FINITE TIME BLOWUP OF 2D BOUSSINESQ AND 3D EULER EQUATIONS WITH $C^{1,\alpha}$ VELOCITY AND BOUNDARY

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Abstract. Inspired by the recent numerical evidence of a potential 3D Euler singularity [28, 29], we prove the finite time singularity for the 2D Boussinesq and the 3D axisymmetric Euler equations in the presence of boundary with $C^{1,\alpha}$ initial data for the velocity (and density in the case of Boussinesq equations). Our finite time blowup solution for the 3D Euler equations and the singular solution considered in [28, 29] share many essential features, including the symmetry properties of the solution, the flow structure, and the sign of the solution in each quadrant, except that we use $C^{1,\alpha}$ initial data for the velocity field. We use the method of analysis proposed in our recent joint work with Huang in [5] and the simplification of the Biot-Savart law derived by Elgindi in [11] for $C^{1,\alpha}$ velocity to establish the nonlinear stability of an approximate self-similar profile. The nonlinear stability enables us to prove that the solution of the 3D Euler equations or the 2D Boussinesq equations with $C^{1,\alpha}$ initial data will develop a finite time singularity. Moreover, the velocity field has finite energy before the singularity time.

1. Introduction

The three-dimensional (3D) incompressible Euler equations in fluid dynamics describe the motion of ideal incompressible flows in the absence of external forcing. It has been used to model ocean currents, weather patterns, and other fluids related phenomena. Despite their wide range of applications, the question regarding the global regularity of the 3D Euler equations has remained open. The interested readers may consult the excellent surveys [1, 9, 15, 17, 23, 30] and the references therein. The main difficulty associated with the regularity properties of the 3D Euler equations is due to the presence of vortex stretching, which is absent in the 2D Euler equations. To better illustrate this difficulty, we consider the so-called vorticity-stream function formulation:

$$\omega_t + u \cdot \nabla \omega = \omega \cdot \nabla u,$$

where $\omega = \nabla \times u$ is the vorticity vector of the fluid, and $u$ is related to $\omega$ via a Biot-Savart law. It is not difficult to see that $\nabla u$ is related to $\omega$ via a Riesz operator of degree zero and satisfies the property

$$\|\omega\|_{L^p} \leq \|\nabla u\|_{L^p} \leq C_p \|\omega\|_{L^p}, \quad 1 < p < \infty.$$  

Thus, the vortex stretching term $\omega \cdot \nabla u$ formally scales like $\omega^2$. If such nonlinear alignment persists in time, the 3D Euler equations may develop a finite-time singularity. However, due to the nonlocal nature of the vortex stretching term, such nonlinear alignment may deplete itself dynamically (see e.g. [20]). Despite considerable efforts, whether the 3D Euler equations with smooth initial data of finite energy can develop a finite time singularity has been one of the most outstanding open questions in nonlinear partial differential equations.

In [28, 29], Hou and Luo presented some convincing numerical evidence that the 3D Euler equations develop a potential finite time singularity for a class of smooth initial data with finite energy. The potentially singular solutions reported in [28, 29] are concerned with the 3D axisymmetric Euler equations with a solid boundary. The point of the potential singularity is located at the intersection of the solid boundary $r = 1$ and the symmetry plane $z = 0$. The fact that the singularity occurs at a stagnation point could have positively contributed to the formation of the singularity since convection tends to have a stabilizing effect as indicated by
the study in [18, 19]. The presence of the boundary and the odd-even symmetry of the solution along the axial direction also play an important role in generating a stable and sustainable finite time singularity. When viewed in the meridian plane, the point of the potential singularity is a hyperbolic saddle of the flow.

The singularity scenario reported in [28, 29] has generated great interests. In [25], Kiselev and Sverak considered an initial condition possessing similar symmetry properties as the ones proposed by [28, 29] for the 2D Boussinesq equations on a disk. They proved that the gradient of vorticity can achieve double exponential growth for this initial boundary value problem, which is the fastest possible growth rate in time for all times. In [6], Choi-Hou-Kiselev-Luo-Sverak-Yao considered the 1D HL model proposed by Hou and Luo in [21, 28, 29] and proved that the HL model develops a finite time singularity (see also [4] for the finite time singularity for the 1D CKY model). In [24], Kiselev-Ryzhik-Yao-Zlatos considered a family of models with initial data that possess similar symmetry properties as the ones considered in [28, 29]. They showed that any model that is slightly more singular than the 2D Euler would develop a finite time singularity. More results can be found in an excellent survey article [23]. Despite all the previous efforts, there is still lack of theoretical justification of the finite time singularity for the 3D axisymmetric Euler equations reported in [28, 29].

1.1. Main results. In this paper, we prove the finite time singularity of the 3D axisymmetric Euler equations with solid boundary and large swirl for a class of $C^{1,\alpha}$ initial data for the velocity field. The setting of our problem (such as the symmetry properties of the solution, the flow structure, and the sign of the solution in each quadrant) is similar to that considered in [28, 29]. Since the singularity of the 3D axisymmetric Euler equations reported in [28, 29] occurs at the boundary, away from the symmetry axis, it is well known that the 2D Boussinesq equations have the same scaling as that of the 3D axisymmetric Euler equations. Thus, it makes sense to investigate the finite time singularity of the 2D Boussinesq equations.

The main results of this paper are summarized by the following two theorems. In our first main result, we prove finite time blowup of the Boussinesq equations with $C^{1,\alpha}$ initial data for the velocity field and the density.

**Theorem 1.1.** Let $\omega$ be the vorticity and $\theta$ be the density in the 2D Boussinesq equations described by (2.1) - (2.3). There exists $\alpha_0 > 0$ such that for $0 < \alpha < \alpha_0$, the unique local solution of the 2D Boussinesq equations in the upper half plane develops a focusing self-similar singularity in finite time for some initial data $\omega \in C^2_c(\mathbb{R}_+^2), \theta \in C^{1,\alpha}_c(\mathbb{R}_+^2)$. In particular, the velocity field is $C^{1,\alpha}$ with finite energy. Moreover, the self-similar profile $(\omega_\infty, \theta_\infty)$ satisfies $\omega_\infty, \nabla \theta_\infty \in C^\infty$.

In our second result, we prove the finite time singularity formation for the 3D axisymmetric Euler equations with solid boundary and large swirl in a cylinder $D = \{(r,z) : r \leq 1, z \in \mathbb{R}\}$.

**Theorem 1.2.** Consider the 3D axisymmetric Euler equations in the cylinder $r, z \in [0, 1] \times \mathbb{R}$. Let $\omega^0$ be the angular vorticity and $u^0$ be the angular velocity. There exists $\alpha_0 > 0$ such that for $0 < \alpha < \alpha_0$, the unique local solution of the 3D axisymmetric Euler equations given by (9.1) - (9.3) develops a singularity in finite time for some initial data $\omega^0 \in C^\infty_c(D), (u^0)^2 \in C^{1,\alpha}_c(D)$ supported away from the axis $r = 0$ with $u^0 \geq 0$. In particular, the velocity field has finite energy.

**Remark 1.3.** For H"older initial data, the local well-posedness of the solutions follows from the argument in [3] for the 2D Boussinesq equations and [30] for the 3D Euler equations. The Beale-Kato-Majda type blowup criterion still applies to both equations in the specified domain. The time integral of $||\nabla \theta||_{L^\infty}$ controls the breakdown of the solutions in the 2D Boussinesq equations [3] and the time integral of $||\omega||_{L^\infty}$ controls the finite time blowup of the 3D Euler equations [2]. We will control these quantities and show that there exists $T_1, T_2 > 0$ such that $\int_0^{T_1} ||\nabla \theta(t)||_{L^\infty} dt = \infty$ in the 2D Boussinesq equations and $\int_0^{T_2} ||\omega(t)||_{L^\infty} dt = \infty$ in the 3D Euler equations. The solutions remain in the same regularity class as that of the initial data before the blowup time. In particular, the velocity field is in $C^{1,\alpha}$ before the blowup time.

**Remark 1.4.** Both the 2D Boussinesq equations and the 3D axisymmetric Euler equations develop a finite time singularity by the same mechanism. The proofs of Theorems 1.1 and 1.2 are...
essentially the same. The main difference is that we need to control the support of the solution for the 3D Euler equation so that it does not touch the symmetry axis before the blowup time. We also need to establish the elliptic estimate near the singularity point. We will mainly focus on the proof of Theorem 1.1 and the proof of Theorem 1.2 is provided in Section 9.

We remark that the driving mechanism for the finite time singularity that we consider in this paper is essentially the same as that for the 3D axisymmetric Euler equations with solid boundary considered in [28,29]. In both cases, it is the strong compression of the angular velocity $u^\theta$ (swirl) toward the symmetry plane $z=0$ along the axial ($z$) direction on the boundary $r=1$ that creates a large gradient in $u^\theta$. Then the nonlinear forcing term $\partial_z (u^\theta)^2$ induces a rapid growth in the angular vorticity $\omega^\theta$, ultimately leading to a finite time blowup. One advantage of using $C^{1,\alpha}$ initial data for the velocity field is that we can obtain an analytic form of an approximate self-similar blowup solution for small $\alpha$.

The proofs of the above theorems follow the method of analysis described in our recent joint work with Huang in [5]. We first reformulate the problem using an equivalent dynamic rescaling formulation and construct an accurate approximate steady state solution of the dynamic rescaling equations. We use the simplified leading order approximation of the Biot-Savart law derived by Elgindi [11] for $C^{1,\alpha}$ velocity field to derive a leading order system for our dynamic rescaling equations. One of our main contributions is to prove that this leading order system is nonlinearly stable (see Sections 5 and 6). The nonlinear stability of the leading system requires some new ideas that are not required by the analysis of the 3D Euler equations without swirl [11]. Moreover, we show that the lower order terms can be controlled by the damping of the leading order system and the approximate self-similar profile is nonlinear stable.

The key in our stability analysis is to construct appropriately chosen singularly weighted Sobolev norms to prove the linear and nonlinear stability of the approximate steady state. The choice of the singular weights is crucial for us to extract the nearly optimal damping effect from the stretching term in the linearized operator around the approximate steady state solution. Once we obtain the nonlinear stability, we can prove that the perturbation from the approximate steady state will be trapped in a small ball (in the energy norm) for all times. Since we have nonlinear stability, we can truncate the far field of the approximate steady state in the angular vorticity and angular velocity (or density for the 2D Boussinesq equations) so that the corresponding velocity field has finite energy with a perturbation that can be made arbitrarily small. Then the same argument still applies and we can prove the finite time blowup of this truncated approximate steady state solution with finite energy velocity field.

Our analysis has benefited from several important observations made by Elgindi in his recent paper on the finite time blowup of the 3D Euler equations in the free space without swirl [11] and his earlier paper with Jeong [10], although the driving mechanism for finite time singularity in our setting is quite different from that considered by Elgindi. One of the most important benefits of using $C^{1,\alpha}$ data for the velocity field is that the transport term is substantially weakened by choosing a small $\alpha$ [10]. This makes it much easier to control some of the nonlocal terms in the weighted energy estimates. We have also used the elliptic estimates derived by Elgindi in [11].

Our analysis also shows that the coupled system between $u^\theta$ and $\omega^\theta$ introduces a number of new difficulties that are not present in the singularity analysis considered in [11]. First of all, we have to consider a coupled evolution system between $\omega^\theta$ and $u^\theta$, while in the case without swirl, one just needs to consider a scalar evolution equation for $\omega^\theta$. Secondly, we need to construct an explicit form of an approximate steady state solution for our dynamic rescaling equations, which is not available prior to our work. Thirdly, the presence of large swirl introduces additional technical difficulties in proving stability of the linearized operator around the approximate steady state solution. We need to use a combination of weighted Sobolev norm and $L^\infty$ norm to prove nonlinear stability due to the fact that the approximate steady state for the angular velocity (swirl) or density does not decay in certain direction.

The coupled system also introduces several nonlocal terms that are difficult to obtain a sharp estimate. It is crucial for us to exploit the cancellation among various nonlocal terms so that they can be controlled using the limited damping effect from the local part of the linearized operator in the leading order system. The analysis of these nonlocal terms also requires careful
estimates and is sensitive to some absolute constants, see Sections 5 and 6. In the coupled system for \( \omega^\theta \) and \( u^\theta \), there is an extra nonlocal term (the \( c_\omega \) term) due to the normalization condition. This additional nonlocal term requires us to choose our singular weights carefully in a two-stage manner near the origin along the \( R \) direction. Such difficulties are not present in the 3D axisymmetric Euler equations in the free space without swirl.

As we mentioned earlier, the driving blowup mechanism for the smooth initial data considered in \[12\] is due to the nonlinear interaction between the large gradient of \( u^\theta \) (swirl) and \( \omega^\theta \). Thus, the driving mechanism for the finite time singularity in this scenario is very different from that studied by Elgindi in \[11\]. It is well known that for the 3D axisymmetric Euler equations with smooth initial data and without swirl, the quantity \( \omega^\theta/r \) enjoys a conservation property and the 3D axisymmetric Euler equations have global regularity. For the \( C^\alpha \) angular vorticity with small \( \alpha \), we no longer have the conservation property of the quantity \( \omega^\theta/r \) without swirl since \( \omega^\theta/r \) is not well defined. We would like to point out that the finite time singular solution considered in \[11\] has infinite energy. After we completed our work, we learned from Dr. Elgindi that the stability of the self-similar blowup solutions in \[11\] and the construction of finite-energy \( C^{1, \alpha} \) solutions that become singular in finite time have been established recently in \[12\].

1.2. Review of other related works. In the recent works \[13, 14\], Elgindi and Jeong proved finite time singularity formation for the 2D Boussinesq and 3D axisymmetric equations in a physical domain with a corner and \( C^{0, \alpha} \) data. With the presence of the corner, the behavior of the solutions near the corner can be characterized by an exact 1D system that blows up in finite time. The domain we study in this paper does not have a corner. In the case of the 3D Euler equations, our physical domain includes the symmetry axis. We need to obtain strong control of the solution in the entire domain instead of just at the singularity point. In comparison, the domain studied in \[14\] does not include the symmetry axis.

In \[16, 26\], the authors studied a modified 2D Boussinesq equations with \( \theta_x \) in (2.1) replaced by \( \theta/x \) and using a simplified Biot-Savart law. In these works, the simplified Biot-Savart law has a positive kernel and the authors have been able to prove finite time blowup for smooth initial data using a functional argument. It seems difficult to extend these arguments to the original 2D Boussinesq equation since the difference between the simplified Biot-Savart kernel and the original Biot-Savart kernel has the same order as the original Biot-Savart kernel in terms of scaling. In this paper, we do not make any modification of the 2D Boussinesq equations.

The rest of the paper is organized as follows. In Sections 2-4, we provide some basic set-up for our analysis, including the derivation of the leading order system, the dynamic rescaling formulation, the reformulation using the polar coordinate \((R, \beta)\), and the construction of the approximate self-similar solution. Section 5 is devoted to the linear stability analysis of the leading order system. In Section 6, we perform higher order estimates of the leading order system as part of the nonlinear stability analysis. Sections 7 and 8 are devoted to the nonlinear stability analysis of the original system. In Section 9, we extend our analysis for the 2D Boussinesq equations to the 3D axisymmetric Euler equations. Some concluding remarks are provided in Section 10 and some technical estimates are provided in the Appendix.

1.3. Notations. We assume that \( \alpha < 1/10 \). We use \( \beta \) to denote the angle between \( x, y \) and \( r \) to denote the radial \[ \beta = \arctan(y/x), \quad r = \sqrt{x^2 + y^2}. \]

For \( \alpha > 0 \), we denote \[ R = r^\alpha. \]

We denote \[ \Omega(R, \beta, t) = \omega(x, y, t), \quad \eta(R, \beta, t) = (\theta_x)(x, y, t), \quad \xi(R, \beta, t) = (\theta_y)(x, y, t). \]

We use \( \langle \cdot, \cdot \rangle, \| \cdot \|_{L^2} \) to denote the inner product in \((R, \beta)\) and its associated \(L^2\) norm

\[
(1.3) \quad \langle f, g \rangle = \int_0^\infty \int_0^{\pi/2} f(R, \beta)g(R, \beta) R dR d\beta, \quad \|f\|_{L^2} = \sqrt{\int_0^\infty \int_0^{\pi/2} f^2(R, \beta) R dR d\beta}.
\]

We remark that we use \( dRd\beta \) in the definition of the inner product rather than \( RdRd\beta \).
We use the notation $A \lesssim B$ if there is some absolute constant $C > 0$ with $A \leq CB$, and denote $A \asymp B$ if $A \lesssim B$ and $B \lesssim A$. The notation $\tilde{\cdot}$ is reserved for the approximate steady states, e.g. $\Omega$ denotes the approximate steady state for $\Omega$. We will use $C, C_1, C_2$ for some absolute constant, which may vary from line to line. We use $K_1, K_2, ...$ and $\mu_1, \mu_2, ...$ to denote some absolute constant which does not vary.

2. Derivation of the leading order system

In this section, we will derive the leading order system that we will use for our analysis later in the paper. We first recall that the 2D Boussinesq equations on the upper half space are given by the following system:

\[
\begin{align*}
\omega_t + u \cdot \nabla \omega &= \theta_x, \\
\theta_t + u \cdot \nabla \theta &= 0,
\end{align*}
\]

where the velocity field $u = (u, v) : \mathbb{R}_+^2 \times [0, T) \to \mathbb{R}_+^2$ is determined via the Biot-Savart law

\[
-\Delta \psi = \omega, \quad u = -\psi_y, \quad v = \psi_x,
\]

with no flow boundary condition

\[
\psi(x, 0) = 0 \quad x \in \mathbb{R}
\]

and $\psi$ is the stream function. The reader should not confuse the vector field $u$ with its first component $u_x$.

The 2D Boussinesq equations have the following scaling-invariant property. If $(\omega, \theta)$ is a solution pair to (2.1)-(2.3), then

\[
\omega_{\lambda, \tau}(x, t) = \frac{1}{\tau} \omega \left( \frac{x}{\lambda}, \frac{t}{\tau} \right), \quad \theta_{\lambda, \tau}(x, t) = \frac{\lambda}{\tau^2} \theta \left( \frac{x}{\lambda}, \frac{t}{\tau} \right)
\]

is also a solution pair to (2.1)-(2.3) for any $\lambda, \tau > 0$.

Next, we will derive the leading order system for the solutions in low Hölder continuous space. There are three reductions to derive the system.

Firstly, we will construct solutions $\omega, \theta_x \in C^\alpha$ with small $\alpha$. In this case, the transport term in (2.5), (2.6) is relatively small for small $\alpha$ in the dynamic rescaling formulation. Secondly, we use the simplification of the Biot-Savart law derived by Elgindi in [11] to derive the leading order terms in the 2D Biot-Savart law. Finally, we will construct initial data $\theta$ so that $\theta$ is anisotropic and $\theta_y$ is relatively small compared to $\theta_x$. We will prove that this property is preserved dynamically. The fact that $\theta$ is anisotropic enables us to focus on the $\omega, \theta_x$ equations (2.5)-(2.6) since the contributions of $\theta_y$ to the whole system (2.5)-(2.7) is of lower order.

2.1. The setup. We search solution for (2.1)-(2.3) with the following symmetry

\[
\omega(x, y) = -\omega(x, -y), \quad \theta(x, y) = \theta(-x, y)
\]

for all $x, y \geq 0$. Accordingly, the stream function $\psi$ (2.3) is odd with respect to $x$

\[
\psi(x, y) = -\psi(-x, y).
\]

It is easy to see that the equations (2.1)-(2.3) preserve these symmetries during time evolution. With these symmetries, it suffices to solve (2.1)-(2.3) on $(x, y) \in [0, \infty) \times [0, \infty)$ with the following boundary conditions

\[
\psi(x, 0) = \psi(0, y) = 0
\]

for the elliptic equation (2.3).

Taking $x, y$ derivative on (2.2), respectively, and recalling (2.1), we yield

\[
\begin{align*}
\omega_t + u \cdot \nabla \omega &= \theta_x, \\
\theta_{xt} + u \cdot \nabla \theta_x &= -u_x \theta_x - v_x \theta_y, \\
\theta_{yt} + u \cdot \nabla \theta_y &= -u_y \theta_x - v_y \theta_y.
\end{align*}
\]

Under the odd symmetry assumption, we have $u(0, y) = 0$. If the initial data $\theta(0, y) = 0$, this property is preserved. Therefore, we can recover $\theta$ from $\theta_x$ by integration. We will perform a-priori estimate of the above system, which is formally a closed system about $(\omega, \theta_x, \theta_y)$. 

\[
\text{Finite time blowup of 2D Boussinesq equations 5}
\]
2.2. Reformulation using polar coordinates. In this section, we reformulate (2.5)-(2.7) using the polar coordinates as \[11\]. We introduce 
\[ r = \sqrt{x^2 + y^2}, \quad \beta = \arctan(y/x), \quad R = r^\alpha \]
for some small $\alpha$ and $\Psi = \frac{1}{2\pi} \psi$. Notice that 
\[ r \partial_r = \alpha R \partial_R. \]
We have
\[
\begin{align*}
\partial_r &= \cos(\beta) \partial_r - \frac{\sin(\beta)}{r} \partial_\beta = \frac{\cos(\beta)}{r} \alpha R \partial_R - \frac{\sin(\beta)}{r} \partial_\beta, \\
\partial_\beta &= \sin(\beta) \partial_r + \frac{\cos(\beta)}{r} \partial_\beta = \frac{\sin(\beta)}{r} \alpha R \partial_R + \frac{\cos(\beta)}{r} \partial_\beta.
\end{align*}
\]
Then using (2.3), we derive
\[
\begin{align*}
u &= \frac{1}{\pi} \int_R^\infty \int_0^{\pi/2} \sin(2\beta) \Omega(s, \beta) ds d\beta.
\end{align*}
\]
For $\omega \in C^\alpha$ with sufficiently small $\alpha > 0$, the leading order term in $\Psi$ is given by the first term on the right hand side. The lower order terms (l.o.t.) are relatively small compared to the first term and we will control them later using the elliptic estimate. We will perform the $L^2$ estimate for the solution of (2.10) and one can see that the a-priori estimate blows up as $\alpha \to 0$. For $\alpha = 0$, (2.10) becomes
\[
\begin{align*}
\Psi(R, 0) = \Psi(R, \pi/2) = 0.
\end{align*}
\]
For the transport term in (2.5)-(2.7), we use (2.8) to derive
\[
\begin{align*}
\begin{align*}
\begin{align*} u \partial_x + v \partial_y &\to - (\alpha R \partial_\beta \Psi) \partial_R + (2 \Psi + \alpha R \partial_R \Psi) \partial_\beta.
\end{align*}
\end{align*}
\end{align*}
\]
2.3. Reductions of the Biot-Savart law and the velocity $u^r$, $u^z$. Following [11], we can decompose the modified stream function $\Psi$ as
\[
\begin{align*}
\Psi = \frac{1}{\pi \alpha} \sin(2\beta) \Omega_{12}(\Omega) + \text{lower order terms},
\end{align*}
\]
\[
\begin{align*}
\Omega_{12}(\Omega) = \int_R^\infty \int_0^{\pi/2} \frac{\sin(2\beta) \Omega(s, \beta)}{s} ds d\beta.
\end{align*}
\]
For $\omega \in C^\alpha$ with sufficiently small $\alpha > 0$, the leading order term in $\Psi$ is given by the first term on the right hand side. The lower order terms (l.o.t.) are relatively small compared to the first term and we will control them later using the elliptic estimate. We will perform the $L^2$ estimate for the solution of (2.10) and one can see that the a-priori estimate blows up as $\alpha \to 0$. For $\alpha = 0$, (2.10) becomes
\[
\begin{align*}
L_0(\Psi) = -\partial_\beta \Psi - 4 \Psi,
\end{align*}
\]
with boundary conditions $\Psi(R, 0) = \Psi(R, \pi/2) = 0$, which is self-adjoint and has kernel $\sin(2\beta)$. In this case, to solve $L_0(\Psi) = \Omega$, a necessary and sufficient condition is that $\Omega$ is orthogonal to $\sin 2\beta$. Imposing this constraint when we perform the elliptic estimate leads to the leading order term in $\Psi$ (2.12). Dropping the order $\alpha$ terms in (2.8), (2.9) and the lower order terms in (2.12) as that in [11], we can extract the leading order term of the velocity $u, v$
\[
\begin{align*}
u &= -2r \cos \beta \pi \alpha \Omega_{12}(\Omega) + l.o.t., \quad v = 2r \sin \beta \pi \alpha \Omega_{12}(\Omega) + l.o.t.,
\end{align*}
\]
\[
\begin{align*}
u_x &= -v_y = -\frac{2}{\pi \alpha} L_{12}(\Omega) + l.o.t., \quad u_y = l.o.t., \quad v_x = l.o.t..
\end{align*}
\]
The complete calculation and the formulas of the lower order terms are given in (8.6)-(8.8). Similarly, the leading order term in the transport terms (2.11) is
\[
\begin{align*}
- (\alpha R \partial_\beta \Psi) \partial_R + (2 \Psi + \alpha R \partial_R \Psi) \partial_\beta = -\frac{2}{\pi} \cos(2\beta) L_{12}(\Omega) R \partial_R + \frac{2}{\pi \alpha} \sin(2\beta) L_{12}(\Omega) \partial_\beta + l.o.t..
\end{align*}
\]
Later, we will prove that the self-similar blowup is non-linearly stable and we will control the above lower order terms using the elliptic estimate. These terms will be treated as a small perturbation and are harmless to the self-similar blowup.
2.4. Decoupling and simplifying the system. We will look for solution \( \theta \) of (2.5)-(2.7) such that \( \theta_x \in C^\alpha, \theta_y \) is odd, and \( \theta_y \) is relatively small compared to \( \theta_x \), i.e., \( \theta \) is not isotropic. The reason we do this comes from the following key observation. For instance, if \( \theta_x = \frac{x^\alpha}{1 + (x^2 + y^2)^{\alpha/2}} \) for \( x, y \geq 0 \), then for \( x, y \) close to 0, we have

\[
(2.15) \quad \theta \approx \frac{1}{1 + \alpha} \cdot \frac{x^{1+\alpha}}{1 + (x^2 + y^2)^{\alpha/2}}, \quad |\theta_y| \approx \frac{\alpha}{1 + \alpha} \cdot \frac{xy}{x^2 + y^2} \cdot \frac{x^\alpha (x^2 + y^2)^{\alpha/2}}{(1 + (x^2 + y^2)^{\alpha/2})} \approx \alpha \theta_x.
\]

Compared to \( \theta_x \), \( \theta_y \) is relatively small. Moreover, \( \theta_y \) is weakly coupled with \( \theta_x \), \( \omega \) in (2.5)-(2.7) since

\[
u_y = l.o.t., \quad v_x = l.o.t.,
\]

according to (2.13). We can decouple (2.6), (2.7) as follows

\[
\theta_{xt} + u \cdot \nabla \theta_x = -u_x \theta_x + l.o.t., \quad \theta_{yt} + u \cdot \nabla \theta_y = -u_y \theta_y + l.o.t.,
\]

These key observations motivate us to focus on the system (2.6)-(2.7) about \( \omega, \theta_x \).

We introduce a variable \( \eta(R, \beta, t) = (\theta_x)(x, y, t) \). Using the calculations of \( u_x, v_y \), we use an argument similar to that in [11]. We are looking for approximate solutions \( \Omega \), \( \eta \) in (2.5)-(2.7) as

\[
(2.16) \quad \Omega_t = \frac{2}{\pi} \cos(2\beta)L_{12}(\Omega) R \partial_R \Omega + \frac{2}{\pi \alpha} \sin(2\beta)L_{12}(\Omega) \partial_\beta \Omega = \eta + l.o.t.,
\]

\[
\eta_t = \frac{2}{\pi} \cos(2\beta)L_{12}(\Omega) R \partial_R \eta + \frac{2}{\pi \alpha} \sin(2\beta)L_{12}(\Omega) \partial_\beta \eta = \frac{2}{\pi \alpha} L_{12}(\Omega) \eta + l.o.t.,
\]

where the equations are evaluated at \( (R, \beta) \) with \( R = (x^2 + y^2)^{\alpha/2}, \beta = \arctan(y/x) \). Notice that the first transport term looks much smaller than the other transport term and the nonlinear term which contains a \( 1/\alpha \) factor. It motivates us to neglect it. For the second transport term, we use an argument similar to that in [11]. We are looking for approximate solutions \( \Omega, \eta \) of the form

\[
\Omega(R, \beta, t) = \Gamma(\beta) \Omega_*(R, t), \quad \eta(R, \beta, t) = \Gamma(\beta) \eta_*(R, t), \quad \Gamma(\beta) = (\cos(\beta))^{\alpha}.
\]

For \( \beta \in [0, \pi/2] \), we gain a small factor \( \alpha \) from the angular derivative:

\[
|\sin(2\beta) \partial_\beta \Gamma(\beta)| = |2\alpha \sin^2(\beta)(\cos(\beta))^{\alpha}| \leq 2 \alpha \Gamma(\beta).
\]

Hence, the angular transport term becomes smaller compared to the nonlinear term.

We remark that in the dynamic rescaling formulation, \( \eta \) is comparable to the nonlinear term \( \alpha^{-1}L_{12}(\Omega) \eta \). Therefore, we drop the transport terms and the lower order terms in (2.16) to derive a leading order system about \( \Omega, \eta \)

\[
(2.17) \quad \Omega_t = \eta, \quad \eta_t = \frac{2}{\pi \alpha} L_{12}(\Omega) \eta, \quad L_{12}(\Omega) = \int_0^{\pi/2} \frac{\Omega(s, \beta) \sin(2\beta)}{s} ds d\beta.
\]

It is not difficult to see that if the initial data \( \Omega, \eta \) are non-negative and are odd with respect to \( x \), the solutions preserve these properties during evolution. In the first equation, \( \Omega \) tends to align with \( \eta \) during evolution. Then the nonlinear term in the second equation is of order \( \eta^2 \), which is the driving force of finite time singularity of the leading order system.

3. Self-similar solution of the leading order system

The leading order system (2.17) is crucial in our analysis and it captures the leading behavior of the blowup solution of the Boussinesq equations (2.1)-(2.3). In this section, we construct the self-similar solution of the leading order system (2.17) for \( \Omega, \eta \). Notice that \( L_{12}(\Omega) \) does not depend on the angular component \( \beta \). We look for a self-similar solution in the form

\[
\Omega(R, \beta, t) = (T - t)^{c_\omega} \Omega_*(\frac{R}{(T - t)^{c_{\omega c_1}}} \Gamma(\beta), \quad \eta(R, \beta, t) = (T - t)^{c_{\omega - c_1}} \eta_*(\frac{R}{(T - t)^{c_{\omega c_1}}} \Gamma(\beta),
\]

where \( c_\omega, c_1, c_\theta \) are the scaling parameters. The reason we use scaling \( (T - t)^{c_{\omega c_1}} \) in the space variable \( R \) is that \( R = r^\alpha \) and \( \frac{R}{(T - t)^{c_{\omega c_1}}} = \left( \frac{r}{(T - t)^{c_{\omega c_1}}} \right)^\alpha, \) where \( r = \sqrt{x^2 + y^2} \). \( (T - t)^{c_{\omega c_1}} \) corresponds...
to the scaling of the original variables \(x, y\) and \((T - t)\) is the scaling of \(\theta\) in \(2.17\). See \(2.4\) for the scaling-invariant of the Boussinesq equations.

Plugging the self-similar solutions ansatz into \(2.17\), we obtain

\[
\begin{align*}
- (T - t)^{-1} c_\omega \Omega_\ast(z) \Gamma(\beta) + (T - t)^{-1} a \alpha \Omega_\ast(z) \Gamma(\beta) + (T - t)^{-1} c_\theta \eta_\ast(z) \Gamma(\beta) &= (T - t)^{-1} \Omega_\ast(z) \Gamma(\beta), \\
- (T - t)^{-1} c_\beta \eta_\ast(z) \Gamma(\beta) &= (T - t)^{-1} \Omega_\ast(z) \Gamma(\beta),
\end{align*}
\]

\(3.1\)

which implies

\[
(\rho - \epsilon_c - 1) c_\omega - c_\theta - 1 = c_s + c_\theta - c_\beta,
\]

which implies

\[
c_\omega = -1, \quad c_\theta = c_\beta + 1.
\]

Denote

\[
c = \frac{2}{\pi} \int_0^\pi \Gamma(\beta) \sin(2\beta) d\beta.
\]

Plugging the relations among the scaling parameters into \(3.1\) and factorizing the temporal variable, we derive

\[
\begin{align*}
\alpha \epsilon_c \beta \partial_\beta \Omega_\ast(z) \Gamma(\beta) &= -\Omega_\ast(z) \Gamma(\beta) + \eta_\ast(z) \Gamma(\beta), \\
\alpha \epsilon_c \beta \partial_\beta \eta_\ast(z) \Gamma(\beta) &= -2\eta_\ast(z) \Gamma(\beta) + \frac{c}{\alpha} \eta_\ast(z) \Gamma(\beta) \int_s^\infty \frac{\Omega_\ast(s)}{s} ds.
\end{align*}
\]

\(3.2\)

We can factorize the angular part \(\Gamma(\beta)\) to further simplify the above equations. Surprisingly, the above equations have explicit solutions of the form

\[
\Omega_\ast(z) = \frac{a z}{(b + z)^2}, \quad c_\beta = \frac{1}{\alpha}
\]

(recall that \(z \geq 0\)). We determine \(\eta_\ast\) from the first equation in \(3.2\)

\[
\eta_\ast(z) = \alpha \epsilon_c \beta \partial_\beta \Omega_\ast + \Omega_\ast = z \partial_\beta \Omega_\ast + \Omega_\ast = \frac{2 ab z}{(b + z)^3},
\]

Then \((\Omega_\ast, \eta_\ast)\) solves \(3.2\) exactly if and only if

\[
z \partial_\beta \eta_\ast + 2 \eta_\ast - \frac{c}{\alpha} \eta_\ast \int_s^\infty \frac{\Omega_\ast(s)}{s} ds = 0
\]

which is equivalent to

\[
0 = z \left( -\frac{6 ab z}{(b + z)^3} + \frac{2 ab}{(b + z)^3} \right) + \frac{4 ab z}{(b + z)^3} - \frac{c}{\alpha} \frac{2 ab z}{(b + z)^3 b + z} = -\frac{2 ab (-3 ab + ac) z}{\alpha (b + z)^2}.
\]

Hence, we obtain

\[
a = \frac{3 ab}{c}.
\]

Using the above formula, we can derive the solutions \((\Omega_\ast, \eta_\ast)\) of \(2.17\). We remark that there is a free parameter \(b\) in the solutions \((\Omega_\ast, \eta_\ast)\). After we impose a normalization condition, e.g. the derivative of \(\Omega_\ast\) at \(z = 0\), we can determine \(b\). For simplicity, we choose \(b = 1\) and then \(a\) becomes \(a = 3 \alpha / c\). Consequently, we obtain the following result

Lemma 3.1. The leading order system \((2.17)\) admits a family of self-similar solutions

\[
\Omega(R, \beta, t) = \frac{\alpha}{c} \frac{1}{T - t} \Gamma(\beta) \Omega_\ast \left( \frac{R}{T - t} \right), \quad \eta(R, \beta, t) = \frac{\alpha}{c} \frac{1}{T - t} \Gamma(\beta) \eta_\ast \left( \frac{R}{T - t} \right),
\]

for some \(T > 0\), where

\[
\Omega_\ast(z) = \frac{3 z}{(1 + z)^2}, \quad \eta_\ast = \frac{6 z}{(1 + z)^3}, \quad c = \frac{2}{\pi} \int_0^{\pi/2} \Gamma(\beta) \sin(2\beta) d\beta \neq 0.
\]
We will choose $\Gamma(\beta) = (\cos(\beta))^\alpha$ in the later discussion.

**Properties of $\theta_x, \omega$.** The self-similar profile of the leading order system (2.17) $(\Omega, \eta_u)$ is indeed isotropic in $x, y$ direction. Moreover, $\theta_x$ and $\omega$ are positive in the first quadrant. For $\Gamma(\beta) = (\cos(\beta))^\alpha$, the self-similar profile of $\theta_x$ in the first quadrant is

$$\theta_x = C \alpha \Gamma(\beta) \frac{R}{(1 + R)^3} = C \alpha \left(\frac{|x|}{(1 + (x^2 + y^2)^{\alpha/2})^3}\right),$$

for some constant $C$. If $x^2 + y^2$ is small, the formal argument (2.15) shows that $\theta_x$ is relatively small compared to $\theta_x$. We will estimate it precisely in Lemma A1 in the Appendix. $\theta_x, \omega$ can be extended to $\mathbb{R}^2$ by an odd extension in the $y$ direction. In this case, $\theta_x$ and $\omega$ have a jump at the boundary $y = 0$.

**Hyperbolic flow field.** The leading order of the flow structure corresponding to the self-similar solution of the leading order system can be obtained using (2.13)

$$L_{12}(\Omega)(R, \beta, t) = \frac{\pi \alpha^3}{2} \left[\frac{1}{T - t} + \frac{3}{T - t + 2R}\right] = \frac{\pi \alpha^3}{2} \left[\frac{1}{T - t} + \frac{3}{T - t + 2R}\right] + R, \quad u(x, y, t) = -\frac{3\pi \cos \beta}{(T - t + 2R)} + l.o.t., \quad v(x, y, t) = \frac{3\pi \sin \beta}{(T - t + 2R)} + l.o.t..$$

In the first quadrant, the flow is clockwise since $u < 0, v > 0$. Moreover, the odd symmetry of $\omega$ implies that the flow is hyperbolic near the origin. The hyperbolic flow field, the sign property and the odd symmetry of the vorticity we design are consistent with that in the work of Hou and Luo [29], in which a potential finite time singularity for the 3D axisymmetric Euler equations is observed at a stagnation point on the boundary.

4. THE DYNAMIC RESCALING FORMULATION AND THE APPROXIMATE STEADY STATE

To prove Theorem 1.1 regarding the finite time self-similar blowup, we use the strategy developed in [3]. We reformulate the problem of proving finite time self-similar singularity into the problem of establishing the nonlinear stability of an approximate self-similar profile using the dynamic rescaling equation. In this section, we reformulate the problem using the dynamic rescaling equation and construct an approximate steady state based on the self-similar solution of the leading order system.

4.1. Dynamic rescaling formulation. Let $\omega(x, t), \theta(x, t), u(x, t)$ be the solutions of (2.1)- (2.3). Then it is easy to show that

(4.1) $\tilde{\omega}_x(x, \tau) + c_l(x) \tilde{\omega} + \tilde{\omega}_x \cdot \nabla \tilde{\omega} = 0$, $\tilde{\theta}_x(x, \tau) + c_l(x) \tilde{\theta} + \tilde{\theta}_x \cdot \nabla \tilde{\theta} = 0$,

where $u = (u, v)^T = \nabla^\perp(-\Delta)^{-1} \tilde{\omega}$, $\tilde{x} = (x, y)^T$.

(4.2) $C_\omega(\tau) = \exp \left(\int_0^\tau c_\omega(s) d\tau\right)$, $C_l(\tau) = \exp \left(\int_0^\tau -c_l(s) ds\right)$, $C_\theta = \exp \left(\int_0^\tau c_\theta(s) d\tau\right)$, $t(\tau) = \int_0^\tau C_\omega(s) d\tau$ and the rescaling parameter $c_l(\tau), c_\theta(\tau), c_\omega(\tau)$ satisfies

(4.3) $c_\omega(\tau) = c_l(\tau) + 2c_\omega(\tau)$.

Recall that the Boussinesq equations have scaling-invariant property (2.14) with two parameters. We have the freedom to choose the time-dependent scaling parameters $c_l(\tau)$ and $c_\omega(\tau)$ according to some normalization conditions. After we determine the normalization conditions for $c_l(\tau)$ and $c_\omega(\tau)$, the dynamic rescaling equation is completely determined and the solution of the dynamic rescaling equation is equivalent to that of the original equation using the scaling relationship described in (4.1)-(4.3), as long as $c_l(\tau)$ and $c_\omega(\tau)$ remain finite.

We remark that the dynamic rescaling formulation was introduced in [27, 32] to study the self-similar blowup of the nonlinear Schrödinger equations. This formulation is also called the...
modulation technique in the literature and has been developed by Merle, Raphael, Martel, Zaag and others. It has been a very effective tool to analyze the formation of singularities for many problems like the nonlinear Schrödinger equation \([22, 33]\), the nonlinear heat equation \([34]\), the generalized KdV equation \([31]\), and other dispersive problems. It has recently been applied to prove singularity formation in fluid dynamics, see e.g. \([4, 5, 8, 11]\).

If there exists \(C > 0\) such that for any \(\tau > 0\), \(c_\omega(\tau) \leq -C < 0\) and the solution \(\tilde{\omega}\) is nontrivial, e.g. \(|\tilde{\omega}(\tau, \cdot)|_{L^\infty} \geq c > 0\) for all \(\tau > 0\), then we have

\[
C_\omega(\tau) \leq e^{-C\tau}, \quad t(\infty) \leq \int_0^\infty e^{-Ct} dt = C^{-1} < +\infty,
\]

and that \(|\omega(C(t) x, t(\tau))| = C_\omega(\tau)^{-1}|\tilde{\omega}(x, \tau)| \geq e^{C\tau}|\tilde{\omega}(x, \tau)|\) blows up at finite time \(T = t(\infty)\). If \((\tilde{\omega}(\tau), \bar{\theta}(\tau), c_\omega(\tau), c_\theta(\tau))\) converges to a steady state \((\omega_\infty, \theta_\infty, c_{\omega, \infty}, c_{\theta, \infty})\) of \((4.2)\) as \(\tau \to \infty\), one can verify that

\[
\omega(x, t) = \frac{1}{1 - t} \omega_\infty \left( \frac{x}{1 - t^{c_{\omega, \infty}/c_\omega}} \right), \quad \theta(x, t) = \frac{1}{1 - t^{c_{\theta, \infty}/c_\omega}} \theta_\infty \left( \frac{x}{1 - t^{c_{\omega, \infty}/c_\omega}} \right)
\]

is a self-similar solution of \((2.4) - (2.5)\). To simplify our presentation, we still use \(t\) to denote the rescaled time in the rest of the paper and drop \(\tilde{\cdot}\) in \((4.2)\).

4.2. Reformulation using the \((R, \beta)\) coordinates. Taking \(x, y\) derivative on the \(\theta\) equation in \((4.2)\), we obtain a system similar to \((2.5) - (2.7)\).

\[
\omega_t + (c_\omega \mathbf{x} + \mathbf{u}) \cdot \nabla \omega_x = c_\omega \omega + \theta_x,
\]

\[
\theta_{xt} + (c_\omega \mathbf{x} + \mathbf{u}) \cdot \nabla \theta_x = (c_\theta - c_t - u_x) \theta_x - v_x \theta_y,
\]

\[
\theta_{yt} + (c_\omega \mathbf{x} + \mathbf{u}) \cdot \nabla \theta_y = (c_\theta - c_t - v_y) \theta_y - u_y \theta_x,
\]

where we have dropped \(\tilde{\cdot}\) to simplify the notations. We make a change of variable \(R = r^\alpha, \beta = \arctan(y/x)\) and introduce

\[\Omega(R, \beta, t) = \omega(x, y, t), \quad \eta(R, \beta, t) = (\theta_x)(x, y, t), \quad \xi(R, \beta, t) = (\theta_y)(x, y, t)\]

in \((4.6)\) as we did in Section 2. Notice that the stretching term and the damping term satisfy

\[c_\omega \mathbf{x} \cdot \nabla \omega(x, y, t) = c_\omega \mathbf{r} \cdot \mathbf{\omega}(r, \beta, t) = \alpha c_\omega \mathbf{R} \partial_R \Omega(R, \beta, t), \quad c_\omega \omega(x, y, t) = c_\omega \Omega(R, \beta, t),\]

and similar relations hold for \(\theta_x, \theta_y\). The reformulated system \((4.5)\) under \((R, \beta)\) coordinate reads

\[
\Omega_t + \alpha c_\omega \mathbf{R} \partial_R \Omega + (\mathbf{u} \cdot \nabla) \Omega = c_\omega \Omega + \eta
\]

\[
\eta_t + \alpha c_\omega \mathbf{R} \partial_R \eta + (\mathbf{u} \cdot \nabla) \eta = (2c_\omega - u_x) \eta - v_x \xi
\]

\[
\xi_t + \alpha c_\omega \mathbf{R} \partial_R \xi + (\mathbf{u} \cdot \nabla) \xi = (2c_\omega - v_y) \xi - u_y \eta,
\]

with the Biot-Savart law in the \((R, \beta)\) coordinate \((2.9)\) and \((2.10)\), where we have used \(c_\theta - c_t = 2c_\omega\) \((1.4)\). For now, we do not expand \(\mathbf{u} \cdot \nabla\) using \((2.11)\) and \(u_x, u_y, v_x, v_y\) due to their complicated expressions. Using the same argument as that in Section 2.3 the leading terms in \((4.6)\) are given by

\[
\Omega_t + \alpha c_\omega \mathbf{R} \partial_R \Omega = c_\omega \Omega + \eta + \text{l.o.t.},
\]

\[
\eta_t + \alpha c_\omega \mathbf{R} \partial_R \eta = (2c_\omega + \frac{2}{\pi \alpha} L_{12}(\Omega)) \eta + \text{l.o.t.},
\]

\[
\xi_t + \alpha c_\omega \mathbf{R} \partial_R \xi = (2c_\omega - \frac{2}{\pi \alpha} L_{12}(\Omega)) \xi + \text{l.o.t.},
\]

where we have dropped the transport terms and simplified \(u_x, u_y, v_x, v_y, u/x, v/y\) using \((2.13)\). We remark that the first two equations in \((4.7)\) are exactly the dynamic rescaling formulation of the leading order system \((2.17)\).
4.3. Constructing an approximate steady state. Notice that the system \([4.7]\) captures the leading order terms in the system \([4.6]\) and that the self-similar profile of \((2.17)\) corresponds to the steady state of the first two equations in \((4.7)\) after neglecting the lower order terms. It motivates us to use the self-similar solutions of \((2.17)\) in Lemma 3.1 as the building block to construct the approximate steady state of \([4.6]\). Firstly, we construct

\[
\bar{\Omega}(R, \beta) = \frac{\alpha}{c} \Gamma(\beta) \left( \frac{3R}{1+R} \right)^2, \quad \bar{\eta}(R, \beta) = \frac{\alpha}{c} \Gamma(\beta) \left( \frac{6R}{1+R} \right)^3, \quad \bar{c}_l = 1 + 3, \quad \bar{c}_\omega = -1,
\]

\[(4.8)\]

\[
\Gamma(\beta) = (\cos(\beta))^{\alpha}, \quad c = \frac{2}{\pi} \int_0^{\pi/2} \Gamma(\beta) \sin(2\beta) d\beta.
\]

Notice that \((\bar{\Omega}, \bar{\eta})\) is a solution of \((5.2)\) with \(c_1 = \frac{1}{\alpha}\). We modify \(\bar{c}_l\) so that the approximate error vanishes quadratically near \(R = 0\), which will be discussed later. The corresponding \(\bar{\theta}\) can be obtained by integrating \(\bar{\theta}_x\) with condition \(\bar{\theta}(0, y) = 0\), which is discussed in Appendix A.3 and \(\bar{u}, \bar{v}\) are obtained from the Biot-Savart law \((2.9), (2.10)\). We can derive the leading order terms using \((2.12)\) and \((2.13)\)

\[
L_{12}(\bar{\Omega}) = \int_R^\infty \int_0^{\pi/2} \frac{\bar{\Omega}(s, \beta) \sin(2\beta)}{s} ds = \frac{\pi}{2} \frac{3\alpha}{1 + R} \Psi = \frac{\sin(2\beta)}{2} \left( \frac{3}{1 + R} + l.o.t., \right.
\]

\[
\bar{u}_x = -\bar{v}_y = -\frac{2}{\pi \alpha} L_{12}(\bar{\Omega}) + l.o.t. = \frac{3}{1 + R} + l.o.t., \quad \bar{u}_y, \bar{v}_x = l.o.t.
\]

We will explain later why we choose the above \(\Gamma(\beta)\). Lemma A.1 in the Appendix shows that \(\Gamma(\beta)\) is essentially equal to the constant 1 in some weighted norm.

We define the error of the approximate steady state below

\[
\bar{F}_\omega \triangleq \bar{c}_\omega \bar{\Omega} + \bar{\eta} - \alpha \bar{c}_l R \bar{\partial}_x \bar{\Omega} - (\bar{u} \cdot \nabla) \bar{\Omega},
\]

\[
\bar{F}_\eta \triangleq (2\bar{c}_\omega - \bar{u}_x) \bar{\eta} - \bar{v}_x \bar{\xi} - \alpha \bar{c}_l R \bar{\partial}_y \bar{\eta} - (\bar{u} \cdot \nabla) \bar{\eta},
\]

\[
\bar{F}_\xi \triangleq (2\bar{c}_\omega - \bar{v}_y) \bar{\xi} - \bar{u}_y \bar{\eta} - \alpha \bar{c}_l R \bar{\partial}_x \bar{\xi} - (\bar{u} \cdot \nabla) \bar{\xi}.
\]

\[(4.10)\]

The criteria to choose \(\Gamma\) in \((4.8)\) is that \(F_{\omega}, F_\eta, F_\xi\) vanish quadratically near \(R = 0\) since we will perform energy estimates with a singular weight in the later sections. Using the formula \((2.11)\) for \(\bar{u} \cdot \nabla\) and \((4.9)\), one can obtain the following expansion of \(\bar{F}_\omega\) near \(R = 0\)

\[
\bar{F}_\omega = -3\alpha R \bar{\partial}_x \bar{\Omega} - (\bar{u} \cdot \nabla) \bar{\Omega} = \frac{9\alpha R}{c} (\alpha \Gamma \cos(2\beta) - \sin(2\beta) \bar{\partial}_x \Gamma - \alpha \Gamma) + O(R^2),
\]

where we have used the explicit formula \((4.8)\) in the first equality and the factor 3 comes from \(\bar{c}_l = \frac{1}{\alpha} + 3\) in \((4.8)\). In order for \(\bar{F}_\omega\) to vanish quadratically near \(R = 0\), we have no choice but to set the coefficient in the \(O(R)\) term to be zero, which gives

\[
\alpha \Gamma \cos(2\beta) - \sin(2\beta) \bar{\partial}_x \Gamma - \alpha \Gamma = 0.
\]

To solve the above first order ODE for \(\Gamma\), we choose the boundary condition \(\Gamma(\pi/2) = 0\) and requires \(\Gamma(\beta) > 0\) for \(\beta \in (0, \pi/2]\). The solution of this ODE is exactly given by the formula of \(\Gamma(\beta)\) in \((4.8)\). As we can see, such choice of \(\Gamma\) is unique and is a consequence of the condition that \(\bar{F}_\omega = O(R^2)\) near \(R = 0\). This condition plays an essential role in our stability analysis for the approximate self-similar profile. With this \(\Gamma(\beta)\), we also have \(\bar{F}_\eta, \bar{F}_\xi = O(R^2)\) near \(R = 0\). We justify these rigorously in Section 8.

Remark 4.1. There is another possible choice of \(\Gamma(\beta)\) in \((4.8)\) by changing the scaling parameters \(c_1, c_\omega\) to \(\bar{c}_1 = \frac{1}{\alpha}, \bar{c}_\omega = -1\) in \((4.8)\). In this case, in order for \(\bar{F}_\omega\) to vanish quadratically near \(R = 0\), one will obtain \(\Gamma(\beta) = (\sin(2\beta))^{\alpha/2}\). For this \(\Gamma\), the approximate profile \(\bar{\Omega}, \bar{\eta}\) vanishes both on \(x\) and \(y\) axes. We do not use this profile since the corresponding \(\bar{\theta}_y = \bar{\xi}\) behaves like \(y^{\alpha/2 - 1}\) near \(y = 0\), which is singular. In this case, \(-\bar{v}_x \bar{\xi}\) in \((4.3)\) is more singular than other terms, which is difficult to control.
12  JIAJIE CHEN AND THOMAS Y. HOU

4.4. Normalization conditions. For initial data \( \tilde{\Omega} + \Omega, \tilde{\eta} + \eta, \tilde{\xi} + \xi \) of (4.10), we treat \( \Omega, \eta, \xi \) as perturbation and choose time-dependent scaling parameters \( c_l + \tilde{c}_l \), \( c_\omega + \tilde{c}_\omega \) as follows

\[
(4.11) \quad c_\omega(t) = \frac{-2}{\pi \alpha} \frac{1}{L_{12}(\Omega(t))}L_{12}(\Omega(t))(0), \quad c_l(t) = \frac{1}{\pi \alpha} \frac{2}{L_{12}(\Omega(t))}L_{12}(\Omega(t))(0) = \frac{1}{\pi \alpha} \frac{2}{L_{12}(\Omega(t))}L_{12}(\Omega(t))(0).
\]

Here, \( c_l(t), c_\omega(t) \) are treated as the perturbation of the scaling parameters \( \tilde{c}_l, \tilde{c}_\omega \). Suppose that \( F_{l_1}(t), F_{\eta}(t), F_{\xi}(t) \) are the time-dependent update in (4.10), i.e.

\[
F_{l_1}(t) = (c_\omega + \tilde{c}_\omega)(\Omega + \tilde{\Omega}) + (\eta + \tilde{\eta}) - \alpha(c_l + \tilde{c}_l)R\partial_R(\Omega + \tilde{\Omega}) + (\langle u + \tilde{u}, \nabla \rangle)(\Omega + \tilde{\Omega}),
\]

and so on. The reason we choose (4.11) is that we want \( F_{l_1}(t), F_{\eta}(t), F_{\xi}(t) \) vanishes quadratically near \( R = 0 \) for any perturbation \( \Omega(t), \eta(t), \xi(t) \) that vanishes quadratically near \( R = 0 \), so that we can choose a singular weight to analyze the stability of the approximate steady state. We will provide rigorous estimates for these terms in Section 8.

