pi} \int_0^{2\pi} f^2 \right)^{1/2},$$

where we have used orthogonality of $\sin kx, \cos kx$ and $\|f\|_{L^2}^2 = \pi \sum_{k \leq n} (c_k^2 + d_k^2)$ in the last equality. It is easy to see that f^2 is again a polynomial in $\sin\beta, \cos\beta$ with degree $\leq 2n$. We fix $M > 2n$. For any $0 \leq k < M$, it is easy to obtain

$$\frac{1}{2\pi} \int_0^{2\pi} e^{ikx} = \frac{1}{M} \sum_{j=1}^M \exp(i \frac{2kj}{M} \pi) = \delta_{k0}.$$

Using the above identity, we establish

$$\|g\|_{L^2}^2 = \frac{2\pi}{M} \sum_{j=1}^M |g(\frac{2j\pi}{M})|^2,$$

for any polynomial g in $\sin\beta, \cos\beta$ with degree $< M/2$. Hence, we prove

$$\|f\|_\infty \leq \left(\frac{2n}{M} \sum_{k=1}^M f^2(\frac{2j\pi}{M}) \right)^{1/2}.$$

The advantage of the above estimate is that to obtain the sharp bound of $C_{i,j,k,l}$, we only need to evaluate $C_{i,j,k,l}, f = \partial_\beta^2 C_{i,j,k,l}$ on finite many points.

C.3.3. *From polar coordinates to the Cartesian coordinate.* We want to obtain the piecewise estimate of $F_{p,q} = \partial_x^p \partial_y^q (A(r)g(\beta))$ on $Q_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, $1 \leq i, j \leq n$. Firstly, we partition the (r, β) coordinate into $r_1 < r_2 < \dots < r_{n_1}$, $0 = \beta_0 < \beta_1 < \dots < \beta_{n_2} = \frac{\pi}{2}$. Then we apply the methods in Section C.3 to bound $F_{p,q}$ on $S_{ij} \triangleq [r_i, r_{i+1}] \times [\beta_j, \beta_{j+1}]$. We cover Q_{ij} by $S_{k,l}$ and transfer the bound from (r, β) coordinate to (x, y) coordinate

$$\max_{x \in Q_{ij}} |F_{p,q}(x)| \leq \max_{S_{k,l} \cap Q_{ij} \neq \emptyset} \|F_{p,q}(r, \beta)\|_{L^\infty(S_{k,l})}$$

For $(r, \beta) \in Q_{i,j}$, we get

$$r \in [(x_i^2 + y_j^2)^{1/2}, (x_{i+1}^2 + y_{j+1}^2)^{1/2}], \quad \beta \in [\arctan \frac{y_j}{x_{i+1}}, \arctan \frac{y_{j+1}}{x_i}].$$

Therefore, we yield the necessary conditions for $Q_{i,j} \cap S_{k,l} \neq \emptyset$:

$$x_{i+1}^2 + y_{j+1}^2 \geq r_k^2, \quad x_i^2 + y_i^2 \leq r_u^2, \quad \arctan \frac{y_{j+1}}{x_i} \geq \beta_l, \quad \arctan \frac{y_j}{x_{i+1}} \leq \beta_{l+1}.$$

Given $Q_{i,j}$, we maximize $\|F_{p,q}\|_{L^\infty(S_{k,l})}$ over (k, l) satisfying the above bounds to control $\|F_{p,q}\|_{L^\infty(Q_{i,j})}$.

C.4. **Estimates of the residual error.** Let $\chi_{\bar{\varepsilon}} = 1 + O(|x|^4)$ be the cutoff function in (D.6). Firstly, we decompose the error of solving the Poisson equations $\bar{\varepsilon} = \bar{\omega} - (-\Delta)\bar{\phi}^N$ as follows

$$(C.18) \quad \begin{aligned} \bar{\varepsilon} &= \bar{\varepsilon}_1 + \bar{\varepsilon}_2, \quad \bar{\varepsilon}_2 = \bar{\varepsilon}_{xy}(0)\Delta\left(\frac{x^3 y}{2}\chi_{\bar{\varepsilon}}\right), \quad \mathbf{u}(\bar{\varepsilon}_2) = \nabla^\perp(-\Delta)^{-1}\bar{\varepsilon}_2 = \frac{1}{2}\bar{\varepsilon}_{xy}(0)\nabla^\perp(x^3 y \chi_{\bar{\varepsilon}}), \\ \mathbf{u}(\bar{\varepsilon}) &= \mathbf{u}(\bar{\varepsilon}_1) + \mathbf{u}(\bar{\varepsilon}_2) = \mathbf{u}_A(\bar{\varepsilon}_1) + (\hat{\mathbf{u}}(\bar{\varepsilon}_1) + \mathbf{u}(\bar{\varepsilon}_2)) \triangleq \mathbf{u}_A(\bar{\varepsilon}_1) + \mathbf{u}_{loc}(\bar{\varepsilon}), \end{aligned}$$

where $\hat{\mathbf{u}}$ is the approximation term for \mathbf{u} defined in Section 4.3 in Part I [14]. We perform the above correction near 0 so that $\bar{\varepsilon}_1 = O(|x|^3)$ near 0. We perform a similar decomposition for $(\nabla \mathbf{u})_A$. Note that we do not have $\partial_{x_i} \mathbf{u}_A = (\partial_{x_i} \mathbf{u})_A$. Using the above decomposition and the notation (3.4), we can rewrite the residual error \bar{F}_i (2.14) with rank-one correction as follows

$$\bar{F}_i - D_i^2 \bar{F}_i(0) f_{\chi,i} = \bar{F}_{loc,i} + \mathcal{B}_{op,i}((\mathbf{u}_A(\bar{\varepsilon}_1), (\nabla \mathbf{u})_A(\bar{\varepsilon}_1)), \bar{W}),$$

where $D^2 = (\partial_{xy}, \partial_{xy}, \partial_x^2)$ is defined in (3.23) and $\bar{F}_{loc,i}$ is defined below in (C.19). Since $\mathbf{u}_A(\bar{\varepsilon}_1) = O(|x|^3)$, $(\nabla \mathbf{u})_A(\bar{\varepsilon}_1) = O(|x|^2)$ (see Section 4.3 in Part I [14] for these properties of $\mathbf{u}_A = \mathbf{u} - \hat{\mathbf{u}}$), from (3.4) and (C.18), we get

$$\mathcal{B}_{op,i}((\mathbf{u}_A(\bar{\varepsilon}_1), (\nabla \mathbf{u})_A(\bar{\varepsilon}_1)), \bar{W}) = O(|x|^3), \quad u_x(\bar{\varepsilon}_2)(0) = 0, \quad u_{x,A}(\bar{\varepsilon}_1)(0) = 0.$$

Using these properties of $\mathcal{B}_{op,i}$, we define $\bar{F}_{loc,i}$ as follows

(C.19)

$$\bar{F}_{loc,i} = II_i - D_i^2 II_i(0) f_{\chi,i}, \quad II_i = \bar{F}_i - \mathcal{B}_{op,i}((\mathbf{u}_A(\bar{\varepsilon}_1), (\nabla \mathbf{u})_A(\bar{\varepsilon}_1)), \bar{W}),$$

$$\mathbf{u}(\bar{\omega}) = \bar{\mathbf{u}} = \bar{\mathbf{u}}^N + \mathbf{u}_{loc}(\bar{\varepsilon}) + \mathbf{u}_A(\bar{\varepsilon}_1), \quad \bar{c}_\omega = \bar{c}_\omega^N + u_x(\bar{\varepsilon}_1)(0), \quad \bar{c}_\omega^N \triangleq \frac{\bar{c}_l}{2} + \bar{u}_x^N(0), \quad c_\omega(\bar{\varepsilon}_1) \triangleq u_x(\bar{\varepsilon}_1)(0),$$

$$II_1 = -(\bar{c}_l x + \bar{\mathbf{u}}^N + \mathbf{u}_{loc}(\bar{\varepsilon})) \cdot \nabla \bar{\omega} + \bar{\theta}_x + (\bar{c}_\omega^N + \bar{c}_\omega(\bar{\varepsilon}_1)) \bar{\omega},$$

$$II_2 = -(\bar{c}_l x + \bar{\mathbf{u}}^N + \mathbf{u}_{loc}(\bar{\varepsilon})) \cdot \nabla \bar{\theta}_x + 2(\bar{c}_\omega^N + \bar{c}_\omega(\bar{\varepsilon}_1)) \bar{\theta}_x - (\bar{u}_x^N + u_{x,loc}(\bar{\varepsilon})) \bar{\theta}_x - (\bar{v}_x^N + v_{x,loc}(\bar{\varepsilon})) \bar{\theta}_y,$$

$$II_2 = -(\bar{c}_l x + \bar{\mathbf{u}}^N + \mathbf{u}_{loc}(\bar{\varepsilon})) \cdot \nabla \bar{\theta}_y + 2(\bar{c}_\omega^N + c_\omega(\bar{\varepsilon}_1)) \bar{\theta}_y - (\bar{u}_y^N + u_{y,loc}(\bar{\varepsilon})) \bar{\theta}_x - (\bar{v}_y^N + v_{y,loc}(\bar{\varepsilon})) \bar{\theta}_y,$$

where $f_{\chi,i}$ is defined in (D.6), and we have used $\bar{c}_\theta = \bar{c}_l + 2\bar{c}_\omega$ (2.14), (2.11) for \bar{c}_ω . The above decomposition is essentially the same as (3.12). We apply the functional inequalities in Section 4 to estimate the nonlocal terms $\mathbf{u}_A(\bar{\varepsilon}_1)$, $(\nabla \mathbf{u})_A(\bar{\varepsilon}_1)$, and combine the estimate of $\mathcal{B}_{op,i}((\mathbf{u}_A, (\nabla \mathbf{u})_A), \bar{W})$ with the energy estimate. See Section 5.8 in Part I [14] for more details about the decompositions and estimates.

The terms II_i depend on the profile $\bar{\omega}, \bar{\theta}, \bar{\varepsilon}$ locally. Using the decomposition (C.18), we can further decompose the above II_i as follow

$$II_i = II_i^N + II_i(\bar{\varepsilon}_1) + II_i(\bar{\varepsilon}_2), \quad II_i(\bar{\varepsilon}_1) = \mathcal{B}_{op,i}(\hat{\mathbf{u}}(\bar{\varepsilon}_1), \widehat{\nabla \mathbf{u}}(\bar{\varepsilon}_1), \bar{W}), \quad II_i(\bar{\varepsilon}_2) = \mathcal{B}_{op,i}(\mathbf{u}(\bar{\varepsilon}_2), \nabla \mathbf{u}(\bar{\varepsilon}_2), \bar{W}),$$

where II_i^N contain the terms in II_i except the $u_{loc}, u(\bar{\varepsilon}_1)$ terms.

For $\hat{\mathbf{u}}(\bar{\varepsilon}_1)$, it is a finite rank operator on $\bar{\varepsilon}_1$, and we can write it as

$$\hat{\mathbf{u}}(\bar{\varepsilon}_1) = \sum_{i=1}^n a_i(\bar{\varepsilon}_1) \bar{g}_i(x) \triangleq C_{\mathbf{u}0}(x) u_x(\bar{\varepsilon}_1)(0) + \tilde{\mathbf{u}}(\bar{\varepsilon}_1), \quad a_i(\bar{\varepsilon}_1) = \int_{\mathbb{R}_2^{++}} \bar{\varepsilon}_1(y) q_i(y) dy,$$

for some functions $\bar{g}_i(x)$ and $q_i(y)$, where $C_{\mathbf{u}0}(x)$ is given in (4.5), and $\tilde{\mathbf{u}}(\bar{\varepsilon}_1)$ denotes other modes with $O(|x|^3)$ vanishing order near 0. See Section 4.3 in [14] for definition. We can obtain more regular estimates, e.g. C^3 estimates, of $\hat{\mathbf{u}}(\bar{\varepsilon}_1)$ since $\bar{g}_1(x)$ is smooth. Similarly, we decompose $\widehat{\nabla \mathbf{u}}(\bar{\varepsilon}_1)$. We obtain piecewise estimates of $\partial_x^i \partial_y^j \bar{\varepsilon}_1$, $i + j \leq 1$ following the methods in Section 3.6 and Section 8 in the supplementary material II [12] (attached to [11]) and then the above integrals on $\bar{\varepsilon}_1$. The main term in $\hat{\mathbf{u}}(\bar{\varepsilon}_1)$ is $C_{\mathbf{u}0} u_x(0)$ with

$$(C.20) \quad \begin{aligned} u_x(\bar{\varepsilon}_1)(0) &= u_x(\bar{\varepsilon})(0) = -\frac{4}{\pi} \int_{\mathbb{R}_2^{++}} \bar{\varepsilon}(y) \frac{y_1 y_2}{|y|^4} dy, \\ u_x(\bar{\varepsilon}_2)(0) &= -\varepsilon_{xy}(0)/2 \cdot \partial_y(x^3 y \chi_{\bar{\varepsilon}})|_{(0,0)} = 0. \end{aligned}$$

Since the kernel $\frac{y_1 y_2}{|y|^4}$ has a slow decay for large $|y|$ (not in L^1), we need to estimate $u_x(\bar{\varepsilon})(0)$ carefully, using Simpson's rule. See Section 6.4.2 in supplementary material II for Part II [12].

Using the above decomposition, we further decompose $\hat{\mathbf{u}}(\bar{\varepsilon}_1)$

$$II_i(\bar{\varepsilon}_1) = u_x(\bar{\varepsilon})(0) \mathcal{B}_{op,i}(C_{\mathbf{u}0}(x), C_{\nabla \mathbf{u}0}(x), \bar{W}) + \mathcal{B}_{op,i}(\tilde{\mathbf{u}}, \widetilde{\nabla \mathbf{u}}, \bar{W}) \triangleq II_{i,M}(\bar{\varepsilon}_1) + II_{i,R}(\bar{\varepsilon}_1).$$

Since D_i^2 is linear, we estimate each term $g_i - D_i^2 g_i(0) f_{\chi,i}$ for $g_i = II_{i,M}(\bar{\varepsilon}_1), II_{i,R}(\bar{\varepsilon}_1), II_i^N, II_i(\bar{\varepsilon}_2)$ to bound $\mathcal{F}_{loc,i}$. To estimate $II_{i,R}$, since $\tilde{\mathbf{u}}(\bar{\varepsilon}_1) = O(|x|^3)$ near 0, (see Section 4.3 in [14]), we get $D_i^2 II_{i,R}(\bar{\varepsilon}_1) = O(|x|^3)$ and estimate

$$\tilde{\mathbf{u}}(\bar{\varepsilon}_1) \rho_{10}, \partial_i \tilde{\mathbf{u}}(\bar{\varepsilon}_1) \rho_{20}, \widetilde{\nabla \mathbf{u}}(\bar{\varepsilon}_1) \rho_{20}, \partial_i \widetilde{\nabla \mathbf{u}}(\bar{\varepsilon}_1) \rho_3, \rho_4 \tilde{\mathbf{u}}(\bar{\varepsilon}_1)$$

for ρ_{i0} (A.2) with $\rho_{i0} \sim |x|^{-4+i}$, $i \leq 3$ near 0 using the C^3 bounds of $\tilde{\mathbf{u}}, \widetilde{\nabla \mathbf{u}}$. Note that $\partial_i \tilde{\mathbf{u}} \neq \widetilde{\partial_i \mathbf{u}}$. The former is the derivative of $\tilde{\mathbf{u}}$, and the later is the approximation term for $\partial_i \mathbf{u}$. With the above weighted estimate, we can bound a typical terms, e.g. $\widetilde{u_x \theta_x} \varphi_2$ in $II_{i,R}(\bar{\varepsilon}_1) \varphi_2$ as follows

$$\widetilde{u_x \theta_x} \varphi_2 = \widetilde{u_x} \rho_{20} \cdot (\bar{\theta}_x \frac{\varphi_2}{\rho_{20}}), \quad \partial_x (\widetilde{u_x \theta_x}) \rho = (\partial_x \widetilde{u_x} \bar{\theta}_x + \widetilde{u_x} \partial_x \bar{\theta}_x) \rho = \partial_x \widetilde{u_x} \rho_3 \cdot \frac{\bar{\theta}_x \rho}{\rho_3} + \widetilde{u_x} \rho_{20} \cdot \frac{\partial_x \bar{\theta}_x \rho}{\rho_{20}},$$

where φ_2 is given in (A.2). Each term A, B in the above products $A \cdot B$ is regular and we estimate each term and then the product to bound weighted L^∞ and C^1 norm of $II_{i,R}(\bar{\varepsilon}_1)$.

The remaining part in $II_i^N, II_{i,M}(\bar{\varepsilon}_1), II_i(\bar{\varepsilon}_2)$ depends on $(\bar{\phi}^N, \bar{\omega}, \bar{\theta})$ locally and are given functions. To estimate the weighted L^∞ and $C^{1/2}$ norms of $g_i - D_i^2 g_i(0) f_{\chi,i} = O(|x|^3)$ with $g = II_{i,M}(\bar{\varepsilon}_1), II_i(\bar{\varepsilon})$, we follow the methods in Sections 3.6, 3.7 with $\partial_t \bar{\omega} = \partial_t \bar{\theta} = 0$.

Estimate in the far-field. Since $\bar{\omega}, \bar{\theta}$ are supported globally, we need to estimate the error in the far-field. Recall the formulas of $\bar{\omega}, \bar{\omega}_1, \bar{\theta}, \bar{\theta}_1$ from (C.1). We consider $|x|_\infty \geq R_1 \geq 10^{12} > 10\alpha_2$ beyond the support of $\bar{\omega}_2, \bar{\theta}_2, \bar{\phi}_2^N, \bar{\phi}_3^N, \bar{\phi}_{cor}^N$ (C.1) so that $\chi(r) = 1$ (D.4) and

$$\bar{\omega} = \bar{\omega}_1 = \bar{g}_1(\beta) r^{\bar{\alpha}_1}, \quad \bar{\theta} = \bar{\theta}_1 = r^{1+2\bar{\alpha}_1} \bar{g}_2(\beta), \quad \bar{\phi}^N = \bar{\phi}_1^N = r^{2+\bar{\alpha}_1} \bar{f}(\beta).$$

We estimate the angular derivatives of $f(\beta), g_i(\beta)$ using the methods in Section C.2.1. Using the above representation, $x \cdot \nabla r^\beta = r \partial_r r^\beta = \beta r^\beta$, $x \cdot \nabla(\partial \bar{\theta}_1) = 2\bar{\alpha}_1(\partial \bar{\theta}_1)$, $x \cdot \nabla \bar{\omega}_1 = \bar{\alpha}_1 \bar{\omega}_1$, $\bar{c}_\omega = \bar{c}_\omega^N + \bar{c}_\omega^\varepsilon$ (3.11), and separating \mathbf{u}^N and \mathbf{u}_{loc} in (C.19), for $|x|_\infty \geq 10^{12}$, we obtain

$$\begin{aligned} \bar{\mathcal{F}}_{loc,1} &= \left((\bar{c}_\omega^N - \bar{c}_l \bar{\alpha}_1) \bar{\omega}_1 - \bar{\mathbf{u}}^N \cdot \nabla \bar{\omega}_1 + \bar{\theta}_{1,x} \right) + \bar{c}_\omega^\varepsilon \bar{\omega}_1 - \mathbf{u}_{loc} \cdot \nabla \bar{\omega}_1 \triangleq I_{11} + I_{12}, \\ \bar{\mathcal{F}}_{loc,2} &= \left((2\bar{c}_\omega^N - 2\bar{c}_l \bar{\alpha}_1) \bar{\theta}_{1,x} - \partial_x (\bar{\mathbf{u}}^N \cdot \nabla \bar{\theta}_1) \right) + 2\bar{c}_\omega^\varepsilon \bar{\theta}_{1,x} - \mathbf{u}_{loc} \cdot \nabla \bar{\theta}_{1,x} - u_{x,loc} \bar{\theta}_x - v_{x,loc} \bar{\theta}_y \triangleq I_{21} + I_{22}, \\ \bar{\mathcal{F}}_{loc,3} &= \left((2\bar{c}_\omega^N - 2\bar{c}_l \bar{\alpha}_1) \bar{\theta}_{1,y} - \partial_x (\bar{\mathbf{u}}^N \cdot \nabla \bar{\theta}_1) \right) + 2\bar{c}_\omega^\varepsilon \bar{\theta}_{1,y} - \mathbf{u}_{loc} \cdot \nabla \bar{\theta}_{1,y} - u_{y,loc} \bar{\theta}_x - v_{y,loc} \bar{\theta}_y \triangleq I_{31} + I_{32}, \end{aligned}$$

where we have simplified $\mathbf{u}_{loc}(\bar{\varepsilon})$ as \mathbf{u}_{loc} and used $f_{\chi,i} = 0$ (D.6), $\bar{\mathcal{F}}_{loc,i} = II_i$ (C.19) since $f_{\chi,j}$ is supported near 0. The terms I_{11}, I_{21}, I_{31} are local with the form $r^\gamma q(\beta)$ for some angular function q and decay rate γ . We estimate their piecewise L^∞ and derivative bounds using

(B.14). From our choice of $\bar{\alpha}_1$ (C.1), $\bar{c}_\omega^N - \bar{c}_l \bar{\alpha}_1$ is very small. Thus the first term in I_{11}, I_{21}, I_{31} is small. The second term in I_{11}, I_{21}, I_{31} has faster decay rates $r^{2\bar{\alpha}_1}, r^{3\bar{\alpha}_1}$ and is also very small.

Estimate of the velocity approximation. From (C.18), since $\bar{\varepsilon}_2$ is supported near 0, we get $\mathbf{u}_{loc} = \hat{\mathbf{u}}(\varepsilon_1)$. For I_{j2} in the above decomposition in the far-field, it remains to estimate

$$(C.21) \quad \bar{c}_\omega^e \bar{\omega} - \hat{\mathbf{u}}(\bar{\varepsilon}_1) \cdot \nabla \bar{\omega}, \quad 2\bar{c}_\omega^e \bar{\theta}_x - \hat{\mathbf{u}}_x(\bar{\varepsilon}_1) \cdot \nabla \bar{\theta} - \hat{\mathbf{u}}(\bar{\varepsilon}_1) \cdot \nabla \bar{\theta}_x, \quad 2\bar{c}_\omega^e \bar{\theta}_y - \hat{\mathbf{u}}_y(\bar{\varepsilon}_1) \cdot \nabla \bar{\theta} - \hat{\mathbf{u}}(\bar{\varepsilon}_1) \cdot \nabla \bar{\theta}_x.$$

Note that $c_\omega(\bar{\varepsilon}_1) = c_\omega(\bar{\varepsilon})$ (C.20) and $c_\omega(\bar{\varepsilon}) = \bar{c}_\omega^e$ in our notation. For any $a \in \mathbb{R}$, we estimate

$$A(f, g) = ag - \hat{\mathbf{u}}(f) \cdot \nabla g, \quad B_i(f, g) = 2a\partial_i g - \hat{\mathbf{u}}(f) \cdot \nabla \partial_i g - \widehat{\partial_i \mathbf{u}}(f) \cdot \nabla g, \quad i = 1, 2.$$

for $|x|_\infty \geq R_1$. From Sections 4.3.2–4.3.3 in Part I [14], for $|x|_\infty \geq R_1$, $\hat{\mathbf{u}}, \widehat{\nabla \mathbf{u}}$ reduce to

$$\begin{aligned} \hat{u}(f) &= x_1 I_{far}(f), & \hat{v}(f) &= -x_2 I_{far}(f), & \widehat{\partial_1 u}(f) &= I_{far}(f), & \widehat{\partial_2 v}(f) &= -I_{far}(f), \\ \widehat{\partial_2 u}(f) &= \widehat{\partial_1 v}(f) = 0, & I_{far}(f) &\triangleq -\frac{4}{\pi} \int_{\max(y_1, y_2) \geq R_n} \frac{y_1 y_2}{|y|^4} \omega(y) dy, \end{aligned}$$

where $R_n = 1024 \cdot 64h_x$ is the largest threshold. Denote $b = I_{far}(f)$. A direct calculation yields

(C.22)

$$\begin{aligned} A(f, g) &= (a - b)g + b(g - x_1 \partial_1 g + x_2 \partial_2 g), \\ B_1(f, g) &= 2a\partial_1 g - b\partial_1 g - bx_1 \partial_{11} g + bx_2 \partial_{12} g = (2a - 2b)\partial_1 g + b(\partial_1 g - x_1 \partial_{11} g + x_2 \partial_{12} g), \\ B_2(f, g) &= 2a\partial_2 g + b\partial_2 g - bx_1 \partial_{12} g + bx_2 \partial_{22} g = (2a - 2b)\partial_2 g + b(3\partial_2 g - x_1 \partial_{12} g + x_2 \partial_{22} g). \end{aligned}$$

Therefore, we only need to bound the functions following Section C.2, e.g. $g - x_1 \partial_1 g + x_2 \partial_2 g$ and g , and the functional $b(f)$ and a . We apply these estimates for (C.21) with $a = \bar{c}_\omega^e, f = \bar{\varepsilon}_1, g = \bar{\omega}, \bar{\theta}$.

APPENDIX D. ESTIMATE OF EXPLICIT FUNCTIONS

In this section, we estimate the derivatives of several explicit or semi-explicit functions using induction, including several cutoff functions used in the estimates and the weight in the stream function (C.8).

D.1. Estimate of the radial functions.

D.1.1. *Estimate of the cutoff function.* We estimate the derivatives of the cutoff function

$$(D.1) \quad \chi_e(x) = \left(1 + \exp\left(\frac{1}{x} + \frac{1}{x-1}\right)\right)^{-1},$$

where e is short for *exponential*. In our verification, it involves high order derivatives of χ_e . Although χ_e is explicit, its formula is complicated and is difficult to estimate. Instead, we use the structure of $\partial_x^i \chi_e$ and induction to estimate $\partial_x^i \chi_e$. Denote

$$p(x) = \frac{1}{x} + \frac{1}{x-1}, \quad f = \frac{1}{1+x}, \quad \chi_e = f(e^p).$$

Firstly, we use induction to derive

$$d_x^k \chi_e = \sum_{i=1}^k (\partial^i f)(e^p) e^{ip} Q_{k,i}(x),$$

where $Q_{k,i} = 0$ for $i > k, i < 0$. A direct calculation yields

$$\begin{aligned} \partial \sum_{i=1}^k \partial^i f e^{ip} Q_{k,i}(x) &= \sum_{i=1}^k (\partial^{i+1} f)(e^p) \cdot p' e^p e^{ip} Q_{k,i} + (\partial^i f) \partial_x (e^{ip} Q_{k,i}) \\ &= \sum_{i=1}^k (\partial^{i+1} f)(e^p) \cdot e^{(i+1)p} p' Q_{k,i} + (\partial^i f) e^{ip} (ip' Q_{k,i} + Q'_{k,i}). \end{aligned}$$

Comparing the above two equations, we derive

$$Q_{k+1,i} = p' Q_{k,i-1} + ip' Q_{k,i} + Q'_{k,i}.$$

The first few terms in $Q_{k,i}$ are given by

$$Q_{0,0} = 1, \quad Q_{1,1} = p', \quad Q_{1,0} = 0.$$

It is not difficult to see that $Q_{k,i}$ is a polynomial of $\partial_x^j p, j \leq k$ with non-negative coefficients. We derive the expression of $Q_{k,i}$ in terms of $\partial_x^j p, j \leq k$ symbolically. Thus, using triangle inequality, we only need to bound $\partial_x^j p$. We have

$$|\partial_x^n p(x)| = n!|x^{-n-1} + (x-1)^{-n-1}| \leq n!(|z|^{-n-1} + 2^{n+1}), \quad z = \min(|x|, |1-x|).$$

If n is even, x^{-n-1} and $(x-1)^{-n-1}$ have different sign, and we get better estimate

$$|\partial_x^n p(x)| \leq n! \max(|x|^{-n-1}, |x-1|^{-n-1}) = n! \cdot z^{-n-1}.$$

Substituting the above bounds into the formula of $Q_{k,i}$, we can obtain the upper bound $Q_{k,i}^u(x)$ for $Q_{k,i}(x)$, which is a polynomial of z^{-1} with positive coefficient. Since each term in $Q_{k,i}$ is given by $c_{i_1, i_2, \dots, i_m} \prod_{j=1}^m \partial_x^{i_j} p$ with $\sum i_j = k$, the above estimate implies

$$|c_{i_1, i_2, \dots, i_m} \prod_{j=1}^m \partial_x^{i_j} p| \leq c_{i_1, i_2, \dots, i_m} \prod_{j=1}^m i_j! (|z|^{-i_j-1} + 2^{i_j+1}).$$

Since $m \leq k$, the highest order of z^{-1} in the upper bound is bounded by $2k$. Thus, we obtain that $Q_{k,i}^u$ is a polynomial in z^{-1} with $\deg Q_{k,i}^u \leq 2k$. Next, we bound

$$|e^{ip} Q_{k,i}| \leq e^{ip} Q_{k,i}^u.$$

For $k \leq 20, x \geq 1 - \frac{1}{2k} \geq \frac{1}{2}, z^{-1} = |x-1|^{-1} \geq 2k$, a direct calculation implies that $e^{ip(x)} Q_{k,i}^u(x)$ is decreasing. In fact, for $l \leq 2k$, we have $z = |x-1| = 1-x$ and

$$\begin{aligned} \partial_x(\exp(ip(x))(1-x)^{-l}) &= \exp(ip(x))(ip'(1-x)^{-l} + l(1-x)^{-l-1}) \\ &= \exp(ip(x)) \left(-\frac{i}{x^2} - \frac{i}{(x-1)^2} + l(1-x)^{-1} \right) (1-x)^{-l} \leq 0. \end{aligned}$$

In the last inequality, we have used $-\frac{i}{1-x} + l \leq -2ki + 2k \leq 0$.

Note that $|(\partial_x^i f)(e^p)| = i!(1+e^p)^{-i-1} \leq i!$. Thus, for $x \in [x_l, x_u]$ with x_l close to 1, we get

$$|\partial_x^k \chi_e(x)| \leq \sum_{i=1}^k |(\partial_x^i f)(e^p)| e^{ip(x)} Q_{k,i}^u(x) \leq \sum_{i=1}^k i! \frac{e^{ip(x)}}{(1+e^p)^{i+1}} Q_{k,i}^u(x) \leq \sum_{i=1}^k i! e^{ip(x_l)} Q_{k,i}^u(x_l).$$

For x away from 1, we use monotonicities of p, Q^u and the above estimate to estimate piecewise bounds of $\partial_x^k \chi_e(x)$. Using the above derivatives bound, the symbolic formula of $\partial_x^k \chi_e$, and the refined second order estimate in Section C.2.1, we can obtain sharp bounds for $\partial_x^k \chi_e$. Remark that we only apply the above estimate to $k \leq 15$.

D.1.2. Estimate of polynomial decay functions. For cutoff function $\chi_e(\frac{|x|-a}{b})$ based on the exponential cutoff function (D.1), it has rapid change from $|x| \leq a$ to $|x| \geq a+b$, which is not very smooth in the computational domain if there are not enough mesh for x with $a \leq |x| \leq b$. We apply these cutoff functions to the far-field, e.g. $|x| \geq 10$, where the mesh is relatively sparse. Thus, we need another function similar to a cutoff function that has a slower change than the exponential cutoff function. We consider

$$(D.2) \quad \chi(x) = \frac{x^7}{(1+x^2)^{7/2}}, \quad x \in \mathbb{R}_+.$$

and will use its rescaled version, e.g., $\chi(\frac{x-a}{b})$, in our verification.

Firstly, we use induction to derive

$$\partial_x^k \chi = \frac{p_k(x)}{(1+x^2)^{7/2+k}}, \quad p_0 = x^7.$$

where $p_k(x)$ is a polynomial. A direct calculation yields

$$\partial_x^{k+1} \chi = \frac{p_k'(x)(1+x^2) - (\frac{7}{2} + k) \cdot 2xp_k(x)}{(1+x^2)^{7/2+k+1}}.$$

Comparing the above two formulas, we yield

$$p_{k+1} = p'_k(1+x^2) - (7+2k)xp_k(x).$$

The first few terms are given by $p_0 = x^7, p_1 = 7x^6$. Using the recursive formula and $\deg p_1 = 6$, we yield

$$(D.3) \quad \deg p_{k+1} \leq \deg p_k + 1, \quad \deg p_k \leq k + 5, \quad k \geq 1.$$

Since p_k is a polynomial, the above recursive formula shows that p_{k+1} is also a polynomial.

To estimate $\partial_x^k \chi$, we decompose p_k into the positive and the negative parts. Suppose that $p_k = \sum_i a_i x^i$. We have

$$p_k = p_k^+ - p_k^-, \quad p_k^+ = \sum a_i^+ x^i, \quad p_k^- = \sum a_i^- x^i.$$

For $x \geq 0$, p_k^+, p_k^- are increasing. Thus, for $x \in [x_l, x_u]$, we get

$$|\partial_x^k \chi| \leq \frac{\max(p_k^+(x_u) - p_k^-(x_l), p_k^-(x_u) - p_k^+(x_l))}{(1+x_l^2)^{7/2+k}}.$$

Next, we estimate $\partial_x^k \chi$ for large x . For $x \geq 2, k \geq 1$ and any polynomial $q(x)$ with non-negative coefficients and $\deg q \leq k + 5$, we yield

$$xq' \leq (k+5)q, \quad \frac{q'(1+x^2)}{(7+2k)xq} \leq \frac{(1+x^2)(k+5)}{(7+2k)x^2} \leq \frac{5(k+5)}{4(7+2k)} < 1.$$

The first inequality follows by comparing the coefficients of xq' and $(k+5)q$, which are nonnegative. It follows

$$\partial_x \frac{q}{(1+x^2)^{7/2+k}} = \frac{q'(1+x^2) - (7/2+k)2xq}{(1+x^2)^{7/2+k+1}} \leq 0, \quad k \geq 1, \quad x \geq 2.$$

Thus $\frac{q}{(1+x^2)^{7/2+k}}$ is decreasing. For $k \geq 1$ and $x \geq x_l \geq 2$, using (D.3) and the monotonicity, we yield

$$|\partial_x^k \chi| \leq \frac{p_k^+(x) + p_k^-(x)}{(1+x^2)^{7/2+k}} \leq \frac{p_k^+(x_l) + p_k^-(x_l)}{(1+x_l^2)^{7/2+k}}$$

For $k = 0$, the estimate is trivial: $\chi(x) \leq 1$. Using these higher order derivative bounds, we can use the discrete values of $\partial_x^k \chi$ and the bound for $\partial_x^{k+2} \chi$ to obtain sharp bounds of $\partial_x^k \chi$.

Note that $\chi_1(x-a) = \frac{(x-a)_+^7}{(1+(x-a)^2)^{7/2}}$ is only $C^{6,1}$. Suppose that $a \in [x_l, x_u]$. Since χ_1 is smooth on $x \leq a$ and on $x \geq a$, we can still use first order estimate to estimate $\partial_x^k \chi_1$ as follows

$$|\partial_x^k \chi_1(x)| \leq \max_{\alpha \in \{l, u\}} |\partial_x^k \chi_1(x_\alpha)| + \max(\|\partial_x^{k+1} \chi_1\|_{L^\infty[x_l, a]}, \|\partial_x^{k+1} \chi_1\|_{L^\infty[a, x_u]}) |x_u - x_l|.$$

D.1.3. Radial cutoff function. Now, we construct the radial cutoff functions for the far-field approximation terms of ω and ϕ as follows

$$(D.4) \quad \chi(r) = \chi_1(1 - \chi_2) + \chi_2, \quad \chi_1(r) = \chi_{rati}\left(\frac{r - a_1}{l_1^{1/2}}\right), \quad \chi_2(r) = \chi_{exp}\left(\frac{r - a_2}{9a_2}\right),$$

$$a_1 = 10, \quad l_1 = 50000, \quad a_2 = 10^5,$$

where χ_{exp} and χ_{rati} are defined in (D.1) and (D.2), respectively. Using the estimates of χ_{rati}, χ_{exp} established in the last two sections, the Leibniz rule (A.6), and (C.12), we can evaluate χ on the grid points and estimate its derivative bounds.

D.2. Cutoff function near the origin. For the cutoff function $\kappa(x)$ used in Section 3, we choose it as follows

$$(D.5) \quad \kappa(x; a, b) = \kappa_1\left(\frac{x}{a}\right)\left(1 - \chi_e\left(\frac{x}{b}\right)\right), \quad \kappa_1(x) = \frac{1}{1+x^4}, \quad \kappa_*(x) = \kappa\left(x; \frac{1}{3}, \frac{3}{2}\right),$$

where χ_e is the cutoff function chosen in (D.1). We mostly use the cutoff κ_* . Since $\chi_e(y) = 1$ for $y \geq 1$ and $\chi_e(y) = 0$ for $y \leq 0$. The above cutoff function is supported in $x \leq a_2$. Using Taylor expansion, we have the following properties for κ

$$\kappa_1(x/a_1) = 1 + O(x^4), \quad \kappa(x) = 1 + O(x^4).$$

For the cutoff functions χ_{NF} in Section 4.2.1 in Part I [14], $\chi_{\bar{\varepsilon}}$ in (C.18), and $\chi_{\hat{\varepsilon}}$ in (3.42), we choose

$$(D.6) \quad \begin{aligned} \chi_{\bar{\varepsilon}}(x, y) &= \kappa(x; \nu_{\bar{\varepsilon},1}, \nu_{\bar{\varepsilon},2})\kappa(y; \nu_{\bar{\varepsilon},1}, \nu_{\bar{\varepsilon},2}), \quad \nu_{\bar{\varepsilon},1} = 1/192, \quad \nu_{\bar{\varepsilon},2} = 3/2, \\ \chi_{\hat{\varepsilon}}(x, y) &= \kappa_*(x)\kappa_*(y), \quad \chi_{NF}(x, y) = \kappa(x; 2, 10)\kappa(y; 2, 10), \\ f_{\chi,1} &= \Delta\left(\frac{xy^3}{6}\chi_{NF}(x, y)\right), \quad f_{\chi,2} = xy\chi_{NF}(x, y), \quad f_{\chi,3} = \frac{x^2}{2}\chi_{NF}(x, y), \end{aligned}$$

For the cutoff function in the stream function (C.1), we choose

$$(D.7) \quad \chi_{\phi} = \kappa_2\left(\frac{x}{\nu_{4,1}}\right)\left(1 - \chi_e\left(\frac{x}{\nu_{4,2}}\right)\right), \quad \kappa_2(x) = \frac{1}{1+x^2}, \quad \nu_{4,1} = 2, \quad \nu_{4,2} = 128.$$

For $\kappa_1(x), \kappa_2(x)$, we use induction to obtain

$$\partial_x^k \kappa_1(x) = \frac{P_k^+(x) - P_k^-(x)}{(1+x^4)^{k+1}}, \quad \partial_x^k \kappa_2(x) = \frac{R_k^+(x) - R_k^-(x)}{(1+x^2)^{k+1}},$$

for some polynomials P_k^{\pm}, R_k^{\pm} with non-negative coefficients, and the same method as that in Section D.1.2 to estimate the derivatives of $\partial_x^i \kappa_1(x)$. The estimate of κ_1 is simpler since κ_1 has a simpler form. Using the Leibniz rule (A.6) and the triangle inequality, we can obtain estimate $\partial_x^l \kappa_1(x)$ in $[a, b]$. Then we use these derivative estimates for $\partial_x^{l+2} \kappa_1(x)$, evaluate $\kappa(x; a_1, a_2)$ on the grid points, and then use (C.12) to obtain a sharp estimate of $\partial_x^l \kappa_1(x)$ on $[a, b]$. The same method applies to estimate κ_2, χ_{ϕ} .

For large x , e.g. $x \geq 100$, the above estimates can lead to a very large round off error. Instead, for $a \geq 2, a \in \mathbb{Z}_+$, we use the Taylor expansion

$$F_a = \frac{1}{1+x^a} = \sum_{k \geq 0} (-1)^k x^{-a(k+1)}, \quad \partial_x^i F_a = \sum_{k \geq 0} (-1)^{k+i} C_{i,k} x^{-a(k+1)-i}, \quad C_{i,k} = \prod_{0 \leq j \leq i-1} (a(k+1)+j).$$

We want to bound $|\partial_x^i F_a| \leq C_{i,0}(1+C_{\varepsilon})x^{-a-i}$ for $x \geq x_l = 100, i \leq 20$. For $k \leq 20$, we bound

$$C_{i,k} x^{-a(k+1)-i} \leq C_{i,k} x_l^{-(a-1)k} x^{-a-i-k} \leq C_{i,0} \varepsilon_1^{-a-i-k}, \quad \varepsilon_1 \triangleq \max_{i \leq 20, k \leq 20} x_l^{-(a-1)k} C_{i,k} C_{i,0}^{-1}.$$

For the tail part $k > 20$, we consider $G(k) = k \log x - i \log(1+k)$. Since $x > 21, i \leq 20$, we get

$$\partial_k G = \log x - \frac{i}{1+k} \geq \log x - 1 > \log 4 - 1 > 0, \quad G(k) \geq G(21) = 21 \log x - i \log 21 > 0.$$

It follows $x^k > (1+k)^i$. Using $\frac{a(k+1)+j}{a+j} \leq 1+k, C_{i,k} \leq C_{i,0}(1+k)^i$, and $a \geq 2$, we further get

$$C_{i,k} x^{-a(k+1)-i} \leq x^{-k-a-i} C_{i,k} x^{-k} \leq x^{-k-a-i} C_{i,0} (1+k)^i x^{-k} \leq C_{i,0} x^{-k-a-i}, \quad k > 20.$$

Combining the above estimates and $x \geq x_l > 10$, we obtain

$$|\partial_x^i F_a| \leq C_{i,0} x^{-a-i} C_a, \quad C_a \leq 1 + \varepsilon_1 \sum_{k=1}^{20} x^{-k} + \sum_{k \geq 21} x^{-k} \leq 1 + \frac{\varepsilon_1 x^{-1}}{1-x^{-1}} + \frac{x^{-21}}{1-x^{-1}} \leq 1 + \frac{\varepsilon_1}{x_l - 1} + x_l^{-20}.$$

D.3. Estimate of $\rho_p(y)$. We estimate the weight $\rho_p(y)$ (C.9) in the representation of the stream function. Using symbolic computation, e.g., Matlab or Mathematica, we yield

$$\begin{aligned}\partial_x^9 \rho_p(y) &= \frac{f_2(y) - f_1(y)}{(g(y))^9}, \quad g(y) = 2 + 2y + y^2, \\ f_1 &= 288y^2 + 672y^3 + 504y^4, \quad f_2 = 16 + 168y^6 + 72y^7 + 9y^8.\end{aligned}$$

Since $f_1, f_2, g \geq 0$ are increasing in $y \geq 0$, for $y \in [y_l, y_u]$, we yield

$$|\partial_x^9 \rho_p(y)| \leq \frac{\max(f_2(y_u) - f_1(y_l), f_1(y_u) - f_2(y_l))}{(g(y_l))^9}.$$

We have a trivial estimate similar to (C.12)

$$(D.8) \quad \max_{x \in I} |f(x)| \leq \max(|f(x_l)|, |f(x_u)|) + \frac{h}{2} \|f_x\|_{L^\infty(I)},$$

which is useful if we do not have bound for f_{xx} .

Based on the above estimates, using the estimates (C.12), (D.8), ideas in Section C.2.1, and evaluating ρ_p on some grid points, we can obtain piecewise sharp bounds for $\partial_x^i \rho_p$ for $i \leq 8$.

APPENDIX E. PIECEWISE $C^{1/2}$ AND LIPSCHITZ ESTIMATES

In this section, we estimate the piecewise $C^{1/2}$ bound and Lipschitz bound for a function.

E.1. Hölder estimate of the functions. In the following two sections, we estimate the Hölder seminorms $[f]_{C_x^{1/2}}$ or $[f]_{C_y^{1/2}}$ of some function f , e.g. $f = (\partial_t - \mathcal{L})\widehat{W}$ in (3.28), based on the previous L^∞ estimates. We will develop two approaches.

Below, we will assume $x, y \in \mathbb{R}_2^{++}$ since our function $f(x)$ defined on $x \in \mathbb{R}_2^+(x_2 \geq 0)$ is either even or odd in x_1 and we can reduce essentially all estimates to the case of \mathbb{R}_{++}^2 using symmetry. Suppose that we have bounds for $\partial_x f, \partial_y f$ and f . Firstly, we consider the $C_x^{1/2}$ estimate. For $x_1 < y_1$ and $x_2 = y_2$, we have

$$I = \frac{|f(x) - f(y)|}{|x - y|^{1/2}} \leq |x - y|^{1/2} \frac{1}{|x - y|} \int_{x_1}^{y_1} |f_x(z_1, x_2)| dz_1.$$

We further bound the average of f_x piecewisely using the method in Appendix E.2 to obtain the first estimate. We have a second estimate

$$\begin{aligned}|I| &= \left| \int_{x_1}^{y_1} f_x(z_1, x_2) dz \right| \cdot \frac{1}{|x - y|^{1/2}} \leq \|f_x x^{1/2}\|_\infty \int_{x_1}^{y_1} z_1^{-1/2} dz_1 \cdot \frac{1}{|x - y|^{1/2}} \\ &\leq \|f_x x^{1/2}\|_\infty 2 \frac{y_1^{1/2} - x_1^{1/2}}{|x - y|^{1/2}} = \|f_x x^{1/2}\|_\infty \frac{2\sqrt{y_1 - x_1}}{\sqrt{x_1} + \sqrt{y_1}}.\end{aligned}$$

We also have a trivial L^∞ estimate

$$|I| \leq \|f_x^{-1/2}\|_\infty \frac{x_1^{1/2} + y_1^{1/2}}{|x - y|^{1/2}}, \quad |I| \leq \|f\|_\infty \frac{2}{|x - y|^{1/2}}.$$

Similar L^∞ and Lipschitz estimates apply to $\|f\|_{C_y^{1/2}}$.

Near the origin, optimizing the above estimates, for $x_2 = y_2$, we obtain

$$\left| \frac{f(x) - f(y)}{|x - y|^{1/2}} \right| \leq \min(\|f_x x^{1/2}\|_\infty 2t, \|f_x^{-1/2}\|_\infty t^{-1}), \quad t = \frac{\sqrt{y_1 - x_1}}{\sqrt{x_1} + \sqrt{y_1}}.$$

In the Y -direction, $x_1 = y_1, x_2 \leq y_2$, we use

$$I_Y = \left| \frac{f(x) - f(y)}{|x - y|^{1/2}} \right| \leq \frac{1}{|x_2 - y_2|^{1/2}} \int_{x_2}^{y_2} |f_y(x_1, z_2)| |z|^{1/2} \cdot |z|^{-1/2} dz_2 \leq \|f_y |x|^{1/2}\|_\infty \frac{|x_2 - y_2|^{1/2}}{|x|^{1/2}} \triangleq At,$$

$$I_Y \leq (|f(x)x_1| + |f(y)x_1|^{1/2}) \left(\frac{x_1}{|x|}\right)^{1/2} \cdot \frac{|x|^{1/2}}{|x_2 - y_2|^{1/2}} \triangleq Bt^{-1}, \quad t \triangleq \frac{|x_2 - y_2|^{1/2}}{|x|^{1/2}}, \quad I_Y \leq \min(At, Bt^{-1})$$

Since $x_1 \leq |x|$, A, B are not singular near $x = 0$. We derive the piecewise bounds for A, B and then optimize two estimates to estimate I_Y .

From the above estimates, to obtain sharp Hölder estimate of f , we estimate the piecewise bounds of $f, f x_1^{-1/2}, f|x|^{-1/2}, f_x, f_y, f_x|x_1|^{1/2}, f_y|x|^{1/2}$, which are local quantities. These estimates can be established using the piecewise bounds of $\partial_x^i \partial_y^j f$ and the methods in Section 8 in the supplementary material II [12].

E.1.1. The second approach of Hölder estimate. We develop an additional approach to estimate $I(f) = \frac{|f(x)-f(z)|}{|x-z|^{1/2}}$ that is sharper if $|x-z|$ is not small and f is smooth. We need the grid point values and derivative bounds of f .

We estimate $I(f) = \frac{|f(x)-f(z)|}{|x-z|^{1/2}}$ for $x \in [x_l, x_u], z \in [z_l, z_u]$. Denote by \hat{f} the linear approximation of f with $\hat{f}(x_i) = f(x_i)$ on the grid point x_i . We have the following Lemma.

Lemma E.1. *Suppose that f is linear on $[x_l, x_u], [z_l, z_u]$ and $x_l \leq x_u \leq z_l \leq z_u$. Then we have*

$$\max_{x \in [x_l, x_u], z \in [z_l, z_u]} \frac{|f(x) - f(z)|}{|x - z|^{1/2}} = \max_{\alpha, \beta \in \{l, u\}} \frac{|f(x_\alpha) - f(z_\beta)|}{|x_\alpha - z_\beta|^{1/2}}.$$

The above Lemma shows that for the linear interpolation of f , the maximum of the Hölder norm is achieved at the grid point.

Proof. Denote by M the right hand side in the Lemma. Clearly, it suffices to prove that the left hand side is bounded by M . We fix $x \in [x_l, x_u], z \in [z_l, z_u]$. Suppose that

$$x = a_l x_l + a_u x_u, \quad z = b_l z_l + b_u z_u, \quad a_l + a_u = 1, \quad b_l + b_u = 1,$$

for $a_l, b_l \in [0, 1]$. Denote

$$m_{\alpha\beta} = a_\alpha b_\beta, \quad \alpha, \beta \in \{l, u\}.$$

Since $f(x)$ is linear on $[x_l, x_u]$ and $[z_l, z_u]$, we get

$$f(x) = a_l f(x_l) + a_u f(x_u), \quad f(z) = b_l f(z_l) + b_u f(z_u).$$

For any function g linear on $[x_l, x_u], [z_l, z_u]$, e.g., $g(x) = 1, g(x) = x, g(x) = f(x)$, we have

$$(E.1) \quad g(x) = \sum_{\alpha, \beta \in \{l, u\}} m_{\alpha\beta} g(x_\alpha), \quad g(z) - g(x) = \sum_{\alpha, \beta \in \{l, u\}} m_{\alpha\beta} (g(z_\beta) - g(x_\alpha)),$$

Using the above identities and the triangle inequality and the definition of M , we yield

$$|f(x) - f(z)| = \left| \sum_{\alpha, \beta \in \{l, u\}} m_{\alpha\beta} (f(x_\alpha) - f(z_\beta)) \right| \leq \sum_{\alpha, \beta \in \{l, u\}} m_{\alpha\beta} M |x_\alpha - z_\beta|^{1/2}.$$

Using the Cauchy-Schwarz inequality, $|x_\alpha - z_\beta| = z_\beta - x_\alpha$ and (E.1), we establish

$$\begin{aligned} |f(x) - f(z)| &\leq \sum_{\alpha, \beta \in \{l, u\}} m_{\alpha\beta} \sum_{\alpha, \beta \in \{l, u\}} m_{\alpha\beta} M |x_\alpha - z_\beta|^{1/2} = \sum_{\alpha, \beta \in \{l, u\}} m_{\alpha\beta} M |x_\alpha - z_\beta|^{1/2} \\ &= M \left(\sum_{\alpha, \beta \in \{l, u\}} m_{\alpha\beta} (z_\beta - x_\alpha) \right)^{1/2} = M (z - x)^{1/2}. \end{aligned}$$

The desired result follows. \square

We generalize Lemma E.1 to 2D as follows.

Lemma E.2. *Let $I_x = [x_l, x_u], I_z = [z_l, z_u], I_y = [y_l, y_u]$ with $x_l \leq x_u \leq z_l \leq z_u$. Suppose that f is linear on $I_x \times I_y$ and $I_z \times I_y$. Then we have*

$$\max_{x \in I_x, z \in I_z, y \in I_y} \frac{|f(x, y) - f(z, y)|}{|x - z|^{1/2}} = \max_{\alpha, \beta, \gamma \in \{l, u\}} \frac{|f(x_\alpha, y_\gamma) - f(z_\beta, y_\gamma)|}{|x_\alpha - z_\beta|^{1/2}}.$$

Proof. Note that the function $I(x, z, y) = \frac{f(x, y) - f(z, y)}{|x - z|^{1/2}}$ is linear in y . We get

$$|I(x, z, y)| = \max(|I(x, z, y_l)|, |I(x, z, y_u)|).$$

Applying Lemma E.1 completes the proof. \square

Let \hat{f} be the linear interpolation of f . Suppose that $x \in I_x, z \in I_z, y \in I_y$ with $x_u \leq z_l$. Using the above estimates and notations, we can bound $I(f)$ as follows

$$\begin{aligned} I(f) &= \frac{|f(z, y) - f(x, y)|}{|x - z|^{1/2}} \leq \frac{|\hat{f}(x, y) - f(x, y)| + |\hat{f}(z, y) - f(z, y)|}{|x - z|^{1/2}} + \max_{\alpha, \beta, \gamma \in \{l, u\}} \frac{|f(x_\alpha, y_\gamma) - f(z_\beta, y_\gamma)|}{|x_\alpha - z_\beta|^{1/2}} \\ &\leq \left(\frac{h_x^2}{8} \|f_{xx}\|_{I_x \times I_y} + \frac{h_y^2}{8} (\|f_{yy}\|_{I_x \times I_y} + \|f_{yy}\|_{I_z \times I_y}) + \frac{h_z^2}{8} \|f_{xx}\|_{I_z \times I_y} \right) |x - z|^{-1/2} + M. \end{aligned}$$

E.2. Piecewise derivative bounds. In this section, we discuss how to obtain the sharp bound of $\frac{p(b) - p(a)}{b - a}$ using piecewise derivative bounds of p .

