# STABLE NEARLY SELF-SIMILAR BLOWUP OF THE 2D BOUSSINESQ AND 3D EULER EQUATIONS WITH SMOOTH DATA II: RIGOROUS NUMERICS 

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#### Abstract

This is Part II of our paper in which we prove finite time blowup of the 2D Boussinesq and 3D axisymmetric Euler equations with smooth initial data of finite energy and boundary. In Part I of our paper 13, we establish an analytic framework to prove stability of an approximate self-similar blowup profile by a combination of a weighted $L^{\infty}$ norm and a weighted $C^{1 / 2}$ norm. Under the assumption that the stability constants, which depend on the approximate steady state, satisfy certain inequalities stated in our stability lemma, we prove stable nearly self-similar blowup of the 2D Boussinesq and 3D Euler equations with smooth initial data and boundary. In Part II of our paper, we provide sharp stability estimates of the linearized operator by constructing space-time solutions with rigorous error control. We also obtain sharp estimates of the velocity in the regular case using computer assistance. These results enable us to verify that the stability constants obtained in Part I 13 indeed satisfy the inequalities in our stability lemma. This completes the analysis of the finite time singularity of the axisymmetric Euler equations with smooth initial data and boundary.


## 1. Introduction

The three dimensional incompressible Euler equations are one of the most fundamental nonlinear partial differential equations that govern the motion of the ideal inviscid fluid flow. It is closely related to the incompressible Navier-Stokes equations. Due to the presence of nonlinear vortex stretching, the global regularity of the 3D incompressible Euler equations with smooth initial data and finite energy has been one of the longstanding open questions in nonlinear partial differential equations. Let $\mathbf{u}$ be the divergence free velocity field and we define $\boldsymbol{\omega}=\nabla \times \mathbf{u}$ as the vorticity vector. The 3D Euler equations governing the vorticity $\boldsymbol{\omega}$ are given by

$$
\begin{equation*}
\boldsymbol{\omega}_{t}+\mathbf{u} \cdot \nabla \boldsymbol{\omega}=\boldsymbol{\omega} \cdot \nabla \mathbf{u} \tag{1.1}
\end{equation*}
$$

where $\mathbf{u}$ is related to $\boldsymbol{\omega}$ via the Biot-Savart law. The velocity gradient $\nabla \mathbf{u}$ formally has the same scaling as vorticity $\boldsymbol{\omega}$. Thus the vortex stretching term, $\boldsymbol{\omega} \cdot \nabla \mathbf{u}$, has a nonlocal quadratic nonlinearity in terms of vorticity. Although many experts tend to believe that the 3D Euler equations would form a finite time singularity from smooth initial data, the nonlocal nature of the vortex stretching term could lead to dynamic depletion of nonlinearity, thus preventing a finite time blowup, see e.g. 19, 23, 36. The interested readers may consult the excellent surveys $[18,30,34,38,43]$ and the references therein.

Our work is inspired by the computation of Luo-Hou 41,42 in which they presented some convincing numerical evidence that the 3D axisymmetric Euler equations with smooth initial data and boundary develop a potential finite time singularity. In Part I of our paper [13], we establish an analytic framework and obtan some essential stability estimates to prove finite time singularity of the 2D Boussinesq and 3D axisymmetric Euler equations with smooth initial data and boundary. The main results of this paper are stated by the two informal theorems below. The more precise and stronger statement of Theorem 1 can be found in Theorem 3 in Section 2.

Theorem 1. Let $\theta$, $\mathbf{u}$ and $\omega$ be the density, velocity and vorticity in the 2D Boussinesq equations (2.3) -(2.5), respectively. There is a family of smooth initial data $\left(\theta_{0}, \omega_{0}\right)$ with $\theta_{0}$ being even and $\omega_{0}$ being odd, such that the solution of the Boussinesq equations develops a singularity in finite time $T<+\infty$. The velocity field $\mathbf{u}_{0}$ has finite energy. The blowup solution $(\theta(t), \omega(t))$ is nearly self-similar in the sense that $(\theta(t), \omega(t))$ with suitable dynamic rescaling is close to an
approximate blowup profile $(\bar{\theta}, \bar{\omega})$ up to the blowup time. Moreover, the blowup is stable for initial data $\left(\theta_{0}, \omega_{0}\right)$ close to $(\bar{\theta}, \bar{\omega})$ in some weighted $L^{\infty}$ and $C^{1 / 2}$ norm.

Theorem 2. Consider the $3 D$ axisymmetric Euler equations in the cylinder $r, z \in[0,1] \times \mathbb{T}$. Let $u^{\theta}$ and $\omega^{\theta}$ be the angular velocity and angular vorticity, respectively. The solution of the 3D Euler equations (2.1)-(2.2) develops a nearly self-similar blowup (in the sense described in Theorem 11) in finite time for some smooth initial data $\omega_{0}^{\theta}$, $u_{0}^{\theta}$ supported away from the symmetry axis $r=0$. The initial velocity field has finite energy, $u_{0}^{\theta}$ and $\omega_{0}^{\theta}$ are odd and periodic in $z$. The blowup is stable for initial data $\left(u_{0}^{\theta}, \omega_{0}^{\theta}\right)$ that are close to the approximate blowup profile $\left(\bar{u}^{\theta}, \bar{\omega}^{\theta}\right)$ after proper rescaling subject to some constraint on the initial support size.

We first review some main ideas in our stability analysis of the linearized operator presented in Part I [13]. We use the 2D Boussinesq system as an example. Let $\bar{\omega}, \bar{\theta}$ be an approximate steady state of the dynamic rescaling formulation. We denote $W=\left(\omega, \theta_{x}, \theta_{y}\right)$ and decompose $W=\bar{W}+\widetilde{W}$ with $\bar{W}=\left(\bar{\omega}, \bar{\theta}_{x}, \bar{\theta}_{y}\right)$. We further denote by $\mathcal{L}$ the linearized operator around $\bar{W}$ that governs the perturbation $\widetilde{W}$ in the dynamic rescaling formulation (see Section (2):

$$
\begin{equation*}
\widetilde{W}_{t}=\mathcal{L}(\widetilde{W}) \tag{1.2}
\end{equation*}
$$

We decompose the linearized operator $\mathcal{L}$ into a leading order operator $\mathcal{L}_{0}$ plus a finite rank perturbation operator $\mathcal{K}$, i.e $\mathcal{L}=\mathcal{L}_{0}+\mathcal{K}$. The leading order operator $\mathcal{L}_{0}$ is constructed in such way that we can obtain sharp stability estimates using weighted estimates and sharp functional inequalities.