5. Linear stability

In this Section, we first linearize the dynamic rescaling formulation in the \((R, \beta)\) coordinates around the approximate steady state \((\Omega, \tilde{\eta}, \tilde{\xi}, \tilde{c}_l, \tilde{c}_\omega)\). Then we establish the linear stability of the leading terms in the linearized system. Throughout this section, we use \( \Omega, \eta, \xi, c_l, c_\omega \) to denote the perturbations around the approximate profile \( (\tilde{\Omega}, \tilde{\eta}, \tilde{\xi}, \tilde{c}_l, \tilde{c}_\omega) \) and assume that \( \Omega \in L^2(\varphi), \eta \in L^2(\psi), \xi \in L^2(\psi) \) for some singular weights \( \varphi, \psi \) to be determined later.

5.1. Linearized system. We linearize (4.10) around \((\tilde{\Omega}, \tilde{\eta}, \tilde{\xi}, \tilde{c}_l, \tilde{c}_\omega)\) (4.8) and derive the equations for the perturbation \( \Omega, \eta, \xi \) as follows

\[
(5.1) \quad \Omega_t + (1 + 3\alpha)R\partial_R \Omega + (\tilde{u} \cdot \nabla) \Omega = -\Omega + \Omega - c_\omega(R - \partial_R \tilde{\Omega}) + (\alpha c_\omega R \partial_R - \langle u, \nabla \rangle) \tilde{\Omega} + \tilde{F}_\Omega + N_\Omega,
\]

\[
\eta_t + (1 + 3\alpha)R\partial_R \eta + (\tilde{u} \cdot \nabla) \eta = (\eta - \tilde{u}_x)\eta - \eta \xi + c_\omega(\eta - \tilde{\eta})R\partial_R \eta - \eta \xi + \tilde{F}_\eta + N_\eta,
\]

\[
\xi_t + (1 + 3\alpha)R\partial_R \xi + (\tilde{u} \cdot \nabla) \xi = (-2 + \tilde{v}_y)\xi + c_\omega(\eta - \tilde{\eta})R\partial_R \xi + \xi - \tilde{u}_y \eta - \tilde{u}_y \eta + \tilde{F}_\xi + N_\xi,
\]

where we have used \( \tilde{c}_l = 1/\alpha + 3, \tilde{c}_\omega = 1 \). \( \alpha c_l(t) = c_\omega(t) - \alpha c_\omega(t) \) (4.11) and \(-\alpha c_l R \partial_R \tilde{\eta} = -c_\omega R \partial_R \eta + c_\omega R \partial_R \eta \) for \( \eta = \tilde{\eta}, \xi, \tilde{\xi} \). The error \( \tilde{F}_\Omega, \tilde{F}_\eta, \tilde{F}_\xi \) are defined in (4.10) and the nonlinear terms are defined below

\[
N_\Omega = c_\omega \Omega + \eta - \alpha c_l R \partial_R \Omega - \langle u, \nabla \rangle \Omega,
\]

\[
N_\eta = (2c_\omega - u_x)\eta - v_x \xi - \alpha c_\omega R \partial_R \eta - \langle u, \nabla \rangle \eta,
\]

\[
N_\xi = (2c_\omega - v_y)\xi - u_y \eta - \alpha c_\omega R \partial_R \xi - \langle u, \nabla \rangle \xi.
\]

We focus on the linearized equation of (5.1). From (2.14) and (1.9), we have

\[
(5.3) \quad 3\alpha R \partial_R + \bar{u} \cdot \nabla = 2\tilde{\Omega} \bar{\partial}_\beta + \{ -\alpha R \bar{\partial}_\beta \tilde{\Omega} \partial_R + \alpha R \partial_R \tilde{\Omega} \bar{\partial}_\beta \} = \frac{3 \sin(2\beta)}{1 + R} \bar{\partial}_\beta + l.o.t.
\]

We will justify the above decomposition using integration by parts to avoid loss of derivatives. We will also show that

\[
(5.4) \quad (\alpha c_\omega R \partial_R - \langle u, \nabla \rangle) \tilde{\Omega}, \quad (\alpha c_\omega R \partial_R - \langle u, \nabla \rangle) \tilde{\eta}, \quad (\alpha c_\omega R \partial_R - \langle u, \nabla \rangle) \tilde{\xi}
\]

in (4.11) are lower order terms. Moreover, we will justify that \( \tilde{\xi} \) is small and is of order \( \alpha^2 \) in Lemma A.7, so that we can treat \( v_x \tilde{\xi} \) as a lower order term in the \( \eta \) equation.

Using (2.14), (4.1), (5.3), (5.4) and then collecting the lower order terms with a small factor \( \alpha \), the error terms \( \tilde{F} \) and the nonlinear terms \( N \) in the remaining term \( \mathcal{R} \), we derive the leading
order terms in the linearized equations
\begin{align}
(5.5) \quad \Omega_t + R\partial_R \Omega + \frac{3 \sin(2\beta)}{1 + R} \partial_\beta \Omega &= -\Omega + \eta + c_\omega(\Omega - R\partial_R \tilde{\Omega}) + \mathcal{R}_\Omega, \\
(5.6) \quad \eta_t + R\partial_R \eta + \frac{3 \sin(2\beta)}{1 + R} \partial_\beta \eta &= (-2 + \frac{3}{1 + R})\eta + \frac{2}{\pi \alpha} L_{12}(\Omega) \bar{\eta} + c_\omega(2\bar{\eta} - R\partial_R \bar{\eta}) + \mathcal{R}_\eta, \\
(5.7) \quad \xi_t + R\partial_R \xi + \frac{3 \sin(2\beta)}{1 + R} \partial_\beta \xi &= (-2 - \frac{3}{1 + R})\xi - \frac{2}{\pi \alpha} L_{12}(\Omega) \bar{\xi} + c_\omega(2\bar{\xi} - R\partial_R \bar{\xi}) + \mathcal{R}_\xi,
\end{align}
where the full expansion of \(\mathcal{R}\) is given in (8.10) and their estimates are deferred to Section 8. In the following subsections, we establish the linear stability for (5.5)-(5.7). The contribution of \(\mathcal{R}\) is small. Using this property, we can further establish the nonlinear stability of the approximate profile \(\tilde{\Omega}\) using a bootstrap argument.

We introduce the following notation \(\Omega\) is defined in (5.8) and \(\bar{\Omega}\) is defined in (5.12). Define the linear operators
\begin{align}
\mathcal{L}_1(\Omega, \eta) &\triangleq -D_R \Omega - \frac{3}{1 + R} D_\beta \Omega + \eta + c_\omega(\Omega - D_R \bar{\Omega}), \\
\mathcal{L}_2(\Omega, \eta) &\triangleq -D_R \eta - \frac{3}{1 + R} D_\beta \eta + (-2 + \frac{3}{1 + R})\eta + \frac{2}{\pi \alpha} L_{12}(\Omega) \bar{\eta} + c_\omega(\bar{\eta} - D_R \bar{\eta}), \\
\mathcal{L}_3(\Omega, \xi) &\triangleq -D_R \xi - \frac{3}{1 + R} D_\beta \xi + (-2 - \frac{3}{1 + R})\xi - \frac{2}{\pi \alpha} L_{12}(\Omega) \bar{\xi} + c_\omega(3\bar{\xi} - D_R \bar{\xi}),
\end{align}
where \(\mathcal{L}_{12}(\Omega)\) is defined in (5.8) and \(\bar{\Omega}, \bar{\eta}\) are defined in (4.8). Define the local part of \(\mathcal{L}_i\) by eliminating \(c_\omega, L_{12}(\Omega)\)
\begin{align}
\mathcal{L}_{10}(\Omega, \eta) &\triangleq -D_R \Omega - \frac{3}{1 + R} D_\beta \Omega + \eta, \quad \mathcal{L}_{20}(\eta) &\triangleq -D_R \eta - \frac{3}{1 + R} D_\beta \eta + (-2 + \frac{3}{1 + R})\eta, \\
\mathcal{L}_{30}(\xi) &\triangleq -D_R \xi - \frac{3}{1 + R} D_\beta \xi + (-2 - \frac{3}{1 + R})\xi.
\end{align}

With the above notations, (5.5)-(5.7) can be reformulated as
\begin{align}
\Omega_t = \mathcal{L}_1(\Omega, \eta) + \mathcal{R}_\Omega, \quad \eta_t = \mathcal{L}_2(\Omega, \eta) + \mathcal{R}_\eta, \quad \xi_t = \mathcal{L}_3(\xi) + \mathcal{R}_\xi,
\end{align}
where we have used the following identities to rewrite
\begin{align}
\frac{2L_{12}(\Omega)}{\pi \alpha} \bar{\eta} + c_\omega(2\bar{\eta} - D_R \bar{\eta}) = \frac{2L_{12}(\Omega)}{\pi \alpha} \bar{\eta} + c_\omega(\bar{\eta} - D_R \bar{\eta}), &\quad \frac{2L_{12}(\Omega)}{\pi \alpha} \bar{\xi} + c_\omega(2\bar{\xi} - D_R \bar{\xi}) = -\frac{2L_{12}(\Omega)}{\pi \alpha} \bar{\eta} + c_\omega(3\bar{\xi} - D_R \bar{\xi})
\end{align}
in (5.5)-(5.7).

5.1.1. Key observations. There are several key observations that play a crucial role in our analysis. Firstly, the leading order terms in the \(\Omega, \eta\) equations (5.5)-(5.6) do not couple the \(\xi\) term, which is consistent with our derivation for the leading order system (2.17).

Secondly, in the \(\xi\) equation, the coupling between \(\Omega\) and \(\xi\) through the nonlocal term \(L_{12}(\Omega)\) and \(c_\omega\) (4.11) is weak due to the fact that \(\xi\) is much smaller than \(\Omega, \bar{\eta}\). After removing these nonlocal terms (5.7) only involves local terms about \(\xi\). By choosing a suitable singular weight, we will show that \(\xi\) is linearly stable up to the weak nonlocal term.
Thirdly, all the nonlocal terms in (5.5)-(5.6), e.g. $c_\omega, L_{12}(\Omega)$, have coefficients with small angular derivative. For example, using (4.8), we have

$$c_\omega(\Omega - R\partial_R\Omega) = c_\omega \cdot \frac{\alpha}{\gamma} \Gamma(\beta) \frac{6R^2}{(1+R)^3}.$$  

We can apply the weighted angular derivative to gain a small factor $\alpha$

$$|\sin(2\beta)\partial_\beta \Gamma(\beta)| = |2\alpha \sin^2(\beta) \Gamma(\beta)| \leq 2\alpha \Gamma(\beta).$$

5.1.2. The angular transport term. To understand the effect of the angular transport term in (5.5)-(5.7), we choose a weight $\varphi(R, \beta) = A(R)(\sin(\beta)^{-\gamma_1}(\cos(\beta))^{-\gamma_2}$ and then perform the $L^2$ estimate and use integration by parts to obtain

$$\frac{1}{2} \frac{d}{dt} \langle \Omega^2, \varphi \rangle = -\left\langle \frac{3\sin(2\beta)}{1+R} \partial_\beta \Omega, \Omega \varphi(R, \beta) \right\rangle + \text{other terms (o.t.)} = \left\langle \frac{3\sin(2\beta)\varphi}{2(1+R)^2}, \Omega^2 \varphi \right\rangle + \text{o.t..}$$

It is not difficult to show that

$$\frac{3\sin(2\beta)\varphi}{2(1+R)^2} \bigg|_{R=0} = 3(1 - \gamma_1) \cos^2(\beta) - 3(1 - \gamma_2) \sin^2(\beta).$$

Suppose that $\gamma_1, \gamma_2 \leq 1$. On one hand, if $\beta$ is small, the angular transport term contributes a growing factor $3(1 - \gamma_1) > 0$ to the energy norm. On the other hand, if $\beta$ is close to $\pi/2$, the above term is negative and we gain a damping factor $-3(1 - \gamma_2)$. It is likely that this damping effect can lead to other design of a singular weight to establish the linear stability, which will be discussed in Section 5.2. In this paper, we do not use this damping effect in the stability analysis.

To establish the linear stability, it is natural to first establish the (weighted) $L^2$ estimate of (5.5)-(5.7). However, the above argument shows that for small $\beta > 0$ the angular transport term destabilizes the profile of the singularity using the singular weights $A(R)(\sin(\beta)^{-\gamma_1}(\cos(\beta))^{-\gamma_2}$ with $\gamma_1 \leq 1$. A possible approach to address this issue in the estimate is to choose $\gamma_1$ close to or larger than 1, i.e. a very singular weight in the $\beta$ direction is desired. In [11], $\gamma_1$ is chosen to be close to $1$ so that such growing factor is minimized. However, for (5.5)-(5.7), due to the presence of the nonlocal term, e.g. $c_\omega(\Omega - R\partial_R\Omega)$, which only vanishes of order $\sin(2\beta)^{\alpha/2}$ near $\beta = 0, \pi/2$, if we use a very singular weight for the angular component $\beta$, such nonlocal term will be very difficult to control. In this case, it seems very difficult to design a suitable norm for weighted $L^2$ estimate and establish the linear stability.

To handle the angular transport term in the $L^2$ estimate, we observe that $\sin(2\beta)\partial_\beta \Omega$ is small since $\Omega$ varies slowly in $\beta$. We expect that a similar smallness result holds for the perturbation term $\sin(2\beta)\partial_\beta \Omega$ and we will justify it in the next subsection. This observation motivates us not to perform integration by parts for the angular transport term in the weighted $L^2$ estimate.

5.2. Estimates of $L_{10}, L_{20}, L_{30}$. We first introduce several singular weights that will be used throughout the paper.

**Definition 5.2.** Define $\varphi_1, \psi_1$ by

$$\varphi_1 \equiv \frac{(1+R)^4}{R^4} \sin(2\beta)^{-\sigma}, \quad \varphi_2 \equiv \frac{(1+R)^4}{R^4} \sin(2\beta)^{-\gamma},$$

$$\psi_1 \equiv \frac{(1+R)^4}{R^4} (\sin(\beta) \cos(\beta))^{-\sigma}, \quad \psi_2 \equiv \frac{(1+R)^4}{R^4} \sin(\beta)^{-\sigma} \cos(\beta)^{-\gamma},$$

where $\sigma = \frac{90}{107}, \gamma = 1 + \frac{19}{10}.$

We will apply the weight $\varphi_1$ to $\Omega, \eta$ and the weight $\psi_1$ to $\xi$. $\varphi_1$ and $\psi_1$ are essentially the same. It is easy to see that $\varphi_1 \lesssim \varphi_2$, $\psi_1 \lesssim \psi_2$. We choose $\psi_2$ less singular than $\varphi_2$ for $\beta$ close to 0 since $\xi$ does not decay in $R$ when $R \sin(\beta)^{\alpha}$ is fixed and $\beta$ is small. See Lemma A.7 about the estimate of $\xi$.

Recall $L_{10}, L_{20}, L_{30}$ in Definition 5.1. The following Lemmas will be used repeatedly.
Lemma 5.3. For some $\delta > 0$, consider the weights
\begin{equation}
\varphi(R, \beta) = \frac{(1 + R)^{4}}{R^{4}} (\sin(2\beta))^{-\delta}, \quad \psi(R, \beta) = \frac{(1 + R)^{4}}{R^{4}} (\sin(\beta))^{-\delta_{1}} (\cos(\beta))^{-\delta_{2}}.
\end{equation}
Assume $\varphi^{1/2} \Omega, \varphi^{1/2} \eta \in L^{2}$. We have
\begin{equation}
\langle L_{10}(\Omega, \eta), \Omega \varphi \rangle + \langle L_{20}(\eta, \eta \varphi) \rangle \leq \left( -\frac{1}{4} + 3|1 - \delta| \right) \langle \Omega^{2}, \varphi \rangle + \langle \eta^{2}, \varphi \rangle.
\end{equation}
Assume that $\psi^{1/2} \xi \in L^{2}$. Then it holds true that
\begin{equation}
\langle L_{30}(\xi, \xi \psi) \rangle \leq \left( -\frac{1}{2} + 3(|1 - \delta_{1}| \vee |1 - \delta_{2}|) \right) \langle \xi^{2}, \psi \rangle,
\end{equation}
where $a \vee b \triangleq \max(a, b)$.

Remark 5.4. The constant $-\frac{1}{4}$ in (5.15) can be improved to $-\frac{1}{4} + \varepsilon$ for any $\varepsilon > 0$ by considering $\lambda_{\varepsilon} \langle L_{10}(\Omega, \eta), \Omega \varphi \rangle + \langle L_{20}(\eta, \eta \varphi) \rangle$ for some $\lambda_{\varepsilon} > 0$, and $-\frac{1}{4}$ in (5.10) can be improved to $-\frac{3}{4}$. Yet, we do not need these sharper estimates.

Proof of Lemma 5.3. By definition of $\varphi, \psi$, we have
\begin{equation}
\frac{(3 \sin(2\beta))\varphi}{2(1 + R)^{2}} = \frac{3}{2(1 + R)} (\sin(2\beta))^{-\delta} (\sin(2\beta))^{-\delta_{1}} (\cos(\beta))^{-\delta_{2}} \leq 3[1 - \delta],
\end{equation}
\begin{equation}
\frac{(3 \sin(2\beta))\psi}{2(1 + R)^{2}} = \frac{3}{2(1 + R)} (\sin(\beta))^{-\delta_{1}} (\sin(\beta))^{-\delta_{2}}.
\end{equation}
Using integration by parts for the transport terms in $L_{10}$ (5.10) and the above calculation, we get
\begin{equation}
\langle L_{10}(\Omega, \eta), \Omega \varphi \rangle = \langle \frac{(R \varphi_{R})}{2 \varphi} + \frac{21(\sin(2\beta))\varphi}{2(1 + R)^{2}}, \Omega^{2} \varphi \rangle - \langle \Omega^{2}, \varphi \rangle + \langle \Omega, \eta \varphi \rangle \leq \left( -\frac{1}{2} - \frac{2}{1 + R} + 3|1 - \delta| - 1, \Omega^{2} \varphi \right) + \langle \Omega, \eta \varphi \rangle = \left( -\frac{1}{2} - \frac{2}{1 + R} + 3|1 - \delta|, \Omega^{2} \varphi \right) + \langle \Omega, \eta \varphi \rangle.
\end{equation}
Similarly, using integration by parts for the transport terms in $L_{20}$ (5.10) and the above calculation, we get
\begin{equation}
\langle L_{20}(\eta, \eta \varphi) \rangle = \langle \frac{(R \varphi_{R})}{2 \varphi} + \frac{21(\sin(2\beta))\varphi}{2(1 + R)^{2}}, (D_{\beta} \eta)^{2} \varphi \rangle + \langle -2 + \frac{3}{1 + R}, \eta^{2} \varphi \rangle \leq \langle 2 \frac{R}{1 + R} - \frac{3}{2} + 3|1 - \delta| + \frac{3}{1 + R}, \eta^{2} \varphi \rangle = \left( -\frac{1}{2} - \frac{R}{1 + R} + 3|1 - \delta|, \eta^{2} \varphi \right).
\end{equation}
We estimate the interaction term between $\Omega, \eta$. Note that
\begin{equation}
4\left( \frac{1}{4} + \frac{2}{1 + R} \right)(\frac{1}{4} + \frac{R}{1 + R}) > \frac{2}{1 + R} + \frac{R}{1 + R} \geq 1.
\end{equation}
Using the Cauchy-Schwarz inequality implies
\begin{equation}
\langle \Omega, \eta \varphi \rangle \leq \left( \frac{1}{4} + \frac{2}{1 + R} \right) \Omega^{2} \varphi \rangle + \left( \frac{1}{4} + \frac{R}{1 + R}, \eta^{2} \varphi \rangle.
\end{equation}
Combining the above estimates, we prove
\begin{equation}
\langle L_{10}(\Omega, \eta), \Omega \varphi \rangle + \langle L_{20}(\Omega, \eta), \eta \psi \rangle \leq \left( -\frac{1}{2} - \frac{2}{1 + R} + 3|1 - \delta|, \Omega^{2} \varphi \rangle + \left( -\frac{1}{2} - \frac{R}{1 + R} + 3|1 - \delta|, \eta^{2} \varphi \rangle \right.
+ \left. \frac{1}{4} + \frac{2}{1 + R} \Omega^{2} \varphi \rangle + \left( \frac{1}{4} + \frac{R}{1 + R}, \eta^{2} \psi \rangle \right) \leq \left( -\frac{1}{4} + 3|1 - \delta| \right) (\langle \Omega^{2}, \varphi \rangle + \langle \eta^{2}, \varphi \rangle)).
\end{equation}
Recall \( \mathcal{L}_{30} \) in Definition 5.1. To prove of (5.16), we use the computations (5.17)-(5.18) to obtain
\[
\langle \mathcal{L}_{30}(\xi), \xi \psi \rangle = \left\langle \frac{(R \psi)_R}{2\psi} + \frac{(3 \sin(2\beta) \psi)_{\beta}}{2(1 + R) \psi} (D_\beta \eta)^2 \psi \right\rangle + \left\langle (-2 - \frac{3}{1 + R}), \eta^2 \psi \right\rangle
\]
\[
\leq \left( \frac{2R}{1 + R} - \frac{3}{2} + 3(1 - \delta_1 \vee |1 - \delta_2|) \right) + \left( -2 - \frac{3}{1 + R} \right), \eta^2 \psi \leq \left( -\frac{1}{2} + 3(1 - \delta_1 \vee |1 - \delta_2|) \right) \eta^2 \psi.
\]

\[\square\]

5.3. \( L^2 \) estimate of the angular derivative \( D_\beta \Omega, D_\beta \eta \). In the next few subsections, we separate the estimates of the system of \( \Omega, \eta \) (5.5)-(5.6) and the equation of \( \xi \) (5.7) since \( \xi \) does not appear in (5.9)-(5.12).

Instead of first performing the weighted \( L^2 \) estimate of the system, we perform the weighted \( L^2 \) estimate of the angular derivative. The reason for doing so is that the linear system of \( D_\beta \Omega, D_\beta \eta \) is local and stable up to some small nonlocal terms of order \( \alpha^{1/2} \) and lower order terms of order \( O(\alpha) \).

**Definition 5.5.** Define an energy \( E(\beta, 1) \geq 0 \) and a remaining term \( \mathcal{R}(\beta, 1) \) by
\[
E(\beta, 1)(\Omega, \eta) \triangleq \langle (D_\beta \Omega)^2, \varphi_2 \rangle + \langle (D_\beta \eta)^2, \varphi_2 \rangle \rangle^{1/2}, \quad \mathcal{R}(\beta, 1) \triangleq \langle D_\beta \mathcal{L}_1 \Omega, \beta \mathcal{L}_2 \Omega, \eta \rangle, \langle D_\beta \mathcal{L}_2 \Omega, \eta \rangle, \langle D_\beta \varphi_2 \rangle.
\]

To simplify the notations, we drop \( \Omega, \eta \) in \( E(\beta, 1) \). The main result in this subsection is the following.

**Proposition 5.6.** Assume that \( \varphi_2^{1/2} D_\beta \Omega, \varphi_2^{1/2} D_\beta \eta \in L^2 \). We have
\[
\langle D_\beta \mathcal{L}_1 (\Omega, \eta), \varphi_2 \rangle + \langle D_\beta \mathcal{L}_2 (\Omega, \eta), \varphi_2 \rangle \leq -\left( \frac{1}{2} - \alpha \right) (E(\beta, 1))^2 + \alpha \mathcal{L}_{12}(\Omega, \eta) + \mathcal{L}_{12}(\Omega) R^{-1} ||R(\varphi_2)||_2^2.
\]

where \( \mathcal{L}_1, \mathcal{L}_2 \) are defined in Definition 5.1.

We will use the following basic property of \( D_\beta = \sin(2\beta) \partial_\beta, \Gamma(\beta) = \cos(\beta) \alpha \) repeatedly
\[
\Gamma(\beta) = -2 \alpha \sin^2(\beta) \cos^2(\beta) \leq 2 \alpha \sin(\beta) \Gamma(\beta), \quad |D_\beta \Gamma(\beta)| \leq 2 \alpha \sin(\beta) \Gamma(\beta).
\]

**Proof.** Notice that the angular transport term in (5.5)-(5.6) can be written as \( \frac{3}{1 + R} D_\beta \) and that \( D_\beta \) commutes with the derivatives in (5.5)-(5.6) and \( \mathcal{L}_{10}, \mathcal{L}_{20} \). We have
\[
D_\beta \mathcal{L}_1 (\Omega, \eta) = D_\beta (\mathcal{L}_{10} (\Omega, \eta) + c_\omega D_\beta (\Omega - R \partial_R \Omega)) = \mathcal{L}_{10} (D_\beta \Omega, D_\beta \eta) + c_\omega D_\beta (\Omega - R \partial_R \Omega),
\]
\[
D_\beta \mathcal{L}_2 (\Omega, \eta) = D_\beta (\mathcal{L}_{20} (\Omega, \eta) + \frac{2}{\pi \alpha} \mathcal{L}_{12} (\Omega) \eta + c_\omega (\Omega - R \partial_R \Omega)) = \mathcal{L}_{10} (D_\beta \Omega, D_\beta \eta) + \frac{2}{\pi \alpha} \mathcal{L}_{12} (\Omega) D_\beta \eta + c_\omega D_\beta (\eta - R \partial_R \eta),
\]
where we have used (5.9). Applying Lemma 5.3 with \( \varphi = \varphi_2 \) and \( \delta = \gamma = 1 + \frac{\alpha}{16} \), we derive
\[
\langle \mathcal{L}_{10} (D_\beta \Omega, D_\beta \eta), (D_\beta \varphi_2) \rangle + \langle \mathcal{L}_{20} (D_\beta \Omega, D_\beta \eta), (D_\beta \varphi_2) \rangle
\]
\[
\leq \left( -\frac{1}{4} + 3(1 - \delta_1) \right) \langle (D_\beta \Omega)^2, \varphi_2 \rangle + \langle (D_\beta \eta)^2, \varphi_2 \rangle \leq \left( -\frac{1}{4} + \alpha \right) \langle (D_\beta \Omega)^2, \varphi_2 \rangle + \langle (D_\beta \eta)^2, \varphi_2 \rangle.
\]

Recall \( c_\omega = -\frac{2}{\pi \alpha} L_{12}(\Omega) \). Using (5.10) in Lemma 5.3 and the Cauchy-Schwarz inequality, we obtain
\[
||c_\omega D_\beta (\Omega - R \partial_R \Omega), (D_\beta \varphi_2) || + ||c_\omega D_\beta (\eta - R \partial_R \eta), (D_\beta \varphi_2) || \leq \alpha^{1/2} ||L_{12}(\Omega)(0)|| \langle ((D_\beta \Omega)^2 + (D_\beta \eta)^2, \varphi_2) \rangle^{1/2}.
\]

Recall the notation \( \mathcal{L}_{12} \). Applying Lemma 5.2 and (A.7) in Lemma A.5 we derive
\[
\langle \left( \frac{2}{\pi \alpha} \mathcal{L}_{12} (\Omega) D_\beta \eta \right)^2, \varphi_2 \rangle^{1/2} \leq \alpha ||L_{12}(\Omega R^{-1})||^2_{L^2(\mathcal{R})}.
\]
Therefore, using the Cauchy-Schwarz inequality, we yield
\[(5.25) \quad \left(\frac{2}{\pi}\tilde{L}_{12}(\Omega)D_{\beta\gamma}D_{\beta}(\eta)\varphi_2\right) \lesssim C_{\alpha} \left\|\tilde{L}_{12}(\Omega)R^{-1}\right\|_{L^2(R)} \left\|(D_{\beta\gamma},\varphi_2)\right\|^{1/2}.
\]
Combining (5.23), (5.24), (5.25) and adding the inner product about two terms in (5.22), we prove
\[
\langle D_{\beta}L_1(\Omega, \eta), (D_{\beta}\Omega, \varphi_2) \rangle + \langle D_{\beta}\bar{L}_2(\Omega, \eta), (D_{\beta\eta}, \varphi_2) \rangle \leq \frac{(1 - \alpha)}{4} \langle (D_{\beta}\Omega)^2, \varphi_2 \rangle + (D_{\beta\eta})^2, \varphi_2 \rangle
\]
\[\quad + C\alpha^{1/2} \left\|\tilde{L}_{12}(\Omega)(0)\right\| \left\|((D_{\beta}\Omega)^2, \varphi_2) + (\langle D_{\beta\eta}, \varphi_2 \rangle)\right\|^{1/2} + C\alpha^{1/2} \left\|\tilde{L}_{12}(\Omega)R^{-1}\right\|_{L^2(R)} \left\|(D_{\beta\eta})^2, \varphi_2 \right\|^{1/2},
\]
where $C$ is some absolute constant. Using the notation $E(\beta, 1)$, the Cauchy-Schwarz inequality concludes the proof of Proposition 5.6 (notice that $-1/4 < -1/5$).

5.4. $L^2$ estimate of $\Omega, \eta$ with a less singular weight. In this subsection, we establish the stability of the weighted $L^2$ estimate of $\Omega, \eta$ with a less singular weight. We begin with a less singular weight in the angular component. Based on this stability estimate, we can further establish the stability with a more singular weight, which will be established in the next subsection. The motivation for starting with a less singular weight is to fully exploit the cancellations among the nonlocal terms, e.g. $c_\omega$ and $\bar{L}_{12}(\Omega)$, and the local terms, e.g. $\eta, \Omega$, which is motivated by our previous joint work with Huang [5] for the De Gregorio model of the 3D Euler equation, where we showed that the nonlocal vortex stretching term $u_\omega \omega$ is harmless to the stability of the profile. In addition, we will establish the damping for $c_\omega, \bar{L}_{12}(\Omega)$.

In the following analysis, we do not perform integration by parts to handle the angular transport term in (5.5)-(5.6) due to the observation in Section 5.1.2. This term can be controlled by an interpolation between $\Omega$ and $D_{\beta}\Omega, \eta$ and $D_{\beta}\eta$. Notice that we already have stability for $D_{\beta}\Omega, D_{\beta}\eta$ from (5.20) up to a small factor.

5.4.1. The worst scenario and the cancellation of the system. Due to the complexity of the nonlocal system (5.5)-(5.6), we identify the worst scenario where the perturbations $\Omega, \eta$ can grow fast and look for possible cancellation among various terms, so that we can estimate the interaction sharply. In (5.5), there is a coupling term $\eta$. $\Omega$ can grow fast if $\Omega$ and $\eta$ have a strong alignment.

Suppose that $\Omega$ and $\eta$ are aligned and have the same sign. From (5.8) and (4.8), we know that the nonlocal vortex stretching term is given by
\[(c_\omega + \bar{L}_{12}(\Omega))\bar{\varphi}_2 = -C \int_0^R \int_0^{\pi/2} \frac{\Omega(\beta, s) \sin(2\beta)}{s} d\beta \cdot \Gamma(\beta) \frac{R}{(1 + R)^4},
\]
for some positive constant $C$. Formally, the above term has a sign that is different from that of $\Omega$. In the case where $\eta$ and $\Omega$ have the same sign, the above term has a sign that is different from that of $\eta$ and we expect that it does not contribute to the growth of $\eta$ in (5.6). In the case where $\eta$ and $\Omega$ have different signs, $\bar{L}_{12}(\Omega)\bar{\varphi}_2$ contributes to the growth of $\eta$ in (5.6), while $\eta$ does not contribute to the growth of $\Omega$ in (5.5). In both cases, there is cancellation between $\eta$ in (5.5) and $\bar{L}_{12}(\Omega)\bar{\varphi}_2$ in (5.6), which motivates us to exploit this cancellation in the energy estimate.

For the nonlocal term about $c_\omega$ in (5.5), using (4.11) and (4.8), we have
\[c_\omega(\Omega - R\partial_R \Omega) = -C \int_0^R \int_0^{\pi/2} \Omega(\beta, s) \sin(2\beta) \frac{d\beta \cdot \Gamma(\beta) R^2}{(1 + R)^4},
\]
for some positive constant $C$. The above term has a sign that is different from that of $\Omega$ and we expect that it has a stabilizing effect to the whole system. Similarly, for the nonlocal term involving $c_\omega$ in (5.6), we have
\[c_\omega(\bar{\varphi}_2 - R\partial_R \bar{\varphi}_2) = -C \int_0^R \int_0^{\pi/2} \Omega(\beta, s) \sin(2\beta) \frac{d\beta \cdot \Gamma(\beta) R^2}{(1 + R)^4},
\]
for some constant $C > 0$. It also has a sign that is different from that of $\eta$ in the case where $\eta$ and $\Omega$ have the same signs. Formally, it should not contribute much to the growth of $\eta$ in (5.6).
This observation motivates us to exploit possible cancellation between \( \eta \) in (5.5) and the above term in (5.6).

We remark that the time dependent normalization conditions (4.11) are important in our analysis and have some stabilizing effect in the dynamic rescaling formulation. We will derive

\[ (5.27) \]

which is exactly the first identity in (5.28). The second identity in (5.28) is a direct consequence of (5.5)-(5.6), where the normalization conditions (4.11) provide a damping factor for \( L_{12}(\Omega)(0) \).

**Definition 5.7.** To exploit the cancellation of the system, we define the following weights

\[
\psi_0 \equiv \frac{6}{8c \pi} \left( R^{-3} + \frac{3 \lambda + R}{2 R^2} \right) = \frac{3}{16} \left( \frac{(1 + R)}{R^4} + \frac{3 (1 + R)}{2 R^3} \right) \Gamma(\beta)^{-1}, \\
\varphi_0 \equiv \frac{(1 + R)}{R^3} \sin(2 \beta), \quad \rho \equiv R^{-3} + R^{-2},
\]

where \( \bar{\eta}, \Gamma(\beta) = \cos^\alpha(\beta) \) are given in (4.8).

Compared to \( \varphi_2 \) in (5.13), the above weights are less singular in the \( R, \beta \) components. The main result in this section is the following

**Proposition 5.8.** Define an energy \( E(R, 0) \) and a remaining term \( \mathcal{R}(R, 0) \)

\[
E(R, 0) = (\Omega^2, \varphi_0) + (\eta^2, \psi_0) + \mu_0 L^2_{12}(\Omega)(0)^{1/2}, \quad \mathcal{R}(R, 0) = (\mathcal{R}_\Omega, \varphi_0) + (\mathcal{R}_\eta, \psi_0) + \mu_0 L_{12}(\Omega)(0)/\langle \mathcal{R}_\Omega, \sin(2 \beta)\rangle + \mu_0 = \frac{81}{4 \pi c}.
\]

Assume that \( \Omega, \eta \) satisfies that \( E(R, 0), E(\beta) < +\infty \). For some absolute constant \( \mu_1 \), we have

\[
\frac{1}{2} \frac{d}{dt}(E(R, 0)^2 + \mu_1 E(\beta, 1)^2) \leq -\left( \frac{1}{9} - C \alpha \right)(E(R, 0)^2 + \mu_1 E(\beta, 1)^2)) - (4 - C\alpha) L^2_{12}(\Omega)(0) - \left( \frac{1}{4} - C \alpha \right) \left\| \tilde{L}_{12}(\Omega) R^{1/2} \right\|_{L^2(R)}^2 + \mathcal{R}(R, 0) + \mu_1 \mathcal{R}(\beta, 1),
\]

where the energy \( E(\beta, 1) \) and the remaining term \( \mathcal{R}(\beta, 1) \) are defined in (5.19).

To exploit the cancellation between \( \eta \) and \( \tilde{L}_{12}(\Omega) \bar{\eta} \) in (5.3)-(5.6), we use the following result.

**Lemma 5.9.** For \( k \in [3/2, 4] \) and any \( \lambda > 0 \), we have

\[
(\sin(2 \beta) \Omega \tilde{L}_{12}(\Omega), R^{-k}) = -\frac{k - 1}{2} \left\| \tilde{L}_{12}(\Omega) R^{-k/2} \right\|_{L^2(R)}^2,
\]

\[
(\sin(2 \beta) \Omega + \lambda \tilde{L}_{12}(\Omega))^2, R^{-k}) = \langle (\sin(2 \beta) \Omega)^2, R^{-k} \rangle - ((k - 1) \lambda - \pi \lambda^2) \left\| \tilde{L}_{12}(\Omega) R^{-k/2} \right\|_{L^2(R)}^2.
\]

**Proof.** From the definition of \( \tilde{L}_{12}(\omega)(R) \) in (5.8), we know that it does not depend on \( \beta \) and

\[
\int_0^{\pi/2} \Omega(s, \beta) \sin(2 \beta) d\beta = -\left( \partial_{\tilde{R}} \tilde{L}_{12}(\Omega) \right) R.
\]

Using integration by parts, we obtain

\[
\langle \sin(2 \beta) \Omega \tilde{L}_{12}(\Omega), R^{-k} \rangle = \int_0^\infty (-\partial_{\tilde{R}} \tilde{L}_{12}(\Omega)(R) \tilde{L}_{12}(\Omega) R^{-k} dR = -\frac{k - 1}{2} \int_0^\infty \tilde{L}_{12}(\Omega)^2 R^{-k} dR,
\]

which is exactly the first identity in (5.28). The second identity in (5.28) is a direct consequence of \( \left( \tilde{L}_{12}^2(\Omega), R^{-k} \right) = \frac{\pi}{2} \left\| \tilde{L}_{12}(\Omega) R^{-k/2} \right\|_{L^2(R)}^2 \) and the first identity.

Next, we proceed to prove Proposition 5.8.

**Remark 5.10.** The careful calculations and estimates to be presented below can be easily verified using Mathematica since we have simple and explicit formulas.
Recall $L_1, L_2$ in Definition 5.1. A direct calculation with weights $\varphi_0, \psi_0$ implies (5.29)
\[
\langle L_1(\Omega, \eta), \Omega \varphi_0 \rangle = -\langle R \partial \Omega, \Omega \varphi_0 \rangle - \langle \Omega^2, \varphi_0 \rangle + \langle \eta, \Omega \varphi_0 \rangle + c_\omega \langle \tilde{\Omega}, R \partial R \tilde{\Omega}, \Omega \varphi_0 \rangle - \left\langle \frac{3}{1 + R} D_\beta \Omega, \Omega \varphi_0 \right\rangle,
\]
\[
\langle L_2(\Omega, \eta) \psi_0 \rangle = -\langle R \partial R \eta, \eta \psi_0 \rangle + \langle -2 + \frac{3}{1 + R}, \eta^2 \psi_0 \rangle + \left\langle \frac{2}{\pi \alpha} \tilde{L}_{12}(\Omega) \tilde{\eta}, \eta \psi_0 \right\rangle + c_\omega \langle \eta - R \partial R \eta, \eta \psi_0 \rangle - \left\langle \frac{3}{1 + R} D_\beta \eta, \eta \psi_0 \right\rangle,
\]
where we have used the notation $D_\beta = \sin(2\beta) \partial_\beta$ to simplify the formula. We treat the sum of the first two terms on the right hand side as the damping terms.

### 5.4.2. The damping terms
We first handle the first two terms on the right hand side of the $L_1$ equation in (5.29). Using integration by parts for $\partial R$ and (5.20), we derive (5.30)
\[
-\langle R \partial R \Omega, \Omega \varphi_0 \rangle - \langle \Omega^2, \varphi_0 \rangle = \left\langle \frac{1}{2} \left( \frac{(1 + R)^3}{R^3} - \frac{(1 + R)^3}{R^3} \right) \right\rangle - \langle -2 + \frac{3}{1 + R}, \eta^2 \psi_0 \rangle = \left\langle -2 R^{-3} - 3 R^{-2} + 1 - \frac{1 + R^3}{R^3}, \Omega^2 \sin(2\beta) \right\rangle = \left\langle -2 R^{-3} - \frac{9}{2} R^{-2} - 3 R^{-1} - \frac{1}{2}, \Omega^2 \sin(2\beta) \right\rangle.
\]
For $\eta$, using integration by parts and (5.20), we have
\[
-\langle R \partial R \eta, \eta \psi_0 \rangle + \langle -2 + \frac{3}{1 + R}, \eta^2 \psi_0 \rangle = \left\langle \frac{1}{2} \left( \frac{3}{16} \frac{(1 + R)^3}{R^2} + \frac{3}{2} \frac{(1 + R)^4}{R^3} \right) \right\rangle \Gamma(\beta)^{-1} + \langle -2 + \frac{3}{1 + R}, \eta \psi_0, \eta^2 \rangle \equiv \langle I + II, \eta^2 \Gamma(\beta)^{-1} \rangle.
\]
Recall $\psi_0$ in (5.20). A direct calculation implies
\[
I = \frac{3}{32} \left( \frac{(1 + R)^3}{R^3} + \frac{3}{2} \frac{(1 + R)^4}{R^2} \right) = \frac{3}{32} \left( \frac{3(1 + R)^2}{R^3} - \frac{3(1 + R)^3}{R^4} + \frac{6(1 + R)^3}{R^2} - \frac{3(1 + R)^4}{R^3} \right),
\]
\[
II = \left( -2 + \frac{3}{1 + R} \right) \frac{3}{32} \left( 2 \frac{(1 + R)^3}{R^4} + \frac{3(1 + R)^4}{R^5} \right) = \frac{3(1 + R)^2}{32 R^4} \left( -2 - 2 R + 3(2 + 3 R(1 + R)) \right),
\]
\[
I + II = \frac{3(1 + R)^2}{32 R^4} \left( -3 - 3 R + 3 R^3 + (1 - 2 R)(2 + 3 R + 3 R^2) \right) = \frac{3(1 + R)^2}{32 R^4} \left( -1 - 4 R - 3 R^2 - 3 R^3 \right).
\]

It follows that
\[
(5.31) \quad -\langle R \partial R \eta, \eta \psi_0 \rangle + \langle -2 + \frac{3}{1 + R}, \eta^2 \psi_0 \rangle = - \left\langle \frac{3(1 + R)^2}{32 R^4} (1 + 4 R + 3 R^2 + 3 R^3), \eta^2 \Gamma(\beta)^{-1} \right\rangle.
\]

### 5.4.3. Estimate of interaction between $\Omega$ and $\eta$
We combine the estimate of $\langle \Omega, \eta \psi \rangle$ and $\langle \frac{2}{\pi \alpha} \tilde{L}_{12}(\Omega) \tilde{\eta}, \eta \psi_0 \rangle$ to exploit the cancellation. Using (4.5) and (5.20), we can compute
\[
I \equiv \left\langle \frac{2}{\pi \alpha} \tilde{L}_{12}(\Omega) \tilde{\eta}, \eta \psi_0 \right\rangle = \left\langle \frac{9}{4 \pi c} \tilde{L}_{12}(\Omega), \eta \left( \frac{1}{R^3} + \frac{3}{2} \frac{1 + R}{R^2} \right) \right\rangle,
\]
\[
II \equiv \langle \Omega, \eta \varphi_0 \rangle = \left\langle \Omega \sin(2\beta), \eta \left( \frac{1}{R^3} + \frac{3}{2} \frac{1 + R}{R^2} + 1 \right) \right\rangle,
\]
where $c$ is defined in (4.8) and satisfies $c = \frac{2}{3} + O(\alpha)$ (see Lemma A.11). Applying the Cauchy-Schwarz inequality, we yield (5.32)

$$I + II = \langle \Omega \sin(2\beta) + \frac{9}{4\pi} \tilde{L}_{12}(\Omega), \eta R^{-3} \rangle + \langle \Omega \sin(2\beta) + \frac{9}{8\pi c} \tilde{L}_{12}(\Omega), 3\eta \frac{R + 1}{R^2} \rangle + \langle \Omega \sin(2\beta), \eta \rangle$$

$$\leq \frac{4}{3} \langle \Omega \sin(2\beta) + \frac{9}{4\pi c} \tilde{L}_{12}(\Omega), R^{-3} \rangle + \frac{1}{4} \left(\frac{3}{4} \langle \eta^2, R^{-3} \rangle \right)$$

$$+ 6 \langle \Omega \sin(2\beta) + \frac{9}{8\pi c} \tilde{L}_{12}(\Omega), R^{-2} \rangle + \frac{3^2}{4 \cdot 6} \langle \eta^2, \frac{(1 + R)^2}{R^2} \rangle$$

$$+ \frac{1}{3} \langle \Omega^2, \frac{1 + R}{R} \sin(2\beta)^2 \rangle + \frac{3}{4} \langle \eta^2, \frac{R}{1 + R} \rangle = \sum_{i=1}^{6} J_i.$$

We apply Lemma [5.9] with $k = 2, 3$ to simplify $J_1, J_3$ defined above:

$$J_1 + J_3 = \langle \Omega^2 \sin(2\beta)^2, \left(\frac{4}{3} R^{-3} + 6 R^{-2}\right) \rangle - \frac{4}{3} \left(2 \cdot \frac{9}{4\pi c} - \frac{\pi}{2} \frac{9^2}{(4\pi c)^2}\right) \|\tilde{L}_{12}(\Omega), R^{-3/2}\|_{L^2(R)}^2$$

$$- 6 \left(\frac{9}{8\pi c} - \frac{\pi}{2} \frac{9^2}{(8\pi c)^2}\right) \|\tilde{L}_{12}(\Omega), R^{-1}\|_{L^2(R)}^2 \|L_{2l}(R) \| \cong M_1 + M_2 + M_3.$$

We further simplify $M_2, M_3$ defined above. Using Lemma A.1, we have $|\pi c - 2| \lesssim \alpha$ and

$$- \frac{4}{3} \left(\frac{9}{4\pi c} - \frac{\pi}{2} \frac{9^2}{4(\pi c)^2}\right) \leq -\frac{4}{3} \left(2 - \frac{\pi}{2} \frac{9^2}{8}\right) + C\alpha < -\frac{1}{4} + C\alpha,$$

$$- \frac{4}{3} \left(\frac{9}{8\pi c} - \frac{\pi}{2} \frac{9^2}{2(8\pi c)^2}\right) \leq -\frac{6}{16} \left(1 - \frac{\pi}{2} \frac{9}{16}\right) + C\alpha < -\frac{1}{4} + C\alpha,$$

for some absolute constant $C$. It follows that $M_2 + M_3 \leq (1 - \frac{1}{4} + C\alpha)(\|\tilde{L}_{12}(\Omega), R^{-3/2}\|_{L^2(R)}^2 + \|\tilde{L}_{12}(\Omega), R^{-1}\|_{L^2(R)}^2) = (\frac{1}{4} + C\alpha)(\|\tilde{L}_{12}(\Omega), R^{-1/2}\|_{L^2(R)}^2$, where we have used the notation $\rho$ defined in (5.20). Therefore, we yield the damping for $\tilde{L}_{12}(\Omega)$.

From the above estimate, we see that the nonlocal vortex stretching term $\tilde{L}_{12}(\Omega)\eta$ or $I$ is indeed harmless to the stability, which is achieved by designing a suitable weight.

Using (5.32), the above estimate of $M_2 + M_3$ in $J_1 + J_3$ and $\sin(2\beta)^2 \leq \sin(2\beta)$, we prove

$$\langle \frac{2}{\pi} \tilde{L}_{12}(\Omega)\eta, \eta \varphi_0 \rangle + \langle \Omega, \eta \varphi_0 \rangle = I + II \leq \langle \frac{4}{3} R^{-3} + 6 R^{-2} + \frac{1 + R}{3R^2}, \Omega^2 \sin(2\beta) \rangle$$

$$+ \frac{3}{16} R^{-3} + \frac{3}{8} \left(\frac{1 + R}{R^4}\right) + \frac{3}{41 + R}, \eta^3 \rangle - \left(\frac{1}{4} - C\alpha\right)(\|\tilde{L}_{12}(\Omega), R^{-1/2}\|_{L^2(R)}^2.\|L_{2l}(R)\|^2.$$

5.4.4. Estimate of the projection $c_{\omega}$ in the $\Omega$ equation. We estimate the terms involving $c_{\omega}$ in (5.29) in this subsection. Notice that $c_{\omega}$ defined in (4.11) is the projection of $\Omega$ onto some function. Using (4.8) and (5.20), we can calculate

$$\langle \Omega - R\partial_{\Omega} \Omega, \Omega \varphi_0 \rangle = \left\langle \frac{\alpha}{c} \Gamma(\beta), \frac{6R^2}{(1 + R)^3}, \Omega \left(\frac{1 + R}{R^3}\right)^{\sin(2\beta)} \right\rangle$$

$$= \frac{6\alpha}{c} \left\langle \frac{1}{R}, \Omega, \sin(2\beta) \Gamma(\beta) \right\rangle.$$

We show that the projection is almost equal to $L_{12}(\Omega)(0)$. Notice that

$$\frac{1}{c} \frac{\Omega}{R} \sin(2\beta) \Gamma(\beta) - \frac{\pi}{2} \frac{\Omega}{R} \sin(2\beta) = \frac{1}{c} \left\langle \frac{\Omega}{R}, \sin(2\beta)(\Gamma(\beta) - 1) \right\rangle + \left(\frac{\pi}{2} - \frac{\Omega}{R}, \sin(2\beta) \right\rangle$$

$$= \frac{1}{c} \left\langle \Omega - R\partial_{\Omega} \Omega, \Omega \varphi_0 \right\rangle = 6|I + II| \lesssim \alpha(\Omega^2, \varphi_0)^{1/2}.$$

Using Lemma A.1, (5.20) and the Cauchy-Schwarz inequality, we have

$$|I| \lesssim \alpha \frac{1}{R} |\Omega| \sin(2\beta)^{1/2} \lesssim \alpha(\Omega^2, \left(\frac{1 + R}{R^3}\right)^{\sin(2\beta)} \frac{1}{R^3}, \left(\frac{R^3}{(1 + R)^3}\right)^{1/2} \lesssim \alpha(\Omega^2, \varphi_0)^{1/2},$$

$$|II| \lesssim \alpha \frac{1}{R} (\Omega, \sin(2\beta)) \lesssim \alpha(\Omega^2, \left(\frac{1 + R}{R^3}\right)^{\sin(2\beta)} \frac{1}{R^3}, \left(\frac{R^3}{(1 + R)^3}\right)^{1/2} \lesssim \alpha(\Omega^2, \varphi_0)^{1/2}.$$

It follows that

$$\left|\frac{1}{\alpha} \left\langle \Omega - R\partial_{\Omega} \Omega, \Omega \varphi_0 \right\rangle - 6 \pi \frac{\Omega}{R} \sin(2\beta) \right| \leq 6|I + II| \lesssim \alpha(\Omega^2, \varphi_0)^{1/2}.$$
Recall the definition of $c_\omega$ in (4.11). Using the above estimate and then the formula of $L_{12}(\Omega)(0)$ (2.12), we have

\begin{equation}
(5.34)
c_\omega(\bar{\Omega} - R \partial_\Omega \bar{\Omega}, \Omega \varphi_0) = -\frac{2}{\pi} L_{12}(\Omega)(0) \cdot \frac{1}{\alpha} \langle \bar{\Omega} - R \partial_\Omega \bar{\Omega}, \Omega \varphi_0 \rangle
\end{equation}

\begin{equation}
\leq -\frac{2}{\pi} L_{12}(\Omega)(0) \cdot \frac{\varphi_0}{2 R} \sin(2\beta) + C_\alpha |L_{12}(\Omega)(0)| |\Omega^2, \varphi_0|^{1/2}
\end{equation}

\begin{equation}
= -6(L_{12}(\Omega)(0))^2 + C_\alpha |L_{12}(\Omega)(0)| |\Omega^2, \varphi_0|^{1/2} \leq -6(L_{12}(\Omega)(0))^2 + C_\alpha (L_{12}^4(\Omega)(0) + |\Omega^2, \varphi_0|).
\end{equation}

We see that $c_\omega(\bar{\Omega} - R \partial_\Omega \bar{\Omega})$ contributes a negative term, which stabilizes the system as we expect in Section 5.4.1.