Suppose that $|p'(y)| \leq C_i, y \in I_i = [y_i, y_{i+1}]$. For any $a \in I_k, b \in I_l, a < b$, we have the bound

$$\begin{aligned} |p(b) - p(a)| &\leq \int_a^b |p'(y)| dy \leq |y_{k+1} - a| C_k + |b - y_l| C_l + \sum_{k+1 \leq m \leq l-1} C_m (y_{m+1} - y_m) \\ &= (y_{k+1} - a) C_k + (b - y_l) C_l + M_{kl} (y_l - y_{k+1}) \mathbf{1}_{l \geq k+1}, \end{aligned}$$

where M_{kl} is defined below:

$$(E.2) \quad M_{kl} = |y_l - y_{k+1}|^{-1} \left(\sum_{k+1 \leq m \leq l-1} C_m |y_{m+1} - y_m| \right).$$

Next, we want to bound $\frac{|p(b) - p(a)|}{|b - a|}$. If $l - k \leq 1$, we get

$$|p(b) - p(a)| \leq (b - a) \max(C_k, C_l).$$

Otherwise, if $l \geq k + 2$, we have

$$|p(b) - p(a)| \leq (y_{k+1} - a)(C_k - M_{kl}) + (b - y_l)(C_l - M_{kl}) + M_{kl}(b - a).$$

Since $\frac{y_{k+1} - a}{b - a}$ is decreasing in a and b , $\frac{b - y_l}{b - a}$ is increasing in b and a , we get

$$0 \leq \frac{y_{k+1} - a}{b - a} \leq \frac{y_{k+1} - y_k}{y_l - y_k}, \quad 0 \leq \frac{b - y_l}{b - a} \leq \frac{y_{l+1} - y_l}{y_{l+1} - y_{k+1}}.$$

Using the above estimates, for $a \in I_k, b \in I_l$, we obtain

$$(E.3) \quad \frac{|p(b) - p(a)|}{|b - a|} \leq \max(C_k - M_{kl}, 0) \frac{y_{k+1} - y_k}{y_l - y_k} + \max(C_l - M_{kl}, 0) \frac{y_{l+1} - y_l}{y_{l+1} - y_{k+1}} + M_{kl}.$$

For uniform mesh, i.e. $y_{i+1} - y_i = h$, we can simplify the above estimate as follows

$$\frac{|p(b) - p(a)|}{|b - a|} \leq \frac{(\max(C_k - M_{kl}, 0) + \max(C_l - M_{kl}, 0))}{l - k} + M_{kl}, \quad M_{kl} = \frac{1}{l - k - 1} \sum_{k+1 \leq m \leq l-1} C_m.$$

The same argument applies to obtain piecewise bounds of $J(a, b) = \frac{p(b) - p(a)}{b - a}$. We use piecewise upper bounds $p'(y) \leq C_i, y \in I_i = [y_i, y_{i+1}]$ and obtain the same upper bounds as (E.3). To get lower bounds of $J(a, b)$, we use piecewise lower bounds $p'(y) \geq C_i$ and (E.2) to get

$$\frac{p(b) - p(a)}{b - a} \geq \min(C_k - M_{kl}, 0) \frac{y_{k+1} - y_k}{y_l - y_k} + \min(C_l - M_{kl}, 0) \frac{y_{l+1} - y_l}{y_{l+1} - y_{k+1}} + M_{kl}.$$

APPENDIX F. NOTATIONS

For the reader's convenience, we collect the main notations used in this paper.

Weights. We use the following weights defined in (A.1), (A.2), (A.3) for the estimates

$$\psi_1, \psi_2, \psi_3, \psi_{du}, \psi_u, \quad \varphi_1, \varphi_{g1}, \varphi_{elli}, \varphi_{evo, i}, \quad i = 1, 2, 3, \quad \rho_{10}, \rho_{20}, \rho_3, \rho_4.$$

We use $f_\lambda(x) = f(\lambda x)$ for rescaled function (4.2).

Cutoff functions. We use various cutoff functions to construct the approximate solutions.

$\chi_{ij}, i = 1, 2, 3, j = 1, 2$ are defined in (3.17).

$\chi_{\bar{\varepsilon}}, \chi_{\hat{\varepsilon}}, f_{\chi, i}, i = 1, 2, 3$ are defined in (D.6), (D.7).

Operators. We use \mathcal{L} to denote various linear operators. \mathcal{L}_i is the full linearized operator around the approximate steady state. We decompose \mathcal{L}_i into $\mathcal{L}_i^e, \mathcal{L}_i^{\bar{e}}, \mathcal{L}_i^N$ (3.12).

$\mathcal{B}_{op,i}$ (3.4), (3.5) denotes bilinear operators related to the linearized operators.

\mathcal{R} . (3.7), (3.8) denotes residual error in the construction of the approximate solution to the linearized equations.

Velocity and kernels. We use K_i to denote the kernels of the velocity, e.g. $K_1, K_2, K_f, f = u, v, u_x, v_x, u_y$ (4.1). We use K^{sym} for the symmetrized kernel (4.25), K_{ux0}, K_{00} (4.5) for the kernel of the approximation terms near $x = 0$

We use $f = u, v, u_x, u_y, v_x, v_y$ to denote the original velocity and its derivatives, \hat{f} for its finite rank approximation, and $f_A = f - \hat{f}$. See the beginning of Section 4.1.

Regions for integrals. We use $B_{lm}(r)$ (4.17) to denote different grids and $R(\cdot)$ to denote various singular region: $R(x, k)$ (4.18), $R_s(x, k), R_{s,i}(x, k)$ (4.19), $R^\pm(x, k)$ (4.20), $R(x, k, \alpha), \alpha = N, E, S, W$ (4.21)

Approximate profiles and solutions. We use ω, η, ξ, ϕ to denote the vorticity, θ_x, θ_y (θ is the density (2.3)), and the stream functions, respectively. We use \bar{f} to denote the approximate profile for f , e.g. $\bar{\omega}, \bar{\theta}$, and use \hat{f} to denote the numeric solution, e.g. \widehat{W} (3.34) and \hat{G} (3.6).

We use $\bar{F}_\omega, \bar{F}_\theta, \bar{F}_i$ (2.14) to denote the residual error of the profile.

Mesh. To construct the approximate profile, we use the adaptive mesh y_i (C.2). To estimate the integrals $\int f(x, y)dy$ in Section 4, we use mesh y_i (4.11) with mesh size h_x, h (4.14).

Differential operators. We denote (3.23) $D^2 = (D_1^2, D_2^2, D_3^2) = (\partial_{xy}, \partial_{xy}, \partial_x^2)^T$.

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**SUPPLEMENTARY MATERIAL FOR “STABLE NEARLY SELF-SIMILAR
BLOWUP OF THE 2D BOUSSINESQ AND 3D EULER EQUATIONS WITH
SMOOTH DATA II: RIGOROUS NUMERICS”**

JIAJIE CHEN AND THOMAS Y. HOU

ABSTRACT. In this supplementary material, we provide all the remaining details in the estimate of integrals related to the nonlocal terms $\mathbf{u}, \nabla \mathbf{u}$. In Section 5, we estimate the integrals in the upper bounds of the sharp Hölder estimate established in Section 3 in [3]. In Section 6, we provide detailed formulas in the estimate of $\nabla \mathbf{u}, u/|x_1|$ in the singular regions and some explicit integrals. In Section 7, we generalize the estimate of integrals of $\mathbf{u}(x), \nabla \mathbf{u}(x)$ in Section 4 in Part II [2] from $x = O(1)$ to x either close to 0 or x very large. In Section 8, we use grid points and piecewise derivative bounds estimated in Appendix in Part II [2] to construct three interpolating polynomials in 2D with rigorous error estimates. Then we use these interpolating polynomials to obtain sharp piecewise estimates of functions, e.g. the residual error.

5. COMPUTATION OF THE SHARP CONSTANTS FOR IN THE HÖLDER ESTIMATES

In this section, we estimate the integrals in the upper bounds of the sharp Hölder estimate established in Section 3 in [3].

5.1. Integrals related to the kernels. Firstly, we derive some analytic integral formulas. The kernels associated with ∇u are given by

$$(5.1) \quad G(y) = -\frac{1}{2} \log |y|, \quad K_1(y) \triangleq G_{xy} = \frac{y_1 y_2}{|y|^4}, \quad K_2(y) \triangleq G_{xx} = -G_{yy} = \frac{1}{2} \frac{y_1^2 - y_2^2}{|y|^4},$$

and we drop the factor $\frac{1}{\pi}$ for simplicity.

5.1.1. Basic Lemmas. We use the following Lemma from Appendix B.1 in Part I [3] to estimate K_2 .

Lemma 5.1. *Suppose that $f \in L^\infty$, is Hölder continuous near 0. For $0 < a, b < \infty$ and $Q = [0, a] \times [0, b], [0, a] \times [-b, 0], [-a, 0] \times [0, b]$, or $[-a, 0] \times [-b, 0]$, we have*

$$P.V. \int_Q K_2(y) f(y) dy = \lim_{\varepsilon \rightarrow 0} \int_{Q \cap |y_1| \geq \varepsilon} K_2(y) f(y) dy - \frac{\pi}{8} f(0) = \lim_{\varepsilon \rightarrow 0} \int_{Q \cap |y_2| \geq \varepsilon} K_2(y) f(y) dy + \frac{\pi}{8} f(0).$$

We have the following estimate for the Green function from Appendix B.1 in Part II [2].

Lemma 5.2. *Denote $r = (x^2 + y^2)^{\frac{1}{2}}$ and $G(x, y) = -\frac{1}{2} \log r$. For any $i, j \geq 0$ with $i + j \geq 1$, we have*

$$|\partial_x^i \partial_y^j G(x, y)| \leq \frac{1}{2} (i + j - 1)! \cdot r^{-i-j}.$$

Consider the odd extension of ω in y from \mathbb{R}_2^+ to \mathbb{R}_2

$$(5.2) \quad W(y) = \omega(y) \text{ for } y_2 \geq 0, \quad W(y) = -\omega(y_1, -y_2) \text{ for } y_2 < 0.$$

W is odd in both y_1 and y_2 variables.

We have the following indefinite integral formulas for K_1, K_2 (5.1)

$$(5.3) \quad \begin{aligned} \int K_1(y) dy_1 dy_2 &= -\frac{1}{2} \log |y| + C, & \int K_2(y) dy &= \frac{1}{2} \arctan \frac{y_1}{y_2} + C = -\frac{1}{2} \arctan \frac{y_2}{y_1} + C, \\ \int K_1(y) dy_1 &= -\frac{1}{2} \frac{y_2}{|y|^2} + C, & \int K_1(y) dy_2 &= -\frac{1}{2} \frac{y_1}{|y|^2} + C, \\ \int K_2(y) dy_1 &= -\frac{1}{2} \frac{y_1}{|y|^2} + C, & \int K_2(y) dy_2 &= \frac{1}{2} \frac{y_2}{|y|^2} + C. \end{aligned}$$

For $K(y) = -\frac{y_i}{2|y|^2} = \partial_{y_i} G$ (5.1), we have

$$(5.4) \quad \begin{aligned} \int \partial_{y_2} G dy &= \int G dy_1 = \frac{1}{4} (2y_1 - 2y_2 \arctan \frac{y_1}{y_2} - y_1 \log(y_1^2 + y_2^2)) + C, \\ \int \partial_{y_1} G dy &= \int G dy_2 = \frac{1}{4} (2y_2 - 2y_1 \arctan \frac{y_2}{y_1} - y_2 \log(y_1^2 + y_2^2)) + C. \end{aligned}$$

5.1.2. *Formulas for integrals with a half power.* We introduce

$$(5.5) \quad f_s(t) \triangleq \int_0^t \frac{s^2}{1+s^4} ds = \frac{1}{2} \int_0^{t^{1/2}} \frac{1}{1+s^2} s^{1/2} ds.$$

Using changes of variables $s = ps$ and $s = t^2$, we yield

$$(5.6) \quad \begin{aligned} \int_a^b \frac{p}{p^2 + s^2} s^{1/2} ds &= p^{1/2} \int_{a/p}^{b/p} \frac{1}{1+s^2} s^{1/2} ds = 2p^{1/2} \int_{\sqrt{a/p}}^{\sqrt{b/p}} \frac{t^2}{1+t^4} dt \\ &= 2p^{1/2} (f_s(\sqrt{b/p}) - f_s(\sqrt{a/p})). \end{aligned}$$

Clearly f_s is increasing. Using the Beta function $B(x, 1-x) = \frac{\pi}{\sin(\pi x)}$ and $t = s^{1/4}$ we have

$$f_s(\infty) = \frac{1}{4} \int_0^\infty \frac{t^{1/2}}{1+t} t^{-3/4} dt = \frac{1}{4} \int_0^\infty \frac{t^{-1/4}}{1+t} dt = \frac{1}{4} B\left(\frac{3}{4}, \frac{1}{4}\right) = \frac{1}{4} \frac{\pi}{\sin \frac{\pi}{4}} = \frac{\pi}{2\sqrt{2}}.$$

For $a, b, c, d \geq 0$, we introduce

$$(5.7) \quad F_{K_2, h}(a, b, c, d) = \int_{[a, b] \times [c, d]} |K_2(y)| y_2^{1/2} dy.$$

Clearly, it suffices to study $F_{K_2, h}(0, a, 0, b)$ for any $a, b > 0$. Then we have

$$(5.8) \quad F_{K_2, h}(a, b, c, d) = |F_{K_2, h}(a, c) + F_{K_2, h}(b, d) - F_{K_2, h}(a, d) - F_{K_2, h}(b, c)|.$$

Denote $m = \min(a, b)$. Using (5.3) and (5.6), we yield

$$(5.9) \quad \begin{aligned} \int_0^m \int_0^m |K_2(y)| y_2^{1/2} dy &= \int_0^m y_2^{1/2} dy_2 \left(\int_{y_2}^m K_2(y) - \int_0^{y_2} K_2(y) \right) dy_1 \\ &= \frac{1}{2} \int_0^m \left(-\frac{y_1}{|y|^2} \Big|_{y_2}^m + \frac{y_1}{|y|^2} \Big|_0^{y_2} \right) y_2^{1/2} dy_2 = \frac{1}{2} \int_0^m \left(\frac{2y_2}{2y_2^2} - \frac{m}{m^2 + y_2^2} \right) y_2^{1/2} dy_2 = \sqrt{m} - \sqrt{m} f_s(1). \end{aligned}$$

Next, we consider the integral in $R = [0, a] \times [0, b] \setminus [0, m]^2$. If $a > b$, we get $y_1 > y_2, K_2(y) > 0$ in R and

$$\begin{aligned} \left| \int_b^a \int_0^b K_2(y) y_2^{1/2} dy \right| &= \left| \int_0^b \left(-\frac{y_1}{|y|^2} \Big|_b^a \right) y_2^{1/2} dy_2 \right| = \left| \frac{1}{2} \int_0^b \left(\frac{b}{b^2 + y_2^2} - \frac{a}{a^2 + y_2^2} \right) y_2^{1/2} dy_2 \right| \\ &= |b^{1/2} f_s(1) - a^{1/2} f_s(\sqrt{b/a})|. \end{aligned}$$

If $a < b$, we yield $y_1 < y_2, K_2(y) < 0$ in R and

$$\left| \int_0^a \int_a^b -K_2(y) y_2^{1/2} dy \right| = \left| \frac{1}{2} \int_a^b \frac{y_1}{|y|^2} \Big|_0^a y_2^{1/2} dy_2 \right| = \left| \frac{1}{2} \int_a^b \frac{a}{a^2 + y_2^2} y_2^{1/2} dy_2 \right| = a^{1/2} |f_s(\sqrt{b/a}) - f_s(1)|.$$

For $0 < a < b$, we introduce

$$(5.10) \quad F_{diag}(a, b) \triangleq \int_{[a, b]^2} |K_2(y)| dy = \frac{1}{2} \log\left(\frac{b}{a}\right) - \left(\frac{\pi}{4} - \arctan \frac{a}{b}\right).$$

To prove the identity, since $K_2(y)$ is symmetric in y_1, y_2 , using (5.3), we yield

$$\begin{aligned} \int_{[a,b]^2} |K_2(y)| dy &= 2 \int_a^b \int_a^{y_1} K_2(y) dy = \int_a^b \frac{y_2}{|y|^2} \Big|_a^{y_1} dy_1 = \int_a^b \frac{1}{2y_1} - \frac{a}{y_1^2 + a^2} dy_1 \\ &= \frac{1}{2} \log \frac{b}{a} - \arctan \frac{y_1}{a} \Big|_a^b = \frac{1}{2} \log \frac{b}{a} - (\arctan \frac{b}{a} - \frac{\pi}{4}) = \frac{1}{2} \log \frac{b}{a} - (\frac{\pi}{4} - \arctan \frac{a}{b}). \end{aligned}$$

Denote some constants

$$(5.11) \quad \begin{aligned} C_{K_2, up} &\triangleq \int_0^1 \int_{y_1}^1 |K_2(y)| \cdot \left| \frac{y_1^2}{y_2} - y_2 \right|^{1/2} dy, \quad C_{K_2, low} \triangleq \int_0^1 \int_0^{y_1} |K_2(y)| \cdot |y_2|^{1/2} dy_2, \\ C_{K_2} &\triangleq C_{K_2, up} + C_{K_2, low}. \end{aligned}$$

Using (5.3), (5.5), and $1 \geq y_1 \geq y_2^{1/2} \geq y_2$, we can simplify the integral in $C_{K_2, low}$ as follows

$$(5.12) \quad \begin{aligned} C_{K_2, low} &= \int_0^1 y_2^{1/2} dy_2 \int_{y_2^{1/2}}^1 K_2(y) dy_1 = \int_0^1 \left(-\frac{1}{2} \frac{y_1}{|y|^2} \Big|_{y_2^{1/2}}^1 \right) y_2^{1/2} dy_2 \\ &= \frac{1}{2} \int_0^1 \left(\frac{y_2^{1/2}}{y_2^2 + y_2} - \frac{1}{y_2^2 + 1} \right) y_2^{1/2} dy_2 = \frac{1}{2} (\log(2) - 2f_s(1)). \end{aligned}$$

For $C_{K_2, up}$, we follow the strategies in Section 5.2. In a small region $[0, \varepsilon]^2$ near the singularity $y = 0$, since $0 \leq y_2 - \frac{y_1^2}{y_2} \leq y_2$, we bound the integrand by $|K_2(y)| |y_2|^{1/2}$ and follow (5.9) to get

$$\int_0^\varepsilon \int_{y_1}^\varepsilon |K_2(y)| |y_2|^{1/2} dy = \left| \int_0^\varepsilon y_2^{1/2} dy_2 \int_0^{y_2} K_2(y) dy_1 \right| = \frac{1}{2} \int_0^\varepsilon y_2^{1/2} \frac{y_1}{|y|^2} \Big|_0^{y_2} dy_2 = \frac{1}{2} \int_0^\varepsilon \frac{y_2^{3/2}}{2y_2^2} dy_2 = \frac{1}{2} \varepsilon^{1/2}.$$

We introduce f_h similar to f_s

$$(5.13) \quad f_h(a) \triangleq \int_0^a \frac{s^{3/2}}{1+s^2} ds = 2 \int_0^{a^{1/2}} \frac{s^4}{1+s^4} ds.$$

Using a change of variables, we obtain

$$(5.14) \quad \int_0^a \frac{s^{3/2}}{p^2 + s^2} ds = p^{1/2} \int_0^{a/p} \frac{s^{3/2}}{1+s^2} ds = p^{1/2} f_h(a/p).$$

5.1.3. *Integral formulas for K_1, K_2 .* We discuss how to compute the integrals

$$J_{ij}(a, b, c, d) = \int_{[a,b] \times [c,d]} |K(y) y_1^i y_2^j| dy, \quad 1 \leq i+j, \quad K = K_l, \quad K = \partial_k K_l.$$

Due to symmetry of $|K(y) y_1^i y_2^j|$, we can assume that $0 \leq a \leq b, 0 \leq c \leq d$. For $K = K_1 = \frac{y_1 y_2}{|y|^4}$, the computation is simple since K_1 has a fixed sign in each quadrant and $|K_1(y) y_1^i y_2^j| = K_1(y) y_1^i y_2^j$. For $K = K_2, \nabla K$, we have

$$K_2 = \frac{1}{2} \frac{y_1^2 - y_2^2}{|y|^4}, \quad \partial_1^3 G = \frac{y_1(-y_1^2 + 3y_2^2)}{|y|^6}, \quad \partial_1^2 \partial_2 G = \frac{y_2(-3y_1^2 + y_2^2)}{|y|^6},$$

where G is the Green function (5.1) and is harmonic. The above kernels can change sign in $[a, b] \times [c, d]$, and the sign is determined by $S(y) = m^2 y_1^2 - y_2^2$ for some $m > 0$. We fix a kernel $K = K_2$ or $\partial_m K_n$ for some $m, n = 1, 2$ and i, j . We have

$$F(y) \triangleq K(y) y_1^i y_2^j, \quad |F| = |K(y) y_1^i y_2^j| = |m^2 y_1^2 - y_2^2| P(y),$$

for some functions $P(y)$ having a fixed sign in \mathbb{R}_2^{++} . For example, $K_2(y) y_1 y_2 = (y_1^2 - y_2^2) \frac{y_1 y_2}{2|y|^4}$, $P = \frac{y_1 y_2}{2|y|^4}$. Denote the indefinite integral

$$(5.15) \quad I_y(y_1, y_2) \triangleq \int F(y) dy_2, \quad I_{xy} \triangleq \int F(y) dy_1 dy_2, \quad I_{diag} \triangleq \int I_y(y_1, m y_1) dy_1,$$

which can be derived using symbolic computation. The constant in the indefinite integral formula will be cancelled when we apply it to evaluate an integral in a specific domain. Next, we evaluate

$$S_{diag}^l(p, q) \triangleq \int_p^q \int_{mp}^{my_1} |F(y)| dy, \quad S_{diag}^u(p, q) \triangleq \int_p^q \int_{my_1}^{mq} |F(y)| dy, \quad S_{diag} = S_{diag}^l + S_{diag}^u.$$

The domain of the integrals S_{diag}^α , $\alpha = l, u$ is a triangle, and S_{diag}^l, S_{diag}^u denote the integral in the lower and the upper triangle $[p, q] \times [mp, mq]$, respectively. We first evaluate S_{diag}^l . Since $y_2 \leq my_1$ in the domain, $F(y)$ has a fixed sign in the domain of S_{diag}^l and

$$(5.16) \quad S_{diag}^l(p, q) = \left| \int_p^q I_y(y_1, \cdot) \Big|_{mp}^{my_1} dy_1 \right| = \left| \int_p^q I_y(y_1, my_1) - I_y(y_1, mp) dy_1 \right| = \left| I_{diag}(\cdot) \Big|_p^q - I_{xy}(y_1, mp) \Big|_p^q \right|.$$

Similarly, we get

$$(5.17) \quad S_{diag}^u = \left| \int_p^q I_y(y_1, mq) - I_y(y_1, my_1) dy_1 \right| = \left| I_{xy}(y_1, mq) \Big|_p^q - I_{diag}(\cdot) \Big|_p^q \right|,$$

and then yield the formula for S_{diag} .

Next, we consider the integral of $|F(y)|$ in a general domain $Q = [a, b] \times [c, d] \subset \mathbb{R}_2^{++}$. We can decompose the regions into several parts according to the intersection between Q and the line $y_2 = my_1$, and then apply the integral formulas S_{diag} and I_{xy} (5.15).

Fixed sign: (1) $c \geq mb$, (6) $d \leq ma$. In these two cases, we have $y_2 \geq my_1$ for any $y \in Q$ or $y_2 \leq my_1$ for any $y \in Q$. Thus, $F(y)$ has a fixed sign in Q and we can use the analytic formula I_{xy} (5.15) to evaluate S .

(2) $ma \leq c \leq mb, mb \leq d$. We decompose Q as follows

$$[a, b] \times [c, d] = [a, c/m] \times [c, d] \cup [c/m, b] \times [mb, d] \cup [c/m, b] \times [c, mb] \triangleq Q_1 \cup Q_2 \cup Q_3.$$

(3) $ma \leq c \leq mb, d \leq mb$. We decompose Q as follows

$$[a, b] \times [c, d] = [a, c/m] \times [c, d] \cup [d/m, b] \times [c, d] \cup [c/m, d/m] \times [c, d] \triangleq Q_1 \cup Q_2 \cup Q_3.$$

(4) $c \leq ma, mb \leq d$. We decompose Q as follows

$$Q = [a, b] \times [c, ma] \cup [a, b] \times [mb, d] \cup [a, b] \times [ma, mb] = Q_1 \cup Q_2 \cup Q_3.$$

(5) $c \leq ma, ma \leq d \leq mb$. We decompose Q as follows

$$Q = [a, d/m] \times [c, ma] \cup [d/m, b] \times [c, d] \cup [a, d/m] \times [ma, d] \triangleq Q_1 \cup Q_2 \cup Q_3.$$

In each case (2)-(5), for Q_1, Q_2 , $(my_1)^2 - y_2^2$ and F have fixed signs and we use (5.15) to evaluate the integral. For Q_3 , we apply (5.16), (5.17). See Figure 1 for an illustration of decompositions in different cases.

We need to estimate the integral

$$\int_Q |F| dy, \quad |F| = |\partial_2(K_2(y)y_2)| = \left| \frac{y_1^4 - 6y_1^2y_2^2 + y_2^4}{2|y|^6} \right|,$$

in $Q = [a, b] \times [c, d]$. Without loss of generality, we assume $Q \subset \mathbb{R}_2^{++}$. We define

$$p_1 = y_1^4 + y_2^4, \quad p_2 = 6y_1^2y_2^2, \quad b(Q) = \max(p_1^u - p_2^l, p_2^u - p_1^l), \quad f(Q) = 1 - (p_1^l \leq p_2^u)(p_2^l \leq p_1^u).$$

Since p_i is increasing in y_1, y_2 , it is easy to obtain the piecewise upper and lower bounds p_i^l, p_i^u . If $f = 1$, we get $p_1(y) \geq p_2(y)$ or $p_2(y) \geq p_1(y)$ for any $y \in Q$, and $F(y)$ has a fixed sign in Q . Then we can use the analytic integral formula for $\partial_2(K_2(y)y_2)$ to evaluate the integral. If $f = 0$, we bound the integral as follows

$$\int_Q |F(y)| dy \leq b(Q) \int_Q \frac{1}{2|y|^6} dy,$$

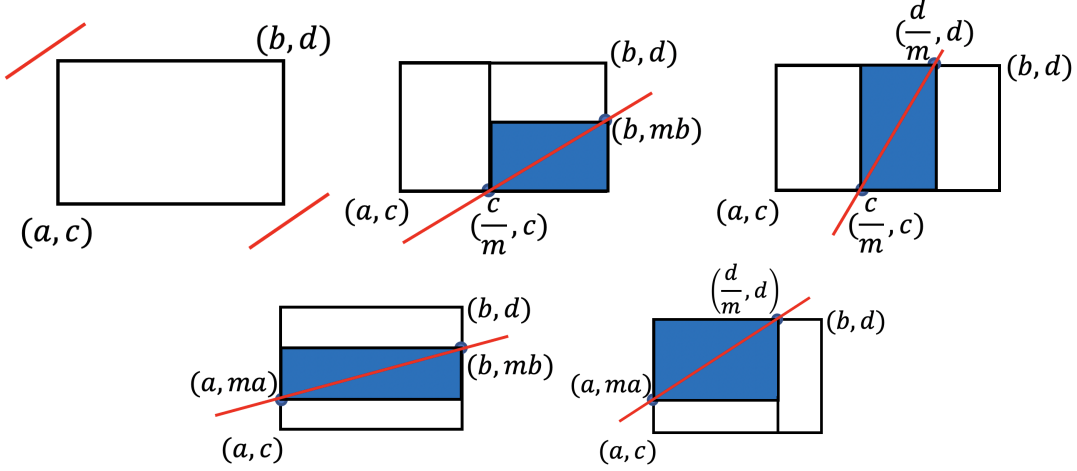


FIGURE 1. Decompositions in different cases. First figure for case (1), (6), the last four figures for case (2)-(5). The blue region denotes Q_3 .

and further evaluate the integral using the analytic formula that can be obtained by symbolic computation.

5.1.4. *Symmetric bounds for $K(y)$.* The kernel $K_l(y)$ (5.1) enjoys the symmetry in y_1, y_2

$$K_l(y_2, y_1) = s_l K_l(y_1, y_2), \quad s_1 = 1, s_2 = -1.$$

Therefore, we have

$$\partial_1^m \partial_2^n (K_l(y_1, y_2)) \cdot y_1^i y_2^j = s_l \partial_1^m \partial_2^n (K_l(y_2, y_1)) \cdot y_1^i y_2^j = s_l (\partial_1^m \partial_2^n K_l(z) z_1^i z_2^j) \Big|_{(z_1, z_2) = (y_2, y_1)}.$$

We need to derive several estimates on $\partial_1 K_l(y_1, y_2) y_1^i y_2^j$. Using the above relations, once we obtain the estimates related to $\partial_1 K_l$, we can obtain the estimate for $\partial_2 K_l$. Moreover, since

$$K_1 = \partial_{12} G, \quad K_2 = \partial_1^2 G, \quad \partial_1 K_1 = \partial_2 K_2, \quad \partial_2 K_1 = \partial_{122} G = -\partial_1^3 G = -\partial_1 K_2,$$

once we derive the estimate related to $\partial_i K_1$, we can derive the estimate for $\partial_{3-i} K_2$ using the above relation.

5.1.5. *Asymptotic integral formulas.* In the estimate of the derivatives of the regular part in $\nabla \mathbf{u}$, we need to estimate several integrals

$$I_{i,j,m,n} \triangleq \int_{Q_{a,b}} |\partial_j K_i(y) y_1^m y_2^n| dy, \quad m+n=2, \quad J_{i,j,l} \triangleq \int_{Q_{a,b}} |\partial_j (K_i(y) y_l)| dy, \quad Q_{a,b} \triangleq [-b, b]^2 \setminus [-a, a]^2$$

for $i, j, l = 1, 2$, where K_i is given in (5.1). When $m+n=2$, the integrand has a singularity of order -1 and it is locally integrable. Thus, we can apply the estimates and the integral formulas developed in Section 5.1.3 to estimate $I_{i,j,m,n}$. Since the integrand is symmetric in y_1, y_2 , we only need to estimate the integral in $[0, b]^2 \setminus [0, a]^2$.

In the second integral, the integrand has a singularity of order -2 and is not integrable near 0. Denote $F_{i,j,l}(y) = \partial_j K_i(y) y_l$. Using the scaling symmetry of K_i , we get $F(\lambda y) = \lambda^{-2} F(y)$ and

$$\begin{aligned} \partial_a J_{i,j,l}(a, b) &= - \int_0^a (|F_{i,j,l}(a, s)| + |F_{i,j,l}(s, a)|) ds = - \int_0^1 (|F_{i,j,l}(a, as)| + |F_{i,j,l}(as, a)|) ads \\ &= - \frac{1}{a} \int_0^1 (|F_{i,j,l}(1, s)| + |F_{i,j,l}(s, 1)|) ds, \end{aligned}$$

where we have used a change of variables $s \rightarrow as$ in the last inequality. We remark that the last integral is independent of a, b and is a constant. Since $J_{i,j,l}(b, b) = 0$, we derive

$$J_{i,j,l}(a, b) = J_{i,j,l}(b, b) - \int_a^b \partial_x J_{i,j,l}(s) ds = C_{i,j,l} \log \frac{b}{a}, \quad C_{i,j,l} \triangleq \int_0^1 (|F_{i,j,l}(1, s)| + |F_{i,j,l}(s, 1)|) ds.$$

It remains to estimate $C_{i,j,l}$. This can be done by using the above identity with $(a, b) = (1, 2)$,

$$C_{i,j,l} = \frac{J_{i,j,l}(1, 2)}{\log 2},$$

and applying the estimates in Section 5.1.3 to $J_{i,j,l}$ in the domain $Q_{1,2}$. In particular, using $|K_i(y_1, y_2)| = |K_i(y_2, y_1)|$ and (5.3)

$$\begin{aligned} \int_0^1 (|K_1(s, 1)| + |K_1(1, s)|) ds &= 2 \int_0^1 K_1(s, 1) ds = \frac{1}{1+s^2} \Big|_0^1 = \frac{1}{2}, \\ \int_0^1 (|K_2(s, 1)| + |K_2(1, s)|) ds &= 2 \int_0^1 K_2(1, s) ds = \frac{s}{s^2+1} \Big|_0^1 = \frac{1}{2}, \end{aligned}$$

we get

$$(5.18) \quad \int_{[-b,b]^2 \setminus [-a,a]^2} |K_i(y)| dy = 2 \log(b/a).$$

5.2. Strategy of estimating the explicit integrals. In the sharp Hölder estimates, we need to estimate several explicit integrals, e.g. (5.34)

$$(5.19) \quad \left| \int_0^\infty \int_{1/2}^\infty \mathbf{1}_{s_2 \leq b} \mathbf{1}_{s_1 \geq f(s_2)} |T(s_1, s_2) - s_1|^{1/2} \Delta(s) ds \right|, \quad \Delta(s) = K(s - s_*) - K(s - s_r),$$

in the Hölder estimate of u_x , where Δ is some kernel singular at s_* , T is the map, and s_r is another fixed point, e.g. $(-1/2, 0)$, outside the domain of the integral. In all cases, the integrand can be written as

$$I(s_1, s_2) = \mathbf{1}_{A(s_1, s_2) \geq 0} |T(s) - s_j|^{1/2} |\Delta(s)|$$

for $j = 1$ or 2 , where $\mathbf{1}_{A(s_1, s_2) \geq 0}$ is some indicator function. To obtain a sharp estimate, we decompose the domain of the integral into the bulk, and the far-field. In the far-field, s is very large. Using the cancellation in the kernel Δ , we have

$$|\Delta(s)| \leq C |s_* - s_r| |\partial_j K(s)|,$$

for some constant C and $j = 1$ or 2 . In the domain $A(s) \geq 0$, we have $0 \leq T(s) \leq s_j$, and apply the trivial bound $|T(s) - s_j| \leq s_j$. This allows us to estimate the far-field of the integral easily. Due to the decay of the integrand, the contribution of the integral in the far-field is very small.

In the bulk, we partition the integral using dense mesh and estimate the integral in each small region $Q = [x_l, x_u] \times [y_l, y_u]$. We study the monotonicity of the function $A(s)$ and $T(s)$ and obtain their piecewise bounds in Q

$$A^l(Q) \leq A(s) \leq A^u(Q), \quad T^l(Q) \leq T(s) \leq T^u(Q), \quad s \in Q.$$

The upper and lower bounds of s_j are given by the endpoints of the domain. It follows

$$(5.20) \quad \begin{aligned} \mathbf{1}_{A^l \geq 0} \leq \mathbf{1}_{A(s) \geq 0} \leq \mathbf{1}_{A^u \geq 0}, \quad |T(s) - s_j| \leq \max_{\alpha, \beta=l, u} |T^\alpha - s^\beta|, \\ S = \int_Q I(s) ds \leq \mathbf{1}_{A^u \geq 0} \max_{\alpha, \beta=l, u} |T^\alpha - s^\beta|^{1/2} \int_Q |\Delta(s)| ds. \end{aligned}$$

In particular, the integral is 0 if $A^u < 0$. The set $A(s) = 0$ has zero measure in our integrals. The integral for $|\Delta(s)|$ can be estimated using the Trapezoidal rule in Section 5.1. The same method applies to estimate more general integrals

$$\prod_{1 \leq i \leq n} \mathbf{1}_{A_i(s) \geq 0} f(s) |\Delta(s)|$$

in Q , for some function $f \geq 0$ and several indicator functions $\mathbf{1}_{A_i(s) \geq 0}$. There are three improvements.

5.2.1. $\Delta(s)$ has a fixed sign. In the support of the integrand $I(s)$, $\Delta(s)$ has a fixed sign. Yet, it may not have a fixed sign in Q since we estimate the indicator function $\mathbf{1}_{A \geq 0} \leq \mathbf{1}_{A^u \geq 0}$. If $A^l \geq 0$, then we have $\mathbf{1}_{A(s) \geq 0} \geq \mathbf{1}_{A^l \geq 0} = 1$, $\Delta(s)$ has a fixed sign in Q and

$$S = \int_Q |T(s) - s_j|^{1/2} |\Delta(s)| ds \leq \max_{s \in Q} |T(s) - s_j|^{1/2} \int_Q |\Delta(s)| ds = \max_Q |T(s) - s_j|^{1/2} \left| \int_Q \Delta(s) ds \right|.$$

For the last integral, we can use the analytic integral formula for $\Delta(s)$.

5.2.2. *Estimate of $|T(s) - s|$ near the singularity.* Near the singularity of $\Delta(s)$ at s_* , we need to exploit the smallness of $|T(s) - s_j| \lesssim |s - s_*|$ to cancel the singularity of $\Delta(s)$, which has order of $|s - s_*|^{-2}$. The previous piecewise bounds for $T(s) - s_j$ in (5.20) do not provide such an order. We estimate

$$(5.21) \quad \mathbf{1}_{A(s) \geq 0} |T(s) - s_j| \leq c_1(Q) |s_1 - s_{*,1}| + c_2(Q) |s_2 - s_{*,2}|.$$

The piecewise constant $c_i(Q)$ can be obtained using the piecewise bounds of ∇T . We will study the piecewise bounds of $T, \nabla T$ in the following section.

5.2.3. *Estimate of integral near the singularity.* Recall from (5.19) that the kernel Δ can be written as $K(s - s_*) - K(s - s_r)$. Near the singularity, $K(s - s_r)$ is regular and its related integral is much smaller. Using (5.21) and the triangle inequality, we get

$$S \leq \max_Q |T(s) - s_j|^{1/2} \int_Q |K(s - s_r)| ds + \int_Q |T(s) - s_j|^{1/2} |K(s - s_*)| ds \triangleq S_1 + S_2.$$

For S_1 , we estimate it using the previous method. For S_2 , we further bound $|T(s) - s_j|$ as follows

$$|T(s) - s_j| \leq c_3(Q) |s_j - s_{*,j}|,$$

for some piecewise constant $c_3(Q)$ depending on T and ∇T . Then we estimate S_2 as follows

$$S_2 \leq c_3(Q) \int_Q |K(s - s_*)| |s_j - s_{*,j}|^{1/2} ds = c_3(Q) \int_{Q-a} |K(s)| |s_j|^{1/2} ds,$$

for $a = (s_{*,1}, 0)$ or $a = (0, s_{*,2})$, and evaluate the integral using the analytic integral formula for $K(s) |s_j|^{1/2}$.

5.2.4. *Partition the domain of the integrals and the piecewise bounds.* To estimate the 2D integral, we will partition the domain $[0, \infty)^2$ using adaptive mesh $x_1 < x_2 < \dots < x_{n_1+1} = \infty, y_1 < y_2 < \dots < y_{n_2+1} = \infty$. The integrals we will estimate depend on 1 or 2 parameters. For example, the integral in (5.34) depends on b , and the integrals in (5.43), (5.49) depend on $[A, B]$. These parameters impose a constraint on the domain of the integral, e.g. $\mathbf{1}_{y_2 \leq b}$ in (5.34), $\mathbf{1}_{y_2 \leq A}$ in (5.49). Using the monotonicity of the integrals (or different parts of the integrals) on these parameters, we only need to estimate the integrals with parameters on the mesh, e.g. $b = y_i$ for (5.34) or $A = y_i, B = x_j$ for (5.49),

$$\int_0^b |F(s_1, s_2)| ds_2 \leq \int_0^{b^u} |F(s_1, s_2)| ds_2, \quad b \in [b^l, b^u].$$

For b, A, B on the grid point, the support of the integrand is the union of the small grid $[x^l, x^u] \times [y^l, y^u]$. Thus we can take the sum of the integrals in these grids to estimate the integrals with parameter b, A, B . For example, after we estimate

$$Q_i = \int_{b_i}^{b_{i+1}} |I(s)| ds,$$

the cumulative sum of $\sum_{j \leq i} Q_j$ provides a piecewise bound for $\int_0^b |I(s)| ds$. This strategy allows us to obtain piecewise bounds for parameters b, A, B and the uniform bound.

5.3. Estimate of the sharp constants for u_x . We first study some properties of the transport map $T(s_1, s_2)$ for u_x . Recall from Section 3.3 [3] that T solves the cubic equation

$$(5.22) \quad 0 = -1 - 8Ts_1 + 16Ts_1(T^2 + Ts_1 + s_1^2) - 8s_2^2(1 - 4s_1T + 2s_2^2).$$

Solving

$$\Delta(y_1, y_2) = 0, \quad \Delta(y) \triangleq K_1(y_1 + 1/2, y_2) - K_1(y_1 - 1/2, y_2) = 0,$$

for $y_1 \geq 0$, we yield the threshold $f(s_2)$ determining the sign of the kernel

$$(5.23) \quad f(s_2) = \left(\frac{\frac{1}{2} - 2s_2^2 + \sqrt{16s_2^4 + 4s_2^2 + 1}}{6} \right)^{1/2}.$$

For $s_1, s_2 > 0$, we establish in Section 3.3 [3]

$$(5.24) \quad \Delta(s) \leq 0, \quad s_1 \in [f(s_2), \infty), \quad \Delta(s) \geq 0, \quad s_1 \in (0, f(s_2)].$$

Firstly, for a fixed $s_2 \neq 0$ and $s_1 > 0$, we show that it has a unique solution on $[0, \infty)$. Dividing s_1 on both side of (5.22) yields

$$(5.25) \quad g(T) \triangleq T^3 + T^2s_1 + T(s_1^2 - \frac{1}{2} + 2s_2^2) - \frac{(4s_2^2 + 1)^2}{16s_1} = 0.$$

Since $g(0) < 0$ and $g(\infty) > 0$, the above equation has at least one real root on \mathbb{R}_+ . We introduce $Z = T + \frac{s_1}{3}$ and can rewrite the above equation in terms of Z

$$(5.26) \quad \begin{aligned} 0 &= (T + \frac{s_1}{3})^3 - \frac{s_1^3}{27} + T(\frac{2}{3}s_1^2 - \frac{1}{2} + 2s_2^2) - \frac{(4s_2^2 + 1)^2}{16s_1} \\ &= Z^3 + Z(\frac{2}{3}s_1^2 - \frac{1}{2} + 2s_2^2) - \left(\frac{(4s_2^2 + 1)^2}{16s_1} + \frac{7}{27}s_1^3 + \frac{s_1}{3}(2s_2^2 - \frac{1}{2}) \right) \\ &\triangleq Z^3 + p(s_1, s_2)Z + q(s_1, s_2). \end{aligned}$$

The discriminant is given by

$$(5.27) \quad \Delta_Z(s_1, s_2) = -(27q(s_1, s_2)^2 + 4p(s_1, s_2)^3).$$

Note that

$$-q \geq \frac{7}{27}s_1^3 - \frac{s_1}{6} + \frac{1}{16s_1} \geq (2\sqrt{\frac{7}{27} \cdot \frac{1}{16}} - \frac{1}{6})s_1 \geq 0,$$

and $-q, p$ are increasing in $|s_2|$. We yield

$$-\Delta_Z(s_1, s_2) \geq -D_Z(s_1, 0) = \frac{(1 - 4s_1^2)^2(27 - 56s_1^2 + 48s_1^4)}{256s_1^2} \geq 0.$$

When $s \neq (\frac{1}{2}, 0)$, the above inequality is strict. Using the solution formula for a cubic equation, we obtain that the cubic equation for T or Z has a unique real root for $s \neq (\pm\frac{1}{2}, 0)$ given by

$$(5.28) \quad Z = r_1 - \frac{p}{3r_1}, \quad r_1 = \left(\frac{-q + \sqrt{q^2 + \frac{4}{27}p^3}}{2} \right)^{1/3}, \quad T = Z - \frac{s_1}{3},$$

where p, q are defined in (5.26). For $s = (\frac{1}{2}, 0)$, it is easy to get that $T = \frac{1}{2} = s_1$ is the unique solution in $(0, \infty)$, which is also given by the above formula.

We have the following basic properties for the map T and the threshold $f(s_2)$ (5.23).

Lemma 5.3. *The map $T(s_1, s_2)$ is increasing in s_2 and decreasing in s_1 . Moreover, we have $f(s_2) \geq \frac{1}{2}$, $f(s_2)$ is increasing in s_2 for $s_2 > 0$, and*

$$(5.29) \quad \begin{aligned} 4s_2^2 + 1 &\geq \max(4s_1T(s_1, s_2), 1), \quad T^2 + Ts_1 + s_1^2 + 2s_2^2 \geq \frac{3}{4}, \quad \text{for } s_1 \geq 0, \\ |T(s_1, s_2) - s_1| &\leq \max_{x \in [f(s_2), s_1]} (|T_x(x, s_2)| + 1) |s_1 - \frac{1}{2}|, \quad s_1 \geq f(s_2). \end{aligned}$$

Proof. The estimate $f(s_2) \geq \frac{1}{2}$ follows from (5.23) and $1 + 4s_2^2 + 16s_2^4 \geq (1 + 2s_2^2)^2$. From (5.23), for $s_2 > 0$, since

$$\frac{d}{ds_2}(6f^2(s_2)) = -4s_2 + \frac{64s_2^3 + 8s_2}{2\sqrt{16s_2^4 + 4s_2^2 + 1}}, \quad (8s_2^2 + 1)^2 > 16s_2^4 + 4s_2^2 + 1,$$

$f(s_2)$ is increasing in $s_2 > 0$. Denote $P = 4s_1T$, $Q = 4s_2^2 + 1$. Using (5.22), we get

$$\begin{aligned} Q^2 &= (4s_2^2 + 1)^2 = 16Ts_1(T^2 + Ts_1 + s_1^2) + 32s_2^2s_1T - 8Ts_1 \\ &\geq 16Ts_1 \cdot 3Ts_1 + 2(Q - 1)P - 2P = 3P^2 + 2PQ - 4P. \end{aligned}$$

If $P \leq 1$, we derive $Q \geq 1 \geq P$. If $P > 1$, solving the above quadratic inequality in Q , we yield

$$Q \geq P + \sqrt{4P^2 - 4P}, \quad \text{or} \quad Q \leq P - \sqrt{4P^2 - 4P}.$$

Note that for $P > 1$, we have $P - \sqrt{4P^2 - 4P} < 1$. Thus, we must have

$$Q \geq P + \sqrt{4P^2 - 4P} \geq P.$$

This proves the first inequality in (5.29). Using (5.22) again, we derive

$$T^2 + Ts_1 + s_1^2 + 2s_2^2 = \frac{1}{16Ts_1}(8Ts_1 + (4s_2^2 + 1)^2) \geq \frac{1}{16Ts_1}(8Ts_1 + 4Ts_1) = \frac{3}{4},$$

where we have used the first inequality in (5.29) that we just proved. The last inequality in (5.29) follows directly from $f(s_2) \geq \frac{1}{2}$ and the mean value theorem.

Since (5.22) is symmetric in T and s_1 , and its has a unique positive real root for any $s_1 > 0$, we get $T(T(s_1, s_2), s_2) = s_1$, or $T \circ T = Id$. Using (5.90) and (5.24), we get

$$\Delta(s_1, s_2) = T_{s_1}(s_1, s_2)\Delta(T(s_1, s_2), s_2).$$

Since $\Delta(s_1, s_2)$ and $\Delta(T(s_1, s_2), s_2)$ have opposite sign, it follows $T_{s_1} \leq 0$.

Taking s_2 derivative on both sides of (5.25), we yield

$$(5.30) \quad \frac{dT}{ds_2}(3T^2 + 2Ts_1 + s_1^2 - \frac{1}{2} + 2s_2^2) + s_2(4T - \frac{4s_2^2 + 1}{s_1}) = 0.$$

Using the first and the second inequality in (5.29), we prove $\frac{dT}{ds_2} \geq 0$. \square

5.3.1. *Piecewise bounds for $T, \nabla T$.* Using the monotonicity of T in Lemma 5.3 and its analytic formula, we can derive the piecewise upper and lower bounds for the map $T(s)$ in $[s_1^l, s_1^u] \times [s_2^l, s_2^u] \subset \mathbb{R}_+^2$,

$$(5.31) \quad T^l = T(s_1^u, s_2^l), \quad T^u = T(s_1^l, s_2^u).$$

Such a formula can lead to a large round off error when s_1/s_2 and s_1 are large. In such a case, we derive another bound. Firstly, we write (5.22), (5.25) as follows

$$T^3 + T^2s_1 + (T - \hat{T})(s_1^2 - \frac{1}{2} + 2s_2^2) = 0, \quad \hat{T} = \frac{(4s_2^2 + 1)^2}{16s_1(s_1^2 - \frac{1}{2} + 2s_2^2)}.$$

The function \hat{T} can be seen as the approximation of T , and we can estimate the piecewise bounds for \hat{T} easily. For $s_1 > 1, s_2 > 0$, since $T > 0, s_1^2 - \frac{1}{2} + 2s_2^2 > 0$, we yield $T \leq \hat{T}$ and

$$(5.32) \quad T = \hat{T} - \frac{T^3 + T^2s_1}{s_1^2 - \frac{1}{2} + 2s_2^2} \geq \hat{T} - \frac{\hat{T}^3 + \hat{T}^2s_1}{s_1^2 - \frac{1}{2} + 2s_2^2}.$$

Using the piecewise bounds for \hat{T} and the above estimates, we can obtain another piecewise bounds for T . Taking s_1 derivative of (5.25), we yield

$$\frac{dT}{ds_1}(3T^2 + 2Ts_1 + s_1^2 - \frac{1}{2} + 2s_2^2) + T^2 + 2s_1T + \frac{(4s_2^2 + 1)^2}{16s_1^2} = 0.$$

From (5.30) and the above formula, we obtain $\partial_{s_i}T = \frac{P_i(T, s_1, s_2)}{Q_i(T, s_1, s_2)}$ for some polynomials P_i, Q_i . Using the above piecewise bounds for T , we can further obtain piecewise bounds for ∇T .

Near the singularity of the integral, we need to obtain a sharp estimate of $T(s) - s_1$. Since $T(f(s_2), s_2) = f(s_2)$, and $f(s_2) \geq 1/2$, for $s_1 \geq f(s_2)$, we yield

$$(5.33) \quad \begin{aligned} |T(s) - s_1| &= |T(s) - s_1 - (T(f(s_2), s_2) - f(s_2))| \leq |s_1 - f(s_2)| \left(\max_{\xi \in [f(s_2), s_1]} |T_{s_1}(\xi, s_2)| + 1 \right) \\ &\leq |s_1 - 1/2| \left(\max_{\xi \in [1/2, s_1]} |T_{s_1}(\xi, s_2)| + 1 \right). \end{aligned}$$

We can estimate the upper bound using the piecewise bounds for ∇T .

5.3.2. *Estimate of the explicit integrals for $[u_x]_{C_x^{1/2}}$.* In this section, we estimate the integral

$$(5.34) \quad \begin{aligned} S(b) &= \left| \int_0^b ds_2 \int_{f(s_2)}^\infty |T(s_1, s_2) - s_1|^{1/2} \Delta(s) ds_1 \right| = \left| \int_0^\infty \int_{1/2}^\infty \mathbf{1}_{s_2 \leq b} \mathbf{1}_{s_1 \geq f(s_2)} |T(s_1, s_2) - s_1|^{1/2} \Delta(s) ds \right|, \\ \Delta(s) &= K_1(s_1 + \frac{1}{2}, s_2) - K_1(s_1 - \frac{1}{2}, s_2), \end{aligned}$$

in the upper bound in Lemma 3.1 [3], and obtain its piecewise bounds for $b \in [1, \infty)$, where $f(s_2)$ and $T(s)$ are determined by (5.23), (5.22). Note that one needs to multiply $S(b)$ by a factor $\frac{4}{\pi}$ to get the constant in Lemma 3.1 [3]. In the second identity, we have used $f(s_2) \geq \frac{1}{2}$ from Lemma 5.3. The kernel satisfies $\Delta(s) \leq 0$ in the domain of the integral.