In Part I [13], we have performed the weighted energy estimates using a combination of weighted $L^{\infty}$ and $C^{1 / 2}$ norm. In our analysis, we decompose $\widetilde{W}=\widetilde{W}_{1}+\widetilde{W}_{2}$, where $\widetilde{W}_{1}$ is the main part of the perturbation, which is essentially governed by the leading order operator $\mathcal{L}_{0}$ with a weak coupling to $\widetilde{W}_{2}$ through nonlinear interaction. The perturbation $\widetilde{W}_{2}$ captures the contribution from the finite rank operator. The key is to show that the energy estimate of the main part $\widetilde{W}_{1}$ satisfies the inequalities stated in our stability Lemma 2.1 (see Section 2). For this purpose, we need to obtain relatively sharp energy estimates for the leading order operator $\mathcal{L}_{0}$ by subtracting a finite rank operator $\mathcal{K}$. Without subtracting the finite rank operator, we would not be able to obtain linear and nonlinear stability of the approximate self-similar profile.

The constants in the weighted energy estimates obtained in Part I [13] depend on the approximate self-similar profile that we constructed numerically in Section 7 of Part I [13] and the singular weights we use. In this paper and in the supplementary material 11 (contained in this paper), we will provide sharp and rigorous upper bounds for these constants by estimating the higher order derivatives and then using interpolation estimates from numerical analysis. We also obtain sharp estimates of the velocity in the regular case by bounding various integrals using numerical integration with computer assistance. These sharp estimates of the constants enable us to prove that the inequalities in our stability lemma hold for our approximate self-similar profile. Thus we can complete the stability analysis of the approximate self-similar profile and complete our blowup analysis for the 2D Boussinesq and 3D Euler equations. See Section 2.2 for more discussion of the main steps in our blowup analysis.

We use the following toy model to illustrate the main ideas of our stability analysis by considering $\mathcal{K}$ as a rank-one operator $\mathcal{K}(\widetilde{W})=a(x) P(\widetilde{W})$ for some operator $P$ satisfying (i) $P(\widetilde{W})$ is constant in space; (ii) $\|P(\widetilde{W})\| \leq c\| \| \widetilde{W} \|$. Given initial data $\widetilde{W}_{0}$, we decompose (1.2) as follows

$$
\begin{align*}
& \partial_{t} \widetilde{W}_{1}(t)=\mathcal{L}_{0} \widetilde{W}_{1}, \quad \widetilde{W}_{1}(0)=\widetilde{W}_{0} \\
& \partial_{t} \widetilde{W}_{2}(t)=\mathcal{L} \widetilde{W}_{2}+a(x) P\left(\widetilde{W}_{1}(t)\right), \quad \widetilde{W}_{2}(0)=0 \tag{1.3}
\end{align*}
$$

It is easy to see that $\widetilde{W}=\widetilde{W}_{1}+\widetilde{W}_{2}$ solves (1.2) with initial data $\widetilde{W}_{0}$ since $\mathcal{L}=\mathcal{L}_{0}+a(x) P$. By construction, the leading operator $\mathcal{L}_{0}$ has the desired structure that enables us to obtain sharp stability estimates. The second part $\widetilde{W}_{2}$ is driven by the rank-one forcing term $a(x) P\left(\widetilde{W}_{1}(t)\right)$.

Using Duhamel's principle, the fact that $P\left(\widetilde{W}_{1}(t)\right)$ is constant in space, we yield

$$
\begin{equation*}
\widetilde{W}_{2}(t)=\int_{0}^{t} P\left(\widetilde{W}_{1}(s)\right) e^{\mathcal{L}(t-s)} a(x) d s \tag{1.4}
\end{equation*}
$$

If $\widetilde{W}_{1}$ is linearly stable in some $L^{\infty}(\varphi)$ space, by checking the decay of $e^{\mathcal{L}(t)} a(x)$ in the energy space for large $t$, we can obtain the stability estimate of $\widetilde{W}_{2}$. Note that $e^{\mathcal{L}(t)} a(x)$ is equivalent to solving the linear evolution equation $v_{t}=\mathcal{L}(v)$ with initial data $v_{0}=a(x)$. We can solve this initial value problem by constructing a space-time solution with rigorous error control.

We remark that our stability analysis is performed mainly for $\widetilde{W}_{1}$ since $\widetilde{W}_{2}$ is driven by $\widetilde{W}_{1}$. The approximation errors in constructing the space-time approximation to $\widetilde{W}_{2}$ can be controlled by the decay estimate of $\widetilde{W}_{1}$. Moreover, the region where we need to modify the linearized operator by a finite rank operator is mainly located in a small sector near the boundary where we have the smallest amount of damping. The total rank is less than 50. In our construction of approximate solution to $\widetilde{W}_{2}$, we need to solve the linear PDE (1.2) in space-time with a number of initial data, which can be implemented in full parallel.

There has been a lot of effort in studying 3D Euler singularities. The most exciting recent development is Elgindi's breakthrough result in which he proved finite time singularity of the axisymmetric Euler equation with no swirl for $C^{\alpha}$ initial vorticity [24] (see also [25]). Earlier efforts include the Constantin-Lax-Majda (CLM) model [20], the De Gregorio (DG) model [21,22], the generalized CML (gCLM) model [49] and the Hou-Li model [35]. See also [5] 8, 14, 20, 26, 28] for the De Gregorio model and for the gCLM model with various parameters. Inspired by their work on the vortex sheet singularity [4, Caflisch and Siegel have studied complexity singularity for 3D Euler equation, see [3, 52] and also [50] for the complex singularities for 2D Euler equation.

In [16], the authors proved the blowup of the Hou-Luo model proposed in [42]. In [15], Chen-Hou-Huang proved the asymptotically self-similar blowup of the Hou-Luo model by extending the method of analysis established for the finite time blowup of the De Gregorio model by the same authors in [14]. In [17,31, 33, 39, the authors proposed several simplified models to study the Hou-Luo blowup scenario [41,42] and established finite time blowup of these models. In [27.29], Elgindi and Jeong proved finite time blowup for the 2D Boussinesq and 3D axisymmetric Euler equations in a domain with a corner using $\dot{C}^{0, \alpha}$ data.

The rest of the paper is organized as follows. In Section 2, we review the analytic framework that we established in Part I 13 and state the assumptions under which we prove the finite time blowup of the 2D Boussinesq and 3D Euler equations with smooth initial data. In Section 3. we discuss the construction of the approximate space-time solution to the linearized operator $\mathcal{L}$. This is crucial to obtain sharp estimates of the perturbed operator $\mathcal{L}-\mathcal{K}$ in the stability analysis. In Section 4 we show how to estimate the $L^{\infty}$ and Hölder norms of the velocity in the regular case. Some technical estimates and derivations are deferred to the Appendix.