5.4.5. Estimate of the projection $c_\omega$ in the $\eta$ equation. Next, we estimate the $c_\omega$ term in the $\eta$ equation (5.29). From (4.18), we know

\begin{equation}
(5.35)\frac{\bar{\eta} - R \partial_\Omega \bar{\eta}}{\eta} = \frac{(1 + R)^3}{6 R} \left( \frac{6 R}{1 + R^3} - R \cdot \frac{6}{1 + R^3} + R \cdot \frac{18 R}{1 + R^3} \right) = \frac{3 R}{1 + R^3}.
\end{equation}

Using the above identity and (5.20), we can compute

\begin{equation}
\langle \bar{\eta} - R \partial_\Omega \bar{\eta}, \bar{\eta} \varphi_0 \rangle = \frac{9 \alpha}{8 c} \langle \eta, \eta \varphi_0 \rangle \left( R^{-3} + \frac{31 + R}{2 R^2} \right) = \frac{27\alpha}{8 c} \left\langle \eta, \frac{1}{1 + R} \right\rangle \cdot \frac{1}{1 + R^2}.
\end{equation}

Using (4.11), we derive

\begin{equation}
(5.36)\frac{c_\omega}{4 \pi c} L_{12}(\Omega)(0) \langle \eta, \frac{1}{1 + R} \rangle \leq -\frac{81}{4 \pi c} L_{12}^2(\Omega)(0) \langle \eta, \frac{1}{1 + R} \rangle \triangleq A_1 + A_2.
\end{equation}

Based on the observation we discussed in Section 5.4.1, we exploit the cancellation between $A_2$ and $\eta$ in $\Omega$ equation (5.5).

An ODE for $L_{12}(\Omega)(0)$. Multiplying $\sin(2\beta)/R$ on both sides of (5.5) and then integrating (5.35), we derive

\begin{equation}
\frac{d}{dt} L_{12}(\Omega)(0) = -\left\langle R \partial_\Omega \Omega, \frac{\sin(2\beta)}{R} \right\rangle - L_{12}(\Omega)(0) + c_\omega \left\langle \bar{\Omega} - R \partial_\Omega \bar{\Omega}, \frac{\sin(2\beta)}{R} \right\rangle
\end{equation}

\begin{equation}
+ \left\langle \eta, \frac{\sin(2\beta)}{R} \right\rangle - \left\langle \frac{3}{1 + R} D_\beta \Omega, \frac{\sin(2\beta)}{R} \right\rangle + \left\langle R \alpha, \frac{\sin(2\beta)}{R} \right\rangle.
\end{equation}

The first term vanishes by an integration by parts argument. Using (4.18) and (4.11), we can compute the third term

\begin{equation}
\langle \bar{\eta} - R \partial_\Omega \bar{\eta}, \frac{\sin(2\beta)}{R} \rangle = \frac{\alpha}{c} c_\omega \int_0^\infty \int_0^{\pi/2} \frac{R^2}{(1 + R)^3} \cdot \frac{\sin(2\beta)}{R} d\beta dR = \frac{\pi \alpha}{2 c} c_\omega \int_0^\infty \frac{R^2}{(1 + R)^3} dR = 3\pi \alpha c_\omega \left( -(1 + R)^{-1} + \frac{1}{2} (1 + R)^{-2} \right) \bigg|_0^\infty = -3 L_{12}(\Omega)(0).
\end{equation}

It follows that

\begin{equation}
\frac{d}{dt} L_{12}(\Omega)(0) = -4 L_{12}(\Omega)(0) - \left\langle \eta, \frac{\sin(2\beta)}{R} \right\rangle - \left\langle \frac{3 \sin(2\beta)}{(1 + R) R}, D_\beta \Omega \right\rangle + \left\langle R \alpha, \frac{\sin(2\beta)}{R} \right\rangle.
\end{equation}

We see that the normalization condition (4.11) contributes a damping factor $-3 L_{12}(\Omega)(0)$ to the above ODE, as we discussed in Section 5.4.1. Multiplying $\frac{81}{4 \pi c} L_{12}(\Omega)(0)$ to both sides, we derive

\begin{equation}
(5.36)\frac{1}{2} \frac{d}{dt} L_{12}^2(\Omega)(0) = \frac{81}{4 \pi c} \left( -4 L_{12}(\Omega)(0) + L_{12}(\Omega)(0) \left\langle \eta, \frac{\sin(2\beta)}{R} \right\rangle - L_{12}(\Omega)(0) \left\langle \frac{3 \sin(2\beta)}{(1 + R) R}, D_\beta \Omega \right\rangle + L_{12}(\Omega)(0) \left\langle R \alpha, \frac{\sin(2\beta)}{R} \right\rangle \right).
\end{equation}
The first term on the right hand side provides damping for \( L_{12}(\Omega)(0) \), which enables us to control \( A_1, A_2 \) defined in (5.33). A key observation is that the projection terms about \( \eta \) in (5.35) and (5.36) have different signs, which enables us to exploit the cancellation of these two terms.

We combine the estimate of \( A_2 \) in (5.35) and the \( \eta \) term in (5.36) to obtain

\[
A_3 \triangleq A_2 + \frac{81}{4\pi c} L_{12}(\Omega)(0) \left\langle \eta, \sin(2\beta) \right\rangle = \frac{81}{8\pi c} L_{12}(\Omega)(0) \left\langle \eta, \frac{1}{R}(-1 + 2\sin(2\beta)) \right\rangle.
\]

Applying the Cauchy-Schwarz inequality yields

\[
A_3 \leq \frac{81}{8\pi c} \left| L_{12}(\Omega)(0) \right| \left\langle \eta^2, \frac{(1 + R)^4}{R^3} \right\rangle^{1/2} \left\langle \frac{R^3}{(1 + R)^4} \frac{1}{R^2}, (1 - 2\sin(2\beta))^2 \right\rangle^{1/2}.
\]

A simple calculation implies that for any \( k > 2 \)

\[
\int_0^\infty (1 + R)^{-k} dR = \frac{1}{k-1}, \quad \int_0^\infty \frac{R}{(1 + R)^k} dR = \frac{1}{k(k-1)(k-2)}.
\]

Choosing \( k = 4 \), we can compute

\[
\int_0^{\pi/2} (1 - 2\sin(2\beta))^2 d\beta = \frac{\pi}{2} - 4 \int_0^{\pi/2} \sin(2\beta) d\beta + 4 \int_0^{\pi/2} (\sin(2\beta))^2 d\beta = \frac{\pi}{2} - 4 + 4 \cdot \frac{\pi}{4},
\]

\[
\left\langle \frac{R^3}{(1 + R)^4} \frac{1}{R^2}, (1 - 2\sin(2\beta))^2 \right\rangle = \left( \frac{3\pi}{2} - 4 \right) \int_0^\infty \frac{R}{(1 + R)^3} dR = \frac{1}{6} \left( \frac{3\pi}{2} - 4 \right).
\]

As a result, we have

\[
A_3 \leq \frac{81}{8\pi c} \sqrt{\frac{1}{6} \left( \frac{3\pi}{2} - 4 \right)} \left| L_{12}(\Omega)(0) \right| \left\langle \eta^2, \frac{(1 + R)^4}{R^3} \right\rangle^{1/2}.
\]

For \( A_1 \) in (5.35), we apply the Cauchy-Schwarz inequality directly to yield

\[
A_1 = -\frac{27}{4\pi c} L_{12}(\Omega)(0) \left\langle \eta, \frac{1}{(1 + R)R^2} \right\rangle \leq \frac{27}{4\pi c} \left| L_{12}(\Omega)(0) \right| \left\langle \eta^2, \frac{(1 + R)^3}{R^4} \right\rangle^{1/2} \left\langle \frac{R^4}{(1 + R)^3} \frac{1}{(1 + R)^2 R^3} \right\rangle^{1/2}.
\]

Using (5.38), we can calculate

\[
\left\langle \frac{R^4}{(1 + R)^3} \frac{1}{(1 + R)^2 R^3} \right\rangle = \frac{\pi}{2} \int_0^\infty (1 + R)^{-5} = \frac{\pi}{8}.
\]

It follows that

\[
A_1 \leq \frac{27}{4\pi c} \sqrt{\frac{\pi}{8}} \left| L_{12}(\Omega)(0) \right| \left\langle \eta^2, \frac{(1 + R)^3}{R^4} \right\rangle^{1/2}.
\]

Combining the identities (5.33), (5.37), the estimates of \( A_1, A_3 \) (5.39)-(5.40) and then using the Cauchy-Schwarz inequality, we prove

\[
\omega \left\langle \bar{\eta} - R\partial_\eta \bar{\eta}, \eta \psi_0 \right\rangle + \frac{81}{4\pi c} L_{12}(\Omega)(0) \left\langle \eta, \sin(2\beta) \right\rangle = A_1 + A_2 + \frac{81}{4\pi c} L_{12}(\Omega)(0) \left\langle \eta, \sin(2\beta) \right\rangle
\]

\[
= A_1 + A_3 \leq b_1 \left| L_{12}(\Omega)(0) \right| \left\langle \eta^2, \frac{(1 + R)^4}{R^3} \right\rangle^{1/2} + b_2 \left| L_{12}(\Omega)(0) \right| \left\langle \eta^2, \frac{(1 + R)^3}{R^4} \right\rangle^{1/2}
\]

\[
\leq \frac{1}{32} \left( \eta^2, \frac{(1 + R)^4}{R^3} \right) + \frac{9}{128} \left( \eta^2, \frac{(1 + R)^3}{R^4} \right) + L_{12}(\Omega)(0) \left( \frac{b_1^2}{4 \times 1/32} + \frac{b_2^2}{4 \times 9/128} \right),
\]

where \( b_1, b_2 \) denote the constants in (5.39)-(5.40), i.e.

\[
b_1 \triangleq \frac{81}{8\pi c} \sqrt{\frac{1}{6} \left( \frac{3\pi}{2} - 4 \right)}, \quad b_2 \triangleq \frac{27}{4\pi c} \sqrt{\frac{\pi}{8}}.
\]

Using Lemma A.1 for the estimate of \( c \) and a direct calculation yield

\[
\frac{b_1^2}{1/8} + \frac{b_2^2}{9/32} - \frac{81}{4\pi c} \cdot 4 \cdot 6 - 8 = \frac{81}{8\pi c} \left( \frac{1}{6} \left( \frac{3\pi}{2} - 4 \right) \right) + \frac{32}{9} \left( \frac{27}{4\pi c} \right)^2 \frac{\pi}{8} - \frac{81}{\pi c} - 6
\]

\[
\leq \frac{4}{3} \left( \frac{81}{16} \right)^2 \left( \frac{3\pi}{2} - 4 \right) + \frac{4\pi}{9} \left( \frac{27}{8} \right)^2 - \frac{81}{2} - 6 + C\alpha < -6 + C\alpha,
\]

\[
\text{(5.42)}
\]
where we have used Lemma A.1 to replace \( c \pi \) by 2 in the first inequality and \( C > 0 \) is some absolute constant. Combining the estimates of \( c_\omega \) terms, or equivalently \( L_{12}(\Omega)(0) \), (5.34), (5.41) and the damping term of \( L_{12}^2(\Omega)(0) \) in (5.35), we prove (5.43)

\[
E_{\omega}(\Omega - R\partial_t \bar{\Omega}, \Omega \varphi_0) + E_{\omega}(\bar{\eta} - R\partial_t \bar{\eta}, \eta \psi_0) + \frac{81}{4\pi c} L_{12}(\Omega)(0) \left( \frac{\sin(2\beta)}{R} \right) - \frac{81}{4\pi c} \cdot 4 L_{12}^2(\Omega)(0)
\]

\[
\leq (\eta^2, \frac{1}{32} \frac{(1 + R)^4}{R^3} + \frac{9}{128} \frac{(1 + R)^3}{R^4}) + L_{12}^2(\Omega)(0) \left( \frac{b_2^4}{1/8} + \frac{b_3^2}{9/32} \frac{81}{\pi c} - 6 \right) + C \alpha |L_{12}(\Omega)(0)||\Omega^2, \varphi_0|^{1/2}
\]

\[
\leq 3 \left( \eta^2, \frac{1}{6} \frac{(1 + R)^4}{R^3} + \frac{3}{8} \frac{(1 + R)^3}{R^4} \right) + L_{12}^2(\Omega)(0) (-6 + C \alpha) + C \alpha |L_{12}(\Omega)(0)||\Omega^2, \varphi_0|^{1/2},
\]

where we have used (5.42) to derive the last inequality.

5.4.6. Estimate of the angular transport term. From the definition of the weights (5.13), (5.26), we have

\[
\varphi_0 \lesssim \varphi_2, \quad (1 + R)^{-1} \psi_0 \lesssim \psi_2, \quad \left( \frac{3 \sin(2\beta)}{(1 + R)^2} \right)^2, \varphi_2^{-1} \lesssim 1.
\]

Therefore, we can estimate the angular transport terms in (5.29), (5.30) as follows

\[
- \frac{3D_{\beta} \Omega}{1 + R}, \Omega \varphi_0 \lesssim \frac{||D_{\beta} \Omega \varphi_2^{1/2}||_2||\Omega \varphi_2^{1/2}||_2}, \quad - \frac{3D_{\beta} \eta}{1 + R}, \eta \psi_0 \lesssim \frac{||D_{\beta} \eta \varphi_2^{1/2}||_2||\eta \varphi_2^{1/2}||_2},
\]

\[
- \frac{81}{4\pi c} L_{12}(\Omega)(0) \left( \frac{3 \sin(2\beta)}{1 + R}, D_{\beta} \Omega \right) \lesssim |L_{12}(\Omega)(0)||\Omega \varphi_2^{1/2}||_2,
\]

where we have used \( c^{-1} \lesssim 1 \) (see Lemma A.1). Using the energy notations \( E(\beta, 1) \) (5.19) and \( E(R, 0) \) (5.27), we further derive (5.44)

\[
- \frac{3D_{\beta} \Omega}{1 + R}, \Omega \varphi_0 - \frac{3D_{\beta} \eta}{1 + R}, \eta \psi_0 - \frac{81}{4\pi c} L_{12}(\Omega)(0) \left( \frac{3 \sin(2\beta)}{1 + R}, D_{\beta} \Omega \right) \leq K_1 E(R, 0) E(\beta, 1),
\]

for some absolute constant \( K_1 \). We remark that the absolute constants \( K_1, K_2, \ldots \) do not change from line to line.

5.4.7. Completing the estimates with a less singular weight. Combining the estimates (5.29)-(5.33) (5.36), (5.43), (5.44) and using the notations \( E(R, 0), \mathcal{R}(R, 0) \) (5.27), we obtain (5.45)

\[
\frac{1}{2} \frac{d}{dt} E(R, 0)^2 = \frac{1}{2} \frac{d}{dt} \left( \Omega^2, \varphi_0 \right) + (\eta^2, \psi_0) + \frac{81}{4\pi c} L_{12}^2(\Omega)(0)
\]

\[
\leq \left( \Omega^2 \sin(2\beta), -2R^2 - \frac{9}{2} R^{-2} - 3R^{-1} - \frac{1}{2} + \frac{4}{3} R^{-3} + 6R^{-2} + \frac{1 + R}{3R} \right)
\]

\[
+ \left( \eta^2, -\frac{3(1 + R)^2}{32R^4}(1 + 4R + 3R^2 + 3R^3) \Gamma'(\beta)^{-1} + \left( \frac{3}{16} R^{-3} - \frac{3(1 + R)^2}{8 R^2} + \frac{3R}{4(1 + R)} \right) \right) + \left( -\frac{1}{4} + C \alpha \right) ||\tilde{L}_{12}(\Omega)\rho^{1/2}||_2^2 \mathcal{R}(R)
\]

\[
+ L_{12}^2(\Omega)(0) (-6 + C \alpha) + C \alpha |L_{12}(\Omega)(0)||\Omega^2, \varphi_0| + K_1 E(R, 0) E(\beta, 1) + \mathcal{R}(R, 0).
\]

Denote by \( D(\Omega), D_1(\eta), D_2(\eta) \) the quantities involving \( \Omega^2, \eta^2 \) on the right hand side of (5.45) as follows

\[
D(\Omega) \triangleq -2R^2 - \frac{9}{2} R^{-2} - 3R^{-1} - \frac{1}{2} + \frac{4}{3} R^{-3} + 6R^{-2} + \frac{1 + R}{3R},
\]

\[
D_1(\eta) \triangleq -\frac{3(1 + R)^2}{32R^4}(1 + 4R + 3R^2 + 3R^3),
\]

\[
D_2(\eta) \triangleq \left( \frac{3}{16} R^{-3} - \frac{3(1 + R)^2}{8 R^2} + \frac{3R}{4(1 + R)} \right) + \left( -\frac{1}{4} + C \alpha \right) ||\tilde{L}_{12}(\Omega)\rho^{1/2}||_2^2 \mathcal{R}(R)
\]

\[
+ \left( \frac{3}{16} R^{-3} - \frac{3(1 + R)^2}{8 R^2} + \frac{3R}{4(1 + R)} \right) + \frac{3}{16} \left( \frac{1 + R)^4}{R^3} + \frac{3(1 + R)^3}{R^4} \right).
\]
Next, we simplify $D(\Omega), D_1(\eta), D_2(\eta)$ and show that

\[(5.46) \quad \langle \Omega^2 \sin(2\beta), D(\Omega) \rangle \leq -\frac{1}{6} \langle \Omega^2, \varphi_0 \rangle, \quad \langle \eta^2, D_1(\eta) \Gamma^{-1}(\beta) + D_2(\eta) \rangle \leq -\frac{1}{8} \langle \eta^2, \psi_0 \rangle,
\]

where $\varphi_0, \psi_0$ are the $L^2$ weights defined in (5.26). Recall $\varphi_0$ defined in (5.26). To prove the first inequality, it suffices to prove

$$D(\Omega) = -2R^{-3} - \frac{9}{2}R^{-2} - 3R^{-1} - \frac{1}{2} + \frac{4}{3}R^{-3} + 6R^{-2} + \frac{1 + R}{3R} \leq -\frac{(1 + R)^3}{6R^3},$$

which is equivalent to proving

$$(-2 + \frac{4}{3} + \frac{1}{6})R^{-3} + (-\frac{9}{2} + 6 + \frac{1}{2})R^{-2} + (-3 + \frac{1}{3} + \frac{1}{2})R^{-1} + (\frac{1}{2} + \frac{1}{3} + \frac{1}{6}) \leq 0.$$

It is further equivalent to

$$\frac{49}{36}R^{-3} + 2R^{-2} - \frac{13}{6}R^{-1} \leq 0,$$

which is valid since $2\sqrt{\frac{1}{2} \times \frac{13}{6}} > 2$. Hence, we prove the first inequality in (5.46).

For the second inequality in (5.46), firstly, we use $\Gamma(\beta)D_2(\eta) \leq D_2(\eta)$ ($\Gamma(\beta) = \cos^\alpha(\beta)$ (4.8)) to obtain

\[(5.47) \quad D(\eta) \triangleq D_1\eta + D_2(\eta) \Gamma^{-1}(\beta) \leq D_1(\eta) + D_2(\eta)\]

$$= \frac{3}{16} \left\{ \frac{(1 + R)^2}{2R^4} (1 + 4R + 3R^2 + 3R^3 + R^{-3}) + 2 \frac{(1 + R)^2}{R^2} + \frac{4R}{1 + R} + \frac{1 + R}{6} \frac{4}{R^3} + \frac{3}{8} \frac{(1 + R)^3}{R^3} \right\}.$$

Recall the definition of $\psi_0$ in (5.26). To prove the second inequality in (5.46), it suffices to prove

$$\Gamma(\beta)^{-1}D(\eta) = \Gamma(\beta)^{-1}D_1(\eta) + D_2(\eta) \leq -\frac{1}{8} \psi_0,$$

which is equivalent to

\[(5.48) \quad D(\eta) \leq \frac{3}{16} \left( -\frac{1}{8} \frac{(1 + R)^3}{R^4} - \frac{3}{16} \frac{(1 + R)^4}{R^3} \right).\]

We split the negative term in the upper bound of $D(\eta)$ in (5.47) as follows

$$\frac{1}{2R^4} (1 + 4R + 3R^2 + 3R^3) = -\frac{(1 + R)^2}{2R^4} \{ (1 + R) + (3R^2) + R(1 + R)^2 + R(2 - 2R + 2R^2) \}$$

$$= \frac{(1 + R)^3}{2R^4} \frac{3}{2R^2} (1 + R)^2 - \frac{(1 + R)^4}{2R^3} (1 - R + R^2).$$

It follows that

$$D(\eta) \leq \frac{3}{16} \left\{ -\frac{(1 + R)^3}{2R^4} \left( \frac{1}{2} + \frac{3}{8} \right) + \frac{(1 + R)^4}{R^3} \left( -\frac{1}{2} \frac{1}{6} \right) + \frac{1 + R}{2R^2} \frac{(1 + R)^2}{R^3} - \frac{(1 + R)^2}{R^3} (1 - R + R^2) \right\} + \frac{1}{2} \frac{(1 + R)^3}{R^3} + \frac{4R}{1 + R}.$$

Observe that

$$\frac{(1 + R)^4}{3R^3} + \frac{1}{2} \frac{(1 + R)^4}{R^3} = -\frac{(1 + R)^4}{3R^3} + \frac{1}{2} \frac{(1 + R)^4}{R^3} \leq -\frac{3}{16} \frac{(1 + R)^4}{R^3},$$

$$\frac{(1 + R)(1 + R^3)}{R^3} + \frac{1}{2} \frac{(1 + R)^2}{R^2} = -\frac{1}{2} \frac{(1 + R)^2}{R^2} - (1 + R) + \frac{4R}{1 + R} - \frac{1}{R^2} \leq 0,$$

where we have used $\frac{7}{48} \frac{(1 + R)^2}{R} \geq \frac{7}{48} \times 4 \geq 1/2$ to derive the first inequality. Therefore, we prove (5.48), which further implies the second inequality in (5.40).

For $L^2(\Omega)(0)$ in (5.40), we use Lemma 4.1 about $c(\epsilon \pi = 2 + O(\alpha))$ to get

$$-6 + C\alpha \leq -\frac{1}{8} \times \frac{81}{8} - 4 + C\alpha \leq -\frac{1}{8} \times \frac{81}{4\pi c} - 4 + C\alpha,$$
which implies
\begin{equation}
(5.49) \quad (-6 + C\alpha)L_{12}^2(\Omega)(0) \leq -\frac{1}{8} \cdot \frac{81}{4\pi c} L_{12}^2(\Omega)(0) - (4 - C\alpha)L_{12}^2(\Omega)(0),
\end{equation}
where $C$ is some absolute constant and may vary from line to line. Observe that
\begin{equation}
(5.50) \quad K_1 E(R, 0)E(\beta, 1) \leq \frac{1}{100}E(R, 0)^2 + 100K_1^2E^2(\beta, 1).
\end{equation}

Recall $E(R, 0)$ in (5.27). Finally, substituting the estimates (5.46)-(5.50) in (5.45), we prove
\begin{equation}
\frac{1}{2} \frac{d}{dt} E(R, 0)^2 \leq -\left(-\frac{1}{6} - C\alpha\right)\langle \Omega^2, \varphi_0 \rangle - \frac{1}{8} \langle \eta^2, \psi_0 \rangle - \frac{1}{8} \cdot \frac{81}{4\pi c} L_{12}^2(\Omega)(0) - (4 - C\alpha)L_{12}^2(\Omega)(0)
\end{equation}
where we have used 
\begin{equation}
\leq -\frac{1}{9} + C\alpha E^2(R, 0) - (4 - C\alpha)L_{12}^2(\Omega)(0) - \left(\frac{1}{4} - C\alpha\right)||\tilde{L}_{12}(\Omega)\beta^{1/2}||_{L^2(R)} + \frac{1}{100}E(R, 0)^2 + 100K_1^2E^2(\beta, 1) + E(R, 0),
\end{equation}
where we have used $-\frac{1}{9} + C\alpha \leq \frac{1}{100}$, $-\frac{1}{8} \leq C\alpha$ to derive the last inequality.

5.4.8. Linear stability with a less singular weight. Using the reformulation (5.11), and the notations $E(\beta, 1)$ and $E(\beta, 1)$ defined in (5.10), we have
\begin{equation}
\frac{1}{2} \frac{d}{dt} E(\beta, 1)^2 = \langle D_{\beta}L_1(\Omega, \eta), (D_{\beta}L_1(\Omega, \eta)\varphi_2 + (D_{\beta}L_2(\Omega, \eta), (D_{\beta}L_2(\Omega, \eta)\varphi_2 &+ \mathcal{R}(\beta, 1).
\end{equation}

Now we combine (5.20) and (5.51) to establish the linear stability of (5.5)-(5.6) with the less singular weight (5.20). Firstly, we choose an absolute constant $\mu_1$ such that
\begin{equation}
100K_1^2 \leq \frac{1}{20}\mu_1,
\end{equation}
where the absolute constant $K_1$ is determined in (5.44). From (5.20), we have $R^{-2} \leq \rho$. Hence,
\begin{equation}
||\tilde{L}_{12}(\Omega)R^{-1}||_{L^2(R)}^2 \leq ||\tilde{L}_{12}(\Omega)\beta^{1/2}||_{L^2(R)}^2.
\end{equation}

Combining Proposition 5.5 (5.51), the formulation (5.52), and the above estimates, we establish the estimate for $E(R, 0)^2 + \mu_1E(\beta, 1)^2$
\begin{equation}
\frac{1}{2} \frac{d}{dt} \left((E(R, 0)^2 + \mu_1E(\beta, 1)^2) \right) \leq -\left(-\frac{1}{9} - C\alpha\right)\left((E(R, 0)^2 + \mu_1E(\beta, 1)^2)\right)
\end{equation}
where we have used
\begin{equation}
\leq -\frac{1}{9} + C\alpha E^2(R, 0) - (4 - C\alpha)L_{12}^2(\Omega)(0) - \left(\frac{1}{4} - C\alpha\right)||\tilde{L}_{12}(\Omega)\beta^{1/2}||_{L^2(R)} + (R, 0)^2 + \mu_1E(\beta, 1).
\end{equation}

The proof of Proposition 5.5 is now complete.

5.5. $L^2$ estimate of $\Omega, \eta$ with a more singular weight. With the linear stability (5.53) with a less singular weight, we can proceed to perform the $L^2$ estimate with a more singular weight.

Definition 5.11. Define an energy $E(R, 1)$ and a remaining term $\mathcal{R}(R, 1)$ by
\begin{equation}
(5.54) \quad E(R, 1) \triangleq \left((\Omega^2, \varphi_1) + (\eta^2, \varphi_1)\right)^{1/2}, \quad \mathcal{R}(R, 1) \triangleq (\mathcal{R}_\Omega, \Omega \varphi_1) + (\mathcal{R}_\eta, \eta \varphi_1),
\end{equation}
where $\varphi_1, \psi_1$ are given in Definition 5.2.

The main result in this Section is the following.

Proposition 5.12. Assume that $\Omega \varphi_1^{1/2}, \eta \varphi_1^{1/2} \in L^2$. We have that
\begin{equation}
\langle L_1(\Omega, \eta), \Omega \varphi_1 \rangle + \langle L_2(\Omega, \eta), \eta \varphi_1 \rangle \leq -\frac{1}{6}E(R, 1)^2 + K_3 \left(L_{12}^2(\Omega)(0) + \left|\tilde{L}_{12}(\Omega)R^{-1}\right|_{L^2(R)}^2\right),
\end{equation}
where $L_1, L_2$ are defined in Definition 5.7. $K_3 > 0$ is some fixed absolute constant.
Proof of Proposition 5.12: A direct calculation yields
\[
\langle L_1(\Omega, \eta), \Omega \varphi_1 \rangle = \langle L_{10}(\Omega, \eta), \Omega \varphi_1 \rangle + c_\omega(\Omega - R \partial_R \Omega, \Omega \varphi_1),
\]
(5.55)
\[
\langle L_2(\Omega, \eta), \eta \varphi_1 \rangle = \langle L_{20}(\eta), \eta \varphi_1 \rangle + \frac{2}{\pi \alpha}(\bar{L}_{12}(\eta \bar{\eta}, \eta \varphi_1) + c_\omega(\eta - R \partial_R \eta, \eta \varphi_1).
\]
Applying Lemma 5.3 with \( \varphi = \varphi_1 \) and \( \delta = \sigma = \frac{99}{100} \), we yield
\[
\langle L_{10}(\Omega, \eta), \Omega \varphi_1 \rangle + \langle L_{20}(\eta), \eta \varphi_1 \rangle \leq \frac{1}{\frac{1}{4} + 3(1 - \sigma)}((\Omega^2, \varphi_1) + (\eta^2, \varphi_1)) \leq -\frac{1}{\frac{1}{5}}((\Omega^2, \varphi_1) + (\eta^2, \varphi_1)).
\]
Recall \( c_\omega = -\frac{2}{\pi \alpha}L_{12}(\Omega)(0) \) (A.11). Using (A.10) in Lemma A.3 and the Cauchy-Schwarz inequality, we obtain
\[
|c_\omega((\bar{\Omega} - R \partial_R \Omega)\Omega, \varphi_1)| + |c_\omega((\bar{\eta} - R \partial_R \eta)\eta, \varphi_1)| \lesssim |L_{12}(\Omega)(0)(\Omega^2, \varphi_1) + (\eta^2, \varphi_1))^{1/2}.
\]
For \( \bar{L}_{12}(\Omega) \) in (5.55), using the Cauchy-Schwarz inequality, we derive
\[
\left\langle \frac{2}{\pi \alpha} \bar{L}_{12}(\Omega \bar{\eta}, \eta \varphi_1) \right\rangle \lesssim \langle \bar{L}_{12}(\Omega)\eta^2, \varphi_1 \rangle^{1/2} (\eta^2, \varphi_1)^{1/2} \lesssim \||\bar{L}_{12}(\Omega)R^{-1}\||_{L^2(R)} \langle \eta^2, \varphi_1 \rangle^{1/2},
\]
where we have applied Lemma A.2 and (A.9) in Lemma A.3 in the second inequality.

Using the Cauchy-Schwarz inequality and the energy notation \( E(R, 1) \) (5.54), we complete the proof of Proposition 5.12.

5.6. \( L^2 \) estimate of \( D_\beta \xi \) and \( \xi \). The estimates of \( \xi \) are simpler since the main terms in the equation of \( \xi \) (5.7) do not couple with \( \Omega, \eta \) directly. We use the weights \( \psi_1, \psi_2 \) in Definition 5.2.

Proposition 5.13. Suppose that \( \psi_1^{1/2} \xi, \psi_2^{1/2}D_\beta \xi \in L^2 \). We have
\[
\langle L_3(\Omega, \xi), \xi \psi_1 \rangle \leq \langle L_3(\Omega, \xi), \xi \psi_1 \rangle - \left( -\frac{1}{3} + C\alpha \right) \langle \xi^2, \psi_1 \rangle + C\alpha \left( \bar{L}_{12}(\Omega)(0) + ||\bar{L}_{12}(\Omega)R^{-1}||_{L^2(R)}^2 \right),
\]
(5.56)
\[
\langle D_\beta L_3(\Omega, \xi), (D_\beta \xi) \psi_2 \rangle \leq \langle D_\beta L_3(\Omega, \xi), (D_\beta \xi) \psi_2 \rangle - \left( -\frac{1}{3} + C\alpha \right) \langle (D_\beta \xi)^2, \psi_2 \rangle + C\alpha \left( \bar{L}_{12}(\Omega)(0) + ||\bar{L}_{12}(\Omega)R^{-1}||_{L^2(R)}^2 \right).
\]
(5.57)

Proof of Proposition 5.13: Since \( D_\beta \) commutes with \( L_3 \) (see Definition 5.1) and \( \bar{L}_{12}(\Omega) \) does not depend on \( \beta \), a direct calculation implies
\[
\langle L_3(\Omega, \xi), \xi \psi_1 \rangle = \langle L_{30}(\xi), \xi \psi_1 \rangle - \frac{2}{\pi \alpha} \langle \bar{L}_{12}(\Omega)\xi, \xi \psi_1 \rangle + c_\omega(3\xi - D_R \xi, \xi \psi_1)
\]
\[
\langle D_\beta L_3(\Omega, \xi), (D_\beta \xi) \psi_2 \rangle = \langle D_{30}(D_\beta \xi), (D_\beta \xi) \psi_2 \rangle - \frac{2}{\pi \alpha} \langle \bar{L}_{12}(\Omega)D_\beta \xi, (D_\beta \xi) \psi_2 \rangle + c_\omega(3\xi - D_R \xi, \psi_2).
\]
(5.10)

Applying (5.10) in Lemma 5.3 with \( \psi = \psi_1 \) (a constant multiple of \( \psi \) does not change the estimate in (5.10)) and with \( \psi = \psi_2 \) (see Definition 5.2), respectively, we derive
\[
\langle L_{30}(\xi), \xi \psi_1 \rangle \leq \left( -\frac{1}{2} + 3|1 - \sigma| \right) \langle \xi^2, \psi_1 \rangle < -\frac{3}{8} \langle \xi^2, \psi_1 \rangle,
\]
\[
\langle D_{30}(D_\beta \xi), (D_\beta \xi) \psi_2 \rangle \leq \left( -\frac{1}{2} + 3|1 - \gamma| \right) \langle (D_\beta \xi)^2, \psi_2 \rangle \leq \left( -\frac{3}{8} + \alpha \right) \langle (D_\beta \xi)^2, \psi_2 \rangle,
\]
where \( \gamma = 1 + \frac{\alpha}{\pi \alpha}, \sigma = \frac{99}{100} \). Using the Cauchy-Schwarz inequality, we yield
\[
\left| -\frac{2}{\pi \alpha} \langle \bar{L}_{12}(\Omega)\xi, \xi \psi_1 \rangle \right| \lesssim \alpha^{-1} \langle \bar{L}_{12}(\Omega) \psi_1 \rangle^{1/2} \langle \xi^2, \psi_1 \rangle^{1/2} \lesssim \alpha ||\bar{L}_{12}(\Omega)R^{-1}||_{L^2(R)} \langle \xi^2, \psi_1 \rangle^{1/2},
\]
where we have applied Lemma A.2 and (A.9) in Lemma A.7 to derive the second inequality.

Using the Cauchy-Schwarz inequality, (5.11) and Lemma A.7, we obtain
\[
c_\omega(3\xi - D_R \xi, \xi \psi_1) \leq \frac{1}{\alpha} L_{12}(\Omega)(0)||((3\xi - D_R \xi)^2, \psi_1 \rangle^{1/2} \langle \xi^2, \psi_1 \rangle^{1/2} \lesssim \alpha ||L_{12}(\Omega)(0)|| \langle \xi^2, \psi_1 \rangle^{1/2}.
\]
(5.59)

Plugging (5.59) in (5.58) and using the Cauchy-Schwarz inequality, we prove (5.58).
The proof of (5.57) is completely similar. We apply estimates similar to those in (5.60)–(5.61) and Lemmas A.2, A.7 to control the $c_i$ and $\tilde{L}_{12}(\Omega)$ terms. Combining these estimates, using the second inequality in (5.59) and then the Cauchy-Schwarz inequality proves (5.57). □

5.6.1. The $L^2$ energy. Using the reformulation (5.11), we have

$$
\frac{1}{2} \frac{d}{dt} (\langle \Omega^2, \varphi_1 \rangle + \langle \eta^2, \varphi_1 \rangle) = \langle L_1(\Omega, \eta, \varphi_1) + \langle L_2(\Omega, \eta, \varphi_1) + \langle R_\Omega, \varphi_1 \rangle,
$$

$$
\frac{1}{2} \frac{d}{dt} (\xi^2, \psi_1) = \langle L_3(\xi, \psi_1) + \langle R_\xi, \xi \psi_1 \rangle, \frac{1}{2} \frac{d}{dt} (D_\beta \xi^2, \psi_2) = \langle D_\beta L_3(\xi), (D_\beta \xi) \psi_2 \rangle + \langle D_\beta R_\xi, D_\beta \xi \psi_2 \rangle.
$$

Recall the energy $E(R, 1)$ and the remaining term $R(R, 1)$ in Definition 5.11

$$
E(R, 1) = (\langle \Omega^2, \varphi_1 \rangle + \langle \eta^2, \varphi_1 \rangle)^{1/2}, \quad R(R, 1) = \langle R_\Omega, \Omega \varphi_1 \rangle + \langle R_\eta, \eta \varphi_1 \rangle.
$$

Combining the above reformulation, Propositions 5.5, 5.12, 5.13 and $R^2 \leq \rho$ (5.20), we know that there is some absolute constant $\mu_2$, which is small enough, e.g. $\mu_2 K_3 < \frac{1}{100}$, such that the following estimate holds

$$
\frac{1}{2} \frac{d}{dt} (E(R, 0)^2 + \mu_1 E(\beta, 1)^2 + \mu_2 E(R, 1)^2 + \langle \xi^2, \psi_1 \rangle + \langle (D_\beta \xi)^2, \psi_2 \rangle)
$$

$$
\leq - \left( \frac{1}{9} - C_\alpha \right) (E(R, 0)^2 + \mu_1 E(\beta, 1)^2 + \mu_2 E(R, 1)^2 + \langle \xi^2, \psi_1 \rangle + \langle (D_\beta \xi)^2, \psi_2 \rangle)
$$

$$
- (3 - C_\alpha)L_{12}(\Omega)(0) - \left( \frac{1}{5} - C_\alpha \right) \left\| L_{12}(\Omega)^{1/2} \right\|_{L^2(R)}^2 + R_0(\Omega, \eta, \xi),
$$

where $R_0$ is defined below. We define the following $L^2$ energy and the remaining term $R_0$\footnote{In fact, $E_0$ contains a $L^2$ norm of the angular derivative $D_\beta \Omega, D_\beta \eta, D_\beta \xi$.}

$$
E_0(\Omega, \eta, \xi) \triangleq (E(R, 0)^2 + \mu_1 E(\beta, 1)^2 + \mu_2 E(R, 1)^2 + \langle \xi^2, \psi_1 \rangle + \langle (D_\beta \xi)^2, \psi_2 \rangle)^{1/2},
$$

$$
R_0(\Omega, \eta, \xi) \triangleq \langle R_\Omega, \Omega \rangle + \mu_1 \langle R_\beta, \beta \rangle + \mu_2 \langle R_1, \Omega \rangle + \langle R_\xi, \xi \psi_1 \rangle + \langle D_\beta R_\xi, (D_\beta \xi) \psi_2 \rangle,
$$

where $(E(R, 0), R(\beta, 0)), (E(\beta, 1), R(\beta, 1)), (E(R, 1), R(\Omega, 1))$ are defined in (5.27), (5.19) and (5.54), respectively, and $\mu_i$ are some fixed absolute constants.

We do not need the extra damping for $\tilde{L}_{12}(\Omega)^{1/2}$ and $L_{12}(\Omega)(0)$ in (5.62) due to Lemma A.3 and the fact that $E_0$ is stronger than $\| \Omega \|_{L^2(R)}^2$. Using (A.3), we know that $C_\rho |L_{12}(\Omega)^{1/2}|_{L^2(R)}$, $C_\rho |L_{12}(\Omega)(0)|^2$ can be bounded by $C_\rho E_0^2$. Hence, using the notation $E_0, R_0$, we derive the following result from (5.62).

**Corollary 5.14.** Let $E_0(\Omega, \eta, \xi), R_0(\Omega, \eta, \xi)$ be the energy and the remaining term defined in (5.63). Under the assumptions of Propositions 5.6, 5.12 and 5.13 we have

$$
\frac{1}{2} \frac{d}{dt} E_0^2 \leq - \left( \frac{1}{9} - C_\alpha \right) E_0^2 + R_0.
$$

6. Higher order estimates and the energy functional

In this section, based on the $L^2$ estimates established in Corollary 5.14, we proceed to perform the higher order estimates in the spirit of Propositions 5.12, 5.13 so that we can complete the nonlinear analysis. In subsection 6.1 we perform the weighted $H^1$ estimates of $\dot{L}_i$ and illustrate how to apply several lemmas to control different terms in $D_\Omega L_i$. In subsection 6.2, 6.3 we use a similar argument to establish weighted $H^2$ and $H^3$ estimates. Finally, we perform weighted $L^\infty$ estimates of $\xi (\theta_y)$ and its derivatives in subsection 6.4. Since $\xi(x, y)$ does not decay in the $x$ direction when $y$ is fixed (see the estimates of $\tilde{\xi}$ in Lemma A.7), we cannot obtain the decay estimate for its perturbation $\xi$. Hence, in order to obtain the $L^\infty$ control of $\xi$ and its derivatives, which will be used later to estimate the nonlinear terms, we cannot apply a $H^k \hookrightarrow L^\infty$ type Sobolev embedding. We need to perform the $L^\infty$ estimates of $\xi$ and its derivative directly. This difficulty is absent in (11) by removing the swirl.
6.1. \( H^1 \) estimates. We remark that the \( H^1 \) estimate with angular derivatives is already established in Section 5.3 about \( D_{\beta} \Omega, D_{\beta} \eta \) and Section 5.6 about \( D_{\beta} \xi \). Recall the weighted differential operator \( D_R = R \partial_R \) in Definition 5.1. We define an energy and a remaining term

\[
E(R, 2)(\Omega, \eta, \xi) \triangleq \left( \langle D_R \Omega^2 \rangle^2, \varphi_1 \right) + \langle (D_R \xi)^2 \rangle^2, \psi_1 \right) \right)^{1/2}, \]

\[
\mathcal{R}(R, 2)(\Omega, \eta, \xi) \triangleq \langle D_R \mathcal{R}_1, D_R \Omega \varphi_1 \rangle + \langle D_R \mathcal{R}_2, D_R \eta \varphi_1 \rangle + \langle D_R \mathcal{R}_3, D_R \xi \psi_1 \rangle,
\]

where \( \varphi_1, \psi_1 \) are defined in (5.13).

Proposition 6.1. Under the assumption of Corollary 5.14 and that \( \varphi_1^{1/2} D_R \Omega, \varphi_1^{1/2} D_R \eta, \psi_1^{1/2} D_R \xi \in L^2 \), we have

\[
\langle D_R \mathcal{L}_1(\Omega, \eta), (D_R \Omega) \varphi_1 \rangle + \langle D_R \mathcal{L}_2(\Omega, \eta), (D_R \eta) \varphi_1 \rangle + \langle D_R \mathcal{L}_3(\xi), (D_R \xi) \psi_1 \rangle \leq -\frac{1}{6} E^2(R, 2) + K_4 E_0^2,
\]

where \( K_4 \) is some fixed absolute constant and \( E_0, E(R, 2) \) are defined in (5.63) and (6.1).

Proof. Since \( D_R \) commutes with \( D_R, D_{\beta} \) in \( \mathcal{L}_1, \mathcal{L}_0 \) (see Definition 5.1), we have

\[
D_R \mathcal{L}_1(\Omega, \eta) = \mathcal{L}_{10}(D_R \Omega, D_R \eta) - D_R \mathcal{L}_1, \quad \mathcal{L}_{10}(D_R \Omega, D_R \eta) = \mathcal{L}_{10}(D_R \Omega, D_R \eta) + \sum_{i=1}^2 I_i,
\]

\[
D_R \mathcal{L}_2(\Omega, \eta) = \mathcal{L}_{20}(D_R \eta) - D_R \mathcal{L}_2, \quad \mathcal{L}_{20}(D_R \eta) = \mathcal{L}_{20}(D_R \eta) + \sum_{i=1}^5 I_i,
\]

\[
D_R \mathcal{L}_3(\xi) = \mathcal{L}_{30}(D_R \xi) - D_R \mathcal{L}_3, \quad \mathcal{L}_{30}(D_R \xi) = \mathcal{L}_{30}(D_R \xi) + \sum_{i=1}^3 I_{3i}.
\]

Applying (5.13) with \( \varphi = \varphi_1 \) (see (5.14)), and (5.10) with \( \psi = \psi_1 \) (see (5.13)) in Lemma (6.8) and 3|1 - \sigma| < \frac{1}{30}, we yield

\[
\langle \mathcal{L}_{10}(D_R \Omega, D_R \eta), (D_R \Omega) \varphi_1 \rangle + \langle \mathcal{L}_{20}(D_R \eta), (D_R \eta) \varphi_1 \rangle \leq -\frac{1}{6} \langle (D_R \Omega)^2, \varphi_1 \rangle + \langle (D_R \eta)^2, \varphi_1 \rangle
\]

\[
\langle \mathcal{L}_{20}(D_R \xi), (D_R \xi) \psi_1 \rangle \leq -\frac{3}{8} \langle (D_R \xi)^2, \psi_1 \rangle.
\]

Notice that \( \varphi_2, \psi_2 \) satisfy \( \varphi_1 \leq \varphi_2, \psi_1 \leq \psi_2 \). For the terms not involving \( \tilde{L}_{12}(\Omega), c_\omega \), we use \( E_0 \) defined in (5.63) to control the weighted \( L^2 \) norm of \( D_{\beta} \Omega, D_{\beta} \eta \). It is easy to see that

\[
||I_1 \varphi_1^{1/2}||_{L^2} \lesssim ||D_{\beta} \Omega \varphi_2^{1/2}||_{L^2} \lesssim E_0, \quad ||I_1 \varphi_1^{1/2}||_{L^2} \lesssim ||D_{\beta} \eta \varphi_2^{1/2}||_{L^2} \lesssim E_0, \quad ||I_2 \varphi_1^{1/2}||_{L^2} \lesssim ||\partial_{\delta} \eta \varphi_1^{1/2}||_{L^2} \lesssim E_0,
\]

\[
||I_3 \varphi_1^{1/2}||_{L^2} \lesssim ||D_{\beta} \xi \varphi_2^{1/2}||_{L^2} \lesssim E_0, \quad ||I_3 \varphi_1^{1/2}||_{L^2} \lesssim ||\partial_{\delta} \xi \varphi_1^{1/2}||_{L^2} \lesssim E_0.
\]

Recall \( c_\omega = -\frac{2}{\pi \alpha} L_{12}(\Omega)(0) \). Applying (A.10) in Lemma (A.8) to \( I_2, I_5 \) and (A.20) in Lemma (A.7) to \( I_{3i} \), we obtain

\[
||I_2 \varphi_1^{1/2}||_{L^2} \lesssim ||L_{12}(\Omega)(0)|| \lesssim E_0, \quad ||I_5 \varphi_1^{1/2}||_{L^2} \lesssim ||L_{12}(\Omega)(0)|| \lesssim E_0, \quad ||I_{3i} \varphi_1^{1/2}||_{L^2} \lesssim \alpha |L_{12}(\Omega)(0)| \lesssim \alpha E_0.
\]

Finally, for the \( \tilde{L}_{12}(\Omega) \) terms, we apply Lemma (A.2). To apply Lemma (A.2), we need the \( L^\infty \) norm of some angular integrals, whose estimates are given in (A.9) in Lemma (A.5) about \( \tilde{\Omega}, \tilde{\eta} \) and (A.19) in Lemma (A.7) about \( \tilde{\xi} \). Using these estimates, we obtain

\[
||I_3 \varphi_1^{1/2}||_{L^2} \lesssim ||\tilde{\Omega}_{12}(\Omega) R^{-1}||_{L^2} \lesssim E_0, \quad ||I_4 \varphi_1^{1/2}||_{L^2} \lesssim ||R^{-1} \Omega||_{L^2} \lesssim E_0,
\]

\[
||I_3 \varphi_1^{1/2}||_{L^2} \lesssim \alpha ||\tilde{\Omega}_{12}(\Omega) R^{-1}||_{L^2} \lesssim \alpha E_0, \quad ||I_4 \varphi_1^{1/2}||_{L^2} \lesssim \alpha ||R^{-1} \Omega||_{L^2} \lesssim \alpha E_0.
\]

The result now follows using the Cauchy-Schwarz inequality (notice that \( -\frac{1}{6} < -\frac{1}{8} \), \( \alpha < 1 \)) and applying the energy notation (6.1). \( \square \)
Using the reformulation (5.11), we have
\[
\frac{1}{2} \frac{d}{dt} E^2(R, 2) = \frac{1}{2} \frac{d}{dt} \left( \langle (D_R \Omega)^2, \varphi_1 \rangle + \langle (D_R \eta)^2, \varphi_1 \rangle + \langle (D_R \xi)^2, \psi_1 \rangle \right)^{1/2}
\]
\[
= \langle D_R \mathcal{L}_1(\Omega, \eta), (D_R \varphi_1) \rangle + \langle D_R \mathcal{L}_2(\Omega, \eta), (D_R \eta) \varphi_1 \rangle + \langle D_R \mathcal{L}_3(\xi), (D_R \xi) \psi_1 \rangle + \mathcal{R}(R, 2).
\]
Therefore, it is not difficult to combine the above reformulation, Corollary 6.1 and Proposition 6.1 to prove the following results.

**Corollary 6.2.** Suppose that \( \Omega, \eta, \xi \) satisfy that \( E_0(\Omega, \eta, \xi), E(R, 2)(\Omega, \eta, \xi) < +\infty \), where \( E_0, E(R, 2) \) are defined in (5.63) and (6.1), respectively. Then there exists an absolute constant \( \mu_3 \), such that, the following statement holds true. The \( H^1 \) energy \( E_1 \) and its associated remaining term \( \mathcal{R}_1 \) defined by
\[
(6.2) \quad E_1(\Omega, \eta, \xi) \triangleq \left( E_0^2(\Omega, \eta, \xi) + \mu_3 E^2(R, 2)(\Omega, \eta, \xi) \right)^{1/2}, \quad \mathcal{R}_1(\Omega, \eta, \xi) \triangleq \mathcal{R}_0 + \mu_3 \mathcal{R}(R, 2),
\]
where \( \mathcal{R}_0, \mathcal{R}(R, 2) \) are defined in (5.63) and (6.1), satisfy
\[
\frac{1}{2} \frac{d}{dt} E_1^2 \leq \left( -\frac{1}{10} + C \alpha \right) E_1^2 + \mathcal{R}_1.
\]

**6.2. \( H^2 \) estimates.** We now proceed to perform the \( H^2 \) estimates. Throughout this subsection, we assume that the following quantities are in \( L^2 \),
\[
\varphi_2^{1/2} D_2^\beta \Omega, \varphi_2^{1/2} D_2^\beta D_R \Omega, \varphi_2^{1/2} D_2^\beta \Omega, \varphi_2^{1/2} D_2^\beta D_\beta \Omega, \varphi_2^{1/2} D_2^\beta D_R \eta, \varphi_2^{1/2} D_2^\beta D_\beta \eta, \varphi_2^{1/2} D_2^\beta D_\beta \xi, \varphi_2^{1/2} D_2^\beta D_R \xi, \varphi_2^{1/2} D_2^\beta D_\beta \xi.
\]
We will use weights \( \varphi_1, \psi_1 \) for \( D_R^2 \) derivative, \( \varphi_2, \psi_2 \) for \( D_\beta^2 \) and \( D_R D_\beta \) derivatives in the \( H^2 \) norm to be constructed. Recall \( \mathcal{L}_i, \mathcal{L}_{i0}, D_\beta \) in Definition 5.1. We perform the estimate of the second derivatives in the order of \( D_\beta^2, D_\beta D_R, D_R^2 \).