For very large R_0, R_1 , e.g. $R_0 = R_1 = 10^8$, and small $m_1 > 0$, we decompose the domain integral into four parts

$$(5.35) \quad \begin{aligned} D_1 &= [1/2, R_1] \times [m_1, R_0], \quad D_2 = [1/2, R_1] \times [0, m_1], \\ D_3 &= [R_1, \infty] \times [0, R_0], \quad D_4 = [1/2, \infty] \times [R_0, \infty). \end{aligned}$$

The first part captures the bulk part of the integrals, D_2 captures the integral near the singularity, and D_3 and D_4 capture the far-field of the integral. We follow the strategy in Section 5.2 to estimate the integrals.

Integral in the finite domain. To estimate the integrals in D_1, D_2 , we discretize the domain $[1/2, R_1] \times [0, R_0]$ using

$$1/2 = x_0 < x_2 < \dots < x_{n_1} = R_1, \quad 0 = z_0 < z_1 < \dots < z_{n_2} = R_0.$$

Since $S(b)$ is increasing in b , it suffices to estimate $S(b)$ for $b = z_i$ and $S(\infty)$. For a fixed domain $s \in [x_{i-1}, x_i] \times [z_{j-1}, z_j] \triangleq X_i \times Z_j \triangleq Q_{ij}$, since $f(s_2)$ is increasing (Lemma 5.3), we have

$$S_{ij} = \int_{Q_{ij}} \mathbf{1}_{s_1 \geq f(s_2)} |T(s) - s_1|^{1/2} |\Delta(s)| ds \leq \mathbf{1}_{x_i \geq f(z_{j-1})} \int_{Q_{ij}} |T(s) - s_1|^{1/2} |\Delta(s)| ds.$$

Using the piecewise bounds of $T(s)$ in Q_{ij} (5.31), (5.33), we get

$$|T(s_1, s_2) - s_1| \leq \min \left(\max(T^u - s_1^l, s_1^u - T^l), |s_1 - 1/2| \left(\max_{\xi \in [1/2, x_i] \times Z_j} |T_{s_1}(\xi, s_2)| + 1 \right) \right).$$

For the integral $\int_Q |\Delta(s)|$, we use the two estimates in Section 5.1 based on the Trapezoidal rule and the analytic integral formulas. If $x_{i-1} \geq f(z_j)$, from (5.24), we get

$$s_1 \geq x_{i-1} \geq f(z_j) \geq f(s_2), \quad |\Delta(s)| = -\Delta(s),$$

for any $s \in Q_{ij}$. Hence, we can use the analytic integral formula for $\Delta(s)$ in this case.

Near the singularity $s = (1/2, 0)$, we use the triangle inequality to obtain another estimate

$$S_{ij} \leq \int_{Q_{ij}} |T(s) - s_1|^{1/2} (K_1(s_1 + 1/2, s_2) + K_1(s_1 - 1/2, s_2)) ds \triangleq I_1 + I_2.$$

where we have used $K_1(s_1, s_2) > 0$ for $s_1, s_2 > 0$. The regular part I_1 is estimated using the previous method. Using (5.33), for the singular part I_2 , we have

$$I_2 \leq \left(\max_{\xi \in [1/2, x_i] \times Z_j} |T_{s_1}(\xi, s_2)| + 1 \right) \int_{Q_{ij}} (s_1 - 1/2)^{1/2} K_1(s - 1/2, s_2) ds,$$

and obtain the integral using the analytic integral formula for $K_1(s) s_1^{1/2}$.

Integral in the far-field. The integrals in the far-field D_3, D_4 (5.35) are very small. Next, we estimate the integral in D_3 , for $s_1 \geq f(s_2)$, we derive

$$(5.36) \quad s_1 \geq f(s_2) \geq T(s), \quad |s_1 - T(s)| \leq s_1.$$

Denote $R_m = \min(R_0, R_1)$. In $D_3 \cup D_4$, we have $\max(s_1, s_2) \geq R_m$. Applying Lemma 5.2 with $i + j = 3$ and the Mean Value Theorem yields

$$|\Delta_{in}(s)| = |\partial_1 K_1(\xi, s_2)| \leq \frac{1}{(\xi^2 + s_2^2)^{3/2}} \leq C_R |s|^{-3}, \quad C_R = \left(\frac{R_m}{R_m - 1/2}\right)^3,$$

for some $\xi \in [s_1 - \frac{1}{2}, s_1 + \frac{1}{2}]$, where we have used $|(\xi, s_2)| \geq |s| - 1/2 \geq |s|(1 - \frac{1}{2R_m}) = |s|\frac{R_m}{R_m - 1/2}$. For the integrals in $D = D_3$ or $D = D_3 \cup D_4$ (5.35), using the above estimates, we yield

$$\begin{aligned} S_D &\triangleq \int_D \mathbf{1}_{s_2 \leq b} \mathbf{1}_{s_1 \geq f(s_2)} |T - s_1|^{\frac{1}{2}} |\Delta(s)| ds \leq C_R \int_D \frac{s_1^{1/2}}{|s|^3} ds, \\ S_{D_3} &\leq C_R \int_{R_1}^{\infty} ds_1 \int_0^{R_0} \frac{s_1^{1/2}}{|s|^3} ds \leq C_R \int_{R_1}^{\infty} s_1^{-5/2} R_0 ds_1 = C_R \frac{2}{3} R_1^{-3/2} R_0 = C_R \frac{2}{3} R_1^{-3/2} R_0, \\ S_{D_3, D_4} &\leq C_R \int_{|s| \geq R_m, s \in \mathbb{R}_2^+} \frac{s_1^{1/2}}{|s|^3} ds \leq C_R \int_{|s| \geq R_m, s \in \mathbb{R}_2^+} |s|^{-\frac{5}{2}} ds = C_R \int_{R_m}^{\infty} r^{-\frac{3}{2}} dr \int_0^{\frac{\pi}{2}} d\beta = C_R \pi R_m^{-\frac{1}{2}}. \end{aligned}$$

Note that if $b \leq R_0$, due to the restriction $\mathbf{1}_{s_2 \leq b}$, we only need to estimate the integral in D_3 .

Combining the estimate of the integrals S_{ij} and S_{D_3} , we can estimate $S(b)$ for $b = z_i, i = 1, 2, \dots, n_2$. Adding the contribution from S_{D_4} , we obtain the estimate of $S(\infty)$. Since $S(b)$ is increasing, we obtain piecewise bounds and the uniform bound for $S(b)$ (5.34). We remark that to obtain the constant in Lemma 3.1s [3], one needs to further multiply the constant $\frac{4}{\pi}$.

5.3.3. *Estimate the integrals for $[u_x]_{C_y}^{1/2}$.* In this section, we estimate the integral

$$(5.37) \quad S(a) = \int_0^a \int_0^{\infty} y_1^{1/2} \left| \frac{y_1(1/2 - y_2)}{(y_1^2 + (1/2 - y_2)^2)^2} + \frac{y_1(1/2 + y_2)}{(y_1^2 + (1/2 + y_2)^2)^2} \right| dy,$$

in Lemma 3.2 [3] for the estimate of $[u_x]_{C_y}^{1/2}$. Note that one needs to multiply $S(a)$ by a factor $\frac{\sqrt{2}}{\pi}$ to get the constant in Lemma 3.2. Clearly, S is increasing in a .

Swapping the dummy variables y_1, y_2 and writing $(1/2 - y_1) = -(y_1 - 1/2)$, we yield

$$(5.38) \quad S(a) = \int_0^{\infty} dy_1 \int_0^a y_2^{1/2} \left| \frac{y_2(y_1 - 1/2)}{(y_2^2 + (1/2 - y_1)^2)^2} - \frac{y_2(1/2 + y_1)}{(y_2^2 + (1/2 + y_1)^2)^2} \right| dy_2 = \int_0^a y_2^{1/2} dy_2 \int_0^{\infty} |\Delta(y)| dy_1,$$

where $\Delta(s)$ is defined in (5.24).

Using the sign property (5.24), we can first compute the integral in y_1

$$(5.39) \quad \begin{aligned} \int_0^{\infty} |\Delta(y)| dy_1 &= \left(\int_0^{f(y_2)} - \int_{f(y_2)}^{\infty} \right) \Delta(y) dy_1 = \frac{1}{2} \left(-\frac{y_2}{y_2^2 + (y_1 + \frac{1}{2})^2} + \frac{y_2}{y_2^2 + (y_1 - \frac{1}{2})^2} \right) \left(\Big|_0^{f(y_2)} - \Big|_{f(y_2)}^{\infty} \right) \\ &= g(f(y_2) - 1/2, y_2) - g(f(y_2) + 1/2, y_2), \quad g(y) \triangleq \frac{y_2}{y_1^2 + y_2^2}, \end{aligned}$$

where we have used $A\left(\Big|_a^b - \Big|_c^d\right)$ to denote $(A(b) - A(a)) - (A(d) - A(c))$. The boundary term vanishes at 0 and ∞ . It follows

$$S(a) = \int_0^a y_2^{1/2} \left(g(f(y_2) - 1/2, y_2) - g(f(y_2) + 1/2, y_2) \right) dy_2, \quad g(y) \triangleq \frac{y_2}{y_1^2 + y_2^2}.$$

We partition $[0, \infty)$ using mesh $0 = z_0 < z_1 < \dots < z_{n_2} = R_0$ and ∞ . Since $S(a)$ is increasing in a , it suffices to estimate $S(a)$ for $a = z_i$ and $a = \infty$.

Integral in a finite domain. Clearly, $g(y)$ is decreasing in y_1 for $y_1 > 0$. For the integral in $Z_j = [z_{j-1}, z_j]$, since $f(y_2)$ is increasing in y_2 and $f(y_2) \geq 1/2$ (see Lemma 5.3), we get

$$1/2 \leq f^l = f(z_{j-1}) \leq f(y_2) \leq f(z_j) = f^u, \quad y_2 \in Z_j.$$

Using the above estimate and $d_{y_2} \log |y| = g$, we derive

$$\begin{aligned} S_j &\triangleq \int_{z_{j-1}}^{z_j} y_2^{1/2} \left(g(f(y_2) - \frac{1}{2}, y_2) - g(f(y_2) + \frac{1}{2}, y_2) \right) dy_2 \leq z_j^{1/2} \int_{Z_j} \left(g(f^l - \frac{1}{2}, y_2) - g(f^u + \frac{1}{2}, y_2) \right) dy_2 \\ &= z_j^{1/2} \frac{1}{2} \left(\log(y_2^2 + (f^l - \frac{1}{2})^2) - \log(y_2^2 + (f^u + \frac{1}{2})^2) \right) \Big|_{z_{j-1}}^{z_j}. \end{aligned}$$

We remark that f^l, f^u do not depend on y_2 . Using

$$\begin{aligned} g(f(y_2) - \frac{1}{2}, y_2) - g(f(y_2) + \frac{1}{2}, y_2) &= \frac{y_2}{(f - \frac{1}{2})^2 + y_2^2} - \frac{y_2}{(f + \frac{1}{2})^2 + y_2^2} = \frac{2y_2 f}{((f - \frac{1}{2})^2 + y_2^2)((f + \frac{1}{2})^2 + y_2^2)} \\ &\leq \frac{f^u}{f^l} \frac{2y_2 f^l}{((f^l - 1/2)^2 + y_2^2)((f^l + 1/2)^2 + y_2^2)} = \frac{f^u}{f^l} \left(g(f^l - \frac{1}{2}, y_2) - g(f^l + \frac{1}{2}, y_2) \right), \end{aligned}$$

and computation similar to the above, we obtain another estimate

$$S_j \leq z_j^{1/2} \frac{f^u}{f^l} \frac{1}{2} \left(\log(y_2^2 + (f^l - \frac{1}{2})^2) - \log(y_2^2 + (f^l + \frac{1}{2})^2) \right) \Big|_{z_{j-1}}^{z_j} = z_j^{1/2} \frac{f^u}{2f^l} \log \left(1 - \frac{2f^l}{y_2^2 + (f^l + 1/2)^2} \right) \Big|_{z_{j-1}}^{z_j}.$$

For the integral near the singularity, e.g. when z_j is small, we need to exploit the cancellation between the kernel and $y_2^{1/2}$. Since the more regular part $-g(f(y_2) + 1/2, y_2) \leq 0$, we derive an improved estimate near the singularity

$$S_j \leq \int_{z_{j-1}}^{z_j} y_2^{1/2} g(f(y_2) - 1/2, y_2) dy_2 \leq \int_{z_{j-1}}^{z_j} y_2^{1/2} \frac{1}{y_2} dy_2 = 2(z_j^{1/2} - z_{j-1}^{1/2}).$$

Integral in the far-field. For the integral in the region $y_2 \geq R_0$, we treat it as an error. Using $\xi^2 + y_2^2 \geq 2|\xi y_2|$ and the Mean Value Theorem, we get

$$|g(f(y_2) - 1/2, y_2) - g(f(y_2) + 1/2, y_2)| = |\partial_{y_1} g(\xi, y_2)| = \frac{2|\xi y_2|}{(\xi^2 + y_2^2)^2} \leq \frac{1}{y_2^2},$$

for some $\xi \in [f - 1/2, f + 1/2]$. Using (5.39) and the above estimate, we yield

$$\int_{R_0}^{\infty} y_2^{\frac{1}{2}} \int_0^{\infty} |\Delta(y)| dy_1 = \int_{R_0}^{\infty} y_2^{1/2} \left(g(f(y_2) - \frac{1}{2}, y_2) - g(f(y_2) + \frac{1}{2}, y_2) \right) dy_2 \leq \int_{R_0}^{\infty} y_2^{\frac{1}{2}} y_2^{-2} dy_2 = 2R_0^{-\frac{1}{2}}.$$

5.4. Estimate the sharp constants for u_y, v_x . In this section, we estimate the integrals in the sharp Hölder estimate of u_y, v_x . Recall the monotonicity lemma proved in Section 3.4 [3].

Lemma 5.4. *Suppose $f, g \in L^1$ and $g \geq 0$ is monotone increasing on $[0, \infty]$. For any $0 \leq k \leq b \leq c$, we have*

$$\int_{b-k}^{b+k} |f(x-k)|g(x)dx \leq \int_{b-k}^{c-k} |f(x-k) - f(x+k)|g(x)dx + \int_{c-k}^{c+k} |f(x-k)|g(x)dx.$$

We have the following basic Lemma for the map T (5.92). See the left figure in Figure 2 for an illustration of the sign of the kernel and the map.

Lemma 5.5. *For $y_1, y_2 > 0$, equation (5.92) has a unique solution T in $(0, \infty)$. For $y_1, y_2 > 0$, $T(y)$ is decreasing in y_2 , and $|\frac{d}{dy_2} T(y)| \leq 1$ if $T(y) \leq y_2$; $T(y_1, y_2)$ is decreasing in y_1 for $y_1 \leq 1$, and T is increasing in y_1 for $y_1 > 1$.*

Proof. We fix $y_1, y_2 > 0$. Recall the equation (5.92)

$$(5.40) \quad g(T) \triangleq T^3 + T^2 y_2 + T(y_2^2 + 2 + 2y_1^2) - \frac{(y_1^2 - 1)^2}{y_2} = 0.$$

Since $g(0) < 0$ and $g(\infty) > 0$, there exists at least one real root in $(0, \infty)$. The uniqueness is discussed below (5.92) and follows from the discriminant. Taking y_2 derivative, we yield

$$\frac{dT}{dy_2}(3T^2 + 2Ty_2 + y_2^2 + 2 + 2y_1^2) + T^2 + 2Ty_2 + \frac{(y_1^2 - 1)^2}{y_2^2} = 0,$$

which implies $\frac{dT}{dy_2} < 0$. For $T(y) \leq y_2$, using (5.40), we get

$$T^2 + 2Ty_2 + \frac{(y_1^2 - 1)^2}{y_2^2} = T^2 + 2Ty_2 + \frac{T^3 + T^2y_2 + T(y_2^2 + 2 + 2y_1^2)}{y_2} \leq T^2 + 2Ty_2 + 2T^2 + (y_2^2 + 2 + 2y_1^2),$$

which along with the formula of $\frac{dT}{dy_2}$ implies $|\frac{dT}{dy_2}| \leq 1$.

Taking y_1 derivative, we yield

$$\frac{dT}{dy_1}(3T^2 + 2Ty_2 + y_2^2 + 2 + 2y_1^2) + 4Ty_1 - \frac{4y_1(y_1^2 - 1)}{y_2} = 0.$$

From the above equation for T , we get

$$\begin{aligned} (y_1^2 - 1)^2 &= y_2(T^3 + T^2y_2 + T(y_2^2 + 2 + 2y_1^2)) > (Ty_2)^2, \\ y_1^2 - 1 > Ty_2, \quad \text{for } y_1 > 1, \quad y_1^2 - 1 < -Ty_2, \quad \text{for } y_1 < 1. \end{aligned}$$

Hence, we prove the monotonicity properties of T . \square

The above derivations provide the formula of ∇T in terms of T, y_1, y_2 . We can use the monotonicity of T to obtain its piecewise bounds. For large y_1, y_2 , to reduce the round off error, following the argument in Section 5.3.1 and (5.32) and using (5.40), we obtain another piecewise bounds for T

$$(5.41) \quad \hat{T} \geq T = \hat{T} - \frac{T^3 + T^2y_2}{y_2^2 + 2 + 2y_1^2} \geq \hat{T} - \frac{\hat{T}^3 + \hat{T}^2y_2}{y_2^2 + 2 + 2y_1^2}, \quad \hat{T} \triangleq \frac{(y_1^2 - 1)^2}{y_2(y_2^2 + 2 + 2y_1^2)}.$$

Using the formulas of ∇T and the bounds for T , we can further obtain the piecewise bounds for ∇T . Near the singularity of the kernel $(\pm 1, 0)$, we need a sharp bound for $T(y) - y_2$ by $C|y - 1|$ so that it can cancel the singularity of the kernel with order -2 . Note that the bound based on the piecewise bounds for $T(y) - y_2$ does not provide the order $|y - 1|$ when y is sufficiently close to 1. Recall $s_c(y_1)$ from (5.89). It satisfies $T(y_1, s_c(y_1)) = s_c(y_1)$. Following (5.36), for $y_2 \geq s_c(y_1) \geq T(y)$ we have

$$|y_2 - T(y_1, y_2)| = |y_2 - T(y_1, y_2) - s_c(y_1) - T(y_1, s_c(y_1))| \leq |y_1 - s_c(y_1)| \left(\max_{\xi \in [s_c(y_1), y_2]} |\partial_{y_2} T(y_1, \xi)| + 1 \right).$$

Using the formula (5.89) and

$$(2y_1^2 - 3y_1 + 2)^2 - (y_1^4 - y_1^2 + 1) = 3y_1^4 - 12y_1^3 + (8 + 9 + 1)y_1^2 - 12y_1 + 3 = 3(y_1 - 1)^4 \geq 0,$$

we yield

$$\begin{aligned} (s_c(y_1))^2 &= \frac{-(y_1^2 + 1) + 2(y_1^4 - y_1^2 + 1)^{1/2}}{3} = \frac{-(y_1^2 + 1)^2 + 4(y_1^4 - y_1^2 + 1)}{3(y_1^2 + 1 + 2(y_1^4 - y_1^2 + 1)^{1/2})} \\ &\geq \frac{3(y_1^2 - 1)^2}{3(y_1^2 + 1 + 2(2y_1^2 - 3y_1 + 2))} = \frac{(y_1 - 1)^2(y_1 + 1)^2}{5y_1^2 - 6y_1 + 5}. \end{aligned}$$

Thus, we get

$$y_2 - s_c(y_1) \leq y_2 - |y_1 - 1|f(y_1) \leq y_2, \quad f(y_1) = \frac{y_1 + 1}{(5y_1^2 - 6y_1 + 5)^{1/2}}.$$

Near $y = 1$, $f(y_1) \approx 1$ and the upper bound is $O(|y - 1|)$. Moreover, from the above formula, since $(y_1^4 - y_1^2 + 1)^{1/2} \geq (y_1^2)^{1/2} = y_1$, we have

$$(5.42) \quad (s_c(y_1))^2 = \frac{3(y_1 - 1)^2(y_1 + 1)^2}{3(y_1^2 + 1 + 2(y_1^4 - y_1^2 + 1)^{1/2})} \leq \frac{3(y_1 - 1)^2(y_1 + 1)^2}{3(y_1^2 + 1 + 2y_1)} = (y_1 - 1)^2, \quad s_c(y_1) \leq |y_1 - 1|.$$

5.4.1. *Estimate the explicit integrals for u_y, v_x .* We follow the strategy in Section 5.2 to estimate the integrals in the $C_x^{1/2}$ estimate of v_x, u_y . Recall from Appendix B.1 [3] the estimate of $[v_x]_{C_x^{1/2}}$

$$\begin{aligned}
2^{-1/2}|v_x(-1, x_2) - v_x(1, x_2)| &\leq 2^{-1/2} \frac{1}{\pi} \left((S_{in,x} + S_{1D})[\omega]_{C_x^{1/2}} + 4(S_{in,y} + S_{out})[\omega]_{C_y^{1/2}} \right), \\
S_{in,x} &= \int_{y_1 \notin J_1, y_1 \geq 0} \sqrt{2y_1} |\Delta_{1D}(y_1)| dy_1, \\
S_{in,y} &= S_{up} + S_{low}, \quad S_{in,y,ns} = S_{up,ns}(\varepsilon) + S_{low}, \quad S_{up} = S_{up,ns}(\varepsilon) + S_{in,y,\varepsilon}, \\
(5.43) \quad S_{up,ns}(\varepsilon) &= \int_{y_1 \in \mathbb{R}_+ \setminus [1-\varepsilon, 1+\varepsilon]} \int_{s_c(y_1)}^A |T(y) - y_2|^{\frac{1}{2}} |\Delta(y)| dy_2 dy_1, \\
S_{low} &= \int_{y_1 \in J_1^+ \setminus [1-\varepsilon, 1+\varepsilon]} \int_0^{T(y_1, A)} |y_2|^{\frac{1}{2}} |\Delta(y)| dy_2 dy_1, \\
S_{out} &= \frac{1}{4} \sqrt{2x_2} \int_{\mathbb{R}} \int_{x_2}^B |\Delta(y)| dy = \frac{1}{2} \sqrt{2x_2} \int_{\mathbb{R}_+} \int_{x_2}^B |\Delta(y)| dy, \\
S_{1D} &= 2 \int_{\frac{1}{9}}^9 |\Delta_{1D}| |y_1 - 1|^{1/2} dy_1 + \int_0^{\frac{1}{9}} |\Delta_{1D}| \sqrt{2y_1} dy_1 + (\pi + P(\frac{1}{9})) \sqrt{2},
\end{aligned}$$

and the estimate of $[u_y]_{C_x^{1/2}}$

$$\begin{aligned}
(5.44) \quad 2^{-1/2}|u_y(-1, x_2) - u_y(1, x_2)| &\leq 2^{-1/2} \frac{1}{\pi} \left(\tilde{S}_{1D}[\omega]_{C_x^{1/2}} + 4(S_{up} + S_{out})[\omega]_{C_y^{1/2}} \right), \\
\tilde{S}_{1D} &= \int_{\mathbb{R}_+} |\Delta_{1D}(y_1)| \sqrt{2y_1} dy_1,
\end{aligned}$$

where S_{out} and S_{up} are given above,

$$\begin{aligned}
(5.45) \quad A &= \min(x_2, B), \quad P(k) \triangleq - \int_k^9 |\Delta_{1D}(y_1)| dy_1, \quad J_1 = [-9, 9], \quad J_1^+ = [0, 9], \\
\Delta_{1D}(y_1) &= g_A(y_1 + 1) \mathbf{1}_{|y_1+1| \leq B} - g_A(y_1 - 1) \mathbf{1}_{|y_1-1| \leq B}, \quad g_b(y) = \frac{b}{y^2 + b^2},
\end{aligned}$$

$s_c(y_1)$ is given below, and $S_{in,y,\varepsilon,\pm}$ estimates the following integrals near the singularity

$$\begin{aligned}
(5.46) \quad I_{\pm}(\varepsilon) &\triangleq \int_{J_\varepsilon} \int_0^A K_2(y_1 \pm 1, y_2) \tilde{W}(y) dy, \quad I(\varepsilon) \triangleq \int_{J_\varepsilon} \int_0^A \Delta(y) \tilde{W}(y) dy = I_+(\varepsilon) - I_-(\varepsilon), \\
|I_{\pm}(\varepsilon)| &\leq S_{in,y,\varepsilon,\pm} \cdot [\omega]_{C_y^{1/2}}, \quad |I(\varepsilon)| \leq S_{in,y,\varepsilon}, \quad \tilde{W}(y) = \omega(y_1, x_2 - y_2) - \omega(y_1, x_2), \\
J_\varepsilon &\triangleq [1 - \varepsilon, 1 + \varepsilon].
\end{aligned}$$

We will estimate $S_{in,y,\delta,\pm}$ in Section 5.4.3. Note that the above variables, e.g. $S_{out}, S_{in,x}$, are different from [3] by a constant. Here, we have restricted the domain of the integral to \mathbb{R}_2^{++} due to symmetry. The upper bounds for v_x, u_y are the same as [3]. The subscript *ns* in (5.43) is short for *non-singular*. For $s_1, s_2 > 0$, the map T and $s_c(s_1)$ are obtained from

$$\begin{aligned}
\Delta(s) &= K_{2,B}(s_1 + 1, s_2) - K_{2,B}(s_1 - 1, s_2), \quad \Delta(s_1, s_c(s_1)) = 0, \\
\int_T^{y_2} \Delta(y_1, s_2) ds_2 &= 0, \quad K_{2,B}(s) = \frac{1}{2} \frac{s_1^2 - s_2^2}{|s|^4} \mathbf{1}_{|s_1|, |s_2| \leq B}.
\end{aligned}$$

When $0 \leq s_1 \leq B - 1$, we have

$$(5.47) \quad \Delta(s) = \Delta_{in}(s) = K_2(s_1 + 1, s_2) - K_2(s_1 - 1, s_2), \quad s_c(s_1) = s_{c,in}(s_1),$$

$s_{c,in}$ is given by (5.89), and the map T is given by (5.92), which is denoted as T_{in} . See the left figure in Figure 2 for an illustrations of the sign and the map in the inner regions. For $s_1 \in [B - 1, B + 1]$, we get

$$(5.48) \quad T_{out} = \frac{(y_1 - 1)^2}{y_2}, \quad \Delta(s) = \Delta_{out}(s) = -K_2(s_1 - 1, s_2), \quad s_c(s_1) = s_1 - 1.$$

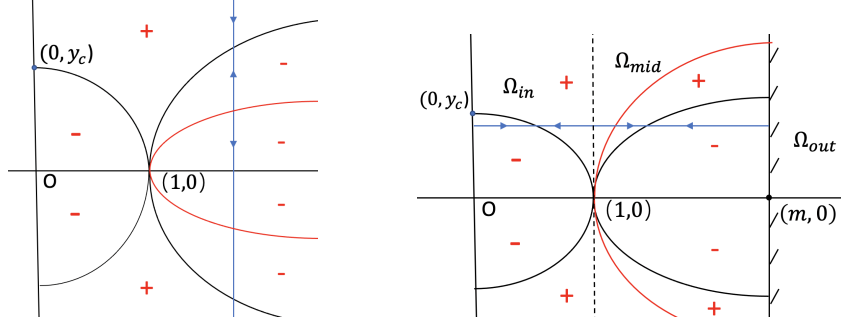


FIGURE 2. Illustration of the sign of the kernel $\Delta(y)$ and transportation plan. The sign of $\Delta(y)$ in different regions is indicated by \pm . The blue arrows indicate the direction of 1D transportation plan. Left for $C_x^{1/2}$ estimate: The black curve and the red curve represents $y_2 = \pm s_{c,in}(y_1)$, $y_2 = \pm T_{in}(y_1, A)$ for $y_1 \geq 0$, respectively. Right for $C_y^{1/2}$ estimate: The black curve is for $y_2 = \pm s_c(y_1)$, or equivalently $y_1 = h_c^-(|y_2|)$ (two left black curves) and $y_1 = h_c^+(y_2)$ (two right black curves). The red curve represents $y_1 = T(m, |y_2|)$. Note that these curves do not agree with the actual functions.

For $y_1 \geq 1$ or $y_1 \leq 1$ and $y_2 > 0$, the piecewise bounds for $T_{out}(y)$ are trivial, and T_{out} is decreasing in y_2 .

The above integral depends on two parameters x_2, B . Denote $A = \min(x_2, B)$. Our goal is to obtain a uniform bound for all $0 \leq A \leq B \leq \infty, B \geq 2$. We partition these two parameters $0 = A_1 < \dots < A_{n_1} < A_{n_1+1} = \infty, 2 \leq B_1 < B_2 < \dots < B_{n_2+1} = \infty$, and estimate the bound for $A \in [A^l, A^u], B \in [B^l, B^u]$. We discuss how to estimate each part below.

5.4.2. *Bulk part $S_{in,y}$.* Using the above notation and the localization of the kernel, we have (5.49)

$$\begin{aligned} S_{in,y,ns}(\varepsilon) &= \int_{[0, B-1] \setminus J_\varepsilon} \int_0^A \left(\mathbf{1}_{y_2 \geq s_c(y_1)} |T_{in}(y) - y_2|^{1/2} + \mathbf{1}_{y_1 \leq 9} \mathbf{1}_{y_2 \leq T_{in}(y_1, A)} y_2^{1/2} \right) |\Delta_{in}(y)| dy \\ &\quad + \int_{B-1}^{B+1} \int_0^A \left(\mathbf{1}_{y_2 \geq s_c(y_1)} |T_{out}(y) - y_2|^{1/2} + \mathbf{1}_{y_1 \leq 9} \mathbf{1}_{y_2 \leq T_{out}(y_1, A)} y_2^{1/2} \right) |\Delta_{out}(y)| dy \\ &\triangleq S_{in,y1,ns} + S_{in,y2}, \end{aligned}$$

where we have combined the integrals

$$\mathbf{1}_{y_1 \in J_1} \mathbf{1}_{y_2 \in [s_c(y_1), A]} + \mathbf{1}_{y_1 \notin J_1} \mathbf{1}_{y_2 \in [s_c(y_1), A]}$$

related to $|T - y_2|^{1/2}$ in $S_{in,y}$ in (5.43). Since $T(y_1, y_2)$ is decreasing in y_2 (see Lemma 5.5, for $A \in [A_l, A_u], B \in [B_l, B_u]$), clearly, we have

$$\begin{aligned} S_{in,y1,ns} &\leq \int_{[0, B^u-1] \setminus [1-\varepsilon, 1+\varepsilon]} \int_0^{A^u} \left(\mathbf{1}_{y_2 \geq s_c(y_1)} |T_{in}(y) - y_2|^{1/2} + \mathbf{1}_{y_1 \leq 9} \mathbf{1}_{y_2 \leq T_{in}(y_1, A^l)} y_2^{1/2} \right) |\Delta_{in}(y)| dy, \\ S_{in,y2} &\leq \int_{B^l-1}^{B^u+1} \int_0^{A^u} \left(\mathbf{1}_{y_2 \geq s_c(y_1)} |T_{out}(y) - y_2|^{1/2} + \mathbf{1}_{y_1 \leq 9} \mathbf{1}_{y_2 \leq T_{out}(y_1, A^l)} y_2^{1/2} \right) |\Delta_{out}(y)| dy. \end{aligned}$$

We choose our mesh aligning with $y_1 = 9, y_1 = 1 \pm \varepsilon, y_2 = A_u$ to partition the integrals so that for each small domain $Q = [y_1^l, y_1^u] \times [y_2^l, y_2^u]$, the restrictions are automatically imposed i.e.

$$\mathbf{1}_{y_1 \leq 9} \mathbf{1}_{y_2 \leq A_u} \mathbf{1}_Q \mathbf{1}_{[1-\varepsilon, 1+\varepsilon]}(y_1) = \mathbf{1}_Q, \text{ or } 0.$$

We follow the strategy in Section 5.2 to handle the indicators, the integral $\int_Q |\Delta_\alpha(s)|$, and $|T_\alpha - y_2|^{1/2}, \alpha = in, out$. The analytic integral formula for $K_2(s)$ and $K_2(s)s_2^{1/2}$ is given in Section 5.1, and the estimate of $|T_{in} - y_2|^{1/2}$ is given in Section 5.4 after Lemma 5.5. The

estimate of the second part S_{in,y_2} is much easier since the integrand is supported away from the singularity $(1, 0)$ since $B \geq 3$ and the map T_{out} (5.48) and $s_c(y_1)$ (5.89) are simple.

For $S_{in,y_1,ns}$, we apply the above strategy to estimate the integrals in $[0, 1 - \varepsilon] \times [0, A]$, $[1 + \varepsilon, \infty)$. For the first integrand in S_{in,y_2} , if $B^u - B^l$ is large, which is the case for large B since we partition the domain of B using adaptive mesh, the above estimate for S_{in,y_2} is not sufficient. Note that the second integrand in S_{in,y_2} vanishes for large B since it is supported in $y_1 \leq 9$.

Additional estimate for S_{in,y_2} . We consider another estimate for S_{in,y_2} by exploiting the boundedness of the interval $y_1 \in [B - 1, B + 1]$

$$\begin{aligned} I &= \int_{B-1}^{B+1} \int_0^A \mathbf{1}_{y_2 \geq s_c(y_1)} |T_{out}(y) - y_2|^{1/2} |\Delta_{out}(y)| dy \\ &= \frac{1}{2} \int_{B-1}^{B+1} \int_{y_1-1}^A \left| \frac{(y_1-1)^2}{y_2} - y_2 \right|^{1/2} |K_2(y_1-1, y_2)| dy = \frac{1}{2} \int_{B-2}^B \int_{y_1}^A \left| \frac{y_1^2}{y_2} - y_2 \right|^{1/2} |K_2(y)| dy, \end{aligned}$$

in S_{in,y_2} in (5.49), where we have used $s_c(y_1) = y_1 - 1$ (5.48), and a change of variables $y_1 \rightarrow y_1 + 1$. Since $A \leq B$ (5.45) and

$$B - 2 \leq y_1 \leq y_2 \leq A \leq B \leq y_1 + 2, \quad y_1 + y_2 \leq \sqrt{2}(y_1^2 + y_2^2)^{1/2}, \quad |y| \geq \sqrt{2}(B - 2),$$

in the support, we get

$$\begin{aligned} \left| \frac{y_1^2}{y_2} - y_2 \right|^{1/2} |K_2(y)| &= \left| \frac{(y_2 - y_1)(y_2 + y_1)}{y_2} \right|^{1/2} \frac{(y_2 - y_1)(y_2 + y_1)}{|y|^4} \leq (2 \cdot 2)^{1/2} 2\sqrt{2}|y|^{-3} \\ &\leq 4\sqrt{2}(\sqrt{2})^{-3}(B - 2)^{-3} = 2(B - 2)^{-3}. \end{aligned}$$

Since $A \leq B, B_l \leq B$, we yield

$$I \leq \frac{1}{2} \int_{B-2}^B \int_{y_1}^A 2(B - 2)^{-3} dy \leq (B - 2)^{-3} \int_{B-2}^B (B - y_1) dy_1 = 2(B - 2)^{-3} \leq 2(B_l - 2)^{-3}.$$

5.4.3. Near the singularity. Near $s_* = (1, 0)$, the integrand in $S_{in,y}$ (5.43), (5.46) is singular of order $|x|^{-3/2}$ and quite complicated. To ease our computation, we use another estimate and separate the estimate of two kernels in $\Delta(s) = K_2(s_1 + 1, s_2) - K_2(s_1 - 1, s_2)$. Below, we estimate $I_{\pm}(\varepsilon)$ from (5.46) and derive the bound $S_{in,y,\varepsilon,\pm}$. We fix ε and then drop ε for simplicity.

(a) Regular part. Since the kernel $K_2(y_1 + 1, y_2)$ is regular, using $K_2(y) = \frac{1}{2} \frac{y^2 - y_2^2}{|y|^4}$, (5.46),

$$(5.50) \quad |\tilde{W}| \leq y_2^{1/2} [\omega]_{C_y^{1/2}}, \quad \int_{1-\varepsilon}^{1+\varepsilon} |K_2(y_1 + 1, y_2)| dy_1 \leq \int_{1-\varepsilon}^{1+\varepsilon} \frac{1}{2} |y|^{-2} dy_1 \leq \frac{\varepsilon}{(2 - \varepsilon)^2 + y_2^2},$$

we get

$$(5.51) \quad |I_+(\varepsilon)| \leq \int_{1-\varepsilon}^{1+\varepsilon} \int_0^A |K_2(y_1 + 1, y_2)| y_2^{1/2} dy [\omega]_{C_y^{1/2}} \leq [\omega]_{C_y^{1/2}} \int_0^{A^u} \frac{\varepsilon y_2^{1/2}}{(2 - \varepsilon)^2 + y_2^2} dy_2,$$

where we have used the fact that the integral is increasing in A and $A \in [A^l, A^u]$ in the last inequality. Following the strategy (d) in Section 5.2, we partition the domain of the integral using the same mesh as that for $S_{in,y}$. In each interval $[y_2^l, y_2^u]$, we have

$$(5.52) \quad \int_{y^l}^{y^u} \frac{\varepsilon y_2^{1/2}}{(2 - \varepsilon)^2 + y_2^2} dy \leq \sqrt{y^u} \frac{\varepsilon}{2 - \varepsilon} \arctan\left(\frac{y_2}{2 - \varepsilon}\right) \Big|_{y^l}^{y^u}.$$

(b) **Singular part.** Denote

$$a = \min(A, \varepsilon), \quad Q_a \triangleq [-a, a] \times [0, a], \quad Q_a^+ \triangleq [0, a]^2.$$

Recall $I_-(\varepsilon)$ from (5.46). We have

$$\begin{aligned} I_-(\varepsilon) &\triangleq \left| \int_{1-\varepsilon}^{1+\varepsilon} \int_0^A K_2(y_1 - 1, y_2) \tilde{W}(y) dy \right| = \left| \int_{-\varepsilon}^{\varepsilon} \int_0^A K_2(y) \tilde{W}(y_1 + 1, y_2) dy \right| \\ &\leq \left| \int_{Q_a} K_2(y_1, y_2) \tilde{W}(y_1 + 1, y_2) dy \right| + \left| \int_{[-\varepsilon, \varepsilon] \times [0, A] \setminus Q_a} K_2(y_1, y_2) \tilde{W}(y_1 + 1, y_2) dy \right| \triangleq I_{-,1} + I_{-,2}. \end{aligned}$$

For the second part, using the bound for \tilde{W} (5.50), we get

$$|I_{-,2}| \leq \left| \int_{[-\varepsilon, \varepsilon] \times [0, A] \setminus Q(a)} |K_2(y_1, y_2)| y_2^{1/2} dy \right| = 2 \int_{[0, \varepsilon] \times [0, A] \setminus Q_a^+} |K_2(y)| y_2^{1/2} dy.$$

If $A \geq A' \geq \varepsilon$, we yield $a = \varepsilon, A \leq A^u$. Using $K_2(y) = \frac{1}{2} \frac{y_1^2 - y_2^2}{|y|^4} \leq 0$ for $y_1 \leq \varepsilon \leq y_2$, we yield

$$I_{-,2} \leq 2 \int_0^\varepsilon \int_\varepsilon^{A^u} |K_2(y)| y_2^{1/2} dy = \left| \int_\varepsilon^{A^u} \frac{y_1}{|y|^2} \Big|_{y_2=0}^\varepsilon y_2^{1/2} dy_2 \right| = \int_\varepsilon^{A^u} \frac{\varepsilon}{\varepsilon^2 + y_2^2} y_2^{1/2} dy_2.$$

We use the same method as (5.52) and $\frac{\varepsilon}{\varepsilon^2 + y_2^2} = \frac{d}{dy_2} \arctan(y_2/\varepsilon)$ to estimate the integral.

We choose ε from the mesh for A . Then when $A' < \varepsilon$, we get $A^u \leq \varepsilon$. (This is similar to $a < b, a, b \in \mathbb{Z}$ implies $a+1 \leq b$.) Since $A \leq A^u \leq \varepsilon$, we get $K_2(y) = \frac{1}{2} \frac{y_1^2 - y_2^2}{|y|^4} \geq 0$ in $[A, \varepsilon] \times [0, A]$,

$$\begin{aligned} I_{-,2} &\leq 2 \int_A^\varepsilon \int_0^A K_2(y) y_2^{1/2} dy = \int_0^A -\frac{y_1}{|y|^2} \Big|_A^\varepsilon y_2^{1/2} dy_2 = \int_0^A \left(\frac{A}{A^2 + y_2^2} - \frac{\varepsilon}{\varepsilon^2 + y_2^2} \right) y_2^{1/2} dy_2 \\ &= 2A^{1/2} f_s(1) - 2\varepsilon^{1/2} f_s((A/\varepsilon)^{1/2}) \leq 2A_u^{1/2} f_s(1) - 2\varepsilon^{1/2} f_s((A_l/\varepsilon)^{1/2}), \end{aligned}$$

where we have used (5.5), (5.6).

For $I_{-,1}$ and a fixed x_2 , using the scaling relation, (5.46) $[\tilde{W}(ay_1, ay_2)]_{C_y^{1/2}(Q_1)} = a^{1/2} [\tilde{W}]_{C_y^{1/2}(Q_a)} \leq a^{1/2} [\omega]_{C_y^{1/2}}$, and (5.86), we get

$$|I_{-,1}| \leq \left| \int_{Q_1} K_2(y_1, y_2) \tilde{W}(ay_1 + 1, ay_2) dy \right| \leq 2C_{K_2} [\tilde{W}(ay_1 + 1, ay_2)]_{C_y(Q_1)} \leq 2 \min(\varepsilon, A)^{1/2} C_{K_2} [\omega]_{C_y^{1/2}}.$$

5.4.4. *Estimate of S_{out} .* In this section, we estimate S_{out} in (5.43), which is much easier than that of $S_{in,y}$. Note that if $B < x_2$, we have $S_{out} = 0$. Thus, we consider $x_2 < B$, and get $A = \min(x_2, B) = x_2$. Using (5.47) and (5.48), we yield

(5.53)

$$S_{out}(A, B) \triangleq \frac{\sqrt{2x_2}}{2} \int_{\mathbb{R}} \int_{x_2}^B |\Delta(y)| dy = \frac{\sqrt{2A}}{2} \int_A^B ds_2 \left(\int_0^{B-1} |\Delta_{in}(s)| ds_1 + \int_{B-1}^{B+1} \Delta_{out}(s) ds_1 \right).$$

For a fixed y_2 , applying Lemma 5.4 with $k = 1, f(x) = K(x, y_2), b = B, c = B^u$, and $B \leq B_u$ we obtain

(5.54)

$$\int_{B-1}^{B+1} |K_2(s-1, s_2)| ds_1 \leq \int_{B-1}^{B^u-1} \left| K_2(s-1, s_2) - K_2(s+1, s_2) \right| + \int_{B^u-1}^{B^u+1} |K_2(s-1, s_2)| ds_1.$$

As a result, $S_{out}(A, B)$ is increasing in B and we get $S_{out}(A, B) \leq S_{out}(A, B^u)$. Denote $I_\varepsilon = [1 - \varepsilon, 1 + \varepsilon]$. Firstly, using $\Delta(s) = \Delta_{in}(s)$ for $y_1 \in I_\varepsilon, y_2 \in [A, B]$, we have

$$S_{out} \leq \frac{\sqrt{2A^u}}{2} \int_{A^l}^{B^u} \int_{y_1 \in [0, B^u+1] \setminus I_\varepsilon} |\Delta(s)| ds + \frac{\sqrt{2A}}{2} \int_A^B \int_{I_\varepsilon} |\Delta_{in}(s)| ds \triangleq S_{out,1} + S_{out,2}.$$

We do not bound the integral region $[A, B]$ in $S_{out,2}$ at this moment. The first part is away from the singularity. We partition the domain and apply the strategy in Section 5.2 to estimate it.

In particular, the integral in $S_{out,1}$ with $y_1 \in [B^u - 1, B^u + 1]$ is given by

$$I = \int_{A^l}^{B^u} \int_{B^{u-1}}^{B^{u+1}} |K_2(s_1 - 1, s_2)| ds = \int_{A^l}^{B^u} \int_{B^{u-2}}^{B^u} |K_2(s_1, s_2)| ds.$$

We decompose the domain of the integral as follows

$$\begin{aligned} [B^u - 2, B^u] \times [A^l, B^u] &= [B^u - 2, B^u] \times [A^l, B^u - 2] \cup [B^u - 2, B^u]^2 \triangleq Q_1 \cup Q_2, \quad A^l \leq B^u - 2, \\ [B^u - 2, B^u] \times [A^l, B^u] &= [B^u - 2, A^l] \times [A^l, B^u] \cup [A^l, B^u]^2 \triangleq Q_1 \cup Q_2, \quad A^l > B^u - 2. \end{aligned}$$

In Q_1 , since $y_1^2 - y_2^2$ has a fixed sign, $K_2(s)$ has a fixed sign. We apply the analytic formula for $K_2(s)$ to evaluate the integral in Q_1 . In Q_2 , we use the formula (5.10).

For $S_{out,2}$, we denote $A_{\varepsilon,l} = \max(\varepsilon, A^l)$. We decompose $S_{out,2}$ as follows

$$S_{out,2} \leq \frac{\sqrt{2A}}{2} \int_{A_{\varepsilon,l}}^B \int_{1-\varepsilon}^{1+\varepsilon} |\Delta_{in}(s)| ds + \frac{\sqrt{2A}}{2} \int_A^\varepsilon \int_{1-\varepsilon}^{1+\varepsilon} |\Delta_{in}(s)| ds \triangleq I_1 + I_2.$$

Note that $I_2 = 0$ if $A \geq \varepsilon$. From (5.42), in the support of the integral I_1 , we have $y_2 \geq \varepsilon \geq |y_1 - 1| \geq s_c(y_1)$ (5.42) and yield that $\Delta(s)$ has a fixed sign. We estimate I_1 as follows

$$I_1 \leq \frac{\sqrt{2A^u}}{2} \left| \int_{A_{\varepsilon,l}}^{B^u} \int_{1-\varepsilon}^{1+\varepsilon} \Delta(s) ds \right|,$$

and evaluate the integral using the analytic integral formula for $\Delta(s)$. For I_2 , we use the fact that $\sqrt{A} \leq s_2^{1/2}$ in the support of the integral and triangle inequality to get

$$I_2 \leq \frac{\sqrt{2}}{2} \int_A^\varepsilon \int_{1-\varepsilon}^{1+\varepsilon} |K_2(s_1 - 1, s_2)| s_2^{1/2} ds + \frac{\sqrt{2A^u}}{2} \int_{A^l}^\varepsilon \int_{1-\varepsilon}^{1+\varepsilon} |K_2(s_1 + 1, s_2)| ds = I_{21} + I_{22}.$$

For I_{22} , we choose $\varepsilon < \frac{1}{100}$. In the support of the integral, $K_2 > 0$ and we get

$$I_{22} = \frac{\sqrt{2A^u}}{2} \left| \int_{A^l}^\varepsilon \int_{1-\varepsilon}^{1+\varepsilon} K_2(s_1 + 1, s_2) ds \right|,$$

and can evaluate the integral using the analytic integral formula. For I_{21} , we have

$$I_{21} \leq \sqrt{2} \int_{A^l}^\varepsilon \int_0^\varepsilon |K_2(s_1, s_2)| s_2^{1/2} ds \leq \sqrt{2} F_{K_2,h}(0, \varepsilon, A^l, \varepsilon),$$

where $F_{K_2,h}$ is defined in (5.7).

5.4.5. *Integral $S_{in,y,\varepsilon}, S_{in,y,ns}, S_{out}$ in the far-field.* The previous method can handle the case that $A \leq B < \infty$. Next, we consider the remaining case $A \leq B, B^l \leq B \leq \infty$ with B^l sufficiently large, e.g. $B^l = 10^8$. Recall $S_{in,y,ns}$ from (5.43), (5.46), (5.49)

$$(5.55) \quad S_{in,y,ns} = \int_0^A dy_2 \left(\int_0^{B-1} I_{in,1} |\Delta_{in}(y)| dy_1 + \int_{B-1}^{B+1} I_{in,2} |\Delta_{out}(y)| dy_1 \right) \triangleq \int_0^A I_{in}(y_2) dy_2,$$

where $I_{in,1}, I_{in,2}$ denotes the integrand,

$$\begin{aligned} I_{in,1} &= (\mathbf{1}_{y_2 \geq s_c(y_1)} |T_{in}(y) - y_2|^{1/2} + \mathbf{1}_{y_1 \leq 9} \mathbf{1}_{y_2 \leq T_{in}(y_1, A)} y_2^{1/2}) \mathbf{1}_{y_1 \notin J_\varepsilon} \\ I_{in,2} &= (\mathbf{1}_{y_2 \geq s_c(y_1)} |T_{out}(y) - y_2|^{1/2} + \mathbf{1}_{y_1 \leq 9} \mathbf{1}_{y_2 \leq T_{out}(y_1, A)} y_2^{1/2}) \mathbf{1}_{y_1 \notin J_\varepsilon}, \quad J_\varepsilon = [1 - \varepsilon, 1 + \varepsilon]. \end{aligned}$$

We decompose the integral in y_1 as follows

$$\int_0^{B-1} I_{in,1} |\Delta_{in}(y)| dy_1 = \int_0^{B^l-1} I_{in,1} |\Delta_{in}(y)| dy_1 + \int_{B^l-1}^{B-1} I_{in,1} |\Delta_{in}(y)| dy_1 \triangleq I_2 + I_3.$$

Since for $A \geq y_2 \geq s_c(y_1)$, we have $y_2 \geq s_c(y_1) \geq T(y_1, y_2) \geq T(y_1, A)$, clearly, we have

$$(5.56) \quad |I_{in,1}| \leq |y_2|^{1/2} \mathbf{1}_{y_1 \notin J_\varepsilon}, \quad I_{in,2} \leq |y_2|^{1/2} \mathbf{1}_{y_1 \notin J_\varepsilon}.$$

Using (5.56) and combining I_3 and the integral of $I_{in,2}$ in (5.55), we further obtain

$$I_3 + \int_{B-1}^{B+1} I_{in,2} |\Delta_{out}(y)| dy_1 \leq y_2^{1/2} \left(\int_{B^l-1}^{B-1} |\Delta_{in}(y)| dy_1 + \int_{B-1}^{B+1} |\Delta_{out}(y)| dy_1 \right).$$

Using the monotonicity Lemma 5.4, we get

$$I_3 + \int_{B^{-1}}^{B+1} I_{in,2} |\Delta_{out}(y)| dy_1 \leq y_2^{1/2} \int_{B^{l-1}}^{\infty} |\Delta_{in}(y)| dy_1.$$

It follows

$$(5.57) \quad S_{in,y,ns} \leq \int_0^A \int_0^{B^{l-1}} I_{in,1} |\Delta_{in}| dy + \int_0^A y_2^{1/2} \int_{B^{l-1}}^{\infty} |\Delta_{in}(y)| dy_1$$

for any $B \geq B^l$ and a fixed A . The first part is the main part.