## 2. Review of the analytic framework from Part I 13

In this section, we will review some main ingredients in our analytic framework to establish stability analysis that we presented in Part I [13]. We will mainly focus on the 2D Boussinesq equations since the difference between the 3D Euler Euler and 2D Boussinesq equations is asymptotically small. As in our previous works [12, 14, 15, we will use the dynamic rescaling formulation for the 2D Boussinesq equations to study the linear stability for the linearized operator around the approximate steady state of the dynamic rescaling equations. Passing from linear stability to nonlinear stability is relatively easier by treating the nonlinear terms and the residual error as small perturbations to the linear damping terms.

Denote by $\omega^{\theta}, u^{\theta}$ and $\phi^{\theta}$ the angular vorticity, angular velocity, and angular stream function, respectively. The 3D axisymmetric Euler equations are given below:

$$
\begin{equation*}
\partial_{t}\left(r u^{\theta}\right)+u^{r}\left(r u^{\theta}\right)_{r}+u^{z}\left(r u^{\theta}\right)_{z}=0, \quad \partial_{t}\left(\frac{\omega^{\theta}}{r}\right)+u^{r}\left(\frac{\omega^{\theta}}{r}\right)_{r}+u^{z}\left(\frac{\omega^{\theta}}{r}\right)_{z}=\frac{1}{r^{4}} \partial_{z}\left(\left(r u^{\theta}\right)^{2}\right) \tag{2.1}
\end{equation*}
$$

where the radial velocity $u^{r}$ and the axial velocity $u^{\theta}$ are given by the Biot-Savart law:

$$
\begin{equation*}
-\left(\partial_{r r}+\frac{1}{r} \partial_{r}+\partial_{z z}\right) \phi^{\theta}+\frac{1}{r^{2}} \phi^{\theta}=\omega^{\theta}, \quad u^{r}=-\phi_{z}^{\theta}, \quad u^{z}=\phi_{r}^{\theta}+\frac{1}{r} \phi^{\theta} \tag{2.2}
\end{equation*}
$$

with the no-flow boundary condition $\phi^{\theta}(1, z)=0$ on the solid boundary $r=1$ and a periodic boundary condition in $z$. For 3D Euler blowup that occurs at the boundary $r=1$, we know that the axisymmetric Euler equations have scaling properties asymptotically the same as those of the 2D Boussinesq equations 43. Thus, we also study the 2D Boussinesq equations on the upper half space:

$$
\begin{align*}
\omega_{t}+\mathbf{u} \cdot \nabla \omega & =\theta_{x}  \tag{2.3}\\
\theta_{t}+\mathbf{u} \cdot \nabla \theta & =0 \tag{2.4}
\end{align*}
$$

where the velocity field $\mathbf{u}=(u, v)^{T}: \mathbb{R}_{+}^{2} \times[0, T) \rightarrow \mathbb{R}_{+}^{2}$ is determined via the Biot-Savart law

$$
\begin{equation*}
-\Delta \phi=\omega, \quad u=-\phi_{y}, \quad v=\phi_{x} \tag{2.5}
\end{equation*}
$$

where $\phi$ is the stream function with the no-flow boundary condition $\phi(x, 0)=0$ at $y=0$. By making the change of variables $\tilde{\theta} \triangleq\left(r u^{\theta}\right)^{2}, \tilde{\omega}=\omega^{\theta} / r$, we can see that $\tilde{\theta}$ and $\tilde{\omega}$ satisfy the 2D Boussinesq equations up to the leading order for $r \geq r_{0}>0$.
2.1. Dynamic rescaling formulation. Following [12, 14, 15], we consider the dynamic rescaling formulation of the 2D Boussinesq equations. Let $\omega(x, t), \theta(x, t), \mathbf{u}(x, t)$ be the solutions of (2.3)-(2.5). Then it is easy to show that

$$
\begin{align*}
\tilde{\omega}(x, \tau) & =C_{\omega}(\tau) \omega\left(C_{l}(\tau) x, t(\tau)\right), \quad \tilde{\theta}(x, \tau)=C_{\theta}(\tau) \theta\left(C_{l}(\tau) x, t(\tau)\right) \\
\tilde{\mathbf{u}}(x, \tau) & =C_{\omega}(\tau) C_{l}(\tau)^{-1} \mathbf{u}\left(C_{l}(\tau) x, t(\tau)\right) \tag{2.6}
\end{align*}
$$

are the solutions to the dynamic rescaling equations

$$
\begin{equation*}
\tilde{\omega}_{\tau}(x, \tau)+\left(c_{l}(\tau) \mathbf{x}+\tilde{\mathbf{u}}\right) \cdot \nabla \tilde{\omega}=c_{\omega}(\tau) \tilde{\omega}+\tilde{\theta}_{x}, \quad \tilde{\theta}_{\tau}(x, \tau)+\left(c_{l}(\tau) \mathbf{x}+\tilde{\mathbf{u}}\right) \cdot \nabla \tilde{\theta}=c_{\theta} \tilde{\theta} \tag{2.7}
\end{equation*}
$$

where $\tilde{\mathbf{u}}=(\tilde{u}, \tilde{v})^{T}=\nabla^{\perp}(-\Delta)^{-1} \tilde{\omega}, \mathbf{x}=(x, y)^{T}$,

$$
\begin{equation*}
C_{\omega}(\tau)=\exp \left(\int_{0}^{\tau} c_{\omega}(s) d \tau\right), C_{l}(\tau)=\exp \left(\int_{0}^{\tau}-c_{l}(s) d s\right), C_{\theta}=\exp \left(\int_{0}^{\tau} c_{\theta}(s) d \tau\right) \tag{2.8}
\end{equation*}
$$

$t(\tau)=\int_{0}^{\tau} C_{\omega}(\tau) d \tau$ and the rescaling parameters $c_{l}(\tau), c_{\theta}(\tau), c_{\omega}(\tau)$ satisfy 12

$$
\begin{equation*}
c_{\theta}(\tau)=c_{l}(\tau)+2 c_{\omega}(\tau) \tag{2.9}
\end{equation*}
$$