Notice that \( D_\beta \) commutes with \( \mathcal{L}_{i0}, i = 1, 2, 3 \). For the \( \mathcal{L}_{i0} \) part in \( \mathcal{L}_i \), applying Lemma 5.3 with \( \varphi = \varphi_2, \delta = \gamma = 1 + \alpha/10, \psi = \varphi_2 \) (see Definition 5.2 for \( \varphi, \psi_1 \)) we obtain
\[
\langle D_\beta^2 \mathcal{L}_{i0}(\Omega, \eta, \xi), (D_\beta^2 \Omega) \varphi_2 \rangle + \langle D_\beta^2 \mathcal{L}_{i20}(\Omega, \eta, \xi), (D_\beta^2 \eta) \varphi_2 \rangle + \langle D_\beta^2 \mathcal{L}_{i30}(\Omega, \xi) \rangle \langle (D_\beta^2 \xi) \psi_2 \rangle \leq (-\frac{1}{4} + \alpha) \langle (D_\beta^2 \Omega)^2, \varphi_2 \rangle + \langle (D_\beta^2 \eta)^2, \varphi_2 \rangle + \langle (-\frac{3}{8} + \alpha) \langle (D_\beta^2 \xi)^2, \psi_2 \rangle.
\]
For the \( D_R D_\beta \) derivative, since \( D_R \) does not commute with \( \mathcal{L}_{i0} \), we have
\[
(6.3) \quad (D_R D_\beta) \mathcal{L}_{i0}(\Omega, \eta, \xi) - \mathcal{L}_{i0}(D_R D_\beta \Omega, D_R D_\beta \eta, D_R D_\beta \xi) = -D_R \frac{3}{1 + R} D_\beta^2 g + M,
\]
where \( g = \Omega, \eta, \xi \) and we have used the notation \( \mathcal{L}_{i0} = \mathcal{L}_{i0}, \mathcal{L}_{j0} = \mathcal{L}_{j0}, \mathcal{L}_{k0} = \mathcal{L}_{k0} \). \( M \) denotes some terms that involves no higher than the first derivatives of \( \Omega, \eta, \xi \) and have coefficients bounded by some absolute constant. For example, \( M \) contains the term \( D_R(-2 + \frac{3}{1 + R})D_\beta \eta \) in \( D_R D_\beta((2 + \frac{1}{1 + R})\eta) \). \( M \) may vary from line to line but its weighted \( L^2 \) norm can be easily bounded by the \( H^1 \) energy \( E_1 \) (6.2). Applying Lemma 5.3 with \( \varphi = \varphi_2 \) to \( \mathcal{L}_{i0}(D_R D_\beta \Omega, D_R D_\beta \eta, D_R D_\beta \xi), g = \Omega, \eta, \xi \) and \( \psi = \varphi_2 \) to \( \mathcal{L}_{i0}(D_R D_\beta \xi) \) and then using the Cauchy-Schwarz inequality to control the right hand side of (6.3), we yield
\[
\langle D_R D_\beta \mathcal{L}_{i0}(\Omega, \eta, \xi), (D_R D_\beta \Omega) \varphi_2 \rangle + \langle D_R D_\beta \mathcal{L}_{i20}(\Omega, \eta, \xi), (D_R D_\beta \eta) \varphi_2 \rangle + \langle D_R D_\beta \mathcal{L}_{i30}(\xi) \rangle \langle (D_R D_\beta \xi) \psi_2 \rangle \leq (-\frac{1}{5} + \alpha) \langle (D_R D_\beta \Omega)^2, \varphi_2 \rangle + \langle (D_R D_\beta \eta)^2, \varphi_2 \rangle + \langle (D_R D_\beta \xi)^2, \psi_2 \rangle + C \left( E_1^2 + \langle (D_\beta^2 \Omega)^2, \varphi_2 \rangle + \langle (D_\beta^2 \eta)^2, \varphi_2 \rangle + \langle (D_\beta^2 \xi)^2, \psi_2 \rangle \right),
\]
where the constant \(-\frac{1}{5}\) is a result from first applying Lemma 5.3 to \( (\Omega, \eta, \xi) \), which gives two damping factors \(-\frac{3}{8} - \frac{1}{4} < -\frac{1}{5}\). Similarly, for the \( D_R^2 \) derivative, we have for \( g = \Omega, \eta, \xi \)
\[
(6.4) \quad (D_R^2 \mathcal{L}_{i0}(\Omega, \eta, \xi) - \mathcal{L}_{i0}(D_R^2 \Omega, D_R^2 \eta, D_R^2 \xi)) = -2D_R \frac{3}{1 + R} D_R D_\beta g + M.
\]
Applying Lemma [5.3] with $\varphi = \varphi_1$ to $L_{0}(D_R^2 \Omega, D_R^2 \eta), g = \Omega, \eta,$ and $\psi = \psi_1$ to $L_{10}(D_R^2 \xi)$ will give two damping factors $-\frac{1}{4} + \frac{1}{10m}, -\frac{1}{8} + \alpha.$ We then use the Cauchy-Schwarz inequality to yield

$$\langle D_R^2 L_{10}(\Omega, \eta), (D_R^2 \Omega) \varphi_1 \rangle + \langle D_R^2 L_{20}(\eta), (D_R^2 \eta) \varphi_1 \rangle + \langle D_R^2 L_{30}(\xi), (D_R^2 \xi) \psi_1 \rangle$$

$$\leq (-\frac{1}{5} + \alpha) \langle (D_R^2 \Omega)^2, \varphi_1 \rangle + \langle (D_R^2 \eta)^2, \varphi_1 \rangle + \langle (D_R^2 \xi)^2, \psi_1 \rangle + C \left( E_1^2 + \langle (D_R D_\beta \Omega)^2, \varphi_2 \rangle + \langle (D_R D_\beta \eta)^2, \varphi_2 \rangle + \langle (D_R D_\beta \xi)^2, \psi_2 \rangle \right),$$

where we have used $\varphi_1 \lesssim \varphi_2, \psi_1 \lesssim \psi_2$ to obtain $\langle (D_R D_\beta \Omega)^2, \varphi_1 \rangle \lesssim \langle (D_R D_\beta \Omega)^2, \varphi_2 \rangle$ and other similar terms. We have also used $-\frac{1}{4} + \frac{1}{10m}, -\frac{1}{8} < -\frac{1}{4}$ when we have applied the Cauchy-Schwarz inequality.

Notice that the remaining terms in $L_4$ except for $L_{10}$ are the $\tilde{L}_{12}(\Omega)$ terms and $c_\omega$ terms. For the $\tilde{L}_{12}(\Omega)$ terms, we use Lemma [A.2] and then [A.9] in Lemma [A.5] about $\Omega, \tilde{\eta}$ and $[A.19]$ in Lemma $[A.4]$ about $\xi$ to estimate the $L^\infty$ norm of some angular integrals. For the $c_\omega$ terms, we use the estimates in [A.10] in Lemma [A.5] about $\Omega, \tilde{\eta}$ and [A.20] in Lemma [A.7] about $\xi$. We remark that from Proposition [A.2], the norm of $R^{-1}D_R^2 \tilde{L}_{12}(\Omega)$ can be bounded by the norm of $R^{-1}D_R \Omega$, which can be further bounded by $E_1$.

Combining these estimates and using the Cauchy-Schwarz inequality, we obtain

$$\langle D_R^2 L_2(\Omega, \eta), (D_R^2 \Omega) \varphi_2 \rangle + \langle D_R^2 L_2(\eta, \Omega), (D_R^2 \eta) \varphi_2 \rangle + \langle D_R^2 L_2(\Omega, \xi, \eta), (D_R^2 \xi) \psi_2 \rangle$$

$$\leq (-\frac{1}{6} + \alpha) \langle (D_R^2 \Omega)^2, \varphi_2 \rangle + \langle (D_R^2 \eta)^2, \varphi_2 \rangle + \langle (D_R^2 \xi)^2, \psi_2 \rangle + C \mu_2 \Omega^2,$n

$$\langle D_R D_\beta L_2(\Omega, \eta), (D_R D_\beta \Omega) \varphi_2 \rangle + \langle D_R D_\beta L_2(\eta, \Omega), (D_R D_\beta \eta) \varphi_2 \rangle + \langle D_R D_\beta L_2(\Omega, \xi, \eta), (D_R D_\beta \xi) \psi_2 \rangle$$

$$\leq (-\frac{1}{6} + \alpha) \langle (D_R^2 \Omega)^2, \varphi_2 \rangle + \langle (D_R^2 \eta)^2, \varphi_2 \rangle + \langle (D_R^2 \xi)^2, \psi_2 \rangle + \langle (D_R D_\beta \Omega)^2, \varphi_1 \rangle$$

$$+ \langle (D_R D_\beta \eta)^2, \varphi_1 \rangle + \langle (D_R D_\beta \xi)^2, \psi_1 \rangle + (D_R^2 \xi) \psi_1 \rangle$$

$$+ K_5 \left( \langle (D_R D_\beta \Omega)^2, \varphi_1 \rangle + \langle (D_R D_\beta \eta)^2, \varphi_1 \rangle + \langle (D_R D_\beta \xi)^2, \psi_1 \rangle + (D_R D_\beta \xi) \psi_1 \rangle \right),$$

where $K_5$ is some fixed absolute constant and we have used $-\frac{1}{4}, -\frac{1}{5} < -\frac{1}{4}$ when we applied the Cauchy-Schwarz inequality to control the inner product between the $\tilde{L}_{12}(\Omega), c_\omega$ terms and the second derivative terms.

Combining Corollary [6.2] with estimates [6.4], we know that there exist some absolute constants $\mu_2,k$ that can be determined in the order of $k = 0, 1, 2,$ such that the $H^2$ energy functional $E_2$ and its associated remaining term $R_2$ defined below satisfy the estimates stated in Corollary [6.3]

$$E_2^2(\Omega, \eta, \xi) \leq E_1^2 + \sum_{0 \leq k \leq 2} \mu_2,k \left( \langle (D_R^2 D_\beta^2 \eta)^2 \rangle, \varphi_i \rangle + \langle (D_R D_\beta^2 \eta) \rangle, \varphi_i \rangle + \langle (D_R D_\beta^2 \xi) \rangle, \psi_i \rangle \right),$$

$$R_2(\Omega, \eta, \xi) \leq R_1 + \sum_{0 \leq k \leq 2} \mu_2,k \left( \langle (D_R D_\beta^2 \xi) \rangle, \varphi_i \rangle + \langle (D_R D_\beta^2 \eta) \rangle, \varphi_i \rangle + \langle (D_R D_\beta^2 \xi) \rangle, \psi_i \rangle \right),$$

where $E_1, R_1$ are defined in [6.2] and $(\varphi_i, \psi_i) = (\varphi_2, \psi_2)$ for $k = 0, 1$ and $(\varphi_1, \psi_1)$ otherwise.

**Corollary 6.3.** Suppose that $E_2(\Omega, \eta, \xi) < +\infty.$ Then the energy $E_2$ satisfies

$$\frac{1}{2} \frac{d}{dt} E_2^2(\Omega, \eta, \xi) \leq (-\frac{1}{10} + C \alpha) E_2^2 + R_2.$$
for other third derivatives $D^i_R D^j_\beta \xi$, we use weight $\psi_3$. The reason we perform weighted $H^3$ is to establish Proposition 7.17.

In the same spirit of the $H^2$ energy functional $E_2$ and Corollary [6.3] we can show that there exist some absolute constants $\mu_{3,k}$, which can be determined in the order $k = 0, 1, 2, 3$, such that the $H^3$ energy functional $E_3 \geq 0$ and its associated remaining term $R_3$ defined below satisfy the estimates stated in Corollary [6.4]

\[
E_3^2(\Omega, \eta, \xi) \triangleq E_2^2 + \sum_{0 \leq k \leq 3} \mu_{3,k} \left( |(D^k_R D^3_\beta \Omega)^2, \varphi_i| + |(D^k_R D^3_\beta \eta)^2, \varphi_i| + |(D^k_R D^3_\beta \xi)^2, \psi_i| \right),
\]

\[
R_3(\Omega, \eta, \xi) \triangleq R_2 + \sum_{0 \leq k \leq 3} \mu_{3,k} \left( |(D^k_R D^3_\beta \Omega, (D^k_R D^3_\beta \eta) \varphi_i| + |(D^k_R D^3_\beta \xi, (D^k_R D^3_\beta \eta) \psi_i| \right)
\]

\[+ (D^k_D^3_\beta R_\xi, (D^k_R D^3_\beta \xi) \psi_i), \]

where $E_2, R_2$ are defined in [5.5], $(\varphi_i, \psi_i) = (\varphi_3, \psi_3)$ for $k = 0, 1, 2$ and $(\varphi_1, \psi_1)$ otherwise.

**Corollary 6.4.** Suppose that $E_3(\Omega, \eta, \xi) < +\infty$. Then the energy $E_3$ satisfies

\[
\frac{1}{2} \frac{d}{dt} E_3^2(\Omega, \eta, \xi) \leq (-\frac{1}{12} + C\alpha) E_3^2 + R_3.
\]

### 6.4. $L^\infty$ estimates.

For $\Omega, \eta$, the weighted $H^3$ estimates that we have obtained guarantee that $\Omega, \eta \in L^\infty$, which will be established precisely in later sections. For $\xi$, however, since the weight $\psi_2$ (see Definition [5.2]) is less singular in $\beta$ for $\beta$ close to 0, the weighted $H^3$ is not embedded continuously into $L^\infty$. Alternatively, we perform $L^\infty$ estimates of $\xi$ and its derivatives directly. This difficulty is absent in [11] by removing the swirl.

Firstly, the transport term in the $\xi$ equation in (5.1), including the nonlinear part in $N_\xi$, is given by

\[
A(\xi) \triangleq (1 + 3\alpha) D_R \xi + \alpha c_D D_R \xi + (\bar{u} \cdot \nabla) \xi + (u \cdot \nabla) \xi.
\]

The main damping term in the $\xi$ equation is $(-2 - \bar{v}_y)\xi$. [4.9] shows that $-\bar{v}_y = -\frac{3}{1+R} + l.o.t..$ Therefore, we consider

\[
(-2 - \bar{v}_y)\xi = (-2 - \frac{3}{1+R})\xi + \Xi_1, \quad \Xi_1 \triangleq \frac{3}{1+R} - \bar{v}_y\xi.
\]

We further introduce $\Xi_2$ to denote the lower order terms in the $\xi$ equation (5.1)

\[
\Xi_2 = -c_\omega (2 \bar{\xi} - R \partial_R \hat{\xi}) + (\alpha c_D R \partial_R - (u \cdot \nabla) \bar{\xi} - (u_y \bar{\eta} + \bar{u}_y \eta).
\]

Then the $\xi$ equation in (5.1) can be simplified as

\[
\partial_t \xi + A(\xi) = (-2 - \frac{3}{1+R})\xi + \Xi_1 + \Xi_2 + \bar{F}_\xi + N_\sigma,
\]

where we have moved part of the nonlinear term $N_\xi$ defined in (5.2) to the transport term $A(\xi)$ and $N_\sigma$ is given by

\[
N_\sigma = (2c_\omega - v_y)\xi - u_y \eta.
\]

Notice that $-\frac{3}{1+R} \leq 0$. Multiplying $\xi$ on both sides and then performing $L^\infty$ estimate yield

\[
\frac{1}{2} \frac{d}{dt} ||\xi||^2_{L^\infty} \leq -2 ||\xi||^2_{L^\infty} + ||\Xi_1||_{L^\infty} + ||\Xi_2||_{L^\infty} + ||\bar{F}_\xi||_{L^\infty} + ||N_\sigma||_{L^\infty},
\]

where the transport term $A(\xi)$ vanishes.

Before we perform weighted $C^1$ estimates, we rewrite $A(\xi)$ defined in (6.7) as follows

\[
A(\xi) = ((1 + 3\alpha + \alpha c_D) D_R \xi + \frac{3}{1+R} D_\beta \xi) + (((u + \bar{u}) \cdot \nabla - \frac{3}{1+R} D_\beta) \xi) \triangleq A_1(\xi) + A_2(\xi).
\]

We introduce the following weights for the weighted $C^1$ estimates

\[
\phi_1 = \frac{1+R}{R}, \quad \phi_2 = 1 + (R \sin(2\beta)^\alpha) - \frac{3}{R}.
\]
Observe that $D_\beta$ commutes with $A_1$ and $D_R$ commutes with $D_R, D_\beta$. Denote by $[P, Q]$ the commutator $PQ - QP$. A direct calculation shows that

$$
\phi_1 D_R A_1 \xi - A(\phi_1 D_R \xi) = \phi_1 D_R \frac{3}{1 + R} \cdot D_\beta \xi - (1 + 3\alpha + \alpha c_1) D_R \phi_1 \cdot D_R \xi + [\phi_1 D_R, A_2] \xi,
$$

$$
= -\frac{3}{1 + R} D_\beta \xi + (1 + 3\alpha + \alpha c_1) \frac{1}{1 + R} \phi_1 D_R \xi + [\phi_1 D_R, A_2] \xi,
$$

$$
\phi_2 D_\beta A_1 \xi - A(\phi_2 D_\beta \xi) = -A_1(\phi_2 - 1) \cdot D_\beta \xi + [\phi_2 D_\beta, A_2] \xi,
$$

where we have used $A_1(1) = 0$ in the last equality. Hence, using (6.10) and the above calculation, we obtain the equation of $\phi_1 D_R \xi$

$$
\partial_t(\phi_1 D_R \xi) + A(\phi_1 D_R \xi) = \frac{3}{1 + R} D_\beta \xi - (1 + 3\alpha + \alpha c_1) \frac{1}{1 + R} \phi_1 D_R \xi - [\phi_1 D_R, A_1] \xi + \phi_1 D_R (-2 - \frac{3}{1 + R}) \xi + \phi_1 D_R (\Xi_1 + \Xi_2 + \tilde{F}_\xi + N_o).
$$

We remark that $-(1 + 3\alpha) \frac{1}{1 + R} \phi_1 D_R \xi$ is a damping term, though we will not use it. Performing $L^\infty$ estimate for $\phi_1 D_\beta \xi$, we obtain the following estimate, which is similar to (6.12)

$$
\frac{1}{2} \frac{d}{dt} \| \phi_1 D_\beta \xi \|^2 \leq -2 - (1 + 3\alpha) \| \phi_1 D_R \xi \|_\infty^2 + 3\| \phi_1 D_R \xi \|_\infty \| \xi \|_\infty + \| \phi_1 D_R \xi \|_\infty \| \xi \|_\infty \| \phi_1 D_R \xi \|_\infty \| \xi \|_\infty,
$$

where we have used $\| \frac{3}{1 + R} \xi \| \leq 3$ and

$$
\phi_1 D_R \xi \cdot \phi_1 D_R (-2) \frac{3}{1 + R} \xi = \phi_1 D_R \xi \cdot (-2) \frac{3}{1 + R} \xi \leq -2 \phi_1 D_R \xi^2 + 3 \| \phi_1 D_R \xi \|_\infty \| \xi \|_\infty.
$$

Similarly, using (6.10), (6.15) and then performing $L^\infty$ estimate on $\phi_2 D_\beta \xi$, we obtain

$$
\frac{1}{2} \frac{d}{dt} \| \phi_2 D_\beta \xi \|^2 \leq -2 \| \phi_2 D_\beta \xi \|^2 \| \xi \|_\infty + \| \phi_2 D_\beta \xi \|_\infty \| \phi_1 (-1) \cdot D_\beta \xi \|_\infty
$$

$$
+ \| \phi_2 D_\beta \xi \|_\infty \| \phi_2 D_\beta, A_2 \|_\infty \| \phi_2 D_\beta \xi \|_\infty \| \phi_2 D_\beta, (\Xi_1 + \Xi_2 + \tilde{F}_\xi + N_o) \|_\infty,
$$

where we have used

$$
\phi_2 D_\beta \xi \cdot \phi_2 D_\beta (-2) \frac{3}{1 + R} \xi \leq -2 \phi_2 D_\beta \xi^2.
$$

We defer the estimates of the remaining terms in (6.12), (6.10), (6.15), which are small, to Section 5.

6.5. The energy functional and the $H^m$ norm. Using all the energy notations (5.19), (5.27), (5.34), (5.36), (5.31), (5.32), (5.33) and (5.30), we can obtain the full expression of $E_3$ (6.6)

$$
E^2 = \langle \Omega^2, \psi \rangle + \langle \eta^2, \psi \rangle + \frac{81}{4\pi c} L^2_1(\Omega)(0) + \mu_1 \left( (\langle D_\beta \Omega^2 \rangle^2, \psi_2) + (\langle D_\beta \eta \rangle^2, \psi_2) \right) + (\langle D_\beta \xi \rangle^2, \psi_2)
$$

$$
+ \mu_2 \left( (\langle \Omega^2, \varphi_1 \rangle + \langle \eta^2, \psi_1 \rangle + \langle \xi^2, \psi_1 \rangle + \mu_3 \left( (\langle D_\beta \Omega \rangle^2, \varphi_1) + (\langle D_\beta \eta \rangle^2, \psi_1) + (\langle D_\beta \xi \rangle^2, \psi_1) \right)
$$

$$
+ \sum_{l=2,3} \sum_{0 \leq k \leq l} \mu_{l,k} \left( (\langle D_R^{l-k} D_R^{l-k} \Omega \rangle^2, \varphi_i) + (\langle D_R^{l-k} D_R^{l-k} \eta \rangle^2, \varphi_i) + (\langle D_R^{l-k} D_R^{l-k} \xi \rangle^2, \varphi_i) \right),
$$

where $(\varphi_i, \psi_i) = (\varphi_l, \psi_k)$ for $k = l$, $(\varphi_i, \psi_k) = (\varphi_2, \psi_2)$ for $k \neq l$ and $l = 2, 3$.

Recall $\varphi_1, \psi_1$ in Definition 5.2. We define the $H^m(\rho)$ norm with $m \geq 1$ as follows

$$
||f||_{H^m(\rho)} \equiv \sum_{0 \leq k \leq m} ||D_R^k f \rho_1^{1/2}||_{L^2} + \sum_{l=1}^{m+1} ||D_R^l D_R^{l+1} f \rho_2^{1/2}||_{L^2},
$$

where for the $H^3(\varphi)$ norm, $\rho_1 = \varphi_1$; for the $H^3(\psi)$ norm, $\rho_1 = \psi_i, i = 1, 2$. We simplify $H^3(\varphi)$ as $H^3$, Notice that the third term only appears for $m \geq 3$. We apply the $H^3$ norm for $\Omega, \eta$ and the $H^3(\psi)$ norm for $\xi$. We use the $H^m$ norm to establish the elliptic estimate in the next Section. We are only going to use the $H^2, H^2(\psi)$ and $H^3, H^3(\psi)$ norms. Remark that the $H^m$ norm is different from the canonical Sobolev $H^m$ norm.
From the Definition 5.2 of $\varphi_i, \psi_i$, we have a simple relationship between $H^m$ and $H^m(\psi)$.

**Lemma 6.5.** For $\frac{m}{2} \leq \lambda \leq \frac{m}{2}$ and $m \leq 3$, we have

$$\|f\|_{H^m(\psi)} \lesssim \|f\|_{H^m}, \quad \|\sin(\beta)^{\frac{m}{2}}f\|_{H^m} \lesssim \|f\|_{H^m(\psi)}.$$  

The proof follows from several simple inequalities $\psi_i \lesssim \varphi_i, \sin(\beta)^{\frac{m}{2}} \varphi_i \lesssim \psi_i, D_\beta^j \sin(\beta)^{\lambda} \varphi_2 = 2\lambda \cos^2(\beta) \sin(\beta)^{\lambda} \varphi_2 \lesssim \psi_i$ for $i \leq 3$, and expanding the norm.

We also define the corresponding inner products on $H^3$ and $H^3(\psi)$, which are equivalent to $H^3, H^3(\psi)$

$$\langle f, g \rangle_{H^3} \triangleq \mu_1 \langle D_\beta^j f, D_\beta^j g \psi_2 \rangle + \mu_2 \langle f, g \varphi_1 \rangle + \mu_3 \langle D_\beta^j f, D_\beta^j g \varphi_1 \rangle + \sum_{k=2}^3 \mu_{k, k} \|D_\beta^j f \varphi_1^{1/2}\|_{L^2} + \sum_{j \geq 1, 2 \leq i+j \leq 3} \mu_{i+j, i} \langle D_\beta^j D_\beta^j f, D_\beta^j D_\beta^j g \psi_2 \rangle.$$  

(6.21)

Clearly, using these notations and (6.18), (6.1), (6.2), (6.3), (6.0), we have

$$E_3 = \frac{81}{4\pi c} L_1^2(\Omega)(0) + \langle \Omega^2, \varphi_0 \rangle + \langle \eta^2, \psi_0 \rangle + \langle \Omega, \Omega \rangle_{H^3} + \langle \eta, \eta \rangle_{H^3} + \langle \xi, \xi \rangle_{H^3(\psi)},$$  

$$R_3 = \langle R_\Omega, \Omega \varphi_0 \rangle + \langle R_\eta, \eta \psi_0 \rangle + \frac{81}{4\pi c} L_1^2(\Omega)(0) \langle R_\Omega, \sin(2\beta) R^{-1} \rangle + \langle R_\Omega, \Omega \rangle_{H^3} + \langle R_\eta, \eta \rangle_{H^3} + \langle R_\xi, \xi \rangle_{H^3(\psi)}.$$  

We also have the following simple inequality

$$\|\Omega\|_{H^3}^2 + \|\eta\|_{H^3}^2 + \|\xi\|_{H^3(\psi)}^2 \lesssim E_3^2(\Omega, \eta, \xi).$$  

Recall the weights $\phi_1, \phi_2$ defined in (6.14). We introduce the $C^1$ norm

$$\|f\|_{C^1} \triangleq \|f\|_{\infty} + \|\phi_1 D_\beta f\|_{\infty} + \|\phi_2 D_\beta f\|_{\infty}$$  

(6.24)

$$= \|f\|_{\infty} + \|1 + \frac{R}{R} D_\beta f\|_{\infty} + \|1 + (R \sin(2\beta) \alpha - \frac{1}{R_\beta}) D_\beta f\|_{\infty}.$$  

7. Elliptic Regularity Estimates and Estimate of nonlinear terms

In this section, we perform several elliptic regularity estimates and estimates of the nonlinear terms. We will follow the argument in 111 to establish the $H^3$ estimates for the elliptic operator in subsection 7.3 and justify that the leading order term of the (modified) stream function can be written as (2.12) in subsection 7.2. The estimates of nonlinear terms can be done in several ways. To simplify our presentation, we will generalize some estimates derived in 111 in subsection 7.3.

The fact that $\xi$ (see Lemma 6.7) and $\xi$ do not decay in certain direction makes the estimates of nonlinear terms complicated since we cannot apply the same weak Sobolev norm to $\Omega, \eta, \xi$. More precisely, the $H^k(\psi)$ norm for $\xi$ is weaker than the $H^k$ norm for $\Omega, \eta$ (see (6.20)). To compensate this, we use a combination of $C^1$ norm and $H^k(\psi)$ norm for $\xi$. We will establish several estimates for $\xi$ in subsection 7.3. Moreover, estimating the $H^k$ norm of $v_\epsilon \xi$ in the $\eta$ equation (5.1) will be more difficult since $\xi$ is in a weaker Sobolev space. In subsection 7.4, we will estimate the nonlinear term $v_\epsilon \xi$ in the $\eta$ equation (5.1). We will also perform a new estimate of the transport term with weighted $H^3$ data.

**Remark 7.1.** The estimates throughout this section are not sensitive to the absolute constants.

Recall that the Biot-Savart law in $\mathbb{R}^2$ is given by (2.3), which can be reformulated using the polar coordinate as

$$-\partial_r \psi - \frac{1}{r} \partial_\theta \psi - \frac{1}{r^2} \partial_{\beta \beta} \psi = \omega,$$

where $r = \sqrt{x^2 + y^2}, \beta = \arctan(y/x)$. We introduce $R = r^\alpha$ and $\Psi(R, \beta) = \frac{1}{r^\alpha} \psi(r, \beta), \Omega(R, \beta) = \omega(r, \beta)$. It is easy to verify that the above elliptic equation is equivalent to

$$L_\omega(\Psi) \triangleq -\alpha^2 R^2 \partial_{RRR} \Psi - \alpha(4 + \alpha) R \partial_R \Psi - \partial_{\beta \beta} \Psi - 4 \Psi = \Omega.$$  

(7.1)
The boundary condition of \( \Psi \) is given by
\[
(7.2) \quad \Psi(R, 0) = \Psi(R, \pi/2) = 0, \quad \lim_{R \to \infty} \Psi(R, \beta) = 0.
\]

7.1. \( \mathcal{H}^3 \) estimates. Recall the \( \mathcal{H}^m \) norm defined in Section 6.5 is given by
\[
(7.3) \quad \|f\|_{\mathcal{H}^m} \equiv \sum_{0 \leq k \leq m} \|D^k_h f \|_{L^2} + \sum_{i+j \leq m-1} \|D^i_h D^j_\beta f \|_{L^2},
\]
where \( \sigma = 99/100, \gamma = 1 + \alpha/10 \) and we have used the definition of \( \varphi_i, \psi_i \) in Definition 5.2.

**Proposition 7.2.** Assume that \( 0 < \alpha \leq \frac{1}{4}, 1 < \gamma \leq \frac{5}{4} \), and \( \Omega \) satisfies \( \|\Omega\|_{\mathcal{H}^3} < +\infty \) with
\[
(7.4) \quad \int_0^{\pi/2} \Omega(R, \beta) \sin(2\beta) d\beta = 0
\]
for every \( R \). The solution of (7.4) satisfies
\[
\alpha^2 R^2 \partial_{RR} \Psi + \alpha R \partial_{R\beta} \Psi + \|\partial_{\beta\beta} \Psi\|_{\mathcal{H}^3} \leq C \|\Omega\|_{\mathcal{H}^3}
\]
for some absolute constant \( C \) independent of \( \alpha \) and \( \gamma \).

**Remark 7.3.** We need the orthogonality assumption (7.4) since \( \sin(2\beta) \) is in the null space of the self-adjoint operator \( L_0(\Psi) = -\partial_{\beta\beta} \Psi - 4\Psi \) with boundary condition \( \Psi(0) = \Psi(\pi/2) = 0 \), which is the limiting operator in (7.1) as \( \alpha \to 0 \).

We only outline some key steps in the proof. Since the \( \mathcal{H}^2 \) norm is the same as that in [11] and the \( \mathcal{H}^2 \) estimates can be easily extended to the \( \mathcal{H}^3 \) estimates, the complete proof follows from the same argument in [11]. Here, the proof is even simpler since there is no first order angular derivative term in (7.1), i.e. \( \partial_\beta \tan(\beta) \Psi \), which is one of the major difficulties in obtaining the elliptic estimate in [11].

**The Orthogonality condition.** Define
\[
\Psi_*(R) = \int_0^{\pi/2} \Psi(R, \beta) \sin(2\beta) d\beta.
\]
Using (7.1) and (7.4), we derive
\[
\alpha^2 R^2 \partial_{RR} \Psi_* + \alpha(\alpha + 4) R \partial_R \Psi_* = 0,
\]
which is an Euler equation and has an explicit solution
\[
\Psi_*(R) = c_1 + c_2 R^{1 - \frac{4+\alpha}{\alpha}}.
\]
Recall the boundary condition in (7.2) for \( \Psi \). We have \( \Psi_* \to 0 \) as \( R \to \infty \). Since \( R^2 \Psi \) vanishes at \( R = 0 \), we derive \( c_1 = c_2 = 0 \) and \( \Psi_* \equiv 0 \).

Recall the boundary condition in (7.2). We can expand \( \Psi(R, \beta) \) in a series
\[
\Psi(R, \beta) = \sum_{n \geq 1} \Psi_n(R) \cos(2n\beta).
\]
Due to the orthogonality condition \( \Psi_*(R) \equiv 0 \), we have \( \Psi_1(R) \equiv 0 \). Therefore, we have
\[
(7.5) \quad \|\partial_\beta \Psi(R, \beta) g(R)\|^2_2 - 16 \|\Psi(R, \beta) g(R)\|^2_2 = \sum_{n \geq 2} (4n^2 - 16) \|\Psi_n(R) g(R)\|^2_{L^2(R)} \geq 0
\]
for some weight \( g(R) \) such that \( \|\partial_\beta \Psi(R, \beta) g(R)\|_2 \) is finite. In particular, we have
\[
\|\partial_\beta \Psi(R, \beta) g(R)\|^2_2 - 4 \|\Psi(R, \beta) g(R)\|^2_2 \geq \frac{3}{4} \|\partial_\beta \Psi(R, \beta) g(R)\|^2_2.
\]
In [11], the corresponding orthogonality condition is \( \int_0^{\pi/2} \Omega(R, \beta) \cos^2(\beta) \sin(\beta) d\beta = 0 \) for every \( R \), which implies \( \|\Psi_1(R) g(R)\|^2_{L^2(R)} \leq \sum_{n \geq 2} \|\Psi_n(R) g(R)\|^2_{L^2(R)} \). Based on this, the positivity of the operator \( L_\alpha (\text{?}) \) in the \( L^2 \) sense is established. Here, we simply have \( \Psi_1(R) \equiv 0 \).

Based on the above estimates, the proof of Proposition 7.2 follows from the same argument as that in [11].
7.2. The singular term. In general the vorticity $\Omega$ does not satisfy the assumption (7.4) in Proposition 7.2. We look for a correction of $\Omega$ to fulfill (7.4).

Suppose that $\Psi$ is the solution of (7.1). Consider $\bar{\Psi} = \Psi + G\sin(2\beta)$. Notice that if $\alpha = 0$, $\sin(2\beta)$ is the kernel of the operator $L_\alpha$ in (7.1) (it is self-adjoint if $\alpha = 0$). We have

$$L_\alpha(\bar{\Psi}) = \Omega + L_\alpha(G(\sin(2\beta)) = \Omega - (\alpha^2 R^2 \partial R G + \alpha(\alpha + 4) R \partial R G) \sin(2\beta).$$

We look for $G(R)$ that satisfies $G(R) \to 0$ as $R \to +\infty$ and $L_\alpha(\bar{\Psi})$ is orthogonal to $\sin(2\beta)$:

$$0 = \int_0^{\pi/2} \sin(2\beta) \Omega - (\alpha^2 R^2 \partial R G + \alpha(\alpha + 4) R \partial R G) \sin(2\beta))d\beta$$

for every $R$, which implies

$$(7.6) \quad \alpha^2 R^2 \partial R G + \alpha(\alpha + 4) R \partial R G = 4 \Omega_*,$$

where $\Omega_*(R) = \int_0^{\pi/2} R(\Omega, R, \beta) \sin(2\beta)d\beta$ and we have used $\int_0^{\pi/2} \sin^2(2\beta)d\beta = \frac{\pi}{4}$. The above ODE is first order with respect to $\partial R \Omega$ and can be solved explicitly. Multiplying the integrating factor $\frac{1}{\alpha^2} R^{-2 + \frac{4\alpha}{\pi}}$ to both sides and then integrating from 0 to $R$ yield

$$R^{\frac{4\alpha}{\pi}} \partial R G = \frac{4}{\alpha^2 \pi} \int_0^R \Omega_*(t) t^{\frac{4}{\pi} - 1}dt. \quad \text{Imposing the vanishing condition } G(R) \to 0 \text{ as } R \to +\infty, \text{ we yield}$$

$$G = -\frac{4}{\alpha^2 \pi} \int_R^{\infty} \Omega_*(s) s^{-\frac{4\alpha}{\pi}} \int_0^s \Omega_*(t) t^{\frac{4}{\pi} - 1}dt ds.$$

Using integration by parts, we further derive

$$G = \frac{1}{\alpha \pi} \int_R \partial_\alpha (s^{-\frac{4\alpha}{\pi}}) \int_0^s \Omega_*(t) t^{\frac{4}{\pi} - 1}dt ds = -\frac{1}{\alpha \pi} \int_R^{\infty} \Omega_*(s) \frac{s^{\frac{4\alpha}{\pi}}}{s} ds - \frac{1}{\alpha \pi} R^{-\frac{4\alpha}{\pi}} \int_0^R \Omega_*(s) s^{\frac{4\alpha}{\pi} - 1}ds.$$

Using the notation $L_{12}(\Omega)$ (7.2), we can rewrite

$$(7.7) \quad G = -\frac{1}{\alpha \pi} L_{12}(\Omega)(R) - \frac{1}{\alpha \pi} R^{-\frac{4\alpha}{\pi}} \int_0^R \Omega_*(s) s^{\frac{4\alpha}{\pi} - 1}ds = - L_{12}(\Omega)(R) + \bar{G}.$$ 

Although there is a large factor $1/\alpha$ in $\bar{G}$, it can be proved that $||\bar{G}||_{H^3}$ can be bounded by $C ||\Omega||_{H^3}$ using a Hardy-type inequality. We refer the reader to [11] and [10] for more details.

Using Proposition 7.2 and an argument similar to that in [11], we have the following result, which is similar to Theorem 2 in [11].

Proposition 7.4. Assume that $\alpha \leq \frac{1}{4}$ and $\Omega \in H^3$. Let $\Psi$ be the unique $C^2$ solution to (7.1) with boundary condition (7.2). Then we have

$$\alpha^2 ||R^2 \partial R \Psi||_{H^3} + \alpha ||R \partial R \Psi||_{H^2} + ||\partial_\beta \psi - \frac{1}{\alpha \pi} \sin(2\beta) L_{12}(\Omega)||_{H^2} \leq C ||\Omega||_{H^3}$$

for some absolute constant $C$ independent of $\alpha, \gamma$ in the definition of $H^3$ (7.3).

Remark 7.5. The $H^3$ norm of $\alpha D_R \partial_\beta \Psi$ is not included in Theorem 2 in [11]. Yet, the estimate of such term can be derived easily from Proposition 7.2 and the estimate of $G$ defined in (7.7).

7.3. Estimates of nonlinear terms. In this subsection, we generalize several estimates of nonlinear terms derived in [11] to be used in our nonlinear stability estimate in the next section.

We define the $W^{l,\infty}$ norm:

$$(7.8) \quad ||f||_{W^{l,\infty}} \triangleq \sum_{0 \leq k + j \leq l, j \neq 0} \left\| \sin(2\beta)^{-\frac{k}{2}} D_R^k \left( \sin(2\beta) \partial_j \right)^j f \right\|_{L^\infty} + \sum_{0 \leq k \leq l} \left\| D_R^k f \right\|_{L^\infty}.$$ 

Our $W^{l,\infty}$ norm is slightly different from that in [11]. Firstly, for $j = 0$, we remove the weight $(\sin(2\beta))^{-\alpha/5}$, since it is not necessary in the proof. Secondly, in the proof of estimates related to $W^{l,\infty}$, one only needs the weight $(\alpha/10 + \sin(2\beta))^{-1}$ rather than $(\alpha/10 + \sin(2\beta))^{-j}$ for large $j : 0 \leq k + j \leq l, j \neq 0$ in $W^{l,\infty}$. Thirdly, we replace the operator $(R + 1)^k \partial_R^k$ by $D_R^k = (R \partial_R)^k$. 

The reason for doing this is that the stronger weight \((R + 1)^k\) is not necessary in the derivation of the product rule in \([11]\) related to \(W^{l, \infty}\), and that the differential operator \(D_R\) commutes with \(\mathcal{L}_\alpha\) in the elliptic equation \((7.1)\), while \(\partial_R\) does not. Therefore, the higher order elliptic estimates related to \(\partial_R\) can depend on the value of \(\alpha\). We are only going to use these estimates when \(\alpha\) is very small. In Proposition \(7.11\), we show that an embedding estimates related to \(W^{l, \infty}\) does not require to use \((R + 1)^k\) in an essential way.

**Functions in \(W^{l, \infty}\).** From Proposition \(A.6\) in the Appendix, we know that \(\Gamma(\beta), \bar{\Omega}, \bar{\eta} \in W^{l, \infty}\).

**Remark 7.6.** We do not apply the \(W^{l, \infty}\) norm to \(\xi, \xi\).

Recall the \(C^1\) norm in \([6, 21]\). For the \(C^1\) and \(W^{1, \infty}\) norms, we have a simple result.

**Proposition 7.7.** For any \(f, g \in C^1\) and \(\frac{1 + R}{R} p \in W^{1, \infty}\), we have

\[
\|fg\|_{C^1} \leq \|f\|_{C^1} \|g\|_{C^1}, \quad \|p\|_{C^1} \lesssim \|\frac{1 + R}{R} p\|_{W^{1, \infty}}.
\]

The \(W^{4, \infty}\) version of the following result is presented in \([11]\), whose generalization to \(W^{l, \infty}\) is straightforward.

**Proposition 7.8.** Assume that \(f, g \in W^{l, \infty}\). Then we have

\[
\|fg\|_{W^{l, \infty}} \lesssim \|f\|_{W^{l, \infty}} \|g\|_{W^{l, \infty}}.
\]

Recall from \((4.9)\) that \(L_{12}(\bar{\Omega}) = \frac{3\pi\alpha}{2} + \frac{1}{1 + R}\). We define \(\bar{\Psi}\) by

\[
\mathcal{L}_\alpha(\bar{\Psi}) = -\alpha^2 R^2 \partial_R \bar{\Psi} - \alpha(4 + \alpha) R \partial_R \bar{\Psi} - \partial_{\beta\beta} \bar{\Psi} - 4\Psi = \bar{\Omega},
\]

where \(\mathcal{L}_\alpha\) is the operator in \((7.1)\). We have the following estimates.

**Proposition 7.9.** For \(\alpha \leq \frac{1}{4}\), we have

\[
\|\frac{1 + R}{R} \partial_{\beta\beta} (\bar{\Psi} - \frac{\sin(2\beta)}{\pi\alpha} L_{12}(\bar{\Omega}))\|_{W^{l, \infty}} \lesssim \alpha, \quad \|L_{12}(\bar{\Omega})\|_{W^{l, \infty}} \lesssim \alpha,
\]

\[
\alpha \|\frac{1 + R}{R} D_R^2 \bar{\Psi}\|_{W^{l, \infty}} + \alpha \|\frac{1 + R}{R} \partial_{\beta\beta} D_R \bar{\Psi}\|_{W^{l, \infty}} + \|\frac{1 + R}{R} \partial_{\beta\beta} (\bar{\Psi} - \frac{\sin(2\beta)}{\pi\alpha} L_{12}(\bar{\Omega}))\|_{W^{l, \infty}} \lesssim \alpha.
\]

**Proof.** The proof of the first inequality follows from the same argument in \([11]\). Here, the proof is even simpler since there is no first order angular derivative term in \((7.1)\), i.e. \(\partial_\beta(\tan(\beta)\Psi)\) in \([11]\).

Using the formula \((4.9)\), we know \(L_{12}(\bar{\Omega}) = \frac{3\pi\alpha}{2(1 + R)} \cdot \frac{\sin(2\beta)}{\pi\alpha} L_{12}(\bar{\Omega}) = \frac{3\sin(2\beta)}{2(1 + R)}\). Since \(L_{12}(\bar{\Omega})\) does not depend on \(\beta\), the second inequality follows from a direct calculation.

For \(0 \leq i, j \leq 7\), we have

\[
\alpha \left|\frac{1 + R}{R} D_R^{i+1} \partial_{\beta}^j 3\sin(2\beta)\right| + \alpha \left|\frac{1 + R}{R} D_R^{i+2} \partial_{\beta}^j 3\sin(2\beta)\right| \lesssim \alpha.
\]

Using \(\frac{3}{2} + \sin(2\beta) \geq \sin(2\beta)^{1-\frac{1}{2}}\), the definition of \(W^{5, \infty}\) in \((5.3)\) and the first inequality, we complete the proof.

7.3.1. **Some embedding Lemmas.** The \(\mathcal{H}^2\) and \(W^{2, \infty}\) versions of the following result have been proved in \([11]\). We remark that we have modified the weight for the \(R\) variable in the \(W^{l, \infty}\) norm.

**Proposition 7.10.** Assume that \(\frac{(1 + R)^3}{R^2} f \in W^{3, \infty}\), then we have \(f \in \mathcal{H}^3\) and

\[
\|f\|_{\mathcal{H}^3} \lesssim \|\frac{(1 + R)^3}{R^2} f\|_{W^{3, \infty}}.
\]

**Proof.** Recall the definition of \(\varphi_i\) in \((5.13)\) and \(\mathcal{H}^3\) in \((7.3)\), respectively. The main term to consider in \(\|f\|_{\mathcal{H}^3}\) is \(\|D_R^3 f \varphi_i^{1/2}\|_{L^2}\). Observe that

\[
\|D_R^3 f (\frac{(1 + R)^2}{R})^2 \sin(2\beta)^{-\sigma/2}\|_{L^2} \lesssim \|\frac{(1 + R)^3}{R^2} D_R^3 f\|_{\infty}.
\]
Lemma 7.12. We have

\[
\frac{(1 + R)^3}{R^2} D_R f = D_R\left(\frac{(1 + R)^3}{R^2} f\right) - 3D_R\left(\frac{(1 + R)^3}{R^2} f\right) + 3D_R\left(\frac{(1 + R)^3}{R^2} f\right) D_R f - D_R\left(\frac{(1 + R)^3}{R^2} f\right) = I_1 + I_2 + I_3 + I_4.
\]

Notice that \(|D_R\left(\frac{(1 + R)^3}{R^2} f\right)| \leq \frac{(1 + R)^3}{R^2}\) for \(k = 1, 2, 3\). Then by the definition of \(W^{3,\infty}\), \(I_1\) and \(I_4\) can be bounded by \(|\left(\frac{(1 + R)^3}{R^2} f\right)|_{W^{3,\infty}}\). For \(I_2, I_3\), we have

\[
|I_2| \leq \left|\frac{(1 + R)^3}{R^2} D_R f\right|, \quad |I_3| \leq \left|\frac{(1 + R)^3}{R^2} D_R f\right|,
\]

which contains lower order derivatives of \(f\) (compared to \(D_R^3 f\)). The same argument implies that \(|I_2|, |I_3|\) can be further bounded by \(|\left(\frac{(1 + R)^3}{R^2} f\right)|_{W^{3,\infty}}\). Other terms in \(H^3\) norm can be estimated similarly.

We have the following decay estimate.

**Lemma 7.11.** Suppose that \(\xi \in H^2(\psi)\), we have

\[
|\frac{R^{1/2} \sin(2\beta)^{1/4}}{L^\infty} \xi| \lesssim |\xi|_{H^2(\psi)}.
\]

The above estimate also holds for \(\xi \in H^2\) since \(H^2\) is stronger than \(H^2(\psi)\) (see Lemma 6.3).

**Proof.** Using a direct calculation yields

\[
|\sin(2\beta)^{1/2} R^2 \xi|^L \lesssim \|\partial R \partial_\beta (\sin(2\beta)^{1/2} R^2 \xi)^2\|_L = \|\partial_\beta (\sin(2\beta)^{1/2}(2 \xi D_R \xi))^2\|_L,
\]

\[
\lesssim \|\sin(2\beta)^{-1/2} (\xi^2 + 2 \xi D_R \xi)\|_{L^1} + \|\sin(2\beta)^{1/2} (2 \xi D_R \xi + 2 \xi \partial_\beta D_R \xi)\|_{L^1}.
\]

Recall the definition of \(H^2(\psi)\) (6.19) and the weights in Definition 5.2. Using the Cauchy-Schwarz inequality concludes the proof.

**Lemma 7.12.** We have

\[
|f|_{L^\infty} \lesssim \alpha^{-1/2} |f|_{H^2},
\]

\[
|f|_{C^1} = |f|_{L^\infty} + \|\frac{1 + R}{R} D_R f\|_{L^\infty} + \left|(1 + (R \sin(2\beta)^{1/2})^{-1} \frac{d}{dR}) D_R f\right|_{L^\infty} \lesssim \alpha^{-1/2} |f|_{H^3},
\]

provided that the right hand side is bounded.

A similar \(H^2\) version of the above Lemma is presented in [11]. Recall the definition of \(H^3\) and its associated weights in (7.3). The proof of the \(C^1\) estimates follows from the argument in the proof of Lemma 7.11, the Cauchy-Schwarz inequality and

\[
\|\frac{1}{1 + R} \sin(2\beta)^{\gamma/2 - 1}\| \lesssim \alpha^{-1/2}, \quad \|\frac{R^{1 - \frac{\alpha}{2}}}{(1 + R)^2} \sin(2\beta)^{\gamma/2 - 1 - \frac{\alpha}{2}}\| \lesssim \alpha^{-1/2}.
\]

7.3.2. The product rules. In this subsection, we generalize the estimates of nonlinear terms and the transport terms derived in [11] to the \(H^3\) and \(H^3(\psi)\) norm.

Denote the sum space \(X \triangleq H^3 \oplus W^{5,\infty}\) with sum norm

\[
|f|_X \triangleq \inf\{|g|_{H^3} + |h|_{W^{5,\infty}} : f = g + h\}.
\]

We use the following product rules to estimate the nonlinear terms.

**Proposition 7.13.** For all \(f \in X, g \in H^3, \xi \in H^3(\psi) \cap C^1\), we have

\[
|fg|_{H^3} \lesssim \alpha^{-1/2} |f|_X |g|_{H^3},
\]

\[
|f\xi|_{H^3(\psi)} \lesssim \alpha^{-1/2} |f|_X (\alpha^{1/2} |\xi|_{C^1} + ||\xi||_{H^3(\psi)}).
\]

The \(H^2\) version of the above Lemma is presented in [11]. Its generalization to the \(H^3\) estimates in (7.10) is straightforward. We focus on the product rule with \(H^3(\psi)\) norm.
Proof. We prove the second inequality in (7.10). If \( f \in W^{5,\infty} \), applying the same argument in (11) yields
\[
\|f\xi\|_{H^3(\psi)} \lesssim \alpha^{-1/2}\|f\|_{W^{5,\infty}}\|\xi\|_{H^3(\psi)}.
\]

Now, we assume \( f \in H^3 \). We consider the third derivative \( D^3 = D^3_R D^3_L \) terms since other terms are easier. If \( (D^3, \psi_i) = (D^3_R, \psi_1), (D^3_L, \psi_2) \), we use a \( L^2 \times L^\infty \) interpolation
\[
\langle (D^3(f\xi))^2, \psi_i \rangle \lesssim \sum_{k=0,1} \langle (D^k f)^2 (D^{3-k} \xi)^2, \psi_i \rangle + \sum_{k=2,3} \langle (D^k f)^2 (D^{3-k} \xi)^2, \psi_i \rangle
\]
\[
\lesssim \|f\|c_1\|\xi\|_{H^3(\psi)} + \|f\|\|H^3(\psi)\|\|\xi\|_{c_1} \lesssim \alpha^{-1/2}\|f\|H^1\|\xi\|_{H^3(\psi)} + \|f\|\|H^3\|\|\xi\|_{c_1},
\]
where we have applied Lemma 7.12 to \( \|f\|_{c_1} \) and Lemma 6.23 to obtain the last inequality.