Applying a similar argument to S_{out} in (5.53), we yield

$$S_{out}(A, B) = \frac{\sqrt{2A}}{2} \int_A^B ds_2 \left(\int_0^{B-1} |\Delta_{in}(s)| ds_1 + \int_{B-1}^{B+1} |\Delta_{out}(s)| ds_1 \right) \leq \frac{\sqrt{2A}}{2} \int_A^B \int_0^{\infty} |\Delta_{in}(s)| ds,$$

where $a \vee b = \max(a, b)$. Using $\sqrt{2A} \leq 2s_2^{1/2}$ in the support of the integral in S_{out} and $I_{in,1} \leq y_2^{1/2}$, we can bound the integrands in $S_{in,y,ns}, S_{out}$ by $y_2^{1/2} |\Delta(y)|$. Denote

$$R(B^l) \triangleq [0, B^l] \times [0, B^l - 1], \quad A_2 \triangleq \min(A, B^l).$$

For $I(\varepsilon), S_{in,y,\varepsilon}$ (5.46), using $|\tilde{W}| \leq y_2^{1/2} [\omega]_{C_y^{1/2}}$ (5.50) we perform a similar decomposition

$$|I(\varepsilon)| \leq \left| \int_{J_\varepsilon} \int_0^{A_2} \Delta(y) \tilde{W}(y) dy \right| + \left| \int_{J_\varepsilon} \int_{A_2}^A y_2^{1/2} |\Delta_{in}(y)| dy [\omega]_{C_y^{1/2}} \right| \triangleq I(\varepsilon, A_2) + S_{in,y,\varepsilon,out} [\omega]_{C_y^{1/2}}.$$

We treat the integrals in $S_{in,y,ns}, S_{out}$ beyond $R(B^l)$ and $S_{in,y,\varepsilon,out}$ as error \mathcal{R} and yield

$$\begin{aligned} S_{in,y,ns} + S_{in,y,\varepsilon,out} + S_{out} &\leq \left(\int_0^A + \int_A^B \right) \int_0^{\infty} \mathbf{1}_{R(B^l)^c}(y) y_2^{1/2} |\Delta_{in}(y)| dy \\ &+ \int_0^A \int_0^{B^{l-1}} \mathbf{1}_{R(B^l)}(y) I_{in,1} |\Delta_{in}(y)| dy + \frac{\sqrt{2A}}{2} \int_A^B \int_0^{\infty} \mathbf{1}_{R(B^l)}(y) |\Delta_{in}(y)| dy \triangleq \mathcal{R} + M_{in} + M_{out}. \end{aligned}$$

Note that

$$\begin{aligned} [0, A] \cup [A, B] &= [0, B], \quad ([0, A] \times [0, B^l - 1]) \cap R(B^l) = [0, A_2] \times [0, B^l - 1] \triangleq R_2(B^l), \\ ([A, B] \times [0, \infty]) \cap R(B^l) &= ([A, B] \cap [0, B^l]) \times [0, B^l - 1] \subset [A_2, B^l] \times [0, B^l - 1], \end{aligned}$$

where we have used $[A, B] \cap [0, B^l] = \emptyset$ if $A > B^l$ in the last inclusion. Since $M_{out} = 0$ if $A > B^l$, we get

$$S_{in,y,ns} + S_{out} + S_{in,y,\varepsilon,out} \leq \int_{R(B^l)^c} y_2^{1/2} |\Delta_{in}(y)| + \int_0^{A_2} \int_0^{B^{l-1}} I_{in,1} |\Delta_{in}(y)| + \frac{\sqrt{2A_2}}{2} \int_{A_2}^{B^l} \int_0^{B^{l-1}} |\Delta_{in}(y)|.$$

The estimate of the last two integrals and $I(\varepsilon, A_2)$ defined above (see also (5.46)) reduce to the case $A_2 \leq B^l$ studied in the previous section. The first term is very small and treated as an error term. Applying Lemma 5.2 with $i + j = 3$ and mean value theorem yields

$$|\Delta_{in}(s)| = |2\partial_{y_1} K_2(\xi, y_2)| \leq \frac{2}{(\xi^2 + y_2^2)^{3/2}} \leq C_B |y|^{-3}, \quad C_B = 2 \left(\frac{B^l - 1}{B^l - 2} \right)^3,$$

for some $\xi \in [y_1 - 1, y_1 + 1]$, where we have used $|(\xi, y_2)| \geq |y| - 1 \geq |y|(1 - \frac{1}{B^l - 1})$. Using $y_2^{1/2} \leq |y|^{1/2}$, we estimate the remaining part as follows

$$(5.58) \quad \begin{aligned} \int_{R(B^l)^c} y_2^{1/2} |\Delta_{in}(y)| &\leq C_B \int_{|y| \geq B^{l-1}, y_2 \geq 0} |y|^{-5/2} dy \leq C_B \int_{B^{l-1}} r^{-3/2} dr \int_0^{\pi/2} 1 d\beta \\ &= C_B 2(B^l - 1)^{-\frac{1}{2}} \cdot \frac{\pi}{2} = C_B \pi (B^l - 1)^{-\frac{1}{2}}. \end{aligned}$$

5.4.6. *Estimate of $S_{in,x}$ and S_{1D} .* Recall $S_{in,x}, S_{1D}$ from (5.43). Denote $J_2 = [1/9, 9]$. Clearly, we have

$$F(A, B) = S_{in,x} + S_{1D} = \int_{y_1 \notin J_2, y_1 \geq 0} \sqrt{2y_1} |\Delta_{1D}(y_1)| dy_1 + 2 \int_{\frac{1}{9}}^9 |\Delta_{1D}| \cdot |y_1 - 1|^{1/2} dy_1 + (\pi + P(\frac{1}{9}))\sqrt{2},$$

where $\Delta_{1D}(y_1), P(\frac{1}{9})$ are given in (5.45). From the definition, we obtain $\Delta_{1D} < 0$ for $y_1 > 0$. First, we show that F is increasing in B . Using (5.45) and $B \geq 3$, we get

$$\begin{aligned} (5.59) \quad F(A, B) &= \pi\sqrt{2} + \int_0^\infty |\Delta_{1D}(y_1)| h(y_1) dy_1 \\ &= \pi\sqrt{2} + \int_0^{B-1} |g_A(y_1 + 1) - g_A(y_1 - 1)| h(y_1) + \int_{B-1}^{B+1} |g_A(y_1 - 1)| h(y_1) dy_1 \\ h(y_1) &= \sqrt{2y_1} \mathbf{1}_{J_2^c}(y_1) - \sqrt{2} \mathbf{1}_{J_2}(y_1) + 2|y_1 - 1|^{1/2} \mathbf{1}_{J_2}(y_1), \end{aligned}$$

where $J_2 = [1/9, 9]$, and the negative term comes from $P(\frac{1}{9})$. The function $h(y_1)$ satisfies

$$h(y_1) = 2|y_1 - 1|^{1/2} - \sqrt{2}, \quad y_1 \in [1/9, 9], \quad h(y_1) = |2y_1|^{1/2}, \quad y_1 > 9, y_1 \in [0, 1/9].$$

Since $h(9-) = 2\sqrt{8} - \sqrt{2} = 3\sqrt{2} = h(9+)$, and $h(y_1)$ is increasing on $[2, 9]$ and $[9, \infty]$, we obtain that $h(y_1)$ is increasing. Using the monotonicity Lemma 5.4, we get

$$F(A, B) \leq F(A, \infty).$$

Thus, it suffices to consider the case $B = \infty$, where we do not have localization of Δ_{1D} in (5.45).

Now, we fix $A \in [A^l, A^u]$. We partition the integral in (5.59) with $B = \infty$ into

$$D_1 = [0, 1/9], D_2 = [1/9, 1 - \varepsilon], D_3 = [1 - \varepsilon, 1 + \varepsilon], D_4 = [1 + \varepsilon, 9], D_5 = [9, R_0], D_f = [R_0, \infty],$$

for some $\varepsilon < 1/4$. In each D_i , we can simplify the function $h(y_1)$ and $h(y_1)$ is monotone. Thus, we can obtain its piecewise bounds easily. Next, we estimate the integral of Δ_{1D} uniformly for $A \in [A_l, A_u]$. Using $\Delta_{1D}(y_1) < 0$ for $y_1 > 0$ and

$$\begin{aligned} |\Delta_{1D}(y_1, A)| &= \frac{4Ay_1}{((y_1 + 1)^2 + A^2)((y_1 - 1)^2 + A^2)} \leq \frac{A_u}{A_l} \frac{4A^l y_1}{((y_1 + 1)^2 + A_l^2)((y_1 - 1)^2 + A_l^2)} \\ &= -\frac{A_u}{A_l} \Delta_{1D}(y_1, A_l) = \frac{A_u}{A_l} \frac{d}{dy_1} \left(\arctan \frac{y_1 - 1}{A_l} - \arctan \frac{y_1 + 1}{A_l} \right), \end{aligned}$$

we get

$$\int_a^b |\Delta_{1D}(y_1, A)| dy_1 \leq \frac{A_u}{A_l} \left(\arctan \frac{y_1 - 1}{A_l} - \arctan \frac{y_1 + 1}{A_l} \right) \Big|_a^b.$$

We apply the above estimate and the strategy in Section 5.2 to estimate the integral in the finite domains D_1, D_2, D_4, D_5 away from the singularity 1. When A is small and near the singularity 1, the above formula can be ∞ . Denote $P = (y_1 - 1)^2 + A^2, Q = (y_1 + 1)^2 + A^2$. Using the above formula for $|\Delta_{1D}|$, we have

$$\frac{1}{4y_1} \partial_A |\Delta_{1D}| = \partial_A \frac{A}{PQ} = \frac{1}{PQ} \left(1 - A \frac{P'}{P} - A \frac{Q'}{Q} \right) = \frac{1}{PQ} \left(1 - 2A^2 \left(\frac{1}{P} + \frac{1}{Q} \right) \right).$$

For $|y_1 - 1| \geq \sqrt{3}A$, we get $Q \geq P \geq 4A^2$ and yield

$$2A^2(P^{-1} + Q^{-1}) \leq 2A^2 \cdot 2(4A^2)^{-1} = 1, \quad \partial_A |\Delta_{1D}| \geq 0.$$

Therefore, if $\min(|a - 1|, |b - 1|) \geq \sqrt{3}A$, we have an improved estimate

$$\int_a^b |\Delta_{1D}(y_1, A)| \leq \int_a^b |\Delta_{1D}(y_1, A_u)| = - \int_a^b \Delta_{1D}(y_1, A_u),$$

which can be evaluated using the analytic integral formula.

For the term involving $P(1/9)$, using the formula of $\int \Delta_{1D}(s)$, we get

$$P(1/9, A) = \arctan \frac{8}{A} - \arctan \frac{10}{A} - \arctan \frac{-8}{9A} + \arctan \frac{10}{9A},$$

$$\partial_A P(1/9, A) = -\frac{13120(6400 + 6724A^2 + 243A^4)}{(A^2 + 64)(A^2 + 100)(64 + 81A^2)(100 + 81A^2)} < 0.$$

Thus $P(1/9, A)$ is monotone in A and

$$\pi + P(1/9, A) \leq \max(\pi + P(1/9, A_l), \pi + P(1/9, A_u)).$$

In the domain D_3 , for A very small, we handle the singularity near $(A, y_1) = (0, 0)$ as follows

$$I_3 = \int_{1-\varepsilon}^{1+\varepsilon} (g_A(1-y_1) - g_A(1+y_1))|y_1 - 1|^{1/2} dy_1 \leq \int_{1-\varepsilon}^{1+\varepsilon} (g_A(1-y_1))|y_1 - 1|^{1/2} dy_1 = 2 \int_0^\varepsilon \frac{A}{A^2 + y_1^2} y_1^{1/2} dy_1$$

$$= 2A^{1/2} \int_0^{\varepsilon/A} \frac{1}{y_1^2 + 1} y_1^{1/2} dy_1 \leq 2A_u^{1/2} \int_0^{\varepsilon/A_l} \frac{1}{y_1^2 + 1} y_1^{1/2} dy_1 = 4A_u^{1/2} f_s((\varepsilon/A_l)^{1/2}),$$

where f_s is the function defined in (5.5).

In the far-field $y_1 \geq R_0$, we have

$$|\Delta_{1D}(y_1)| = |2\partial_{y_1} g_A(\xi)| = \left| \frac{4A\xi}{(A^2 + \xi^2)^2} \right| \leq \frac{2}{\xi^2 + A^2} \leq 2(y_1 - 1)^{-2},$$

for some $\xi \in [y_1 - 1, y_1 + 1]$. Since $y_1 \leq (y_1 - 1)\frac{R_0}{R_0 - 1}$ for $y_1 \geq R_0$, we yield

$$I_f = \int_{R_0}^\infty |\Delta_{1D}(y_1)| \sqrt{2y_1} dy_1 \leq 2 \left(\frac{2R_0}{R_0 - 1} \right)^{1/2} \int_{R_0}^\infty (y_1 - 1)^{-2+1/2} dy_1 = 4 \left(\frac{2R_0}{R_0 - 1} \right)^{1/2} (R_0 - 1)^{-1/2}.$$

Finally, we handle the case A sufficiently large and can be ∞ . For $A \geq A_l$ with A_l sufficiently large, e.g. $A_l = 10^8$, we use $h(y_1) \leq |2y_1|^{1/2}$ (5.59) and the identity (5.60) below to get

$$F(A, B) \leq F(A, \infty) = \sqrt{2}\pi + \int_0^\infty |\Delta_{in}(y_1)| \sqrt{2y_1} dy_1 = \sqrt{2}\pi + \pi\sqrt{2} \sqrt{\sqrt{A^2 + 1} - A}$$

$$= \sqrt{2}\pi(1 + (\sqrt{A^2 + 1} + A)^{-1/2}) \leq \sqrt{2}\pi(1 + (\sqrt{A_l^2 + 1} + A_l)^{-1/2}),$$

where we have used $\sqrt{A^2 + 1} - A = \frac{1}{\sqrt{A^2 + 1} + A}$.

Combining the above estimates, we complete the estimate of the integrals in (5.43) for $[v_x]_{C_y^{1/2}}$.

The estimate of $[u_y]_{C_y^{1/2}}$ in (5.44) is similar and simpler. We have estimated the terms in S_{low} (5.44) in our estimate of $S_{in,y}$ (5.43). The estimate of \tilde{S}_{1D} is similar. Using the monotonicity Lemma 5.4, the identity (5.60) below, we get

$$\tilde{S}_{1D} \leq \int_0^\infty (g_A(y_1 - 1) - g_A(y_1 + 1))(2y_1)^{1/2} dy_1 = \sqrt{2}\pi \sqrt{\sqrt{1 + A^2} - A}$$

$$= \sqrt{2}\pi(\sqrt{1 + A^2} + A)^{-1/2} \leq \sqrt{2}\pi(\sqrt{1 + A_l^2} + A_l)^{-1/2}.$$

An integral identity. We prove the following identity using the residue formula

$$(5.60) \quad T = \int_0^\infty (g_A(y_1 - 1) - g_A(y_1 + 1))y_1^{1/2} dy_1 = \pi \sqrt{\sqrt{1 + A^2} - A}.$$

Using a change of variable $y_1 = s^2$ and writing the integral on \mathbb{R} using the symmetry, we get

$$T = \int_{-\infty}^\infty s^2 \left(\frac{A}{(s^2 - 1)^2 + A^2} - \frac{A}{(s^2 + 1)^2 + A^2} \right) ds \triangleq T_1 - T_2.$$

The integrand is analytic except a few poles. In the upper half plane, we have poles

$$s_1 = (1 + iA)^{1/2} = re^{i\theta}, \quad s_2 = re^{\pi - \theta} = -\bar{s}_1, \quad s_2^2 = \bar{s}_1^2 = 1 - iA,$$

for some $\theta \in [0, \pi]$. Denote $f(s) = (s^2 - 1)^2 + A^2$. We have $\partial_s f = 4s(s^2 - 1)$. Applying the residue formula, we get

$$T_1 = 2\pi i \left(\frac{As_1^2}{f'(s_1)} + \frac{As_1^2}{f'(s_1)} \right) = \frac{1}{2} \pi i A \left(\frac{s_1}{s_1^2 - 1} + \frac{s_2}{s_2^2 - 1} \right) = \frac{\pi i A}{2} \left(\frac{s_1}{iA} + \frac{-\bar{s}_1}{-iA} \right) = \frac{\pi}{2} (s_1 + \bar{s}_1).$$

Since $(s_1 + \bar{s}_1)^2 = s_1^2 + \bar{s}_1^2 + 2|s_1|^2 = 2 + 2(1 + A^2)^{1/2}$ and $T_1 > 0$, we get $T_1 = \frac{\pi}{2} \sqrt{2 + 2\sqrt{1 + A^2}}$. For T_2 , the computation is similar. Let $-1 + iA = r_2 e^{i\theta_2}$, $\theta_2 > 0$. We have poles in \mathbb{R}_2^+

$$s_3 = r_2 e^{i\theta_2/2}, \quad s_3^2 = -1 + iA, \quad s_4 = \bar{s}_3 = -r_2 e^{-i\theta_2/2}, \quad s_4^2 = r_2^2 e^{-i\theta_2} = -1 - iA,$$

and apply the residue formula to get

$$T_2 = \frac{\pi}{2} (s_3 + \bar{s}_3) = \frac{1}{2} \pi i A \left(\frac{s_3}{s_3^2 + 1} + \frac{s_4}{s_4^2 + 1} \right) = \frac{\pi}{2} (s_3 + \bar{s}_3) = \frac{\pi}{2} (-2 + 2\sqrt{1 + A^2})^{1/2}.$$

Denote $G = (1 + A^2)^{1/2}$. Using the above identities and $G^2 - 1 = A^2$, we prove

$$\begin{aligned} T &= T_1 - T_2 = \sqrt{2} \frac{\pi}{2} ((G + 1)^{1/2} - (G - 1)^{1/2}) = \sqrt{2} \frac{\pi}{2} ((G + 1) + (G - 1) - 2(G^2 - 1)^{1/2})^{1/2} \\ &= \sqrt{2} \frac{\pi}{2} (2G - 2A)^{1/2} = \pi ((1 + A^2)^{1/2} - A)^{1/2}. \end{aligned}$$

5.5. Estimate of the constant for $[u_y]_{C_y^{1/2}}, [v_x]_{C_y^{1/2}}$. Recall from Appendix B.2 in [3] the estimate of $[u_y]_{C_y^{1/2}}, [v_x]_{C_y^{1/2}}$

$$(5.61) \quad \begin{aligned} \frac{|u_y(z) - u_y(x)|}{\sqrt{2}} &\leq \frac{1}{\pi\sqrt{2}} \left((\tilde{C}_{in}(\varepsilon) + \min(\tilde{C}_{mid,1}(\varepsilon), C_{mid,3})) [\eta]_{C_x^{1/2}} + C_{out}(m, \varepsilon) [\eta]_{C_y^{1/2}} \right), \\ \frac{|v_x(z) - v_x(x)|}{\sqrt{2}} &\leq \frac{1}{\pi\sqrt{2}} \left(\tilde{C}_{in}(\varepsilon) [\eta]_{C_x^{1/2}} + C_{out}(1, \delta) [\eta]_{C_y^{1/2}} + \frac{\pi}{2} \sqrt{2} [\eta]_{C_x^{1/2}} \right), \end{aligned}$$

where η is a rotation of the original variable ω and the upper bounds are given by

$$(5.62) \quad \begin{aligned} \tilde{C}_{in}(\varepsilon) &= C_{in,\varepsilon,ns} + C_{in,\varepsilon}, \quad \tilde{C}_{mid,1}(m, \varepsilon) = C_{mid,1,\varepsilon,ns} + C_{mid,1,\varepsilon}, \\ C_{in,\varepsilon,ns} &= 4 \int_{\varepsilon}^{y_c} dy_2 \int_0^{h_c^-(y_2)} |\Delta(y)| |y_1 - T_1(y)|^{1/2} dy_1 + 2 \int_{R_{in,\varepsilon}^{++}} |\Delta(y)| \sqrt{2y_1} dy, \\ C_{mid,1,\varepsilon,ns} &= 4 \int_{\varepsilon}^{s_c(m)} dy_2 \int_{h_c^+(y_2)}^m |\Delta(y)| |y_1 - T_1(y)|^{1/2} dy_1 + 2 \int_{R_{mid,y_2 \geq \varepsilon}^{++}} |\Delta(y)| \sqrt{2y_1} dy, \\ C_{mid,2}(\varepsilon) &= 4 \int_{1+\varepsilon}^{\infty} dy_1 \int_{s_c(y_1)}^{\infty} |\Delta(y)| |y_2 - T(y)|^{1/2} dy_2, \quad \varepsilon_m = \max(m, \varepsilon + 1), \\ C_{out}(m, \varepsilon) &= C_{mid,2}(\varepsilon_m - 1) + 4(I_{K_2,\infty}(m + 1, \varepsilon_m + 1), I_{K_2,\infty}(m - 1, \varepsilon_m - 1)), \\ C_{out}(1, \varepsilon) &= C_{mid,2}(\varepsilon) + 4(I_{K_2,\infty}(2, 2 + \varepsilon) + I_{K_2,\infty}(0, \varepsilon)), \end{aligned}$$

and we have replaced the dummy parameter δ used in [3] by ε . The value of ε can be different for different variables. The subscript ns means non-singular. The functions $C_{mid,3}, C_{in,\varepsilon}, C_{mid,\varepsilon}$ are upper bounds of the following integrals

$$(5.63) \quad \begin{aligned} S_{in,\varepsilon} &\triangleq \lim_{\delta \rightarrow 0} \int_{\delta \leq |y_2| \leq \varepsilon, |y_1| \leq 1} \Delta(y) \eta_m(y) dy, \quad |S_{in}| \leq C_{in,\varepsilon} [\eta]_{C_x^{1/2}}, \\ S_{mid,1,\varepsilon} &\triangleq \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon \leq |y_2| \leq \delta, 1 \leq |y_1| \leq m} \Delta(y) \eta_m(y) dy, \quad |S_{mid,1,\varepsilon}| \leq C_{mid,1,\varepsilon} [\eta]_{C_x^{1/2}}, \\ S_{mid,int} &\triangleq \lim_{\delta \rightarrow 0} \int_{\delta \leq |y_2|, |y_1| \in [1, m]} \Delta(y) \eta_m(y) dy, \quad |S_{mid,int}| \leq C_{mid,3} [\eta]_{C_x^{1/2}}, \end{aligned}$$

and will be established in Sections 5.5.2, 5.5.3. In (5.62), $I_{K_2,\infty}$ is defined in (5.88), and T_1 is the transport map in x direction given in (5.93), T is the previous transport map in y direction

(5.92), s_c is given in (5.89), h_c^\pm are given in (5.89), the kernel Δ and Δ_m are given by

$$(5.64) \quad \begin{aligned} \Delta(y) &= K_2(y_1 + 1, y_2) - K_2(y_1 - 1, y_2), \quad \Delta_m(y_1) = \frac{1}{2}(g_{m+1}(y_2) - g_{m-1}(y_2)), \\ y_m &= \sqrt{m^2 - 1}, \quad y_c = 3^{-1/2}. \end{aligned}$$

Different domains are given by

$$(5.65) \quad \begin{aligned} \Omega_{in} &\triangleq \{y : |y_1| \leq 1\}, \quad \Omega_{mid} \triangleq \{y : |y_1| \in [1, m]\}, \quad \Omega_{out} \triangleq \{y : |y_1| > m\}, \\ R_{in,\varepsilon} &\triangleq \{|y_1| \leq 1, |y_2| \geq y_c\} \cup \{T_1(0, |y_2|) \leq y_1 \leq 1, \varepsilon \leq |y_2| \leq y_c\}, \\ R_{in,\varepsilon}^+ &= R_{in,\varepsilon} \cap [0, 1] \times \mathbb{R}, \quad R_{in,\varepsilon}^{++} = R_{in,\varepsilon} \cap [0, 1] \times \mathbb{R}^+, \\ R_{mid} &\triangleq \{|y_1| \in [1, m], |y_2| \geq s_c(m)\} \cup \{|y_2| < s_c(m), 1 \leq |y_1| \leq T(m, |y_2|)\}, \\ R_{mid}^+ &\triangleq R_{out} \cap \{y_1 \geq 0\}, \quad R_{mid}^{++} \triangleq R_{mid} \cap \mathbb{R}_2^{++}. \end{aligned}$$

To estimate these integrals, we follow the strategy in Section 5.2. We need to derive the piecewise bounds for the map T_1 . For $y_2 > 0$, it is easy to see that $h_c^-(y_2)$ is decreasing and $h_c^+(y_2)$ is increasing by checking the sign of $\frac{d}{dy_2} h_c^\pm(y_2)$. Thus the bounds for h_c^\pm are trivial.

Remark 5.6. Although the bounds in (5.62) look quite complicated, we develop several estimates for the same integral so that for the integrals in different region, especially near the singularity, we can choose the easiest one to compute.

5.5.1. *Basic properties of T_1, y_m, h_c^\pm .* Recall that h_c^\pm determine the sign of $\Delta(y)$

$$(5.66) \quad \Delta(y) < 0, \quad y_1 \in (0, h_c^-), \quad y_1 > h_c^+, \quad \Delta(y) > 0, \quad y_1 \in (h_c^-, h_c^+),$$

for $y_2 > 0$. See the right figure in Figure 2 for an illustrations of different regions in $\{y_1 \geq 0\}$ and the sign of the kernels. We have the following basic property.

Lemma 5.7. *Consider the equation and region D_i*

$$\int_T^{y_1} \Delta(s, y_2) ds = 0, \quad T \geq 0, \quad D_1 = \{y_2 < 3^{-\frac{1}{2}}, y_1 < h_c^-(y_2)\} \cap \mathbb{R}_2^{++}, \quad D_2 = \{y_1 > h_c^+(y_2)\} \cap \mathbb{R}_2^{++}.$$

For $y \in D_1$, there is a unique solution $T(y) \in [h_c^-(y_2), 1]$, and T is decreasing in y_1 and y_2 . For $y \in D_2$, there is a unique solution $T(y) \in [1, h_c^+]$, and T is decreasing in y_1 and increasing in y_2 . In both cases, the solution is given by (5.93).

Proof. Clearly, using (5.93), we know that T solves the equation. Next, we consider the bound of T . For a fixed $y_2 \leq 3^{-1/2}$. Denote

$$F(t) = \int_0^t \Delta(s, y_2) ds = \frac{1}{2} \left(\frac{y_1 - 1}{|y_1 - 1|^2 + y_2^2} - \frac{y_1 + 1}{|y_1 + 1|^2 + y_2^2} \right) \Big|_0^t.$$

Using (5.66), we get $F(t) < 0, t \in [0, h_c^-]$, $F(t)$ is strictly decreasing on $[h_c^-, 1]$, and $F(1) < 0$. Thus, we have a unique solution $T \in [h_c^-, 1]$. Since $y_1 < h_c^-(y_1) < T(y)$, $\Delta(y)$ and $\Delta(T(y), y_2)$ have opposite signs. Taking y_1 derivatives yields $\partial_{y_1} T \leq 0$. The properties of T with $y_1 > 1$ follow by a similar argument and studying $F(t) = \int_t^\infty \Delta(s, y_2) ds$.

From the formula of T (5.93), we have $T^2 = \frac{P(y)}{Q(y)}$ for some polynomials P, Q with P increasing in y_2 and Q decreasing in y_2 . Moreover, $P = QT^2$ and Q have the same sign. For $y_2 > 0$, using (5.89), if $y \in D_1$, we get $y_1 < h_c^-(y_2)$, $y_1^2 < y_2^2 + 1$, $Q < 0$ and $P = T^2 Q < 0$. Thus, $\partial_{y_2}(P/Q) = \frac{P'Q - PQ'}{Q^2} < 0$ and T is decreasing in y_2 . As a result, we have

$$(5.67) \quad y_1 \leq h_c^-(y_2) \leq T(y), \quad |T(y) - y_1| = T(y) - y_1 \leq T(y_1^l, y_2^l) - y_1^l,$$

for $y \in D_1 \cap [y_1^l, y_1^u] \times [y_2^l, y_2^u]$. Since $y_1^l \leq y_1 < h_c^-(y_2) < h_c^-(y_2^l)$, (y_1^l, y_2^l) is still in D_1 and the upper bound is well-defined.

Similarly, if $y_1 \in D_2$, we get $y_1 > 1, P > 0, Q = PT^2 > 0$ and obtain that T is decreasing in y_1 , but increasing in y_2 . Moreover, for $y \in D_2$, we have

$$(5.68) \quad y_1 \geq h_c^+(y_2) \geq T(y), \quad |T(y) - y_1| = y_1 - T(y) \leq y_1^u - T(y_1^u, y_2^l).$$

Since $y_1^u > y_1 > h_c^+(y_2) > h_c^+(y_2^l)$, we get $(y_1^u, y_2^l) \in D_2$ and the upper bound is well-defined. We conclude the proof. \square

We remark that since $\int_0^1 \Delta(s, y_2) ds \neq 0$ and $\int_1^\infty \Delta(s, y_2) ds$, for a fixed y_2 , the total mass of the positive part and the negative part of $\Delta(s, y_2) ds$ are not equal. As a result, we cannot construct the map T_1 for all $y_1 > 0$.

5.5.2. *Estimate the integrals in the inner region.* Recall the integrals from (5.62). We have one parameter $m > 1$ and want to obtain a uniform estimate for all $m > 1$. We follow the strategy (d) to partition the domains of the integrals and this parameter. The singularity of the integral is at $y = (1, 0)$. We follow the strategy in Section 5.2 to modify the estimate of $S_{in, \varepsilon}$ near the singularity, which gives the bound $C_{in, \varepsilon}$ (5.63). For a fixed small $\varepsilon > 0$, denote

$$(5.69) \quad Q_1(\varepsilon) = [0, 1] \times [0, \varepsilon], \quad Q_2(\varepsilon) = [1, m] \times [0, \varepsilon], \quad Q_3(\varepsilon) = [m, \infty) \times [0, \varepsilon].$$

Recall from (5.62) that

$$\tilde{C}_{in} = C_{in, \varepsilon, ns} + C_{in, \varepsilon}.$$

The integral in $C_{in, \varepsilon, ns}$ is restricted to $|y_2| \geq \varepsilon$, while $C_{in, \varepsilon}$ is in $|y_2| \leq \varepsilon$. We estimate $C_{in, \varepsilon, ns}$ using previous method and the strategy in Section 5.2. To estimate $|T(y) - y_1|$, we use the piecewise bounds (5.67). For the integral in $C_{in, \varepsilon, ns}$ in $Q_{1, \varepsilon}^c \cap R_{in, 0}^{++}$, since the kernel $\Delta(y)$ has a fixed sign, it becomes

$$(5.70) \quad C_{in, Q_{1, \varepsilon}^c} \triangleq 2 \int_0^1 \left(\int_{y_c}^\infty \Delta(y) dy_2 \right) \sqrt{2y_1} dy_1 + 2 \int_\varepsilon^{y_c} dy_2 \int_{T_1(0, y_2)}^1 |\Delta(y)| \sqrt{2y_1} dy_1.$$

For the first integral, we first apply the integral formula (5.3) for K_2 , and then follow Section 5.2 by bounding the integral of Δ and $y_1^{1/2}$ piecewisely. See (5.51), (5.52) for an example. For the second integral, we estimate the piecewise integrals of $|\Delta(y)| \sqrt{2y_1}$ and then estimate the indicator function of $y_1 \leq T_1(0, y_2) = \max(1 - 3y_2^2, 0)^{1/2}$ (see (5.93)).

Next, we estimate $S_{in, \varepsilon}$ and the bound $C_{in, \varepsilon}$ (5.62), (5.63). We follow Section 5.2 and estimate two kernels in $\Delta(y) = K_2(y_1 + 1, y_2) - K_2(y_1 - 1, y_2)$ separately. Since Δ is odd in y_1 , we have

$$(5.71) \quad \begin{aligned} S_{in, \varepsilon} &= \lim_{\delta \rightarrow 0} \int_{\delta \leq |y_2| \leq \varepsilon} \int_0^1 \tilde{\eta}_m(y) (K_2(y_1 + 1, y_2) - K_2(y_1 - 1, y_2)) dy \\ &= \int_{|y_2| \leq \varepsilon} \int_0^1 K_2(y_1 + 1, y_2) \tilde{\eta}_m(y) dy - \lim_{\delta \rightarrow 0} \int_{\delta \leq |y_2| \leq \varepsilon} \int_{-1}^0 K_2(y_1, y_2) \tilde{\eta}_m(y_1 + 1, y_2) dy \\ &\triangleq I_1 + I_2, \quad \tilde{\eta}_m(y) \triangleq \eta_m(y) - \eta_m(-y_1, y_2), \end{aligned}$$

where we have used a change of variable $y_1 \rightarrow y_1 + 1$ for the second integral. Using

$$(5.72) \quad |\tilde{\eta}_m(y)| \leq \sqrt{2y_1} [\eta_m]_{C_x^{1/2}} = \sqrt{2y_1} [\eta]_{C_x^{1/2}}, \quad (y_1 + 1) > |y_2|, y \in [0, 1] \times [-\varepsilon, \varepsilon],$$

and the formula for $\int K_2 dy_2$ (5.3), for I_1 , we get

$$(5.73) \quad |I_1| \leq \int_0^1 \left(\int_{|y_2| \leq \varepsilon} K_2(y_1 + 1, y_2) dy_2 \right) |2y_1|^{1/2} dy_1 [\eta]_{C_x^{1/2}} = 2\sqrt{2} \cdot \frac{1}{2} \int_0^1 \frac{\varepsilon y_1^{1/2}}{(y_1 + 1)^2 + \varepsilon^2} dy_1 [\eta]_{C_x^{1/2}}.$$

We can bound the integral using the previous method, see e.g. (5.52). For I_2 , applying the estimate (5.85) in Section 5.5.4 in the domain $[-1, 0] \times \pm[0, \varepsilon]$ and

$$(5.74) \quad |\tilde{\eta}_m(y_1 + 1, y_2)| \leq \sqrt{2|y_1 + 1|} [\eta]_{C_x^{1/2}} \leq \sqrt{2} [\eta]_{C_x^{1/2}}, \quad [\tilde{\eta}_m]_{C_x^{1/2}} \leq 2[\eta_m]_{C_x^{1/2}} \leq 2[\eta]_{C_x^{1/2}},$$

for $y_1 \in [-1, 0]$ from (5.72), we get

$$(5.75) \quad |I_2| \leq 2 \left(\sqrt{2} \frac{1}{2} \arctan(\varepsilon) + 2\varepsilon^{1/2} (f_s(\sqrt{1/\varepsilon}) - f_s(1) + C_{K_2, up}) \right) [\eta]_{C_x^{1/2}},$$

where f_s is defined in (5.6). We get a factor 2 since we have two domains $y_2 \in \pm[0, \varepsilon]$.

5.5.3. *Estimate of the integrals in the middle parts and outer parts.* Other integrals in (5.62) are in the region $\Omega_{mid}(m), \Omega_{out}(m)$ (5.65) and depend on the parameter $m > 1$ and we want to obtain piecewise bound for $m \in [m^l, m^u]$. Recall the region $Q_{2,\varepsilon} = [1, m] \times [0, \varepsilon]$ (5.69).

The integrand in $C_{mid,1,\varepsilon,ns}(m)$ is restricted to $|y_2| \geq \varepsilon$ away from the singularity $(1, 0)$. We estimate it using the strategy in Section 5.2 and the above argument for C_{in} . To control $|T(y) - y_1|$, since $y_1 \geq h_c^+(y_2)$ in the support of the integrand, we use the piecewise bound (5.68). Note that from (5.89), for $y_1 > 1, y_2 > 0, y_2 < s_c(y_1)$ is equivalent to $h_c^+(y_2) < y_1$. See the right black curve in right Figure 2 for an illustration. The integral region in the first term satisfies

$$\{y : h_c^+(y_2) < y_1 < m, \varepsilon_2 y_2 < s_c(m)\} = \{y : h_c^+(y_2) < y_1 < m, y_2 > \varepsilon\},$$

which is increasing in m . Thus, we can bound the first integral in $C_{mid,1,\varepsilon,ns}(m)$ for $m \in [m^l, m^u]$ uniformly by the case of $m = m^u$.

Since $s_c(m)$ is increasing in m , the second integral of $C_{mid,1,\varepsilon,ns}$ in $Q_{2,\varepsilon}^c \cap R_{mid}^{++}$ (5.65) satisfies

$$\begin{aligned} C_{mid,1,Q_{2,\varepsilon}^c}(m) &\leq 2 \int_{\varepsilon}^{s_c(m^u)} \int \mathbf{1}_{1 \leq y_1 \leq T(m,y_2)} |\Delta(y)| |2y_1|^{1/2} dy + 2 \int_1^{m^u} \int_{\max(s_c(m^l), \varepsilon)}^{\infty} |\Delta(y)| dy_2 (2y_1)^{1/2} dy_1 \\ &\triangleq I_1 + I_2. \end{aligned}$$

For $m \in [m^l, m^u]$, we estimate the indicator function in I_1 using Lemma 5.7 for T , and then estimate I_1 following Section 5.2 and a method similar to the above. For the second term, we denote $yc_\alpha = \max(s_c(m^l), \varepsilon)$, $\alpha = l, u$ and decompose the domain into three parts

$$D_1 = [1, m^l] \times [yc_l, \infty], \quad D_2 = [m^l, m^u] \times [yc_u, \infty], \quad D_3 = [m^l, m^u] \times [yc_l, yc_u].$$

For $y \in D_1, D_2$, since $s_c(y_1)$ (5.89) is increasing in y_1 , by definition, we get $y_2 \geq s_c(m_\alpha) > s_c(y_1)$, $\alpha = l, u$. Thus, the kernel Δ has a fixed sign in D_1, D_2 . We estimate the integral I_2 in D_1, D_2 using an argument similar to that in (5.70). The kernel Δ can change sign on D_3 . We estimate I_2 in D_3 using the piecewise integral bound for $|\Delta|$ and $y_1^{1/2} \leq (m^u)^{1/2}$.

For $S_{mid,1,\varepsilon}, C_{mid,1,\varepsilon}$ (5.63), following the estimates (5.72)-(5.75) for $S_{in,\varepsilon}$ and applying the estimate in Section 5.5.4 to the region $[0, m-1] \times [0, \varepsilon]$ when $m-1 \geq \varepsilon$, we obtain

$$\begin{aligned} |S_{mid,1,\varepsilon}| &\leq \int_{|y_2| \leq \varepsilon} \int_1^{m^u} K_2(y_1+1, y_2) |2y_1|^{1/2} dy_1 [\eta]_{C_x^{1/2}} \\ &\quad + 2 \left(\sqrt{2m} \frac{1}{2} \arctan\left(\frac{\varepsilon}{m-1}\right) + 2\varepsilon^{1/2} (f_s(\sqrt{(m-1)/\varepsilon}) - f_s(1) + C_{K_2,up}) \right) [\eta]_{C_x^{1/2}}, \end{aligned}$$

where f_s is defined in (5.6) and is increasing in m . The bound $2m^{1/2}$ follows from $|\eta_m(y_1, y_2) - \eta_m(-y_1, y_2)| \leq \sqrt{2|y_1|} [\eta]_{C_x^{1/2}} \leq \sqrt{2m} [\eta]_{C_x^{1/2}}$ for $y_1 \in [1, m]$, which is used to bound the L^∞ norm in (5.85). Note that $\arctan \frac{\varepsilon}{m-1}$ is decreasing in m and other functions in the upper bound of II are increasing in m . Then we can obtain piecewise upper bounds for $m \in [m^l, m^u]$. We estimate the first integral following (5.70) using analytic formulas for piecewise integral of $K_2(y_1+1, y_2)$ and bounding $y_1^{1/2}$ piecewisely.

Small $m-1$. Different from the integral C_{in} , for $C_{mid,1}$, if m is very close to 1, the above method does not work since we require $m-1 \geq 2\varepsilon$, which relates to the condition $a > b$ in the estimate of $J_{a,b}$ (5.85). In this case, we follow the strategy in Section 5.2.3 to separate two kernels in Δ and estimate $S_{mid,int}, C_{mid,3}$ (5.63). Recall from (5.47) that Δ is odd. To overcome the computational difficulties in this singular case, we symmetrize the integral following (5.71) and then decompose it as follows

$$\begin{aligned} S_{mid,int} &= \int_{|y_2| \geq m-1} \int_1^m \Delta(y) \tilde{\eta}_m(y) dy + \int_{|y_2| \leq m-1} \int_1^m K_2(y_1+1, y_2) \tilde{\eta}_m(y) dy \\ &\quad + \int_{|y_2| \leq m-1} \int_1^m K_2(y_1-1, y_2) \tilde{\eta}_m(y) dy = P_1 + P_2 + P_3. \end{aligned}$$

In the domain of P_1 , since $|y_2| \geq y_1 - 1 \geq s_c(y_1)$ (5.89), (5.42), we get $\Delta(s) \geq 0$. Denote $\gamma = m - 1$. Using $|\tilde{\eta}_m| \leq \sqrt{2m}[\eta]_{C_x^{1/2}}$ (5.74) and the formula (5.3), we get

$$\begin{aligned} |P_1| &\leq 2\sqrt{2m}[\eta]_{C_x^{1/2}} \int_{m-1}^{\infty} \int_1^m \Delta(y) dy = 2\sqrt{2m}[\eta]_{C_x^{1/2}} \cdot \frac{1}{2} \int_1^{1+\gamma} \frac{\gamma^2}{(y_1-1)^2 + \gamma^2} - \frac{\gamma}{(y_1+1)^2 + \gamma^2} dy_1 \\ &\leq \sqrt{2m}[\eta]_{C_x^{1/2}} \int_0^{\gamma} \frac{\gamma}{y_1^2 + \gamma^2} dy_1 = \frac{\pi}{4} \sqrt{2m}[\eta]_{C_x^{1/2}}. \end{aligned}$$

For P_2 , using $|K_2(y_1 + 1, y_2)| \leq \frac{1}{2} \frac{1}{(y_1+1)^2 + y_2^2} \leq \frac{1}{8}$ for $y_1 \in [1, m]$ and (5.74), we get

$$P_2 \leq \frac{1}{8} \cdot 2(m-1)^2 \sqrt{2m}[\eta]_{C_x^{1/2}} = \frac{1}{4} (m-1)^2 \sqrt{2m}[\eta]_{C_x^{1/2}}.$$

For P_3 , applying (5.85) with $a = b = m - 1$ and (5.74), we get

$$\begin{aligned} P_3 &\leq 2 \left(\sqrt{2m} \frac{1}{2} \arctan(1) + 2(m-1)^{1/2} (f_s(1) - f_s(1) + C_{K_2, up}) \right) [\eta]_{C_x^{1/2}} \\ &= 2 \left(\sqrt{2m} \frac{\pi}{8} + 2(m-1)^{1/2} C_{K_2, up} \right) [\eta]_{C_x^{1/2}}. \end{aligned}$$

Combining the above estimates, we yield the upper bounds $C_{mid,3}$ for $S_{mid,int}$ in (5.63). The above bounds are increasing in m and we get $C_{mid,3}(m) \leq C_{mid,3}(m^u)$.

Estimate of C_{out} . Recall $C_{out}(m, \varepsilon), C_{out}(1, \varepsilon)$ from (5.62). For $m \in [m^l, m^u]$, since the region $\Omega_{mid,2}(m)$ and $I_{K_2}(0, m-1), I_{K_2}(2, m+1)$ (see (5.88)) are increasing in m , for a fixed ε , we have $C_{out}(m) \leq C_{out}(m^l)$. We fix a small $\varepsilon > 0$ and get $\max(m^l, 1 + \varepsilon)$. We have estimated the integral of $|\Delta(y)||y_2 - T(y)|^{1/2}$ in $C_{mid,2}(\varepsilon_m)$ (5.62) in Section 5.4, e.g. S_{up} , and $I_{K_2, \infty}$ in (5.88) and the paragraph therein.

Estimate of the integrals in the far-field. To estimate the case of $m \geq m_f$ with $m_f = R_1$ sufficient large, we want to reduce it to the case of $m = m_f$ with an integral in the far-field, which is small. Recall R_{mid}, R_{mid}^+ from (5.65)

$$(5.76) \quad R_{mid}^{++}(m) \triangleq \{y_1 \in [1, m], y_2 \geq s_c(m)\} \cup \{y_2 < s_c(m), 1 \leq y_1 \leq T(m, |y_2|)\}.$$

Note that from definition of s_c, h_c^+ (5.89), for $y_1 > 1, y_2 > 0$, we have

$$(5.77) \quad h_c^+(y_2) < y_1, \iff y_2 < s_c(y_1).$$

See the right black curve in right Figure 2 for an illustration. For $m \geq m_f$, we decompose the domain of integrals in $C_{mid}, S_{mid,1,\varepsilon}$ (5.62), (5.63) into $y_1 \in [1, m_f]$ and $y_1 \in [m_f, m]$ as follows

$$\begin{aligned} R_{mid,low}(m) &\triangleq \{y : h_c^+(y_2) < y_1 < m, y_2 \geq \varepsilon\} = \{y : h_c^+(y_2) < y_1 < m_f, y_2 \geq \varepsilon\} \\ &\quad \cup \{y : h_c^+(y_2) < y_1, m_f < y_1 < m, y_2 \geq \varepsilon\} \triangleq \Omega_{1M} \cup \Omega_{1R}, \\ R_{mid}^{++}(m) &= R_{mid}^{++}(m) \cap \{y : y_1 \in [1, m_f]\} \cup R_{mid}^{++}(m) \cap \{y : y_1 \in [m_f, m]\} \triangleq \Omega_{2M} \cap \Omega_{2R}, \\ [1, m] \times [0, \varepsilon] &= [1, m_f] \times [0, \varepsilon] \cup [m_f, m] \times [0, \varepsilon] \triangleq \Omega_{3M} \cup \Omega_{3R}. \end{aligned}$$

In Ω_{1M} , due to (5.77) and $s_c(y_1)$ is increasing in $y_1 > 1$ (5.89), we get $y_2 < s_c(y_1) < s_c(m_f)$ and $\Omega_{1M} = R_{mid,low}(m_f)$. For Ω_{2M} , we want to show $\Omega_{2M} \subset R_{mid}^{++}(m_f)$. We fix $y \in \Omega_{2M}$. If $y_2 \geq s_c(m_f)$, since $y_1 \in [1, m_f]$ for $y \in \Omega_{2M}$, from (5.76), we get $y \in R_{mid}^{++}(m_f)$. If $y_2 < s_c(m_f)$, from (5.77), we get $y_1 > h_c^+(m_f)$ and y is in the definition of T . See Lemma 5.7. Using $T(y)$ is decreasing in y_1 from Lemma 5.7 and $y_1 < T(m, y_2)$ for $y \in \Omega_{2M}$, we get $y_1 < T(m, y_2) < T(m_f, y_2)$. Thus $\Omega_{2M} \subset R_{mid}^{++}(m_f)$.

For the integral $S_{mid,1,\varepsilon}$ in $|y_1| \in [m_f, m]$, since Δ is odd, using $|\eta_m(y_1, y_2) - \eta_m(-y_1, y_2)| \leq \sqrt{2y_1}[\eta]_{C_x^{1/2}}$ and following (5.71), (5.72), we get

$$\left| \int_{|y_1| \in [m_f, m], |y_2| \leq \varepsilon} \Delta(y) \eta_m(y) dy \right| \leq 2\sqrt{2} \int_{\Omega_{3R}} |\Delta(y)| \sqrt{y_1} dy [\eta]_{C_x^{1/2}}.$$

Thus, the integral in $\tilde{C}_{mid,1}(m, \varepsilon)$ (5.62) in $\Omega_{1M}, \Omega_{2M}, \Omega_{3M}$ can be bounded by the case of $m = m_f$. In Ω_{1R} , since $0 < T < y_1$, we get $|y_1 - T_1(y)|^{1/2} \leq |y_1|^{1/2}$. Since Ω_{iR}, Ω_{iM} are disjoint,

following (5.58), we bound the integral in $\tilde{C}_{mid,1}$ for $|y_1| \geq m_f = R_1$ or $|y_2| \geq R_2$, which cover Ω_{iR} , by

$$4 \int_{y_1 \geq R_1, \text{ or } y_2 \geq R_2} |\Delta(y)| y_1^{1/2} dy \leq 4C_B \pi (B-1)^{-1/2}, \quad B = \min(R_1, R_2) - 1.$$

To estimate the integral in C_{out} in the far-field, e.g. outside a domain $[0, R_1] \times [0, R_2]$ with large $R_1, R_2 \geq 10^8$, we use $|y_2 - T| \leq |y_2|$ in the support of the integral and the estimate (5.58).

5.5.4. *Estimate in a strip.* Near the singularity, we need to obtain a sharp estimate of the integral

$$(5.78) \quad I_{a,b} = \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \int_{Q_{\varepsilon,a,b}} \frac{y_1^2 - y_2^2}{|y|^4} f(y) dy, \quad Q_{\varepsilon,a,b} = (\pm[\varepsilon, a]) \times (\pm[0, b]), \quad a < b.$$

Without loss of generality, we assume $Q_{\varepsilon,a,b} = [\varepsilon, a] \times [0, b]$. Firstly, for a fixed y_1 , we can construct a map $T_3(y) \leq y_1$ by solving

$$\int_T^{y_2} K_2(y) dy = 0, \quad K_2(y) = \frac{y_1^2}{y_2}.$$

Since $K_2(y) > 0$ for $y_2 < y_1$ and $K_2(y) < 0$ for $y_2 > y_1$, applying the transportation lemma, Lemma 3.6 [3], we get

(5.79)

$$I_{a,b} \leq \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^a \int_{y_1}^b |K_2(y)| |T(y) - y_2|^{1/2} dy [f]_{C_y^{1/2}} + \left| \int_0^a \int_0^{y_1^2/b} K_2(y) f(y) dy \right| \triangleq I_1 [f]_{C_y^{1/2}} + I_2.$$

Denote $Q = [0, a] \times [0, a]$. Since $y_1^2/b \leq a^2/b \leq a$, in I_2 , we have $|f(y)| \leq \|f\|_{L^\infty(Q)}$.