To simplify our presentation, we still use $t$ to denote the rescaled time in (2.7) and simplify $\tilde{\omega}, \tilde{\theta}$ as $\omega, \theta$

$$
\begin{equation*}
\omega_{t}+\left(c_{l} x+\mathbf{u}\right) \cdot \nabla \omega=\theta_{x}+c_{\omega} \omega, \quad \theta_{t}+\left(c_{l} x+\mathbf{u}\right) \cdot \nabla \theta=c_{\theta} \theta \tag{2.10}
\end{equation*}
$$

Following [15], we impose the following normalization conditions on $c_{\omega}, c_{l}$

$$
\begin{equation*}
c_{l}=2 \frac{\theta_{x x}(0)}{\omega_{x}(0)}, \quad c_{\omega}=\frac{1}{2} c_{l}+u_{x}(0), \quad c_{\theta}=c_{l}+2 c_{\omega} \tag{2.11}
\end{equation*}
$$

For smooth data, these two normalization conditions play the role of enforcing

$$
\begin{equation*}
\theta_{x x}(t, 0)=\theta_{x x}(0,0), \quad \omega_{x}(t, 0)=\omega_{x}(0,0) \tag{2.12}
\end{equation*}
$$

for all time.
We remark that the dynamic rescaling formulation was introduced in 40,45 to study the self-similar blowup of the nonlinear Schrödinger equations. This formulation is also called the modulation technique in the literature and has been developed by Merle, Raphael, Martel, Zaag and others, see e.g. 1, 2, 37, 44, 46, 48]. Recently, this method has been applied to study singularity formation in incompressible fluids [12, 24] and related models [6] 8, 14 .

The more precise statement of our Theorem is stated as follows.

Theorem 3. Let $\left(\bar{\theta}, \bar{\omega}, \overline{\mathbf{u}}, \bar{c}_{l}, \bar{c}_{\omega}\right)$ be the approximate self-similar profile constructed in Section 7 of Part $I$ [13] and $E_{*}=5 \cdot 10^{-6}$. For even initial data $\theta_{0}$ and odd $\omega_{0}$ of (2.10) satisfying $E\left(\omega_{0}-\bar{\omega}, \theta_{0, x}-\bar{\theta}_{x}, \theta_{0, y}-\bar{\theta}_{y}\right)<E_{*}$, we have

$$
\begin{equation*}
\|\omega-\bar{\omega}\|_{L^{\infty}},\left\|\theta_{x}-\bar{\theta}_{x}\right\|_{L^{\infty}},\left\|\theta_{y}-\bar{\theta}_{y}\right\|_{\infty}<200 E_{*}, \quad\left|u_{x}(t, 0)-\bar{u}_{x}(0)\right|,\left|\bar{c}_{\omega}-c_{\omega}\right|<100 E_{*} \tag{2.13}
\end{equation*}
$$

for all time. In particular, we can choose smooth initial data $\omega_{0}, \theta_{0} \in C_{c}^{\infty}$ in this class with finite energy $\left\|\mathbf{u}_{0}\right\|_{L^{2}}<+\infty$ such that the solution to the physical equations (2.3) -(2.5) with these initial data blows up in finite time $T$.

The energy $E$ is quite complicated, and we refer to Section 2.3 in Part I 13 for its formula.
2.2. The main steps in the proof of Theorem 3. We will follow the framework in [12, 14, 15] to establish finite time blowup by proving the nonlinear stability of an approximate steady state to (2.10). We divide the proof of Theorem 3 into proving the following lemmas. The energy norm below is defined in Section 5 in Part I [13] for energy estimates, and the requirement of smallness is incorporated in the conditions (2.17), e.g. the term $a_{i j, 3}$, in Lemma 2.5.

The upper bar notation is reserved for the approximate steady state, e.g. $\bar{\omega}, \bar{\theta}$. Given the approximate steady state $\bar{\omega}, \bar{\theta}, \bar{c}_{l}, \bar{c}_{\omega}$, we denote by $\overline{\mathcal{F}}_{i}$ and $\bar{F}_{\omega}, \bar{F}_{\theta}$ the residual error

$$
\begin{align*}
& \bar{F}_{\omega}=-\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \bar{\omega}+\bar{\theta}_{x}+\bar{c}_{\omega} \bar{\omega}, \quad \bar{F}_{\theta}=-\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \bar{\theta}+\bar{c}_{\theta} \bar{\theta} \\
& \overline{\mathcal{F}}_{1} \triangleq \bar{F}_{\omega}, \quad \overline{\mathcal{F}}_{2} \triangleq \partial_{x} \bar{F}_{\theta}, \quad \overline{\mathcal{F}}_{3} \triangleq \partial_{y} \bar{F}_{\theta} \tag{2.14}
\end{align*}
$$

We have the following nonlinear stability Lemma for $L^{\infty}$-based energy estimate, which is proved in Appendix A. 1 of Part I 13 .

Lemma 2.1. Suppose that $f_{i}(x, z, t): \mathbb{R}_{++}^{2} \times \mathbb{R}_{++}^{2} \times[0, T] \rightarrow \mathbb{R}, 1 \leq i \leq n$, satisfies

$$
\begin{equation*}
\partial_{t} f_{i}+v_{i}(x, z) \cdot \nabla_{x, z} f_{i}=-a_{i i}(x, z, t) f_{i}+B_{i}(x, z, t)+N_{i}(x, z, t)+\bar{\varepsilon}_{i} \tag{2.15}
\end{equation*}
$$

where $v_{i}(x, z, t)$ are some vector fields Lipschitz in $x, z$ with $\left.v_{i}\right|_{x_{1}=0}=0,\left.v_{i}\right|_{z_{1}=0}=0$. For some $\mu_{i}>0$, we define the energy

$$
E(t)=\max _{1 \leq i \leq n}\left(\mu_{i}\left\|f_{i}\right\|_{L^{\infty}}\right)
$$

Suppose that $B_{i}, N_{i}$ and $\bar{\varepsilon}_{i}$ satisfy the following estimate
$\mu_{i}\left(\left|B_{i}(x, z, t)\right|+\left|N_{i}(x, z, t)\right|+\left|\bar{e}_{i}\right|\right) \leq \sum_{j \neq i}\left(\left|a_{i j}(x, z, t)\right| E(t)+\left|a_{i j, 2}(x, z, t)\right| E^{2}(t)+\left|a_{i j, 3}(x, z, t)\right|\right)$.
If there exists some $E_{*}, \varepsilon_{0}, M>0$ such that

$$
\begin{align*}
& a_{i i}(x, z, t) E_{*}-\sum_{j \neq i}\left(\left|a_{i j}\right| E_{*}+\left|a_{i j, 2}\right| E_{*}^{2}+\left|a_{i j, 3}(x, z, t)\right|\right)>\varepsilon_{0}  \tag{2.17}\\
& \sum_{j \neq i}\left(\left|a_{i j}\right| E_{*}+\left|a_{i j, 2}\right| E_{*}^{2}+\left|a_{i j, 3}(x, z, t)\right|\right)<M
\end{align*}
$$

for all $x, z$ and $t \in[0, T]$. Then for $E(0)<E_{*}$, we have $E(t)<E_{*}$ for $t \in[0, T]$.
Lemma 2.2. There exists a nontrivial approximate steady state ( $\bar{\omega}, \bar{\theta}, \bar{c}_{l}, \bar{c}_{\omega}$ ) to (2.10), (2.11) with $\bar{\omega}, \bar{\theta} \in C^{4,1}$ and residual errors $\overline{\mathcal{F}}_{i}, i=1,2,3(\sqrt{(2.14)}$ sufficiently small in some energy norm.