If \( D^3 = D^3_R D^3_L \) or \( D^3_R D^3_L \), the corresponding singular weight in the \( H^3(\psi) \) norm is \( \psi_2 \). We consider the term \( D^2_R \xi D^2_L f \psi_2^{1/2} \) in the \( L^2 \) estimate of \( D^3(f\xi)\psi_2^{1/2}, \) which is a typical and the most difficult term. The previous \( L^2 \times L^\infty \) estimate fails since \( D^2_R \xi \psi_2^{1/2} \) is not in \( L^2(R, \beta) \). Recall the Definition 5.2 of \( \psi_2, \phi_2 \). Denote
\[
(7.11) \ W = \frac{1 + R^4}{R^4}, \quad P = \sin(\beta)^{-\gamma} \cos(\beta)^{-\gamma}, \quad Q = \sin(2\beta)^{-\gamma}, \quad S = \sin(2\beta)^{-\gamma}, \quad \lambda = \gamma - \sigma.
\]

Clearly, we have \( \phi_2 = WQ, \psi_2 = WP, \psi_1 \parallel WS \) and \( P \lesssim \sin(\beta)^{\gamma}Q \). We use a \( L^2(R, L^\infty(\beta)) \times L^\infty(R, L^2(\beta)) \) estimate
\[
\langle (D^2_R \xi)^2 (D^2_L f)^2, WP \rangle \leq \left[ \left( \sin(\beta)^{\gamma/2} D^2_R \xi (R, \cdot) \right) \|L^\infty(\beta)\| D^2_L f Q^{1/2}(R, \cdot) \|L^2(\beta)\| W \right]^{1/2}
\]
\[
\triangleq \|A(R)^2 B(R)^2 W\| L^1(R).
\]

We further estimate the integrands \( A(R), B(R) \). Using the Poincare inequality, we have
\[
A(R) \lesssim \|\partial_\beta (\sin(\beta)^{\gamma/2} D^2_R \xi (R, \cdot))\| L^1(\beta) + \|\sin(\beta)^{\gamma/2} D^2_R \xi (R, \cdot)\| L^2(\beta) \triangleq A_1(R) + A_2(R).
\]

Using the Cauchy-Schwarz inequality, we can bound the first term as follows
\[
A_1(R) \lesssim \|\sin(\beta)^{\gamma/2-1} D^2_R \xi (R, \cdot)\| L^1(\beta) + \|\sin(\beta)^{\gamma/2} \sin(2\beta)^{-1} D^2_R \xi (R, \cdot)\| L^1(\beta)
\]
\[
\lesssim \|S^{1/2} D^2_R \xi (R, \cdot)\| L^2(\beta) + \|D^2_R \xi (R, \cdot)\| L^2(\beta) \|P^{-1/2} \sin(\beta)^{\gamma/2-1} \|L^2\| = \alpha^{-1/2}(\beta).
\]

Recall \( P, S, \lambda \) defined in (7.11) and \( \gamma = 1 + \frac{\beta}{\lambda} \). A simple calculation yields
\[
\|S^{-1/2} \sin(\beta)^{\gamma/2-1} \|L^2 \lesssim \|\sin(\beta)^{\gamma/2-1}\| L^1(\beta) \lesssim \alpha^{-1/2} ;
\]
\[
\|P^{-1/2} \sin(\beta)^{\gamma/2-1} \|L^2 \lesssim \|\sin(\beta)^{\gamma/2-1} \cos(\beta)^{\gamma/2-1}\| L^2(\beta) \lesssim \alpha^{-1/2}.
\]

Combining the above estimates, we derive
\[
A \lesssim A_1(R) + A_2(R) \lesssim \alpha^{-1/2}(\|S^{1/2} D^2_R \xi (R, \cdot)\| L^2(\beta) + \|D^2_R \xi (R, \cdot)\| L^2(\beta)) + \|D^2_R \xi (R, \cdot)\| L^2(\beta).
\]

Recall \( WS \lesssim \psi_1, WP \lesssim \psi_2 \). Consequently, we have
\[
\|A^2(R) W\| L^1(\beta) \lesssim \alpha^{-1}\|\xi\|_{H^3(\psi)}.
\]

Recall \( B(R) \) in (7.12). Since \( D^2 \beta f Q^{1/2} W^{1/2}, D_R D^2 \beta f Q^{1/2} W^{1/2} \in L^2 \), we have \( \lim \inf_{R \to 0} B(R) = 0 \) and yield
\[
\|B^2\|_{L^\infty(\beta)} \leq \|\partial_R B^2\|_{L^1(\beta)} \lesssim \|\partial_R D^2 \beta f Q^{1/2}\|_{L^2} \|D^2 \beta f Q^{1/2}\|_{L^2} \lesssim \|f\|^2_{L^2},
\]
where we have used \( \partial_R = R^{-1} D_R, R^{-1} \lesssim \sqrt{W^2} \) and \( WQ = \phi_2 \) to obtain the last inequality. Plugging the estimates of \( A \) and \( B \) in (7.12), we yield the desired estimate on \( \|D^2 \xi D^2_L f \psi_2^{1/2}\|_{L^2} \). \( \square \)

\(^3\) The \( L^2(R, L^\infty(\beta)) \times L^\infty(R, L^2(\beta)) \) estimate of the mixed derivatives term in the \( H^3 \) norm is due to Dongyi Wei. We are grateful to him for telling us this estimate. We apply this idea to derive the estimates in the \( H^3(\psi) \) norm.
Proposition 7.16. Assume that $u, \partial_\beta u, D_R u \in \mathcal{H}^3$ and $\Omega \in \mathcal{H}^3, \xi \in \mathcal{H}^3(\psi) \cap C^1$, we have

\[
|\langle \Omega, u D_R \Omega \rangle_{\mathcal{H}^3}| \lesssim \alpha^{-\frac{1}{2}} (||u||_{\mathcal{H}^3} + ||\partial_\beta u||_{\mathcal{H}^3} + ||D_R u||_{\mathcal{H}^3}) ||\Omega||_{\mathcal{H}^3}^2,
\]

\[
|\langle \xi, u D_R \xi \rangle_{\mathcal{H}^3(\psi)}| \lesssim \alpha^{-\frac{1}{2}} (||\xi||_{\mathcal{H}^3} + ||\partial_\beta u||_{\mathcal{H}^3} + ||D_R u||_{\mathcal{H}^3}) (||\xi||_{\mathcal{H}^3(\psi)} + \alpha^{1/2} ||\xi||_{C^1})^2.
\]

Moreover, for all $u, D_R u \in \mathcal{H}^3 \oplus \mathcal{W}^{5,\infty}$ and $\Omega \in \mathcal{H}^3, \xi \in \mathcal{H}^3(\psi) \cap C^1$, we have

\[
|\langle \Omega, u D_\beta \Omega \rangle_{\mathcal{H}^3}| \lesssim \alpha^{-1/2} (||u||_{\mathcal{H}^3} + ||D_R u||_{\mathcal{H}^3}) ||\Omega||_{\mathcal{H}^3}^2,
\]

\[
|\langle \xi, u D_\beta \xi \rangle_{\mathcal{H}^3(\psi)}| \lesssim \alpha^{-1/2} (||u||_{\mathcal{H}^3} + ||D_R u||_{\mathcal{H}^3}) (||\xi||_{\mathcal{H}^3(\psi)} + \alpha^{1/2} ||\xi||_{C^1})^2.
\]

The proof follows from the argument in the proof of Proposition 7.13 and that in [11]. Here, the proof is easier since the data is more regular (than $\mathcal{H}^2$), i.e. $\mathcal{H}^3$ or $\mathcal{H}^3(\psi)$, and then the estimate of several nonlinear terms can be done by applying $L^\infty$ estimate on one term. To estimate the mixed derivative terms, e.g. $\langle D_R^2 D_\beta \xi, D_R^3 D_\beta (u D_\beta \xi) \psi_2 \rangle_1$, one can also apply the $L^2(\Omega, L^\infty(\beta)) \times L^\infty(\Omega, L^2(\beta))$ argument.

The following result is a simple $\mathcal{H}^3, \mathcal{H}^3(\psi)$ generalization of another transport estimate in [11].

Proposition 7.15. Let $\mathcal{H}^3(\rho)$ be either $\mathcal{H}^3$ or $\mathcal{H}^3(\psi)$. For all $g \in \mathcal{H}^3(\rho)$, $u$ with $||D_R^i u||_{L^\infty} < \infty$ for $i \leq 3$ and $||D_R^i D_\beta^j \partial_\beta u||_{L^\infty} < \infty$ for $i + j \leq 2$, we have

\[
|\langle g, u D_R g \rangle_{\mathcal{H}^3(\rho)}| \lesssim \alpha^{-1/2} \left( \sum_{0 \leq i \leq 3} ||D_R^i u||_{L^\infty} + \sum_{i+j \leq 2} ||D_R^i D_\beta^j \partial_\beta u||_{L^\infty} \right) ||g||_{\mathcal{H}^3(\rho)}^2.
\]

The proof follows simply from applying $L^\infty$ estimate on the $u$ term and integration by parts.

7.4. A new estimate of the transport term and the estimate of $v_\xi \xi$. In this subsection, we establish a new estimate of the transport term which is necessary to close the nonlinear estimate and estimate $||v_\xi \xi||_{\mathcal{H}^3}$ which is not covered by Proposition 7.13.

Proposition 7.17. Let $\Psi$ be a solution of (1.1). Suppose that $g, \Omega \in \mathcal{H}^3, \xi \in \mathcal{H}^3(\psi) \cap C^1$. We have

\[
|\langle g, \frac{1}{\sin(2\beta)} D_R \Psi D_\beta g \rangle_{\mathcal{H}^3}| \lesssim \alpha^{-3/2} ||\Omega||_{\mathcal{H}^3} ||g||_{\mathcal{H}^3}^2,
\]

\[
|\langle \xi, \frac{1}{\sin(2\beta)} D_R \Psi D_\beta \xi \rangle_{\mathcal{H}^3(\psi)}| \lesssim \alpha^{-3/2} ||\Omega||_{\mathcal{H}^3} (||\xi||_{\mathcal{H}^3(\psi)} + \alpha^{1/2} ||\xi||_{C^1})^2.
\]

If one apply Proposition 7.13 with $u = \frac{D_R \Psi}{\sin(2\beta)}$, $||D_R u||_{\mathcal{H}^3}$ in the upper bound cannot be bounded by $||\Omega||_{\mathcal{H}^3}$.

Proof. Denote $u = \frac{D_R \Psi}{\sin(2\beta)}$. The estimate of the transport term is similar to that in Proposition 7.13 except that we need to perform integration by parts for the terms $\langle D^3 g, u D^3 D_\beta \psi \rangle$ in the estimate. We focus on a typical and difficult term $\langle D_R^2 D_\beta \xi, D_R^3 D_\beta \xi \psi_2 \rangle$ to see why we can improve the estimate in Proposition 7.15. Other terms can be estimated similarly.

For this term, it suffices to estimate the $L^2$ norm of $D_R^2 u D_\beta^2 \xi \psi_2^{1/2}$. It can be estimated by applying $L^2(R, L^\infty(\beta))$ estimate on $D_R^2 u$, which can be further bounded by $\alpha^{-1/2} ||\Omega||_{\mathcal{H}^3}$ using Proposition 7.4 and $L^\infty(R, L^2(\beta))$ estimate on $D_\beta^2 \xi$. It is similar to the argument in the proof of Proposition 7.13 and we omit the detail. □
Remark 7.18. If one only consider $\mathcal{H}^2$ data, to estimate the mixed derivative term $(D_R D_\beta \xi, D_R u D_\beta^2 \xi_2)$ in the $\mathcal{H}^2$ estimate, the above argument fails since the $L^\infty(R, L^2(\beta))$ estimate of $D_\beta^3 \xi$ cannot be further bounded by $||\xi||_{\mathcal{H}^2(\psi)}$ and $||\xi||_{L^\infty}$. Moreover, $||D_R u||_{L^\infty}$ cannot be bounded by $||\Omega||_{\mathcal{H}^2}$.

The reason we perform the $\mathcal{H}^3$ estimates in Subsection 6.3 is to establish the above transport estimates.

Finally, we estimate the nonlinear term $v_x \xi$ in the $\eta$ equation (5.1).

**Proposition 7.19.** Let $\Psi, \bar{\Psi}$ be a solution of (7.1) with source term $\Omega, \bar{\Omega}$, respectively, and $V_1(\Psi)$ be the operator which is related to $v_x$ and is to be defined in (8.6). Assume that $\xi \in \mathcal{H}^3(\psi) \cap C^1, \Omega \in \mathcal{H}^3$. We have

\[
||V_1(\Psi)\xi||_{\mathcal{H}^3} \lesssim \alpha^{-1/2}||\Omega||_{\mathcal{H}^3}(\alpha^{1/2}||\xi||_{C^1} + ||\xi||_{\mathcal{H}^2(\psi)}),
\]

\[
||V_1(\bar{\Psi})\xi||_{\mathcal{H}^3} \lesssim \alpha^{-1/2}||\xi||_{\mathcal{H}^3(\psi)}.
\]

The difficulty lies in that $\mathcal{H}^3(\psi)$ is weaker than $\mathcal{H}^3$ (see Lemma 6.5). We cannot apply Proposition 7.13 directly to estimate $v_x \xi$. We need to use a key fact that $v_x$ vanishes on $\beta = 0$.

**Proof.** We use the formula of $V_1(\Psi)$ (8.8) to be derived

\[
V_1(\Psi) = \alpha(1 + 2 \cos^2(\beta)) D_R \Psi - \alpha D_R D_\beta \Psi - D_\beta \Psi^* + 2 \Psi^* + \sin^2(\beta) \partial_\beta^2 \Psi^* + \alpha^2 \cos^2(\beta) D_R^2 \Psi
\]

\[
\equiv A(\Psi) + \alpha^2 \cos^2(\beta) D_R^2 \Psi.
\]

where $\Psi^* = \Psi - \frac{\sin(2\beta)}{2}\partial_{\Omega} L_{12}(\Omega)$. We first consider the second inequality in (7.13). Notice that $V_1(\bar{\Psi})$ vanishes on $\beta = 0$. More precisely, Proposition 7.9 implies $\sin(\beta)^{-1/2} V_1(\bar{\Psi}) \in \mathcal{W}^{5, \infty}$. Applying the product rule in $\mathcal{H}^3$ norm in Proposition 7.13, Lemma 6.5 and then Proposition 7.3, we obtain

\[
||V_1(\bar{\Psi})\xi||_{\mathcal{H}^3} \lesssim \alpha^{-1/2}||\xi||_{\mathcal{H}^3(\psi)}.
\]

Next, we consider the first inequality in (7.13). From Proposition 7.4, we know that $\sin(\beta)^{-1/2} A(\Psi) \in \mathcal{H}^3$. Applying Propositions 7.13, 7.4 and Lemma 6.5, we derive

\[
||A(\Psi)\xi||_{\mathcal{H}^3} \lesssim \alpha^{-1/2}||A(\Psi) \sin(\beta)^{-1/2}||_{\mathcal{H}^3}||\xi||_{\mathcal{H}^3(\psi)} \lesssim \alpha^{-1/2}||\Omega||_{\mathcal{H}^3}||\xi||_{\mathcal{H}^3(\psi)}.
\]

Finally, we focus on the term $g \equiv \alpha^2 D_R^2 \Psi$ in $V_1(\Psi)$. We consider the third derivative terms $D^3(R \Psi^\alpha \xi)$ with $D^3 = D^3 R D^3_\beta, i + j = 3$ in the $\mathcal{H}^3$ estimate since other terms are easier. If $D^3 = D^3_\beta$, we need to estimate the $L^2$ norm of $D^3_\beta (g \xi) v_1^{1/2}$. Since $v_1 \approx \psi_1$, the estimate follows from the argument in the proof of Proposition 7.13 and we obtain

\[
||D^3_\beta (g \xi)^{1/2} \phi_2^{1/2}||_{L^2} \lesssim ||\sin(\beta)^{1/2} D_\beta g \cdot \sin(\beta)^{1/4} \xi||_{\mathcal{H}^2} + ||\sin(\beta)^{-1/2} g \sin(\beta)^{1/4} D_\beta \xi||_{\mathcal{H}^2}.
\]

Notice that $\sin(\beta)^{-1/4} \phi_2 \lesssim \phi_1, \psi_1$. Using the idea in the discussion of Lemma 6.1 and expanding the $\mathcal{H}^2$ norm, one can verify easily that

\[
||D^2(D_\beta (g \xi)^{1/2} \phi_2^{1/2})||_{L^2} \lesssim ||\sin(\beta)^{1/2} \partial_\beta g \cdot \sin(\beta)^{1/4} \xi||_{\mathcal{H}^2} + ||\sin(\beta)^{-1/2} g \sin(\beta)^{1/4} D_\beta \xi||_{\mathcal{H}^2}.
\]

Applying the $\mathcal{H}^2$ version of the product rule in Proposition 7.13 (it is given in (11)), Proposition 7.22 to $g = \alpha^2 D_R^2 \Psi$, and Lemma 6.5, we obtain

\[
||D^2(D_\beta (g \xi)^{1/2} \phi_2^{1/2})||_{L^2} \lesssim \alpha^{-1/2}||\sin(\beta)^{1/2} \partial_\beta g \cdot \sin(\beta)^{1/4} \xi||_{\mathcal{H}^2} + \alpha^{-1/2}||\sin(\beta)^{-1/2} g \sin(\beta)^{1/4} D_\beta \xi||_{\mathcal{H}^2} \lesssim \alpha^{3/2}||\Omega||_{\mathcal{H}^3}||\xi||_{\mathcal{H}^3(\psi)}.
\]

Combining the estimates of $A(\Psi)$ and $\alpha^2 D_R^2 \Psi$ completes the proof. \qed
8. Nonlinear stability

In this section, we complete the estimates of the remaining terms \( R_3 \) in Corollary \((6.4)\) and in \((6.12), (6.16), (6.17)\). We will prove the following for the energy \( E_3 \) in \((6.6)\) and \( E(\xi, \infty) \)

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} E_3 & \leq \frac{1}{12} E_3^2 + C_a \alpha^{1/2} (E_3^2 + \alpha \| \xi \|^2_{C^1}) + C \alpha^{-3/2} (E_3 + \alpha^{1/2} \| \xi \|_{C^1})^3 + C \alpha^2 E_3, \\
\frac{1}{2} \frac{d}{dt} E(\xi, \infty) & \leq -E(\xi, \infty)^2 + C \| \xi \|_{C^1} (\alpha^{-1/2} E_3 + \alpha \| \xi \|_{C^1}) \\
& \quad + C \| \xi \|_{C^1} (\alpha^{-1} E_3 + \alpha^{-1} E_3 \| \xi \|_{C^1}) + C \alpha^2 E(\xi, \infty),
\end{align*}
\]

for any initial perturbation \( \Omega, \eta, \xi \) with \( E_3(\Omega, \eta, \xi) < +\infty \) and \( E(\xi, \infty) < +\infty \), where

\[
E(\xi, \infty) \doteq (\| \xi \|^2_{\infty} + \| \phi_2 D_\beta \xi \|^2_{\infty} + \mu_4 \| \phi_1 D R \xi \|^2_{\infty})^{1/2}
\]

for some absolute constants \( \mu_4 \). \( E(\xi, \infty) \) is equivalent to \( \| \xi \|_{C^1} \) \((6.24)\) once we determine the absolute constants \( \mu_4 \).

The major step is the linear stability that gives the damping term \((-\frac{1}{12} + C_\alpha) E_3^2 \) and \((-1 + C_\alpha) E(\xi, \infty)^2 \). We have already established the linear stability in Corollary \((6.4)\) and estimates \((6.12), (6.16), (6.17)\). The remaining terms \( R_3 \) in Corollary \((6.4)\) and in \((6.12), (6.16), (6.17)\) contribute other terms in \((8.1)-(8.2)\). We will further construct an energy \( E^2(\Omega, \eta, \xi) \doteq \alpha E(\xi, \infty)^2 + E^2_3(\Omega, \eta, \xi) \) and these remaining terms are relatively small at the threshold \( E = O(\alpha^2) \). Then we can close the nonlinear estimate.

We will first derive several formulas for later use in subsection \(8.1\). Then we estimate the remaining terms mentioned above. In subsection \(8.2\) and \(8.3\), we will apply the product rules obtained in subsection \(7.4\) to estimate the transport terms and nonlinear terms and then complete the estimate \((8.1)\). We will derive the \( C^1 \) estimate \((8.2)\) in subsection \(8.4\) and prove finite time blowup in subsection \(8.5\). We remark that estimates similar to the \( C^1 \) estimates \((8.2)\) are not required in \((11)\) since there is no swirl.

Notations. Throughout this section, \( \chi \) is the radial cutoff function in Lemma \((A.3)\). We use \( \Psi_*, \Psi_* \) to denote the lower order terms in \( \Psi, \Psi \), i.e.

\[
\Psi_* \doteq \Psi - \frac{\sin(2\beta)}{\pi \alpha} L_{12}(\Omega), \quad \Psi_* \doteq \Psi - \frac{\sin(2\beta)}{\pi \alpha} L_{12}(\Omega).
\]

\( \Psi_* \) and \( \Psi \) enjoys the elliptic estimate in Proposition \((7.2)\) and \( \Psi, \Psi_* \) satisfy Proposition \((7.9)\).

8.1. Formulas of the velocity and related terms. In this subsection, we derive the formulas of the velocity in terms of the stream function in the \( (R, \beta) \) coordinate to be used later and then collect the remaining terms to be estimated in the nonlinear stability analysis.

Denote

\[
\begin{align*}
u & \doteq U(\Psi), \quad v \doteq V(\Psi), \quad u_x \doteq U_1(\Psi), \quad u_y \doteq U_2(\Psi), \quad v_x \doteq V_1(\Psi), \quad v_y \doteq V_2(\Psi).
\end{align*}
\]

The formula of \( U, V \) in terms of \( \Psi \) are given in \((2.8), (2.9)\). We also collect them below. Using \((2.8), (2.9)\), \( D_R = R \partial_R, r \partial_r = \alpha D_R \) and the incompressible condition \( u_x + v_y = 0 \), we compute

\[
\begin{align*}
U(\Psi) & = -2r \sin \beta \Psi - \alpha r \sin \beta D_R \Psi - r \cos \beta \partial_\beta \Psi, \quad V(\Psi) = 2r \cos \beta \Psi + \alpha r \cos \beta D_R \Psi - r \sin \beta \partial_\beta \Psi, \\
U_1(\Psi) & = \frac{1}{2} \alpha^2 \sin(2\beta) D_R^2 \Psi - \frac{\alpha}{2} \sin(2\beta) D_R \Psi - \cos(2\beta) \partial_\beta \Psi - \alpha \cos(2\beta) \partial_\beta D_R \Psi + \frac{\sin(2\beta)}{2} \partial_\beta^2 \Psi, \\
U_2(\Psi) & = \alpha \left( -2 - 2 \sin^2 \beta \right) D_R \Psi - \alpha \partial_\beta D_R \Psi - \partial_\beta \Psi - 2 \Psi - \alpha^2 \sin^2(\beta) D_R^2 \Psi - \cos^2(\beta) \partial_\beta^2 \Psi, \\
V_1(\Psi) & = \alpha \left( 1 + 2 \cos^2 \beta \right) D_R \Psi - \alpha \partial_\beta D_R \Psi - \partial_\beta \Psi + 2 \Psi + \alpha^2 \cos^2(\beta) D_R^2 \Psi + \sin^2(\beta) \partial_\beta^2 \Psi, \\
V_2(\Psi) & = -U_1(\Psi).
\end{align*}
\]

Recall \( \Psi = \frac{\sin(2\beta)}{\pi \alpha} L_{12}(\Omega) + \Psi_* \). For the terms not involving the \( R \)-derivative, e.g. \( \Psi, \partial_\beta \Psi \), we compute the contributions from the leading order part of \( \Psi \), i.e. \( \frac{\sin(2\beta)}{\pi \alpha} L_{12}(\Omega) \), and \( \Psi_* \).
separately, (8.7) 
\[ U(\Psi) = -\frac{2r \cos(\beta)}{\pi \alpha} L_{12}(\Omega) - 2r \sin(\beta) \Psi - \alpha r \sin \beta D_R \Psi - r \cos \beta \partial_{\beta} \Psi \] 
\[ V(\Psi) = \frac{2r \sin(\beta)}{\pi \alpha} L_{12}(\Omega) + 2r \cos \beta \Psi + \alpha r \cos \beta D_R \Psi - r \sin \beta \partial_{\beta} \Psi \] 
\[ U_1(\Psi) = -\frac{2}{\pi \alpha} L_{12}(\Omega) - \frac{\alpha^2}{2} \sin(2\beta) D_R^2 \Psi - \frac{\alpha}{2} \sin(2\beta) D_R \Psi - \cos(2\beta) \partial_{\beta} \Psi - \alpha \cos(2\beta) \partial_{\beta} D_R \Psi \] 
\[ + \frac{\sin(2\beta)}{2} \partial_{\beta}^2 \Psi = -\frac{2}{\pi \alpha} L_{12}(\Omega) + U_1(\Psi, \Psi_*) \] 
\[ V_2(\Psi) = -U_1(\Psi) = \frac{2}{\pi \alpha} L_{12}(\Omega) - U_1(\Psi, \Psi_*) \].

The first term in the formulas of \( U, V, U_1, V_2 \) is the leading order term. Observe that 
\[ -D_\beta \sin(2\beta) - 2 \sin(2\beta) - \cos^2(\beta) \partial^2_{\beta} \sin(2\beta) = 0, \quad -D_\beta \sin(2\beta) + 2 \sin(2\beta) + \cos^2(\beta) \partial^2_{\beta} \sin(2\beta) = 0. \]

For the terms not involving the \( R \)-derivative in \( U_2(\Psi), V_1(\Psi) \) (8.7), the contributions from \( \sin(2\beta)L_{12}(\Omega) \) cancel each other. Hence, we have

\[ U_2(\Psi) = \alpha(1 - 2 \sin^2(\beta)) D_R \Psi - \alpha D_R D_\beta \Psi - D_\beta \Psi - 2 \Psi + \alpha \sin^2(\beta) D_R^2 \Psi - \cos^2(\beta) \partial_{\beta} \Psi, \]
\[ V_1(\Psi) = \alpha(1 + 2 \cos^2(\beta)) D_R \Psi - \alpha D_R D_\beta \Psi - D_\beta \Psi + 2 \Psi + \alpha \cos^2(\beta) D_R^2 \Psi + \sin^2(\beta) \partial_{\beta} \Psi. \]

We decompose \( U, V \) in (8.7), (8.8) so that we can apply the elliptic estimate in Propositions 7.4 (7.9) to \( U(\Psi, \Psi_*), V(\Psi, \Psi_*), U_1(\Psi, \Psi_*), U_2(\Psi), V_1(\Psi) \).

Recall the formula of \( u \cdot \nabla \) in (2.11) 
\[ u \cdot \nabla = -r D_\beta + (2 \Psi + \alpha D_\beta R \Psi) \partial_{\beta}. \]

Since \( \Psi = \frac{\sin(2\beta)}{\pi \alpha} L_{12}(\Omega) + \Psi_* \), \( D_\beta = \sin(2\beta) \partial_{\beta} \), we have

\[ u \cdot \nabla = \left( -\frac{2 \cos(2\beta)}{\pi} L_{12}(\Omega) - \alpha \partial_{\beta} \Psi \right) D_R + \left( \frac{2 \Psi + \alpha D_R \Psi}{\sin(2\beta)} \right) \partial_{\beta} D_R \Psi \]
\[ + \frac{2 \Psi + \alpha D_R \Psi}{\sin(2\beta)} D_R. \]

Using (4.9), we have \( \frac{2}{\pi \alpha} L_{12}(\Omega) = \frac{3}{1 + R} \) and
\[ u \cdot \nabla = \frac{3}{1 + R} D_\beta + T(\Omega). \]

Recall the formulations (5.3)-(5.7) and their equivalence (5.11). We use the notations (8.5) to rewrite \( u_x, u_y \) and so on, and the above computations to expand the remaining terms \( R \) in (5.3)-(5.7). \( R \) consists of three parts: the lower order terms in the linearized equation (denote as \( P \)), the error term \( \tilde{F} \) (1.10) and the nonlinear term \( N \) (5.2). The formula of \( P \) is given by (8.10) 
\[ R = P_0 + F_\Omega + N_\Omega, \quad R_\eta = P_\eta + F_\eta + N_\eta, \quad R_\xi = P_\xi + F_\xi + N_\xi, \]
\[ P_0 = (-3 \alpha D_R - T(\Omega)) \Omega + (\alpha c_\omega D_R - (u \cdot \nabla)) \Omega, \]
\[ P_\eta = (-3 \alpha D_R - T(\Omega)) \eta + (\alpha c_\omega D_R - (u \cdot \nabla)) \eta - (U_1(\Psi) + \frac{3}{1 + R}) \eta - (U_1(\Psi) + \frac{2}{\pi \alpha} L_{12}(\Omega)) \bar{\eta} - (V_1(\Psi) + \xi - V_1(\Psi) \xi), \]
\[ P_\xi = (-3 \alpha D_R - T(\Omega)) \xi + (\alpha c_\omega D_R - (u \cdot \nabla)) \xi + (V_2(\Psi) + \frac{3}{1 + R}) \xi + (V_2(\Psi) + \frac{2}{\pi \alpha} L_{12}(\Omega)) \bar{\xi} - (U_2(\Psi) \xi + U_2(\Psi) \eta). \]

We remark that \( P \) is the difference between the linear part of (5.1) and (5.5)-(5.7).

Recall \( c_\omega = -1, \tilde{c}_\omega = \frac{1}{4} + 3 \) and \( \Omega, \bar{\eta} \) in (4.8). Notice that \( c_1 = \frac{1}{\alpha}, \Omega = \frac{3}{R}, \eta_* = \frac{6 \alpha R}{c_1 (1 + R)^3}, \Gamma = \cos(\beta) \xi \) is a solution of (5.2) and \( \bar{\Omega}, \bar{\eta} \) satisfy \( \bar{\Omega} = \Omega \Gamma(\beta), \bar{\eta} = \eta_\alpha \Gamma(\beta), \):
In subsection 8.4, we estimate the first three terms in $\Omega$. Our main tools in this and the next few subsections are the product rules, the elliptic estimates \[ I \]

Recall cancellation near $\Omega, \eta$ that $\Omega \in H^3$. In subsection 8.2, we estimate the transport terms in the last three terms in $\Omega$. In subsection 8.3, we estimate the nonlinear terms in the last three terms in $\Omega$. We are going to estimate $\Omega, \eta \Delta \rho(x) \frac{\partial}{\partial t} \bar{\Psi}$ in the first three terms and the last three terms in $\Omega$. We will choose initial perturbations $\Omega, \eta, \xi$ in these classes. In subsection 8.2, we estimate the transport terms in $\Omega, \eta, \xi \in H^4$. In subsection 8.3, we estimate the nonlinear terms in $\Omega, \eta, \xi \in H^4$. In subsection 8.4, we estimate the first three terms in $\Omega$.

8.2. Analysis of the transport terms in $P, N, F$.

In this subsection, we estimate the transport terms in $P, N$ and $F$ in $H^4$ or $H^3(\psi)$ norm. Our main tools in this and the next few subsections are the product rules, the elliptic estimates obtained in Section 7 and Lemma A.3 on $L_{12}(\Omega)$. The reader should pay attention to the subtle cancellation near $R = 0$ in the estimates in subsections 8.2.3 and 8.2.4.

8.2.1. Transport terms $I : (-3\alpha D_R - T(\Omega))g$ in $P$. We estimate \[ I_1 = \langle (-3\alpha D_R - T(\Omega))\Omega, \Omega \rangle_{H^3}, I_2 = \langle (-3\alpha D_R - T(\Omega))\eta, \eta \rangle_{H^3}, I_3 = \langle (-3\alpha D_R - T(\Omega))\xi, \xi \rangle_{H^3}. \]

Recall $T(\Omega)$ in (8.9) \[ 3\alpha D_R + T(\Omega) = 3\alpha D_R - \frac{2\cos(2\beta)}{\pi} L_{12}(\Omega) D_R - \alpha \partial_\beta \bar{\Psi} \psi D_R + \frac{1}{\sin(2\beta)} \bar{\Psi} + \alpha D_R \bar{\Psi} D_B. \]

Applying Proposition 7.9 to estimate the above coefficients, then Proposition 7.15 to the $D_B$ transport terms and Proposition 7.16 to the $D_R$ transport terms yield \[ I_1 \lesssim \alpha^{1/2} ||\Omega||_H^2, I_2 \lesssim \alpha^{1/2} ||\eta||_H^2, I_3 \lesssim \alpha^{1/2} ||\xi||_H^2. \]

8.2.2. Transport term $II : -\alpha c_1 R \partial_\rho g - (u \cdot \nabla)g$ in $N$. We are going to estimate \[ \langle (-\alpha c_1 D_R - (u \cdot \nabla))\Omega, \Omega \rangle_{H^3}, \langle (-\alpha c_1 D_R - (u \cdot \nabla))\eta, \eta \rangle_{H^3}, \langle (-\alpha c_1 D_R - (u \cdot \nabla))\xi, \xi \rangle_{H^3}. \]

Recall $c_1 = -\frac{2(1-a)}{\pi a} L_{12}(\Omega)(0)$ in (4.11) and the computation about $u \cdot \nabla$ in (8.5) \[ (-\alpha c_1 D_R - (u \cdot \nabla)) = \frac{2(1-a)}{\pi a} L_{12}(\Omega)(0) + \frac{2\cos(2\beta)}{\pi} L_{12}(\Omega) + \alpha \partial_\beta \bar{\Psi} \psi D_R - \frac{2}{\pi a} L_{12}(\Omega) + \frac{2\bar{\Psi}}{\sin(2\beta)} + \frac{\alpha D_R \bar{\Psi}}{\sin(2\beta)} D_B. \]

For the first two $D_R$ transport terms, we apply Proposition 7.10 and Lemma A.3 to estimate $||D_R^k L_{12}(\Omega)||_{L^\infty}$ for $k \leq 3$. For the third, fourth, fifth $D_B$ transport terms, we apply Proposition 7.13, Proposition 7.4 to $\partial_\beta \bar{\Psi} \psi$, $\frac{\bar{\Psi}}{\sin(2\beta)}$, and Lemma A.3 in Lemma A.3 to $L_{12}(\Omega)$. For the last transport term, we use Proposition 7.17. Hence, we derive \[ |I_1| \lesssim \alpha^{-3/2} ||\Omega||_H^3, |I_2| \lesssim \alpha^{-3/2} ||\Omega||_H^3 ||\eta||_H^3, |I_3| \lesssim \alpha^{-3/2} ||\Omega||_H^3 (||\xi||_H^3 + \alpha^{1/2} ||\xi||_C^1)^2. \]

The largest term is $\frac{2}{\pi a} L_{12}(\Omega) D_B$, which leads to $\alpha^{-3/2}$ in the upper bound.
8.2.3. Transport term III : \((\alpha c_\omega D_R - (u \cdot \nabla)g)\) in \(P\). Next, we estimate
\[
|||\alpha c_\omega D_R - (u \cdot \nabla)\bar{\Omega}|||_{H^3}, \quad |||\alpha c_\omega D_R - (u \cdot \nabla)\bar{\eta}|||_{H^3}, \quad |||\alpha c_\omega D_R - (u \cdot \nabla)\bar{\xi}|||_{H^3(\psi)}.
\]
Recall that \(H^3\) contains a singular weight \(\frac{1 + R^3}{R^4}\). We use the explicit form \(\Gamma(\beta) = \cos(\beta)^n\) and a careful calculation to cancel the singular weight \(R^{-4}\) near \(R = 0\). Using the formula for \(c_\omega\) in (4.11) and the computation in (8.9), we have
\[
(8.13) \quad (\alpha c_\omega D_R - (u \cdot \nabla)g) = \left( -\frac{2}{\pi} L_{12}(\Omega)(0)D_R + \frac{2 \cos(2\beta)}{\pi} L_{12}(\Omega)D_R - \frac{2}{\alpha} L_{12}(\Omega)D_\beta \right) g
+ (\alpha \partial_\beta \Psi_\ast D_R - (\sin(2\beta))^{-1}(2\Psi_\ast + \alpha D_R \Psi_\ast)D_\beta)g \triangleq I(g) + II(g).
\]
Denote \(Q = L_{12}(\Omega) - \chi L_{12}(\Omega)(0)\). We use \(L_{12}(\Omega) = Q + \chi L_{12}(\Omega)(0)\) to rewrite \(I(g)\)
\[
(8.14) \quad I = \frac{2}{\pi} L_{12}(\Omega)(0)(-D_R g + \cos(2\beta) \chi^2 D_R g - \frac{1}{\alpha} \chi D_\beta g) + \frac{2}{\pi} Q(\cos(2\beta) D_R g - \frac{1}{\alpha} D_\beta g) \triangleq I_1 + I_2.
\]
Using (6.21) and the formula of \(g = \bar{\Omega}, \bar{\eta}\) in (4.13), we have
\[
D_\beta \Gamma = -2\alpha \sin^2(\beta) \Gamma, \quad D_\beta g = -2\alpha \sin^2(\beta) g.
\]
It follows that
\[
(8.15) \quad I_1 = \frac{2}{\pi} L_{12}(\Omega)(0)(-D_R g + \cos(2\beta) \chi D_R g + 2\sin^2(\beta) \chi D_R g) = \frac{2}{\pi} L_{12}(\Omega)(0)(-1 - \chi D_R g + 2\sin^2(\beta) \chi (-D_R g + g)).
\]
Since the smooth cutoff function \(\chi\) satisfies \(1 - \chi(R) = 0\) for \(R \leq 1\). \(I_1\) vanishes quadratically near \(R = 0\). For \((g, H^3(\rho)) = (\bar{\Omega}, \bar{H}^3), (\bar{\eta}, \bar{H}^3)\) or \((\bar{\xi}, \bar{H}^3(\psi))\), applying Lemma A.3 to \(g = \bar{\Omega}, \bar{\eta}, \bar{\xi}\) in Lemma A.7 to \(g = \xi\) and using a direct calculation yield
\[
|||I_1(g)|||_{H^3(\rho)} \lesssim |L_{12}(\Omega)(0)(0)(|||1 - \chi|||_{H^3(\rho)} + |||D_R g - g|||_{H^3(\rho)}| \lesssim \alpha|L_{12}(\Omega)(0)| \lesssim \alpha|||\Omega|||_{H^3},
\]
where we have used Lemma A.3 in the last inequality.
Recall \(Q = L_{12}(\Omega) - \chi L_{12}(\Omega)(0)\) and \(I_2, II(g)\) in (8.13), (8.14). For \(g = \bar{\Omega}, \bar{\eta}, \bar{\xi}\) applying the product estimate in Proposition 7.13 we get
\[
|||I_2(g)|||_{H^3} \lesssim \alpha^{-1/2}|||Q|||_{H^3}|||D_R g|||_{W^{5, \infty}} + \alpha^{-1/2}|||D_\beta g|||_{W^{5, \infty}} \lesssim \alpha^{1/2}|||\Omega|||_{H^3},
\]
\[
|||II(g)|||_{H^3} \lesssim \alpha^{-1/2}|||\Omega|||_{H^3}|||D_R g|||_{W^{5, \infty}} + |||D_\beta g|||_{W^{5, \infty}} \lesssim \alpha^{3/2}|||\Omega|||_{H^3},
\]
where we have applied Proposition 7.14 to \(g\) in Lemma A.3 to \(Q\) and Proposition A.6 to \(g = \bar{\Omega}, \bar{\eta}\).
For \(g = \xi\), applying Proposition 7.13 yields
\[
|||I_2(\bar{\xi})|||_{H^3(\rho)} \lesssim \alpha^{-1/2}|||Q|||_{H^3}|||D_R \bar{\xi}|||_{C^3} + |||D_\beta \bar{\xi}|||_{H^3(\rho)} + \alpha^{-1/2}|||D_\beta \bar{\xi}|||_{C^3} + |||D_\beta \bar{\xi}|||_{H^3(\rho)} \lesssim \alpha^{1/2}|||\Omega|||_{H^3},
\]
\[
|||II(\bar{\xi})|||_{H^3(\rho)} \lesssim \alpha^{-1/2}|||\Omega|||_{H^3}|||D_\beta \bar{\xi}|||_{H^3(\rho)} + \alpha^{-1/2}|||D_\beta \bar{\xi}|||_{H^3(\rho)} + \alpha^{-1/2}|||D_\beta \bar{\xi}|||_{H^3(\rho)} \lesssim \alpha^{1/2}|||\Omega|||_{H^3},
\]
where we have used Lemma A.7 to estimate the norm of \(\bar{\xi}\). Hence, we prove
\[
|||\alpha c_\omega D_R - (u \cdot \nabla)\bar{\Omega}|||_{H^3} + |||\alpha c_\omega D_R - (u \cdot \nabla)\bar{\eta}|||_{H^3} + |||\alpha c_\omega D_R - (u \cdot \nabla)\bar{\xi}|||_{H^3(\rho)} \lesssim \alpha^{1/2}|||\Omega|||_{H^3}.
\]
8.2.4. Transport term IV: \((-3\alpha D_R - \bar{u} \cdot \nabla)g\) in \(F_\bar{\Omega}, F_\bar{\eta}, F_\bar{\xi}\). We will prove for \((g, H^3(\rho)) = (\bar{\Omega}, \bar{H}^3), (\bar{\eta}, \bar{H}^3), (\bar{\xi}, \bar{H}^3(\psi))\)
\[
(8.16) \quad |||(-3\alpha D_R - \bar{u} \cdot \nabla)g|||_{H^3(\rho)} \lesssim \alpha^2.
\]
From (4.3), we have \(\frac{2}{\pi} L_{12}(\Omega)(0) = 3\alpha\). Hence, we can apply the decomposition in (8.13)- (8.14) to \((-3\alpha D_R - \bar{u} \cdot \nabla)g\) to get
\[
(8.17) \quad (-3\alpha D_R - \bar{u} \cdot \nabla)g = I_1(g) + I_2(g) + II(g), \quad II(g) = (\alpha \partial_\beta \bar{\Psi}_\ast D_R - (\sin(2\beta))^{-1}(2\bar{\Psi}_\ast + \alpha D_R \bar{\Psi}_\ast)D_\beta)g
I_1(g) = \frac{2}{\pi} L_{12}(\Omega)(0)(-D_R g + \cos(2\beta) \chi D_R g - \frac{1}{\alpha} \chi D_\beta g), \quad I_2(g) = \frac{2}{\pi} Q(\cos(2\beta) D_R g - \frac{1}{\alpha} D_\beta g),
\]
where \(Q = L_{12}(\Omega) - \chi L_{12}(\Omega)(0)\). Notice that the computation (8.13) still holds for \(g = \bar{\Omega}, \bar{\eta}
\[
I_1(g) = \frac{2}{\pi} L_{12}(\Omega)(0)(-1 - \chi) D_R g + 2\sin^2(\beta) \chi (-D_R g + g).
\]
\[\text{The estimate of } I_2(\bar{\xi}), II(\bar{\xi}) \text{ can be improved to } \alpha^{3/2}|||\Omega|||_{H^3} \text{ but we do not need this extra smallness here.} \]
Recall $L_{12}(\bar{\Omega}) = \frac{3\pi}{21 + R^2}$. Notice that $(1 - \chi)D_{Rg}, D_{Rg} - g, QD_{Rg}, QD_{Rg}$ vanish quadratically near $R = 0$. Applying Lemma $A.5$ to $g = \bar{\Omega}, \bar{\eta}$ and using a direct calculation yield

$$||I_1(g)||_{H^3} \lesssim \alpha |L_{12}(\bar{\Omega})(0)| \lesssim \alpha^2, \quad ||I_2(g)||_{H^3} \lesssim \alpha^2.$$  

Since $\xi$ already vanishes quadratically near $R = 0$, using Lemma $A.7$ for $\xi$ and a direct calculation give

$$||I_1(\xi)||_{H^3(\psi)} \lesssim \alpha |L_{12}(\bar{\Omega})(0)| \lesssim \alpha^2, \quad ||I_2(\xi)||_{H^3(\psi)} \lesssim \alpha^2.$$  

For $II(g)$ with $g = \bar{\Omega}, \bar{\eta}$, we apply Propositions $7.10, 7.8$ and the triangle inequality to yield

$$||II(g)||_{H^3} \lesssim \||1 + R)^3 II(g)||_{W^{3,\infty}} \lesssim \||1 + R\alpha \bar{\Psi}||_{W^{3,\infty}} ||(1 + R)^2 D_{Rg}||_{W^{3,\infty}} \lesssim \alpha^2,$$

where we have applied Proposition $7.9$ to $\bar{\Psi}, \bar{\Psi}_s$ and Proposition $A.6$ to $g = \bar{\Omega}, \bar{\eta}$.  

For $II(\xi)$, we use Propositions $7.13, 7,9$ and Lemma $A.7$ to get

$$||II(\xi)||_{H^3(\psi)} \lesssim \alpha^{-1/2}||\bar{\partial}_g \bar{\Psi}_s||_{W^{3,\infty}} (\alpha^{3/2}||D_{R\bar{\xi}}||_{W^{3,\infty}} + \alpha ||D_{R\bar{\xi}}||_{H^3(\psi)})$$

$$+ \alpha^{-1/2}(\cos(2\beta))^{-1}(2\bar{\Psi}_s + \alpha D_{R\bar{\Psi}})||_{W^{3,\infty}} (\alpha^{3/2}||D_{R\bar{\xi}}||_{C^1} + ||D_{R\bar{\xi}}||_{H^3(\psi)}) \lesssim \alpha^{5/2}.$$

### 8.3. Nonlinear forcing terms in $P, N, F$.

The estimates in this subsection are obtained by applying the product estimates in subsection 7.3 directly. The reader should pay attention to the cancellation near $R = 0$ in the estimates in subsection 8.3.2.

#### 8.3.1. Nonlinear forcing term in $P_0, P_2$.

We are going to estimate

$$I_1 = ||(U_1(\bar{\Psi}) + \frac{3}{1 + R})\bar{\eta} - (U_1(\bar{\Psi}) + \frac{2}{\pi \bar{\Omega}} L_{12}(\bar{\Omega})\bar{\eta})||_{H^3}, \quad I_2 = ||V_1(\bar{\Psi})\bar{\xi} + V_1(\bar{\Psi})\bar{\xi}||_{H^3},$$

$$II_1 = ||(-V_2(\bar{\Psi}) + \frac{3}{1 + R})\bar{\xi} + (-V_2(\bar{\Psi}) + \frac{2}{\pi \bar{\Omega}} L_{12}(\bar{\Omega})\bar{\xi})||_{H^3}, \quad II_2 = ||U_2(\bar{\Psi})\bar{\eta} + U_2(\bar{\Psi})\bar{\eta}||_{H^3}.$$  

From (4.9), $\frac{2}{\pi \bar{\Omega}} L_{12}(\bar{\Omega}) = \frac{\pi}{1 + R}$. Recall the formula of $U_i, V_j$ in (8.7) - (8.8). Applying Propositions $7.9, 7.3$ we obtain

$$||U_1(\bar{\Psi}) + \frac{2}{\pi \bar{\Omega}} L_{12}(\bar{\Omega})||_{W^{3,\infty}} = ||-V_2(\bar{\Psi}) + \frac{2}{\pi \bar{\Omega}} L_{12}(\bar{\Omega})||_{W^{3,\infty}} \lesssim \alpha, \quad ||U_2(\bar{\Psi})||_{W^{3,\infty}} \lesssim \alpha,$$

$$||U_1(\bar{\Psi}) + \frac{2}{\pi \bar{\Omega}} L_{12}(\bar{\Omega})||_{H^3} = ||-V_2(\bar{\Psi}) + \frac{2}{\pi \bar{\Omega}} L_{12}(\bar{\Omega})||_{W^{3,\infty}} \lesssim ||\bar{\Omega}||_{H^3}, \quad ||U_2(\bar{\Psi})||_{H^3} \lesssim ||\bar{\Omega}||_{H^3}.$$  

Applying Proposition $7.13$, Lemma $A.6$ to $\bar{\eta}$ and Lemma $A.7$ to $\bar{\xi}$, we yield

$$I_1 \lesssim \alpha^{1/2}||\bar{\eta}||_{H^3} + \alpha^{-1/2}||\bar{\Omega}||_{H^3} \lesssim \alpha^{1/2}||\bar{\eta}||_{H^3} + ||\bar{\Omega}||_{H^3},$$

$$II_1 \lesssim \alpha^{1/2}(\alpha^{1/2}||\bar{\xi}||_{C^1} + ||\bar{\xi}||_{H^3(\psi)}) + \alpha^{-1/2}||\bar{\Omega}||_{H^3} \lesssim \alpha^{3/2}||\bar{\xi}||_{C^1} + ||\bar{\xi}||_{H^3(\psi)} \lesssim \alpha^{3/2}||\bar{\xi}||_{H^3(\psi)},$$

where we have used Lemma $A.7$ in the last inequality. Using Lemma $A.5$ and Proposition $7.18$ we derive

$$II_2 \lesssim ||U_2(\bar{\Psi})\bar{\eta} + U_2(\bar{\Psi})\bar{\eta}||_{H^3} \lesssim \alpha^{-1/2}||\bar{\Omega}||_{H^3} + ||\bar{\Omega}||_{H^3} + ||U_2(\bar{\Psi})||_{W^{3,\infty}}$$

$$\lesssim \alpha^{1/2}(||\bar{\Omega}||_{H^3} + ||\bar{\Omega}||_{H^3} + ||\bar{\Omega}||_{H^3}) \lesssim \alpha^{1/2}(||\bar{\Omega}||_{H^3} + ||\bar{\Omega}||_{H^3}).$$

For $I_2$, we use Proposition $7.19$ and Lemma $A.7$ to obtain

$$I_2 \lesssim \alpha^{1/2}||\bar{\xi}||_{H^3(\psi)} + \alpha^{-1/2}||\bar{\Omega}||_{H^3} \lesssim \alpha^{1/2}||\bar{\xi}||_{H^3(\psi)} + \alpha^{3/2}||\bar{\Omega}||_{H^3}.$$
8.3.2. Nonlinear forcing term in \( N \).

\[ c_\omega \Omega, \ (2c_\omega - U_1(\Psi))\eta - V_1(\Psi)\xi, (2c_\omega - V_2(\Psi))\xi - U_2(\Psi)\eta. \]

Recall the formula of \( U_1, V_2 \) in (6.7). We use the following decomposition

\[ -V_2(\Psi) = U_1(\Psi) = (U_1(\Psi) + \frac{2}{\pi \alpha} L_{12}(\Omega)) - \frac{2}{\pi \alpha} L_{12}(\Omega) = I + II. \]

Applying Proposition 7.3 to \( I \) and Lemma A.3 to \( II \), we obtain

\[ \| V_2(\Psi)\|_X = \| U_1(\Psi)\|_X \lesssim \| I\|_{\mathcal{H}^3} + \alpha^{-1}\| L_{12}(\Omega)\|_X \lesssim \alpha^{-1}\| \Omega\|_{\mathcal{H}^3}. \]

Applying Propositions 7.13, 7.4, we get

\[ \| U_1(\Psi)\|_{\mathcal{H}^3} \lesssim \alpha^{-3/2}\| \Omega\|_{\mathcal{H}^3}\| \eta\|_{\mathcal{H}^3}, \quad \| (V_2(\Psi)\xi)\|_{\mathcal{H}^3(\psi)} \lesssim \alpha^{-3/2}\| \Omega\|_{\mathcal{H}^3}\| \xi\|_{\mathcal{H}^3(\psi)} + \alpha^{1/2}\| \xi\|_{C^1}. \]

Applying Proposition 7.19 to \( V_1 \xi \), Proposition 7.13 and Lemma A.5 to \( U_2 \eta \) yields

\[
\| -V_1(\Psi)\|_{\mathcal{H}^3} \lesssim \alpha^{-1/2}\| \Omega\|_{\mathcal{H}^3}\| \xi\|_{\mathcal{H}^3(\psi)} + \alpha^{1/2}\| \xi\|_{C^1},
\]

\[
\| -U_2(\Psi)\|_{\mathcal{H}^3(\psi)} \lesssim \| U_2(\Psi)\|_{\mathcal{H}^3} \lesssim \alpha^{-1/2}\| \Omega\|_{\mathcal{H}^3}\| \eta\|_{\mathcal{H}^3}.\]

Finally, from (4.11), (4.3), the scalar \( c_\omega \) satisfies \( |c_\omega| \lesssim \alpha^{-1}\| \Omega\|_{\mathcal{H}^3}. \) Hence, we obtain

\[ \| c_\omega \Omega\|_{\mathcal{H}^3} \lesssim \alpha^{-1}\| \Omega\|_{\mathcal{H}^3}, \quad \| c_\omega \eta\|_{\mathcal{H}^3} \lesssim \alpha^{-1}\| \Omega\|_{\mathcal{H}^3}\| \eta\|_{\mathcal{H}^3}, \quad \| c_\omega \xi\|_{\mathcal{H}^3(\psi)} \lesssim \alpha^{-1}\| \Omega\|_{\mathcal{H}^3}\| \xi\|_{\mathcal{H}^3(\psi)}. \]

8.3.3. Nonlinear forcing terms in \( F \). Recall that we have estimated the transport term \(-3\alpha D_R \cdot \bar{u}\bar{\nabla})g\) in \( F_{\Omega}, F_{\eta}, F_{\xi} \) in (8.19). The remaining terms in \( \tilde{F}_\eta \) and \( \tilde{F}_\xi \) (see (4.10), (8.11)) are

\[ \tilde{F}_\eta = \left( -\frac{3}{1 + R} - U_1(\Psi)\right)\bar{\eta} - V_1(\bar{\Psi})\bar{\xi}, \quad \tilde{F}_\xi = (2\tilde{c}_\omega - V_2(\bar{\Psi}))\bar{\xi} - U_2(\bar{\Psi})\bar{\eta} - D_R\bar{\xi}, \]

where we have used \(-\alpha \tilde{c}_\omega D_R = -D_R - 3\alpha D_R \) since \( \tilde{c}_\omega = \frac{1}{\alpha} + 3 \) (4.3). From (4.19), we have \( \frac{2}{\pi \alpha} L_{12}(\Omega) = \frac{1}{1 + R}. \) Using \( U_1, V_2 \) in (6.7), (6.8), \( \tilde{\eta} \) (1.8) and Proposition 7.9, we have

\[ \| \frac{1}{1 + R} (U_1(\Psi) + \frac{3}{1 + R})\|_{\mathcal{W}^{3,\infty}} \lesssim \alpha, \quad \| \frac{1}{1 + R} U_2(\Psi)\|_{\mathcal{W}^{3,\infty}} \lesssim \alpha, \quad \| \frac{(1 + R)^2}{R}\bar{\eta}\|_{\mathcal{W}^{3,\infty}} \lesssim \alpha. \]