For I_2 , we have $K_2(y) > 0$ and

$$\begin{aligned} I_2 &\leq \|f\|_{L^\infty(Q)} \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^a \int_0^{y_1^2/b} K_2(y) dy = \|f\|_{L^\infty(Q)} \int_0^a \frac{1}{2} \frac{y_2}{|y|^2} \Big|_0^{y_1^2/b} dy_1 \\ &= \|f\|_{L^\infty(Q)} \frac{1}{2} \int_0^a \frac{y_1^2/b}{y_1^2 + (y_1^2/b)^2} dy_1 = \|f\|_{L^\infty(Q)} \frac{1}{2} \int_0^a \frac{b}{y_1^2 + b^2} dy_1 = \|f\|_{L^\infty(Q)} \frac{1}{2} \arctan\left(\frac{a}{b}\right). \end{aligned}$$

For I_1 , since $y_2 \geq y_1 \geq T$, we have $|T - y_2| \leq |y_2|$, and

$$(5.80) \quad I_1 \leq \int_0^a \int_a^b |K_2(y)| y_2^{1/2} dy_2 + \int_0^a \int_{y_1}^a K_2(y) |y_2 - \frac{y_1^2}{y_2}|^{1/2} dy_2 \triangleq I_{11} + I_{12}.$$

Using the scaling symmetry and (5.11), we get

$$(5.81) \quad I_{12} = a^{1/2} C_{K_2, up}.$$

In I_{11} , $K_2(y)$ has a fixed sign in the domain of the integral. Integrating y_1 first and then using (5.5), and we yield

$$(5.82) \quad \begin{aligned} I_{11} &= \frac{1}{2} \int_a^b \frac{y_1}{y_1^2 + y_2^2} \Big|_0^a y_2^{1/2} dy_2 = \frac{1}{2} \int_a^b \frac{a y_2^{1/2}}{a^2 + y_2^2} dy_2 = \frac{1}{2} \cdot 2a^{1/2} (f_s(\sqrt{b/a}) - f_s(1)) \\ &= a^{1/2} (f_s(\sqrt{b/a}) - f_s(1)). \end{aligned}$$

In summary, we establish

$$(5.83) \quad |I_{a,b}| \leq \|f\|_{L^\infty(Q)} \frac{1}{2} \arctan\left(\frac{a}{b}\right) + a^{1/2} \left(f_s(\sqrt{b/a}) - f_s(1) + C_{K_2, up} \right) [f]_{C_y^{1/2}}, \quad Q = [0, a]^2.$$

Using the same argument and swapping the variable y_1, y_2 , for $0 < b < a$ and

$$(5.84) \quad J_{a,b} = \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \int_{R_{\varepsilon,a,b}} \frac{y_1^2 - y_2^2}{|y|^4} f(y) dy, \quad R_{\varepsilon,a,b} = (\pm[0, a]) \times (\pm[\varepsilon, b]), \quad a < b,$$

we also obtain

(5.85)

$$|J_{a,b}| \leq \|f\|_{L^\infty(Q)} \frac{1}{2} \arctan\left(\frac{b}{a}\right) + b^{1/2} \left(f_s(\sqrt{a/b}) - f_s(1) + C_{K_2, up} \right) [f]_{C_x^{1/2}}, \quad Q = \pm[0, b] \times \pm[0, b].$$

A special case. Consider a similar integral

$$M(a) \triangleq \int_{[0,a]^2} \frac{y_1^2 - y_2^2}{2|y|^4} \tilde{f}(y) dy, \quad \tilde{f}(y) = f(y) - f(y_1, 0), \quad Q_a \triangleq [0, a]^2.$$

For $f \in C^{1/2}(Q_a)$, the integrand is locally integrable, and we do not need to take the principle value. Applying (5.78), (5.79) with $a = b$, $|\tilde{f}(y)| \leq y_2^{1/2}[f]_{C_y^{1/2}(Q_a)}$, $[\tilde{f}]_{C_y^{1/2}(Q_a)} \leq [f]_{C_y^{1/2}(Q_a)}$, the scaling symmetry, and (5.11), we can estimate $M(a)$ as follows

$$(5.86) \quad |M(a)| \leq [f]_{C_y^{\frac{1}{2}}(Q_a)} \left(\int_0^a \int_{y_1}^a |K_2(y)| \left| \frac{y_1^2}{y_2} - y_2 \right|^{\frac{1}{2}} + \int_0^a \int_0^{y_1^2/a} |K_2(y)| y_2^{\frac{1}{2}} dy \right) \leq a^{\frac{1}{2}} C_{K_2} [f]_{C_y^{\frac{1}{2}}(Q_a)}.$$

The infinite length case. We consider estimating

$$(5.87) \quad I = I_{b,\infty} - I_{a,\infty} = \lim_{\varepsilon \rightarrow 0} \int_{a_\varepsilon}^b \int_0^\infty K_2(y) f(y) dy.$$

Applying the above argument and estimate (5.79), we get

$$(5.88) \quad I \leq I_{K_2,\infty}(a, b) [f]_{C_y^{1/2}}, \quad I_{K_2,\infty}(a, b) \triangleq \int_a^b \int_{y_1}^\infty \frac{1}{2} \left| \frac{y_1^2 - y_2^2}{|y|^4} \right| \left| \frac{y_1^2}{y_2} - y_2 \right|^{1/2} dy.$$

The second term in (5.79) vanishes since $\frac{y_1^2}{\infty} = 0$. Applying a change of variable $y_2 = sy_1$ yields

$$I \leq \frac{1}{2} \int_a^b y_1^{-1/2} dy_1 \int_1^\infty \frac{s^2 - 1}{(s^2 + 1)^2} \left| s - \frac{1}{s} \right|^{1/2} ds [f]_{C_y^{1/2}} = (b^{1/2} - a^{1/2}) \int_1^\infty \frac{s^2 - 1}{(s^2 + 1)^2} \left| s - \frac{1}{s} \right|^{1/2} ds [f]_{C_y^{1/2}}.$$

The integral can be estimated using the strategy in Section 5.2 by partitioning $[0, \infty)$ and the integral formula for $K_2(s, 1)$ (5.3). Note that in the far-field $s \geq R > 1$, we have

$$\int_R^\infty \frac{s^2 - 1}{(s^2 + 1)^2} \left| s - \frac{1}{s} \right|^{1/2} \leq \int_R^\infty s^{-2+1/2} ds = 2R^{-1/2}.$$

5.6. Functions and transportation maps. We present the formulas of the transportation maps and the functions related to the sign of the kernels in the sharp Hölder estimate. Recall

$$K_1 = \frac{y_1 y_2}{|y|^4}, \quad K_2 = \frac{1}{2} \frac{y_1^2 - y_2^2}{|y|^4}.$$

5.6.1. *Sign functions.* Solving $K_2(y_1 + 1, y_2) - K_2(y_1 - 1, y_2) = 0$ for $y_2 \geq 0$, we yield

$$(5.89) \quad \begin{aligned} y_1 &= h_c^\pm(y_2) \triangleq \left(y_2^2 + 1 \pm 2y_2 \sqrt{y_2^2 + 1} \right)^{1/2}, \\ y_2 &= s_{c,in}(y_1) \triangleq \left(\frac{-(y_1^2 + 1) + 2(y_1^4 - y_1^2 + 1)^{1/2}}{3} \right)^{1/2}. \end{aligned}$$

5.6.2. *Transportation maps.*

Map for u_x . For a fixed $s_2 \neq 0$ and $s_1 > 0$, solving

$$(5.90) \quad \int_{T(s)}^{s_1} (K_1(s_1 + 1/2, s_2) - K_1(s_1 - 1/2, s_2)) ds_1 = 0,$$

yields the equation of the transportation map in the x direction

$$(5.91) \quad T^3 + T^2 s_1 + T(s_1^2 - \frac{1}{2} + 2s_2^2) - \frac{(4s_2^2 + 1)^2}{16s_1} = 0.$$

Map for $[u_y]_{C_x^{1/2}}$. For a fixed $y_1 \geq 0$, solving

$$\int_{T(y)}^{y_2} (K_2(y_1 + 1, y_2) - K_2(y_1 - 1, y_2)) dy_2 = 0,$$

yields the equation of the transportation map in the y direction

$$(5.92) \quad T^3 + T^2 y_2 + T(y_2^2 + 2 + 2y_1^2) - \frac{(y_1^2 - 1)^2}{y_2} = 0.$$

We rewrite the above equation as an equation for $W = T + \frac{y_2}{3}$

$$0 = W^3 + W\left(\frac{2y_2^2}{3} + 2 + 2y_1^2\right) - \left(\frac{(y_1^2 - 1)^2}{y_2} + \frac{y_2^3}{27} + \frac{y_2}{3}\left(\frac{2y_2^2}{3} + 2 + 2y_1\right)\right) \triangleq W^3 + p_2(y)W + q_2(y).$$

Since $p_2 > 0$, using the discriminant (5.27), we obtain $-\Delta_W(y) > 0$. Thus the cubic equation of T or W has a unique real root that can be obtained by the formula similar to (5.28).

Map for $[u_y]_{C_y^{1/2}}$. For a fixed $y_1, y_2 \geq 0$, solving

$$\int_{y_1}^{T_1(y)} \Delta(s, y_2) ds = 0$$

with $T \geq 0$ yields the equation of the transportation map in x direction

$$1 - T^2 - y_1^2 + T^2 y_1^2 - 2y_2^2 - T^2 y_2^2 - y_1^2 y_2^2 - 3y_2^4 = 0,$$

or equivalently

$$(5.93) \quad T^2 = \frac{y_1^2 + 2y_2^2 + y_1^2 y_2^2 + 3y_2^4 - 1}{y_1^2 - y_2^2 - 1}.$$

We apply the above map to the following two regions separately

$$y_1 \in [0, 1], \quad y_1 \leq h_c^-(y_2), \quad y_1 \in [1, \infty], \quad y_1 \geq h_c^+(y_2).$$

6. ADDITIONAL L^∞ ESTIMATES OF $\nabla \mathbf{u}$ AND SOME EXPLICIT INTEGRALS

We have discussed the L^∞ estimates of $\nabla \mathbf{u}$ in Section 4 in Part II [2]. In this section, we provide a few more detailed calculations in the singular region. Recall from Section 5.3, 5 in [3] the energy E_1, E_4 for $W_1 = (\omega_1, \eta_1, \xi_1)$, which satisfy

$$(6.1) \quad \begin{aligned} & \max(\|\omega_1 \varphi_1\|_\infty, \sqrt{2} \tau_1^{-1} \|\omega_1 |x_1|^{-1/2} \psi_1\|_\infty, \tau_1^{-1} \|\omega_1 \psi_1\|_{C_{g_1}^{1/2}}) \leq E_1(t), \\ & \max(E_1(t), \mu_{g,1} \|\omega_1 \varphi_{g,1}\|_\infty) \leq E_4(t), \quad \mu_{g,1} = \tau_2 \mu_4, \end{aligned}$$

with weights and parameters given below. In Appendix C.1 in Part I [3], we choose the following parameters for the energy E_1

$$(6.2) \quad \tau_1 = 5, \quad \mu_4 = 0.065, \quad \tau_2 = 0.23,$$

the following weights in the estimate of nonlocal terms

$$(6.3) \quad \begin{aligned} & \psi_1 = |x|^{-2} + 0.5|x|^{-1} + 0.2|x|^{-1/6}, \quad \psi_{du} = \psi_1, \quad \psi_u = |x|^{5/2} + 0.2|x|^{-7/6}, \\ & g_1(h) = g_{10}(h)g_{10}(1, 0)^{-1}, \quad g_{10}(h) = (\sqrt{h_1 + q_{11}h_2} + q_{13}\sqrt{h_2 + q_{12}h_1})^{-1}, \\ & \vec{q}_1 = (0.12, 0.01, 0.25), \end{aligned}$$

and the following weights for ω and the error

$$(6.4) \quad \begin{aligned} & \varphi_1 = x^{-1/2}(|x|^{-2.4} + 0.6|x|^{-1/2}) + 0.3|x|^{-1/6}, \quad \varphi_{g1} = \varphi_1 + |x|^{1/16}, \\ & \varphi_{elli} = |x_1|^{-1/2}(|x|^{-2} + 0.6|x|^{-1/2}) + 0.3|x|^{-1/6}. \end{aligned}$$

We do not write down the full energy since we do not use other norms in E_i in this supplementary material. Below, to simplify the notation, we simplify ω_1 as ω in the energy.

6.1. L^∞ estimate of $\nabla \mathbf{u}$ in the singular region. In this section, we estimate

$$(6.5) \quad \begin{aligned} S &= P.V. \int_{Q_\delta} K(y)(W\psi)(x+y) + s(W\psi)(x), \quad Q_\delta = [-\delta, \delta]^2, \quad Q_\delta^\pm \triangleq [-\delta, \delta] \times \pm[0, \delta], \\ K &= K_1 = \frac{y_1 y_2}{|y|^4}, \quad K_2 = \frac{1}{2} \frac{y_1^2 - y_2^2}{|y|^4}, \quad s = 0, \frac{\pi}{2} \text{ or } -\frac{\pi}{2}, \end{aligned}$$

related to the piecewise L^∞ estimate of $\psi \nabla \mathbf{u}$ using the energy E_1 defined in (6.1) which satisfies

$$(6.6) \quad \|\omega\varphi\|_{L^\infty} \leq E_1, \quad [\omega\psi]_{C_x^{1/2}} \leq \gamma_1 E_1, \quad [\omega\psi]_{C_x^{1/2}} \leq \gamma_2 E_1, \quad \gamma_1 = \tau_1, \quad \gamma_2 = \tau_1 g_1(0, 1)^{-1}, \quad \gamma_1 > \gamma_2.$$

Our goal is to establish the following estimate

$$(6.7) \quad |S| \leq C_1 \|\omega\varphi\|_{L^\infty} + C_2 [\omega\psi]_{C_x^{1/2}} + C_3 [\omega\psi]_{C_y^{1/2}} \leq (C_1 + \gamma_1 C_2 + C_3 \gamma_2) E_1,$$

with constant $C_1 + \gamma_1 C_2 + C_3 \gamma_2$ as small as possible. Below, we focus on the case using the norm $\|\omega\varphi\|_\infty$ with $\varphi = \varphi_1$. The estimate using other norms $\|\omega\varphi\|_\infty, \varphi = \varphi_{g,1}, \varphi_{elli}$ is similar and is discussed in Section 6.3. We adopt the notations from Section 3 in [3], Section 4 in Part II [2]. Here, W is the odd extension of ω in y form \mathbb{R}_2^+ to \mathbb{R}_2 , $\tau_1 = 5$ and $g_1, \psi = \psi_1, \varphi$ are the weights for ω in the energy. We use

$$(K, s) = (K_1, 0), \text{ for } u_x, \quad (K_2, -\frac{\pi}{2}), \text{ for } v_x, \quad (K_2, \frac{\pi}{2}) \text{ for } u_y.$$

Note that one needs to multiply $\frac{1}{\pi}$ to get the estimate for $\psi \nabla \mathbf{u}$. We have discussed the estimate for u_x in Section 4.2 in Part II [2].

We assume that $x_i \in [x_i^l, x_i^u] \in \mathbb{R}_+$, and derive the piecewise bound for $S(x)$. Denote

$$(6.8) \quad F = W\psi, \quad x_{2,\delta} = \min(x_2, \delta), \quad \alpha = x_{2,\delta}/\delta, \quad \alpha^l = \min(x_2^l/\delta, 1), \quad \alpha^u = \min(x_2^u/\delta, 1).$$

6.1.1. *Estimate of u_x .* In the case of u_x , we have $K = K_1, s = 0$ and $K_1(s)$ is odd in s_1, s_2 . Using $|W\psi(x+y) - W\psi(x+(-y_1, y_2))| \leq \sqrt{2y_1} [W\psi]_{C_x^{1/2}}$, we get

$$|S| \leq [W\psi]_{C_x^{1/2}} \int_{[0,\delta] \times [-\delta,\delta]} |K_1(s)| |2s_1|^{1/2} ds = 2\sqrt{2\delta} [\omega\psi]_{C_x^{1/2}} \int_{[0,1]^2} K_1(s) |s_1|^{1/2} ds.$$

Using (5.3) and (5.14), we get

$$\int_{[0,1]^2} K_1(s) |s_1|^{1/2} ds = \int_0^1 -\frac{1}{2} \frac{s_1^{3/2}}{|s|^2} \Big|_0^1 ds_1 = \frac{1}{2} \int_0^1 -\frac{s_1^{3/2}}{s_1^2 + 1} + \frac{s_1^{3/2}}{s_1^2} ds_1 = 1 - \frac{1}{2} f_h(1).$$

The above estimate only involves $[\omega]_{C_x^{1/2}}$ and is used when x is close to the boundary $x_2 = 0$. For x away from the boundary, W is also Hölder continuous in y in $Q(x, r)$, since $\gamma_2 < \gamma_1$ (6.6), we can use the seminorm $[\omega]_{C_y^{1/2}}$ to control S and improve the estimate. We have

$$S = \int_{-\delta}^{\delta} dy_1 \left(\int_{-\delta}^{-x_{2,\delta}} + \int_{-x_{2,\delta}}^{x_{2,\delta}} + \int_{x_{2,\delta}}^{\delta} \right) F(x+y) K_1(y) dy_2 \triangleq I_1 + I_2 + I_3.$$

The domain in I_1 is below the boundary, and the domain in I_2, I_3 is above the boundary, where $F \in C_y^{1/2}$. For I_1, I_3 , we use $[F]_{C_x^{1/2}}^{1/2}$ to control it. For I_2 , we use $[F]_{C_y^{1/2}}^{1/2}$ to control it. Using the fact that $K_1(y)$ is odd in y_1, y_2 (6.8) and (6.6), we get

$$\begin{aligned} S &\leq [F]_{C_x^{1/2}} \int_0^\delta \left(\int_{-\delta}^{-x_{2,\delta}} + \int_{x_{2,\delta}}^\delta \right) |K_1(y)| |2y_1|^{1/2} + [F]_{C_y^{1/2}} \int_{-\delta}^\delta \int_0^{x_{2,\delta}} |K_1(y)| |2y_2|^{1/2} dy \\ &= E_1 2\sqrt{2\delta} \left(\gamma_1 \int_0^1 \int_\alpha^1 K_1(y) y_1^{1/2} dy_1 + \gamma_2 \int_0^1 \int_0^\alpha K_1(y) y_2^{1/2} dy_1 \right) \triangleq E_1 2\sqrt{2\delta} (\gamma_1 I_1 + \gamma_2 I_2), \end{aligned}$$

where α is defined in (6.8), and I_1, I_2 denote the first and the second integral and we have rescaled the integral by changing $y \rightarrow \delta y$. Using (5.3), (6.8), and (5.13), we yield

$$\begin{aligned} I_2 &= \frac{1}{2} \int_0^\alpha -\frac{y_2^{3/2}}{y_1^2 + y_2^2} \Big|_{y_1=0}^1 dy_2 = \frac{1}{2} \int_0^\alpha \left(\frac{1}{y_2^{1/2}} - \frac{y_2^{3/2}}{1 + y_2^2} \right) dy_2 \leq \frac{1}{2} \int_0^{\alpha^u} \left(\frac{1}{y_2^{1/2}} - \frac{y_2^{3/2}}{1 + y_2^2} \right) dy_2 \\ &= \frac{1}{2} (2\sqrt{\alpha^u} - f_h(\alpha^u)), \end{aligned}$$

where the inequality follows from $\frac{1}{y_2^{1/2}} - \frac{y_2^{3/2}}{1 + y_2^2} \geq 0$. For I_1 , using (5.3) and (5.14), we get

$$\begin{aligned} I_1 &= \frac{1}{2} \int_0^1 -\frac{y_1^{3/2}}{|y|^2} \Big|_{y_2=\alpha}^1 dy_1 = \frac{1}{2} \int_0^1 \frac{y_1^{3/2}}{\alpha^2 + y_1^2} - \frac{y_1^{3/2}}{1 + y_1^2} dy_1 \leq \frac{1}{2} \int_0^1 \frac{y_1^{3/2}}{(\alpha^l)^2 + y_1^2} - \frac{y_1^{3/2}}{1 + y_1^2} dy_1 \\ &= \frac{1}{2} (\sqrt{\alpha^l} f_h(\frac{1}{\alpha^l}) - f_h(1)). \end{aligned}$$

If $x_2 \geq \delta$, we get $\alpha^l = 1$ and the first part vanishes $I_1 = 0$.

Improvement for $x_2 \geq \delta$. When $x_2 \geq \delta$, we have $x + Q_\delta \subset \mathbb{R}_2^+$ and $F \in C^{1/2}(x + Q_\delta)$. Denote

$$M = \gamma_2^2 / \gamma_1^2 < 1.$$

Symmetrizing the integrals, using (6.6),

$$\begin{aligned} &|F(x + y) + F(x - y) - F(x_1 - y_1, x_1 + y_2) - F(x_1 + y_1, x_2 - y_2)| \\ &\leq 2 \min([F]_{C_x^{1/2}} |2y_1|^{1/2}, [F]_{C_y^{1/2}} |2y_2|^{1/2}) \leq 2\sqrt{2} \min(\gamma_1 |y_1|^{1/2}, \gamma_2 |y_2|^{1/2}) E_1, \end{aligned}$$

$\gamma_1 > \gamma_2$, and an argument similar to the above, we derive

$$|S| \leq 2\sqrt{2\delta} E_1 \int_{[0,1]^2} K_1(y) \min(\gamma_1 |y_1|^{1/2}, \gamma_2 |y_2|^{1/2}). \triangleq 2\sqrt{2\delta} E_1 I.$$

The threshold between two estimates is $y_2 = y_1/M$ or $y_1 = My_2$. We have

$$\begin{aligned} I &= \int_{[0,1]^2} K_1(y) (\mathbf{1}_{\frac{y_1}{M} \leq y_2} |y_1|^{1/2} \gamma_1 + \mathbf{1}_{\frac{y_1}{M} \geq y_2} |y_2|^{1/2} \gamma_2) dy \\ &= \gamma_1 \int_0^M dy_1 \int_{y_1/M}^1 \frac{y_1 y_2}{|y|^4} y_1^{1/2} dy_2 + \gamma_2 \int_0^1 dy_2 \int_{My_2}^1 \frac{y_1 y_2}{|y|^4} y_2^{1/2} dy_1 \triangleq I_1 + I_2. \end{aligned}$$

For I_1 , using (5.3) and (5.13), we get

$$I_1 = \frac{\gamma_1}{2} \int_0^M -\frac{y_1^{3/2}}{y_1^2 + y_2^2} \Big|_{y_1/M}^1 dy_1 = \frac{\gamma_1}{2} \int_0^M -\frac{y_1^{3/2}}{1 + y_1^2} + y_1^{-1/2} \frac{1}{(1/M)^2 + 1} dy_1 = \gamma_1 \frac{M^{5/2}}{M^2 + 1} - \frac{\gamma_1}{2} f_h(M).$$

For I_2 , using (5.3) and (5.13), we yield

$$I_2 = \frac{\gamma_2}{2} \int_0^1 -\frac{y_2^{3/2}}{y_1^2 + y_2^2} \Big|_{My_2}^1 dy_2 = \frac{\gamma_2}{2} \int_0^1 y_2^{3/2} \left(\frac{1}{(M^2 + 1)y_2^2} - \frac{1}{y_2^2 + 1} \right) dy_2 = \frac{\gamma_2}{2} \left(\frac{2}{M^2 + 1} - f_h(1) \right).$$

Since $M = \gamma_2^2 / \gamma_1^2$, $\gamma_1 M^{1/2} = \gamma_2$, we establish

$$|S| \leq \sqrt{2\delta} E_1 (2\gamma_1 \frac{M^{5/2}}{1 + M^2} - \gamma_1 f_h(M) + 2\gamma_2 \frac{1}{M^2 + 1} - \gamma_2 f_h(1)) = \sqrt{2\delta} E_1 (2\gamma_2 - \gamma_1 f_h(\gamma_2^2 / \gamma_1^2) - \gamma_2 f_h(1)).$$

6.1.2. Estimate of v_x . In the case of v_x, u_y (6.5), we have $K(y) = K_2(y)$ which is even in y_1, y_2 , and $s = -\frac{\pi}{2}$ in the case of v_x , and $s = \frac{\pi}{2}$ for u_y . Firstly, if $x_2 \geq \delta$, we get $x + Q_\delta \subset \mathbb{R}_2^+$ and $\omega\psi \in C^{1/2}(x + Q_\delta)$. Using Lemma 5.1, we rewrite S in the case of v_x as follows

$$\begin{aligned} S &= P.V. \int_{Q_\delta} K_2(y) (F(x + y) - F(x)) dx - \frac{\pi}{2} F(x) = \lim_{\varepsilon \rightarrow 0} \int_{Q_\delta, |y_1| \geq \varepsilon} K_2(y) (F(x + y) - F(x)) dy - \frac{\pi}{2} F(x) \\ &\triangleq S_1 - \frac{\pi}{2} F(x). \end{aligned}$$

The first term S_1 has the form (5.78). Applying (5.79) to four regions $\pm[0, \delta] \times [0, \delta]$ and $[F(x + \cdot) - F(x)]_{C_y^{1/2}} = [F]_{C_y^{1/2}}$, we yield

$$(6.9) \quad \begin{aligned} |S_1| &\leq [F]_{C_y^{1/2}} \int_{-\delta}^{\delta} dy_1 \int_{|y_1| \leq |y_2| \leq \delta} \left| \frac{y_1^2}{y_2} - y_2 \right|^{\frac{1}{2}} |K_2(y)| dy_2 \\ &\quad + \left| \int_{-\delta}^{\delta} dy_1 \int_{|y_2| \leq \frac{y_1^2}{\delta}} K_2(y) (F(x+y) - F(x)) dy_2 \right| \triangleq S_{11} + S_{12}. \end{aligned}$$

For S_{11} , using scaling symmetries, (5.11), and (6.6), we get

$$S_{11} = 4[F]_{C_y^{1/2}} \delta^{1/2} C_{K_2, up} \leq 4\gamma_2 \delta^{1/2} C_{K_2, up} E_1.$$

For S_{12} , using (6.6), $|F(x+y) - F(x)| \leq [F]_{C_x}^{1/2} |y_1|^{1/2} + [F]_{C_y}^{1/2} |y_2|^{1/2} \leq E_1(\gamma_1 |y_1|^{1/2} + \gamma_2 |y_2|^{1/2})$, the symmetries of K_2 in y , and $|y_2| \leq |y_1|$, we yield

$$|S_{12}| \leq 4E_1 \int_0^{\delta} \int_0^{y_1^2/\delta} (\gamma_1 |y_1|^{1/2} + \gamma_2 |y_2|^{1/2}) K_2(y) dy = 4E_1(\gamma_1 I_1 + \gamma_2 I_2),$$

where I_i denotes two integrals. Using the scaling symmetry of K_2 , (5.3), (5.6) and (5.12), we get

$$\begin{aligned} I_2 &= \delta^{1/2} \int_0^1 \int_0^{y_1^2} K_2(y) |y_2|^{1/2} dy = \delta^{1/2} C_{K_2, low}, \\ I_1 &= \delta^{1/2} \int_0^1 \int_0^{y_1^2} K_2(y) y_1^{1/2} dy = \delta^{1/2} \int_0^1 \frac{y_2}{2|y_2|^2} \Big|_0^{y_1^2} dy_1 = \delta^{1/2} \int_0^1 \frac{1}{2(1+y_1^2)} y_1^{1/2} dy_1 = \delta^{1/2} f_s(1). \end{aligned}$$

Combining the above estimates of S_{11} , S_{12} and further bounding $F(x) \leq \psi(x)/\varphi(x)E_1$ (6.6), we establish

$$|S(x)| \leq |S_{11}| + |S_{12}| + \frac{\pi}{2} |F(x)| \leq 4E_1 \delta^{1/2} (\gamma_1 f_s(1) + \gamma_2 (C_{K_2, low} + C_{K_2, up})) + \frac{\pi}{2} \frac{\psi(x)}{\varphi(x)} E_1.$$

The above estimate does not hold for $x_2 < \delta$, i.e. the singular region touches the boundary, since W is discontinuous across the boundary. Before we estimate u_y and the case $x_2 < \delta$, we discuss another L^∞ estimate for S_{12} .

6.1.3. L^∞ estimate in a curved region. Since $K_2(y)F(x+y)\mathbf{1}_{|y_2| \leq y_1^2/\delta} \mathbf{1}_{|y_1| \leq \delta}$ is locally integrable, we can bound it using $\|\omega\varphi\|_\infty$. We focus on a specific quadrant $y_1, y_2 \geq 0$. Then in the integral region $y_2 \leq y_1^2/\delta \leq y_1$, we get $K_2(y) \geq 0$. We partition $[0, 1]$ using mesh $0 = z_0 < \dots < z_m = 1$ and use a change of variables $y = \delta s$, $K_2(\delta s)\delta^2 = K_2(s)$ to get

$$\begin{aligned} |S_{12}^{+++}| &= \left| \int_0^{\delta} \int_0^{y_2^2/\delta} K_2(y) (W\psi)(x+y) dy \right| = \left| \sum_{0 \leq i \leq m-1} \int_{z_i}^{z_{i+1}} \int_0^{s_1^2} K_2(s) (W\psi)(x+\delta s) ds \right| \\ &\leq \sum_{0 \leq i \leq m-1} \|W\varphi\|_{L^\infty} \left\| \frac{\psi}{\varphi} \right\|_{L^\infty(x+R_i)} \left| \int_{z_i}^{z_{i+1}} \int_0^{s_1^2} K_2(s) ds \right|, \quad R_i = [z_i\delta, z_{i+1}\delta] \times [0, z_{i+1}^2\delta]. \end{aligned}$$

The piecewise L^∞ bound $\|\frac{\psi}{\varphi}\|_{L^\infty(x+R_i)}$ for $x \in [x_1^l, x_1^u] \times [x_2^l, x_2^u] \triangleq B_x$ can be obtained by covering the region $B_x + R_i$. See Section 4.1.6 in Part II [2]. Since $K_2(s)$ has a fixed sign in the domain, using (5.3), we get

$$(6.10) \quad \int_{z_i}^{z_{i+1}} \int_0^{s_1^2} K_2(s) ds = \int_{z_i}^{z_{i+1}} \frac{1}{2} \frac{s_1^2}{s_1^2 + s_1^4} ds_1 = \frac{1}{2} \arctan s_1 \Big|_{z_i}^{z_{i+1}}.$$

Combining the above estimates, we obtain a sharp L^∞ estimate for S_{12}^{+++} . When δ is small enough, we do not further partition the domain $\delta \cdot [0, 1]$ and estimate S_{12}^{+++} directly

$$|S_{12}^{+++}| \leq \|W\varphi\|_\infty \left\| \frac{\psi}{\varphi} \right\|_{L^\infty(x+[0, \delta]^2)} \cdot \frac{\pi}{8},$$

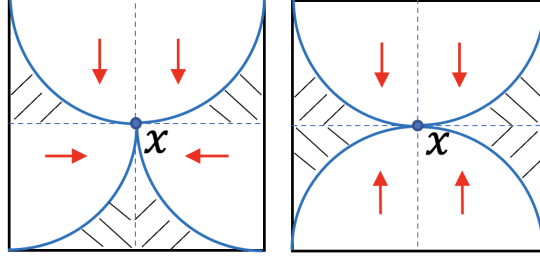


FIGURE 3. Illustration of the estimates. The shaded region represents the curved region, where we estimate the integrals $S_{cur,i}^{\alpha,\beta}$ using the estimate in Section 6.1.3 and $\|\omega\varphi\|_\infty$. For other regions, we estimate the integrals using $C_{x_i}^{1/2}$ seminorm. The red arrow represents the direction of the $C_{x_i}^{1/2}$ seminorm. Left: estimate of u_y, v_x with $x_2 < \delta$. Right: estimate of u_y with $x_2 > \delta$.

where $\pi/4$ is the integral of K_2 in the whole region, which follows from (6.10) with $z_i = 0, z_{i+1} = 1$.

Similar estimates apply to the integrals in other curved regions

$$(6.11) \quad S_{cur,i}^{\alpha,\beta}(x, \delta) \triangleq \int_{R_i \cap \mathbb{R}^\alpha \times \mathbb{R}^\beta} \mathbf{1}_{s \in \mathbb{R}^\alpha \times \mathbb{R}^\beta} K_2(s) (W\psi)(x+s) ds, \quad \alpha, \beta = \pm,$$

$$R_1 = \{|s_1| \leq \delta, |s_2| \leq s_1^2/\delta\}, \quad R_2 = \{|s_2| \leq \delta, |s_1| \leq s_2^2/\delta\}.$$

6.1.4. *Estimate of v_x for $x_2 \leq \delta$.* For $x_2 \leq \delta$, the singular region $x + Q_\delta$ touches the boundary $x_2 = 0$ and $F \notin C_y^{1/2}(x + Q_\delta)$. Using Lemma 5.1, we decompose the integral (6.5) as follows

$$(6.12) \quad S(x) = \lim_{\varepsilon \rightarrow 0} \left(\int_{Q_\delta^+, |y_1| \geq \varepsilon} + \int_{Q_\delta^-, y_2 \leq -\varepsilon} \right) K_2(y) F(x+y) dy - \frac{\pi}{2} F(x) \triangleq S_1 + S_2 + S_3, \quad Q_\delta^\pm \triangleq Q_\delta \cap \mathbb{R}_2^\pm.$$

We get a factor $-\frac{\pi}{4}F(x)$ in the upper part Q^+ and $\frac{\pi}{4}F(x)$ in Q^- from Lemma 5.1, and they are canceled. We first apply estimate (5.79) and then the scaling symmetries of K_2 , (5.11) and (6.11) to get

$$S_1 \leq [F]_{C_y^{1/2}} \int_{-\delta}^{\delta} dy_1 \int_{y_1}^{\delta} K_2(y) \left| \frac{y_1^2}{y_2} - y_2 \right|^{1/2} dy_2 + \int_{-\delta}^{\delta} \int_{y_2 \leq y_1^2/\delta} K_2(y) F(x+y) dy$$

$$= 2[F]_{C_y^{1/2}} \delta^{1/2} C_{K_2, up} + S_{cur,1}^{++} + S_{cur,1}^{-+},$$

where we have a factor 2 since the domain contains 2 quadrants.

For S_2 , $F \notin C_y^{1/2}(x + Q_\delta^-)$. Thus, we apply the estimate for (5.84) using $[F]_{C_x}^{1/2}$ instead. Using an estimate similar to (5.79), (5.11), and (6.11), we get

$$S_2 = [F]_{C_x}^{1/2} \int_{-\delta}^0 dy_2 \int_{|y_2| \leq |y_1| \leq \delta} \left| \frac{y_2^2}{y_1} - y_1 \right|^{1/2} |K_2(y)| dy + \int_{-\delta}^0 \int_{|y_1| \leq y_2^2/\delta} K_2(y) F(x+y) dy$$

$$= 2[F]_{C_x}^{1/2} \delta^{1/2} C_{K_2, up} + S_{cur,2}^{+-} + S_{cur,2}^{--}.$$

The remaining terms $S_{cur,i}^{\alpha,\beta}$ in the above are estimated using the method in Section 6.1.3. The last term $S_3 = \frac{\pi}{2}F(x)$ is estimated directly using $\frac{\pi}{2} \frac{\psi(x)}{\varphi(x)} E_1$ (6.6), (6.8). See the left figure in Figure 3 for various regions. We estimate $S_{cur,i}^{\alpha,\beta}$ in the shaded region.

6.1.5. *Estimate for u_y .* The estimate of u_y is completely similar. In this case, (6.5) becomes

$$S(x) = P.V. \int_{Q_\delta} K_2(y) F(x+y) dy + \frac{\pi}{2} F(x).$$

If $x_2 \geq \delta$, we have $x + Q_\delta \subset \mathbb{R}_2^+$ and $F \in C^{1/2}(x + Q_\delta)$. Applying Lemma 5.1 to 4 quadrants and (5.79) yields

$$\begin{aligned} |S(x)| &= \left| \lim_{\varepsilon \rightarrow 0} \int_{Q_\delta, |y_1| \geq \varepsilon} K_2(y) F(x+y) dy \right| \\ &\leq [F]_{C_y^{1/2}} \int_{-\delta}^{\delta} dy_1 \int_{|y_1| \leq |y_2| \leq \delta} \left| \frac{y_1^2}{y_2} - y_2 \right|^{1/2} |K_2(y)| dy_2 + \left| \int_{-\delta}^{\delta} dy_1 \int_{|y_2| \leq y_1^2/\delta} K_2(y) F(x+y) dy_2 \right| \\ &= 4\delta^{1/2} [F]_{C_y^{1/2}} C_{K_2, up} + S_{cur,1}^{++} + S_{cur,1}^{+-} + S_{cur,1}^{-+} + S_{cur,1}^{--}. \end{aligned}$$

For $x_2 < \delta$, we do not have $F \in C_y^{1/2}(x + Q)$. Using Lemma 5.1 yields another decomposition

$$S(x) = \lim_{\varepsilon \rightarrow 0} \left(\int_{Q_\delta^+, |y_1| \geq \varepsilon} + \int_{Q_\delta^-, y_2 \leq -\varepsilon} \right) K_2(y) F(x+y) dy + \frac{\pi}{2} F(x) \triangleq S_1 + S_2 + S_3.$$

The estimates for S_1, S_2, S_3 are completely the same as those in Section 6.1.4. See the left figure in Figure 3 for an illustration of the estimate in the case of $x_2 \geq \delta$, and right figure for $x_2 > \delta$.

6.2. Estimate of $u(x)/x_1^{1/2}$. In the weighted L^∞ energy estimate in [3], we need to estimate $u(x)/|x_1|^{1/2}$. We have discussed how to estimate $u(x)/|x_1|^{1/2}$ in Appendix B.4 in Part II [2]. In this section, we derive the piecewise bounds for $J_i(B)$

$$J_1(B) = \int_{[-1,0] \times [0,1/B]} K_s(s) ds, \quad J_2(B) = \int_{[0,1/B]^2} K_s(s) ds, \quad K_s(s) = \frac{2(s_1+1)s_2}{|s|^2((s_1+2)^2 + s_2^2)},$$

which captures the singular part in the estimate of $u(x)/x_1$. Clearly, $J_i(B)$ is decreasing in B , for $B \in [B^l, B^u]$, we get $J_i(B) \leq J_i(B^l)$. Since

$$K_s(s) = \frac{1}{2} \left(\frac{s_2}{|s|^2} - \frac{s_2}{(s_1+2)^2 + s_2^2} \right),$$

we can derive its analytic integral formula using (5.4). In particular, we have

$$\begin{aligned} J_1(B) &= \frac{B \left(\log\left(\frac{1}{B^2} + 1\right) - \log\left(\frac{1}{B^2} + 4\right) + \log(4) \right) + 2 \arctan(B) - \arctan(2B)}{2B}, \\ J_2(B) &= J_{21}(B) - \log B, \end{aligned}$$

where the formula for $J_2(B)$ is too lengthy and we refer it to the Mathematica code in [1]. $J_{21}(B)$ is the regular part as $B \rightarrow 0$ and we define it by $J_2(B) - \log B$. Since $J_i(B)$ is decreasing in B , using the above formulas and $J_i(B) \leq J_i(B^l)$, we get the piecewise bounds for $J_i(B)$ in $[B^l, B^u]$. For $B \in [B^l, B^u] = [0, B^u]$, we derive the asymptotic behavior of the formula as $B \rightarrow 0$. For $J_1(B)$, since $2 \arctan B - \arctan(2B) = O(B^3)$, we get

$$\lim_{B \rightarrow 0} J_1(B) = \lim_{B \rightarrow 0} \frac{1}{2} \log\left(\frac{1/B^2 + 1}{1/B^2 + 4}\right) \cdot 4 = \lim_{B \rightarrow 0} \frac{1}{4} \log\left(\frac{4 + 4B^2}{1 + 4B^2}\right) = \frac{1}{2} \log 4 = \log 2.$$

Next, we show that $J_{21}(B)$ is increasing for B close to 0, which allows us to estimate $J_2(B)$ near $B = 0$. Using symbolic computation, we get

$$\partial_B J_{21}(B) = - \frac{-2 \log\left(\frac{1}{B^2} + \frac{2}{B} + 2\right) + 2 \log\left(\frac{(2B+1)^2}{B^2}\right) - 8B + 4 \tan^{-1}(2B) - 4 \tan^{-1}(2B+1) + \pi}{8B^2} \triangleq - \frac{S}{8B^2}.$$

We can rewrite the numerator S as follows

$$S = 2 \log((1+2B)^2) - 2 \log(2B^2 + 2B + 1) - 8B + 4 \tan^{-1}(2B) - 4 \tan^{-1}(2B+1) + \pi.$$

Clearly, we have $S(0) = 0$. Next, we show that $S'(B) \leq 0$:

$$\begin{aligned} S'(B) &= -8 + \frac{8}{1+(2B)^2} - \frac{8}{1+(1+2B)^2} + \frac{8}{1+2B} - \frac{2(4B+2)}{1+2B+2B^2} \\ &\leq -\frac{8}{2+4B+4B^2} + \frac{8}{1+2B} - \frac{2(4B+2)}{1+2B+2B^2} = \frac{8}{1+2B} - \frac{8B+8}{1+2B+2B^2} \leq 0, \end{aligned}$$

where we have used $1+2B+2B^2 - (1+B)(1+2B) = -B \leq 0$. Thus $S(B) \leq S(0) = 0$ and $J_{21}(B)$ is increasing for $B > 0$. Recall from Appendix B.4 in Part II [2] that $B_2 = \frac{\hat{x}_1}{h}$,

where h is the mesh size and \hat{x}_1 is the rescaled x in the computation domain. Using the above monotonicity property, for $\hat{x}_1 \in [z^l, z^u]$, $z^u \leq \frac{h}{2}$ and $\alpha \in (0, 1)$, we have $B_2 \leq \frac{z^u}{h} \leq \frac{1}{2}$ and

$$J_2(B_2)\hat{x}_1^{1-\alpha} = J_{21}\left(\frac{\hat{x}_1}{h}\right)\hat{x}_1^{1-\alpha} - \log\left(\frac{\hat{x}_1}{h}\right)\hat{x}_1^{1-\alpha} \leq J_{21}\left(\frac{z^u}{h}\right)(z^u)^{1-\alpha} + I(\hat{x}_1), \quad I(t) \triangleq \log\left(\frac{h}{t}\right)t^{1-\alpha}.$$

To obtain the piecewise bound for I , taking derivative, we yields

$$\partial_t I = (1-\alpha)t^{-\alpha} \log \frac{h}{t} - t^{-\alpha} = t^{-\alpha}((1-\alpha) \log \frac{h}{t} - 1), \quad \partial_t I > 0, \quad \text{for } t < he^{-1/(1-\alpha)}.$$

Therefore, for $z^u \leq \min(he^{-1/(1-\alpha)}, h/2)$, $I(\hat{x}_1)$ is increasing and thus

$$I(\hat{x}_1) \leq I(z^u), \quad J_2(B_2)\hat{x}_1^{1-\alpha} \leq J_2\left(\frac{z^u}{h}\right)(z^u)^{1-\alpha}.$$

The quantities $J_2(B)\hat{x}_1^{1-\alpha}$ appear in our bound for $\frac{u(x)}{|x_1|^\alpha}$. In our application, we choose $\alpha = 1/2$.

The above estimate allows us to control $\frac{u(x)}{|x_1|^\alpha}$ for small x_1 .

Integral close to the singularity. In addition to $J_i(B)$, we need to estimate the integral

$$\int_a^b \int_c^d K_{du}(x, s) ds, \quad K_{du}(x, s) = \frac{2(s_1 + x_1)s_2}{|s|^2((s_1 + 2x_1)^2 + s_2^2)} \mathbf{1}_{x_1+s_1 \geq 0}, \quad Q = [a, b] \times [c, d],$$

for $x_1 \in [x_1^l, x_1^u] \subset \mathbb{R}^+$, in the region close to the singularity, e.g. $s \in [-kh_0, kh_0]^2 \setminus [-h_0, h_0]^2$. See II_1, II_2 in the Appendix B.4 in Part II [2]. Denote $h = x_1^u - x_1^l$. Without loss of generality, we consider $c, d \geq 0$, $s_1 \in [a, b]$, $s_2 \in [c, d]$. For $s_1 + x_1 \geq 0$,

$$K_{du} \geq 0, \quad |s_1 + 2x_1|^2 \geq s_1^2, \quad |s_1 + 2x_1|^2 - |s_1 + 2x_1^l|^2 = (x_1 - x_1^l)(2s_1 + 2x_1 + 2x_1^l) \geq 0, \quad x_1 \leq x_1^l + h.$$

Estimate I: $a + x_1^l \geq 0$. In this cas, we have $s_1 + x_1 \geq a + x_1^l \geq 0$ uniformly for $s_1 \in [a, b]$ and $x \in [x_1^l, x_1^u]$. Thus, the indicator function is 1 in Q , and we yield

$$0 \leq K_{du} \mathbf{1}_{x_1+s_1 \geq 0} \leq \frac{2(s_1 + x_1^l + h)s_2}{|s|^2((s_1 + 2x_1^l)^2 + s_2^2)} = \frac{1}{2x_1^l} \left(\frac{s_2}{|s|^2} - \frac{s_2}{(s_1 + 2x_1^l)^2 + s_2^2} \right) + \frac{2hs_2}{|s|^2((s_1 + 2x_1^l)^2 + s_2^2)} \triangleq I_1 + I_2.$$

For I_1 , if $x_1^l > 0$, we use the analytic formula (5.4) to evaluate the integral. For I_2 , it is much smaller than the main term. Since $(s_1 + 2x_1^l)^2 + s_2^2$ is regular and $a + x_1^l \geq 0$, we bound it as follows

$$(6.13) \quad \int_Q I_2 ds \leq \max_Q \frac{2h}{(s_1 + 2x_1^l)^2 + s_2^2} \int_Q \frac{s_2}{|s|^2} ds, \\ s_1 + 2x_1^l \geq \max(a + 2x_1^l, x_1^l, 0) \triangleq dis^l, \quad s_2^2 \geq \min(c^2, d^2), \quad cd \geq 0,$$

and evaluate the integral using (5.4). If $x_1^l = 0$, I_1, I_2 reduce to $I_1 = \frac{2s_1 s_2}{|s|^4} = 2K_1(s)$, $I_2 = \frac{2hs_2}{|s|^4}$. We evaluate their integrals using the analytic integral formula (5.3) and

$$\int \frac{s_2}{|s|^4} ds = \frac{1}{2} s_2^{-1} \arctan \frac{s_2}{s_1} + C.$$

Note that the integrand is singular near 0 when $x_1^l = 0$. We only apply it to the region away from the origin $s = 0$.

If $a + x_1^l \leq 0$, since in the support of the integrand, we have

$$0 \leq s_1 + x_1 \leq s_1 + x_1 - a - x_1^l \leq b - a + h, \quad s_1 + 2x_1 \geq \max(x_1^l, a + 2x_1^l, 0) = dis^l.$$

We bound the integrand as follows

$$0 \leq K_{du} \mathbf{1}_{x_1+s_1 \geq 0} \leq \frac{2(b-a+h)s_2}{|s|^2((s_1 + 2x_1)^2) + s_2^2} \mathbf{1}_{x_1+s_1 \geq 0} \leq \frac{2(b-a+h)s_2}{|s|^2((dis^l)^2 + s_2^2)},$$

and then estimate the integral following (6.13).

Estimate II. If $a + x_1^u \geq 0$, using $x_1 + s_1 \leq s_1 + x_1^u$, we have another estimate for K_{du}

$$0 \leq x_1 K_{du} \mathbf{1}_{x_1+s_1 \geq 0} = \frac{1}{2} \left(\frac{s_2}{|s|^2} - \frac{s_2}{(s_1 + 2x_1)^2 + s_2^2} \right) \mathbf{1}_{x_1+s_1 \geq 0} \leq \frac{1}{2} \left(\frac{s_2}{|s|^2} - \frac{s_2}{(s_1 + 2x_1^u)^2 + s_2^2} \right).$$

Since $a + x_1^u \geq 0$, the right hand side is nonnegative for $s_1 \in [a, b]$. We further use (5.4) to evaluate the integral of the upper bound.

If $a + x_1^u \leq 0$, in the support of the integrand, we yield

$$0 \leq s_1 + x_1 \leq s_1 - a + (x_1 - x_1^u) \leq b - a, \quad 0 \leq x_1 K_{du} \mathbf{1}_{x_1+s_1 \geq 0} \leq \frac{2(b-a)x_1^u s_2}{|s|^2((s_1 + 2x_1)^2 + s_2^2)} \mathbf{1}_{x_1+s_1 \geq 0}.$$

Then we use $s_1 + 2x_1 \geq \max(a + 2x_1^l, x_1^l, 0)$ and follow (6.13) to estimate the integral. We also have a simple bound using $x_1 K_{du} \mathbf{1}_{x_1+s_1} \leq \frac{s_2}{2|s|^2}$ and then apply the integral formula (5.4).

We apply the above estimate to obtain sharp estimates of the integral of u (not divided by $\frac{1}{x_1}$). Dividing both sides by $\frac{1}{x_1}$ and using $\frac{1}{x_1} \leq \frac{1}{x_1^l}$, we yield another estimate for the integral of K_{du} . This estimate is better if x_1^u/x_1^l is close to 1.

We remark that if $b + x_1^u \leq 0$, since $s_1 + x_1 \leq b + x_1^u \leq 0$, the integral is 0.

6.3. Estimate using other norms. The estimate using the weights $\|\omega\varphi_{g,1}\|_\infty$ is similar. From (6.1), we have $\|\omega\varphi_{g,1}\|_\infty \leq \mu_{g,1}^{-1} E_4$, $E_1 \leq E_4$. Using the norm $\|\omega\varphi_{g,1}\|_\infty, [\omega\psi_1]_{C_{x_i}^{1/2}}$, we develop another estimate for $\mathbf{u}_A, (\nabla \mathbf{u})_A$. For the singular part S (6.5), using $\mu_{g,1} \leq 0.02$, we estimate

$$(6.14) \quad \begin{aligned} |S| &\leq C_1 \|\omega\varphi_{g,1}\|_{L^\infty} + C_2 [\omega\psi]_{C_x^{1/2}} + C_3 [\omega\psi]_{C_y^{1/2}} \leq (C_1 \mu_{g,1}^{-1} + \gamma_1 C_2 + C_3 \gamma_2) E_4 \\ &\leq \mu_{g,1}^{-1} (C_1 + \mu_{g,1} \gamma_1 C_2 + \mu_{g,1} \gamma_2 C_3) E_4 \leq \mu_{g,1}^{-1} (C_1 + 0.02 \gamma_1 C_2 + 0.02 \gamma_2 C_3) E_4, \end{aligned}$$

similar to (6.7), with constant $C_1 + (0.02\gamma_1)C_2 + (0.02\gamma_2)C_3\gamma_2$ as small as possible. For this purpose, we can apply the estimates in the previous sections with (γ_1, γ_2) replaced by $0.02(\gamma_1, \gamma_2)$. Note that this additional estimate for $\mathbf{u}_A, (\nabla \mathbf{u})_A$ is used to close the nonlinear estimates. The overestimate $\mu_{g,1} \leq 0.02$ only slightly increases the constant in the nonlinear estimates.

To estimate the nonlocal error $\mathbf{u}(\varepsilon), \varepsilon = \bar{\omega} - (-\Delta)\bar{\phi}^N, \varepsilon = \hat{\omega} - (-\Delta)\hat{\phi}^N$ for the approximate steady state or \hat{W}_2 (see Section 5.8 in [3]), we develop similar estimates for $\mathbf{u}_A(\varepsilon), (\nabla \mathbf{u})_A$ using the norm $\|\varepsilon\varphi_{elli}\|_\infty, [\varepsilon\psi_1]_{C_{x_i}^{1/2}}$ (6.3), (6.4). For the singular part S , we estimate it using the norm localized to D containing $x + Q_\delta$ in (6.5)

$$(6.15) \quad \begin{aligned} |S(\varepsilon)| &\leq C_1(x) \|\varepsilon\varphi_{g,1}\|_{L^\infty(D)} + C_2(x) [\varepsilon\psi]_{C_x^{1/2}(D)} + C_3(x) [\varepsilon\psi]_{C_y^{1/2}(D)} \\ &\leq \bar{B}_0 (C_1 + \bar{B}_0^{-1} \bar{B}_1 C_2 + \bar{B}_0^{-1} \bar{B}_2 C_3), \quad \|\varepsilon\varphi_{g,1}\|_{L^\infty(D)} \leq \bar{B}_0, [\varepsilon\psi_1]_{C_{x_i}^{1/2}(D)} \leq \bar{B}_i, \end{aligned}$$

with constant $C_1(x) + \bar{B}_0^{-1} \bar{B}_1 C_2 + \bar{B}_0^{-1} \bar{B}_2 C_3(x)$ as small as possible. Since ε depends on the numerical solution, e.g. $\bar{\omega}, \bar{\phi}^N$, locally, we can bound \bar{B}_i directly. To get a sharp estimate, we can apply the estimate in the previous sections with (γ_1, γ_2) replaced by $(\bar{B}_0^{-1} \bar{B}_1, \bar{B}_0^{-1} \bar{B}_2)$. See Section 4.7 in Part II [2] for more discussions of the localized estimate.

We can generalize the above estimates of combining different norms in the estimates of $\mathbf{u}_A, (\nabla \mathbf{u})_A$ for x very small or very large. See Section 7.5.

6.4. Estimate of some integrals. We discuss the refined estimate of $u_x(0)(\varepsilon)$ and $K_{00}(\varepsilon)$ for the error ε of solving the Poisson equation. The refined estimates of these terms are important for us to show that the error is small.

6.4.1. Estimate of K_{00} . In the estimate of the integrals, near the origin, we use the triangle inequality and we estimate the approximation term

$$I = p_\lambda(\hat{x}) \lambda^{-2} C(\lambda \hat{x}) \int_{|\hat{y}|_\infty \leq kh} K_{00}(\hat{y}) W_\lambda(\hat{y}) d\hat{y} = p(\lambda x) C(x) \int_{|y| \leq \lambda kh} K_{00}(y) W(y) dy,$$

for a fixed k , e.g. $k = 12$, separately since the kernel given below is singular with order $|y|^{-4}$ near $y = 0$

$$K_{00}(y) = \frac{24y_1 y_2 (y_1^2 - y_2^2)}{|y|^8} = \partial_1^3 \partial_2 f(y), \quad f(y) = -\log |y|.$$

We need to estimate the above term for finitely many $\lambda \leq 10$ and $\lambda \leq \lambda_*$ uniformly for x near 0. In the energy estimate for linear stability analysis, we bound it using $\|\omega\varphi\|_\infty$. In the error estimate, such an estimate is not sufficient. Since we can evaluate $\omega(y)$, we exploit the cancellation in the integral. We partition the domain using fine mesh. Denote

$$0 = y_1 < y_2 < \dots < y_N, \quad y_{i,1/2} = (y_i + y_{i+1})/2, \quad Q_{ij} = [y_i, y_{i+1}] \times [y_j, y_{j+1}].$$

We evaluate the integral in Q using Simpson's rule

$$(6.16) \quad \int_{[a_1, a_3] \times [b_1, b_3]} g(y) dy = \sum_{i,j \leq 3} c_i c_j f(a_i, b_j) + \text{err}, \quad |\text{err}| \leq \frac{1}{2880} (h_1^4 \|\partial_x^4 g\|_{L^\infty(Q)} + h_2^4 \|\partial_y^4 g\|_{L^\infty(Q)}),$$

$$h_1 = a_3 - a_1, h_2 = b_3 - b_1 \quad Q = [a_1, a_3] \times [b_1, b_3], \quad a_2 = \frac{a_1 + a_3}{2}, \quad b_2 = \frac{b_1 + b_3}{2}, \quad c = \left[\frac{1}{6}, \frac{4}{6}, \frac{1}{6}\right].$$

We obtain the piecewise derivative bound of $K_{00}(y)W$ using the bound of W established by the method in Appendix C in Part II [2] and (5.2) for K_{00} . The above error estimate is obtained by applying the error estimate of Simpson's rule first in x and then in y .

For the integral very close to origin, e.g. in $Q = [0, D]^2 = [0, y_m]^2$, the above method fails since the kernel is singular. We defer the estimate below. With these estimates, for a fixed λ , we pick l such that $y_l \leq r < y_{l+1}$, $r = \lambda kh$ and decompose the integral into three regions

$$\int_{[0, r]^2} g(y) dy = \left(\int_{[0, y_m]^2} + \int_{[0, y_l]^2 \setminus [0, y_m]^2} + \int_{[0, r]^2 \setminus [0, y_l]^2} \right) g(y) dy \triangleq S_1 + S_2 + S_3, \quad |S_3| \leq \int_{[0, y_{l+1}]^2 \setminus [0, y_l]^2} |g(y)| dy.$$

For S_1 , we apply the estimate near 0 discussed below. For S_2 , we use the above Simpson's rule. Since $[0, r]^2 \setminus [0, y_l]^2$ is small, we treat S_3 as an error and use the piecewise bounds for $g(y) = \omega(y)K_{00}(y)$.