The construction of an approximate self-similar profile with a small residual error stated in Lemma 2.2 is provided in Section 7 of Part I 13 and the properties of ( $\bar{\omega}, \bar{\theta}, \bar{c}_{l}, \bar{c}_{\omega}$ ) are described in Section 2.4 of Part I [13]. We will estimate the local part of the residual error in Appendix C.4. We linearize (2.10) around ( $\bar{\omega}, \bar{\theta}, \bar{c}_{l}, \bar{c}_{\omega}$ ) and perform energy estimate of the perturbation $W=\left(\omega, \theta_{x}, \theta_{y}\right)$ in Section 5 in Part I [13]. In our estimates, we need to control a number of nonlocal terms.

Lemma 2.3. Let $\omega$ be odd in $x_{1}$. Denote $\delta(f, x, z)=f(x)-f(z)$. There exists finite rank approximations $\hat{\mathbf{u}}, \widehat{\nabla \mathbf{u}}$ for $\mathbf{u}(\omega), \nabla \mathbf{u}(\omega)$ with rank less than 50 such that we have the following weighted $L^{\infty}$ and directional Hölder estimate for $f=u, v, \partial_{l} u, \partial_{l} v, x, z \in \mathbb{R}_{2}^{++}, i=1,2, \gamma_{i}>0$

$$
\begin{align*}
&\left|\rho_{f}(f-\hat{f})(x)\right| \leq C_{f, \infty}\left(x, \varphi, \psi_{1}, \gamma\right) \max \left(\|\omega \varphi\|_{\infty}, s_{f} \max _{j=1,2} \gamma_{j}\left[\omega \psi_{1}\right]_{C_{x_{j}}^{1 / 2}\left(\mathbb{R}_{2}^{+}\right)}\right) \\
& \frac{\left|\delta\left(\psi_{f}(f-\hat{f}), x, z\right)\right|}{|x-z|^{1 / 2}} \leq C_{f, i}\left(x, z, \varphi, \psi_{1}, \gamma\right) \max \left(\|\omega \varphi\|_{\infty}, s_{f} \max _{j=1,2} \gamma_{j}\left[\omega \psi_{1}\right]_{C_{x_{j}}^{1 / 2}\left(\mathbb{R}_{2}^{+}\right)}\right) \tag{2.18}
\end{align*}
$$

with $x_{3-i}=z_{3-i}$, where $s_{f}=0$ for $f=u, v, s_{f}=1$ for $f=\partial_{l} u, \partial_{l} v$, the functions $C(x), C(x, z)$ depend on $\gamma$, the weights, and the approximations, the singular weights $\varphi=\varphi_{1}, \varphi_{g, 1}, \varphi_{\text {elli }}, \psi_{\partial u}=$ $\psi_{1}, \psi_{u}$ are defined in (A.2), the weight $\rho_{10}$ for $\mathbf{u}$ and the weight for $\rho_{i j}$ for $\nabla \mathbf{u}$ with $i+j=2$ are given in A.2). In the estimate of $f=u, v$, we do not need the Hölder semi-norm and $s_{f}=0$. Moreover, $C(x), C(x, z)$ are bounded in any compact domain of $\mathbb{R}_{2}^{++}$. We have an additional estimate for $\rho_{4}(u-\hat{u})$ similar to the above with $\rho_{4}$ A.2) singular along $x_{1}=0$.

Furthermore, we have the following estimate using the localized norm. There exists $D_{1}, D_{2}, . . D_{n} \subset$ $\mathbb{R}_{2}^{++}$and $D_{S} \in \mathbb{R}_{2}^{+}$depending on $x$ in the $L^{\infty}$ estimate and $x, z$ in the $C_{x_{i}}^{1 / 2}$ estimate, such that

$$
\begin{aligned}
\left|\rho_{f}(f-\hat{f})(x)\right| & \leq \sum_{j} C_{f, \infty, j}\left(x, \varphi, \psi_{1}, \gamma\right)\|\omega \varphi\|_{L^{\infty}\left(D_{j}\right)}+C_{f, \infty, S}\left(x, \varphi, \psi_{1}, \gamma\right) \max _{l=1,2}\left(\gamma_{l}\left[\omega \psi_{1}\right]_{C_{x_{l}}^{1 / 2}\left(D_{S}\right)}\right), \\
\frac{\left|\delta\left(\psi_{f}(f-\hat{f}), x, z\right)\right|}{|x-z|^{1 / 2}} & \leq \sum_{j} C_{f, i, j}\left(x, z, \varphi, \psi_{1}, \gamma\right)| | \omega \varphi \|_{L^{\infty}\left(D_{j}\right)}+C_{f, i, S}\left(x, z, \varphi, \psi_{1}, \gamma\right) \max _{l=1,2}\left(\gamma_{l}\left[\omega \psi_{1}\right]_{C_{x_{j}}^{1 / 2}\left(D_{S}\right)}\right),
\end{aligned}
$$

for $x_{3-i}=z_{3-i}, \varphi=\varphi_{\text {elli }}$ and the same notation as above, where $C_{f, \infty, S}, C_{f, i, S}=0$ for $f=u, v$. Similarly, we have an estimate for $\rho_{4}(u-\hat{u})$ using localized norm with $C_{f, \infty, S}=0$ similar to the above.

Since the weights $\rho_{10} \sim|x|^{-3}, \psi_{1} \sim|x|^{-2}, \psi_{u}$ are singular near $x=0$, without subtracting the approximation $\hat{f}$ from $f, \rho_{f} f$ is not bounded near $x=0$.