Applying the embedding in Proposition 7.10 and then the algebra property of \( \mathcal{W}^{3,\infty} \) in Proposition 7.8 to \( \tilde{\eta} \) and the above estimates, we get

\[ \| (-\frac{3}{1 + R} - U_1(\Psi))\tilde{\eta}\|_{\mathcal{H}^3} \lesssim \alpha^2, \quad \| U_2(\bar{\Psi})\tilde{\eta}\|_{\mathcal{H}^3(\psi)} \lesssim \| U_2(\bar{\Psi})\|_{\mathcal{H}^3} \lesssim \alpha^2, \]

where we have used (8.20) in the second inequality. Applying the product estimates in Propositions 7.13, 7.19, Proposition 7.9 to \( V_2(\bar{\Psi}) \) and Lemma A.7 to \( \bar{\xi} \), we yield

\[ \| (V_2(\bar{\Psi}) - \frac{3}{1 + R} \bar{\xi})\|_{\mathcal{H}^3(\psi)} \lesssim \alpha^{1/2}\| \bar{\xi}\|_{C^1} + \| \bar{\xi}\|_{\mathcal{H}^3(\psi)} \lesssim \alpha^{5/2}, \]

\[ \| V_1(\bar{\Psi})\bar{\xi}\|_{\mathcal{H}^3} \lesssim \alpha^{1/2}\| \bar{\xi}\|_{\mathcal{H}^3(\psi)} \lesssim \alpha^{5/2}. \]

For the remaining part in \( II \), we simply use \( \tilde{c}_\omega = -1 \) and Lemma A.7 to get

\[ \| 2\tilde{c}_\omega \bar{\xi} - D_R\bar{\xi}\|_{\mathcal{H}^3(\psi)} + \| \frac{3}{1 + R} \bar{\xi}\|_{\mathcal{H}^3(\psi)} \lesssim \alpha^2. \]

Therefore, combining the formula of \( \tilde{F} \) in (4.10), (8.11), the estimate (8.16) and the above estimates of \( I, II \), we prove

\[ \| \tilde{F}_\Omega\|_{\mathcal{H}^3} \lesssim \alpha^2, \quad \| \tilde{F}_\eta\|_{\mathcal{H}^3} \lesssim \alpha^2, \quad \| \tilde{F}_\xi\|_{\mathcal{H}^3(\psi)} \lesssim \alpha^2. \]
8.4. Analysis of the remaining terms in $R_3$. It remains to estimate
\begin{equation}
(8.22) \quad \langle R_\Omega, \Omega \varphi_0 \rangle, \quad \langle R_\eta, \eta \psi_0 \rangle, \quad \frac{81}{4\pi c} L_{12}(\Omega)(0) \langle R_\Omega, \sin(2\beta)R^{-1} \rangle,
\end{equation}
in $R_3$ (8.12). Recall the definition of $\varphi_0, \psi_0$ in Definition 5.20 and $\varphi_1$ in Definition 5.2. Note that $\psi_0(R, \beta)$ grows linearly for large $R$. Clearly, we have
$$
\varphi_0 \lesssim \varphi_1, \quad \psi_0 = \frac{9}{32} R \Gamma(\beta)^{-1} + \frac{3}{16} \left( \frac{(1 + R)^3}{R^3} + \frac{3}{2} \frac{(1 + R)^4}{R^3} - \frac{3}{2} \right) \Gamma(\beta)^{-1} \lesssim \psi_{0,1} + \psi_{0,2}.
$$
Since the weights $\varphi_0, \psi_{0,2}, R^{-1} \sin(2\beta)$ are much weaker than the weights $\varphi_1$, the estimates of
$$
\langle R_\Omega, \Omega \varphi_0 \rangle, \quad \langle R_\eta, \eta \psi_0 \rangle, \quad \frac{81}{4\pi c} L_{12}(\Omega)(0) \langle R_\Omega, \sin(2\beta)R^{-1} \rangle
$$
follows from the same argument as that in the last two sections and a similar bound can be derived. It remains to estimate $\langle R_\eta, \eta \Gamma(\beta)^{-1} \rangle$. Compared to $\varphi_1$, $\Gamma(\beta)^{-1}$ is much less singular in $R$ and $\beta$. We focus on how to control the growing factor $R$. We use the decay estimate of $\tilde{\eta}$ in Lemma A.5 and $\tilde{\xi}$ in Lemma A.7. In particular, for $i + j \leq 7$ we have
\begin{equation}
(8.23) \quad |D^i_R D^j_\beta \tilde{\eta}| \lesssim \alpha(1 + R)^{-2}, \quad |D^i_R D^j_\beta \tilde{\xi}| \lesssim \tilde{\xi} \lesssim \alpha^2(1 + R)^{-2} \sin(\beta)^{-2\alpha}.
\end{equation}
Recall the decomposition of $R_\eta$ in (8.10) and the error $\tilde{F}_\eta$ defined in (8.11). We use argument similar to that in the last subsection to estimate $\langle \tilde{F}_\eta^2, \Gamma(\beta)^{-1} \rangle$. A typical term in $\tilde{F}_\eta$ can be estimated as follows
$$
\langle V_1(\tilde{\Psi})^2, \Gamma(\beta)^{-1} \rangle \lesssim \alpha^2 \langle \alpha^4(1 + R)^{-4} \sin(\beta)^{-4\alpha}, \Gamma(\beta)^{-1} \rangle \lesssim \alpha^6 \lesssim \alpha^4,
$$
where we have applied Proposition 7.9 to estimate $V_1(\tilde{\Psi})$ and used $\alpha < \frac{1}{8}$ (we will choose $\alpha$ sufficiently small). Similarly, we have $\langle \tilde{F}_\eta^2, \Gamma(\beta)^{-1} \rangle \lesssim \alpha^4$. Hence, using the Cauchy-Schwarz inequality, we get
$$
|\langle \tilde{F}_\eta, \eta \Gamma(\beta)^{-1} \rangle| \lesssim \langle \tilde{F}_\eta^2, \Gamma(\beta)^{-1} \rangle^{1/2} \langle \eta^2, \Gamma(\beta)^{-1} \rangle^{1/2} \lesssim \alpha^2 \langle \eta^2, \psi_0 \rangle^{1/2} \lesssim \alpha^2 E_3,
$$
where we have used (8.22) to derive the last inequality.
Recall $P_\eta$ in (8.10), $N_\eta$ in (5.2) and the formula of $u \cdot \nabla$ in (8.9). We use integration by parts and then a $L^\infty$ estimate to estimate the transport terms in $P_\eta, N_\eta$. A typical term in $P_\eta$ can be estimated as follows
$$
|\langle \frac{2}{\pi \alpha} L_{12}(\Omega) D_\beta \eta, \eta \Gamma^{-1} \rangle| = \frac{2}{\pi \alpha} L_{12}(\Omega) \partial_\beta(\sin(2\beta)\Gamma^{-1})(\eta^2 R)| \lesssim \alpha^{-1} |\|L_{12}(\Omega)\|_{L^\infty} \langle \eta^2, \Gamma(\beta)^{-1} \rangle| \lesssim \alpha^{-1} \|\eta \varphi_1^{1/2}\|_{L^2} \|\psi_0\| \lesssim \alpha^{-1} E_3,
$$
where we have used $\Gamma(\beta) = \cos(\beta)\alpha, |\sin(2\beta)\partial_\beta \Gamma^{-1}| \lesssim \Gamma^{-1}$ in the first inequality, Lemma A.3 in the second inequality and (8.22) in the last inequality.
For the nonlinear terms related to $\eta$, i.e., $2c_\omega - U_1(\Psi) \eta$ in $N_\eta$ (5.2) and $-U_1(\Psi) + \frac{3}{1 + R} \eta$ in $P_\eta$ (8.10), we also apply a $L^\infty$ estimate. For example, we have
$$
|\langle 2c_\omega - U_1(\Psi) \rangle \eta, \eta \Gamma(\beta)^{-1} \rangle| \lesssim |2c_\omega - U_1(\Psi)| \|\eta^2, \psi_0\| \lesssim \alpha^{-1} |\|\Omega\|_{H^3} \langle \eta^2, \psi_0 \rangle \lesssim \alpha^{-1} E_3,
$$
where we have used (8.19) and $|c_\omega| = \frac{2}{\pi \alpha} L_{12}(\Omega)(0) \lesssim \alpha^{-1} |\|\Omega\|_{H^3}$ (see Lemma A.3) in the last inequality.
For the terms related to $\tilde{\eta}, \tilde{\xi}$ in $P_\eta$ (8.11), i.e., $(U_1(\Psi) + \frac{3}{\pi \alpha} L_{12}(\Omega)) \tilde{\eta}, V_1(\Psi) \tilde{\xi}$, they can be estimated easily by using the fast decay of $\tilde{\xi}, \tilde{\eta}$ (8.23).
Finally, for the terms related to $\xi, \tilde{\xi}$ in $N_\eta$ (5.2) and $V_1(\Psi) \tilde{\xi}$ in (8.10), we get
$$
|\langle V_1(\tilde{\Psi}) \xi, \eta \Gamma^{-1} \rangle| + |\langle V_1(\Psi) \xi, \eta \Gamma^{-1} \rangle| \lesssim \|\eta \|_{L^1 \Gamma^{-1/2}} \|\xi \|_{L^2} \|\eta \|_{L^\infty} \|\sin(2\theta)\|_{L^\infty} (||V_1(\tilde{\Psi}) \sin(2\theta)^{-1/4} \Gamma^{-1/2}||_{L^2} + ||V_1(\Psi) \sin(2\theta)^{-1/4} \Gamma^{-1/2}||_{L^2} + ||V_1(\Psi) \sin(2\theta)^{-1/4} \Gamma^{-1/2}||_{L^2} + ||V_1(\Psi) \sin(2\theta)^{-1/4} \Gamma^{-1/2}||_{L^2}) \lesssim \|\eta \|_{L^2} \|\xi \|_{L^2} \|\eta \|_{L^\infty} \|\sin(2\theta)\|_{L^\infty} \lesssim \|\tilde{E}_3^2 \|_{L^2} \|\tilde{E}_3^2 \|_{L^2} \lesssim \|E_3^2 \|_{L^2} (\alpha + E_3),
$$
where we have applied Lemma 7.1 in the second inequality, the weighted $L^2$ (with weight $\sin(2\beta)^{-\sigma}, \sigma = \frac{1}{4\alpha}$) version of Proposition 7.4 in the third inequality and a direct computation using (4.8) in the last inequality.

Combining the estimates of $\bar{F}_\eta, P_\eta, N_\eta$, we have

$$\|\langle R_\eta, \eta R^{-1} \rangle\| \lesssim \alpha^{-3/2} E_3^3 + \alpha^{1/2} E_3^2 + \alpha^2 E_3.$$

8.4.1. Completing the $H^4$ and $H^3(\psi)$ estimates. From (6.23), we can use $E_3$ to bound $\|\Omega\|_{H^3}, \|\eta\|_{H^3}, \|\xi\|_{H^3(\psi)}$. Combining the estimates in the last few subsections, we prove

$$\|\langle R_\Omega, \Omega \rangle H^1\|, \|\langle R_\eta, \eta \rangle H^1\|, \|\langle R_\xi, \xi \rangle H^3(\psi)\| \lesssim \alpha^{1/2} (E_3^2 + \alpha \|\xi\|_{C^1}^2) + \alpha^{-3/2} (E_3 + \alpha^{1/2}) \|\xi\|_{C^1}^3 + \alpha^2 E_3,$$

where $E_3$ is defined in (6.6). Combining Corollary 6.4 and the above estimates, we prove (8.1).

8.5. Remaining terms in the $C^1$ estimate of $\xi$.

Recall that we perform $L^\infty$ estimates of $\xi$ and its derivatives in subsection 6.4. In this subsection, we complete the estimate of the remaining terms in these estimates and derive (8.2).

We group together the remaining terms in (6.12), (6.16), (6.17), which remain to be estimated. They can be bounded by

$$\|\xi\|_{C^1} (||\xi||_{C^1} + ||\xi||_{C^1} + ||\bar{F}_\xi||_{C^1} + ||N_o||_{C^1}), \|\xi\|_{C^1} \|\phi_1 D_R A_2\|_{L^\infty}, \|\phi_1 D_R \xi\|_{C^1}, \|\phi_2 D_\beta \xi\|_{L^\infty} \|A_1 (\phi_2 - 1) \cdot D_\beta \xi\|_{L^\infty}.$$

8.5.1. Analysis of $\Xi_1, \Xi_2, N_o$. Recall $\Xi_1, \Xi_2, N_o$ in (6.8), (6.9), (6.11)

$$\Xi_1 = (\frac{3}{1 + R} - V_2 (\bar{\psi})) \xi, \Xi_2 = -V_2 (\bar{\psi}) \bar{\xi} + c_w (2 \bar{\xi} - R \partial_R \bar{\xi}) + (\alpha c_w R \partial_R - (u \cdot \nabla)) \bar{\xi} - (U_2 (\bar{\psi}) \bar{\eta} + U_2 (\bar{\psi}) \eta),$$

$$N_o = \frac{2}{\pi \alpha} \kappa_{L_{12}}(\Omega) = \frac{3}{1 + R}, \quad \kappa_w = -\frac{2}{\pi \alpha} \kappa_{L_{12}}(\bar{\eta}) = -\frac{2}{\pi \alpha} \kappa_{L_{12}}(\bar{\psi} + \bar{\psi}_+), \quad \kappa = \bar{\kappa}_{L_{12}}(\Omega).$$

Then we obtain $V_2 (\bar{\psi}) - \frac{\pi}{\pi + R} = -U_1 (\bar{\psi}, \bar{\psi}_+)$ (see 8.2).

For the transport term $\alpha c_w R \partial_R - (u \cdot \nabla) \bar{\xi}$, we use the decomposition $\bar{\xi} = \bar{\eta} + \bar{\psi}$ with $g = \bar{\xi}$. Then each term in $\Xi_1, \Xi_2, N_o$ depends only on $L_{12}(\Omega), \bar{\eta}, \bar{\xi}$ and their approximate steady state, e.g. $V_2 (\bar{\psi})$. To estimate the $C^1$ norm of the product in $\Xi_1, \Xi_2, N_o$, using Proposition 7.7 we only need to estimate the $C^1$ norm of each single term.

For the terms depending on $\bar{\psi}, \bar{\psi}_+, e.g. V_2 (\bar{\psi}) - \frac{\pi}{\pi + R} \kappa_{L_{12}}(\Omega)$ (see 8.7), we apply Proposition 7.4 and Lemma 7.12 to obtain the $C^1$ estimate. For the terms depending on $\bar{\psi}, \bar{\psi}_+$, we apply Propositions 7.9 and 7.7 to estimate the $C^1$ norm.

For the terms depending on $L_{12}(\Omega)$, we use (A.4) in Lemma A.3 to estimate the $C^1$ norm.

The slightly difficult term is $V_2 (\bar{\psi})$. Using the formula of $V_2 (\bar{\psi})$ in (8.7), (8.8), Propositions 7.4 and Lemmas 7.12 and 7.3 we get

$$(8.24) \|V_2 (\bar{\psi})\|_{C^1} \lesssim \|V_2 (\bar{\psi}) - \frac{2}{\pi \alpha} \kappa_{L_{12}}(\Omega)\|_{C^1} + \frac{2}{\pi \alpha} \|L_{12}(\Omega)\|_{C^1} \lesssim (\alpha^{-1/2} + \alpha^{-1}) \|\Omega\|_{H^3} \lesssim \alpha^{-1} \|\Omega\|_{H^3}.$$

Using (A.7)-(A.8) in Lemma A.7 and Lemma A.12, we have $\|\xi\|_{C^1} + \|D_R \xi\|_{C^1} \lesssim \alpha^2, \|\bar{\eta}\|_{C^1} \lesssim \alpha$. From (8.24), we know $\|\Xi_1\|_{C^1} \lesssim \alpha \|\bar{\xi}\|_{C^1}, \|\Xi_2\|_{C^1} \lesssim \alpha^{1/2} \|\Omega\|_{H^3} + \alpha^{1/2} \|\bar{\eta}\|_{H^3}, \|N_o\|_{C^1} \lesssim \alpha^{-1} \|\xi\|_{C^1} \|\Omega\|_{H^3} + \alpha^{-1} \|\Omega\|_{H^3} \|\bar{\eta}\|_{H^3}$. The largest term in $\Xi_2$ is given by $(U_2 (\bar{\psi}) \bar{\eta} + U_2 (\bar{\psi}) \eta)$, which leads to the above upper bound.
8.5.2. Analysis of $\bar{F}_\xi$. Recall $\bar{F}_\xi$ and $\bar{u} \cdot \nabla$ defined in (4.10) and (8.9).

$$
\bar{F}_\xi = (2\bar{c}_\omega - V_2(\bar{\Omega}))\bar{\xi} - U_2(\bar{\Omega})\bar{\eta} - \alpha c_l R \bar{\eta} \xi - (\bar{u} \cdot \nabla)\bar{\xi},
$$

$$
\bar{u} \cdot \nabla\bar{\xi} = \left( -\frac{2\cos(2\beta)}{\pi} L_{12}(\Omega) - \alpha \partial_\beta \Psi(s) \right) D_R^2 \bar{\xi} + \left( \frac{2}{\pi \alpha} L_{12}(\Omega) + \frac{2\Psi_* + \alpha D_R \Psi}{\sin(2\beta)} \right) D_\beta \bar{\xi}.
$$

For $\bar{\xi}$ terms, we use $||D_R^2 \bar{\xi}||_{C^1} \lesssim \alpha^2, i + j \leq 2$ from (A.17) and (A.18) in Lemma A.7. For other terms, we use $||\bar{\eta}||_{C^1} \lesssim \alpha$ from Lemma A.5 and apply the strategy in the last subsection to estimate the $C^1$ norm. We get

$$
||\bar{F}_\xi||_{C^1} \lesssim \alpha^2.
$$

8.5.3. $||\phi_2 D_\beta, A_2||_{C^1}, ||\phi_1 D_R, A_2||_{C^1}$. Recall $A_2$ defined in (6.13). Using (8.9) and the $C^1$ norm defined in (6.23), we obtain

$$
||\phi D, H \bar{D}|| = ||\phi D H \cdot \bar{D} - H \bar{D} \phi \cdot D \xi|| \leq ||H||_{C^1} ||\xi||_{C^1} + ||H||_{L^\infty} ||\phi||_{L^\infty} ||\phi D\xi||_{L^\infty} \lesssim ||H||_{C^1} ||\xi||_{C^1}.
$$

Applying the strategy in Section 8.5.1 to estimate the $C^1$ norm of $\Psi, \bar{\Psi}, L_{12}(\Omega)$ terms, we obtain

$$
||H_1||_{C^1} \lesssim \alpha^{-1} ||\Omega||_{H^3}, \quad ||H_2||_{C^1} \lesssim ||\Omega||_{H^3} + \alpha, \quad ||H_3||_{C^1} \lesssim \alpha^{-1/2} ||\Omega||_{H^3} + \alpha, \quad ||H_4||_{C^1} \lesssim \alpha^{-1/2} ||\Omega||_{H^3} + \alpha.
$$

The largest term is $\alpha^{-1} L_{12}(\Omega)$ in $H_1$, which is estimated by (A.4) in Lemma A.3 and using $D_\beta L_{12}(\Omega) = 0$.

Combining the above estimates, we conclude that

$$
||D_R, A_2||_{C^1}, ||D_\beta, A_2||_{C^1} \lesssim ||\xi||_{C^1} (\alpha^{-1} ||\Omega||_{H^3} + \alpha).
$$

8.5.4. Analysis of $|\alpha c_l|, ||A_1(\phi_2 - 1) \cdot D_\beta \bar{\xi}||_{L^\infty}$. Using (4.11) and (A.3) in Lemma A.3, we obtain

$$
|\alpha c_l| \leq C \alpha^{-1} L_{12}(\Omega)(0) \leq C \alpha^{-1} ||\Omega||_{H^3}.
$$

Using the formulas of $\phi_2, A_1$ in (6.14), (6.13), we get

$$
|\phi_2(\phi_2 - 1)| = |\phi_2(1 + 3\alpha + \alpha c_l) D_R + \frac{3}{1 + R} D_\beta (R \sin(2\beta)^{\alpha})^{-1/40}|
$$

$$
\leq |\phi_2(1 + 3\alpha + \alpha c_l)| (\alpha^{-1} ||\Omega||_{H^3} + \alpha),
$$

where we have used $D_R (R \sin(2\beta)^{\alpha})^{-1/40} = \frac{1}{40} (R \sin(2\beta)^{\alpha})^{-1/40} D_\beta (R \sin(2\beta)^{\alpha})^{-1/40} \lesssim \alpha (R \sin(2\beta)^{\alpha})^{-1/40}$ in the first inequality. Therefore, we get

$$
||A_1(\phi_2 - 1) \cdot D_\beta \bar{\xi}||_{L^\infty} \leq \left( \frac{1}{40} + C \alpha + C \alpha^{-1} ||\Omega||_{H^3} \right) ||\phi_2 D_\beta \bar{\xi}||_{L^\infty}.
$$

8.5.5. Completing the $C^1$ estimates. From (6.28), we can use $E_3$ to further bound $||\Omega||_{H^3}, ||\eta||_{H^3}, ||\xi||_{H^3(\Psi)}$. Plugging all the above estimates of the remaining terms in (6.12), (6.16), (6.17), we prove

$$
\frac{1}{2} \frac{d}{dt} ||\xi||_{C^1} \leq -2 ||\xi||_{C^1}^2 + C ||\xi||_{C^1} (\alpha^{1/2} E_3 + \alpha ||\xi||_{C^1} + \alpha^{-1} E_3^2 + \alpha^{-1} E_3 ||\xi||_{C^1}) + C \alpha^2 ||\xi||_{C^1},
$$

$$
\frac{1}{2} \frac{d}{dt} ||\phi_2 D_\beta \xi||_{C^1} \leq -\left( 2 - \frac{1}{40} \right) ||\phi_2 D_\beta \xi||_{C^1}^2
$$

$$
+ C ||\xi||_{C^1} (\alpha^{1/2} E_3 + \alpha ||\xi||_{C^1} + \alpha^{-1} E_3^2 + \alpha^{-1} E_3 ||\xi||_{C^1}) + C \alpha^2 ||\phi_2 D_\beta \xi||_{C^1},
$$

$$
\frac{1}{2} \frac{d}{dt} ||\phi_1 D_R \xi||_{C^1} \leq -2 ||\phi_1 D_R \xi||_{C^1}^2 + 3 ||\phi_1 D_R \xi||_{C^1}^2 (||\phi_2 D_\beta \xi||_{C^1} + ||\xi||_{C^1})
$$

$$
+ C ||\xi||_{C^1} (\alpha^{1/2} E_3 + \alpha ||\xi||_{C^1} + \alpha^{-1} E_3^2 + \alpha^{-1} E_3 ||\xi||_{C^1}) + C \alpha^2 ||\phi_1 D_R \xi||_{C^1}.
$$
Hence, for some absolute constant $\mu_4$, e.g. $\mu_4 = \frac{1}{10}$, the energy defined in (8.3) satisfies (8.2).

8.6. **Finite time blowup with finite energy velocity field.**

8.6.1. **The bootstrap argument.** Now, we construct the energy

\[(8.26) \quad E(\Omega, \eta, \xi) = (E_0(\Omega, \eta, \xi)^2 + \alpha E(\xi, \infty)^2)^{1/2}.\]

Adding the estimates (8.1) and $\alpha \times (8.2)$, we have

\[(8.27) \quad \frac{1}{2} \frac{d}{dt} E^2(\Omega, \eta, \xi) \leq -\frac{1}{12} E^2 + K\alpha^{1/2} E^2 + K\alpha^{-3/2} E^3 + K\alpha^2 E,\]

for some universal constant $K$, where we have used the fact that $E(\xi, \infty)$ is equivalent to $||\xi||_{C^1}$, since $\mu_4$ is an absolute constant. We know that there exists a small absolute constant $\alpha_1 < \frac{1}{1000}$ and $K_*$, such that, for any $\alpha < \alpha_1$ and $E = K_*\alpha^2$, we have

\[(8.28) \quad -\frac{1}{12} E^2 + K\alpha^{1/2} E^2 + K\alpha^{-3/2} E^3 + K\alpha^2 E < 0.\]

If $E(\Omega(\cdot, 0), \eta(\cdot, 0), \xi(\cdot, 0)) < K_*\alpha^2$, we have

\[(8.29) \quad E(\Omega(t), \eta(t), \xi(t)) < K_*\alpha^2,\]

for all time $t > 0$, where we have used the time-dependent normalization condition (4.11) for $c(\omega(t), \xi(t))$. Applying Lemma A.3 to $L_{12}(\Omega)(0)$ and Lemma 7.12 to $\Omega, \eta$, we derive

\[\begin{align*}
|c_\omega(t)| = & \frac{2}{\pi\alpha} L_{12}(\Omega)(0) < C\alpha^{-1} ||\Omega||_{4\alpha^3} \leq C\alpha^{-1} E \leq K_9\alpha, & |c_\xi(t)| = & \left| \frac{1 - \alpha}{\pi\alpha} \frac{2}{\alpha} L_{12}(\Omega)(0) \right| < C\alpha^{-2} E \leq K_9, \\
||\Omega||_{L^\infty} + ||\eta||_{L^\infty} < CE \leq C\alpha^2 \leq K_9 \alpha & \min(||\Omega||_{L^\infty}, ||\eta||_{L^\infty}), & ||\xi||_{L^\infty} < C\alpha^{-1/2} E \leq K_9 \alpha^{3/2},
\end{align*}\]

where we have used $||\Omega||_{L^\infty}, ||\eta||_{L^\infty} \geq C\alpha$ according to (4.8) and Lemma A.1 in the last inequality, and $K_9 > 0$ is some absolute constant. We further take

\[(8.30) \quad \alpha_0 = \min(\alpha_1, \frac{3\pi}{4K_*}, \frac{K_1}{4K_{10}^2}, \frac{1}{16(K_9 + 1)^4}),\]

where $K_{10}$ is the constant defined in Lemma A.10. For $\alpha < \alpha_0$, using $\tilde{c}_\omega = -1, \tilde{c}_\xi = \frac{1}{2\alpha} + 3$ and the formula of $\Omega, \eta$ in (8.3), we further yield

\[(8.31) \quad \frac{3}{2} < c_\omega + \tilde{c}_\omega < -\frac{1}{2}, \quad c_\xi + \tilde{c}_\xi > \frac{1}{2\alpha} + 3, \quad ||\Omega + \tilde{\Omega}||_{L^\infty} \approx ||\Omega||_{L^\infty} \approx \alpha, \quad ||\eta + \tilde{\eta}||_{L^\infty} \approx ||\tilde{\eta}||_{L^\infty} \approx \alpha, \quad ||\xi||_{L^\infty} \leq \frac{1}{2}\alpha^{3/4}.\]

8.6.2. **Finite time blowup.** Let $\chi(\cdot) : [0, \infty) \to [0, 1]$ be a smooth cutoff function, such that $\chi(R) = 1$ for $R \leq 1$ and $\chi(R) = 0$ for $R > 2$. We choose perturbation $\tilde{\Omega} = \chi(R/\lambda - 1)\tilde{\Omega}$, $\tilde{\theta} = \tilde{\theta}(x, y)$, $\theta = \theta_x + \xi$ can be obtained accordingly, where $\theta(x, y)$ is recovered from $\theta_x$ by integration (4.14). Obviously, $\Omega, \eta, \xi \equiv 0$ for $R \leq \lambda$. Using Lemma A.10 for $\Omega, \eta, \xi$ and $\alpha < \alpha_0$ (see (8.30)), we obtain that these initial perturbations satisfy $E(\Omega(0), \eta(0), \xi(0)) < 2K_{10}\alpha^{1/2} \leq K_1\alpha^2$ for sufficiently large $\lambda$. We remark that the initial perturbation is of size $C\alpha^{3/2}$ even for extremely large $\lambda$ because $\tilde{\xi}$ does not decay in the $C^1$ norm for large $R$. It is important to add a small weight $\alpha$ in $E(\xi, \infty)$ when we define the final energy in (8.26).

In particular, the initial data $\tilde{\Omega} + \Omega = \chi(R/\lambda)\tilde{\Omega}$ (recall $\Omega(R, \beta) = \omega(x, y)$), $\tilde{\theta} + \theta = \chi(R/\lambda)\tilde{\theta}$ have compact support and thus we have finite energy $||u + \tilde{u}||_{L^2} < +\infty, ||\theta + \tilde{\theta}||_{L^2} < +\infty$. $c_\omega(t), c_\xi(t)$ are determined by (4.11).

Denote by $\omega_{\text{phy}}, \theta_{\text{phy}}$ the corresponding solutions in the original Boussinesq equation (2.1)-(2.2), which are related to the rescaled variables $\omega, \theta$ via the rescaling formula (4.1), (4.3)

\[(8.32) \quad \omega_{\text{phy}}(x, t(\tau)) = \frac{C_\omega(\tau)^{-1} \omega(\tau)}{C_\xi(t(\tau)), \quad \theta_{\text{phy}}(x, t(\tau)) = \frac{C_\theta(\tau)^{-1} \theta(\tau)}{C_\xi(t(\tau))}, \quad C_\omega(\tau) = \exp \left( \int_0^\tau c_\omega(s) + \tilde{c}_\omega(s) \, ds \right), \quad C_\xi(\tau) = \exp \left( - \int_0^\tau c_\xi(s) + \tilde{c}_\xi(s) \, ds \right), \quad t(\tau) = \int_0^\tau C_\omega(\tau) \, d\tau.\]
We remark that the scaling parameters in (4.3) become \((c_\omega + \bar{c}_\omega, c_l + \bar{c}_l)\). Denote

\[
M(\tau) \triangleq \int_0^{\tau(\tau)} ||\nabla \theta_{phy}(s)||_{L^\infty} ds.
\]

Using a change of variable \(s = t(p)\) and \(\partial_x(\theta + \bar{\theta}) = (\eta + \bar{\eta}), \partial_y(\theta + \bar{\theta}) = (\xi + \bar{\xi})\), we obtain

\[
M(\tau) = \int_0^\tau ||\nabla \theta_{phy}(t(p))||_{L^\infty} C_\omega(p) dp = \int_0^\tau C_\omega(p)^{-1} (||(\eta + \bar{\eta})(p)||_{L^\infty} + ||(\xi + \bar{\xi})(p)||_{L^\infty}) dp,
\]

where we have used the formula (8.32) and \(C_0^{-1}(p)C^{-1}_1(p) = C_\omega(p)^{-2}\) according to (4.3), (4.4) in the second equality. Using the bootstrap estimates (8.31) and Lemma A.7 about \(\bar{\xi}\), we obtain

\[
M(\tau) \approx \alpha \int_0^\tau C_\omega(p)^{-1} dp.
\]

Using (8.31) and (8.32), we have \(e^{-3p/2} < C_\omega(p) < e^{-p/2}\). Therefore, we obtain

\[
M(\tau) < +\infty \quad \forall \tau < +\infty, \quad M(\infty) \geq C_\alpha \int_0^\infty e^{p/2} dp = \infty, \quad t(\infty) \leq \int_0^\infty e^{-p/2} dp < +\infty.
\]

Denote \(T^* = t(\infty)\). Applying the BKM type blowup criterion in (3.1), we obtain that the solutions remain in the same regularity class as that of the initial data before \(T^*\) and develop a finite time singularity at \(T^*\). Similarly, by rescaling the time variable, we prove that \(||\omega_{phy}||_{L^\infty}\) and \(||\nabla \theta_{phy}||_{L^\infty}\) blowup at \(T^*\).

**Remark 8.1.** The crucial nonlinear estimate (8.27) and *a priori* estimate (8.29), i.e. the bootstrap estimate for small perturbation, offer strong control on the perturbation and the exact solution before the blowup time. In particular, it allows us to truncate the far field of the approximate steady state, which leads to a small perturbation only, to obtain initial data with finite energy.

### 8.6.3. Convergence to the self-similar solution

Taking the time derivative of (5.1), using the *a priori* estimate (8.29) for the small perturbation and analysis similar to that in the previous Section, we can further perform \(H^2\) estimates on \(\Omega_t, \eta_t, H^2(\psi)\) and \(L^\infty\) estimates on \(\xi_t\). In particular, following the argument in our previous joint work with Huang [5], we can further obtain that there exists an exact self-similar solution \(\Omega_\infty, \eta_\infty \in H^3, \xi_\infty \in H^3(\psi) \cap L^\infty\), such that the solution of the dynamic rescaling equation with initial data constructed in Subsection 8.6.2 converges to \((\Omega_\infty, \eta_\infty, \xi_\infty)\) exponentially fast. The convergence is in the \(H^2\) norm for the variables \(\Omega, \eta\) and both \(H^2(\psi)\) and \(L^\infty\) norm for the variable \(\xi\).

Using the *a priori* estimate (8.29) and Lemma A.7 we have \(||\xi + \xi(t)||_{C^1} \leq C_\alpha^{3/2}\) for all time in the dynamic rescaling equation. Using Lemma A.12 we know that the space \(C^1\) (the weighted \(C^1\) space) can be embedded continuously into the standard Hölder space \(C^{\alpha/40}\). Therefore, the \(C^1\) estimate of \(\xi + \xi(t)\) implies that \(\xi + \xi(t) \in C^{\alpha/40}\) with uniform Hölder norm. Since \(\xi + \xi(t)\) converges to \(\xi_\infty\) in \(L^\infty\), we have \(\xi_\infty \in C^{\alpha/40}\). Finally, using the same argument, the fact that \(\Omega_\infty, \eta_\infty \in H^3\) and the embedding \(H^3 \hookrightarrow C^1\) in Lemma 7.12 we conclude \(\Omega_\infty, \eta_\infty, \xi_\infty \in C^{\alpha/40}\).

Notice that \(c_l + \bar{c}_l > \frac{1}{12}\) from (8.31). Thus, the self-similar blowup is focusing. This completes the proof of Theorem 1.1.

### 9. Finite time blowup of 3D axisymmetric Euler equations with solid boundary

In this Section, we prove Theorem 1.2. Let \(D = \{(r, z) : r \leq 1, z \in \mathbb{R}\}\) be a cylinder. The singularity we are interested in occurs at \((r, z) = (1, 0)\) on the boundary and is away from the symmetry axis \(r = 0\).

We first state the 3D axisymmetric Euler equations in the cylinder \(D\). Then we consider the dynamic rescaling formulation of the equations centered at \((r, z) = (1, 0)\) in subsection 9.1. In the new formulation, the domain is transformed to \((x, y) \in \mathbb{R} \times [0, C_l(\tau)^{-1}]\), where \(C_l(\tau)\) is the rescaling factor. In particular, the boundary \(r = 1\) and the symmetry plane \(z = 0\) correspond to the boundary \(y = 0\) and the symmetry axis \(x = 0\) in the 2D Boussinesq equations, while the symmetry axis \(r = 0\) becomes another boundary \(y = C_l(\tau)^{-1}\) that will go to infinite at the blowup time. Notice that we do not have a boundary condition for this artificial boundary \(y = C_l(\tau)^{-1}\). In order to perform the elliptic estimates in the transformed domain, we first
obtain the far field estimates of the stream function in Lemma 9.1 and Lemma 9.3. Then we perform the elliptic estimates in subsection 9.2 for vorticity supported near \((r, z) = (1, 0)\) (or supported at the near field in the transformed domain) by localizing the elliptic equation.

In subsection 9.3, we will construct initial data with support sufficiently close to \((r, z) = (1, 0)\) and control the evolution of the support so that it does not touch the symmetry axis. With these estimates, the rest of the proof follows essentially the nonlinear stability analysis of the 2D Boussinesq equations and is sketched in the same subsection.

Let \(u\) be the axisymmetric velocity and \(\omega = \nabla \times u\) be the vorticity vector. In the cylindrical coordinates, we have the following representation

\[
u(r, z) = u^r(r, z) e_r + u^\theta(r, z) e_\theta + u^z(r, z) e_z, \quad \omega = \omega^r(r, z) e_r + \omega^\theta(r, z) e_\theta + \omega^z(r, z) e_z,
\]

where \(e_r, e_\theta\) and \(e_z\) are the standard orthonormal vectors defining the cylindrical coordinates,

\[
e_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right)^T, \quad e_\theta = \left(\frac{x_2}{r}, -\frac{x_1}{r}, 0\right)^T, \quad e_z = (0, 0, 1)^T,
\]

and \(r = \sqrt{x_1^2 + x_2^2}\) and \(z = x_3\).

The 3D axisymmetric Euler equations are given below:

\[
\begin{align*}
\partial_t (ru^\theta) + u^r (ru^\theta)_r + u^\theta (ru^\theta)_z &= 0, \\
\partial_t \omega^\theta + u^r \omega^\theta_r + u^\theta \omega^\theta_z &= \frac{1}{r} \partial_z ((ru^\theta)^2).
\end{align*}
\]

The radial and axial components of the velocity can be recovered from the Biot-Savart law

\[
\begin{align*}
- (\partial_{rr} + \frac{1}{r} \partial_r + \partial_{zz}) \tilde{\psi} + \frac{1}{r^2} \tilde{\psi} &= \omega^\theta, \\
u^r &= -\tilde{\psi}_z, \\
v^z &= \tilde{\psi}_r + \frac{1}{r} \tilde{\psi}
\end{align*}
\]

with a no-flow boundary condition on the solid boundary \(r = 1\)

\[
\tilde{\psi}(1, z) = 0.
\]

We first perform an estimate for \(\tilde{\psi}\) obtained from the Biot Savart law (9.2), (9.3).

**Lemma 9.1.** Let \(\tilde{\psi}\) be a solution of \((9.2), (9.3)\) and \(\omega^\theta \in L^2\) with compact support, \(\text{supp}(\omega^\theta) \subset \{(r, z) : (r - 1)^2 + z^2 < 1/4\}\). For \(r > 1/4\), we have

\[
|\tilde{\psi}(r, z)| \lesssim \int |\omega^\theta(r_1, z_1) \log((r - r_1)^2 + (z - z_1)^2) r_1 dr_1 dz_1.
\]

The proof is based on comparing \(\tilde{\psi}\) with the solutions of \((-\partial_{rr} - \partial_{zz} - \frac{1}{r} \partial_r + \frac{1}{r^2}) \psi_\pm = \omega_\pm\) on the whole space, which can be obtained by the Green function of the Laplace equation on \(\mathbb{R}^3\). We defer the proof to Appendix A.4.

If the initial data \(u^\theta\) of \((9.1), (9.3)\) is non-negative, \(u^\theta\) remains non-negative before the blowup, if it exists. Then, \(u^\theta\) can be uniquely determined by \((u^\theta)^2\). We introduce the following variables

\[
\tilde{\theta} \triangleq (ru^\theta)^2, \quad \tilde{\omega} = \omega^\theta / r.
\]

We can reformulate \((9.1), (9.3)\) as

\[
\begin{align*}
\partial_t \tilde{\theta} + u^r \tilde{\theta}_r + u^\theta \tilde{\theta}_z &= 0, \\
\partial_t \tilde{\omega} + u^r \tilde{\omega}_r + u^\theta \tilde{\omega}_z &= \frac{1}{r} \tilde{\omega}_z,
\end{align*}
\]

\[
- (\partial_{rr}^2 + \frac{1}{r} \partial_r + \partial_{zz}^2 - \frac{1}{r^2}) \tilde{\psi} = r \tilde{\omega}, \quad \tilde{\psi}(1, z) = 0, \quad u^r = -\tilde{\psi}_z, \quad u^z = \frac{1}{r} \tilde{\psi} + \tilde{\psi}_r.
\]

### 9.1 Dynamic rescaling formulation

We consider the following dynamic rescaling formulation centered at \(r = 1, z = 0\)

\[
\theta(x, y) = C_\theta(\tau) \tilde{\theta}(1 - C_1(\tau) y, C_1(\tau) x, t(\tau)), \quad \omega(x, y) = C_\omega(\tau) \tilde{\omega}(1 - C_1(\tau) y, C_1(\tau) x, t(\tau)),
\]

\[
\psi = C_\omega(\tau) C_1(\tau)^{-2} \tilde{\psi}(1 - C_1(\tau) y, C_1(\tau) x, t(\tau)),
\]

where \(C_1(\tau), C_\theta(\tau), C_\omega(\tau), t(\tau)\) are given by \(C_\theta = C_\theta^{-1}(0) C_\omega^2(0) \exp(\int_0^\tau C_\theta(s) ds),\)

\[
C_\omega(\tau) = C_\omega(0) \exp \left(\int_0^\tau C_\omega(s) ds\right), \quad C_1(\tau) = C_1(0) \exp \left(\int_0^\tau -C_1(s) ds\right), \quad t(\tau) = \int_0^\tau C_\omega(\tau) d\tau,
\]

\(^5\)This result is due to De Huang. We are grateful to him for telling us this result.
and the rescaling parameter \( c_0(\tau), c_0(\tau), c_0(\tau) \) satisfies \( c_0(\tau) = c_0(\tau) + 2c_0(\tau) \). We remark that 
\( C_0(\tau) \) is determined by \( C_1, C_2 \) via \( C_0 = C_2^2 C_1^{1/2} \). The radial variable \( r \) becomes \( 1 - C(\tau) y \), \( z \) becomes \( C(\tau) x \). We have \( y \leq C(\tau)^{-1} \) since \( r \in [0, 1] \). From now on, we will use \( s = 1 - C(\tau) y \) to denote the original radial variable. \( r \) will be reused later. We have a minus sign for \( \partial_y \\
r \) estimate the stream function away from the support of \( \omega \).

\[ C_9.2. \]

\[
\partial_y \theta = -C_0 C_1(\tau) \tilde{\partial}_r, \quad \partial_y \omega = -C_0 C_1(\tau) \tilde{\omega}_r, \quad \partial_y \psi = -C_0 C_1(\tau)^{-1} \tilde{\psi}_r.
\]

Let \( (\tilde{\theta}, \tilde{\omega}, \tilde{\psi}) \) be a solutions of \( C_9.5 \). It is easy to show that \( \omega, \theta \) satisfy

\[
\theta_x + c_1 \mathbf{x} \cdot \nabla \theta + (-u^r) \theta_y + u^z \theta_x = c_0 \theta, \quad \omega_x + c_1 \mathbf{x} \cdot \nabla \omega + (-u^r) \omega_y + u^z \omega_x = c_0 \omega + \frac{1}{s^4} \theta_x.
\]

The Biot-Savart law in \( C_9.5 \) depends on the rescaling parameter \( C_1, \tau \)

\[
-(\partial_{xx} + \partial_{yy}) \psi + \frac{1}{s} C_1 \partial_x \psi + \frac{1}{s} C_1^2 \partial_y \psi = s \omega, \quad u^r(s, x) = -\psi_x, \quad u^z(s, x) = \frac{1}{s} C_1(\tau) \psi - \psi_y,
\]

where \( s = 1 - C_1(\tau) y \). We introduce \( u = u^z, v = -u^r \). Then, we can further simplify

\[
\theta_t + (c_1 \mathbf{x} + \mathbf{u} \cdot \nabla) \theta = c_0 \theta, \quad \omega_t + (c_1 \mathbf{x} + \mathbf{u} \cdot \nabla) \omega = \theta_x + \frac{1}{s^4} \theta_x,
\]

\[
-(\partial_{xx} + \partial_{yy}) \psi + \frac{1}{s} C_1 \partial_x \psi + \frac{1}{s} C_1^2 \partial_y \psi = s \omega, \quad u(x, y) = -\psi_y + \frac{1}{s} C_1 \psi, \quad v = \psi_x,
\]

with boundary condition \( \psi(x, 0) \equiv 0 \). If \( C_1 \) is extremely small, we expect that the above equations are essentially the same as the dynamic rescaling formulation \( C_4.2 \) of the Boussinesq equations. We look for solutions of \( C_9.8 \) with the following symmetry

\[
\omega(x, y) = -\omega(-x, y), \quad \theta(x, y) = \theta(-x, y).
\]

Obviously, the equations preserve these symmetries and thus it suffices to solve \( C_9.8 \) on \( x, y \geq 0 \) with boundary condition \( \psi(x, 0) = \psi(y, 0) = 0 \) for the elliptic equation.

9.2. The elliptic estimates.

In this Section, we estimate the time-dependent elliptic equation in \( C_9.8 \). We will first estimate the stream function away from the support of \( \omega \). Then we will localize the elliptic equation and exploit the smallness of \( C_1 \) to establish the \( H^3 \) elliptic estimates.

Under the polar coordinates \( r = \sqrt{x^2 + y^2}, \beta = \arctan(y/x) \), \( C_9.8 \) can be reformulated as

\[
-\partial_r \psi - \frac{1}{r} \partial_r \psi - \frac{1}{r^2} \partial_\beta \psi + \frac{1}{s} \sin(\beta) \partial_r \psi + \frac{1}{s} \cos(\beta) \partial_\beta \psi + \frac{1}{s} C_1^2 \psi = s \omega,
\]

where \( s = 1 - C_1 y = 1 - C_1 r \sin(\beta) \). Denote \( R = r^\alpha \) and

\[
\Psi(R, \beta) = \frac{1}{r^2} \psi(r, \beta), \quad \Omega(R, \beta) = \omega(r, \beta), \quad \eta(R, \beta) = (\theta_x)(r, \beta), \quad \xi(R, \beta) = (\theta_y)(r, \beta).
\]

Since we rescale the cylinder \( \{(s, z) : s \leq 1, z \in \mathbb{R}\} \), the domain for \( (x, y) \) is \( x \in \mathbb{R}, y \in [0, C_1^{-1}] \). We focus on the sector \( r \leq C_1^{-1} \), or equivalently \( R \leq C_1^{-1} \), and \( \beta \in [0, \pi/2] \) due to the symmetry of the solutions.

Notice that \( r \partial_r = \alpha R \partial_R = \alpha D_R \). It is easy to verify that the above equation is equivalent to

\[
-\alpha^2 R^2 \partial_R \Psi - \alpha(4 + \alpha) R \partial_R \Psi - \partial_\beta \Psi - 4 \Psi + \frac{C_1 r}{s} \sin(\beta)(2 + \alpha D_R) \Psi + \cos(\beta) \partial_\beta \Psi + \frac{C_1^2 r^2 \Psi}{s} = s \Omega.
\]

We keep the notation \( r = R^{1/\alpha}, s = 1 - C_1 r \sin(\beta) \) to simplify the formulation. The boundary condition of \( \Psi \) is given by (in the domain sector \( R \leq C_1^{-1} \))

\[
\Psi(R, 0) = \Psi(R, \pi/2) = 0.
\]

Definition 9.2. We define the size of support of \( (\theta, \omega) \) of \( C_9.8 \)

\[
S(\tau) = \text{ess inf} \{(\rho : \theta(x, y, \tau) = 0, \omega(x, y, \tau) = 0 \text{ for } x^2 + y^2 \geq \rho^2)\}.
\]
Obviously, the support of $\Omega, \eta$ defined in (9.9) is $S(\tau)^\alpha$. After rescaling the spatial variable, the support of $(\tilde{\theta}, \tilde{\omega})$ of (1.3) satisfies

$$\text{supp } \tilde{\theta}(t(\tau)), \text{ supp } \tilde{\omega}(t(\tau)) \subset \{(s, z) : ((s - 1)^2 + z^2)^{1/2} \leq C_1(\tau)S(\tau)\}.$$ 

We will construct initial data of (9.8) with compact support $S(0) < +\infty$ and use a bootstrap argument to control the support size so that $C_1(\tau)S(\tau)$ remains sufficiently small for all $\tau > 0$.

Since we do not have a boundary condition for $\Psi$ in (9.10) when $R$ is large, we need an estimate for $\Psi$ when $R$ is large. Recall the $L^2$ inner product defined in (1.3).

**Lemma 9.3.** Let $S(\tau)$ be the support size of $\omega(\tau), \theta(\tau)$. Assume $C_1(\tau)S(\tau) < \frac{1}{4}$. For any $M > (2S(\tau))^{\alpha}$, we have

$$\|\Psi(1_{M \leq R \leq (2C_1)^{-\alpha}})\|_{L^2} \lesssim (1 + |\log(C_1M^{1/\alpha})|) \sqrt{\frac{S(\tau)}{M}}^{2/\alpha - 1/2}.$$

**Remark 9.4.** We restrict to $R \leq (2C_1)^{-\alpha}$ since at $R = C_1^{-\alpha}$, the solutions touch the axis.

**Proof.** Recall $\tilde{\omega}(s, z) = \omega^0(s, z)/s$ $(s = 1 - Cy$ denotes the original radial variable). Since the support size satisfies $C_1(\tau)S(\tau) < 1/4$, within the support of $\omega(s, z)$, we have $s \geq 1/2$. Hence, $\tilde{\omega} \approx \omega^0$. Moreover, $R \leq (2C_1)^{-\alpha}$ implies $s \geq 1 - \frac{1}{2} = \frac{1}{4}$. Therefore, for $(x, y)$ with $R = (x^2 + y^2)^{\alpha/2} \leq (2C_1)^{-\alpha}$, we can apply Lemma 9.1 and (9.6) to get

$$|\psi(x, y)| \lesssim C_\omega C_1^{-2} \int |\tilde{\omega}(s_1, z_1)\log((s_1 - (1 - C_1y))^2 + (z_1 - C_1x)^2)| ds_1 dz_1$$

$$= C_\omega \int |\tilde{\omega}(1 - C_1y_1, C_1x_1)\log(C_1^2((y_1 - y)^2 + (x_1 - x)^2))| dy_1 dx_1,$$

where we have used Lemma 9.1 and $s_1 \leq 1$ in the first inequality, and used change of variables $s_1 = 1 - C_1y_1, z_1 = C_1x_1$ in the second identity. From (9.6), $C_\omega \tilde{\omega}(1 - C_1y_1, C_1x_1)$ in the integrand becomes $\omega^0(x_1, y_1)$. For $(x_1, y_1)$ within the support of $\omega$, we have $x_1^2 + y_1^2 < S(\tau)^2$. Hence, for any $x^2 + y^2 > 4S^2(\tau)$, or equivalently, $R > (2S(\tau))^\alpha$, we get

$$(x_1 - x)^2 + (y_1 - y)^2 \approx x^2 + y^2 = r^2 = R^2/\alpha, \quad |\log(C_1^2((y_1 - y)^2 + (x_1 - x)^2))| \lesssim |\log(C_1^2R^2/\alpha)| + 1.$$

Using $\Psi(R, \beta) = \frac{1}{r^2} \psi(x, y) = R^{-2/\alpha}\psi(x, y)$ and the above estimates, we get

$$|\Psi(R, \beta)| \lesssim R^{-2/\alpha}(1 + |\log(C_1^2R^2/\alpha)|) \int |\omega(x_1, y_1)| dx_1 dy_1.$$

Passing to the $(R, \beta)$ coordinates, we have $dx_1 dy_1 = \tilde{r}d\tilde{r}d\tilde{\beta} = \alpha^{-1} \tilde{R}^{2/\alpha - 1}d\tilde{R}d\tilde{\beta}$. Using the Cauchy-Schwarz inequality, we get

$$\int |\omega(x_1, y_1)| dx_1 dy_1 = \alpha^{-1} \int_{R \leq S(\tau)^\alpha} |\Omega(\tilde{R}, \tilde{\beta})|\tilde{R}^{2/\alpha - 1}d\tilde{R}d\tilde{\beta} \lesssim \alpha^{-1}||\Omega||_{L^2}(\int_0^{\pi/2} \int_{0}^{\Omega(\tilde{R}, \tilde{\beta})})^{1/2}\tilde{R}^{2/\alpha - 1}d\tilde{R}d\tilde{\beta})^{1/2} \lesssim \alpha^{-1/2}||\Omega||_{L^2S(\tau)^{2-2\alpha/2}}.$$

It follows that

$$|\Psi(R, \beta)| \lesssim R^{-2/\alpha}(1 + |\log(C_1^2R^2/\alpha)|)\alpha^{-1/2}||\Omega||_{L^2S(\tau)^{2-2\alpha/2}}.$$

Integrating $|\Psi(R, \beta)|^2$ from $M$ to $(2C_1)^{-\alpha}$ yields the desired result. Remark that $\alpha^{-1/2}$ is canceled due to $R^{-4/\alpha}$ in the integrand.

**9.2.1. Localizing the elliptic equation.** We will take advantage of that $C_1(\tau)S(\tau)$ can be extremely small and localize the elliptic equation. Firstly, we assume that $C_1(\tau)S(\tau) < \frac{1}{4}$. Then we have $s = 1 - C_1r \geq \frac{3}{4}, s^{-1} \lesssim 1$.