The above method also provide the piecewise bound of the integral for $\lambda kh \in [y_l, y_{l+1}]$. This allows us to obtain the uniform bound for $\lambda \in [y_m/(kh), \lambda_*]$ by covering the interval. To obtain the uniform estimate for all small $\lambda \leq \lambda_*$, we further apply the method discussed below to estimate the case of $\lambda \leq y_m/(kh)$ uniformly.

Estimate near $y = 0$. Consider $Q = [0, D]^2$. In our case, ω is odd and satisfies $\omega = O(|x|^3)$ near $x = 0$ and we have piecewise C^3 bounds for ω . Using integration by parts, we get

$$\begin{aligned} \int_Q \omega(y) f_{xxxy}(y) dy &= - \int_Q \omega_x(y) f_{xxy} dy + \int_0^D \omega(D, y) f_{xxy}(D, y) dy \\ &= \int_Q \omega_{xx}(D, y) f_{xy}(D, y) dy - \int_0^D \omega_x(y) f_{xy} dy + \int_0^D \omega(D, y) f_{xxy}(D, y) dy \triangleq I + II + III. \end{aligned}$$

The boundary term vanishes on $x = 0$ since $\omega f_{xxy}, \omega_x f_{xy}$ is odd. Denote $M_{ij} = \|\partial_x^i \partial_y^j \omega\|_{L^\infty(Q)}$. Since ω is odd, for $y \in Q$, using $\omega = O(|x|^3)$, Taylor expansion

$$\omega = \int_0^{y_1} \omega_x(z, y_2) dz = \omega_x(0, y_2) y_1 + \int_0^{y_1} \omega_{xxx}(z, y_2) \frac{(y_1 - z)^2}{2} dz, \quad \omega_x(0, y_2) = \int_0^{y_2} \omega_{xyy}(0, z) (y_2 - z) dz,$$

taking derivatives on the above expansions, and using $\int_0^y z^k dz = \frac{y^{k+1}}{k+1}$, we get

$$|\omega(y)| \leq \frac{y_1^3}{6} M_{30} + \frac{y_1 y_2^2}{2} M_{12}, \quad |\omega_x| \leq \frac{y_1^2}{2} M_{30} + \frac{y_2^2}{2} M_{12}, \quad |\omega_{xx}| \leq y_1 M_{30}, \quad M_{ij} \triangleq \|\partial_x^i \partial_y^j \omega\|_{L^\infty(Q)}.$$

Since

$$f_{xy} = \frac{2y_1 y_2}{|y|^4}, \quad f_{xxy} = \frac{2y_2(-3y_1^2 + y_2^2)}{|y|^6},$$

f_{xy}, f_{xxy} have fixed signs in $[0, D]^2, \{D\} \times [0, D]$, respectively.

For I , since ω_{xx} is odd, using the scaling symmetry of the kernel, we get

$$|I| \leq M_{30} \int_Q |f_{xy} y_1| dy = M_{30} \int_Q \frac{2y_1^2 y_2}{|y|^4} dy = M_{30} D \int_{[0, 1]^2} \frac{2y_1^2 y_2}{|y|^4} dy = M_{30} D \cdot y_2 \arctan \frac{y_1}{y_2} \Big|_{[0, 1]^2} = M_{30} \frac{D\pi}{4},$$

where we use $\partial_{12}(y_2 \arctan \frac{y_1}{y_2}) = \partial_2(y_2 \frac{1/y_2}{(y_1/y_2)^2 + 1}) = \partial_2 \frac{y_2^2}{y_1^2 + y_2^2} = -\partial_2 \frac{y_1^2}{y_1^2 + y_2^2} = \frac{2y_1^2 y_2}{|y|^4}$.

For II and III , we use the expansion at (D, y) . We get

$$|II| \leq \frac{M_{30}D^2}{2} \left| \int_0^D f_{xy}(D, y) dy \right| + \frac{M_{12}}{2} \left| \int_0^D f_{xy}(D, y) y^2 dy \right| \triangleq \frac{M_{30}}{2} II_1 + \frac{M_{12}}{2} II_2.$$

Using the scaling symmetry, we get

$$\begin{aligned} II_1 &= D \left| \int_0^1 f_{xy}(1, y) dy \right| = D \left| f_x(1, y) \Big|_0^1 \right| = D |f_x(1, 1) - f_x(1, 0)|, \\ II_2 &= D \left| \int_0^1 \frac{2y^3}{(1+y^2)^2} dy \right| = D \left(\frac{1}{1+y^2} + \log(1+y^2) \right) \Big|_0^1 = D(\log 2 - 1/2). \end{aligned}$$

For III , since $f_{xy}(D, y)$ has a fixed sign in $[0, D]$, we get

$$|III| \leq \frac{M_{30}D^3}{6} \left| \int_0^D f_{xxy}(D, y) dy \right| + \frac{M_{12}}{2} D \left| \int_0^D f_{xxy}(D, y) y^2 dy \right| = \frac{M_{30}}{6} III_1 + \frac{M_{12}}{2} III_2.$$

Since f is harmonic, $f_{xxy} = -f_{yyy}$, using the scaling symmetry and integration by parts, we get

$$\begin{aligned} III_1 &= D \left| \int_0^1 f_{xxy}(1, y) dy \right| = D |f_{xx}(1, 1) - f_{xx}(1, 0)|, \\ III_2 &= D \left| \int_0^1 f_{xxy}(1, y) y^2 dy \right| = D \left| \int_0^1 f_{yyy}(1, y) y^2 dy \right| = D \left| (f_{yy}(1, y) y^2 - 2f_y(1, y) y + 2f) \Big|_0^1 \right|. \end{aligned}$$

All the above terms are linear in D since $y_1^i y_2^j K_{00}(y) dy = D \hat{y}_1^i \hat{y}_2^j K_{00}(\hat{y}) d\hat{y}$ for $y = D\hat{y}$, $i+j = 3$.

6.4.2. *Estimate of $c_\omega(\bar{\omega})$.* Recall from Appendix C in Part II [2] that we represent $\bar{\omega}$ using piecewise polynomials $\bar{\omega}_2$ and the semi-analytic part $\bar{\omega}_1 = \chi(r)r^{-\alpha}g(\beta)$. We discuss the computation of

$$c_\omega(f) = u_x(f)(0) = -\frac{4}{\pi} \int_{\mathbb{R}_2^{++}} K(y) f(y) dy, \quad K(y) = \frac{y_1 y_2}{|y|^4},$$

for f being a single-level B-spline and $f = \bar{\omega}_1$. If f is a multi-level B-spline (see Section 7 in [3]), i.e. $f = \sum_{i \leq n} f_i$ with f_i being a single level B-spline, we estimate $c_\omega(f_i)$ and then use linearity to estimate $c_\omega(f)$. For f being B-spline with supporting points $y_0 < y_1 < \dots < y_n$, its support is contained in $D = [0, y_{n+7}]^2$. We partition the domain into

$$D_i = [0, y_{k_i}]^2, \quad i = 1, 2, \dots, l, \quad y_{k_1} < y_{k_2} < \dots < y_{k_l} = y_{n+7}.$$

For each i , we refine each mesh $Q = [y_{i_1}, y_{i_1+1}] \times [y_{j_1}, y_{j_1+1}]$ in $D_{i+1} \setminus D_i$ $2n_i$ times

$$Q = \cup_{k, l \leq n_i} Q_{kl}, \quad Q_{kl} = [y_{i_1} + (k-1)h_1, y_{i_1} + kh_1] \times [y_{j_1} + (l-1)h_2, y_{j_1} + lh_2], \quad h_1 = \frac{y_{i_1+1} - y_{i_1}}{n_i}, \quad h_2 = \frac{y_{j_1+1} - y_{j_1}}{n_i},$$

and then apply Simpson's rule (6.16) to estimate the integral $\int_{Q_{kl}} f(y) K(y) dy$ by evaluating fK on $y_{i_1} + \frac{ph_1}{2}, y_{j_1} + \frac{qh_2}{2}, 0 \leq p, q \leq 2n_i$ and estimating the error. In Q , using (6.16), the error is given by

$$\frac{1}{2880} \sum_{k, l \leq n_i} \int_{Q_{kl}} h_1^4 \|\partial_x^4(fK)\|_{L^\infty} + h_2^4 \|\partial_y^4(fK)\|_{L^\infty}.$$

Using $K(y) = -\frac{1}{2} \partial_1 \partial_2 \log(|y|)$ and $|\partial_x^i \partial_y^j K(y)| \leq \frac{(i+j+1)!}{2} |y|^{-i-j-2}$ from Lemma 5.2, we get

$$\sum_{k, l} \int_{Q_{kl}} \|\partial_x^4(fK)\|_{L^\infty} \leq \sum_{0 \leq m \leq 4} \binom{4}{m} \|\partial_x^{4-m} f\|_{L^\infty(Q)} h_1 h_2 \frac{(m+1)!}{2} \sum_{kl} \max_{y \in Q_{kl}} |y|^{-m-2}.$$

The bound for $\partial_y^4(fK)$ is similar.

Near 0. Since the integrand is singular, near 0, we use Taylor expansion. We choose $D_0 \subset [0, y_1]^2$. For $f(x_1, x_2)$ odd in x_1 , we get

$$f(x) = \partial_1 f(0)x_1 + \partial_{12} f(0)x_1 x_2 + \frac{\partial_{111} f(0)x_1^3}{6} + \frac{\partial_{122} f(0)x_1 x_2^2}{2} + \varepsilon, \quad |\varepsilon| \leq \frac{1}{24} \sum_{0 \leq i \leq 4} \binom{4}{i} x_1^i x_2^{4-i} \|\partial_x^i \partial_y^{4-i} f\|_{L^\infty(D_0)}.$$

We integrate $K(y)y_1^i y_2^j$ analytically using symbolic computation and then obtain the error estimate and the approximation of the integral by evaluating $\partial_1^i \partial_2^j f(0)$.

Integral for $\bar{\omega}_1$. For $f = \bar{\omega}_1 = \chi(r)r^{-\alpha-1}g(\beta)$, $\chi(r)$ is supported in $[A, \infty)$ and $\chi(r) = 1$ for $r \geq B$, $A = 5$, $B = 2 \cdot 10^6$. Using $K(y) = \frac{\sin(2\beta)}{2r^2}$ and a direct calculation yields

$$\begin{aligned} -\frac{4}{\pi} \int_{\mathbb{R}_{++}^2} K(y)\bar{\omega}_1 dy &= -\frac{2}{\pi} \int_A^\infty \chi(r)r^{-\alpha-1} dr \int_0^{\pi/2} g(\beta) \sin(2\beta) d\beta \\ &= -\frac{2}{\pi} \left(\int_A^B \chi(r)r^{-\alpha-1} dr + \alpha^{-1} B^{-\alpha} \right) \int_0^{\pi/2} g(\beta) \sin(2\beta) d\beta. \end{aligned}$$

We apply 1D Simpson's rule to estimate two 1D integrals.

7. ESTIMATE OF THE VELOCITY NEAR 0 AND IN THE FAR-FIELD

For x very close to the origin or very large, we cannot rescale the integral by choosing finitely many rescaling factors λ_i . See Section 4.5 in Part II [2] for more discussions. Instead, we choose $\lambda = \frac{\max(x_1, x_2)}{x_c}$ and estimate the rescaled integral with a $-d$ -homogeneous kernel K

$$(7.1) \quad p(x) \int K(x-y)W(y)dy = p_\lambda(x) \int K(\hat{x}-\hat{y})\lambda^{2-d}W_\lambda(\hat{y})dy, \quad f_\lambda(x) \triangleq f(\lambda x),$$

uniformly for all small $\lambda \ll 1$ or large $\lambda \gg 1$. The rescaled singularity $\hat{x} = x/\lambda$ satisfies $\max_i \hat{x}_i = x_c$ and is in the bulk of our computation domain. The method is essentially the same as those described in Sections 4.1, 4.2 in Part II [2]. We have the following scaling relation

$$(7.2) \quad \begin{aligned} \|\omega_\lambda \varphi_\lambda\|_\infty &= \|\omega \varphi\|_\infty, \quad [\omega_\lambda \psi_\lambda]_{C_{x_i}^{1/2}} = \lambda^{\frac{1}{2}} [\omega \psi]_{C_{x_i}^{1/2}}, \\ \partial_{x_i} f(x) &= \frac{d\hat{x}_i}{dx_i} \partial_{\hat{x}_i} f_\lambda(\hat{x}) = \frac{1}{\lambda} \partial_{\hat{x}_i} f_\lambda(\hat{x}), \quad \frac{|f(x) - f(z)|}{|x - z|^{1/2}} = \lambda^{-\frac{1}{2}} \frac{|f_\lambda(\hat{x}) - f_\lambda(\hat{z})|}{|\hat{x} - \hat{z}|^{1/2}}. \end{aligned}$$

Denote by y_i the adaptive mesh for computing the approximate steady state designed in Appendix C.1 in Part II [2]

$$(7.3) \quad y_1 < y_2 < \dots < y_N, \quad Q_{ij} = [y_i, y_{i+1}] \times [y_j, y_{j+1}],$$

We have $y_N > 10^{15}$. In Part II [2], we introduce the following notations: $h_x = h/2$,

$$(7.4) \quad |x|_\infty = \max(x_1, x_2), \quad x \in B_{i_1, j_1}(h_x) \subset B_{ij}(h), \quad B_{lm}(r) = [lr, (l+1)r] \times [mr, (m+1)r],$$

and singular regions near x

$$(7.5) \quad \begin{aligned} R(x, k) &= [(i-k)h, (i+1+k)h] \times [(j-k)h, (j+k+1)h], \\ R_s(k) &= [x_1 - kh, x_1 + kh] \times [x_2 - kh, x_2 + kh]. \end{aligned}$$

We introduce the following domain and points

$$(7.6) \quad \begin{aligned} M_{l, i_1, j_1} &= \lambda_l B_{i_1, j_1}(h_x), \quad \Gamma(a) \triangleq \{\hat{x} : \max_i \hat{x}_i = a\} = \cup_{ih_x \leq a} \Gamma_{1,i}(a) \cup \Gamma_{2,i}(a) \\ \Gamma_{1,i}(a) &\triangleq \{a\} \times [(i-1)h_x, ih_x], \quad \Gamma_{2,i}(a) \triangleq [(i-1)h_x, ih_x] \times \{a\}, \\ x_c &= 128h_x, \quad x_{c2} = 256h_x, \quad x_{c3} = 3x_{c2}, \quad m_3 = 2x_{c2}h_x^{-1} = 512. \end{aligned}$$

Domain M_{l, i_1, j_1} is a dyadic mesh, and x_c, x_{ci} can be viewed as the reference centers. We choose a to be a multiple of h_x , and then $\Gamma(a)$ consists of two intervals. In the weighted L^∞ estimate, we rescale $x = \lambda \hat{x}$ such that $\hat{x} \in \Gamma(x_c)$. In the Hölder estimate, we rescale (x, z) with $|x| \leq |z|$ such that $x = \lambda \hat{x}$ with $\hat{x} \in \Gamma(2x_c)$.

We use the L^∞ norm $\|\omega\varphi\|_\infty$ and the Hölder semi-norm $[\omega\psi]_{C_{x_i}^{1/2}}$ to control the piecewise L^∞ norm of \mathbf{u} , $\nabla\mathbf{u}$ and the Hölder semi-norm of the weighted quantities $\psi_u\mathbf{u}$, $\psi\nabla\mathbf{u}$. Denote by a_1, a_n the power of $\varphi(x)$ near 0 and ∞ , b_1, b_n the power for ψ , and c_1, c_n for ψ_u . From the definitions of these weights (6.3), (6.4) we have

$$(7.7) \quad \begin{aligned} \varphi(x) &\geq q_1 r^{a_1} (\cos\beta)^{-\frac{1}{2}} + q_n r^{a_n}, \quad (a_1, a_n) = (-2.9, -1/6), (-2.9, 1/16), (-5/2, -1/6), \\ \psi(x) &\sim B_1 r^{b_1}, r \ll 1, \quad \psi(x) \sim B_n r^{b_n}, r \gg 1, \quad (b_1, b_n) = (-2, -1/6), \\ \psi_u &\sim C_1 r^{c_1}, r \ll 1, \quad \psi_u \sim C_n r^{c_n}, r \gg 1, \quad (c_1, c_n) = (-5/2, -7/6), \end{aligned}$$

for some constants q_i, B_j, C_j , where $r = |x|, \beta = \arctan(\frac{x_2}{x_1})$. Here $\psi = \psi_1$ (6.3) and we drop 1 to simplify the notation. We use the norm $\|\omega\varphi\|_\infty$ with three different weights φ in (6.4) and the above power (a_1, a_n) . We have

$$(7.8) \quad \varphi_\lambda(x) \geq \lambda^{a_\alpha} \varphi_{\infty, \alpha}(x), \quad \alpha = 1, n.$$

We choose $c_n = b_n - 1$ to capture the fact that \mathbf{u} decays one order slower than $\nabla\mathbf{u}$.

7.1. L^∞ estimate. Denote by ϕ the stream function. Recall from Section 4.3 [3] that for λ very small, i.e. x near 0, the velocity with approximation terms is given by

$$(7.9) \quad (\partial_x^i \partial_y^j \phi)_A \triangleq \partial_x^i \partial_y^j (\phi - \phi_{xy}(0)xy - \frac{1}{6}\phi_{xxx}(0)(x^3y - xy^3)),$$

for $i + j = 1, 2$. For example, we have $u_x = -\partial_{xy}\phi, v_x = \partial_{xx}\phi$. See Section 4.3 in [3]. The vorticity ω vanishes like $O(|x|^{2+\alpha})$ for some $\alpha > 0$ near $x = 0$, and $\phi, \phi_{xy}(0), \phi_{xxxy}(0)$ are given by

$$(7.10) \quad \phi(x) = -\frac{1}{\pi} \int_{\mathbb{R}_2} G(x-y)W(y)dy, \quad \phi_{xy}(0) = \frac{4}{\pi} \int_{\mathbb{R}_2^+} \frac{y_1 y_2}{|y|^4}, \quad \phi_{xxxy}(0) = \frac{2}{\pi} \int_{\mathbb{R}_2^+} \omega(y) \frac{24y_1 y_2 (y_1^2 - y_2^2)}{|y|^8} dy,$$

where G is the Green function (5.1), and W is given in (5.2).

For λ large, x near ∞ , we estimate

$$(7.11) \quad (\partial_x^i \partial_y^j \phi)_A = \partial_x^i \partial_y^j (\phi + L_{12}(0)xy), \quad L_{12} = -\frac{4}{\pi} \int_{y \in \mathbb{R}_2^+ \setminus [0, R_n]^2} \frac{y_1 y_2}{|y|^4} \omega(y) dy,$$

for some parameter R_n (the largest threshold) given in Appendix C.2 in [3], where $L_{12}(0)$ approximates $u_x(0) = -\phi_{xy}(0)$. Note that for $|x| \geq R_n$, $(\partial_x^i \partial_y^j \phi)_A$ satisfies the differential relation

$$(7.12) \quad \partial_x^i \partial_y^j (\partial_x^{i_2} \partial_y^{j_2} \phi)_A = (\partial_x^{i+i_2} \partial_y^{j+j_2} \phi)_A, \quad 1 \leq i + j + i_1 + j_1 \leq 2.$$

Using the scaling symmetry of the kernels, we can rewrite the above quantities as integrals of ω and in the form (7.1). We estimate piecewise bounds for $\nabla\mathbf{u}(\lambda\hat{x}), \mathbf{u}(\lambda\hat{x})$ for $\hat{x} \in \Gamma_{i,j}$ and $\lambda \leq \lambda_*$ or $\lambda \geq \lambda_*$ uniformly.

7.1.1. Near field, far field, and the nonsingular part. For the integral in these regions, the integrand is not singular, and we follow the method in Sections 4.1, 4.2 in Part II [2] to estimate them. For example, for a region $Q \subset \mathbb{R}_2^{++}$ away from the singularity, using (7.8), we estimate the integral (7.1) with symmetrized kernel

$$(7.13) \quad K^{sym}(x, y) \triangleq K(x-y) + K(x+y) - K(x_1-y_1, x_2+y_2) - K(x_1+y_1, x_2-y_2),$$

and approximation terms $K = K^{sym} - K_{app}$ as follows

$$(7.14) \quad \left| \int_D K(\hat{x}, \hat{y}) W_\lambda(\hat{y}) d\hat{y} \right| \leq \|W_\lambda \varphi_\lambda\|_\infty \int_D |K(\hat{x}, \hat{y})| \varphi_\lambda^{-1}(\hat{y}) d\hat{y} \leq \lambda^{-a_\alpha} \|\omega\varphi\|_\infty \int_D |K(\hat{x}, \hat{y})| \varphi_{\infty, \alpha}^{-1}(\hat{y}) d\hat{y},$$

for $\alpha = 1, n$. Using the decay estimates of the symmetrized integrals of (7.10) with approximations in Appendix B.1 in Part II [2] and (7.7), one can show that $|K(x, y)| \varphi_{\infty, \alpha}^{-1}(y)$ is integrable away from the singularity. The last integral does not depend on λ and can be estimated using the method in [3]. We track the integral and the power λ^{-a_α} . Thus, the estimates of the integrals in these regions are essentially the same as those for finite rescaling factor λ_i discussed

in Sections 4.1, 4.2 in Part II [2], except that we use the asymptotic behavior of the weights $\varphi_{\infty,1}, \varphi_{\infty,n}$ instead of φ .

For large λ , we need to estimate (7.11). Since $[0, R_n]^2$ does not enjoy the scaling symmetry, $y = \lambda\hat{y} \in [0, R_n]^2$ if and only if $\hat{y} \in [0, R_n/\lambda]^2$, we decompose (7.11) as follows

$$\begin{aligned} L_{12} &= -\frac{4}{\pi} \int_{\mathbb{R}_2^{++} \setminus [0, R_n/\lambda]^2} \frac{\hat{y}_1 \hat{y}_2}{|\hat{y}|^4} \omega_\lambda(\hat{y}) d\hat{y} = -\frac{4}{\pi} \int_{y \in \mathbb{R}_2^{++}} \mathbf{1}_{D_1^c} + \left(\mathbf{1}_{([0, R_n/\lambda]^2)^c}(\hat{y}) - \mathbf{1}_{D_1^c}(\hat{y}) \right) K_0(\hat{y}) \omega_\lambda(\hat{y}) d\hat{y} \\ &\triangleq \hat{L}_{12}(\lambda) + I, \quad K_0(y) = \frac{4y_1 y_2}{|y|^4}, \quad D_1 = [0, k_1 h]^2. \end{aligned}$$

The domain of the rescaled integral in $L_{12}(\lambda)$ does not depend on λ , and we estimate the integrals for $\partial_x^i \partial_y^j \phi - \partial_x^i \partial_y^j (xy) \hat{L}_{12}(\lambda)$ using the above method. The second part I is treated as an error term. Using (7.7), (7.8) and $|\omega_\lambda(\hat{y})| \leq \lambda^{-a_n} q_n^{-1} |\hat{y}|^{-a_n} \|\omega\varphi\|_\infty$, we have

$$|I| = \frac{1}{\pi} \int (\mathbf{1}_{D_1} - \mathbf{1}_{[0, \frac{R_n}{\lambda}]^2}) |K_0(\hat{y}) \omega_\lambda(\hat{y})| d\hat{y} \leq \frac{\|\omega\varphi\|_\infty}{\pi q_n} \lambda^{-a_n} \int |\mathbf{1}_{D_1} - \mathbf{1}_{[0, \frac{R_n}{\lambda}]^2}| \frac{4\hat{y}_1 \hat{y}_2}{|\hat{y}|^4} |\hat{y}|^{-a_n} d\hat{y} \triangleq \frac{\|\omega\varphi\|_\infty}{\pi q_n} \cdot II.$$

Next, we estimate $|II|$ uniformly for $\lambda \geq \lambda_*$. For $0 < c < d$, since $\frac{y_1 y_2}{|y|^\beta}$ is symmetric in y_1, y_2 , we get

$$\begin{aligned} J(c, d) &= \int_{[0, d]^2 \setminus [0, c]^2} \frac{4y_1 y_2}{|y|^{4+a_n}} dy = 8 \int_c^d \int_0^{y_1} \frac{y_1 y_2}{|y|^{4+a_n}} dy = \frac{8}{2+a_n} \int_c^d \frac{-y_1}{|y|^{2+a_n}} \Big|_0^{y_1} dy_1 \\ &= \frac{8(1-2^{-(2+a_n)/2})}{2+a_n} \int_c^d y_1^{-1-a_n} dy_1 = C_{a_n} |d^{-a_n} - c^{-a_n}|, \quad C_\alpha = \frac{8}{(2+\alpha)|\alpha|} (1-2^{-(2+\alpha)/2}). \end{aligned}$$

Applying the above estimate to II , we obtain

$$|II| \leq C_{a_n} \left| \left(\frac{R_n}{\lambda} \right)^{-a_n} - (k_1 h)^{-a_n} \right| \lambda^{-a_n} = C_{a_n} |R_n^{-a_n} - (\lambda k_1 h)^{-a_n}|.$$

If $R_n \leq \lambda_* k_1 h$, for $\lambda \geq \lambda_*$, we have

$$|II| \leq C_{a_n} (R_n^{-a_n} - (\lambda k_1 h)^{-a_n}) \leq C_{a_n} R_n^{-a_n}, \quad a_n > 0, \quad |II| \leq C_{a_n} ((\lambda k_1 h)^{-a_n} - R_n^{-a_n}), \quad a_n < 0.$$

If $R_n > \lambda_* k_1 h$ and $a_n < 0$, for $\lambda \geq \lambda_*$, we have two cases, $R_n \geq \lambda k_1 h$ and $R_n < \lambda k_1 h$. In the first case, since $\lambda \geq \lambda_*$, $a_n < 0$, we get $\lambda^{a_n} \leq \lambda_*^{a_n}$ and

$$|II| \leq C_{a_n} \left(\left(\frac{R}{\lambda} \right)^{-a_n} - (k_1 h)^{-a_n} \right) \lambda^{-a_n} \leq C_{a_n} \left(\left(\frac{R}{\lambda_*} \right)^{-a_n} - (k_1 h)^{-a_n} \right) \lambda^{-a_n}.$$

In the second case, we get

$$|II| \leq C_{a_n} \left((k_1 h)^{-a_n} - (R/\lambda)^{-a_n} \right) \lambda^{-a_n} \leq C_{a_n} (k_1 h)^{-a_n} \lambda^{-a_n}.$$

Combining two cases, we yield

$$|II| \leq C_{a_n} \lambda^{-a_n} \max((k_1 h)^{-a_n}, (R/\lambda_*)^{-a_n} - (k_1 h)^{-a_n}).$$

In our estimate, we do not have the case $R_n > \lambda_* k_1 h$ and $a_n > 0$. We can pick λ_* large enough and the power a_n in the weight to avoid such a case. In all cases, we can estimate

$$|II| \leq C_1 \lambda^{-a_n} + C_2$$

for some C_1, C_2 (can be negatively) independent of λ uniformly for $\lambda \geq \lambda_*$.

7.1.2. Singular part. We focus on the estimate of $\nabla \mathbf{u}$. which is much more difficult. For the singular part, we follow Section 4.2 in Parr II [2] to decompose it as follows

$$\begin{aligned} (\nabla \mathbf{u})_S &= \int_{R(x,k)} K(\hat{x} - \hat{y}) W_\lambda(\hat{y}) d\hat{y} = \int_{R(x,k) \setminus R_s(b)} + \int_{R_s(b) \setminus R_s(a)} + \int_{R_s(a)} K(\hat{x} - \hat{y}) W_\lambda(\hat{y}) d\hat{y} \\ &\triangleq I + II + III, \end{aligned}$$

for fixed $k, b \geq 1$ and a to be chosen, where $R(x, k), R_s(k)$ are singular regions defined in (7.5). The integrand in I is nonsingular and we estimate it by $\|\omega\varphi\|_\infty$ using the same method as that

in Section 7.1.1 and Sections 4.1, 4.2 in Part II [2], For II , using L^∞ estimate, (7.8), (5.18), and $R_s(b) \subset R(b)$, we get

$$|II| \leq \|W\varphi\|_{L^\infty} \lambda^{-a\alpha} \|\varphi_{\infty,\alpha}^{-1}\|_{L^\infty(R(b))} \int_{R_s(b) \setminus R_s(a)} |K(\hat{x}-\hat{y})| d\hat{y} = \|\omega\varphi\|_{L^\infty} \lambda^{-a\alpha} \|\varphi_{\infty,\alpha}^{-1}\|_{L^\infty(R(b))} 2 \log\left(\frac{b}{a}\right).$$

Following the estimate of the singular part in Section 4.2 in Part II [2], we get

$$III = \int_{R_s(a)} K(\hat{x}-\hat{y})(\psi_\lambda^{-1}(\hat{x}) - \psi_\lambda^{-1}(\hat{y})) \omega_\lambda \psi_\lambda(\hat{y}) dy + \psi_\lambda^{-1}(\hat{x}) \int_{R_s(a)} K(\hat{x}-\hat{y}) \omega_\lambda(\hat{y}) \psi_\lambda(\hat{y}) d\hat{y} = III_1 + III_2.$$

For III_1 , we follow Section 4.2 in Part II [2] using Taylor expansion and the L^∞ estimate

$$\begin{aligned} \psi_\lambda^{-1}(\hat{y}) - \psi_\lambda^{-1}(\hat{x}) &= (\nabla \psi_\lambda^{-1})(\hat{x}) \cdot (\hat{y} - \hat{x}) + P_e, \quad |P_e| \leq C |\nabla^2 \psi_\lambda^{-1}| |\hat{y} - \hat{x}|^2, \quad |W_\lambda \psi_\lambda(x)| \leq \frac{\psi_\lambda}{\varphi_\lambda}(\hat{x}) \|\omega\varphi\|_{L^\infty}, \\ \int_{R_s(a)} |K(\hat{x} - \hat{y})(\hat{x}_1 - \hat{y}_1)^i (\hat{x}_2 - \hat{y}_2)^j| d\hat{y} &= 4 \int_{[0,a]^2} |K(s) s_1^i s_2^j| ds = 4a^{i+j} \int_{[0,1]^2} |K(s) s_1^i s_2^j| ds, \quad i + j \geq 1, \end{aligned}$$

where the last integral can be evaluated using the methods in Section 5.1.3. In particular, we gain a small factor $|a|$ in the estimate of III_1 . We refer detailed estimate of III_1 to Section 4.2 in Part II [2]. To factorize out the dependence of λ in the above estimates, we need the following estimates for the weights

$$(7.15) \quad \left| \frac{\partial^i}{\partial \hat{x}_1} \frac{\partial^j}{\partial \hat{x}_2} \psi_\lambda^{-1}(\hat{x}) \right| \leq \lambda^{a\alpha} (\partial_1^i \partial_2^j \psi)_{\infty,\alpha}(\hat{x}), \quad \frac{\psi_\lambda(\hat{x})}{\varphi_\lambda(\hat{x})} \leq \lambda^{b\alpha - a\alpha} \left(\frac{\psi}{\varphi} \right)_{\infty,\alpha}(\hat{x}).$$

uniformly for $\lambda \leq \lambda_1$ for $\alpha = 1$, i.e. x close to 0, and $\lambda \geq \lambda_n$ for $\alpha = n$, i.e. x in the far-field, which have been established in Appendix A.2, A.3 in Part II [2]. Note that we do not pick up extra λ power in the asymptotic analysis when we take derivatives $\partial_{\hat{x}_i}^j$ on ψ_λ^{-1} . For example, if $\psi(x) = |x|^{-2}$, $\partial_{\hat{x}_1} \psi_\lambda^{-1}(\hat{x}) = \partial_{\hat{x}_1} \lambda^2 |\hat{x}|^2 = \lambda^2 2\hat{x}_1$. Using these estimates, we derive

$$III_1 \leq \lambda^{-a\alpha} |a| C(\hat{x}),$$

for some constant $C(\hat{x})$.

For III_2 , using the estimates in Section 6.1, we yield

$$|\psi_\lambda(\hat{x}) III_2| \leq C_1(\hat{x}) \left\| \frac{\psi_\lambda}{\varphi_\lambda} \right\|_{L^\infty(R(b))} \|\omega_\lambda \varphi_\lambda\|_{L^\infty} + C_2(\hat{x}) |a|^{1/2} [\omega_\lambda \psi_\lambda]_{C_x^{1/2}} + C_3(\hat{x}) |a|^{1/2} [\omega_\lambda \psi_\lambda]_{C_y^{1/2}},$$

for some $C_i(\hat{x})$ independent of the weights φ and λ , which can be derived using the same method as that in Section 6.1. Using the scaling relation (7.2), we yield

$$|\psi_\lambda(\hat{x}) III_2| \leq C_1(\hat{x}) \lambda^{a\alpha - b\alpha} \|\omega\varphi\|_{L^\infty} + C_2(\hat{x}) |\lambda a|^{1/2} [\omega\psi]_{C_x^{1/2}} + C_3(\hat{x}) |\lambda a|^{1/2} [\omega\psi]_{C_y^{1/2}},$$

where we have changed $C_1(\hat{x})$ to track the constant $\left\| \frac{\psi_\lambda}{\varphi_\lambda} \right\|_{L^\infty(R(b))}$.

Using the above estimates and (6.6), we can bound the singular part as follows

$$(7.16) \quad \begin{aligned} (\nabla \mathbf{u})_S &\leq \lambda^{-a\alpha} (C_1(\hat{x}) + C_2(\hat{x}) \log \frac{b}{a} + C_3(\hat{x}) |a|) \|\omega\varphi\|_\infty + \lambda^{-b\alpha + \frac{1}{2}} (C_{41}(\hat{x}) [\omega\psi]_{C_x^{1/2}} + C_{42}(\hat{x}) [\omega\psi]_{C_y^{1/2}}) \\ &\leq E_1 \left(\lambda^{-a\alpha} (C_1(\hat{x}) + C_2(\hat{x}) \log \frac{b}{a} + C_3(\hat{x}) |a|) + \left(\gamma_1 C_{41}(\hat{x}) + \gamma_2 C_{42}(\hat{x}) \right) \lambda^{-b\alpha + 1/2} \right), \end{aligned}$$

for $x = \lambda\hat{x}$ close to 0, $\alpha = 1$, and for $x = \lambda\hat{x}$ in the far-field, $\alpha = n$, with \hat{x} in each $\Gamma_{1,i}, \Gamma_{2,i}$ (7.6).

We perform the above estimate for a list of $a, a_1 < a_2 < \dots < a_N \leq b$ and uniform estimate for $ah \leq h \leq bh$. We will optimize a to obtain a sharp estimate in Section 7.3.1.

7.1.3. *Summary of the estimates and scaling.* Combining the estimates in Section 7.1.1, e.g. (7.14), and (7.16), we obtain the estimate for $\nabla \mathbf{u}$. From the above estimates, we can apply the same estimates as those in [3] and Section 6.1 except that we need to perform the estimates using the asymptotic properties of the weights (7.7), (7.8), (7.15) uniformly for small λ or large λ , instead of the weights $\varphi^{-1}, \frac{\psi}{\varphi}, \partial_1^i \partial_2^j (\psi^{-1})$ in the estimates for the finite λ case. Moreover, we need to track the power in λ . In particular, we can obtain piecewise estimates for $\mathbf{u}, \nabla \mathbf{u}$ (7.1) for $x \in \Gamma_{i,j}(x_c)$ (7.6) and $\lambda \leq \lambda_*$ or $\lambda \geq \lambda_*$ uniformly. The estimates weighted by $\psi_\lambda(\hat{x})$ can be obtained similarly.

For $\mathbf{u}(\lambda \hat{x})$, the estimate is much easier since the kernel is locally integrable. Using the argument in Section 7.1.1, the scaling relation (7.1) with $d = 1$, and the same method in [3] except that we use the weight $\varphi_{\infty,\alpha}$ (7.8) instead of φ , we can bound the integral and $\mathbf{u}(\lambda \hat{x})$ by

$$C(\hat{x})\lambda^{-a_\alpha+1}\|\omega\varphi\|_\infty.$$

We get the power 1 since the kernel in \mathbf{u} is -1 homogeneous and we use the scaling relation (7.1) with $d = 1$.

Remark 7.1. To track the λ power in the estimates (7.14), (7.16), we have the power λ^{-a_α} in the coefficient of $\|\omega\varphi\|_\infty$ since $\varphi^{-1}(\lambda x)$ has asymptotic scaling property λ^{-a_α} (7.8), and $\lambda^{-b_\alpha+1/2}$ in the coefficient of $[\omega\psi]_{C_x^{1/2}}$ since $\psi^{-1}(\lambda x)$ has asymptotic scaling property λ^{-b_α} (7.7), (7.8) and we get an extra $\lambda^{1/2}$ from the scaling law (7.2). In the weighted estimate, e.g. $\psi_\lambda \nabla \mathbf{u}$, we will obtain an additional power λ^{b_α} in the upper bound since the weight ψ_λ has asymptotics λ^{-b_α} . In the estimate of \mathbf{u} , we gain the factor λ in $\lambda^{-a_\alpha+1}$ since \mathbf{u} is 1 order more regular than $\nabla \mathbf{u}$.

7.2. **Hölder estimates of $\nabla \mathbf{u}, \mathbf{u}$.** Denote

$$(7.17) \quad \begin{aligned} x_{c2} &= 2x_c, \quad \mu = 1/8, \quad \Omega_1 \triangleq [x_{c2}, (1+\mu)x_{c2}] \times [0, x_{c2}] \cup [0, (1+\mu)x_{c2}] \times \{x_{c2}\} \\ \Omega_2 &\triangleq \{x_{c2}\} \times [0, (1+\mu)x_{c2}] \cup [0, x_{c2}] \times [x_{c2}, (1+\mu)x_{c2}]. \end{aligned}$$

Firstly, we choose λ and rescale $x = \lambda \hat{x}$ with $|\hat{x}|_\infty = x_{c2}$. In the $C_x^{1/2}$ estimate, for $z = \lambda \hat{z}$ with $x_1 \leq z_1, z_2 = x_2$, if $|\hat{x} - \hat{z}| \leq \mu x_{c2}$, we have $\hat{z} \in \Omega_1$. In the $C_y^{1/2}$ estimate, for $z = \lambda \hat{z}$ with $x_2 \leq z_2, x_1 = z_1$, we have $z \in \Omega_2$. For $|\hat{x} - \hat{z}| > \mu x_{c2}$, we apply the triangle inequality to estimate $\nabla \mathbf{u}(x) - \nabla \mathbf{u}(z)$ directly. In Section 4.5 in Part II [2], we discuss the estimate of the derivatives of the regular part of the integrand in (7.1)

$$\partial_{\hat{x}_i} J(\hat{x}, \hat{y}), \quad J = K(\hat{x}, \hat{y})p_\lambda(\hat{x}), \quad \text{or } J = K^C(\hat{x}, \hat{y})(p_\lambda(\hat{x}) - p_\lambda(\hat{y})) + K^{NC}(\hat{x}, \hat{y})p_\lambda(\hat{x}),$$

with weight $p(x) = \psi(x)$, where K^C, K^{NC} are parts of the symmetrized kernel K^{sym} (7.13) determined by the distance between the y and the singularity x . See Section 4.1.5 in Part II [2]. Using such estimates for $\partial_{\hat{x}_i} J$, we can estimate the integral of $\partial_{\hat{x}_i} J$ in terms of $\|\omega\varphi\|_\infty$ using the method in Section 7.1.1 and the method in Section 4.3 in Part II [2] with weight $\varphi_{\infty,\alpha}$ (7.8). In particular, these parts are bounded by

$$C(\hat{x})\lambda^{b_\alpha-a_\alpha}\|\omega\varphi\|_\infty.$$

We have the power λ^{b_α} from the weight $p_\lambda = \psi_\lambda$ (7.7). Here, $\alpha = 1$ means that we estimate $x = \lambda \hat{x}$ for very small x and λ , and $\alpha = n$ means that we estimate $x = \lambda \hat{x}$ for very large x and λ . We use the same notations below.

Remark 7.2. We estimate $\partial_{\hat{x}_i} J$ rather than $\partial_{x_i} J$. We have the scaling relation (7.2) between these two quantities.

The remaining part is the integral in the singular region (see I_5 in Section 4.3.4 in Part II [2])

$$I_5(\hat{x}, k_2) = \int_{R(k_2)} K(\hat{x} - \hat{y})(\psi_\lambda(\hat{x}) - \psi_\lambda(\hat{y}))W_\lambda(\hat{y})d\hat{y}.$$

In Part II [2], we discuss the case of finite λ and choose $\lambda = 1$ for simplicity. The case of general λ is given above using the rescaling relation (7.1). For the Hölder estimate, in Sections 4.3

in Part II [2], we decompose I_5 into several parts and estimate the piecewise L^∞ norm or the piecewise derivatives bounds for each part. For each part, we can bound it using the weights

$$\psi_\lambda(\hat{x})\partial_{\hat{x}_1}^i\partial_{\hat{x}_2}^j\psi_\lambda^{-1}(\hat{x}), \quad \psi_\lambda(\hat{x})/\varphi_\lambda(\hat{x}),$$

the norm $\|W_\lambda\varphi_\lambda\|_{L^\infty}$ and the semi-norm $[\omega_\lambda\psi_\lambda]_{C_{x_i}^{1/2}}$. The above weights are the same as those in (7.15), and we establish the uniform estimates in Appendix A.2, A.3 in Part II [2]. Using the scaling relation (7.2), we can bound each part by

$$C_1(\hat{x})\lambda^{b_\alpha-a_\alpha}\|\omega\varphi\|_{L^\infty} + C_2(\hat{x})\lambda^{1/2}[\omega\psi]_{C_{x_i}^{1/2}},$$

for $l = 1$ or 2 . In the $C_{x_i}^{1/2}$ estimate, we only need the semi-norm $[\omega\psi]_{C_{x_i}^{1/2}}$. The above estimates allow us to estimate different parts in the Hölder estimate in Section 4.3 in Part II [2]. Similar to the L^∞ estimate in Section 7.1, we can apply essentially the same estimate in [3] for the case of finite λ to the case of $\lambda \rightarrow 0$ or $\lambda \rightarrow \infty$ except that we need to use the asymptotic properties of the weights (7.7), (7.8), (7.15) instead of using $\varphi_\lambda^{-1}, \partial_x^i\partial_y^j\psi_\lambda^{-1}$ etc for a fixed λ , and track the power.

Similarly, in the Hölder estimate of $\psi_{u,\lambda}(x)\mathbf{u}(\lambda x)$ (see Section 4.3.8 in Part II [2]), we can bound the piecewise L^∞ norm or derivatives of each part by

$$(7.18) \quad C(\hat{x})\lambda^{1+c_\alpha-a_\alpha}\|\omega\varphi\|_\infty,$$

where c_α is the leading order power of $\psi_{u,\lambda}$ (7.7). We refer to Remark 7.1 for tracking the power in the upper bounds.

Once we obtain the above bounds for each parts, we assemble the Hölder estimate following Section 4.6 in Part II [2] and the scaling relation (7.2). See Section 7.4.

7.3. Assemble L^∞ estimate of $\mathbf{u}, \nabla\mathbf{u}$ in \mathbb{R}_2^{++} . We use the method in Section 4.2 in Part II [2] to estimate the piecewise bounds of $\mathbf{u}(x), \nabla\mathbf{u}(x)$ for \hat{x} in each $h_x \times h_x$ grid $B_{i_1j_1}(h_x) \subset [0, 2x_c]^2 \setminus [0, x_c]^2$

$$x = \lambda\hat{x}, \quad \hat{x} \in [0, 2x_c]^2 \setminus [0, x_c]^2 = \cup_{i_1, j_1} B_{i_1j_1}(h_x),$$

with finitely many $\lambda = \lambda_i, i = 1, 2, \dots, n_1$. This allows us to bound $\mathbf{u}, \nabla\mathbf{u}$ for $x \in \lambda_{n_1}[0, 2x_c]^2 \setminus \lambda_1[0, x_c]^2$. To assemble the estimate in \mathbb{R}_2^{++} , we first pass the estimates from the dyadic mesh to the mesh (7.3) for computing the approximate steady state. For each mesh Q_{ij} (7.3) with $\lambda_1x_c \leq \max(y_i, y_j) \leq \max(y_{i+1}, y_{j+1}) \leq 2\lambda_{n_1}x_c$, we can cover Q_{ij} by finitely many dyadic mesh $M_{i_1j_1}$ (7.6). Then we use

$$\|f\|_{L^\infty(Q_{ij})} \leq \max_{Q_{ij} \cap M_{i_1j_1} \neq \emptyset} \|f\|_{L^\infty(M_{i_1j_1})}$$

to obtain the piecewise bound on Q_{ij} . In our implementation, we loop over l, i_1, j_1 and update the bound in Q_{ij} if $Q_{ij} \cap M_{i_1j_1} \neq \emptyset$. The same method applies to piecewise bounds of weighted velocity $\rho_{10}\mathbf{u}, \psi_1\nabla\mathbf{u}$ for some weights since we have piecewise bounds for the weights. See Appendix A.2, A.3 in Part II [2].

7.3.1. Near-field and far-field. We need to further bound $\mathbf{u}, \nabla\mathbf{u}$ for $x \in \lambda_1[0, x_c]^2$ and $|x|_\infty \geq 2\lambda_nx_c$. We focus on the weighted estimate $(p\nabla\mathbf{u})(x)$. The estimate for \mathbf{u} is similar. To estimate $\mathbf{u}, \nabla\mathbf{u}$ in the far field, we use the estimate in Section 7.1 with $\lambda \geq \lambda_*$, $\lambda_*x_c \leq 2\lambda_nx_c$.

We focus on the estimate of the singular part (7.16) in the far-field. The estimates of other parts are easier. In the first estimate, we assume that x is in the computational domain $x = \lambda\hat{x} \in [y_i, y_{i+1}] \times [y_j, y_{j+1}] = Q_{ij}$ with $\lambda = \frac{|x|_\infty}{x_c}$ (7.4) and $|\hat{x}|_\infty = x_c$. We can cover the range of \hat{x} by finitely many intervals $\Gamma_{1,i_1}, \Gamma_{2,j_1}$ in (7.6). For \hat{x} in each interval, from (7.16), we have

$$(7.19) \quad |(\nabla\mathbf{u})_S(x)| \leq E_1 \left(\lambda^{-a_n} (C_0 + C_1 \log \frac{b}{a} + C_2|a|) + C_3|a|^{1/2}\lambda^{-b_n+1/2} \right) \triangleq g(\lambda, a) \cdot E_1,$$

for some constants C_i depending on the interval, where a_n, b_n are the slowest decay power in φ, ψ (7.7). Suppose that

$$(7.20) \quad p(x) \leq c|x|^\alpha.$$

We can obtain the upper and lower bound of $\lambda/|x| = O(1)$, $\lambda, |x|$ for $x \in Q_{ij}$. We first bound λ^α by $\max(\lambda_l^\alpha, \lambda_u^\alpha)$ and then optimize the estimate (7.19) over finite many $a = a_i$, established in (7.16). In the second estimate, we minimize the upper bound in (7.19) by choosing

$$(7.21) \quad a = \min(h, \min(f(\lambda_l), f(\lambda_u))), \quad f(\lambda) = \left(\frac{2C_1}{C_3}\right)^2 \lambda^{2(b_n - a_n) - 1}$$

With the above choice of a , the upper bound in (7.19) is of order $\lambda^{-a_n} \log \lambda$. We further use the piecewise bounds for $\lambda/|x|$, $|x|$, and (7.20) to obtain piecewise bounds of $p \nabla \mathbf{u}(x)$ for $x \in Q_{ij}$ and \hat{x} in one of the intervals $\Gamma_{1,i_1}, \Gamma_{2,j_1}$. Taking the maximum of different cases yields the piecewise estimates in Q_{ij} .

For x sufficiently large $\max x_i \geq R$, x is outside the domain spanned by Q_{ij} . We take $\lambda_* = \frac{R}{x_c}$. In this case, we estimate $p(x) \nabla \mathbf{u}(x)$ uniformly for x with $|x|_\infty \geq R$ (7.4). We assume (7.20) for all x with $\max x_i \geq R$ and require $\alpha < \min(a_n, 0)$. We optimize (7.19) by choosing $a = f(\lambda)$ (7.21).

Recall the power (a_α, b_α) of the weights (7.7). Since $2(b_n - a_n) - 1 \leq -\frac{1}{2}$ and λ is sufficiently large, we have $a = f(\lambda) \leq h$ in our application. With the above choice of a , we get

$$C_3 a^{1/2} \lambda^{-b_n + 1/2} = C_3 \frac{2C_1}{C_3} \lambda^{-b_n + 1/2 + b_n - a_n - 1/2} = 2C_1 \lambda^{-a_n},$$

and $g(\lambda, a)$ (7.19) reduces to

$$g(\lambda, a) = \lambda^{-a_n} (C_0 + C_1 \log \frac{b}{a} + C_2 \left(\frac{2C_1}{C_3}\right)^2 \lambda^{2(b_n - a_n) - 1} + 2C_1).$$

Estimate of the logarithm term. We need to further control the term $\lambda^{-a_n} \log \frac{b}{a}$. Denote

$$F(\lambda, b, C, d, \beta) = \log(b/A(\lambda)) \lambda^C, \quad A(\lambda) = d \lambda^{-\beta},$$

for some $\beta > 0 > C$. We maximize it over $\lambda \geq \lambda_*$. Clearly, we have

$$\frac{d}{d\lambda} F(\lambda) = \lambda^{C-1} C \log \frac{b}{A(\lambda)} - \lambda^C \frac{A'}{A} = \lambda^{C-1} C \log \frac{b}{A(\lambda)} + \beta \lambda^{C-1} = \lambda^{C-1} (C \log \frac{b}{A(\lambda)} + \beta).$$

The critical point λ_c is given by

$$A(\lambda_c) = b e^{\beta/C}, \quad \lambda_c = \left(\frac{d}{A(\lambda_c)}\right)^{1/\beta} = \left(\frac{d}{b}\right)^{1/\beta} e^{-1/C}.$$

For $\lambda \leq \lambda_c$, $F(\lambda)$ is increasing. For $\lambda \geq \lambda_c$, $F(\lambda)$ is decreasing. Thus for $\lambda \geq \lambda_*$, we get

$$F(\lambda, b, C, d, e) \leq F(\max(\lambda_c, \lambda_*), b, C, d, e).$$

Finally, using $|x| \geq \max_i(x_i) = \lambda x_c$, $\alpha \leq \min(a_n, 0)$, $\lambda \geq \lambda_*$, and $p \leq c|x|^\alpha$ (7.20), we obtain

$$\begin{aligned} |p(\nabla \mathbf{u})| &\leq c x_c^\alpha g(\lambda, b) \lambda^\alpha E_1 \leq c x_c^\alpha E_1 \left((C_0 + 2C_1) \lambda_*^{\alpha - a_n} \right. \\ &\quad \left. + C_2 \left(\frac{2C_1}{C_3}\right)^2 \lambda_*^{2(b_n - a_n) - 1 + \alpha - a_n} + C_1 F(\lambda, b, \alpha - a_n, \left(\frac{2C_1}{C_3}\right)^2, 2(-b_n + a_n) + 1) \right), \end{aligned}$$

where $F(\lambda)$ is further bounded using the above estimate.

Estimate of the regular part. From Section 7.1.1, the regular part of $\nabla \mathbf{u}$ with approximation terms can be bounded by

$$|(\nabla \mathbf{u})_R| \leq C_1(\hat{x}) \lambda^{-a_n} + C_2(\hat{x}),$$

where $C_i(\hat{x})$ are some piecewise constants and $C_2(\hat{x})$ can be negative. For a weight satisfying (7.20) and $x \in Q_{ij} = [y_i, y_{i+1}] \times [y_j, y_{j+1}]$, it is easy to obtain the upper and lower bounds for $r = |x|$ and $\lambda = \frac{\max x_i}{x_c}$. Then using

$$(7.22) \quad \min(C t_l^\alpha, C t_u^\alpha) \leq C t^\alpha \leq \max(C t_l^\alpha, C t_u^\alpha), \quad (fg)_u = \max(f_l g_l, f_l g_u, f_u g_l, f_u g_u),$$

for any C, α , we obtain the upper bound in Q_{ij} . For x sufficiently large $\max x_i \geq R$, we assume that the weight p satisfies (7.20) with $\alpha < a_n$ so that $|p(\nabla \mathbf{u})_R|$ decays for large x . In this case, we have $r_l = R, r_u = \infty$, $\lambda_l = \frac{R}{x_c}, \lambda_u = \infty$, and the estimates follow from (7.22).