Based on these finite rank approximations, we can decompose the perturbations.
Lemma 2.4. There exists $m<50$ approximate solutions $\hat{F}_{i}$ to the linearized equations $\partial_{t} W=$ $\mathcal{L} W$ of (2.10) around $\left(\bar{\omega}, \bar{\theta}, \bar{c}_{l}, \bar{c}_{\omega}\right)$ in Lemma 2.2 from given initial data $\bar{F}_{i}(0)$ with residual error $\mathcal{R}$ small in the energy norm. Further we can decompose the perturbation $W=W_{1}+\widehat{W}_{2}$ with the following properties. (a) $\hat{W}_{2}$ is constructed based on $\widehat{F}_{i}$, see Section 4.2 .4 of Part $I$ [13]; (b) $W_{1}$ satisfies equations with the leading order linearized operator $(\mathcal{L}-\mathcal{K}) W_{1}$ up to the small residual error $\mathcal{R}$ for some finite rank operator $\mathcal{K}$, and $W_{1}$ depends on $\widehat{W}_{2}$ weakly at the linear level via $\mathcal{R}$. The functionals $a_{i}\left(W_{1}\right), a_{n l, i}(W)$ in the construction of $\widehat{W}_{2}$ and $\mathcal{K}$ (see Section 4.2 .4 of Part I [13]) are related to the finite rank approximations in Lemma 2.3.

Moreover, there exists an energy $E_{4}(t)$ for $W_{1}, W$ (see Section 5.6.3. of Part I [13]) that controls the weighted $L^{\infty}$ and $C^{1 / 2}$ seminorm of $W_{1}$ such that under the bootstrap assumption $E_{4}(t)<E_{* 0}$ with $E_{* 0}>0$, we can establish nonlinear energy estimates for $E_{4}(t)$ using the estimates in Lemma 2.3 .

If the bounds in Lemma 2.3 are tight, and the residual error in the constructions of $(\bar{\omega}, \bar{\theta}), \widehat{F}_{i}$ are small enough, we can use Lemma 2.1 to obtain nonlinear stability.
Lemma 2.5. For $E_{*}=5 \cdot 10^{-6}$, the coefficients in the nonlinear energy estimates of $E_{4}(t)$ satisfy the conditions (2.17), and the statements in Theorem 3 hold true.

The main purpose of Part II of our paper is the following. Firstly, we obtain sharp estimates of the constants in Lemma 2.3, which only depends on the weights. Secondly, we construct the approximate $\hat{F}_{i}(t)$ in Lemma 2.4 numerically, and estimate its piecewise derivatives and the local residual error in Section 3. Thirdly, we estimate piecewise bounds of the approximate steady steady in Appendix C the singular weights in Appendix A, some explicit functions related to the approximate solutions in Appendix D. We remark that all of these estimates and constants depend on the given weights, some operators and functions, e.g. the approximate steady state
and the specific initial conditions. With these estimates and constants, we obtain the concrete values of the inequalities in (2.17) and Lemma [2.5, which are given in Appendix D in Part I [13]. We further verify the inequalities for the stability conditions in Lemma 2.5 .

Let us comment the above lemmas. Firstly, our energy estimate is based on weighted $L^{\infty}$ functional spaces, which is crucial for extracting the damping terms for the energy estimate. See Section 2.7 of Part I 13 for the motivations. Given $\omega \in C^{1 / 2}$, we have $\mathbf{u} \in C^{3 / 2}, \nabla \mathbf{u} \in C^{1 / 2}$. To establish the nonlinear stability conditions (2.17) in Lemma 2.5, we need sharp constants in the estimates in Lemma 2.3. We use some techniques from optimal transport to obtain sharp $C^{1 / 2}$ estimate of $\nabla \mathbf{u}$ in Section 3 of Part I [13]. This corresponds to the limiting case in the $C_{x_{i}}^{1 / 2}$ estimate in Lemma 2.3 for a fixed $x$ with $|x-z| \rightarrow 0$ and captures the most singular part in the estimates in Lemma [2.3. The constants in the sharp $C^{1 / 2}$ estimate established in Part I [13] are given by several integrals. In Section 5 in the supplementary material II [11] (contained in this paper), we estimate these integrals.

Other parts of the estimates in Lemma 2.3 are more regular since we work with the regular part of the velocity integral with a desingularized kernel. Given $\omega \in C^{1 / 2}$, we can reduce the estimates of these more regular terms to estimate some explicit $L^{1}$ integrals. We can obtain sharp estimates of these more regular integrals using some numerical quadrature with computer assistance. See Section 4 .

By designing $\mathcal{K}$ to approximate the nonlocal terms, we can obtain much better linear stability estimates for $\mathcal{L}-\mathcal{K}$. After we have shown that the stability conditions (2.17) are satisfied, we have nonlinear stability estimates $E_{4}(t)<E_{*}$ for all $t>0$ using Lemma 2.1, which implies the bounds in Theorem 3. The remaining steps of obtaining finite time blowup from smooth initial data and finite energy follows [14] and a rescaling argument. We remark that the variable $\widehat{W}_{2}$ in Lemma 2.4 (see full definition in Section 4.2 .4 of Part I [13]) plays an auxiliary role, and we do not perform energy estimate on $\widehat{W}_{2}$ directly.

Note that all the nonlocal terms in the linearized equations are not small. Without the sharp $C^{1 / 2}$ estimate, with the choice of energy $E_{4}$, the stability conditions in (2.17) and Lemma 2.5 fail in the weighted Hölder estimate. Without the finite rank approximations for the nonlocal terms in Lemma 2.3, 2.4 the stability conditions for weighted $L^{\infty}$ estimate also fail.

Rigorous numerics. The codes for the computations can be found in 9. The codes are implemented in MatLab with package INTLAB [51] for interval arithmetic. The estimates of the constants in Lemma 2.3, integrals in Section 4, and the constructions and estimates of the approximate space-time solutions in Lemma 2.4 and in Section 3 are performed in parallel using the Caltech High Performance Computing. Other computer-assisted estimates and main part of the verifications are done in Mac Pro (Rack,2019) with 2.5 GHz 28 -core Intel Xeon W processor and $768 \mathrm{~GB}(6 \times 128 \mathrm{~GB})$ of DDR4 ECC memory.

## 3. Constructing and estimating the approximate solution to the linearized EQUATIONS

As we described in Section 2 of Part I [13] (see also the Introduction), we need to construct the approximate solutions to $e^{\mathcal{L} t} F_{0}$ for several initial data $\bar{F}_{i}, \bar{F}_{\chi, i}$. In this section, we discuss how to construct these space-time solutions numerically with some vanishing properties at the origin with rigorous error control.