Let $\chi_1(\cdot) : [0, \infty) \to [0, 1]$ be a smooth cutoff function, such that $\chi_1(R) = 1$ for $R \leq 1$, $\chi_1(R) = 0$ for $R \geq 2$ and $(D_R\chi_1)^2 \lesssim \chi_1$. This assumption can be satisfied if $\chi_1 = \chi_0$ where $\chi_0$ is another smooth cutoff function. Denote $\chi_\lambda(R) = \chi_1(R/\lambda)$. It is easy to verify that

$$\frac{D_R\chi_\lambda}{\chi_\lambda} \lesssim 1_{\lambda \leq R \leq 2\lambda},$$

$$\frac{D_R^k\chi_\lambda}{\chi_\lambda} \lesssim 1_{\lambda \leq R \leq 2\lambda}.$$
for $k \leq 5$, where we have used the property $|D_R^2 \chi_k| \lesssim \chi_1$ in the first inequality. Denote

$$\Psi_\lambda = \Psi \chi_\lambda, \quad \Omega_\chi = \Omega \chi_\lambda.$$ 

At this moment, we just simplify $\chi_\lambda$ as $\chi$. Observe that $R^2 \partial_{RR} + R \partial_R = D_R^2$ and

$$\alpha D_R(\chi \Psi) = \alpha D_R \chi \Psi + \alpha D_R \Psi, \quad \alpha^2 D_R^2(\chi \Psi) = \alpha^2 \chi D_R^2 \Psi + 2 \alpha D_R \chi \Psi + \alpha^2 D_R^2 \chi \Psi.$$ 

Multiplying $\chi$ on both sides of (9.10) and using a direct calculation yield

$$- \alpha^2 D_R^2 \Psi_\chi - 4 \alpha D_R \Psi_\chi - \partial_\beta \Psi_\chi - 4 \Psi_\chi = \Omega_\chi + Z_\chi, \quad Z_\chi = Z_1 + Z_2,$$

where $Z_\chi = Z_1 + Z_2$, $Z_1$ and $Z_2$ are given below

$$Z_1 = - \frac{C_1 r}{s} (\sin(\beta)(2 \Psi_\chi + \alpha D_R \Psi_\chi) + \cos(\beta) \partial_\beta \Psi_\chi) - \frac{C_2 r^2}{s^2} \Psi_\chi,$$

$$Z_2 = \frac{C_1 \sin(\beta) r}{s} \alpha D_R \Psi - (\alpha^2 D_R^2 \chi + 4 \alpha D_R \chi) \Psi - 2 \alpha^2 D_R \Psi D_R \Psi.$$ 

Recall that $R = r^\alpha, s = 1 - C_l y = 1 - C_l r \sin(\beta)$.

Recall $L_{12}(f)(0)$ from (2.12). Firstly, for sufficiently smooth $\Omega, \Psi$ and $\Omega_\chi$ that vanishes at least linear near $R = 0$, we show that $L_{12}(Z_\chi)(0)$ is independent of the cutoff radial $\lambda$ for $\chi \geq S(\tau)^\alpha$.

From $\chi \geq S(\tau)^\alpha$, we have $\Omega = \Omega_\chi = \chi_\lambda$. For $\chi \geq S(\tau)^\alpha$, we have

$$L_{12}(Z_\chi)(0) = 4 \alpha \int_0^{\pi/2} \Psi(0, \beta) \sin(2\beta) d\beta.$$ 

We have the following $L^2$ estimate for $\Psi_\chi$.

**Lemma 9.5.** There exists $\alpha_2 > 0$ such that if $\alpha < \alpha_2, C_l S < 4^{-1/\alpha - 1}$, for $\chi = \frac{1}{4} C_l^{-\alpha}$, the solution of (9.13) satisfies

$$\alpha^2 \|D_R \Psi_\chi\|_{L^2}^2 + \alpha \|\Psi_\chi\|_{L^2}^2 + \alpha \|\partial_\beta \Psi_\chi\|_{L^2}^2 \lesssim \alpha^{-1} \|\Omega\|_{L^2}.$$ 

**Remark 9.6.** Under the above assumption, we have $\chi \geq 4^{\alpha} S(\tau)^\alpha$ and thus $\Omega_\chi = \Omega \chi_\lambda = \Omega$.

**Proof.** We simplify $\chi_\lambda$ as $\chi$. Multiplying (9.13) by $\Psi_\chi$ and integrating by parts, we get

$$I \triangleq \alpha^2 \|R \partial_\beta \Psi_\chi\|_{L^2}^2 + \frac{4 \alpha - \alpha^2}{2} \|\Psi_\chi\|_{L^2}^2 + \|\partial_\beta \Psi_\chi\|_{L^2}^2 - 4 \|\Psi_\chi\|_{L^2}^2 = \langle \Omega, \Psi_\chi \rangle + \langle Z_1, \Psi_\chi \rangle + \langle Z_2, \Psi_\chi \rangle.$$ 

Using the Fourier series expansion with basis $\{\sin(2n\beta)\}_{n \geq 1}$, one can verify that

$$||\partial_\beta \Psi_\chi||_{L^2} \geq \frac{4}{\alpha} ||\Psi_\chi||_{L^2},$$

which is sharp with equality when $\Psi_\chi = \sin(2\beta)$. Therefore, multiplying the above inequality by $1 - \frac{2}{\alpha}$ and then applying it to the left hand side of (9.16) yields

$$I \geq \alpha^2 \|D_R \Psi_\chi\|_{L^2}^2 + \frac{2 \alpha - \alpha^2}{2} ||\Psi_\chi||_{L^2}^2 + \alpha \|\partial_\beta \Psi_\chi\|_{L^2}^2 \geq \alpha^2 \|D_R \Psi_\chi\|_{L^2}^2 + \frac{\alpha}{4} ||\Psi_\chi||_{L^2}^2 + \frac{\alpha}{4} ||\partial_\beta \Psi_\chi||_{L^2}^2,$$

where we have used $\alpha \leq 1$.

Within the support of $\chi = \chi_\lambda$, we have $R \leq 2\lambda$. By assumption, we have $\lambda = \frac{1}{4} C_l^{-\alpha} > 4^{\alpha} S^{\alpha}$.

It follows that

$$C_l R 1_{R \leq 2\lambda} \leq C_l R \frac{4}{\alpha} 1_{R \leq 2\lambda} \leq C_l (2\lambda) \frac{4}{\alpha} = 2^{-\frac{4}{\alpha}} \lesssim \alpha^2, \quad |\log(C_l \lambda)^{\frac{4}{\alpha}}| \lesssim \alpha^{-1}, \quad 4^{\alpha} S^{\alpha} \leq \lambda.$$ 

Since $s^{-1} \lesssim 1$, we get

$$\|Z_1\|_{L^2} \lesssim \alpha^2 \|\Psi_\chi\|_{L^2} + \|\alpha D_R \Psi_\chi\|_{L^2} + \|\partial_\beta \Psi_\chi\|_{L^2} \lesssim \alpha^2 \alpha^{-1/2} f^{1/2} \lesssim \alpha^{3/2} f^{1/2}.$$
We perform integration by parts for the last term in $Z_2$

$$-2\alpha^2\langle D_{RX}D_{R}\Psi, \Psi_X \rangle = \alpha^2\langle (D_{RX})_RX, \Psi^2 \rangle = \alpha^2\langle (D_{RX})_R^2 + \chi D_{RX}^2 + \chi D_{RX}, \Psi^2 \rangle.$$  

Using the above identity, (9.11) for $|D_{RX}|$ and (9.17), we obtain

$$\langle |Z_2, \Psi_X| \rangle \lesssim (\alpha^2 + \alpha) \|\Psi_{1R_{\lambda}}\|_{L^2}^2 \lesssim \alpha \|\Psi_{1R_{\lambda}}\|_{L^2}^2.$$  

Since $\lambda > 4^\alpha S(\tau)^\alpha$ (see (9.17)), we can apply Lemma 9.3 and (9.17) to get

$$\langle |Z_2, \Psi_X| \rangle \lesssim \alpha (1 + (\log(C\lambda^{1/\alpha}))^2(S^\alpha/\lambda)^{4/\alpha-1})\|\Omega\|_{L^2} \lesssim \alpha^{-1}\|\Omega\|_{L^2}.$$  

Plugging the estimates of $Z_1, Z_2$ and $\|\Psi_X\|_{L^2} \lesssim \alpha^{-1/2}I^{1/2}$ into (9.10) and then using the Cauchy-Schwarz inequality we prove

$$I \leq C\alpha^{-1/2}I^{1/2}\|\Omega\|_{L^2} + C\alpha \cdot I + C\alpha^{-1}\|\Omega\|_{L^2}.$$  

Now we choose $\alpha = \frac{1}{\alpha_2 + \epsilon}$. Then for $\alpha < \alpha_2$, we have $C\alpha < \frac{1}{2}$. Solving the above inequality yields $I \lesssim \alpha^{-1}\|\Omega\|_{L^2}$. \hfill \Box

9.2.2. Localized $H^3$ estimates.

**Proposition 9.7.** Let $\Psi$ be the solution of (9.10) with term $\Omega(\tau)$ and $W = \frac{(1+R)^2}{R^2}$. There exists $\hat{\alpha} < \alpha_2$, such that, if $\alpha < \hat{\alpha}$, $C_1 S < \alpha \cdot \delta^{1/\alpha-1}$, for $\lambda = \frac{1}{8}C^{-\alpha}_1$, we have

$$\alpha^2\|R\partial_{R\beta}\Psi_X\|_{L^2} + \alpha\|R\partial_{R\beta}\Psi_X\|_{L^2} + \alpha\|\partial_{\beta\beta}(\Psi_X - \frac{\sin(2\beta)}{\alpha\pi})\|_{L^2} \lesssim \alpha^{-1}\|\Omega\|_{L^2}.$$  

where $Z_\lambda$ is defined in (1.13) and $\chi_\lambda$ is the cutoff function. Moreover, for $\nu \geq S(\tau)^\alpha$, $L_{12}(Z_{\lambda})(0)$ does not depend on $\nu$ and satisfies

$$\|L_{12}(Z_{\lambda})(0)\| = \|L_{12}(Z_{\lambda})(0)\| \lesssim (4^{-\alpha/2} \alpha^{-1} + (8C_1^\alpha S\alpha^{2} \alpha^{-1/2}))\|\Omega\|_{L^2}.$$  

Proof. Notice that the elliptic equation (9.13) under the $(R, \beta)$ is localized to $R \leq 2R \lesssim \frac{1}{\delta}C^{-\alpha}_1$, which is away from the axis. Therefore, $\Psi_X$ is a solution of (4.4) in the whole space $R \geq 0$, $\beta \in [0, \pi/2]$ with term $\Omega_\lambda + Z_\lambda$. We can apply the elliptic estimate in Proposition 7.3 in the weighted $L^2$ case, which can be proved using the same argument, to obtain

$$I \equiv \alpha^2\|R\partial_{R\beta}\Psi_X\|_{L^2} + \alpha\|R\partial_{R\beta}\Psi_X\|_{L^2} + \alpha\|\partial_{\beta\beta}(\Psi_X - \frac{\sin(2\beta)}{\alpha\pi})\|_{L^2} \lesssim \|\Omega_\lambda + Z_\lambda\|_{L^2}.$$  

Under the assumption $C_1 S < \alpha^{8^{-1/\alpha-1}}$, $\lambda = \frac{1}{8}C^{-\alpha}_1$, we have $(2S)^\alpha < \frac{1}{8}C^{-\alpha}_1 \leq \lambda$. Thus, $\Omega_\lambda = \chi_\lambda \Omega = \Omega$. Recall $Z_\lambda = Z_1 + Z_2$ in (9.14) and $r = R^{1/\alpha}$. Within the support of $\chi$, we have

$$C_{\tau} \Omega \Omega = C_1 R^{1/\alpha} - 2\lesssim C_1(2\lambda)^{1/\alpha} \leq \frac{1}{4} - \frac{1}{\alpha}.$$  

We can apply Lemma 9.3 to estimate the $L^2(W^2)$ norm of $Z_1$

$$\|Z_1\|_{L^2} \lesssim \|C_{\tau} W_X \|_{L^\infty}(\|\Psi_X\|_{L^2}) \lesssim \|D_{R\lambda} \Psi_X\|_{L^2} + \|\partial_{\beta} \Psi_X\|_{L^2} \lesssim \|\Omega\|_{L^2}.$$  

Recall $Z_2$ defined in (9.14). Notice that the support of $Z_2$ lies in $\lambda \leq R \leq 2\lambda$ due to the $D_{RX}$ term. Within this annulus, we get $W \lesssim 1$. Due to the smallness of $C_\tau$ from (9.19), we have

$$\|Z_2\|_{L^2} \lesssim \alpha \|\Psi_{1R^2\lambda}\|_{L^2} + \alpha^2 \|D_{RX} D_R \Psi_X\|_{L^2}.$$  

Since $\lambda \geq (2S(\tau))^{\alpha}$, applying Lemma 9.3 to the $\Psi$ terms in $Z_2$ and Lemma 9.3 to $D_R \Psi_X$ (note that $2\lambda = 4^{-1}C^{-\alpha}_1$), we get

$$\|Z_2\|_{L^2} \lesssim (\alpha \cdot \alpha^{-1}(8C_1^\alpha S\alpha^{2} \alpha^{-1} + \alpha^{1/2})\|\Omega\|_{L^2} \lesssim \alpha^{1/2}\|\Omega\|_{L^2},$$  

where we have used the assumption $\lambda = \frac{1}{8}C^{-\alpha}_1$ and $C_1 S < \alpha^{8^{-1/\alpha-1}}$ to obtain

$$\log(C_1 \lambda^{1/\alpha}) = \log(8^{1/\alpha}) \lesssim \alpha^{-1}, \quad (8C_1^\alpha S\alpha^{2} \alpha^{-1} \leq 8^{1/\alpha}\alpha C_1 S \leq \alpha.$$  

Plugging (9.20)-(9.22) into (9.18) and using $4^{-1/\alpha-1} \lesssim 1$, we prove

$$I \lesssim \|\Omega_\lambda W\|_{L^2} \lesssim \|\Omega W\|_{L^2},$$  

56 JIAJIE CHEN AND THOMAS Y. HOU
Based on this estimate, we can refine the estimate of $Z_2$ in (9.21). Using (9.12), we can obtain

$$\alpha^2||\chi D^2_R\Psi W||_{L^2} \lesssim \alpha^2||D^2_R\Psi W||_{L^2} + \alpha^2||D^2_R\chi \Psi||_{L^2} + \alpha^2||\Psi 1_{\lambda \leq \rho \leq 2\lambda}||_{L^2},$$

where we have used $|D^2_R\chi| \lesssim 1_{\lambda \leq \rho \leq 2\lambda}$. Using integration by parts, we get

$$J \triangleq (\alpha^2 D^2_R(\chi^2), (D^2_R\Psi^2)) = -\alpha^4(\partial_R(R(D^2_R)^2)D^2_R\Psi - \alpha^4(R(D^2_R)^2\partial_R D^2_R\Psi, \Psi).$$

Using the Cauchy-Schwarz inequality and (9.11), we yield

$$J \lesssim \alpha^2(\alpha^2(D^2_R(\chi^2), (D^2_R\Psi^2)))^{1/2} + \alpha^2||D^2_R\Psi||_{L^2})||\Psi 1_{\lambda \leq \rho \leq 2\lambda}||_{L^2}.$$ 

Recall from (9.18), (9.24) that $\alpha^2||D^2_R\Psi W||_{L^2} \lesssim I$. The first two terms in (9.25) are bounded by $||\Omega W||_{L^2} + J^{1/2}$. Hence, we derive

$$J \lesssim \alpha^2(J^{1/2} + ||\Omega W||_{L^2} + \alpha^2||\Psi 1_{\lambda \leq \rho \leq 2\lambda}||_{L^2})||\Psi 1_{\lambda \leq \rho \leq 2\lambda}||_{L^2}.$$ 

Applying Lemma 9.3 and (9.23) to the $\Psi$ terms, we further derive

$$J \lesssim \alpha(\frac{1}{2} + ||\Omega W||_{L^2} + \alpha^2||\Psi 1_{\lambda \leq \rho \leq 2\lambda}||_{L^2})||\Psi 1_{\lambda \leq \rho \leq 2\lambda}||_{L^2}.$$ 

Using $||\Omega||_{L^2} \lesssim ||\omega||_{L^2}$ and then solving the inequality for $J$, we prove

$$J \lesssim \alpha(8C_1^2 S^\alpha)^{2/\alpha}||\Omega W||_{L^2}^2.$$ 

Combining the above estimate of $J$ and (9.24), (9.28), we yield

$$||Z_2 W||_{L^2} \lesssim (\alpha^2(8C_1^2 S^\alpha)^{1/2 - \frac{1}{\alpha}} + (8C_1^2 S^\alpha)^{\frac{1}{\alpha} - 1})||\Omega W||_{L^2} \lesssim \alpha||\Omega W||_{L^2},$$

Using Lemma A.3 and (9.20) and the above refined estimate, we have

$$||L_2(Z_{\chi}) - \chi L_2(Z_{\chi}(0))W||_{L^2} \lesssim ||Z_{\chi} W||_{L^2} \lesssim ||Z_{\chi} W||_{L^2} + ||Z_{\chi} W||_{L^2} \lesssim \alpha||\Omega W||_{L^2},$$

where we have used $4^{-1/\alpha - 1} \lesssim \alpha$. Combining (9.18), (9.24) and the above estimate, we complete the proof of the first estimate. We remark that we only need the bound $||Z_{\chi} W||_{L^2} \lesssim \alpha||\Omega W||_{L^2}$ from (9.27) in this estimate.

Using Lemma A.3 and (9.20) and (9.27), we prove

$$||L_2(Z_{\chi})|| \leq ||Z_{\chi} W||_{L^2} \lesssim ||Z_{\chi} W||_{L^2} + ||Z_{\chi} W||_{L^2} \lesssim (4^{-\frac{1}{\alpha} - 1} + (8C_1^2 S^\alpha)^{\frac{1}{\alpha} - \frac{1}{2}})||\Omega W||_{L^2}.$$ 

Using (9.15), we yield that $L_2(Z_{\chi})$ is independent of $\nu$ for $\nu \geq S(\tau)^\alpha$. \hfill \Box

**Proposition 9.8.** Let $\Psi$ be the solution of (9.11). If $\alpha < \alpha_2, \lambda = \frac{1}{128} C_{-\alpha}, C_{1S} < C(\alpha) < \alpha \cdot 128^{-1/\alpha}$ for some constant $C(\alpha)$, then we have

$$\alpha^2||R^2\partial_{RR}\Psi_{\chi}\phi_1||_{H^3} + \alpha||R\partial_{RR}\Psi_{\chi}\phi_1||_{H^2} + ||\partial_{\beta\beta}(\Psi_{\chi} - \frac{\sin(2\beta)}{\alpha\pi}(L_2(\Omega) + \chi L_2(Z_{\chi}(0))))||_{H^3} \lesssim ||\Omega||_{H^3},$$

$$|L_2(Z_{\chi})|| \lesssim 4^{-\frac{1}{\alpha}} ||\Omega^\frac{1}{\alpha} + \frac{R}{R}||_{L^2},$$

provided that the norm of $\Omega$ on the right hand side is bounded. Moreover, $L_2(Z_{\chi})$ does not depend on $\nu$ for $\nu \geq S(\tau)^\alpha$.

**Remark 9.9.** $L_2(Z_{\chi})$ is used to correct $\Psi$ so that $\Psi_{\chi} - \frac{\sin(2\beta)}{\alpha\pi}(L_2(\Omega) - \chi L_2(Z_{\chi}(0)))$ vanishes near $R = 0$.

**Proof.** Recall the $H^3$ norm defined in (7.3). We establish the $L^2(\phi_1)$ elliptic estimates in a smaller region $\lambda = \frac{1}{16} C_{1S}^{\alpha}$ (denote by $P_1$ this elliptic estimate) to illustrate an induction-type procedure. The $L^2(\phi_2)$ elliptic estimates in Proposition 9.1 can be regarded as the base case, which is denoted as $P_0$.

To establish $P_k, k \geq 1$, in step I, we use the $P_k$ version of the elliptic estimates in Proposition 7.3 with source term $\Omega + Z_{\chi}$, i.e. the $L^2(\phi_1)$ estimates for $k = 1$, to obtain

$$\alpha^2||R^2 \partial_{RR}\Psi_{\chi} \phi_1^{1/2}||_{L^2} + \alpha||R\partial_{RR}\Psi_{\chi} \phi_1^{1/2}||_{L^2}$$

$$+ ||\partial_{\beta\beta}(\Psi_{\chi} - (\pi \alpha)^{-1} \sin(2\beta)(L_2(\Omega + Z_{\chi})) \cdot \phi_1^{1/2}||_{L^2} \lesssim ||(\Omega + Z_{\chi})\phi_1^{1/2}||_{L^2}.$$ 

(9.29)
In step II, we apply Lemma 9.3 to the $L_{12}(\cdot)$ terms and the elliptic estimate we have obtained, i.e. $\mathcal{P}_i$, $i \leq k - 1$, to control the $Z_k$ terms. In particular, for $k = 1$, $\mathcal{P}_i$, $i \leq k - 1$ is just $\mathcal{P}_0$ or Proposition 9.7. One can obtain the following estimates using $\mathcal{P}_1$, $i \leq k - 1$

$$\text{(9.30)} \quad ||Z_1 \varphi_1^{1/2}||_{L^2} \lesssim ||\Omega \varphi^{1/2}||_{L^2}, \quad ||\sin(2\beta) \alpha \pi (L_{12}(Z_\chi) - \chi_1 L_{12}(Z_\chi)(0)) \varphi_1^{1/2}||_{L^2} \lesssim ||\Omega \varphi^{1/2}||_{L^2}.$$  

Recall $Z_\chi = Z_1 + Z_2$ and (9.14). The above estimates (and similar estimates appeared in $\mathcal{P}_l$, $l > k$) hold due to the following three reasons. Firstly, $Z_1, Z_2$ defined in (9.14) only contains the first order derivative $D_R \partial_\beta$ of $\Psi$, which are lower order compared with the leading terms in (9.13). Hence, we can apply the previous elliptic estimates, e.g. $\mathcal{P}_0$ or Proposition 9.7 for $k = 1$, to estimate the norm of higher order derivatives of $Z_\chi$ or the norm of $Z_\chi$ with more singular weight. When we estimate the $\Psi$ terms in $Z_1, Z_2$ that do not involve $D_R$ derivative, e.g. $\partial_\beta \Psi, \Psi$, we decompose $\Psi$ into $\Psi - \frac{\sin(2\beta)}{2\beta} (L_{12}(\Omega) + \chi_1 L_{12}(Z)(0))$ and $\frac{\sin(2\beta)}{2\beta} (L_{12}(\Omega) + \chi_1 L_{12}(Z_\chi)(0))$. Then we apply the elliptic estimates to estimate the first part, Lemma 9.3 and Proposition 9.7 for $L_{12}(Z_\chi)(0)$ to the second part.

Secondly, $Z_1$ and $Z_2$ contains small factors. For $Z_1$ defined in (9.14), within the support of $\chi_\lambda$, we have a small parameter $C_\ell r$, which is bounded by $C_\ell (2\lambda)^{1/\alpha} \lesssim 8^{-1/\alpha}$ since $\lambda = \frac{1}{16} C_\ell^{-\alpha}$. Clearly, $\alpha^{-k} C_\ell r \chi_\lambda \lesssim 4^{-1/\alpha}$ for any absolute constant $k > 0$. For $Z_2$ defined in (9.14), the first term in $Z_2$ also contains the small factor $C_\ell r$, the second and the fourth terms contains a small factor $\alpha^2$ and the third term contains $\alpha$. The small factors $\alpha, \alpha^2$ are from the commutators in (9.12) and the boundedness of $D_R^{1/\alpha}$. These small factors cancel $\alpha^{-1}$ in the $\alpha^{-1} L_{12}(\cdot)$ term when applying the elliptic estimate. These two properties imply the first inequality in (9.30).

Thirdly, using Lemma 9.3 and (9.6) in its proof, the $H^l$ norm of $L_{12}(f) - \chi L_{12}(f)(0)$ can be bounded by the $L^2$ norm of $D_R^{1/2} \frac{(1 + R^2)}{R^2} i \leq \max(l - 1, 0)$. In (9.30), we use $f = L_{12}(Z_\chi) - \chi_1 Z_\chi(0)$. Applying Lemma 9.3 to $\chi \leq \chi \leq 2, \Psi$ and the elliptic estimates $\mathcal{P}_i, i \leq k - 1$ to $D_R^{1/2} \Psi$ with $m \geq 1$ in the support of $\chi$, we obtain that

$$||Z_2 \frac{(1 + R^2)}{R^2} ||_{L^2}, ||D_R Z_2 \frac{(1 + R^2)}{R^2} ||_{L^2} \lesssim \alpha ||\Omega \varphi^{1/2}||_{L^2}.$$  

Similar estimates holds for $D_R^2 Z_2, D_R^3 Z_2$ and will be used in establising $\mathcal{P}_i, l > 1$. This estimate and the small factors $C_\ell r, \alpha, \alpha^2$ discussed earlier (in the second reason) will cancel $\frac{1}{\alpha}$ in the second inequality in (9.30).

Therefore, combining (9.29) and (9.30), we obtain the $L^2(\varphi_1)$ elliptic estimate, i.e. $\mathcal{P}_1$. Repeating this argument, we can obtain the $H^l$ elliptic estimates.

To estimate $L_{12}(Z_{\chi_\lambda})(0)$, we use the argument in the proof of Proposition 9.7. Choosing $C_1 \Omega$ small enough, we get the desired estimate for $L_{12}(Z_{\chi_\lambda})(0)$.

We have a result similar to Proposition 9.9.

**Proposition 9.10.** Let $\tilde{\Omega}_0(t)$ be the solution of (9.10) with source term $\tilde{\Omega}_0 = \Omega_{\chi_\nu}$. If $\alpha < \alpha_2, \lambda = \frac{1}{128} C_1^{-\alpha}, (2\nu)^{1/\alpha} C_1 < C(\alpha) < \alpha \cdot 128^{-1/\alpha - 1}$ for some constant $C(\alpha)$, then we have

$$\alpha ||\frac{1 + R}{R} D_R^2 \bar{\Psi}_{0, \chi_\lambda}||_{W^{s, \infty}} + \alpha ||\frac{1 + R}{R} D_R \bar{\Psi}_{0, \chi_\lambda}||_{W^{s, \infty}}$$

$$+ ||\frac{1 + R}{R} \partial_\beta (\bar{\Psi}_{0, \chi_\lambda} - \frac{\sin(2\beta)}{2\beta} (L_{12}(\Omega) + \chi_1 L_{12}(Z_{\chi_\lambda})(0)))||_{W^{s, \infty}} \lesssim \alpha,$$

$$||L_{12}(Z_{\chi_\lambda})(0)|| \lesssim 4^{-\tilde{\pi}}.$$  

Moreover, $L_{12}(Z_{\chi_\lambda})(0)$ does not depend on $\nu$ for $\nu \geq S(\tau)\alpha$ and enjoys the above estimate for $L_{12}(Z_{\chi_\lambda})$.

**Remark 9.11.** Although $\tilde{\Omega}_0 = \Omega_{\chi_\nu}$ is time-independent, the equation (9.2) is not and $\tilde{\Psi}_0(t)$ depends on how we rescale the space. The factor $2\nu$ is the support size of $\tilde{\Omega}_0$.

The proof follows from the argument in the proof of Propositions 7.9, 7.7 and 9.8.

**9.3. Nonlinear stability.** We apply the nonlinear stability analysis of the 2D Boussinesq equations to prove Theorem 7.2.
9.3.1. Bootstrap assumption on the support size. Recall $\alpha_2$ defined in Proposition 9.8. We first require $\alpha < \alpha_2$.

We impose the first bootstrap assumption: for $t \geq 0$, we have

$$C(t) \max(S(t), S(0)) < C(\alpha) < C(t) \alpha \cdot 128^{-\frac{1}{2}} - 1,$$

where $C(\alpha)$ is the constant in Proposition 9.8. Under the above Bootstrap assumption, the support of $\omega, \theta$ does not touch the symmetry axis and the assumption in Proposition 9.8 is satisfied.

9.3.2. Approximate steady state and the normalization condition. We localize $\bar{\Omega}, \bar{\theta}$ defined in (4.38) to construct the approximate steady state for (9.8)

$$\bar{\Omega}_0 \equiv \chi_\nu \bar{\Omega}, \quad \bar{\theta}_0 \equiv \chi_\nu x J(\bar{\eta}),$$

where $\chi_\nu = \chi_1(R/\nu)$ and we have applied the integral operator $J(f)$ in Lemma A.10. Clearly, the support size of $\bar{\Omega}_0, \bar{\theta}_0$ is 2$\nu$. Using the computation in (A.33), we have

$$\bar{\eta}_0 = \partial_x(\chi_\nu \bar{\theta}) = \alpha \cos^2(\bar{\beta})D_R \chi_\nu J(\bar{\eta}) + \chi_\nu \bar{\eta}, \quad \xi_\nu(R, \beta) = \partial_\bar{y}(\chi_\nu \bar{\theta}) = \alpha \sin(\beta) \cos(\beta)D_R \chi_\nu J(\bar{\eta}) + \chi_\nu \xi,$$

Let $\bar{\Psi}_0(t)$ be the solution of (9.13) with source term $\bar{\Omega}_0$. Applying Lemma A.10 and the analysis in its proof, we know that $\bar{\Omega}_0, \bar{\eta}_0, \xi_0$ enjoys the same estimates as that of $\Omega, \eta, \xi$ in Lemmas A.5 and A.7.

We need to adjust the time-dependent normalization condition for $c_\omega(t), c_\xi(t)$. Firstly, we choose the time-dependent cutoff radial $\lambda(t) = \frac{1}{128}(C(t))^{-\alpha}$ according to Proposition 9.8.

Define $\bar{Z}_{\lambda(0)}(t)$ according to (4.31), or equivalently (9.15), with $\Psi = \bar{\Psi}_0(t), \Omega = \bar{\Omega}_0$ and $\chi = \chi_{\lambda(0)}$. It does not depend on the cutoff radial as long as $\lambda(0) \geq (2\nu)^{\alpha}$, where $2\nu$ is the size of support of $\bar{\Omega}_0$. We use the following conditions

$$\bar{c}_\omega(t) = -1 - \frac{2}{\pi \alpha}L_{12}(\bar{\Omega}_0 - \bar{\Omega} + \bar{Z}_{\chi_{\lambda(0)}}(0))(0), \quad \bar{\xi}_\nu(t) = \frac{1}{\alpha} + 3 - \frac{1 - \alpha}{\pi \alpha}L_{12}(\bar{\Omega}_0 - \bar{\Omega} + \bar{Z}_{\chi_{\lambda(0)}}(0))(0).$$

We remark that $\bar{c}_\omega(t), \bar{\xi}_\nu(t)$ is time-dependent. Without the $Z$ term, the above conditions for $\bar{c}_\omega, \bar{c}_\xi$ are the same as that in (4.31) with a correction due to the difference between the profiles $(\bar{\Omega}, \bar{\eta})$ in (4.8) and $\bar{\Omega}_0, \bar{\eta}_0$ in (9.32)-(9.33). For this difference, we use (4.11) to correct $\bar{c}_\nu, \bar{c}_\xi$.

For any perturbation $\Omega(t)$, we use the following conditions for $c_\omega(t), c_\xi(t)$

$$c_\omega(t) = -\frac{2}{\pi \alpha}L_{12}(\Omega(t) + Z_{\chi_{\lambda(t)}}(t))(0), \quad c_\xi(t) = \frac{1 - \alpha}{\alpha}c_\omega(t).$$

Without the $Z$ term, the above conditions for $c_\omega(t), c_\xi(t)$ are the same as that in (4.11).

We add the $Z$ terms in (9.34), (9.35) since the behavior of $\Psi$, which is the solution of (9.10), is characterized by $L_{12}(\Omega + Z_{\chi})(0)$ for $R$ close to 0 according to the elliptic estimate in Proposition 9.8. For the 2D Boussinesq equation, we use $L_{12}(\Omega)(0)$ to determine $c_\omega, c_\xi$ since it also characterizes the behavior of $\Psi$ near $R = 0$ according to Proposition 7.3.

We choose the above conditions so that the error of the approximate steady state vanishes quadratically near $R = 0$ and that the update of $\Omega(t), \eta(t)(\omega, \theta_x)$ in equation (9.5) also vanishes quadratically near $R = 0$ if the initial perturbation $\Omega(\cdot, 0), \eta(\cdot, 0)(\theta_x(0))$ vanishes quadratically. We also determine $\bar{c}_\omega, \bar{c}_\xi$ in (4.3) and $c_\omega, c_\xi$ in (4.11) based on this principle.

9.3.3. Estimate of the lower order terms. The equations (9.8) are slightly different from (4.6) for the Boussinesq systems. We show how to estimate their differences. Suppose that $\Omega(t), \theta(t)$ are the perturbations and the support size of $\bar{\Omega}_0 + \Omega(t), \bar{\theta}_0 + \theta(t)$ is $S(t)$.

Assume that the bootstrap assumption (9.31) holds true. For the term $\frac{1 - s^4}{s^4} \theta_x$, within the support of $\omega, \theta$, we have $r \leq S(t), s = 1 - C_I \gamma \sin(\beta) \in [3/4, 1]$. We get

$$\left| \frac{1 - s^4}{s^4} \right| \lesssim 1 - s \leq C_I r \leq C_I S(t),$$

which is an extremely small factor. Since $r = R^{1/\alpha}$, the factor $C_I r, 1 - s^4$ vanishes with an order much higher than $R^2$ near $R = 0$. Hence, $\frac{1 - s^4}{s^4} \theta_x$ is a smooth (near $R = 0$) small error term.
For the term $\frac{1}{4}C_t \Psi$ in $u = -\psi_y + \frac{1}{4}C_t \Psi$ defined in \eqref{9.3}. Under the $(R, \beta)$ coordinates, it becomes $\frac{1}{4}C_t \Psi(R, \beta)$. Compared to $-\psi_y = -(\Psi_y)$ in \eqref{2.20}, $\frac{1}{4}C_t \Psi(R, \beta)$ vanishes on $\beta = 0, \pi/2$ and contains a small smooth factor $C_t R^{1/\alpha}$ within the support of $\omega, \theta$.

The last difference is the elliptic estimate between Propositions \ref{6.4} and \ref{9.8}. Notice that in \eqref{9.8}, we only use $\Psi(R, \beta)$ for $(R, \beta)$ within the support of $\omega, \theta$. We have $\Psi_\chi_{\lambda_1}(R, \beta) = \Psi(R, \beta)$ for $\lambda(t) = \frac{C_t R^{1/\alpha}}{2 \pi R}$, $R \leq S(t)$. Finally, $\chi_1 L_{12}(Z_{\chi_{\lambda_1}})$ in Proposition \ref{9.8} only affects the equation near $R = 0$. Since $\frac{1}{1+R} \Omega \in L^2$, using the estimate in Proposition \ref{9.8} we get

$$
(9.36) \quad |L_{12}(Z_{\chi_{\lambda_1}})(0)| = |L_{12}(\bar{Z}_{\chi_{\lambda_1}})(0)| \lesssim \alpha 4^{-1/\alpha}, \quad |L_{12}(Z_{\chi_{\lambda_1}})(0)| \lesssim 4^{-1/\alpha} ||\frac{1}{1+R} \Omega||_{L^2},
$$

where we have used $\lambda(t) \geq \lambda(0)$ to obtain the first identity, and used \eqref{4.3}, \eqref{9.2} and $||\frac{1}{1+R} \Omega||_{L^2} \lesssim \alpha$ to obtain the first inequality.

Using the estimate in Section 8 one can easily estimate these lower order terms in $H^3, H^3(\psi)$ or $C^1$ norm accordingly and obtain a small norm bounded by $C(1 + \alpha^{-\kappa})(4^{-1/\alpha} + C_1 S)$, where $\kappa, C > 0$ are some absolute constants.

9.3.4. Nonlinear stability. Notice that the domain of the dynamic rescaling equation is $R \in [0, C_t^{-1}]$ rather than $R \geq 0$. We cannot apply directly the estimates in Sections 8-9 because in these estimates, we linearized the equations around $\Omega, \eta, \xi$ which are defined globally.

We consider the system of $\theta_x, \theta_y, \omega$ obtained from \eqref{9.8} and then linearize it around the approximate steady state $\Omega_0, \eta_0, \xi_0, \bar{c}_\omega, \bar{c}_t$ to obtain a system similar to \eqref{5.3}-\eqref{5.7} with $\Omega, \bar{c}_\omega, \bar{c}_t, \frac{2}{\pi \alpha} L_{12}(\bar{\Omega})$ replaced by $\Omega_0, \eta_0, \xi_0, \frac{2}{\pi \alpha} L_{12}(\Omega_0)$. We also put the lower order terms discussed in Section 9.3.3 into the remaining terms $R_\Omega, R_\eta, R_\xi$.

According to Lemma \ref{A.10}, we know that $\Omega_0, \eta_0, \xi_0$ converges to $\bar{\Omega}, \bar{\eta}, \bar{\xi}$ in the $H^3, H^3(\psi)$ norm as $\nu \to \infty$ ($\nu$ is the cutoff radial in \eqref{0.32}). Moreover, we can easily generalize the $H^3, H^3(\psi)$ convergence to the higher order convergence. We choose the same weights and the same energy norm as that in Section 8. Then for sufficient large $\nu$, due to these convergence results, under the bootstrap assumption \eqref{9.31}, we can obtain the following $H^3, H^3(\psi)$ estimates similar to that in Corollary \ref{6.3}

$$
\frac{1}{2} \frac{d}{dt} E_3^2(\Omega, \eta, \xi) \leq \left( - \frac{1}{13} + C_\alpha \right) E_3^2 + R_3,
$$

where we have a slightly weaker estimate ($\frac{1}{13} < \frac{1}{12}$) due to the small difference between $(\Omega_0, \eta_0, \xi_0)$ and $(\bar{\Omega}, \bar{\eta}, \bar{\xi})$.

Recall the equation \eqref{6.10} for the 2D Boussinesq equation in the $C^1$ estimate of $\xi$. The damping part in \eqref{6.10} is $(-2 - \frac{2}{\pi \alpha}) \xi$. For the 3D Euler equation, it is replaced by $(-2 - \frac{2}{\pi \alpha} L_{12}(\bar{\Omega}) \xi)$. For sufficient large $\nu$, using the convergence results, we can obtain estimates similar to \eqref{6.12}, \eqref{6.10}, \eqref{6.17} with slightly larger constants, e.g. $-2, 3$ are replaced by $-2 + \frac{1}{100}, 3 + \frac{1}{100}$.

There exists a large absolute constant $\nu_0$, such that for $\nu > \nu_0$, $\nu$ satisfies the above requirements and that for $\nu > \nu_0$, we have

$$
(9.37) \quad \left| \frac{2}{\pi \alpha} L_{12}(\bar{\Omega} - \Omega_0)(0) \right| \leq \frac{1}{100}.
$$

Note that $\bar{\Omega}, \Omega_0$ contains a small factor $\alpha$ (see \eqref{4.3}). Since the estimates of the remaining terms in $H^3, H^3(\psi), C^1$ estimates are not sensitive to the absolute constants, we can apply the estimates in Section 8 and the argument in Section 9.3.3 to estimate the lower order terms in \eqref{9.8}. Therefore, for $\nu > \nu_0$, under the bootstrap assumption, we can obtain the following nonlinear estimate for compactly supported perturbations $\Omega(t), \eta(t), \xi(t)$ around $(\Omega_0, \eta_0, \xi_0)$, which is similar to \eqref{8.27},

$$
(9.38) \quad \frac{1}{2} \frac{d}{dt} E^2(\Omega, \eta, \xi) \leq - \frac{1}{13} E^2 + C(\alpha^{1/2} E^2 + \alpha^{-3/2} E^3 + \alpha^2 E) + C(\alpha, C_1(t), S(t))(E^2 + E + E^3),
$$

where the last term is from the estimates of the lower order terms that we have analyzed in previous section and $C(\alpha, C_1(t), S(t)) = C(1 + \alpha^{-\kappa})(4^{-1/\alpha} + C_1(t) S(t))$ for some universal
constant $C$. Under the bootstrap assumption (9.31), we further obtain
\begin{equation}
C(\alpha, C(t), S(t)) \lesssim (1 + \alpha^{-\kappa})4^{-1/\alpha} \lesssim \alpha^3.
\end{equation}

Combining (9.38), (9.39), we obtain that there exist $\alpha_3$ with $0 < \alpha_3 < \alpha_2$ ($\alpha_2$ is the constant in Proposition 9.8 and an absolute constant $\bar{K} > 0$, such that if $E(\Omega(0), \eta(0), \xi(0)) < \bar{K}^2$, under the bootstrap assumption (9.31) we have
\begin{equation}
E(\Omega(t), \eta(t), \xi(t)) < \bar{K}^2.
\end{equation}

Recall $c_\omega, c_1, \bar{c}_\omega, \bar{c}_1$ defined in (9.31), (9.35). Using (9.36), (9.37), $|L_{12}(\Omega)(0)| \lesssim ||\Omega||_{\mathcal{H}^3} \lesssim E \lesssim \alpha^2$, we obtain
\begin{equation}
|c_\omega + \bar{c}_\omega + 1| < \frac{1}{100} + C4^{-1/\alpha} + C\alpha, \quad c_1 + \bar{c}_1 > \frac{1}{\alpha} + 3 - \frac{1}{100\alpha} - C4^{-1/\alpha} \alpha^{-1} - C.
\end{equation}

We can further choose $\alpha_4$ with $0 < \alpha_4 < \alpha_3$, such that for $\alpha < \alpha_4$,
\begin{equation}
-\frac{3}{2} < c_\omega + \bar{c}_\omega < -\frac{1}{2}, \quad c_1 + \bar{c}_1 > \frac{3}{4\alpha}.
\end{equation}

9.3.5. Growth of the support. Finally, we estimate the growth of the support $S(\tau)$ of the solutions $\Omega + \bar{\Omega}, \theta + \bar{\theta}$. Denote $\hat{u}(t) = u(t) + \hat{u}(t), \hat{\Psi}(t) = \Psi(t) + \hat{\Psi}(t), \hat{c}_1(t) = c_1(t) + \bar{c}_1(t)$.

Applying (2.8)-(2.9) and (2.11) to $\hat{\Psi}$, we can rewrite the transport term $\hat{u} \cdot \nabla$ as
\begin{equation}
\hat{u} \cdot \nabla = (\partial_\beta \hat{\psi} + C_1 s^{-1} \hat{\psi}) \partial_\beta + \partial_\eta \hat{\psi} \partial_\eta = \left(\frac{\alpha C_1 r \cos(\beta)}{s} R \hat{\psi} - \alpha R \partial_\beta \hat{\psi}\right) \partial_\beta + (\hat{\psi} \partial_\beta) \partial_\beta - \frac{C_1 r \sin(\beta)}{s} \hat{\psi},
\end{equation}
where $r = R^{1/\alpha}, s = 1 - C_1 r \sin(\beta)$. The above formula is different from (2.11) due to the extra term $C_1 s^{-1} \hat{\psi} \partial_\beta$. Notice that $\hat{c}_1 \hat{\mathbf{x}} \cdot \nabla$ becomes $\alpha \hat{c}_1 \partial_\beta$ under the $(R, \beta)$ coordinates. For a point which is inside the support of $\Omega, \theta(R, \beta)$ and have coordinate $(R(t), \beta(t))$, its trajectory under the flow ($\hat{c}_1 \hat{\mathbf{x}} \cdot \nabla$ is governed by
\begin{equation}
\frac{d}{dt} R(t) = (\alpha \hat{c}_1 R(t)) + \frac{\alpha C_1 R(t) \cos(\beta)}{s(t)} R(t) \hat{\Psi}(R(t), \beta(t)) - \alpha R(t) \partial_\beta \hat{\Psi}(R(t), \beta(t)),
\end{equation}
where the relation between $\hat{c}_1(t), C_1(t)$ is given in (9.7).

**Lemma 9.12.** Under the assumption of Proposition 9.8 and that $\Omega \in \mathcal{H}^3$, for $R \leq S(t)$, we have
\begin{equation}
||(1 + R^{1/3}) \hat{\Psi}(R, \beta)| + |(1 + R^{1/3}) \partial_\beta \hat{\Psi}(R, \beta)| \lesssim \alpha^{-1} ||\Omega||_{\mathcal{H}^2} + 1 \lesssim \alpha^{-1} E(\Omega(t), \eta(t), \xi(t)) + 1.
\end{equation}

Recall the weights $\varphi_i$ defined in Definition 5.2 for the $\mathcal{H}^3$ norm. Note that the elliptic estimates in Proposition 9.8 with radial weight $\frac{1 + R^4}{R^3}$ in the $\mathcal{H}^3$ norm replaced by $\frac{1 + R^2}{R^3}$ can be obtained in the same way. Since $\bar{\Omega} + \Omega$ is in this modified $\mathcal{H}^3$ space ( $\Omega$ vanishes linearly near $R = 0$), we can apply its associated elliptic estimate to $\hat{\Psi}(t)$. Then the proof follows from an argument similar to that in the proof of Lemma 7.11 and applying this elliptic estimate to $\hat{\Psi}$ and Lemma A.3 to $L_{12}(\Omega)$. We have the desired decay property for $\hat{\Psi}$ due to the radial weight $\frac{1 + R^2}{R^3}$.

Now we assume that the initial data satisfies $E(\Omega(0), \eta(0), \xi(0)) < \bar{K}^2$. Under the bootstrap assumption (9.31), we have a priori estimates (9.40), (9.41).

Plugging the bootstrap assumption (9.31), (9.40) and Lemma 9.12 in (9.42), we derive
\begin{equation}
\frac{d}{dt} R(t) \leq \alpha \hat{c}_1 R(t) + C_1(\alpha^{-1} E + 1) R(t)^{2/3} \leq \alpha \hat{c}_1 R(t) + C_1 R(t)^{2/3},
\end{equation}
where we have used $C_1(t) r(t) \lesssim C_1(t) S(t) < \frac{1}{4\alpha}, s^{-1} \lesssim 2$. From the formula of $C_1(t)$, we know $\frac{d}{dt} C_1(t) = -\hat{c}_1(t) C_1(t)$. Multiplying $C_1^\alpha(t)$ on both sides, we get
\begin{equation}
\frac{d}{dt} C_1^\alpha R(t) \leq C_1(\alpha^{-1} E + 1) R(t)^{2/3} = C_1(\alpha^2 R^{2/3}) C_1(t)^{\alpha/3}.
\end{equation}
From the a priori estimate (9.41) and the formula of $C_t$ in (9.7), we know $C_t^\alpha(t) \leq C_t^\alpha(0) \exp(-\frac{4}{3})$. Then solving this ODE, we yield

$$(C_t^\alpha(R(t)))^{1/3} \leq (C_t^\alpha(0) \exp(S(0)^{\alpha}))^{1/3} + C\alpha \int_0^\infty C_t^{\alpha/3}(0) \exp(-\frac{b}{6}) db \leq C_t^\alpha(0) S(0)^{\alpha/3} + C\alpha.$$ 

Taking the suprement over $(R(t), \beta(t))$ within the support of $\Omega, \theta$, we prove

$$(9.43) \quad C_t(t)S(t) \leq C(\alpha, S(0))C_t(0).$$

9.3.6. Finite time blowup. For fixed $\alpha < \alpha_4, \nu > \nu_0$, we choose zero initial perturbation $\Omega(0) = 0, \eta(0) = 0, \xi(0) = 0$. Then the initial data is $(\Omega_0, \theta_0)$ defined in (9.32) which has compact support with support size $S(0) = 2\nu$. We choose initial rescaling $C_t(0)$ such that $C(\alpha, S(0))C_t(0) < C(\alpha)/2$. Using the a priori estimates (9.20), (9.21) and (9.43), we know that the bootstrap assumption in (9.31) can be continued and thus these estimates hold true for all time.

Since $-\frac{1}{2} < c_\omega + \tilde{c}_\omega < -\frac{1}{2} (9.41)$ and the solutions $\omega, \theta$ are close to $\bar{\omega}, \bar{\theta}$ for all time in the dynamic rescaling equation, using the argument in Subsection 8.6 and the BKM blowup criterion in [2], we prove that the solutions remain in the same regularity class as that of the initial data before $T^* < +\infty$ and develop a finite time singularity at $T^*$, where $T^* = t(\infty) = \int_0^{\infty} C_\omega(\tau) d\tau < +\infty$.

Since $\bar{\theta}_0 + \theta(t) \geq 0$ and the support of $\omega, \theta$ is away from the axis, we can recover $u^\theta, \omega^\theta$ from $\theta, \omega$ via (9.4), (9.6). Since $u^\theta, \omega^\theta$ have compact support, the solutions have finite energy.

10. CONCLUDING REMARKS

We have proved finite time self-similar blowup of the 2D Boussinesq and the 3D axisymmetric Euler equations with solid boundary and large swirl using $C_t^\alpha$ initial data with small $\alpha$ for $(\omega, \nabla \theta)$ in the case of the 2D Boussinesq equations and for $(\omega^\theta, \nabla (u^\theta)^2)$ in the case of the 3D Euler equations, respectively. In particular, we showed that the velocity field is in $C^{1,\alpha}$ and has finite energy. Moreover, our solution for the 3D axisymmetric Euler equations is in $C^\infty$ before the blowup time except on the symmetry plane $z = 0$. Similarly, the solution for the 2D Boussinesq equations is also in $C^\infty$ before the blowup time except on the symmetry axis $x = 0$. It is likely that one can choose other singular weights in the analysis based on the guideline discussed in Subsection 5.1.2 and perturb the approximate steady state to construct an initial data that is $C^\infty$ except at the origin.

Our work was inspired by the numerical evidence of finite time singularity for the 3D axisymmetric Euler equations with solid boundary in [28, 29]. Our singular solution and the finite time blowup solution considered in [28, 29] share many essential features except that the regularity of our initial data is in $C^{1,\alpha}$ while the initial data considered in [28, 29] is in $C^\infty$. The driving mechanisms for the finite time singularity for the two scenarios are essentially the same. It is generally believed that the presence of the boundary and the odd-even symmetry properties of the solution along the axial direction have played an important role in generating a stable and sustainable finite time singularity. With the presence of the solid boundary, we were able to construct an approximate steady state solution for the dynamic rescaling equations with an explicit expression. More importantly, the presence of the boundary and the odd-even properties of the solution along the axial direction enabled us to prove linear and nonlinear stability of the approximate steady state solution by using appropriately constructed singular weights.

The results presented in this paper can be easily extended to prove finite time singularity of two closely relate problems. First of all, the same analysis can be applied to prove finite time blowup of the 3D axisymmetric Euler equations in a domain outside the cylinder $(r, z) : r \geq 1, z \in \mathbb{R}$. The proof is easier since such domain does not contain the symmetry axis.

Secondly, our method of analysis can be applied to prove the finite time blowup of the following modified 2D Boussinesq equation on the whole space for $C^\alpha$ initial data $\omega, \frac{\theta}{x}$ with small $\alpha$:

$$\omega_t + u \cdot \nabla \omega = \theta/x, \quad \theta_t + u \cdot \nabla \theta = 0, \quad u = \nabla^\perp (-\Delta)^{-1} \omega.$$ 

The above modified Boussinesq equations with a simplified Biot-Savart law have been studied in [16], [29]. Note that the above equations are a closed system for $\omega, \theta/x$. We can derive the
corresponding dynamic rescaling formulation for the above system and reformulate problem using the \((R, \beta)\) variables. We consider the equations for the variable \(\Omega(R, \beta) = \omega(x, y), \eta(R, \beta) = (\theta/x)(x, y)\). The approximate steady state for \(\Omega, \eta\) is similar to \((4.8)\) with \(\cos(\beta)^{\alpha}\) replaced by \((\sin(2\beta))^{\alpha/2}\), which is \(C^a\) globally on \(\mathbb{R}^2\). Moreover, the scaling parameters are \(\tilde{c}_l = 1/\alpha, \tilde{c}_w = -1\).

The leading order part of the linearized equation of this system is exactly the same as that in \((5.5)-(5.6)\). The same analysis in Section \(6.8\) applies to the above system and the proof is much easier since the \(\theta_v\) variable appeared in \((5.7)-(5.7)\) is not present in this system. Moreover, we do not have a term similar to \(v_x\theta_y\) so that we can use the profile with \(\Gamma(\beta) = (\sin(2\beta))^{\alpha/2}\).

We would like to point out that the results presented in this paper do not provide a full justification of the finite time singularity of the 3D axisymmetric Euler equations with solid boundary considered in \([28, 29]\). The method of analysis presented here relies heavily on the assumption that the initial velocity field is in \(C^{1,\alpha}\) with a small \(\alpha\). Under this assumption, several important nonlocal terms in the perturbation analysis can be made arbitrarily small by choosing a sufficiently small \(\alpha\). For smooth initial data considered in \([28, 29]\), it is almost impossible to obtain an analytic expression of an accurate approximate steady state solution for the dynamic rescaling equations. Even if we use a numerically constructed approximate steady state solution, there are still substantial difficulties in designing appropriately chosen singular weighted norms to prove nonlinear stability of this approximate steady state solution. The standard weighted energy estimates for the nonlocal terms are simply too crude to control the nonlocal terms. The control of these nonlocal terms is crucial for us to prove the linear stability.