The weighted estimate of $\nabla \mathbf{u}$ in the near field and the estimate for \mathbf{u} are similar.

Estimate of $u_A/|x|^{1/2}$. In the energy estimate, we need to control $u_A/|x|^{1/2}$ in the far-field, where $u_A = -\partial_y(\phi + L_{12}(0)xy)$ (7.11). We want to control $\frac{u_A}{x^{1/2}}r^\gamma$ for some $\gamma < -1$ and $|x|_\infty \geq R$ (7.4). Firstly, using the previous methods and $u_{x,A} = u_{A,x}$ (7.12), we obtain

$$|u_{A,x}(x)|r^{\gamma+1} \leq C_2, \quad |u_A(x)|r^\gamma \leq C_1, \quad |x|_\infty \geq R.$$

Denote $r = |x|, \beta = \arctan \frac{x_2}{x_1}$. For $x_1 \geq x_2$, we get $\beta \leq \pi/4, x_1 = |x|_\infty \geq R, x_1 \geq r/\sqrt{2}$. Thus, we get

$$(7.23) \quad \frac{|u_A|}{|x_1|^{1/2}}r^\gamma \leq \frac{C_1}{(r \cos \beta)^{1/2}}, \quad \beta \in [0, \frac{\pi}{2}], \quad \frac{|u_A|}{|x_1|^{1/2}}r^\gamma \leq \min(\frac{C_1}{R^{1/2}}, \frac{C_1 2^{1/4}}{r^{1/2}}), \quad \beta \leq \frac{\pi}{4}.$$

Using the estimate for $u_{A,x}$ and $\gamma + 1 < 0$, we yield

$$|u_A| = \int_0^{x_1} |u_{A,x}(z, x_2)| dx_1 \leq \int_0^{x_1} C_2 |z, x_2|^{-\gamma-1} dx_1 \leq C_2 x_1 r^{-\gamma-1}, \quad \frac{|u_A| r^\gamma}{|x_1|^{1/2}} \leq C_2 \frac{|x_1|^{1/2}}{r} \leq \frac{C_2 (\cos \beta)^{1/2}}{r^{1/2}}.$$

Combining the above two estimates, for $x_2 > x_1$, we establish

$$\frac{|u_A| r^\gamma}{|x_1|^{1/2}} \leq r^{-1/2} \min(\frac{C_1}{(\cos \beta)^{1/2}}, C_2 (\cos \beta)^{1/2}) \leq R^{-1/2} \min(\sqrt{C_1 C_2}, \frac{C_1}{(\cos \beta^u)^{1/2}}, C_2 (\cos \beta^l)^{1/2}).$$

Combining (7.23) and the above estimate, we establish the estimate for $u_A/|x_1|^{1/2}$.

7.4. Assemble the Hölder estimate in \mathbb{R}_2^+ . In the remaining part of this section, $\mathbf{u}, \nabla \mathbf{u}$ denote the velocity with approximations, i.e. $\mathbf{u}_A, (\nabla \mathbf{u})_A$. We do not write down the approximation terms to simplify the notations.

In Section 4.6 in Part II [2], we discuss how to assemble the estimates of different parts of (7.1) to obtain the $C_{x_i}^{1/2}$ estimate $\delta_i(f, x, z)$ (7.24)

$$(7.24) \quad \delta_i(f, x, z) \triangleq \frac{|f(x) - f(z)|}{|x - z|^{1/2}}, \quad z_i > x_i, \quad z_{3-i} = x_{3-i},$$

for f being $\mathbf{u}, \nabla \mathbf{u}$ with the approximation terms, $x = \lambda \hat{x}, z = \lambda \hat{z}$ with $\hat{x} \in [0, 2x_c]^2 \setminus [0, x_c]^2, |\hat{z} - \hat{x}| \leq 2\nu x_c$, and $\lambda = \lambda_1, \lambda_2, \dots, \lambda_{n_1}$. For x, z with $2\nu x_c \leq |\hat{x} - \hat{z}| \leq 2\nu_2 x_c$, we get $|x - z| \geq 2\nu \lambda x_c \geq \nu |x|_\infty$ and apply triangle inequality to estimate $\delta_i(f, x, z)$ for \hat{x}, \hat{z} in a larger domain. Then, in the $C_x^{1/2}$ estimate, we obtain the piecewise estimate of $\delta_i(f, x, z)$ on the dyadic mesh (7.6)

$$(7.25) \quad x \in M_{lpq}, \quad z \in M_{lrq}, \quad p \leq r \leq p + m_1, \quad \hat{x} \in B_{pq}(h_x) \subset [0, 2x_c]^2 \setminus [0, x_c]^2.$$

In the above meshes, l denotes the scaling factor λ_l . We have the same subindex q for x, z since $x_2 = z_2$. Next, we pass the estimate from the dyadic mesh to the mesh Q_{ij} (7.3).

7.4.1. The Hölder estimate in the bulk. Recall M_{l, i_1, j_1} from (7.6). We focus on the $C_x^{1/2}$ estimate. For $x \in Q_{i_1 j_1} \subset \lambda_{n_2} [0, 2x_c]^2 \setminus \lambda_1 [0, x_c]^2$, and $z \in Q_{i_2 j_1}$ close to x $i_1 \leq i_2 \leq i_1 + m$, by covering $Q_{i_1 j_1}, Q_{i_2 j_1}$, we obtain

$$(7.26) \quad \max_{x \in Q_{i_1 j_1}, z \in Q_{i_2 j_1}} |\delta_1(f, x, z)| \leq \max_l \left(\max_{x \in M_{lpq} \cap Q_{i_1 j_1} \neq \emptyset, z \in M_{lsp} \cap Q_{i_2 j_1} \neq \emptyset} \delta_1(f, x, z) \right).$$

Suppose that $Q_{i_1 j_1} \subset \cup M_{lpq}$. For each $x \in M_{lpq}$, we have $\hat{x} \in B_{pq}, \hat{z}_2 = \hat{x}_2 \in I_q$. Denote

$$I_q = [qh_x, (q+1)h_x], \quad S_z \triangleq \{a \in \mathbb{R}^2 : a_2 \in \lambda_l I_q\} \cap Q_{i_2 j_1}, \quad C_z \triangleq \cup_{p \leq p_2 \leq p+m_1} M_{lp_2 q}.$$

The set S_z is the location of $z \in Q_{i_2 j_1}$ since $z_2 = x_2$, and C_z is the location of z we have computed $\delta_1(f, x, z)$ in the dyadic mesh (7.25). For each $Q_{i_2 j_1}$ $i_2 \leq i_1 + m$, we have two cases

$$(a) \quad y_{i_2+1} \leq \lambda_l(p+m_1+1)h_x, \quad (b) \quad y_{i_2+1} > \lambda_l(p+m_1+1)h_x,$$

where y is the mesh (7.3). In case (a), z is close to x or i_2 close to i_1 , e.g. $i_2 = i_1$. We estimate $\delta_1(f, x, z)$ using the estimate of $\delta_1(f, x, z)$ on the dyadic mesh (7.25) C_z we have computed.

In case (b), the mesh C_z does not cover S_z , the range of z with $z_1 \in [y_{i_2}, y_{i_2+1}], z_2 \in I_q$. For $z_1 \in [y_{i_2}, \lambda_l(p+m_1+1)h_x]$, C_z covers parts of S_z (or none), and we use the estimate on the

dyadic mesh (7.25) to estimate $\delta_1(f, x, z)$. For $z_1 \in [\max(y_{i_2}, \lambda_l(p + m_1 + 1)h_x), y_{i_2+1}]$, since $x \in Q_{i_1 j_1} \cap M_{lpq}$ and we choose $\lambda_l = 2^{k_l}$ for $k_l \in Z$, we have

$$(7.27) \quad x_1 \leq \lambda_l(p + 1)h_x, \quad \max(y_{i_1}, y_{j_1}) \leq |x|_\infty = \lambda_l |\hat{x}|_\infty \leq \lambda_l 2x_c, \quad \lambda_l \geq 2^{\lceil \log_2(\frac{\max(y_{i_1}, y_{j_1})}{2x_c}) \rceil} \triangleq \lambda_{i_1, j_1},$$

where $\lceil a \rceil$ is the smallest integer no less than a . It follows

$$z_1 - x_1 \geq \lambda_l m_1 h_x \geq \frac{m_1 h_x}{2x_c} |x|_\infty \geq \frac{m_1 h_x}{2x_c} \max(y_{i_1}, y_{j_1}), \quad z_1 - x_1 \geq \max(y_{i_2} - y_{i_1+1}, 0).$$

We apply the triangle inequality, the piecewise bound of $f = \nabla \mathbf{u}$ in $Q_{i_2 j_1}, Q_{i_1 j_1}$ established in Section 7.3, and the above lower bound on $|x - z|$ to estimate $\delta_1(f, x, z)$. Since $x_1 \in \lambda_l h_x [p, p + 1]$ ($x \in M_{lpq}$), conditions $y_{i_2+1} > \lambda_l(p + m_1 + 1)h_x, x \in Q_{i_1 j_1}$, imply

$$y_{i_2+1} - y_{i_1} \geq y_{i_2+1} - x_1 \geq y_{i_2+1} - \lambda_l(p + 1)h_x \geq \lambda_l m_1 h_x \geq m_1 h_x 2^{\lceil \log_2(\frac{\max(y_{i_1}, y_{j_1})}{2x_c}) \rceil}.$$

Recall λ_{i_1, j_1} from (7.27). For $i_1 \leq i_2$, we introduce

$$D(i_1, i_2, j_1) = \max(\lambda_{i_1, j_1} h_x, \max(y_{i_2} - y_{i_1+1}, 0)), \quad S \triangleq \{(i_1, i_2, j_1) : y_{i_2+1} - y_{i_1} \geq \lambda_{i_1, j_1} h_x m_1\}.$$

From the above discussion, we can modify the estimate (7.26) as follows

$$\max_{x \in Q_{i_1 j_1}, z \in Q_{i_1, j_2}} |\delta_1(f, x, z)| \leq \max \left\{ \max_i \left(\max_{x \in M_{lpq} \cap Q_{i_1 j_1} \neq \emptyset, z \in M_{lp_2q} \cap Q_{i_2 j_1} \neq \emptyset, p \leq p_2 \leq p + m_1} \delta_1(f, x, z) \right), \right. \\ \left. \mathbf{1}_S(i_1, i_2, j_1) \frac{\|f\|_{\infty(Q_{i_1 j_1})} + \|f\|_{\infty(Q_{i_2 j_1})}}{D_{i_1, i_2, j_1}^{1/2}} \right\}.$$

Note that $D(i_1, i_2, j_1), \lambda_{i_1, j_1}, S$, and the second bound in the maximum are independent of the dyadic estimates, and only depend on the piecewise estimates on mesh Q_{ij} . The above estimate allows us to obtain a piecewise $C_x^{1/2}$ estimates for $x \in Q_{i_1, j_1}, z \in Q_{i_2, j_1}$ with $|i_1 - i_2| \leq m$ and x in the bulk of the domain. Similarly, we obtain the piecewise bounds for the $C_y^{1/2}$ estimate.

7.4.2. The Hölder estimate in the far-field and near-field for small distance. We focus on the estimate for $\nabla \mathbf{u}$. The estimate of \mathbf{u} is similar and easier. Firstly, in the $C_{x_i}^{1/2}$ estimate using (6.6) and the estimates in Section 7.2, we can decompose (7.1) into several parts and bound the piecewise L^∞ norm or derivatives in \hat{x} for each part in a small domain $B_{i_1 j_1}(h_x) \subset \Omega_i$ (7.4), (7.17) by

$$C_1(\hat{x}) \lambda^{b_\alpha - a_\alpha} \|\omega \varphi\|_{L^\infty} + C_2(\hat{x}) \lambda^{1/2} [\omega \psi]_{C_{x_i}^{1/2}} \leq \lambda^{1/2} (C_1(\hat{x}) \lambda^{b_\alpha - a_\alpha - 1/2} + \gamma_i C_2(\hat{x})) E_1.$$

Recall the power a_α, b_α of the weights (7.7). We have

$$b_1 - a_1 - \frac{1}{2} \geq -2 + \frac{5}{2} - \frac{1}{2} \geq 0, \quad b_n - a_n - \frac{1}{2} \leq -\frac{1}{6} + \frac{1}{6} - \frac{1}{2} \leq -\frac{1}{2} \leq 0.$$

In the near field $\lambda \leq \lambda_* \ll 1$ or the far field $\lambda \geq \lambda_*$, using $\lambda^{b_\alpha - a_\alpha - 1/2} \leq \lambda_*^{b_\alpha - a_\alpha - 1/2}$, we can bound the estimate of each part uniformly in λ by

$$\lambda^{1/2} (C_1(\hat{x}) \lambda_*^{b_\alpha - a_\alpha - 1/2} + \gamma_i C_2(\hat{x})) E_1.$$

The power $\lambda^{1/2}$ is factorized out in the estimates. Thus we can treat the Hölder estimate $\delta_i(f, \lambda \hat{x}, \lambda \hat{z})$ (7.24) for $\lambda \geq \lambda_*$ or $\lambda \leq \lambda_*$ in the same way as that for finite λ case using the λ -independent bound $(C_1(\hat{x}) \lambda_*^{b_\alpha - a_\alpha - 1/2} + \gamma_i C_2(\hat{x})) E_1$ and tracking the power $\lambda^{1/2}$. Using the scaling relation (7.2) and the estimates in Section 4.6 in Part II [2], we obtain the weighted Hölder estimate $\delta_i(f, \lambda \hat{x}, \lambda \hat{z})$ (7.24) for $f = \psi \nabla \mathbf{u}$ and $|\hat{x}|_\infty = x_{c2}, |\hat{z} - \hat{x}| \leq \mu x_{c2}$ uniformly for $\lambda \leq \lambda_*$ or $\lambda \geq \lambda_*$. Note that the power $\lambda^{1/2}$ in the above estimate and that in (7.2) are exactly canceled.

Large distance. The above argument applies to obtain Hölder estimate for x, z close relative to $|x|$. To estimate $\frac{|f(\lambda \hat{x}) - f(\lambda \hat{z})|}{|\lambda(\hat{x} - \hat{z})|^{1/2}}$ with large $|\hat{x} - \hat{z}|, |\hat{x}| = x_{c2}$, we need to bound $f(\lambda \hat{z})$ for large \hat{z} . We focus on $C_x^{1/2}$ estimate of $\nabla \mathbf{u}$. Other estimates are similar.

L^∞ **estimate.** From the above discussions, we seek a bound

$$(7.28) \quad |f(\lambda\hat{z})| \leq \lambda^\beta C(\hat{z}),$$

for $f = \psi \nabla \mathbf{u}$ with approximation terms uniformly in $\lambda \geq \lambda_*$, $\beta \leq \frac{1}{2}$ or $\lambda \leq \lambda_*$, $\beta \geq \frac{1}{2}$, and \hat{z} in a large domain, e.g. $[0, x_{c3}]^2 \setminus [0, x_{c2}]^2$ (7.6).

Recall the estimate of the regular parts, e.g. (7.14), and singular parts (7.16) in Section 7.1. We first optimize the bound in (7.16) with a fixed $\lambda = \lambda_*$ over $a = a_1, a_2, \dots, a_N$. Denote by a_* be the minimizer among a_i . Choosing $a = a_*$ in (7.16), we can bound $(\nabla \mathbf{u})_S$ by

$$E_1(\lambda^{-a_\alpha} C_1(\hat{x}) + \lambda^{-b_\alpha + 1/2} C_2(\hat{x})),$$

for another piecewise constants $C_i(\hat{x})$. For the nonsingular parts in Section 7.1.1, for $\lambda \in [\lambda_l, \lambda_u]$, we can bound it by

$$E_1(C_3(\hat{x})\lambda^{-a_\alpha} - C_4(\hat{x})) \leq E_1\lambda^{-a_\alpha}(C_3(\hat{x}) - C_4(\hat{x}) \min(\lambda_l^{a_\alpha}, \lambda_u^{a_\alpha})) = C_5(\hat{x})\lambda_l^{a_\alpha} E_1.$$

If $\lambda_u = \infty$, we simply drop the term $-C_4(\hat{x})$. Using this method, we can bound $\nabla \mathbf{u}$ by

$$(7.29) \quad E_1 \cdot \sum_{i \leq N} C_i(\hat{x})\lambda^{\beta_i}, \quad N = 2, \quad \beta = (-a_\alpha, -b_\alpha + 1/2), \quad |\hat{x}| = x_c, \quad x = \lambda\hat{x},$$

for some piecewise constant $C_1(\hat{x}), C_2(\hat{x})$. We do not further optimize the estimate (7.16) over all small a since the above estimate is good enough for our purpose and is simpler. For \mathbf{u} , we can obtain similar estimates and we only have the term $C_1(\hat{x})\lambda^{-a_\alpha + 1}$. Next, for the weight

$$(7.30) \quad \psi_\lambda(x) \leq \psi_{\infty, u}(x)\lambda^{b_\alpha},$$

we estimate the piecewise bound

$$(\psi \nabla \mathbf{u})_\tau(\hat{z}) \leq C_{1,ij}\tau^{b_\alpha - a_\alpha} + C_{2,ij}\tau^{1/2}$$

for \hat{z} in each grid $B_{ij}(h_x) \subset [0, x_{c3}]^2 \setminus [0, x_{c2}]^2$, $\tau \leq \lambda_*$, $\alpha = 1$ or $\tau \geq \lambda_*$, $\alpha = n$, which gives the desired estimate (7.28). To apply (7.29), we need to rescale $\tau\hat{z} = \lambda\tilde{z}$ with $|\tilde{z}|_\infty = x_c$. Without loss of generality, we assume $\hat{z}_1 \geq \hat{z}_2$. For $\hat{z} \in B_{ij}(h_x)$, we have

$$\lambda = \frac{|\tau\hat{z}|_\infty}{x_c} = \frac{\tau\hat{z}_1}{x_c} \in \tau[ih_x/x_c, (i+1)h_x/x_c].$$

Applying (7.29) and $\psi_\tau(\hat{z}) \leq \tau^{b_\alpha} \psi_{\infty, u}(\hat{z})$ (7.30), we yield

$$(7.31) \quad |(\psi \nabla \mathbf{u})(\tau\hat{z})| \leq E_1 \sum_{l \leq N} \tau^{\beta_l + b_\alpha} C_l(\tilde{z}) \max((ih_x/x_c)^{\beta_l}, ((i+1)h_x/x_c)^{\beta_l}) \psi_{\infty, u}(\hat{z}), \quad \tilde{z} = \frac{\hat{z} \cdot x_c}{|\hat{z}|_\infty}.$$

We cover the range of \tilde{z} by the intervals $\Gamma_{p,q}(x_c)$ (7.6) and apply the piecewise bound of $C_i(\hat{x})$ in $\Gamma_{p,q}(x_c)$ to bound $C_i(\tilde{z})$. Similarly, we derive the piecewise bound of $(\psi \nabla \mathbf{u})_\tau(\hat{z})$ for $\hat{z} \in \Gamma(x_{c2})$ (7.6). Since $\hat{z}/2 \in \Gamma(x_{c2}/2) = \Gamma(x_c)$ and $\tau\hat{z} = (2\tau) \cdot \hat{z}/2$, applying the above estimates, we yield

$$|(\psi \nabla \mathbf{u})(\tau\hat{z})| \leq E_1 \sum_{l \leq N} \tau^{\beta_l + b_\alpha} C_l(\tilde{z}) 2^{\beta_l} \psi_{\infty, u}(\hat{z}).$$

Using the above estimate, we obtain the uniform bound for $z = \lambda\hat{z}$ with $|\hat{z}| = x_{c2}$ and $\lambda \geq \lambda_*$

$$(7.32) \quad \max_{\hat{z} \in \Gamma_{k,i}(x_{c2}), i} \frac{|(\psi \nabla \mathbf{u})(\lambda\hat{z})|}{|\lambda\hat{z}|_\infty^{1/2}} \leq E_1 \max_{\hat{z} \in \Gamma_{k,i}(x_{c2}), i} \sum_{l \leq N} \lambda^{\beta_l + b_\alpha - 1/2} \frac{C_l(\hat{z})}{|x_{c2}|^{1/2}} \psi_{\infty, u}(\hat{z}) 2^{\beta_l} \triangleq E_1 \cdot F_{ha,k}(\lambda).$$

From (7.29), it is easy to check that the power $\lambda^{\beta_l + b_\alpha - 1/2}$ and $F_i(\lambda)$ are increasing in λ for $\lambda \leq \lambda_*$, $\alpha = 1$ and decreasing in λ for $\lambda \geq \lambda_*$, $\alpha = n$. The functions $F_{ha,k}(\lambda)$ allow us to control $\psi \nabla \mathbf{u}(z)$ for $z_2 \geq z_1$ and $z_1 \leq z_2$ uniformly in λ .

7.4.3. *The Hölder estimate with large distance.* Now, we are in a position to perform $C_x^{1/2}$ estimate $\delta_1(f, x, z)$ with large $|x - z|$ relative to $|x|$. Denote $f = \psi \nabla \mathbf{u}$. Firstly, using (7.31), we can obtain piecewise bound of $f(\lambda \hat{z})$ for $\hat{z} \in [0, x_{c3}]^2 \setminus [0, x_{c2}]^2$. Using the triangle inequality, we can bound

$$\delta_1(f, \lambda \hat{x}, \lambda \hat{z}), \quad |\hat{x}|_\infty = x_{c2}, \quad \hat{z}_2 = \hat{x}_2, \quad \hat{z}_1 - \hat{x}_1 \leq m_3 h_x,$$

uniformly in λ , where m_3 a parameter given in (7.6). Note that we have estimated $\delta_1(f, \lambda \hat{x}, \lambda \hat{z})$ based on Hölder regularity of ω for $|\hat{z} - \hat{x}| \leq \mu |x_{c2}|$ at the beginning of Section 7.4.2. In such a case, we have two estimates and we will optimize them. These allow us to estimate $\delta_1(f, \hat{x}, \hat{z})$ with $|\hat{x} - \hat{z}| \leq m_3 h_x = x_{c3} - x_{c2}$.

Finally, we consider the case

$$(7.33) \quad |\hat{z} - \hat{x}| \geq m_3 h_x \geq x_{c2} = |\hat{x}|_\infty.$$

We consider the far-field estimate $\lambda \geq \lambda_*$. The estimate in the near-field is similar. We have two estimates. In the first estimate, (1) if $\hat{x}_1 \geq \hat{x}_2$, we get $\hat{z}_1 \geq \hat{x}_1 + m_3 h_x > \hat{x}_2 = \hat{z}_2$; (2) if $\hat{x}_1 \leq \hat{x}_2$, we get $\hat{z}_1 \geq \hat{x}_1 + m_3 h_x$. In case (1), we use (7.32) with $F_{ha,l}$ to bound $f_\lambda(\hat{x})$. In both cases, we use $\max_{s=1,2}(F_{ha,s}(\lambda \hat{z}))$ to bound $f(\lambda \hat{z})$. Using the triangle inequality and $|\hat{z} - \hat{x}| \geq |m_3 h_x|^{1/2}$, we can bound $\delta_1(f, x, z)$.

We have an improved estimate for $x \in Q_{i_1 j_1}, z \in Q_{i_2 j_2}$ (7.3). Suppose that $|x|_\infty \geq |x|_{\infty,l}, |z| \geq |z|_{\infty,l}$. We consider two scenarios (1) $x_1 \geq x_2$, (2) $x_2 \leq x_1$. In case (1), we get $z_1 \geq x_1 \geq x_2 = z_2$. Suppose that

$$(7.34) \quad |\hat{z} - \hat{x}| = \hat{z}_1 - \hat{x}_1 \geq \tau \hat{x}_1.$$

From (7.33), we have $\tau \gtrsim 1$. For example, if $x_1 \in [y_i, y_{i+1}], z_1 \in [y_j, y_{j+1}], j \geq i+1$ for the mesh (7.3), we get $\frac{\hat{z}_1 - \hat{x}_1}{\hat{x}_1} \geq \frac{y_j}{y_{i+1}} - 1$. Using (7.32) and the fact that $F_{ha,\cdot}(\lambda)$ is decreasing, we yield

$$\begin{aligned} \delta_1(f, \lambda \hat{x}, \lambda \hat{z}) &\leq \frac{F_{ha,l}(\frac{|x|_{\infty,l}}{x_{c2}}) |x|_\infty^{1/2} + F_{ha,l}(\frac{|z|_{\infty,l}}{x_{c2}}) |x|_\infty^{1/2}}{|x - z|^{1/2}} \cdot E_1 \\ &\leq \left(F_{ha,1}(\frac{|x|_{\infty,l}}{x_{c2}}) (\frac{\hat{x}_1}{|\hat{z} - \hat{x}|})^{1/2} + F_{ha,1}(\frac{|z|_{\infty,l}}{x_{c2}}) (\frac{\hat{z}_1}{|\hat{z} - \hat{x}|})^{1/2} \right) E_1. \end{aligned}$$

From (7.34), we derive

$$\frac{\hat{x}_1}{\hat{z}_1 - \hat{x}_1} \leq \tau^{-1}, \quad \hat{z}_1 \geq (1 + \tau) \hat{x}_1, \quad \hat{z}_1 - \hat{x}_1 \geq (1 - \frac{1}{\tau + 1}) \hat{z}_1 = \frac{\tau}{\tau + 1} \hat{z}_1, \quad \frac{\hat{z}_1}{\hat{z}_1 - \hat{x}_1} \leq \frac{\tau + 1}{\tau},$$

and the upper bounds of $\frac{\hat{x}_1}{\hat{z}_1 - \hat{x}_1}, \frac{\hat{z}_1}{\hat{z}_1 - \hat{x}_1}$ are decreasing in τ . Combining the above two estimates, we yield the estimate of $\delta_1(f, x, z)$.

In case (2) $x_1 \leq x_2$, since $z_1 \geq x_1 + x_{c3} - x_{c2} \geq 2x_{c2} > x_2 = z_2$, we get $|z|_\infty = z_1$ and

$$(7.35) \quad \delta_1(f, \lambda \hat{x}, \lambda \hat{z}) \leq E_1 \max_{s=1,2} F_{ha,s}(\frac{|x|_{\infty,l}}{x_{c2}}) (\frac{\hat{x}_2}{|\hat{z}_1 - \hat{x}_1|})^{\frac{1}{2}} + E_1 \max_{s=1,2} F_{ha,s}(\frac{|z|_{\infty,l}}{x_{c2}}) (\frac{\hat{z}_1}{|\hat{z}_1 - \hat{x}_1|})^{\frac{1}{2}}.$$

We further bound the ratio. Suppose that $\hat{x}_2/\hat{x}_1 \in [y_l/x_u, y_u/x_l]$. Since $\hat{x}_2 = |\hat{x}| = x_{c2}$ and $m_3 h_x \geq \hat{x}_2$ (7.33), we get

$$\frac{\hat{z}_1}{\hat{x}_2} \geq \frac{\hat{x}_1 + m_3 h_x}{\hat{x}_2} \geq \frac{x_l}{y_u} + \frac{m_3 h_x}{x_{c2}}, \quad \frac{\hat{z}_1}{\hat{x}_1} \geq 1 + \frac{m_3 h_x}{\hat{x}_1} \geq 1 + \max(\frac{m_3 h_x}{x_{c2}}, \frac{\hat{x}_2}{\hat{x}_1}) \geq 1 + \max(\frac{m_3 h_x}{x_{c2}}, \frac{y_l}{x_u}).$$

Combining the above estimates and using

$$\frac{\hat{x}_2}{|\hat{z}_1 - \hat{x}_1|} = \frac{\hat{x}_2/\hat{z}_1}{1 - \hat{x}_1/\hat{z}_1}, \quad \frac{\hat{z}_1}{|\hat{z}_1 - \hat{x}_1|} = \frac{1}{1 - \hat{x}_1/\hat{z}_1},$$

we obtain the bound for $\delta_1(f, \lambda \hat{x}, \lambda \hat{z})$. From the estimates of the above two cases, if $|\hat{z}_1 - \hat{x}_1| \gg |\hat{x}|_\infty$, we have $|\delta_1(f, x, z)| \leq C \max_s F_{ha,s}(\frac{|z|_{\infty,l}}{x_{c2}})$ with $C \approx 1$.

Covering the large domain. Using the above Hölder estimate $\delta_1(f, \lambda\hat{x}, \lambda\hat{z})$ for \hat{x}, \hat{z} uniformly in λ and the covering argument in Section 7.4.1, we can obtain piecewise estimate

$$\delta_1(f, x, z), \quad x \in Q_{i_1 j_1}, \quad z \in Q_{i_2 j_1}, \quad i_1 \leq i_2 \leq i_1 + m,$$

on the mesh (7.3) for large $\max(i_1, j_1)$. For $x \in Q_{i_1 j_1}, z \in Q_{i_2 j_1}$, we can derive the upper and lower bounds for x_i, z_i and can estimate the ratio \hat{z}_1/\hat{x}_1 and \hat{x}_2/\hat{x}_1 used in the above estimates (7.34),(7.35) for large $|\hat{z}_1 - \hat{x}_1|$.

Infinite region. To estimate $\delta_1(f, x, z)$ with $|x|_\infty \geq R$ with sufficiently large R , we simply apply the estimate in Section 7.4.2 for small distance $|\hat{x} - \hat{z}|$, the estimate above (7.33) for $|\hat{x} - \hat{z}| \leq m_3 h_x$, and the first estimate below (7.33) for $|\hat{x} - \hat{z}| \geq m_3 h_x$. Since we do not have upper bound for x, z , we do not apply the improved estimate in Section 7.4.3.

The $C_y^{1/2}$ estimates for $\nabla \mathbf{u}$ are completely similar.

Hölder estimate of \mathbf{u} . The Hölder estimates of \mathbf{u} are completely similar. From (7.18),(7.2), and Section (7.2), we can obtain Hölder estimate of $\psi_{u,\lambda} \tilde{\mathbf{u}}_\lambda$ as follows

$$\frac{|\psi_{u,\lambda} \mathbf{u}_\lambda(\hat{x}) - \psi_{u,\lambda} \mathbf{u}_\lambda(\hat{z})|}{|\lambda\hat{x} - \lambda\hat{z}|^{1/2}} \leq \lambda^{c_\alpha - a_\alpha + 1/2} C(\hat{x}, \hat{z}),$$

uniformly for $\lambda \leq \lambda_*, \alpha = 1$ or $\lambda \geq \lambda_*, \alpha = n$ for $|\hat{x}|_\infty = x_{c2}, |\hat{x} - \hat{z}| \leq \mu x_{c2}$, see (7.17). We lose a power $\lambda^{1/2}$ due to (7.2). The power c_1, c_n are given in (7.7). In particular, we have

$$(7.36) \quad c_1 - a_1 + \frac{1}{2} \geq -\frac{5}{2} + \frac{5}{2} + \frac{1}{2} > 0, \quad c_n - a_n + \frac{1}{2} \leq -\frac{7}{6} + \frac{1}{2} + \frac{1}{2} < 0.$$

In the near-field $\lambda \leq \lambda_*$ or in the far-field $\lambda \geq \lambda_*$, we have a uniform estimate $\lambda^{c_\alpha - a_\alpha + 1/2} \leq \lambda_*^{c_\alpha - a_\alpha + 1/2}$. For a large distance, we apply L^∞ estimate similar to that in Section 7.4.3. We can obtain a piecewise L^∞ estimate similar to the estimate above (7.32) as follows

$$|(\psi_{\mathbf{u}})(\lambda\hat{z})| \leq E_1 \lambda^{c_\alpha - a_\alpha + 1} C(\hat{z}), \quad \hat{z} \in [0, x_{c3}]^2 \setminus [0, x_{c2}]^2,$$

and another estimate similar to (7.32). For \mathbf{u} , since the kernel is locally integrable, we only have one term related to $\|\omega\varphi\|_\infty$ in the above bound. From the power law (7.36) and the scaling relation (7.2), using the argument for the Hölder estimate of $\nabla \mathbf{u}$, we obtain Hölder estimate $\delta_i(f, \lambda\hat{x}, \lambda\hat{z})$ for large $|\hat{x} - \hat{z}|$ uniformly in λ .

7.5. Estimate using other norms. Similar to Section 6.3, we estimate $\mathbf{u}_A, (\nabla \mathbf{u})_A$ using another combination of norms, e.g. $\|\omega\varphi_{1,g}\|_\infty, [\omega\psi_1]_{C_{x_i}^{1/2}}$. We use the estimates in previous sections of Section 7 to obtain the estimates in the near-field and the far-field and bound different norms using the energy (6.14) or the direct estimate for the error (6.15). In addition to the L^∞ estimate mentioned in Section 6.3, we use $\|\varepsilon\varphi_{ell i}\|_\infty, [\varepsilon\psi_1]_{C_{x_i}^{1/2}}$ for the Hölder estimate of $\mathbf{u}_A(\varepsilon), (\nabla \mathbf{u})_A(\varepsilon)$, where $\varepsilon = \bar{\omega} - \bar{\phi}^N, \varepsilon = \hat{\omega} - \hat{\phi}^N$ is the error of solving the Poisson equations. We use these estimates to control the nonlocal error.

8. ESTIMATING THE PIECEWISE BOUNDS OF FUNCTIONS

In this appendix, we estimate $\|f\|_{L^\infty(D)}$ in the domain $D = [a, b] \times [c, d]$ given the grid point values of f in D and the derivatives bound $\|\partial_x^i \partial_y^j f\|_{L^\infty}$. We want to obtain an error term as small as possible without evaluating f on too many grid points and its high order derivatives, which are expensive for some complicated function f , e.g. the residual error $f = (\partial_t - \mathcal{L})\widehat{W}$ in Section 3 in Part II [2]. Based on these L^∞ estimates, we further develop the Hölder estimate in Appendix E in Part II [2]. We use these estimates to verify the smallness of the residual error, e.g. $f = (\partial_t - \mathcal{L})\widehat{W}$, in suitable energy norm.

We will develop three estimates based on the Newton polynomial, the Lagrangian interpolating polynomial, and the Hermite interpolation. Each method has its own advantages. For the Newton and the Lagrangian method, to obtain 4-th order error estimates, we only need to evaluate 4×4 grid point values of f .

(a) For the Newton method, we have a sharp error bound with a much smaller constant than that of the Lagrangian method.

(b) For the Lagrangian method, it is easier and more efficient to estimate the Lagrangian interpolating polynomials for grid points (x_i, y_j) in a general position.

In some situation, we need to estimate both f and ∇f . We use grid point values $f(x_i, y_j)$ and $\nabla f(x_i, y_j)$ to build 4-th and 5-th order interpolating polynomials based on the Hermite interpolation. The 4-th order error estimate is as sharp as the Newton method. Moreover, in the 4-th order Newton or Lagrangian interpolation, we need $f \in C^4[x_0, x_3]$. When f is only piecewise smooth in $[x_i, x_{i+1}]$, $i \leq 3$, we cannot use these two methods. Instead, we evaluate $f(x_i)$, $f'(x_i)$ and construct the Hermite interpolation in each interval $[x_i, x_{i+1}]$. One disadvantage is that the estimate of the interpolating polynomials is more complicated and takes longer time in practice.

We do not pursue higher order error estimates since most of these estimates are applied to estimate the residual errors, e.g. $(\partial_t - \mathcal{L})\widehat{W}$ in Section 3 [2], which is only piecewise smooth, and we do not use very small h in the whole computational domain.

8.1. Estimates based on the Newton polynomials. Given $x_0, x_1, x_2, \dots, x_k$, we first define the divided differences recursively

$$f[x] = f(x), \quad f[x, y] = \frac{f(y) - f(x)}{y - x}, \quad f[x_i, x_1, \dots, x_{j+1}] = \frac{f[x_{i+1}, x_{i+2}, \dots, x_{j+1}] - f[x_i, x_{i+1}, \dots, x_j]}{x_{j+1} - x_i}.$$

8.1.1. *The Newton polynomials in 1D.* We first discuss how to bound $f(x)$ in 1D. We consider the domain $[a, b]$ and denote

$$z_0 = a, \quad h = (b - a)/3, \quad z_i = z_0 + ih.$$

Denote

$$(8.1) \quad \begin{aligned} D_{1,i}f &= f(z_{i+1}) - f(z_i), \quad i = 0, 1, 2, & D_{2,i}f &= D_{1,i+1}f - D_{1,i}f, \quad i = 0, 1, \\ D_{3,i}f &= D_{2,i+1}f - D_{2,i}f = f(z_3) - 3f(z_2) + 3f(z_1) - f(z_0). \end{aligned}$$

Let $\{x_i\}_{i=0}^3$ be a permutation of $\{z_i\}_{i=0}^3$. We construct the Newton polynomial

$$(8.2) \quad \begin{aligned} P(x) &= (f[x_0] + f[x_0, x_1](x - x_0)) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &\quad + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2) \triangleq l(x) + q(x) + c(x), \end{aligned}$$

where $l(x), q(x), c(x)$ denote the linear, quadratic, and the cubic parts, respectively. We remark that the above Newton polynomial agrees with the Lagrangian polynomial interpolating $(z_i, f(z_i))$.

By standard error analysis of the Newton interpolation, the error part $R(x)$ can be bounded as follows

$$(8.3) \quad |f(x) - P(x)| \leq \frac{1}{24} \|\partial_x^4 f\|_{L^\infty[a,b]} \max_{x \in [a,b]} |\Pi_{0 \leq i \leq 3}(x - x_i)| = \frac{1}{24} \|\partial_x^4 f\|_{L^\infty[a,b]} \frac{(b-a)^4}{81}.$$

To obtain the last equality, using the definition of x_i, z_i , we write $z = a + th$, $t \in [0, 3]$ and get

$$(8.4) \quad \max_{x \in [a,b]} |\Pi_{0 \leq i \leq 3}(x - x_i)| = \max_{z \in [a,b]} |\Pi_{0 \leq i \leq 3}(z - z_i)| = \max_{t \in [0,3]} h^4 |\Pi_{0 \leq i \leq 3}(t - i)| \leq h^4,$$

where we have used (8.71) in Lemma 8.1 in the last inequality.

To bound $f(x)$, given the derivative bound of f and the above estimate, we only need to control $P(x)$. We choose different permutation $\{x_i\}_{i=0}^3$ of $\{z_i\}_{i=0}^3$ for z in different part of $[a, b]$:

$$\begin{aligned} x_i &= z_i, \quad z \in [z_0, z_1], & (x_0, x_1, x_2, x_3) &= (z_2, z_1, z_0, z_3), \quad z \in [z_1, z_2], \\ (x_0, x_1, x_2, x_3) &= (z_3, z_2, z_1, z_0), \quad z \in [z_2, z_3]. \end{aligned}$$

Let I_z be the interval with endpoints x_0, x_1 . We have $z \in I_z$. Since $l(x)$ in (8.2) is linear with $l(x_i) = f(x_i)$ and $|(x - x_0)(x - x_1)| \leq \frac{(x_1 - x_0)^2}{4}$, we get

$$|l(z)| \leq \max(|f(x_0)|, |f(x_1)|), \quad \max_{z \in I_z} |q(z)| \leq |f[x_0, x_1, x_2]| \frac{(x_1 - x_0)^2}{4} = |f[x_0, x_1, x_2]| \frac{h^2}{4}.$$

Since x_0, x_1, x_2 are three consecutive points with distance h and $I \triangleq [\min(x_0, x_1, x_2), \max(x_0, x_1, x_2)]$ covers z , we have

$$(8.5) \quad \max_{z \in I} |(z - x_0)(z - x_1)(z - x_2)| = \max_{z \in [0, 2h]} |z(z - h)(z - 2h)| \leq \frac{2}{3\sqrt{3}}h^3,$$

where we have used (8.69) in Lemma 8.1 in the last inequality.

Next, we use (8.1) to simplify $f[x_i, x_{i+1}, \dots, x_j]$. For each case, a direct calculation yields

$$\begin{aligned} z \in [z_0, z_1] : |f[z_0, z_1, z_2]| &= \left| \frac{f[z_2, z_1] - f[z_1, z_0]}{z_2 - z_0} \right| = \left| \frac{1}{2h^2}(D_{1,1}f - D_{1,0}f) \right| = \frac{1}{2h^2}|D_{2,0}f|, \\ z \in [z_1, z_2] : |f(z_2, z_1, z_0)| &= \left| \frac{f[z_0, z_1] - f[z_1, z_2]}{z_0 - z_2} \right| = \left| \frac{1}{2h^2}(D_{1,1}f - D_{1,0}f) \right| = \frac{1}{2h^2}|D_{2,0}f|. \end{aligned}$$

Similarly, for $z \in [z_2, z_3]$, we get

$$|f(z_3, z_2, z_1)| = \frac{1}{2h^2}|D_{2,1}f|.$$

A direct calculation yields $|f[x_0, x_1, x_2, x_3]| = \frac{1}{6h^3}|D_3f|$ for any permutation $\{x_i\}$ of $\{z_i\}$. Thus, for $c(x)$ (8.2), using (8.5) and the above estimate, we get

$$\max_{z \in [a, b]} |c(x)| \leq \frac{2}{3\sqrt{3}}h^3|f[x_0, x_1, x_2, x_3]| = \frac{2}{3\sqrt{3}}h^3 \cdot \frac{1}{6h^3}|D_3f| = \frac{1}{9\sqrt{3}}|D_3f|.$$

Combining the above estimates, we obtain

$$(8.6) \quad |P(x)| \leq \max \left(\max_{i=0,1,2} |f(z_i)| + c_1|D_{2,0}f|, \max_{i=1,2,3} |f(z_i)| + c_1|D_{2,1}f| \right) + c_2|D_3f|, \quad c_1 = \frac{1}{8}, \quad c_2 = \frac{1}{9\sqrt{3}}.$$

8.1.2. *A quadratic interpolation.* We also need a cubic interpolation in the Hermite interpolation in Section 8.4. Given $x_0 < x_1 < x_2$ with $x_2 - x_1 = x_1 - x_0 = h$, we define

$$(8.7) \quad N_2(f, x_0, x_1, x_2)(x) \triangleq f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1),$$

and construct $P(x) = N_2(f, x_0, x_1, x_2)(x)$.

We have an error estimate similar to (8.3) for $x \in [x_0, x_2]$, $h = x_1 - x_0$

$$(8.8) \quad |P(x) - f(x)| \leq \frac{\|\partial_x^3 f\|_{L^\infty[x_0, x_2]}}{6} \max_{x \in [x_0, x_2]} |\Pi_{i=0,1,2}(x - x_i)| \leq \frac{\|\partial_x^3 f\|_{L^\infty[x_0, x_2]} 2h^3}{6 \cdot 3\sqrt{3}} = \frac{\|\partial_x^3 f\|_{L^\infty[x_0, x_2]} h^3}{9\sqrt{3}},$$

where we have used (8.5) in the last inequality.

Using the same estimates in Section 8.1.1 for the linear part and quadratic part, we obtain

$$(8.9) \quad \begin{aligned} x \in [x_0, x_1] : |P(x)| &\leq \max(|f(x_0)|, |f(x_1)|) + \frac{1}{8}|f(x_0) - 2f(x_1) + f(x_2)|, \\ x \in [x_0, x_1] : |P(x)| &\leq \max(|f(x_0)|, |f(x_1)|) + \frac{1}{8}|f(x_0) - 2f(x_1) + f(x_2)|. \end{aligned}$$

Note that using the notation (8.1), we have $|D_{2,0}f| = |f(x_0) - 2f(x_1) + f(x_2)|$.

8.1.3. *Generalization to 2D.* Denote

$$D = [a, b] \times [c, d], \quad x_i = a + ih_1, \quad y_j = c + jh_2, \quad h_1 = (b - a)/3, \quad h_2 = (d - c)/3.$$

Suppose that $f(x_i, y_j)$ and $\|\partial_x^k \partial_y^l f\|_{L^\infty}$, $k + l \leq 4$ are given. Firstly, we treat y as a parameter and interpolate $f(x, y)$ in x . Denote

$$\begin{aligned} D_{i,1}f(y) &= f(x_{i+1}, y) - f(x_i, y), \quad 0 \leq i \leq 2, \quad D_{i,2}f(y) = D_{i+1,1}f(y) - D_{i,1}f(y), \quad 0 \leq i \leq 1, \\ D_3f(y) &= D_{2,1}f(y) - D_{2,0}f(y). \end{aligned}$$

Applying (8.3), (8.6), we get

$$\begin{aligned} \max_{x \in [a, b]} |f(x, y)| &\leq \max \left(\max_{i=0,1,2} |f(x_i, y)| + c_1|D_{2,0}f(y)|, \max_{i=1,2,3} |f(x_i, y)| + c_1|D_{2,1}f(y)| \right) \\ &\quad + c_2|D_3f(y)| + \frac{1}{24}h_1^4 \|\partial_x^4 f\|_{L^\infty(D)}. \end{aligned}$$

Note that $f(x_i, y), D_{i,j}f(y)$ are 1D functions in y and their grid point values on y_j can be obtained from $f(x_i, y_j)$. We further apply (8.3),(8.6) to estimating $\|g\|_{L^\infty[c,d]}$ with $g = f(x_i, y), D_{i,j}f(y)$. Maximizing the above estimate over y yields the bound for f .

8.2. Estimates based on the Lagrangian interpolation. If the grid points x_i are not equispaced, the estimates of the Newton polynomials can be more complicated. We develop another estimate based on the Lagrangian interpolating polynomials. Although these two interpolating polynomials on $(x_i, f(x_i))$ are equivalent, the Lagrangian formulation is easier to estimate.

Firstly, let $p_i(x), q_j(y)$ be the Lagrange interpolating polynomials associated to the points $x_1 < \dots < x_k \in [a, b], y_1 < y_2 < \dots < y_k \in [c, d]$

$$(8.10) \quad p_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}, \quad q_i(y) = \prod_{j \neq i} \frac{y - y_j}{y_i - y_j}.$$

For any $(x, y) \in D$, we consider the following decomposition by first interpolating f in x and then in y

$$(8.11) \quad \begin{aligned} f(x, y) &= \sum_{i=1}^k f(x_i, y) p_i(x) + R_1(x, y) = \sum_{i=1}^k \left(\sum_{j=1}^k f(x_i, y_j) q_j(y) + R_2(x_i, y) \right) p_i(x) + R_1(x, y) \\ &= \sum_{i,j=1}^k p_i(x) q_j(y) f(x_i, y_j) + \left(\sum_i R_2(x_i, y) p_i(x) + R_1(x, y) \right) \triangleq I + II. \end{aligned}$$

By standard error analysis of the Lagrange interpolation, the error part $R_1(x, y), R_2(x, y)$ can be bounded as follows

$$(8.12) \quad \begin{aligned} |R_1(x, y)| &\leq \frac{1}{k!} \|\partial_x^k f(x, y)\|_{L^\infty(D)} \max_{x \in [a, b]} \left| \prod_{i=1}^k (x - x_i) \right|, \\ |R_2(x_i, y)| &\leq \frac{1}{k!} \|\partial_y^k f(x, y)\|_{L^\infty(D)} \max_{y \in [c, d]} \left| \prod_{i=1}^k (y - y_i) \right|. \end{aligned}$$

Denote

$$(8.13) \quad C_1 = \max_{x \in [a, b]} \sum_{i=1}^k |p_i(x)|, \quad a_{ij} = f(x_i, y_j).$$

Note that the value C_1 only depends on the ratio $\frac{x_{i+1} - x_i}{b - a}, i = 1, \dots, k$, since from (8.10), we have

$$p_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j} = \prod_{j \neq i} \frac{t - t_j}{t_i - t_j}, \quad t = \frac{x - a}{b - a}, \quad t_j = \frac{x_j - a}{b - a}.$$

We will choose the grid points with $\frac{x_{i+1} - x_i}{b - a} = \frac{y_{i+1} - y_i}{d - c}$ so that we also have $C_1 = \max_{y \in [c, d]} \sum_{i=1}^k |q_j(y)|$. See (8.17). We have the following trivial estimate for any c_j

$$(8.14) \quad \left| \sum_i p_i(x) c_i \right| \leq \sum_i |p_i(x)| \max |c_i| \leq C_1 \max |c_i|, \quad \left| \sum_j q_j(y) c_j \right| \leq C_1 \max |c_j|.$$

Next, we estimate I . Since $\sum_i p_i(x) = \sum_j q_j(y) = 1$, we expect that $I \approx f(x_i, y_j) + O(\max(h_1, h_2))$, where $h_1 = b - a, h_2 = d - c$. Thus, for some m to be chosen, we further decompose it into the mean and the variation and apply (8.14) to obtain

$$\begin{aligned} |I| &= \left| m + \sum_{i,j=1}^k p_i(x) q_j(y) (a_{ij} - m) \right| \leq |m| + \max_{i,j} |a_{ij} - m| \sum_i |p_i(x)| \sum_j |q_j(y)| \\ &= |m| + C_1^2 \max_{i,j} |a_{ij} - m|. \end{aligned}$$

We use the following trivial equality for b_1, b_2, \dots, b_n

$$(8.15) \quad \max_i |b_i - b| = \frac{1}{2} (\max_i b_i - \min_i b_i), \quad b = \frac{1}{2} (\max_i b_i + \min_i b_i),$$

which can be proved by ordering b_i . Thus, we optimize the estimate of I by choosing $m = \frac{1}{2} (\max_{i,j} a_{ij} + \min_{i,j} a_{ij})$.

We can obtain a sharper estimate as follows

$$(8.16) \quad \begin{aligned} |I| &= \left| \sum_j q_j(y) \left(\bar{a}_j + \sum_i p_i(x)(a_{ij} - \bar{a}_j) \right) \right| = \left| \sum_j q_j(y) (\bar{a}_j + S_j(x)) \right| \\ &\leq |\bar{a}| + \left| \sum_j q_j(y) (\bar{a}_j - \bar{a}) \right| + \left| \sum_j q_j(y) S_j(x) \right|, \quad S_j(x) = \sum_i p_i(x)(a_{ij} - \bar{a}_j). \end{aligned}$$

For a fixed j , we choose

$$\bar{a}_j = \frac{\max_i a_{ij} + \min_i a_{ij}}{2}, \quad \bar{a} = \frac{\max_j \bar{a}_j + \min_j \bar{a}_j}{2}.$$

Applying (8.14), (8.15), and the definition of $S_j(x)$ in (8.16), we yield

$$\left| \sum_j q_j(y) S_j(x) \right| \leq C_1 \cdot \max_j |S_j(x)|, \quad |S_j(x)| \leq C_1 \max_i |a_{ij} - \bar{a}_j| = \frac{C_1}{2} \left| \max_i a_{ij} - \min_i a_{ij} \right|.$$

Similarly, we have

$$\left| \sum_j q_j(x) (\bar{a}_j - \bar{a}) \right| \leq C_1 \max_j |\bar{a}_j - \bar{a}| = \frac{C_1}{2} (\max_j \bar{a}_j - \min_j \bar{a}_j).$$

Combining two parts, we yield an improved estimate for $|I|$

$$|I| \leq \frac{1}{2} \left| \max_j \bar{a}_j + \min_j \bar{a}_j \right| + \frac{C_1}{2} (\max_j \bar{a}_j - \min_j \bar{a}_j) + \frac{C_1^2}{2} \max_j \left| \max_i a_{ij} - \min_i a_{ij} \right|.$$

The above estimate is better if $a_{ij} = f(x_i, y_j)$ is smooth in x . Similarly, we can first sum over $p_i(x)$ and then sum over $q_j(y)$ in (8.16) to obtain another improved estimate.