The linearized equations associated with $\mathcal{L}$ read

$$
\begin{align*}
\partial_{t} \omega & =-\left(\bar{c}_{l} x+\bar{u}\right) \cdot \nabla \omega+\eta+\bar{c}_{\omega} \omega-\mathbf{u} \cdot \nabla \bar{\omega}+c_{\omega} \bar{\omega}=\mathcal{L}_{1}(\omega, \eta, \xi) \\
\partial_{t} \eta & =-\left(\bar{c}_{l} x+\bar{u}\right) \cdot \nabla \eta+\left(2 \bar{c}_{\omega}-\bar{u}_{x}\right) \eta-\bar{v}_{x} \xi-\mathbf{u}_{x} \cdot \nabla \bar{\theta}-\mathbf{u} \cdot \nabla \bar{\theta}_{x}+2 c_{\omega} \bar{\theta}_{x}=\mathcal{L}_{2}(\omega, \eta, \xi),  \tag{3.1}\\
\partial_{t} \xi & =-\left(\bar{c}_{l} x+\bar{u}\right) \cdot \nabla \xi+\left(2 \bar{c}_{\omega}+\bar{u}_{x}\right) \xi-\bar{u}_{y} \eta-\mathbf{u}_{y} \cdot \nabla \bar{\theta}-\mathbf{u} \cdot \nabla \bar{\theta}_{y}+2 c_{\omega} \bar{\theta}_{y}=\mathcal{L}_{3}(\omega, \eta, \xi),
\end{align*}
$$

with normalization condition

$$
\begin{equation*}
c_{\omega}=u_{x}(0), \quad c_{l} \equiv 0 \tag{3.2}
\end{equation*}
$$

[^0]Although $\eta, \xi$ represent $\theta_{x}, \theta_{y}$ in the Boussinesq equations, we will consider initial data $\left(\omega_{0}, \eta_{0}, \xi_{0}\right)$ with $\partial_{y} \eta_{0} \neq \partial_{x} \xi_{0}$. Thus, we do not have the relation $\partial_{y} \eta=\partial_{x} \xi$ and will treat $\eta, \xi$ as two independent variables. The solutions $\omega, \eta$ are odd, $\xi$ is even with $\xi(0, y)=0$. We consider initial data $\left(\omega_{0}, \eta_{0}, \xi_{0}\right)=O\left(|x|^{2}\right)$ near $x=0$. Using a direct calculation, we obtain that these vanishing conditions are preserved

$$
\begin{equation*}
\omega(t, x), \eta(t, x), \quad \xi(t, x)=O\left(|x|^{2}\right) \tag{3.3}
\end{equation*}
$$

We introduce the bilinear operator $B_{o p, i}((\mathbf{u}, M), G)$ for $(\mathbf{u}, M), G=\left(G_{1}, G_{2}, G_{3}\right)$

$$
\begin{align*}
& \mathcal{B}_{o p, 1}=-\mathbf{u} \cdot \nabla G_{1}+M_{11}(0) G_{1}, \quad \mathcal{B}_{o p, 2}=-\mathbf{u} \cdot \nabla G_{2}+2 M_{11}(0) G_{2}-M_{11} G_{2}-M_{21} G_{3} \\
& \mathcal{B}_{o p, 3}=-\mathbf{u} \cdot \nabla G_{3}+2 M_{11}(0) G_{3}-M_{12} G_{2}-M_{22} G_{3} \tag{3.4}
\end{align*}
$$

If $M=\nabla \mathbf{u}, M_{11}=u_{x}, M_{12}=u_{y}, M_{21}=v_{x}, M_{22}=v_{y}$, then we drop $M$ to simplify the notation

$$
\begin{align*}
\mathcal{B}_{o p, 1}(\mathbf{u}, G) & =-\mathbf{u} \cdot \nabla G_{1}+u_{x}(0) G_{1}, \quad \mathcal{B}_{o p, 2}=-\mathbf{u} \cdot \nabla G_{2}+2 u_{x}(0) G_{2}-u_{x} G_{2}-v_{x} G_{3} \\
\mathcal{B}_{o p, 3} & =-\mathbf{u} \cdot \nabla G_{3}+2 u_{x}(0) G_{3}-u_{y} G_{2}-v_{y} G_{3} \tag{3.5}
\end{align*}
$$

The main result in this section is the following. Given $n$ initial data $\bar{G}_{i}=\left(\bar{G}_{1,1}, \bar{G}_{i, 2}, \bar{G}_{i, 3}\right)$ and $n$ functions $c_{i}(t),, i=1,2, . ., n$ Lipschitz and bounded in $t$, we construct approximate spacetime solution $\hat{W}_{i}=\left(\hat{W}_{i, 1}, \hat{W}_{i, 2}, \hat{W}_{i, 3}\right), \hat{G}$ and the approximate stream functions $\left(\hat{\phi}_{i}^{N}, \hat{\phi}^{N}\right)$ and the error $\hat{\varepsilon}_{1}$ associated with $\hat{W}_{i, 1}, \hat{G}_{1}$
$\hat{G}=\sum_{i \leq n} \int c_{i}(t-s) \hat{W}_{i}(s) d s, \quad \hat{\phi}^{N}=\sum_{i \leq n} \int c_{i}(t-s) \hat{\phi}_{i}^{N}(s) d s, \quad \hat{\varepsilon}=\sum_{i \leq n} \int c_{i}(t-s)\left(\hat{W}_{i, 1}+\Delta \hat{\phi}_{i}^{N}\right)(s) d s$,
with residual error

$$
\begin{equation*}
\mathcal{R}=\sum_{i \leq n} c_{i}(t)\left(\hat{W}_{i}(0)-\bar{W}_{i}\right)+\int_{0}^{t} c(t-s)\left(\partial_{t}-\mathcal{L}\right) \hat{W}_{i}(s) d s \tag{3.7}
\end{equation*}
$$

vanishing $O\left(|x|^{3}\right)$ near $x=0$. Moreover, we can decompose $\mathcal{R}$ as follows

$$
\begin{align*}
& \mathcal{R}_{j}(t)=\mathcal{R}_{l o c, 0, j}(t)+I_{j}-D_{j}^{2} I_{j}(0) \chi_{j, 2}, \quad \mathcal{R}_{l o c, j}=\sum_{i \leq n} \int_{0}^{t} c_{i}(t-s) \mathcal{R}_{n u m, i, j}(s) d s  \tag{3.8}\\
& \mathcal{R}_{n u m, i, j}=O\left(|x|^{3}\right), \quad I_{j}=\mathcal{B}_{o p, j}(\mathbf{u}(\bar{\varepsilon}), \hat{G})+\mathcal{B}_{o p, j}\left(\mathbf{u}(\hat{\varepsilon}),\left(\bar{\omega}, \bar{\theta}_{x}, \bar{\theta}_{y}\right)\right.
\end{align*}
$$