Recently, in collaborator with De Huang, we have been able to prove the finite time self-similar singularity of the HL model with \(C^\infty\) initial data by using the method of analysis presented in \([5]\) and a computer assisted analysis. We are now working to extend this computer assisted analysis to prove the finite time self-similar singularity of the 2D Boussinesq and 3D axisymmetric Euler equations in the presence of boundary with smooth initial data in the same setting as that considered in \([28, 29]\). We will report these results in a forthcoming paper.

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APPENDIX A.

In Appendix \(A.1\) we estimate \(\Gamma(\beta)\) and the constant \(c\) appeared in the approximate profile \((4.8)\). In Appendix \(A.2\) we will establish several estimates of \(L_{12}(\Omega)\) that are used frequently in the nonlinear stability analysis. Notice that we only have the formula of \(\tilde{\eta} = \tilde{\theta}_x\) in \((4.8)\). We need to recover \(\theta, \xi = \theta_y\) from \(\tilde{\eta}\) via integration. Yet, we do not have a simple formula to perform integration. Alternatively, we derive useful estimates for \(\xi\) in Appendix \(A.3\). Some estimates of \(\Omega, \tilde{\eta}\) are also obtained there. In Appendix \(A.4\) we show that the truncation of the approximate steady state would contribute only to a small perturbation under the norm we use, and we prove Lemma \(9.1\).

A.1. Estimates of \(\Gamma(\beta)\) and the constant \(c\).

Lemma A.1. For \(x \in [0, 1]\), the following estimate holds uniformly for \(\lambda \geq 1/10\),

\[
(1 - x^c) x^\lambda \leq \frac{K}{\lambda}.
\]

Consequently, for \(\beta \in [0, \pi/2]\), \(2 \geq \lambda \geq 1/10\), we have

\[
|\Gamma(\beta) - 1| \lesssim (\cos(2\beta))^{\lambda} \leq |(\cos^2(\beta) - 1)(\cos(\beta))^{\lambda}| \lesssim \alpha,
\]

and

\[
|c - \frac{2}{\pi}| = \left| \frac{2}{\pi} \int_0^{\pi/2} (\Gamma(\beta) - 1) \sin(2\beta) d\beta \right| \leq 2\alpha.
\]
Proof. Using change of a variable $t = x^\kappa$, it suffices to show that for $t \in [0, 1], (1 - t)t^{\lambda/k} \leq \frac{\lambda}{\kappa}$. Notice that $\lambda \geq 1/10$ and $t \leq 1$. Using Young’s inequality, we derive

$$(1 - t)t^{\lambda/k} = \frac{\kappa}{\lambda} \cdot (\frac{\lambda}{\kappa} (1 - t))t^{\lambda/k} \leq \frac{\kappa}{\lambda} \left( \frac{\lambda}{\kappa} (1 - t) + \frac{\lambda}{\kappa} t \right)^{1+\lambda/k} = \frac{\kappa}{\lambda} \left( \lambda + \kappa \right)^{1+\lambda/k} \leq \frac{\kappa}{\lambda},$$

which implies (A.1). The remaining inequalities in the Lemma follows directly from (A.1).

A.2. Inequalities about $L_{12}(\Omega)$. Recall $\hat{L}_{12}(\Omega) = L_{12}(\Omega) - L_{12}(\Omega)(0)$. To estimate $\hat{L}_{12}(\Omega)g$ in $\mathcal{L}_t$, we use the following simple Lemma.

Lemma A.2. Let $g$ be some function depending on $\hat{\Omega}, \hat{\eta}, \hat{\xi}$ and $\varphi$ be some weights. We have

$$
\langle \hat{L}_{12}^2(\Omega)g^2, \varphi \rangle \lesssim \left\| R^{-1}\hat{L}_{12}(\Omega) \right\|_{L^2(R)}^2 \int_0^{\pi/2} R^2 g^2(R, \beta) \varphi(R, \beta) d\beta \biggr\|_{L^\infty(R)},
$$

(A.2)

$$
\langle (D^k_{\hat{R}}\hat{L}_{12}(\Omega))^2 g^2, \varphi \rangle \lesssim \left\| R^{-1}D^{k-1}_{\hat{R}}\Omega \right\|_{L^2}^2 \int_0^{\pi/2} R^2 g^2(R, \beta) \varphi(R, \beta) d\beta \biggr\|_{L^\infty(R)},
$$

for $k \geq 1$, provided that the upper bound is well-defined, where $D_{\hat{R}} = \hat{R}\partial_{\hat{R}}$.

Proof. The first inequality follows directly from that $\hat{L}_{12}(\Omega)$ does not dependent on $\beta$. Recall the definition of $\hat{L}_{12}(\Omega)$ in (5.8) and $D_{\hat{R}} = \hat{R}\partial_{\hat{R}}$. Notice that for $k \geq 1$, we have

$$
D^k_{\hat{R}}\hat{L}_{12}(\Omega) = -\int_0^{\pi/2} D^{k-1}_{\hat{R}}\Omega(R, \beta) \sin(2\beta) d\beta.
$$

Using the Cauchy-Schwarz inequality, we prove

$$
\langle (D^k_{\hat{R}}\hat{L}_{12}(\Omega))^2 g^2, \varphi \rangle = \int_0^{\infty} \left( \int_0^{\pi/2} (D^{k-1}_{\hat{R}}\Omega(R, \beta) \sin(2\beta) d\beta)^2 \right)^{\frac{1}{2}} \int_0^{\pi/2} g^2 \varphi d\beta \right) dR
$$

$$
\lesssim \int_0^{\infty} \left( \int_0^{\pi/2} (D^{k-1}_{\hat{R}}\Omega)^2 d\beta \right)^{\frac{1}{2}} \int_0^{\pi/2} g^2 \varphi d\beta \lesssim \left\| R^{-1}D^{k-1}_{\hat{R}}\Omega \right\|_{L^2} \int_0^{\pi/2} R^2 g^2(R, \beta) \varphi(R, \beta) d\beta \biggr\|_{L^\infty(R)}.
$$

Lemma A.3. Let $\chi(\cdot) : [0, \infty) \to [0, 1]$ be a smooth cutoff function, such that $\chi(R) = 1$ for $R \leq 1$ and $\chi(R) = 0$ for $R \geq 2$. For $k = 1, 2, \ldots$, we have

$$
\left\| L_{12}(\Omega) \right\|_{L^\infty} \lesssim \left\| \frac{1 + R}{R} \Omega \right\|_{L^2}, \quad \left\| \hat{L}_{12}(\Omega)(R^2 + R^{-3})^{1/2} \right\|_{L^2(R)} \lesssim \left\| \Omega \left( \frac{1 + R^2}{R^2} \right) \right\|_{L^2},
$$

(A.3)

$$
\left\| \frac{1 + R^k}{R^k} L_{12}(\Omega) - L_{12}(\Omega)(0) \chi \right\|_{H^3} + \left\| D_{\varphi} L_{12}(\Omega) - L_{12}(\Omega)(0) \chi \right\|_{H^3} \lesssim \left\| \Omega \right\|_{H^3},
$$

$$
\left\| D^k_{\varphi} L_{12}(\Omega) \right\|_{H^3} + \left\| D^k_{\varphi}(L_{12}(\Omega) - \chi L_{12}(\Omega)(0)) \right\|_{H^3} \lesssim \left\| \Omega \right\|_{H^3},
$$

(A.4)

$$
\left\| (1 + R)\partial_{\varphi} D^k_{\varphi} L_{12}(\Omega) \right\|_{H^3} + \left\| (1 + R)\partial_{\varphi} D^k_{\varphi}(L_{12}(\Omega) - \chi L_{12}(\Omega)(0)) \right\|_{H^3} \lesssim \left\| \Omega \right\|_{H^3},
$$

$$
\left\| L_{12}(\Omega) \right\|_{X} + \left\| D_{\varphi} L_{12}(\Omega) \right\|_{X} \lesssim \left\| \Omega \right\|_{H^3},
$$

where $X \equiv H^3 \oplus W^{5, \infty}$ is defined in (7.9).

Remark A.4. We subtract $\chi L_{12}(\Omega)(0)$ near $R = 0$ since $L_{12}(\Omega)$ does not vanishes at $R = 0$.

Proof. Recall $L_{12}(\Omega)$ in (2.12) and $\hat{L}_{12}(\Omega)$ in (5.8). Using the Cauchy-Schwarz and the Hardy inequality, we get

$$
\left\| L_{12}(\Omega) \right\|_{L^\infty} \lesssim \left\| \Omega \right\|_{L^2} \left( \frac{1}{R} \right)^l \left\| \frac{1 + R}{R} \Omega \right\|_{L^2} \left( \frac{1}{1 + R} \right)^l \left\| L^2(\Omega) \right\|_{L^2(R)} \lesssim \left\| \frac{1 + R}{R} \Omega \right\|_{L^2},
$$

(A.5)

$$
\left\| \frac{1}{R^l} \hat{L}_{12}(\Omega) \right\|_{L^2(R)} \lesssim \int_0^{\infty} \frac{1}{R^l} \hat{L}_{12}(\Omega) dR \lesssim \int_0^{\infty} \frac{1}{R^{2l-2}} (\partial_{\varphi} \hat{L}_{12}(\Omega))^2 dR \lesssim \left\| \Omega^2, R^{-2l} \right\|,
$$
for $l = 1, \frac{3}{2}, 2$, which implies the first two inequalities in (A.3). For $k = 1, 2$, observe that
\[
\| (1 + R^k_kL(12\Omega - L_{12}(\Omega)(0)\chi) \|_{L^2(R)} \lesssim \left\| \frac{1 + R^k_kL(12\Omega - L_{12}(\Omega)(0)\chi) \|_{L^2(R)} + \left\| \frac{1 + R^k_kL(12\Omega)(1 - \chi) \|_{L^2(R)} \right. \right. \\
\lesssim \left\| \frac{1 + R^k_kL(12\Omega) \|_{L^2(R)} + \| L_{12}(\Omega) \|_{L^2(R)} \lesssim \left| \frac{1 + R^k_kL(12\Omega)(1 - \chi) \|_{L^2(R)} \right. \right.
\]
where we have used (A.5) in the last inequality. Denote $\Omega_*$ in subsection A.3.1, we estimate $\Omega_* = \int_0^{\pi/2} \Omega d\beta$. From (2.12), we know
\[
L_{12}(\Omega)(R) = \int_R^\infty \frac{\Omega_*(S)}{S} dS = \int_R^\infty K(R, S) \Omega_*(S) dS, \quad K(R, S) = \frac{1}{S} \frac{1}{1 + R \leq S}.
\]
The $L^2$ boundedness of $L_{12}$ is standard. Notice that $K$ is homogeneous of degree $-1$, i.e. $K(\lambda R, \lambda S) = \lambda^{-1} K(R, S)$ for $\lambda > 0$. Using change of a variable $S = Rz$, we get
\[
L_{12}(\Omega)(R) = \int_0^\infty \frac{1}{R} K(1, z) \Omega_*(Rz) Rdz = \int_0^\infty K(1, z) \Omega_*(Rz) dz.
\]
Then, the Minkowski inequality implies
\[
\| L_{12}(\Omega) \|_{L^2} \leq \int_0^\infty K(1, z) \| \Omega_*(Rz) \|_{L^2(R)} dz \leq \int_0^\infty K(1, z) z^{-1/2} \| \Omega \|_{L^2} dz \leq \| \Omega \|_{L^2} \int_{z \geq 1} z^{-3/2} dz \lesssim \| \Omega \|_{L^2}.
\]
We complete the proof of (A.3). Notice that $D_R L_{12}(\Omega) = -\Omega_*$, $\| D_R^k \|_{L^2} \lesssim 1$ for $1 \leq k \leq 4$ and $D_\beta L_{12}(\Omega) = 0, D_\beta \chi = 0$. Using that $\sin(2\beta)^{-1}$ in the weight $\varphi_1 = \sin(2\beta)^{-1}$ is integrable in the $\beta$ direction and (A.3), we yield
\[
\| (L_{12}(\Omega) - L_{12}(\Omega)(0)\chi) \varphi_1^{1/2} \|_{L^2} + \| D_R^k \|_{L^2(R)} - L_{12}(\Omega)(0)\chi) \varphi_1^{1/2} \|_{L^2} \lesssim \| (L_{12}(\Omega) - L_{12}(\Omega)(0)\chi) \|_{L^2} + \| D_R^k \|_{L^2(R)} - L_{12}(\Omega)(0)\chi) \|_{L^2} \lesssim \| \Omega \|_{H^3},
\]
which implies the first estimate in (A.4). From the definition of $L_{12}(\Omega)$ in (2.12), we have $D_R L_{12}(\Omega) = L_{12}(D_R \Omega)$. Notice that $\| D_R^k \|_{L^2(R)} \lesssim 1$. Using (A.3), we prove for $k \leq 3$
\[
\| D_R^k \|_{L^2(R)} \|_{L^2} + \| L_{12}(\Omega)(0)\chi \|_{L^2(R)} \lesssim \| \Omega \|_{H^3},
\]
which implies the second estimate in (A.4). Similarly, since $\partial_\beta D_R^k \|_{L^2(R)} = \partial_\beta L_{12}(D_R \Omega) = -R^{-1} \partial_\beta \Omega_*(R)$, where $\Omega_*(R) = \int_0^{\pi/2} \Omega(R, \beta) d\beta$, and that $l \leq 2$, we have
\[
\| \partial_\beta D_R^k \|_{L^2(R)} \|_{L^2(R)} \lesssim \| \partial_\beta R^{-1} \partial_\beta \Omega_*(R) \|_{L^2(R)} \lesssim \| \Omega \|_{H^3},
\]
which along with the second estimate in (A.4) and $| \partial_\beta D_R^k \|_{L^2(R)} \|_{L^2(R)} \|_{L^2(R)} \lesssim \| \Omega \|_{H^3}$ completes the proof of the third estimate in (A.4).

Since $\chi L_{12}(\Omega)(0)$ does not depend on $\beta$, we apply the first two estimates in (A.4) to yield
\[
\| D_R \|_{L^2(\Omega)} \|_{L^2} \lesssim \| \Omega \|_{H^3} + \| L_{12}(\Omega)(0)\chi \|_{H^3} + \| D_R \|_{L^2(\Omega)} \|_{L^2(\Omega)} \|_{L^2(\Omega)} \lesssim \| \Omega \|_{H^3}
\]
for $i = 0, 1$. We complete the proof of (A.4). \( \square \)

**A.3. Estimate of the approximate self-similar solution.** In appendix A.3.1, we estimate some norm of $\hat{\Omega}, \hat{\chi}$ using the explicit formulas. For $\xi$, it is given by an integration of $\hat{\eta}$ that does not have an explicit formula. We estimates $\xi$, its derivatives and some norm in subsection A.3.2.
Lemma A.5. The following results apply to any \( k \leq 3, 0 \leq i + j \leq 3, j \neq 1 \). (a) For \( f = \bar{\Omega}, \bar{\eta}, \bar{\Omega} - D_R \bar{\Omega}, \bar{\eta} - D_R \bar{\eta} \), we have

\[
|D^k_R f| \lesssim f, \quad |D'_R D''_R f| \lesssim \alpha \sin(\beta) f.
\]

(b) Let \( \phi_i \) be the weights defined in (5.13). For \( g = \bar{\Omega}, \bar{\eta} \), we have

\[
\int_0^{\pi/2} R^2 (D^k_R g)^2 \phi_1 d\beta \lesssim \alpha^2, \quad \int_0^{\pi/2} R^2 (D'_R D''_R g)^2 \phi_2 d\beta \lesssim \alpha^3.
\]

uniformly in \( R \) and

\[
\langle (D^k_R (g - D_R g))^2, \phi_1 \rangle \lesssim \alpha^2, \quad \langle (D'_R D''_R (g - D_R g))^2, \phi_2 \rangle \lesssim \alpha^3.
\]

Proof. Recall \( D_\beta = \sin(2\beta) \partial_\beta, D_R = R \bar{\partial}_R \). Using \( \Gamma(\beta) = \cos(\beta)^\sigma, (5.21) \) and a direct calculation gives

\[
|D^j_R \Gamma(\beta)| \lesssim \alpha \sin(\beta) \Gamma(\beta), \quad |D^j_R D'_R \Gamma(\beta)| \lesssim \frac{R}{(1 + R)^m}, \quad |D^j_R D''_R \Gamma(\beta)| \lesssim \frac{R^2}{(1 + R)^m}.
\]

for \( 1 \leq j \leq 5, 0 \leq i \leq 5 \) and \( m = 2, 3, 4 \). Combining these estimates and the formulas in (A.7) implies (A.8). As a result, we have the following pointwise estimates for \( g = \bar{\Omega} \) or \( \bar{\eta} \)

\[
|D^k_R g| \lesssim g \lesssim \alpha \Gamma(\beta) \frac{R}{(1 + R)^2}, \quad |D^j_R D'_R \Gamma(\beta)| \lesssim \alpha \sin(\beta) \Gamma(\beta) \frac{R}{(1 + R)^2}, \quad |D^j_R D''_R (g - D_R g)| \lesssim \alpha \sin(\beta) (g - D_R g) \lesssim \alpha^2 \sin(\beta) \frac{R^2 \Gamma(\beta)}{(1 + R)^3}.
\]

for \( k \leq 3, i + j \leq 3, j \neq 0 \), where we have used \( c \approx \frac{2}{\pi} \) in Lemma A.1. Recall \( \phi_i \) in Definition 5.2

\[
\phi_1 \triangleq (1 + R)^4 R^{-4} \sin(2\beta)^{-\sigma}, \quad \phi_2 \triangleq (1 + R)^4 R^{-4} \sin(2\beta)^{-\gamma}.
\]

Notice that for \( \sigma = \frac{99}{100}, \gamma = 1 + \frac{\alpha}{10} \), we have

\[
\int_0^{\pi/2} \Gamma(\beta)^2 \sin(2\beta)^{-\sigma} d\beta \lesssim 1, \quad \int_0^{\pi/2} \alpha^2 \sin(\beta)^2 \Gamma^2(\beta) \sin(2\beta)^{-\gamma} d\beta \lesssim \alpha^2 \int_0^{\pi/2} \cos(\beta)^{2 \gamma - 1 - \alpha/10} d\beta \lesssim \alpha.
\]

Combining the pointwise estimates, the estimates of the angular integral and a simple calculation then gives (A.9), (A.10).

Recall the \( \mathcal{W}^l_{\infty} \) norm in (7.8). We have

**Proposition A.6.** It holds true that \( \Gamma(\beta), \bar{\Omega}, \bar{\eta} \in \mathcal{W}^7_{\infty} \) with

\[
||\Gamma(\beta)||_{\mathcal{W}^7_{\infty}} \lesssim 1, \quad ||\frac{(1 + R)^2}{R} \bar{\Omega}||_{\mathcal{W}^7_{\infty}} + ||\frac{(1 + R)^2}{R} \bar{\eta}||_{\mathcal{W}^7_{\infty}} \lesssim \alpha,
\]

\[
||D_\beta \bar{\Omega}||_{\mathcal{W}^7_{\infty}} + ||D_\beta \bar{\eta}||_{\mathcal{W}^7_{\infty}} \lesssim \alpha^2.
\]

Proof. The proof follows directly from the calculation (A.11) and \( \sin(\beta) \Gamma(\beta) \sin(2\beta)^{-\alpha/5} \lesssim 1 \).
A.3.2. Estimates of $\tilde{\xi}$. Recall that the approximate self-similar profile $\tilde{\eta}$ is given by

$$\tilde{\eta}(x, y) = \frac{6R}{c (1 + R)^3} \cos^\alpha(\beta) = \frac{6\alpha}{c (1 + (x^2 + y^2)^{\alpha/2})^3}. \tag{A.12}$$

We also use $\tilde{\eta}(x, y)$ to denote the above expression. Throughout this section, we use the following notation

$$R = (x^2 + y^2)^{\alpha/2}, \quad \beta = \arctan(y/x), \quad S = (z^2 + y^2)^{\alpha/2}, \quad \tau = \arctan(y/z), \tag{A.13}$$

where $z$ will be used in the integral. $\tilde{\theta}(x, y), \tilde{\xi}(R, \theta) = \tilde{\eta}_x(x, y)$ can be obtained from $\tilde{\eta}(x, y)$ (or $\tilde{\eta}_x$) as follows

$$\tilde{\theta} = \int_0^x \tilde{\eta}(z, y) dz, \quad \tilde{\xi} = \tilde{\eta}_y = \int_0^x \tilde{\eta}_y(z, y) dz, \tag{A.14}$$

where we have used $\tilde{\theta}(0, y) = 0$. Observe that

$$\tilde{\eta}_y(z, y) = \frac{6\alpha}{c} \frac{3\alpha y}{y^2 + z^2} \frac{(z^2 + y^2)^{\alpha/2} z}{(1 + (z^2 + y^2)^{\alpha/2})^4}$$

$$= -\frac{1}{3} \frac{3\alpha y z}{y^2 + z^2} \frac{(z^2 + y^2)^{\alpha/2}}{1 + (z^2 + y^2)^{\alpha/2}} \tilde{\eta}(z, y) = -\frac{1}{3} \frac{13\alpha \sin(2\tau) S}{2 (1 + S)} \tilde{\eta}, \tag{A.15}$$

where we have used the notation $S, \tau$ defined in (A.13). Hence, we get

$$\tilde{\xi} = \int_0^x -\frac{1}{3} \frac{13\alpha \sin(2\tau) S}{2 (1 + S)} \tilde{\eta} dz = \int_0^x \frac{1}{z} \frac{3\alpha \sin(2\tau) S}{2 (1 + S)} \tilde{\eta} dz. \tag{A.16}$$

These integrals cannot be calculated explicitly for general $\alpha$. We have the following estimates for $\tilde{\xi}$.

**Lemma A.7.** Assume that $0 \leq \alpha \leq \frac{1}{1000}$. For $R \geq 0, \beta \in [0, \pi/2]$ and $0 \leq i + j \leq 5$, we have

$$|D_R^i D_\beta^j \tilde{\xi}| \lesssim -\tilde{\xi}, \quad |D_R^i D_\beta^j (3\tilde{\xi} - R\partial_R \tilde{\xi})| \lesssim -\tilde{\xi}, \tag{A.17}$$

$$|\tilde{\xi}| \lesssim \alpha^2 \frac{(x^2 + y^2)^{\alpha/2}}{(1 + (x^2 + y^2)^{\alpha/2})} \frac{y^\alpha}{(1 + y^\alpha)} \min \left(1, \frac{1 + \alpha}{y^3 + 1}\right) \lesssim \alpha^2 \left(1 + \frac{\alpha \sin^\alpha(\beta)}{1 + \beta} + \frac{\cos^{\alpha+1}(\beta)}{1 + \beta} \right), \tag{A.18}$$

$$-\tilde{\xi} \lesssim \alpha^2 \cos(\beta), \quad ||\tilde{\xi}||_{C^1} \lesssim \frac{1}{R} (1 + (R \sin(2\beta)^{\alpha})^{-2}) \tilde{\xi}_{L^\infty} \lesssim \alpha^2, \tag{A.19}$$

where $|| \cdot ||_{C^1}$ is defined in (6.24). Let $\psi_1, \psi_2$ be the weights defined in (5.13). We have

$$\int_0^{\pi/2} R^2 (D_R^i D_\beta^j \tilde{\xi})^2 \psi_k d\beta \lesssim \alpha^4 \tag{A.19}$$

uniformly in $R$, and

$$|(D_R^i D_\beta^j (3\tilde{\xi} - R\partial_R \tilde{\xi}))^2, \psi_k) \lesssim \alpha^4, \quad (D_R^i D_\beta^j \tilde{\xi}^2, \psi_k) \lesssim \langle \tilde{\xi}^2, \psi_k \rangle \lesssim \alpha^4, \tag{A.20}$$

where $(D_R^i D_\beta^j, \psi_k)$ represents $(D_R^i, \psi_1)$ for $0 \leq i \leq 5$, and $(D_R^i D_\beta^j, \psi_2)$ for $i + j \leq 5, j \geq 1$.

**Remark A.8.** Using (A.19), we have $-\tilde{\xi} \geq 0$ for $R \geq 0, \beta \in [0, \pi/2]$.

We have several commutator estimates which enable us to exchange the derivative and integration in (A.19) so that we can estimate $D_R^i D_\beta^j \tilde{\xi}$ easily.

Recall the relation between $\partial_x, \partial_y$ and $\partial_R, \partial_\beta$ in (2.28). We have the following relation

$$D_R = R\partial_R = \frac{1}{\alpha} (x\partial_x + y\partial_y), \quad D_\beta = \sin(2\beta)\partial_\beta = 2y\partial_y - 2\alpha \sin^2(\beta)D_R. \tag{A.21}$$

The first relation holds because $R = r^\alpha, \partial_R = \frac{1}{\alpha} r \partial_r$, and the second relation is obtained by multiplying $\partial_y = \frac{\sin(\beta)}{r} \alpha D_R + \frac{\cos(\beta)}{r} \partial_\beta$ by $y$ and then using $y/r = \sin(\beta), x/r = \cos(\beta)$. 
Lemma A.9. Suppose that \( f(0, y) = 0 \) for any \( y \). Denote
\[
I(f)(x, y) = \int_0^x \frac{1}{z} f(z, y) dz.
\]
We have
\[
D_R I(f)(x, y) = I(D_S f)(x, y),
\]
\[
D_\beta I(f)(x, y) - I(D_\tau f)(x, y) = -2\alpha \sin^2(\beta) \cdot I(D_S f) + 2\alpha I(\sin^2(\tau) D_S f),
\]
where \( R, \beta, S, \tau \) are defined in (A.13), provided that \( f \) is sufficiently smooth.

Proof. Notice that \( y \partial_y \) commutes with the \( z \) integral. From (A.21), it suffices to prove
\[
x \partial_x I(f)(x, y) = I(z \partial_x f).
\]
A directly calculation yields
\[
x \partial_x I(f)(x, y) = x \partial_x \left( \int_0^x \frac{1}{z} f(z, y) dz \right) = f(x, y), \quad I(z \partial_x f)(x, y) = \int_0^x \frac{1}{z} \cdot z \partial_z f(z, y) dz = f(x, y).
\]
It follows (A.23). Using the fact that both \( y \partial_y \) and \( R \partial_R \) commute with the \( z \) integral and the formula of \( D_\beta \) twice, we derive
\[
D_\beta I(f)(x, y) = (2y \partial_y - 2\alpha \sin^2(\beta) D_R) I(f) = I(2y \partial_y f) - 2\alpha \sin^2(\beta) I(D_S f)
\]
\[
= I(D_\tau f + 2\alpha \sin^2(\tau) D_S f) - 2\alpha \sin^2(\beta) I(D_S f) = I(D_\tau f) + 2\alpha I(\sin^2(\tau) D_S f) - 2\alpha \sin^2(\beta) I(D_S f).
\]
(A.24) follows by rearranging the above identity. \( \Box \)

Next, we prove Lemma A.7.

Proof of Lemma A.7. Step 1. Recall \( D_R = R \partial_R, D_\beta = \sin(2\beta) \partial_\beta \). First, we show that
\[
|D_R^i D_\beta^j \xi| \lesssim \alpha \int_0^x \frac{1}{z} \sin(2\tau) \frac{S}{1+S} \eta(z, y) dz \lesssim -\xi
\]
for \( 0 \leq i + j \leq 5 \). Using \( \Gamma(\beta) = \cos(\beta)^a \) (A.21) and a direct calculation yields
\[
|D_R^i D_\beta^j \xi| \lesssim \alpha \sin(\beta) \Gamma(\beta), \quad |D_\beta \sin(2\beta)| \lesssim \sin(2\beta)
\]
for \( i \leq 5 \). Denote
\[
f(S, \tau) = \frac{3\alpha}{2} \sin(2\tau) \frac{S}{1+S} \eta = \frac{9\alpha^2}{c} \sin(2\tau) \Gamma(\tau) \frac{S^2}{(1+S)^4}.
\]
We remark that \( f = -z \eta(z, y) \) according to (A.15). Obviously, \( f(S, \tau) \geq 0 \). Using the above estimates, we get
\[
|D_\beta^i D_\tau^j f| \lesssim f
\]
for \( i + j \leq 5 \). Notice that (A.16) implies \( \xi = -I(f) \) and that \( I(\cdot) \) (A.22) is a positive linear operator for \( x \geq 0 \). We further derive
\[
|I(D_S^i D_\tau^j f)| \leq I(|D_S^i D_\tau^j f|) \lesssim I(f)
\]
for \( i + j \leq 5 \). Using (A.23) and the above estimates, we yield
\[
|D_R^i \xi| = |D_R^i I(f)| = |I(D_R^i f)| \lesssim I(f).
\]
For other derivatives \( D_R^i D_\beta^j \) with \( j \geq 1, i + j \leq 5 \), we estimate \( D_\beta^j \xi \), which is representative. Using (A.24), we have
\[
D_\beta^j \xi = D_\beta^j I(f) = D_\beta \left( I(D_\tau f) - 2\alpha \sin^2(\beta) \cdot I(D_S f) + 2\alpha I(\sin^2(\tau) D_S f) \right)
\]
\[
= I(D_\tau^j f) - 2\alpha \sin^2(\beta) \cdot I(D_S D_\tau f) + 2\alpha I(\sin^2(\tau) D_S D_\tau f)
\]
\[
+ D_\beta \left( -2\alpha \sin^2(\beta) \cdot I(D_S f) \right) + D_\beta \left( 2\alpha I(\sin^2(\tau) D_S f) \right) = J_1 + J_2 + J_3 + J_4 + J_5.
\]
For \( J_1, J_2, J_3 \), we simply use \( \sin^2(\beta), \sin^2(\tau) \leq 1 \) and (A.29) to obtain
\[
I_1, J_2, J_3 \lesssim I(|D_R^i D_\tau^j f|) \lesssim I(f)
\]
(A.30)
for \((i, j) = (0, 2), (1, 1), (1, 1)\) respectively. For \(J_1\), if \(D_\beta\) acts on \(\sin^2(\beta)\), we obtain \(\alpha D_\beta(\sin^2(\beta)) \cdot I(D_S f)\), which can be bounded as before using (A.29). For the remaining parts in \(J_1\) and \(J_5\), \(D_\beta\) acts on \(I(\cdot)\) and we can use (A.24) again to obtain several terms. Each term can be bounded using (A.29) and an argument similar to (A.30). The estimates of other derivatives \(D_R^i D_\beta^j\) can be done similarly. We omit these estimates. Since the right hand side of (A.25) is
\[
\frac{2}{\xi} I(f) = -\frac{2}{\xi} \zeta = -\zeta,
\]
the above estimates imply (A.25).

Step 2. The estimate (A.25) can be generalized to \(i + j \leq 6\) easily. Hence, we get
\[
|D_R^i D_\beta^j (3\zeta - R\partial_R \zeta)| \lesssim |D_R^i D_\beta^j \zeta| + |D_R^{i+1} D_\beta^j \zeta| \lesssim -\zeta,
\]
for any \(i + j \leq 5\), which proves (A.17).

Step 3: Pointwise estimate. In this step, we prove (A.18). From (A.16), we know that the first inequality in (A.18) is equivalent to
\[
\int_0^x \frac{y}{y^2 + z^2} \frac{z^\alpha (y^2 + z^2)^{\alpha/2}}{(1 + (y^2 + z^2)^{\alpha/2})^3} \, dz \lesssim \frac{y^\alpha}{(1 + y^\alpha)^3} \min \left(1, \frac{x^{1+\alpha}}{y^{1+\alpha}}\right).
\]
For \(z \in [0, x]\), we have \(z^2 + y^2 \leq x^2 + y^2\). Since \(\frac{y}{x}\) is increasing with respect to \(t \geq 0\), we yield
\[
\frac{y^2 + z^2}{1 + (y^2 + z^2)^{\alpha/2}} \lesssim \frac{y^2}{1 + (y^2 + z^2)^{\alpha/2}}.
\]
Therefore, it suffices to prove
\[
(A.31) \quad J(x, y) \triangleq \int_0^x \frac{y^2 + z^2}{(1 + (y^2 + z^2)^{\alpha/2})^3} \, dz \lesssim \frac{y^\alpha}{(1 + y^\alpha)^3} \min \left(1, \frac{x^{1+\alpha}}{y^{1+\alpha}}\right).
\]
Case 1: \(x \leq 1 + y\). Observe that
\[
J \leq \frac{1}{(1 + y^\alpha)^3} \int_0^x \frac{y^2 + z^2}{(1 + (y^2 + z^2)^{\alpha/2})^3} \, dz \lesssim \frac{y^\alpha}{(1 + y^\alpha)^3} \int_0^x \frac{t^\alpha}{1 + t^2} \, dt,
\]
where we have used change of a variable \(z = yt\) to derive the identity. Since \(\alpha \leq 1/10\), we get
\[
\int_0^x \frac{t^\alpha}{1 + t^2} \, dt \lesssim \int_0^\infty \frac{t^\alpha}{1 + t^2} \, dt \lesssim 1, \quad \int_0^x \frac{t^\alpha}{1 + t^2} \, dt \lesssim \int_0^x \frac{t^\alpha}{1 + t^2} \, dt \lesssim \frac{x^{1+\alpha}}{y^{1+\alpha}}.
\]
Combining the above estimates, we prove (A.31) for \(x \leq 1 + y\).

Case 2: \(x > 1 + y\). Firstly, we have
\[
J(x, y) = \int_0^{1+y} \frac{y^2 + z^2}{(1 + (y^2 + z^2)^{\alpha/2})^3} \, dz + \int_{1+y}^x \frac{y^2 + z^2}{(1 + (y^2 + z^2)^{\alpha/2})^3} \, dz \triangleq J_1 + J_2.
\]
We apply the result in Case 1 to estimate \(J_1\)
\[
J(1+y, y) \lesssim \frac{y^\alpha}{(1 + y^\alpha)^3} \min \left(1, \frac{(1 + y)^{1+\alpha}}{y^{1+\alpha}}\right) \lesssim \frac{y^\alpha}{(1 + y^\alpha)^3}.
\]
For \(J_2\), we have
\[
J_2 \lesssim \int_{1+y}^x \frac{y^2 + z^2}{(1 + (y^2 + z^2)^{\alpha/2})^3} \, dz = y^{-2\alpha} \int_{1+y}^x \frac{t^{-2\alpha}}{1 + t^2} \, dt \lesssim y^{-2\alpha} \int_{1+y}^\infty \frac{t^{-2\alpha - 2}}{1 + t^2} \, dt
\]
\[
\lesssim y^{-2\alpha} \left(\frac{1 + y}{y}\right)^{-1 - 2\alpha} = y^\alpha \left(\frac{1 + y}{y}\right)^{1+2\alpha} \lesssim \frac{y^\alpha}{(1 + y^\alpha)^3},
\]
where we have used change of a variable \(z = yt\) to derive the first identity. Noting that \(x \geq y\) in this case. We conclude
\[
J(x, y) = J_1 + J_2 \lesssim \frac{y^\alpha}{(1 + y^\alpha)^3} \lesssim \frac{y^\alpha}{(1 + y^\alpha)^3} \min \left(1, \frac{x^{1+\alpha}}{y^{1+\alpha}}\right).
\]
Combining the above two cases, we prove (A.31), which implies the first inequality in (A.18).
Finally, we prove the second inequality in (A.18). Using the notation (A.13), we have
\[ R = (x^2 + y^2)^{\alpha/2}, \quad \frac{(x^2 + y^2)^{\alpha/2}}{1 + (x^2 + y^2)^{\alpha/2}} = \frac{R}{1 + R}, \quad y^\alpha = R \sin^\alpha(\beta), \quad \frac{y^\alpha}{(1 + y^\alpha)^3} = \frac{R \sin^\alpha(\beta)}{(1 + R \sin^\alpha(\beta))^3}. \]

For \( x \leq y \), we have \( \beta \geq \pi/4 \), \( 1 \leq \sin(\beta) \), \( x^2 + y^2 \leq y^2 \). Hence,
\[ \frac{y^\alpha}{(1 + y^\alpha)^3} y^{1+\alpha} \leq \frac{R \sin^\alpha(\beta)}{(1 + R)^3} (1 + (x^2 + y^2)^{\alpha/2})^3 y^{1+\alpha} = \frac{R \sin^\alpha(\beta)}{(1 + R)^3} \cos^{1+\alpha}(\beta) \sin^{1+\alpha}(\beta) \leq \frac{R \sin^{1+\alpha}(\beta)}{(1 + R)^3}. \]

Combining the above identity and the estimate, we prove the second inequality in (A.18). The last inequality in (A.18) follows directly from (A.17) and the first two inequalities in (A.18).

Step 4: Estimates of the integral Now, we are in a position to prove (A.19) and (A.20). We are going to prove
\[ \int_0^{\pi/2} \xi^2(R, \beta) \psi_k d\beta \lesssim \frac{\alpha^4}{(1 + R)^2}. \]
Clearly, (A.19) and (A.20) follow from the above estimate and (A.17).

Notice that \( \psi_i \) defined in (A.13) satisfies
\[ \psi_1, \psi_2 \lesssim \frac{(1 + R)^4}{R^4} \sin(\beta)^{-\alpha} \cos(\beta)^{-\gamma}, \]
where \( \gamma = 1 + \frac{\alpha}{10}, \sigma = \frac{\alpha}{100}. \) Using (A.18), \( 1 + R \sin^\alpha(\beta) \geq (1 + R) \sin^\alpha(\beta) \), we yield
\[ (1 + R)^2 \int_0^{\pi/2} |\xi^2| \psi_k d\beta \lesssim (1 + R)^2 \frac{\alpha^4 R^4}{(1 + R)^2} \left\{ \int_0^{\pi/4} \sin^{2\alpha}(\beta) d\beta + \int_{\pi/4}^{\pi/2} \cos^{2\alpha + 2}(\beta) d\beta \right\} \]
\[ \lesssim \frac{\alpha^4 R^4}{(1 + R)^2} \left\{ \int_0^{\pi/4} \sin(\beta)^{-4\alpha} \sin(\beta)^{-\sigma} \cos(\beta)^{-\gamma} d\beta + \int_{\pi/4}^{\pi/2} \cos(\beta)^{2 + 2\alpha} \sin(\beta)^{-\sigma} \cos(\beta)^{-\gamma} d\beta \right\} \]
\[ \lesssim \alpha^4 \left( \int_0^{\pi/4} \sin(\beta)^{-4\alpha} d\beta + \int_{\pi/4}^{\pi/2} \cos(\beta)^{2 + 2\alpha - \gamma} d\beta \right) \lesssim \alpha^4, \]
where we have used \( \alpha \leq \frac{1}{1000} \), \( 4\alpha + \sigma < \frac{199}{200}, 2 + 2\alpha - \gamma \geq 1 \), to derive the last inequality which does not depend on \( \alpha \) for \( \alpha \leq \frac{1}{1000} \). It follows (A.32).

A.4. Other Lemmas. We use the following Lemma to construct small perturbation.

Lemma A.10. Let \( \chi(\cdot) : [0, \infty) \to [0, 1] \) be a smooth cutoff function, such that \( \chi(R) = 1 \) for \( R \leq 1 \) and \( \chi(R) = 0 \) for \( R \geq 2 \). Denote
\[ (A.34) \quad \chi_\lambda(R) = \chi(R/\lambda), \quad \Omega_\lambda = \chi_\lambda \Omega, \quad \eta_\lambda = \partial_x(\chi_\lambda \theta), \quad \xi_\lambda = \partial_y(\chi_\lambda \theta), \]
where \( \theta \) is obtained in (A.13). We have
\[ \lim_{\lambda \to +\infty} \left[ \Omega_\lambda - \Omega \right|_{\partial \Omega} + \left[ (1 + R)(\eta_\lambda - \eta) \right|_{\partial \Omega} + \left[ \xi_\lambda - \xi \right|_{\partial \mathcal{P}(\psi)} = 0, \quad \lim_{\lambda \to +\infty} \left[ \xi_\lambda - \xi \right|_{\partial \mathcal{C}_1} \leq K_{10} \alpha^2, \]
where \( K_{10} > 0 \) is some absolute constant. In particular, we also have
\[ \lim_{\lambda \to +\infty} L^2_{\lambda}(\Omega_\lambda - \Omega)(0) + \langle (\Omega_\lambda - \Omega)^2, \varphi_0 \rangle + \langle (\eta_\lambda - \eta)^2, \psi_0 \rangle = 0. \]

We need a Lemma similar to Lemma A.9.

Lemma A.11. Suppose that \( f(0, y) = 0 \) for any \( y \). Denote \( J(f)(x, y) = \frac{1}{2} \int_0^y f(z, y) dz \). We have
\[ D_R I(f)(x, y) = I(D_S f)(x, y), \]
\[ D_\beta I(f)(x, y) - I(D_\tau f)(x, y) = -2\alpha \sin^2(\beta) \cdot I(D_S f) + 2\alpha I(\sin^2(\tau) D_S f), \]
where \( R, \beta, S, \tau \) are defined in (A.13), provided that \( f \) is sufficiently smooth.

The first identity follows from a direct calculation and the proof of the second is similar to that in Lemma A.9. We omit the proof.
Proof of Lemma A.10 Step 1: Estimate of $\bar{\theta}$. Using (A.14) and the operator $J$ in Lemma A.11 we get $\frac{\partial}{\partial x} J(\bar{\eta})$. We have the following estimate for $\bar{\theta}$

(A.37) $|D^i_R D^j_R \bar{\theta}| = |D^i_R D^j_R J(\bar{\eta})| \lesssim J(\bar{\eta}) = \frac{1}{x} \int_0^x \bar{\eta}(z, y) dz \lesssim \bar{\eta}$,

for $0 \leq i + j \leq 5$. The proof of the first inequality follows from Lemma A.11 and the argument in the proof of (A.21). The proof of the second inequality is similar to that of (A.31) by considering $x \leq 1 + y$ and $x > 1 + y$. We omit the proof.

Step 2: Estimate of $\bar{\eta}_λ - \bar{\eta}$, $\xi_λ - \xi$. Recall $\bar{\eta}_λ = \partial_x (\chi \bar{\theta})$, $\xi_λ = \partial_y (\chi \bar{\theta})$ and the formula of $\partial_x, \partial_y$ (2.8). A direct calculation yields

(A.38) $\bar{\eta}_λ(R, \beta) - \bar{\eta} = \alpha \frac{\cos(\beta)}{r} D_R \chi \lambda \cdot \bar{\theta} + (\chi \lambda - 1) \bar{\eta} = \alpha \cos^2(\beta) D_R \chi \lambda \cdot J(\bar{\eta}) + (\chi \lambda - 1) \bar{\eta}$,

$\xi_λ(R, \beta) - \xi = \alpha \frac{\sin(\beta)}{r} D_R \chi \lambda \cdot \bar{\theta} + (\chi \lambda - 1) \bar{\xi} = \alpha \sin(\beta) \cos(\beta) D_R \chi \lambda \cdot J(\bar{\eta}) + (\chi \lambda - 1) \bar{\xi}$,

where we have used $\partial_x \bar{\theta} = \bar{\eta}$, $\partial_y \bar{\theta} = \bar{\xi}$, $(r \cos(\beta))^{-1} \bar{\theta} = \frac{2}{\xi} \bar{\theta} = J(\bar{\eta})$. From (A.34), we have

$$D \chi \lambda = 0, \quad |D \chi \lambda| = (R/\lambda)|\chi'(R/\lambda)| \lesssim 1.$$  

Similarly, we have

(A.39) $|D_R^k \chi \lambda| \lesssim 1,$

for $k = 1, 2, 3, 4$. Notice that $\partial_R \chi \lambda$, $(\chi \lambda - 1) = 0$ for $R \leq \lambda$. From the formula of $\bar{\eta}$ and (A.20) in Lemma A.11, we know $(\chi_λ - 1)(1 + R) \bar{\eta} \in \mathcal{H}^3$ ($\bar{\eta}$ decays $R^{-2}$ for large $R$) and $\bar{\xi} \in \mathcal{H}^3(\psi)$. Using the estimates of $J(\bar{\eta})$ in (A.37), we also have $(\chi_λ - 1)J(\bar{\eta}) \in \mathcal{H}^3 \subset \mathcal{H}^3(\psi)$. Therefore, applying (A.38), (A.39) to $\chi \lambda$ and the Dominated Convergence Theorem yields

$$\lim_{\lambda \to \infty} (1 + R)|\bar{\eta}_λ - \bar{\eta}||_\mathcal{H}^3 = 0, \quad \lim_{\lambda \to \infty} ||\bar{\xi}_λ - \bar{\xi}||_\mathcal{H}^3(\psi) = 0.$$

Similarly, we have

$$\lim_{\lambda \to \infty} ||\bar{\Omega}_λ - \bar{\Omega}||_{\mathcal{H}^3} = 0.$$

Using (A.37), (A.39) and the fact that $\bar{\eta}$ decays for large $R$ (see (4.8), we have

$\lim_{\lambda \to \infty} ||\sin(\beta) \cos(\beta) D_R \chi \lambda \cdot J(\bar{\eta})||_{\mathcal{C}^1} = 0.$

Using (A.17), (A.18) in Lemma A.11 and (A.39), we conclude

$$||(\chi_λ - 1)\bar{\xi}||_{\mathcal{C}^2} \lesssim \alpha^2.$$

We complete the proof of (A.36).

Recall that the $\mathcal{H}^3$ norm is stronger than $L^2(\varphi_1)$. Using Lemma A.3 for $L_{12}(\Omega)(0)$, the fact that $\varphi_0 \lesssim \varphi_1, \psi_0 \lesssim (1 + R)\varphi_1$ (see Definition 5.2, 5.7, and the limit obtained in (A.36), we prove Lemma A.4).

Let $C^\#$ be the standard Hölder space. Recall the $C^1$ norm defined in (6.24). We have the following embedding.

**Lemma A.12.** Suppose that $f \in C^1(R, \beta)$ and $f(R, \pi/2) = 0$ for $R \geq 0$. We have

$$||f||_{C^\#} \leq C_\alpha ||f||_{C^1}$$  

for some constant $C_\alpha$ depending on $\alpha$ only.

**Proof.** Recall the relation between the Cartesian coordinate $(x, y)$ and the polar coordinate $(r, \beta, (R, \beta))$. Since $f$ vanishes on the axis $\beta = \frac{\pi}{2}$. It suffices to prove that $f$ is Hölder in $\mathbb{R}^2_{x+}$. Let $(R_1, \beta_1), (R_2, \beta_2)$ be arbitrary two different points in $\mathbb{R}^2_{x+}$, i.e., $R_1, R_2 \geq 0, \beta_1, \beta_2 \in [0, \pi/2]$, and $r_1 = R_1^{1/\alpha}, r_2 = R_2^{1/\alpha}$. Without loss of generality, we assume $R_1 \leq R_2, \beta_1 \leq \beta_2$ and $||f||_{C^1} = 1$. From (6.24), we have $|f| \leq 1, |\partial_R f| \leq \frac{1}{1 + R}$, $|\partial_\beta f| \leq R^{1/40} \sin(2\beta)^{\alpha/40-1}$. Using $\sin(2\beta)^{\alpha/40-1} \lesssim (\sin(\beta)^{\alpha/40-1} + \cos(\beta)^{\alpha/40-1}) \lesssim (\beta^{\alpha/40-1} + (\pi/2 - \beta)^{\alpha/40-1})$. 

and the estimates of the derivatives, we obtain

$$|f(R_1, \beta_1) - f(R_1, \beta_2)| \leq \int_{\beta_1}^{\beta_2} |\partial_\beta f(R_1, \beta)| d\beta \leq CR_1^{\frac{1}{40}} \int_{\beta_1}^{\beta_2} \left( \beta^{\frac{1}{2}} + (\frac{\pi}{2} - \beta)^{\frac{1}{2}} \right) d\beta$$

$$\leq C_\alpha R_1^{\frac{1}{40}} \left( \beta^{\frac{1}{2}} - \beta_1^{\frac{1}{2}} + (\frac{\pi}{2} - \beta_1)^{\frac{1}{2}} - (\frac{\pi}{2} - \beta_2)^{\frac{1}{2}} \right) \leq C_\alpha R_1^{\frac{1}{40}} |\beta_2 - \beta_1|^{\frac{1}{40}},$$

where we have used $C_\alpha R_1^{\frac{1}{40}} \leq (1 + R_2 - R_1)$ and $\log(1 + x) \leq x^{1/40}$ for $x \geq 0$ in the last inequality. The distance $d$ between two points is

$$d^2 = (r_1 \cos(\beta_1) - r_2 \cos(\beta_2))^2 + (r_1 \sin(\beta_1) - r_2 \sin(\beta_2))^2 = (r_1 - r_2)^2 + 2r_1r_2(1 - \cos(\beta_1 - \beta_2))$$

$$= |R_1^{1/\alpha} - R_2^{1/\alpha}|^2 + 4R_1^{1/\alpha} R_2^{1/\alpha} \sin\left(\frac{1}{2}(\beta_1 - \beta_2)\right)^2 \geq C_\alpha (|R_1 - R_2|^{2/\alpha} + R_1^{2/\alpha} |\beta_1 - \beta_2|^2),$$

where we have used $R_1 \leq R_2$ in the last inequality. Using the triangle inequality and the above estimates, we conclude $|f(R_1, \beta_1) - f(R_2, \beta_2)| \leq C_\alpha d^{\frac{1}{40}}.$

□

**Proof of Lemma 6.4.** To simplify the notation, we simplify $\omega^\theta$ as $\omega$. Denote $\omega_\pm = \max(\pm \omega, 0)$

$$\mathcal{L} = -\partial_r r + \frac{1}{r} \partial_r + \partial_{zz} + \frac{1}{r^2} \partial_{\beta^2}, \quad \Delta = \partial_r r + \frac{1}{r} \partial_r + \partial_{zz} + \frac{1}{r^2} \partial_{\beta^2},$$

$$\psi_\pm(r, z) = \frac{1}{4\pi} \int_0^\infty \int_{-\infty}^{\infty} \int_0^{2\pi} \cos(\beta) \omega_\pm(r_1, z_1) (z - z_1) \partial_1 dr_1 dz_1 d\beta.$$

Clearly, $\psi_\pm$ solve the Laplace equation on $\mathbb{R}^3$: $-\Delta(\sin(\beta) \psi_\pm(r, z)) = \omega_\pm(r, z) \sin(\beta)$, which can be verified easily by the Green function of $-\Delta$.

Let $\psi$ be a solution of (3.2)-(3.3). It is easy to verify that

$$-\Delta(\psi \sin(\beta)) = \sin(\beta) \mathcal{L} \psi = \omega \sin(\beta).$$

Consider the domain $D^+ = \{(r, z, \beta) : r \in [0, 1], z \in \mathbb{R}, \beta \in [0, \pi]\}$, which is a half of the cylinder.

For $(r, z, \beta) \in D^+$, we have $\sin(\beta) \geq 0$ and

$$-\Delta(\psi \pm \psi^+ \sin(\beta)) = (\omega - \omega^+ \sin(\beta)) \leq 0.$$

Recall that $\tilde{\psi}$ satisfies the zero boundary condition (3.3): $\tilde{\psi}(1, z) = 0$ and that $\psi^+$ is nonnegative. Hence, on the boundary of $D^+$, i.e. $\beta = 0, \pi$ or $r = 1$, we have

$$(\psi - \psi^+) \sin(\beta) = 0 \quad \text{for} \quad \beta = 0, \pi, \quad (\psi - \psi^+) \sin(\beta) \leq 0 \quad \text{for} \quad r = 1,$$

where we have used $\sin(\beta) \geq 0$ in $D^+$. Moreover, since $\omega$ has compact support and $\sup(\omega) \subset \{(r, z) : (r - 1)^2 + z^2 < 1/4\}$, it is not difficult to verify that $\tilde{\psi}$ and $\psi$ decay in $z$ direction. Applying the maximal principle to (4.40), we yield $(\tilde{\psi}(r, z) - \psi^+(r, z)) \sin(\beta) \leq 0$ in $D^+$, which further implies $\tilde{\psi}(r, z) \leq \psi^+(r, z)$ for $r \leq 1, z \in \mathbb{R}$. Similarly, we have $\tilde{\psi} + \psi^- \geq 0$. Hence $|\tilde{\psi}| \leq \psi_+ + \psi_-$.

Finally, within the support of $\omega$, i.e. $|r_1 - 1|^2 + z_1^2 < 1/4$, and $\beta \in [-\pi, \pi]$, we have

$$(z - z_1)^2 + r_1^2 - 2 \cos(\beta) r_1 r_1 = (z - z_1)^2 + (r - r_1)^2 + 4 \sin(\beta) / 2 r_1 r_1 = (((z - z_1)^2 + (r - r_1)^2)^{1/2} + |\beta|^2)^2.$$

Integrating the $\beta$ variable in the integral about $\psi_\pm$ completes the proof. □

**REFERENCES**


References


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