We apply the above method to the fourth and third order estimate of f on $[a, b] \times [c, d]$. We choose

$$(x_1, x_2, x_3, x_4) = (a, a + \frac{1}{3}h_1, a + \frac{2}{3}h_1, b), \quad (y_1, y_2, y_3, y_4) = (c, c + \frac{1}{3}h_2, c + \frac{2}{3}h_2, d),$$

for the fourth order estimate, and

$$(8.17) \quad (x_1, x_2, x_3) = (a + \frac{h_1}{32}, a + \frac{h_1}{2}, a + \frac{31}{32}h_1), \quad (y_1, y_2, y_3) = (c + \frac{h_2}{32}, c + \frac{h_2}{2}, c + \frac{31}{32}h_2),$$

for the third order estimate, where $h_1 = b - a$, $h_2 = d - c$. To estimate the constant C_1 (8.13) in each case, since the interpolation polynomial $p_i(x)$ (8.10) has degree at most 3 and C_1 only depends on the ratio $\frac{x_{i+1} - x_i}{b - a}$, we can first assume $I = [a, b] = [0, 3]$ or $[0, 1]$ and partition I into N small sub-intervals I_j with $|I_j| = |I|/N$. Then we estimate the piecewise bound of $p_i(x)$ in I_j using (8.6) and then $\sum |p_j|$ using the triangle inequality. We get

$$C_1^{(4)} \leq 1.635, \quad C_2^{(3)} \leq 1.276$$

for the fourth order and third order case, respectively. To estimate the 3rd order error term in (8.12), we use (8.6) again to estimate the third order polynomial $p(t)$ below and yield

$$\begin{aligned} \max_{x \in [a, b]} \prod_{i=1}^3 |x - x_i| &= h_1^3 \max_{t \in [0, 1]} |p(t)| = C_3 h_1^3, \quad \max_{x \in [a, b]} \prod_{i=1}^3 |y - y_i| = C_3 h_2^3, \quad p(t) = (t - \frac{1}{32})(t - \frac{1}{2})(t - \frac{31}{32}), \\ C_3^{(3)} &\leq 0.0397. \end{aligned}$$

For the fourth order error term, since $\{x_i\}$ are equi-spacing, following (8.4), we get

$$\max_{x \in [a, b]} \prod_{i=1}^4 |x - x_i| = (\frac{h_1}{3})^4 \max_{t \in [0, 3]} |t(t-1)(t-2)(t-3)| \leq (\frac{h_1}{3})^4.$$

Using (8.12), (8.14), and the above estimate, for the 3rd order estimate, we get

$$|II| \leq C_2^{(3)} \max_i \|R_2(x_i, y)\|_{L^\infty} + \|R_1\|_{L^\infty} \leq \frac{1}{6} \cdot C_3^{(3)} (C_2^{(3)} \|\partial_y^3 f\|_{L^\infty(D)} h_2^3 + \|\partial_x^3 f\|_{L^\infty(D)} h_1^3).$$

We can also first interpolate f in y and then in x (8.11) to obtain another form of II . Similar estimates yield

$$|II| \leq \frac{1}{6} \cdot C_3^{(3)}(C_2^{(3)}) \|\partial_x^3 f\|_{L^\infty(D)} h_1^3 + \|\partial_y^3 f\|_{L^\infty(D)} h_2^3.$$

We minimize the above two estimates to bound II . Similar arguments apply to the fourth order estimate.

8.3. Hermite interpolation in 1D. We first discuss the Hermite interpolation in 1D, and then generalize it to 2D. Consider $x_0 < x_1 < x_2$ with $x_2 - x_1 = x_1 - x_0$. Denote by p_i, q_i the cubic polynomials such that

$$\begin{aligned} p_i(x_i) &= 1, & p_i(x_{1-i}) &= 0, & i &= 0, 1, & p'_i(x_j) &= 0, & j &= 0, 1, \\ q'_i(x_i) &= 1, & q'_i(x_{1-i}) &= 0, & i &= 0, 1, & q_i(x_j) &= 0, & j &= 0, 1, \end{aligned}$$

and

$$(8.18) \quad l_i = f(x_i), \quad m_i = h f'(x_i), \quad h = x_1 - x_0.$$

We consider the 4-th and 5-th order Hermite interpolations for f

$$(8.19) \quad \begin{aligned} H_4(f, x_0, x_1)(x) &= \sum_{i=0,1} (f(x_i)p_i(x) + f'(x_i)q_i(x)), \\ H_5(f, x_0, x_1, x_2)(x) &= H_4(x) + (f(x_2) - H_4(x_2)) \frac{(x-x_0)^2(x-x_1)^2}{(x_2-x_0)^2(x_2-x_1)^2}. \end{aligned}$$

For simplicity, we drop the dependence of f, x_i . Note that the coefficients of the polynomials p_i, q_i depend on x_0, x_1, x_2 only. It is easy to see that

$$H_4(x_i) = H_5(x_i) = f(x_i), \quad H'_4(x_i) = H'_5(x_i) = f'(x_i), \quad i = 0, 1, \quad H_5(x_2) = f(x_2).$$

8.3.1. Estimates of the interpolation error. For $[x_0, x_2]$, we have the standard error estimate

$$(8.20) \quad |f(x) - H_4(x)| \leq \frac{1}{24} \|\partial_x^4 f\|_{L^\infty([x_0, x_2])} (x-x_0)^2(x-x_1)^2, \quad x \in [x_0, x_2].$$

For $x \in [x_0, x_1]$, we have the following error estimates with further simplifications

$$(8.21) \quad \begin{aligned} |f(x) - H_4(x)| &\leq \frac{1}{24} \|\partial_x^4 f\|_{L^\infty([x_0, x_1])} (x-x_0)^2(x-x_1)^2 \leq \frac{h^4}{384} \|\partial_x^4 f\|_{L^\infty([x_0, x_1])}, \\ |f(x) - H_5(x)| &\leq \frac{\|\partial_x^5 f\|_{L^\infty([x_0, x_2])}}{120} |(x-x_0)^2(x-x_1)^2(x-x_2)| \leq \frac{h^5}{1200} \|\partial_x^5 f\|_{L^\infty([x_0, x_2])}, \end{aligned}$$

where we have used (8.67), (8.72) in Lemma 8.1 in the last inequality. The proof of the first inequality is standard and easier. We consider the second estimate. For any $t \in [x_0, x_2]$, denote

$$R_t(x) = f(x) - H_5(x) - (f(t) - H_5(t)) \frac{(x-x_0)^2(x-x_1)^2(x-x_2)}{(t-x_0)^2(t-x_1)^2(t-x_2)}.$$

Clearly, we have $R_t(x_i) = 0, i = 0, 1, 2, R'_t(x_i) = 0, i = 0, 1, R_t(t) = 0$. Thus, R_t has 6 zeros. For $f \in C^{4,1}$, applying the Rolle's theorem repeatedly up to $\partial_x^4 f$, we yield $\xi_1 \neq \xi_2 \in (x_0, x_2)$ with

$$0 = \partial_x^4 f(\xi_i) - C_1 - \frac{(f(t) - H_5(t))(120\xi_i - C_2)}{(t-x_0)^2(t-x_1)^2(t-x_2)},$$

where we have used $\partial_x^4 H_5(x) = C_1, \partial_x^4(x-x_0)^2(x-x_1)^2(x-x_2) = 120x - C_2$. Rewriting the above identities and computing the difference, we obtain

$$|f(t) - H_5(t)| = \left| \frac{\partial_x^4 f(\xi_2) - \partial_x^4 f(\xi_1)}{120(\xi_2 - \xi_1)} (t-x_0)^2(t-x_1)^2(t-x_2) \right| \leq \frac{\|\partial_x^5 f\|_{L^\infty[x_0, x_2]}}{120} |(t-x_0)^2(t-x_1)^2(t-x_2)|.$$

Since t is arbitrary, this proves the second estimate in (8.21). The first estimate can be proved similarly. Next, we estimate the interpolating polynomials H_4, H_5 .

We should compare the first estimate (8.21) with (8.3). In (8.3), $x_1 - x_0 = \frac{b-a}{3} = h$ and the upper bound is $\frac{1}{24} \|\partial_x^4 f\|_\infty h^4$. In (8.21), for $x \in [x_0, x_1]$, using $|(x-x_0)(x-x_1)| \leq \frac{(x_1-x_0)^2}{4}$, we obtain an extra small factor $\frac{1}{16}$. This is one of the main advantages of the Hermite interpolation.

8.3.2. *Estimate H_4, H_5 .* We consider $x \in [x_0, x_1]$. Recall l_i, m_i from (8.18). We have

$$(8.22) \quad \begin{aligned} G_4(t) &\triangleq H_4(x_0 + th) = H_4(x), \quad t = \frac{x - x_0}{h}, \\ G_4(t) &= \left(l_0(1-t) + l_1 t \right) + \left(t^2(t-1)(m_1 - (l_1 - l_0)) + (t-1)^2 t(m_0 - (l_1 - l_0)) \right) \\ &\triangleq I(t) + II(t). \end{aligned}$$

To show that G_4 defined via the first identity has the second expression, we only need to verify that the expression satisfies $G_4(i) = l_i = f(x_i), \partial_t G_4(i) = m_i = h f'(x_i), i = 0, 1$, which is obvious. Then both expressions are Hermite polynomials interpolating $(x_i, f(x_i)), (x_i, f'(x_i))$, and thus they must be the same. To estimate $H_4(x)$ on $[x_0, x_1]$, we only need to estimate $G_4(t)$ on $[0, 1]$. The estimate of the linear part is trivial

$$|I(t)| = |l_0(1-t) + l_1 t| \leq \max(|l_0|, |l_1|).$$

The second part II is treated as error and we want to obtain a sharp constant. Denote

$$(8.23) \quad M_1 = \max(|m_1 - m_0|, |m_1 - (l_1 - l_0)|, |m_0 - (l_1 - l_0)|).$$

For $0 \leq t \leq \frac{1}{2}$, using $t(t-1)^2 \leq \frac{4}{27}$ (8.70), we have

$$\begin{aligned} II(t) &= t(t-1)(t(m_1 - (l_1 - l_0)) + (t-1)(m_0 - (l_1 - l_0))) \\ &= t(t-1)(t(m_1 - m_0) + (2t-1)(m_0 - (l_1 - l_0))), \\ |II(t)| &\leq M_1 |t(t-1)| (t + |1-2t|) = M_1 t(1-t)^2 \leq \frac{4}{27} M_1. \end{aligned}$$

Similarly, for $t \in [1/2, 1]$, writing $m_0 - (l_1 - l_0) = m_0 - m_1 + (m_1 - (l_1 - l_0))$, we get

$$\begin{aligned} II(t) &= t(t-1)((t-1)(m_0 - m_1) + (2t-1)(m_1 - (l_1 - l_0))), \\ |II(t)| &\leq |t(t-1)| (|t-1| + |2t-1|) M_1 = t(1-t)(1-t+2t-1) M_1 = t^2(1-t) M_1 \leq \frac{4}{27} M_1. \end{aligned}$$

To obtain the last inequality, we apply (8.70) with $s = 1-t$. Therefore, we prove

$$(8.24) \quad |H_4(x)| = |G_4(t)| \leq \max(|l_0|, |l_1|) + \frac{4}{27} \max(|m_1 - m_0|, |m_1 - (l_1 - l_0)|, |m_0 - (l_1 - l_0)|).$$

For H_5 , we estimate the extra term in (8.19). Since $x_2 - x_1 = x_1 - x_0 = h$, we have

$$H_4(x_2) = G_4(2) = -l_0 + 2l_1 + 4(m_1 - (l_1 - l_0)) + 2(m_0 - (l_1 - l_0)) = l_0 + 2m_0 + 4(m_1 - (l_1 - l_0)).$$

Since $x = x_0 + th \in [x_0, x_1]$, we have $t \in [0, 1]$, $|t(1-t)| \leq \frac{1}{4}$,

$$(8.25) \quad \frac{(x - x_0)^2 (x - x_1)^2}{(x_2 - x_0)^2 (x_1 - x_0)^2} = \frac{t^2 (t-1)^2}{4} \leq \frac{1}{64}.$$

We obtain

$$(8.26) \quad \max_{x \in [x_0, x_1]} |H_5| \leq \max_{x \in [x_0, x_1]} |H_4(x)| + \frac{1}{64} |f(x_2) - H_4(x_2)|.$$

8.3.3. *Estimate derivatives in 1D.* In this subsection, we discuss how to estimate $\partial f(x)$ with fourth order error term using the Hermite interpolation $\partial H_5(x)$. We consider ∂_x without loss of generality. Firstly, since $f(x) - H_5(x)$ has five zeros: two zeros at x_0 , two zeros at x_1 , and one zero at x_2 , we know that $\partial_x(f(x) - H_5(x))$ has four zeros: $x_0 < \xi < x_1 < \eta$. Using the Rolle's theorem and an argument similar to that in Section 8.3.1, we get

$$|\partial_x(f(x) - H_5(x))| \leq \frac{1}{24} \|\partial_x^5 f\|_{L^\infty[x_0, x_2]} |(x - x_0)(x - x_1)(x - \xi)(x - \eta)|.$$

Next, for $x \in [x_0, x_1]$, we simplify the upper bound. Clearly, we have $|x - \xi| \leq \max(|x - x_0|, |x_1 - x|)$, $|x - \eta| = \eta - x \leq x_2 - x$. We yield

$$\begin{aligned} p(x) &\triangleq |(x - x_0)(x - x_1)(x - \xi)(x - \eta)| \leq |(x - x_0)(x - x_1)(x - x_2)| \max(|x - x_0|, |x - x_1|) \\ &= h^4 |t(1-t)(2-t)| \max(|1-t|, t) \triangleq h^4 q(t), \quad t = (x - x_0)h^{-1}. \end{aligned}$$

If $t \leq \frac{1}{2}$, we denote $s = (1-t)^2 \in [0, 1]$ and get

$$q(t) = t(1-t)^2(2-t) = (2t-t^2)s = (1-s)s \leq \frac{1}{4}.$$

If $t \in [1/2, 1]$, we get

$$q(t) = t(1-t) \cdot t(2-t) \leq \frac{1}{4} \cdot 1 = \frac{1}{4}.$$

Thus, for $x \in [x_0, x_1]$, we prove the error estimate

$$(8.27) \quad |\partial_x(f(x) - H_5(x))| \leq \frac{h^4}{96} \|\partial_x^5 f\|_{L^\infty[x_0, x_2]}.$$

Next, we estimate $\partial_x H_5$. Recall $t = \frac{x-x_0}{h}$. Using (8.19) and the chain rule, we get

$$(8.28) \quad \begin{aligned} H_5(x) &= H_4(x) + (f(x_2) - H_4(x_2)) \frac{t^2(t-1)^2}{4} = G_4(t) + (f(x_2) - H_4(x_2)) \frac{t^2(t-1)^2}{4}, \\ \partial_x H_5(x) &= \frac{1}{h} (\partial_t G_4(t) + (f(x_2) - H_4(x_2)) \partial_t \frac{t^2(t-1)^2}{4}) \triangleq \frac{1}{h} (I(t) + II(t)). \end{aligned}$$

We estimate two parts separately. Using (8.22), we get

$$(8.29) \quad I(t) \triangleq \partial_t G_4 = (l_1 - l_0) + (3t^2 - 2t)(m_1 - (l_1 - l_0)) + (3t^2 - 4t + 1)(m_0 - (l_1 - l_0)).$$

Note that $l_1 - l_0, m_1, m_0$ are approximations of $hf'(x)$ and have cancellations. We discuss different $t \in [0, 1]$ to exploit the cancellations.

Denote $a \vee b = \max(a, b)$. If $t \leq \frac{1}{3}$, we get

$$I(t) = (4t - 3t^2)(l_1 - l_0) + (3t^2 - 4t + 1)m_0 + (2t - 3t^2)(l_1 - l_0 - m_1).$$

The first two terms are the main terms, and the last term is the error term. Since $t \leq \frac{1}{3}$, we get

$$4t - 3t^2 > 0, \quad 3t^2 - 4t + 1 = (1-3t)(1-t) \geq 0, \quad 0 \leq 2t - 3t^2 = 3t\left(\frac{2}{3} - t\right) \leq \frac{1}{3},$$

where the last inequality is equivalent to $3(t - \frac{1}{3})^2 \geq 0$. It follows

$$(8.30) \quad |I(t)| \leq (|l_1 - l_0| \vee m_0) \cdot (4t - 3t^2 + 3t^2 - 4t + 1) + \frac{1}{3} |l_1 - l_0 - m_1| = |l_1 - l_0| \vee m_0 + \frac{1}{3} |l_1 - l_0 - m_1|.$$

The estimate of $t \in [2/3, 1]$ is similar by swapping t and $1-t$, m_0 and m_1 , p_0 and p_1 . We get

$$(8.31) \quad |I(t)| \leq |l_1 - l_0| \vee m_1 + \frac{1}{3} |l_1 - l_0 - m_0|.$$

For $t \in [1/3, 2/3]$, we rewrite $I(t)$ (8.29) as follows

$$I(t) = l_1 - l_0 + (4t - 3t^2 - 1)(l_1 - l_0 - m_0) + (2t - 3t^2)(l_1 - l_0 - m_1).$$

Since $t \in [1/3, 2/3]$, we get

$$4t - 3t^2 - 1 = (3t - 1)(1 - t) \geq 0, \quad 2t - 3t^2 = t(2 - 3t) \geq 0,$$

$$0 \leq 4t - 3t^2 - 1 + 2t - 3t^2 = 6t - 6t^2 - 1 = 6t(1-t) - 1 \leq 6 \cdot \frac{1}{4} - 1 = \frac{3}{2} - 1 \leq \frac{1}{2}.$$

Thus, we obtain

$$(8.32) \quad |I(t)| \leq |l_1 - l_0| + \frac{1}{2} (|l_1 - l_0 - m_0| \vee |l_1 - l_0 - m_1|).$$

Combining three cases (8.30)-(8.32), we obtain the bound for $\partial_t G_4$

$$(8.33) \quad \frac{1}{h} |\partial_t G_4| \leq \max \left(\max_{i=0,1} (|l_1 - l_0| \vee m_i + \frac{1}{3} |l_1 - l_0 - m_{1-i}|), |l_1 - l_0| + \frac{1}{2} (|l_1 - l_0 - m_0| \vee |l_1 - l_0 - m_1|) \right).$$

For the second term in (8.28), using (8.69) with $2t \in [0, 2]$ (we consider $x \in [x_0, x_1]$), we get

$$(8.34) \quad \partial_t \frac{t^2(t-1)^2}{4} = t(t-1)\left(t - \frac{1}{2}\right), \quad |t(t-1)\left(t - \frac{1}{2}\right)| = \frac{1}{8} |2t(1-2t)(2-2t)| \leq \frac{1}{8} \cdot \frac{2}{3\sqrt{3}} \leq \frac{1}{12\sqrt{3}},$$

which implies

$$(8.35) \quad \left| \frac{1}{h} II(t) \right| \leq \frac{1}{12\sqrt{3}h} |f(x_2) - H_4(x_2)|.$$

Combining the estimates (8.30)-(8.35), we obtain the estimate for $\partial_x H_5$.

8.3.4. *A 4-th order interpolation in 1D.* We will also use a 4-th order interpolation $H_4(x)$ to approximate f and $\partial_x f$. Below, we restrict the derivation to $x \in [x_0, x_1]$. We have estimated $f - H_4$ in (8.21) and H_4 in (8.24). For the derivatives, we have

$$\partial_x f = \partial_x(f - H_4) + \partial_x H_4 = \partial_x(f - H_4) + \frac{1}{h} \partial_t G_4 \triangleq II_1 + II_2.$$

We have estimated $II_2 = \partial_t G_4$ in (8.29) and (8.33). For II_1 , following the argument at the beginning of Section 8.3.3 and (8.35) and using Rolle's theorem, we get

$$|\partial_x(f(x) - H_4(x))| \leq \frac{1}{6} \|\partial_x^4 f\|_{L^\infty[x_0, x_1]} |(x - x_0)(x - x_1)(x - \xi(x))| \triangleq \frac{1}{6} \|\partial_x^4 f\|_{L^\infty[x_0, x_1]} Q(x),$$

for some $\xi(x) \in [x_0, x_1]$. Without loss of generality, we assume that $x \leq \frac{x_0 + x_1}{2}$. Using $h = x_1 - x_0$, $|x - \xi(x)| \leq \max(x_1 - x, x - x_0) = x_1 - x$, $x = x_0 + t \cdot h$, and (8.70), we get

$$(8.36) \quad \begin{aligned} Q(x) &\leq (x - x_0)(x_1 - x)^2 = h^3 t(1 - t)^2 \leq \frac{4}{27} h^3, \\ |\partial_x(f(x) - H_4(x))| &\leq \frac{4}{27} \cdot \frac{1}{6} h^3 \|\partial_x^4 f\|_{L^\infty[x_0, x_1]} = \frac{2h^3}{81} \|\partial_x^4 f\|_{L^\infty[x_0, x_1]}. \end{aligned}$$

8.4. **Hermite interpolation in 2D.** The estimate in 2D is more involved. Consider

$$x_0 < x_1 < x_2, y_0 < y_1 < y_2, x_2 - x_1 = x_1 - x_0, y_2 - y_1 = y_1 - y_0.$$

The domain D can be decomposed into 4 blocks with size $h_1 \times h_2$. For the 5-th order interpolation, we interpolate f in $(x, y) \in [x_0, x_1] \times [y_0, y_1]$ without loss of generality. For the 4-th order interpolation, we only need the value of f in 1 block $[x_0, x_1] \times [y_0, y_1]$. Denote

$$(8.37) \quad \begin{aligned} D &= [x_0, x_2] \times [y_0, y_2], \quad h_1 = x_1 - x_0, \quad h_2 = y_1 - y_0, \\ t &= (x - x_0)h_1^{-1}, \quad s = (y - y_0)h_2^{-1}. \end{aligned}$$

8.4.1. *Estimate the L^∞ norm.* We first consider the estimate of $\|f\|_{L^\infty}$. We treat y as a parameter and use (8.19) to construct the 4-th and 5-th order interpolations in x : $H_4(x, y), H_5(x, y)$. Using the formula in (8.22), we have

$$(8.38) \quad \begin{aligned} G_5(t, s) &\triangleq H_5(x, y) = H_4(x, y) + (f(x_2, y) - H_4(x_2, y)) \frac{t^2(t-1)^2}{4}, \\ H_4(x, y) &= f(x_0, y)(1-t) + f(x_1, y)t + t^2(t-1)(hf'(x_1, y) - (f(x_1, y) - f(x_0, y))) \\ &\quad + (t-1)^2 t(hf'(x_0, y) - (f(x_1, y) - f(x_0, y))). \end{aligned}$$

Using (8.21), we have the error bound in x

$$(8.39) \quad |H_5(x, y) - f(x, y)| \leq \frac{1}{1200} \|\partial_x^5 f\|_{L^\infty(D)} h_1^5.$$

We need to further interpolate $G_5(t, s), H_5(x, y)$ in the y direction. To achieve the overall 5-th order error, we do not need to apply a high order interpolation to each coefficient. For the linear term, we apply the 5-th order Hermite interpolation to $f(x_i, y)$ in y using $f(x_i, y_j), j = 0, 1, 2$ and $\partial_y f(x_i, y_j), j = 0, 1$ for $i = 0, 1$ and denote it by $A_i^{(5)}(y)$, i.e.

$$(8.40) \quad A_i^{(5)}(y) = H_5(f(x_i, \cdot), y_0, y_1, y_2)(y)$$

using the notation in (8.19). Applying (8.21), we have the error bound in y

$$(8.41) \quad |f(x_i, y) - A_i^{(5)}(y)| \leq \frac{1}{1200} \|\partial_y^5 f\|_{L^\infty(D)} h_2^5.$$

Denote

$$(8.42) \quad M_i(y) \triangleq hf'(x_i, y) - (f(x_1, y) - f(x_0, y)).$$

For $M_i(y)$, $i = 1, 2$, it is of order h_1^2 . Thus, we apply the cubic interpolation in Section 8.1.2 to these functions in the y direction on grids y_0, y_1, y_2 and denote it by $Q_i(y)$. Using the notation (8.7), we have

$$(8.43) \quad Q_i(y) = N_2(M_i, y_0, y_1, y_2)(y).$$

Applying (8.8), we have the error bound

$$(8.44) \quad |M_i(y) - Q_i(y)| \leq \frac{h_2^3}{9\sqrt{3}} \|\partial_y^3 M_i(y)\|_{L^\infty([y_0, y_2])} \leq \frac{h_2^3 h_1^2}{18\sqrt{3}} \|\partial_x^2 \partial_y^3 f\|_{L^\infty(D)},$$

where we have used the following estimate with $c = a, b, g = \partial_y^3 f(\cdot, y)$ in the last inequality (8.45)

$$\left| (b-a)g'(c) - (g(b) - g(a)) \right| = \left| \int_a^b g'(c) - g'(s) ds \right| \leq \|g''\|_{L^\infty[a, b]} \int_a^b |c-s| ds = \frac{(b-a)^2}{2} \|g''\|_{L^\infty[a, b]}.$$

The last term $f(x_2, y) - H_4(x_2, y)$ is already very small and of order h_1^4 . From (8.38), since ∂_y commutes with $t = (x - x_0)h^{-1}$ and ∂_x , for a fixed y , $\partial_y H_4(x, y)$ is the Hermite polynomials for $\partial_y f(x, y)$ in x . We use the first order estimate in y and then (8.20) with $\partial_y f$ and $x = x_2$ to obtain

$$(8.46) \quad \begin{aligned} |f(x_2, y) - H_4(x_2, y)| &\leq \max_{j=0,1} |f(x_2, y_j) - H_4(x_2, y_j)| + \frac{h_2}{2} |\partial_y(f(x_2, y) - H_4(x_2, y))| \\ &\leq \max_{j=0,1} |f(x_2, y_j) - H_4(x_2, y_j)| + \frac{h_2 h_1^4}{2 \cdot 6} \|\partial_y \partial_x^4 f\|_{L^\infty(D)}, \end{aligned}$$

where we have used $x_2 - x_0 = 2h_1, x_1 - x_0 = h_1, \frac{(x_2 - x_0)^2 (x_1 - x_0)^2}{24} = \frac{4h_1^2 \cdot h_1^2}{24} = \frac{h_1^4}{6}$ in (8.20).

Estimate the 2D interpolating polynomials for f . We obtain the 2D interpolating polynomials in $(x, y) \in [x_0, x_1] \times [y_0, y_1]$ for $H_4(x, y)$ as follows (8.47)

$$P_5(x, y) = \left(A_0^{(5)}(y)(1-t) + A_1^{(5)}(y)t \right) + \left(t^2(t-1)Q_1(y) + t(t-1)^2Q_0(y) \right) = I(t, y) + II(t, y).$$

We restrict $t \in [0, 1], x \in [x_0, x_1], y \in [y_0, y_1]$. Applying (8.39) for $H_5 - f$, (8.46) for $H_4 - H_5$ in (8.38), (8.41)-(8.44) for $H_4 - P_5$, $t + (1-t) = 1$, we have the following error bound (8.48)

$$(8.48) \quad \begin{aligned} |P_5(x, y) - f(x, y)| &\leq |H_4(x, y) - H_5(x, y)| + |H_5(x, y) - f(x, y)| + |H_4(x, y) - P_5(x, y)|, \\ |H_4 - H_5| &\leq \frac{1}{64} |f(x_2, y) - H_4(x_2, y)| \leq \frac{1}{64} \max_{j=0,1} |f(x_2, y_j) - H_4(x_2, y_j)| + \frac{h_2 h_1^4}{768} \|\partial_y \partial_x^4 f\|_{L^\infty(D)}, \\ |H_5 - f| + |H_4 - P_5| &\leq \frac{1}{72\sqrt{3}} h_1^2 h_2^3 \|\partial_x^2 \partial_y^3 f\|_{L^\infty(D)} + \frac{1}{1200} (h_1^5 \|\partial_x^5 f\|_{L^\infty(D)} + h_2^5 \|\partial_y^5 f\|_{L^\infty(D)}), \end{aligned}$$

where the constant $\frac{1}{64}$ in $H_4 - H_5$ is from (8.25), $\frac{1}{768}$ comes from (8.46) and $\frac{1}{768} = \frac{1}{64} \cdot \frac{1}{16}, \frac{1}{72\sqrt{3}}$ from $\frac{1}{18\sqrt{3}}(t^2|t-1| + (t-1)^2t) = \frac{1}{18\sqrt{3}}(1-t)t \leq \frac{1}{72\sqrt{3}}$ (8.44). To estimate $P_5(x, y)$ (8.47), we use estimates similar to (8.24). The estimate of the linear part is trivial

$$(8.49) \quad |I(t, y)| \leq \max_{i=0,1} \|A_i^{(5)}(y)\|_{L^\infty[y_0, y_1]}, \quad y \in [y_0, y_1].$$

Since $A_i(y)$ (8.40) is the Hermite polynomial in y , we can use the method in Section 8.3 to estimate it. For II , following the derivations between (8.23) to (8.24) and by considering two cases: $t \leq \frac{1}{2}$ and $t > \frac{1}{2}$, we obtain

$$(8.50) \quad |II(t, y)| \leq \frac{4}{27} \max(|Q_1(y) - Q_0(y)|, |Q_1(y)|, |Q_0(y)|).$$

Since $Q_i, Q_1 - Q_0$ are quadratic interpolating polynomials of $M_i, M_1 - M_0$ (8.42) on y_0, y_1, y_2 , we can use (8.9) to estimate the $L^\infty[y_0, y_1]$ norm.

8.4.2. *Estimates of the $\partial f(x, y)$ in $2D$.* Now, we consider how to estimate $\partial f(x, y)$ using the Hermite interpolation. We consider ∂_x without loss of generality. Recall the notation (8.37). We first fix y as a parameter and interpolate $f(x, y)$ in x using the same method as (8.38). Using the error estimate (8.27), we yield

$$(8.51) \quad |\partial_x(f(x, y) - H_5(x, y))| \leq \frac{h_1^4}{96} \|\partial_x^5 f\|_{L^\infty(D)}.$$

Using the computation (8.28), (8.29), (8.34) in Section 8.3.3 and the notation (8.18) for m_i, l_i , we have

$$(8.52) \quad \begin{aligned} h_1 \partial_x H_5(x, y) &= h_1 \partial_x H_4(x, y) + (f(x_2, y) - H_4(x_2, y))t(t-1)(t-\frac{1}{2}), \\ h_1 \partial_x H_4(x, y) &= f(x_1, y) - f(x_0, y) + (3t^2 - 2t)(hf'(x_1, y) - (f(x_1, y) - f(x_0, y))) \\ &\quad + (3t^2 - 4t + 1)(hf'(x_0, y) - (f(x_1, y) - f(x_0, y))) \\ &= f(x_0, y) - f(x_1, y) + (3t^2 - 2t)M_1(y) + (3t^2 - 4t + 1)(M_0(y)), \end{aligned}$$

where $t = \frac{x-x_0}{h_1}$ (8.37), and we have used M_i (8.42) to simplify the presentation.

Next, we interpolate the above functions in y . We want to achieve an overall 4-th order approximation for $\partial_x H_4(x, y)$. For $f(x_1, y) - f(x_0, y)$, we use the 4-th order Hermite interpolation in y based on the grid point values $f(x_1, y_j) - f(x_0, y_j), \partial_y(f(x_1, y_j) - f(x_0, y_j)), j = 0, 1$ and denote it as $B(y)$, i.e.

$$(8.53) \quad B(y) \triangleq H_4(f(x_1, \cdot) - f(x_0, \cdot), y_0, y_1),$$

using the notation (8.19). By (8.21), we have the error estimate

$$(8.54) \quad \left| \frac{B(y)}{h_1} - \frac{f(x_1, y) - f(x_0, y)}{h_1} \right| \leq \frac{h_2^4}{384} \left\| \partial_y^4 \frac{f(x_1, y) - f(x_0, y)}{h_1} \right\|_{L^\infty([y_0, y_1])} \leq \frac{h_2^4}{384} \left\| \partial_y^4 \partial_x f \right\|_{L^\infty(D)}.$$

For M_i , we apply the same quadratic interpolation Q_i in y (8.43). Using the error bound (8.44) and (8.68) in Lemma 8.1, we obtain

$$\begin{aligned} & h_1^{-1} \left| (3t^2 - 2t)M_1(y) + (3t^2 - 4t + 1)M_0(y) - (3t^2 - 2t)Q_1(y) - (3t^2 - 4t + 1)Q_0(y) \right| \\ & \leq h_1^{-1} (|3t^2 - 2t| + |3t^2 - 4t + 1|) \max_{i=1,2} |Q_i - M_i| \leq \frac{h_2^3 h_1}{18\sqrt{3}} \|\partial_x^2 \partial_y^3 f\|_{L^\infty(D)}. \end{aligned}$$

For $f(x_2, y) - H_4(x_2, y)$, we use the same estimate (8.46), which along with (8.34) implies

$$(8.55) \quad \begin{aligned} \frac{1}{h_1} \left| (f(x_2, y) - H_4(x_2, y))t(t-1)(t-\frac{1}{2}) \right| &\leq \frac{1}{12\sqrt{3}h_1} \max_{j=0,1} |f(x_2, y_j) - H_4(x_2, y_j)| \\ &\quad + \frac{h_2 h_1^3}{12 \cdot 12\sqrt{3}} \|\partial_y \partial_x^4 f\|_{L^\infty(D)}. \end{aligned}$$

Estimate the 2D interpolating polynomials for $\partial_x f$. Now, we use (8.53), (8.43) to construct the interpolating polynomials for $h_1 \partial_x H_4$

$$(8.56) \quad S_4(x, y) = B(y) + (3t^2 - 2t)Q_1(y) + (3t^2 - 4t + 1)Q_0(y).$$

Combining the estimate (8.54) and using the triangle inequality, we can estimate the error $\frac{1}{h_1} S_4(x, y) - f(x, y)$.

It remains to estimate $S_4(x, y)$. We further decompose the above approximation as the linear part and the nonlinear part in y . The linear part in y is the main term, and we want to obtain a sharper estimate. Since B is the 4-th order Hermite interpolation in y (8.53), we can apply the decomposition (8.22) into the linear and the nonlinear parts to B . Since Q_i is the quadratic interpolation of M_i (8.42), (8.43), we can apply the decomposition (8.7) into the linear and the quadratic terms to Q_i

$$(8.57) \quad S_4 = S_{lin} + S_{nlin}, \quad S_\sigma(t, y) \triangleq B_\sigma(y) + (3t^2 - 2t)Q_{1,\sigma}(y) + (3t^2 - 4t + 1)Q_{0,\sigma}(y),$$

where $\sigma \in \{lin, nlin\}$, f_{lin}, f_{nlin} denote the linear part and nonlinear part in y , respectively.

Since S_{lin} is linear in y and $B(y_j) = f(x_1, y_j) - f(x_0, y_j)$, $Q_i(y_j) = M_i(y_j)$ for $j = 0, 1$ (the interpolating polynomials agree with the functions on the grid points), we get

$$(8.58) \quad \begin{aligned} |S_{lin}(t, y)| &\leq \max_{i=0,1} \left| B_{lin}(y_i) + (3t^2 - 2t)Q_{1,lin}(y_i) + (3t^2 - 4t + 1)Q_{0,lin}(y_i) \right|, \\ &= \max_{i=0,1} \left| f(x_1, y_i) - f(x_0, y_i) + (3t^2 - 2t)M_1(y_i) + (3t^2 - 4t + 1)M_0(y_i) \right|. \end{aligned}$$

Recall the definition of M_i in (8.42). The polynomials inside $|\cdot|$ is the 1D polynomial in t with the same form as $I(t)$ (8.29). We estimate it using (8.33), (8.35) and following Section 8.3.3.

For the nonlinear part, we have

$$(8.59) \quad S_{nlin} = B_{nlin}(y) + (3t^2 - 2t)Q_{1,nlin}(y) + (3t^2 - 4t + 1)Q_{0,nlin}(y),$$

we recall the definition of B (8.53) and Q (8.43). The estimate of B_{nlin} follows the method of estimating $II(t)$ in (8.22), (8.24) with l_i, m_i replaced by

$$\tilde{l}_i = f(x_1, y_i) - f(x_0, y_i), \quad \tilde{m}_i = h_2 \partial_y (f(x_1, y_i) - f(x_0, y_i)),$$

which gives

$$(8.60) \quad |B_{nlin}(y)| \leq \frac{4}{27} \max(|\tilde{m}_0, \tilde{m}_1|, |\tilde{m}_0 - (\tilde{l}_1 - \tilde{l}_0)|, |\tilde{m}_1 - (\tilde{l}_1 - \tilde{l}_0)|).$$

The estimate of the Q_{nlin} follows that in (8.9) and Section 8.1.2, which gives

$$|Q_{i,nlin}| \leq \frac{1}{8} |M_i(y_0) - 2M_i(y_1) + M_i(y_2)|.$$

Using (8.68) in Lemma 8.1, for $t \in [0, 1]$, we prove

$$\begin{aligned} &|(3t^2 - 2t)Q_{1,nlin}(y) + (3t^2 - 4t + 1)Q_{0,nlin}(y)| \\ &\leq (|3t^2 - 2t| + |3t^2 - 4t + 1|) \max_{i=0,1} |Q_{i,nlin}(y)| \leq \frac{1}{8} \max_{i=0,1} |M_i(y_0) - 2M_i(y_1) + M_i(y_2)|. \end{aligned}$$

Thus, we yield the estimate of S_{nlin} . Combining the above estimates of S_{lin} and S_{nlin} and using (8.57), we obtain the estimate of S_4 . We remark that one needs to further divide the above bounds by $\frac{1}{h_1}$ to get the bound for $\frac{S_4}{h_1}$.

8.4.3. The fourth order estimate. Denote $D = [x_0, x_1] \times [y_0, y_1]$. We only need the value of f in D . The above method interpolates $f(x, y)$ with a fifth order error estimate. We also use a simpler fourth order method to estimate $f(x, y)$. Recall the interpolation polynomials H_4 and M_i in x defined in (8.38), (8.42). We consider the following interpolation for $f(x, y)$

$$(8.61) \quad P_4(x, y) = \left(A_0^{(4)}(y)(1-t) + A_1^{(4)}(y)t \right) + \left(t^2(t-1)l_1(y) + t(t-1)^2l_0(y) \right) = I(t, y) + II(t, y),$$

similar to (8.47), where l_i is a linear interpolation of M_i in y , $A^{(4)}$ is the 4-th order Hermite interpolation (8.19) of $f(x_i, y)$ in y using $f(x_i, y_j), j = 0, 1$ and $\partial_y f(x_i, y_j), j = 0, 1$ for $i = 0, 1$

$$(8.62) \quad A_i^{(4)}(y) = H_4(f(x_i, \cdot), y_0, y_1)(y).$$

For a fixed t , applying the linear interpolation error bound for $M_i - l_i$ and (8.45), we yield

$$(8.63) \quad |l_i - M_i| \leq \frac{h_2^2}{8} \|\partial_{yy} M_i\|_{L^\infty(D)} \leq \frac{h_2^2 h_1^2}{16} \|\partial_x^2 \partial_y^2 f\|_{L^\infty(D)}, \quad i = 1, 2,$$

which along with (8.21) for interpolation error $A_i^{(4)} - f(x_i, y)$ in y , $t(1-t) \leq \frac{1}{4}$ implies

$$|H_4 - P_4| \leq \frac{h_2^4}{384} \|\partial_y^4 f\|_{L^\infty(D)} + (|t^2(t-1)| + |t(1-t)^2|) \max_i |l_i - M_i| \leq \frac{h_2^4}{384} \|\partial_y^4 f\|_{L^\infty(D)} + \frac{h_1^2 h_2^2}{64} \|\partial_x^2 \partial_y^2 f\|_{L^\infty(D)}.$$

Applying (8.21) for the interpolation error $H_4 - f$ in x , we obtain

$$|P_4 - f| \leq |H_4 - f| + |H_4(x, y) - P_4(x, y)| \leq \frac{1}{384} (h_1^4 \|\partial_x^4 f\|_{L^\infty(D)} + h_2^4 \|\partial_y^4 f\|_{L^\infty(D)}) + \frac{h_1^2 h_2^2}{64} \|\partial_x^2 \partial_y^2 f\|_{L^\infty(D)}.$$

To estimate P_4 , we follow the estimates (8.49) and (8.50)

$$|I(y)| \leq \max_{i=0,1} |A_i^{(4)}(y)|_{L^\infty[y_0, y_1]}, \quad |II(y)| \leq \frac{4}{27} \max(|l_1(y) - l_0(y)|, |l_0(y)|, |l_1(y)|).$$

The estimate of the 4-th order interpolation polynomial $A_i^{(4)}(y)$ in y follows (8.24). Since $l_i(y), l_1(y) - l_0(y)$ is linear in y , the maximum of each term is achieved at the endpoint $y = y_0, y = y_1$, and we can bound $l_i, l_1(y) - l_0(y)$ and $II(t)$.

Estimate ∂f . We use $\partial H_4(x, y)$ to approximate ∂f , which has the formula (8.52). We further approximate $h_1 \partial_x H_4$ by constructing interpolation similar to (8.56)

$$\tilde{S}_4 = B(y) + (3t^2 - 2t)l_1(y) + (3t^2 - 4t + 1)l_0(y),$$

with Q_i in (8.56) replaced by the linear interpolation of M_i (8.52), $l_i, B(y)$ is the same as (8.53). Using (8.54), (8.63), and (8.68), we yield

$$\begin{aligned} |\partial_x f - \frac{\tilde{S}}{h_1}| &\leq |\frac{\tilde{S}_4}{h_1} - \partial_x H_4| + |\partial_x f - \partial_x H_4| = I + II, \\ |I| &\leq \left| \frac{B(y)}{h_1} - \frac{f(x_1, y) - f(x_0, y)}{h_1} \right| + \frac{1}{h_1} (|3t^2 - 2t| + |3t^2 - 4t + 1|) \max_i |M_i - l_i| \\ &\leq \frac{h_2^4}{384} \|\partial_y^4 \partial_x f\|_{L^\infty(D)} + \frac{h_1 h_2^2}{16} \|\partial_x^2 \partial_y^2 f\|_{L^\infty(D)}, \end{aligned}$$

where $D = [x_0, x_1] \times [y_0, y_1]$. For II , we apply (8.36). It remains to estimate \tilde{S} . Using the notation in (8.57), $Q_{i,lin} = l_i, B = B_{lin} + B_{nlin}$, we obtain

$$\tilde{S}_4 = B + (3t^2 - 2t)Q_{1,lin} + (3t^2 - 4t + 1)Q_{0,lin} = S_{lin} + B_{nlin},$$

and then estimate \tilde{S}_4 using (8.58) and (8.60). The estimate for $\partial_y f$ is similar.

8.5. Weighted estimates of a function using derivatives. In our estimate of the residual error or some norms, we need to estimate $F(x)\rho(x)$ near $x = 0$ with a singular weight ρ . In this section, we discuss how to use the estimate of the derivatives of F to estimate the weighted norm of F . Note that $\partial_x^i \partial_y^j F$ can be estimated by the methods in Sections 8.1-8.4.

Typical behavior of ρ near $x = 0$ is $|x|^{-k}, k = 2, \frac{5}{2}, 3$ or $|x|^{-k} x_1^{-1/2}$. See (6.4), (6.3). By decomposing $\rho = |x|^{-i} x_1^{-j/2} \rho_m$, where ρ_m is regular near $x = 0$, we only need to estimate $F|x|^{-i} x^{-j/2}$. Denote

$$E_x(F, i)(x, y) \triangleq \frac{1}{x^{i+1}} \int_0^x F(z, y)(x-z)^i dz, \quad E_y(F, i)(x, y) \triangleq \frac{1}{y^{i+1}} \int_0^y F(x, z)(y-z)^i dz.$$

Using Fubini's Theorem, we get

$$E_x(E_y(F, j), i) = E_y(E_x(F, i), j) = \frac{1}{x^{i+1} y^{j+1}} \int_0^x \int_0^y F(x-t)^i (y-s)^j dt ds \triangleq E_{ij}(F)(x, y).$$

Using the piecewise bound of F , we can bound these functions easily. See (8.66).

For $F(x, y)$ odd in x and $\nabla^k F(0) = 0$ for $k \leq 2$, we have the following identities

$$\begin{aligned} F(x, y) &= \int_0^y F_y(x, z) dz + F(x, 0) \\ &= \int_0^x \int_0^y F_{xyy}(t, s)(y-s) dt ds + y \int_0^x F_{xxy}(t, 0)(x-t) dt + \frac{1}{2} \int_0^x F_{xxx}(t, 0)(x-t)^2 dt. \end{aligned}$$

Using the average operators, we yield

$$|F(x, y)| \leq E_{xy}(|F_{xyy}|, 0, 1)xy^2 + x^2 y E_x(|F_{xxy}(\cdot, 0)|, 1) + \frac{x^3}{2} E_x(|F_{xxx}(\cdot, 0)|, 2)(x, 0).$$

Denote $\beta = \arctan(\frac{y}{x}), r = (x^2 + y^2)^{1/2}$. Using these estimates, for $a + b = 3, b \leq 1$, we get

$$(8.64) \quad \begin{aligned} \frac{|F(x, y)|}{r^a x^b} &\leq \frac{1}{2} E_x(|F_{xxx}|, 2)(x, 0) \cos^{3-b}(\beta) + E_{xy}(|F_{xyy}|, 0, 1) \cos \beta^{1-b} \sin^2 \beta \\ &\quad + E_x(|F_{xxy}(\cdot, 0)|, 1) \cos^{2-b} \sin \beta. \end{aligned}$$

Since we have the bounds for these coefficients, e.g., $E_{xy}(|F_{xyy}|, 0, 1)$, by maximizing $\beta \in [0, \pi/2]$, we obtain the bounds for $\frac{F}{r^3}$ and $\frac{F}{r^{5/2} x^{1/2}}$.

Similarly, we can bound $\frac{\partial_x^i \partial_y^j F}{r^k}$, $i + j + k \leq 3$. For odd F , we have

$$(8.65) \quad \begin{aligned} |F| &= \left| \int_0^x F_x(z, y) dz \right| \leq E_x(|F_x|, 0)x, \\ |F| &= \left| \int_0^y F_y(x, z) dz + F(x, 0) \right| = \left| \int_0^x \int_0^y F_{xy}(t, s) dx dy + \int_0^x F_{xx}(t, 0)(x-t) dt \right| \\ &\leq E_{xy}(|F_{xy}|, 0, 0)xy + E_x(|F_{xx}(\cdot, 0)|, 1)(x, 0)x^2, \quad \nabla^k F(0) = 0, k = 0, 1, \\ |F_x| &= \left| \int_0^y F_{xyy}(x, z)(y-z) dz + y \int_0^x F_{xxy}(z, 0) dz + \int_0^x F_{xxx}(z, 0)(x-z) dz \right| \\ &\leq E_y(|F_{xyy}|, 1)y^2 + xyE_x(|F_{xxy}(\cdot, 0)|, 0) + x^2E_x(|F_{xxx}(\cdot, 0)|, 1), \quad \nabla^k F(0) = 0, k \leq 2. \end{aligned}$$

Using estimate similar to (8.64), we can bound $\frac{F}{|x|^2}$, $\frac{F}{|x|^{3/2}|x_1|^{1/2}}$, $\frac{F}{|x|}$.

Weighted derivatives. Similarly, we estimate $\frac{\partial_x F}{|x|^2}$, $\frac{\partial_x F x_j^{1/2}}{|x|^{5/2}}$. For odd F with $\nabla^k F = 0$, $k \leq 2$, using (8.65) with F replaced by F_y , we get

$$|F_y| \leq E_{xy}(|F_{xyy}|, 0, 0)xy + E_x(|F_{xxy}(\cdot, 0)|, 1)(x, 0)x^2.$$

Then, we can use the method in (8.64) to estimate

$$\frac{|x_j|^\alpha \partial_{x_i} F}{|x|^{2+\alpha}} = (g(\beta))^\alpha \frac{\partial_{x_i} F}{|x|^2}, \quad g(\beta) = \cos \beta, j = 1, \text{ or } \sin \beta, j = 2.$$

We also need to estimate $\frac{F_x}{|x|}$ and $\frac{F_y}{|x|}$. Using (8.65) with F replaced by F_y , we get

$$|F_y| \leq E_x(|F_{xy}|, 0)x.$$

For $\frac{F_x}{|x|}$, we have two cases. If $F(x, 0) \equiv 0$, we yield

$$|F_x| \leq E_y(|F_{xy}, 0|)y.$$

Without the vanishing conditions, we require $\nabla F(0, 0) = 0$ and yield

$$\begin{aligned} |F_x(x, y)| &= \left| \int_0^y F_{xy}(x, z) dz + F_x(x, 0) \right| = \left| \int_0^y F_{xy}(x, z) dz + \int_0^x F_{xx}(z, 0) dz \right| \\ &\leq E_y(|F_{xy}|, 0)y + E_x(|F_{xx}(\cdot, 0)|, 0)x. \end{aligned}$$

Then we apply the method in (8.64) to estimate $\frac{\partial_{x_i} F}{|x|}$.

Piecewise bound for powers. Let $x_i = ih$, $I_i = [x_{i-1}, x_i]$. Suppose that $0 \leq f$ with $f(x) \leq f_i$, $x \in I_i$, $i \geq 1$. We can obtain a piecewise bound for

$$(8.66) \quad I_k(f) = \frac{1}{x^{k+1}} \int_0^x f(z)(x-z)^k dz = \frac{1}{x^{k+1}} \int_0^x f(x-z)z^k dz = \int_0^1 f(x(1-t))t^k dt,$$

from above for $x \in I_m = [(m-1)h, mh]$, where we have used a change of variable $z = tx$. We fix $x \in I_m = [(m-1)h, mh]$. We partition $[0, 1]$ into $J_i = [(i-1)/m, i/m]$, $1 \leq i \leq m$. Since for $t \in [(i-1)/m, i/m]$, we have $(m-1)(m-i) = m^2 - m(i+1) + i \geq m(m-i-1)$ and

$$x(1-t) \geq (m-1)h(1-\frac{i}{m}) = \frac{(m-1)(m-i)h}{m} \geq (m-i-1)h, \quad x(1-t) \leq mh(1-\frac{i-1}{m}) = (m-i+1)h.$$

Thus for $t \in J_i$, we get $x(1-t) \in I_{m-i} \cup I_{m-i+1}$, $I_0 = \emptyset$, $f(x(1-t)) \leq \max(f_{m-i}, f_{m-i+1})$. It follows

$$I_k(f) = \sum_{1 \leq j \leq m} \int_{J_j} f(x(1-t))t^k dt \leq \frac{1}{k+1} \sum_{1 \leq i \leq m} \frac{1}{m^{k+1}} (i^{k+1} - (i-1)^{k+1}) \max(f_{m-i}, f_{m-i+1}), \quad f_0 = 0.$$

8.6. Estimate of some explicit polynomials. We use the following bounds for some polynomials in the error estimate of the interpolation.

Lemma 8.1. *We have the following estimates*

$$(8.67) \quad |(t-a)(t-b)| \leq \frac{(b-a)^2}{4}, \quad t \in [a, b],$$

$$(8.68) \quad |3t^2 - 2t| + |3t^2 - 4t + 1| \leq 1, \quad t \in [0, 1],$$

$$(8.69) \quad |t(t-1)(t-2)| \leq \frac{2}{3\sqrt{3}}, \quad t \in [0, 2],$$

$$(8.70) \quad t(t-1)^2 \leq \frac{4}{27}, \quad t \in [0, 1],$$

$$(8.71) \quad |t(t-1)(t-2)(t-3)| \leq 1, \quad t \in [0, 3],$$

$$(8.72) \quad |t^2(t-1)^2(t-2)| \leq \frac{1}{10}, \quad t \in [0, 1].$$

Proof. The proof of (8.67) follows from the inequality of arithmetic and geometric means (AM-GM) or a direct calculation.

For (8.68), firstly, we note that $|a| + |b| = |a+b|$ or $|a-b|$. It suffices to prove $|a+b| \leq 1$ and $|a-b| \leq 1$ for $a = 3t^2 - 2t, b = 3t^2 - 4t + 1$. Since $t^2 - t \in [-1/4, 0], 2t - 1 \in [-1, 1]$, we have

$$a + b = 6t^2 - 6t + 1 \in [-1/2, 1], \quad a - b = 2t - 1 \in [-1, 1],$$

which implies $|a+b|, |a-b| \leq 1$. We prove the desired result.

Denote $s = t - 1 \in [-1, 1]$. Then using the AM-GM inequality, we have

$$t^2(t-1)^2(t-2)^2 = (t-1)^2(t^2-2t)^2 = \frac{1}{2}2s^2(1-s^2)^2 \leq \frac{1}{2}\left(\frac{2s^2+2(1-s^2)}{3}\right)^3 = \frac{4}{27}.$$

Taking the squart root on both sides proves (8.69).

To prove (8.70), applying the AM-GM inequality, we get

$$t(1-t)^2 = \frac{1}{2}2t(1-t)^2 \leq \frac{1}{2}\left(\frac{2t+2(1-t)}{3}\right)^3 = \frac{4}{27}.$$

Denote $s = t(3-t) \in [0, \frac{9}{4}]$. Then we have $|s-1|^2 \in [0, 2]$ and

$$|(t-1)(t-2)t(t-3)| = |s(t^2-3t+2)| = |(2-s)s| = |1-(s-1)^2| \leq 1,$$

which implies (8.71).

To prove (8.72), we use (8.67) with $a = 0, b = 1$ and (8.69) to obtain

$$|t^2(t-1)^2(t-2)| \leq \frac{1}{4} \frac{2}{3\sqrt{3}} = \frac{1}{6\sqrt{3}} < \frac{1}{10},$$

where the last inequality follows from $(6\sqrt{3})^2 = 108 > 100 = 10^2$. □

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