where $\chi_{j 2}$ is given in (D.5), and $\bar{\varepsilon}=\bar{\omega}-(-\Delta) \bar{\phi}^{N}$ is the error of the approximate stream function for $(-\Delta)^{-1} \bar{\omega}, \mathcal{R}_{n u m, j}(t, x)$ depends on $\hat{W}_{i}, \hat{\phi}_{i}$ in $x$ locally. We have absorbed the initial error in $\mathcal{R}_{n u m}$. We derive the above decompositions and estimates of $\mathcal{R}_{l o c, 0, j}, \hat{G}, \hat{\phi}^{N}, \hat{\varepsilon}$, in Section 3.5. 3.7. See (3.34), (3.32). We combine the estimate of the nonlocal error in $I_{j}$ and perturbation in Section 5.8 in Part I [13]. Furthermore, we track the piecewise bounds of the following quantities

$$
\begin{align*}
& \int_{0}^{\infty}\left|\partial_{x}^{k} \partial_{y}^{l} F(t)\right| d t, F=\hat{W}_{i, j}, \quad F=\hat{\phi}_{i}^{N}, F=\hat{\phi}_{i}^{N}-\partial_{x y} \phi_{i}^{N}(0) x y, F=\hat{W}_{i, 1}+\Delta \phi_{i}^{N}  \tag{3.9}\\
& F=c_{j} \hat{W}_{i, j}-x \partial_{x} \hat{W}_{i, j}+y \partial_{y} \hat{W}_{i, j}-D_{j}^{2} \hat{W}_{i, j}(0) f_{\chi, j}, D^{2}=\left(\partial_{x y}, \partial_{x y}, \partial_{x x}\right), c=(1,1,3)
\end{align*}
$$

for $j=1,2,3, i=1,2, . ., n$, where $f_{\chi, j}$ is defined in (D.6). We track the $C^{2}$ bound of $\hat{W}_{i, j}$ and $C^{4}$ bounds for others following (3.34), and use these bounds to control $\widehat{W}_{2}$ in Lemma 2.4 and use them in the nonlinear energy estimates in Section 5 in Part I [13].

In practice, we choose the initial data $\bar{F}_{i}$ given in Appendix C.2.1 in Part I [13], and $c_{i}(t)$ some functionals of the perturbation $W_{1}, \hat{W}_{2}$ related to the finite rank perturbation.

Numerical methods. We solve (3.1) using the numerical method outlined in Section 7 of Part I [13] to obtain the solution $\left(\omega_{k}, \eta_{k}, \xi_{k}\right)$ at discrete time $t_{k}$. Since $\xi$ is even with $\xi(0, y)=0$, we write $\xi=x \zeta$ for an odd function $\zeta$. We use the adaptive mesh discussed in Appendix C. 1 to discretize the spatial domain. Then we represent $\omega, \eta, \zeta$ using the piecewise 6 -th order B-spline (C.5). See Appendix C.1. To solve the stream function $-\Delta \phi=\omega$ numerically, we use the B-spline based finite element method and obtain the numerical approximation $\phi^{N}$ for $(-\Delta)^{-1} \omega$. Then we can construct the velocity $\mathbf{u}^{N}=\nabla^{\perp} \phi^{N}$.

The gradients of several initial conditions $\bar{F}_{i}$ are relatively large and the linearized equations (3.1) involve $\nabla \hat{W}$. To obtain a better approximation of the solution, we represent $\omega, \eta, \zeta$ using a finer mesh $Y \times Y$ with $Y$ being smaller than the mesh $y$ by a factor of three in Appendix C.1] Since solving the Poisson equation is the main computational cost in each time step, we still represent $\phi^{N}$ using the coarse mesh $y \times y$ and solve it from source term with grid points value $\omega\left(y_{i}, y_{j}\right)$.

In the temporal variable, we use a third order Runge-Kutta method to update the PDE. To reduce the round-off error near $x=0$, where we require a very small error in solving the linear PDE, we use a multi-level representation. We refer more details to Section 7 in Part I 13. To keep the residual error smooth near $x=0$, we apply a weak numerical filter near $x=0$ every three steps. We do not add the semi-analytic part in constructing $\left(\omega_{k}, \eta_{k}, \xi_{k}\right)$ for efficiency consideration and that the far-field behavior of the solutions is changing over time.

After we obtain the numerical solution $\left(\omega_{k}, \eta_{k}, \xi_{k}, \phi_{k, 1}^{N}\right)$ at discrete time, we will perform two rank-one corrections and interpolate the solution in time using a cubic polynomial to obtain the approximate space time solution $\hat{W}$, and estimate residual error in the energy space $a$-posteriori.
3.1. A posteriori error estimates: decomposition of errors. Since we cannot solve the Poisson equation exactly, we decompose the stream function $\bar{\phi}, \phi$ as follows

$$
\begin{equation*}
\bar{\phi}=(-\Delta)^{-1} \bar{\omega}=\bar{\phi}^{N}+\bar{\phi}^{e}, \quad \phi=(-\Delta)^{-1} \omega=\phi^{N}+\phi^{e} \tag{3.10}
\end{equation*}
$$

where $\bar{\phi}^{N}, \phi^{N}$ constructed using finite element method are the numeric approximation of the stream function, and the short hands $N$, e denote numeric, error, respectively. We use similar notations below for other nonlocal terms since we cannot construct them exactly. We will construct $\bar{\phi}^{N}, \phi^{N}$ numerically and treat $\bar{\phi}^{e}, \phi^{e}$ as error. The reader should not confuse $\phi^{N}$ with the $N$-th power of $\phi$. We will never use power of $\phi$ throughout the paper. Similarly, we denote by $\mathbf{u}^{N}, \mathbf{u}^{e}$ the velocities corresponding to $\phi^{N}, \phi^{e}$. For example, we have

$$
\begin{equation*}
\mathbf{u}^{N}=\nabla^{\perp} \phi^{N}, \quad \mathbf{u}^{e}=\nabla^{\perp} \phi^{e}=\nabla^{\perp}(-\Delta)^{-1}\left(\omega-(-\Delta) \phi^{N}\right), \quad c_{\omega}^{N}=u_{x}^{N}(0), \quad c_{\omega}^{e}=u_{x}^{e}(0) \tag{3.11}
\end{equation*}
$$

The above decomposition leads to the following decomposition of the operator $\mathcal{L}$

$$
\begin{aligned}
\mathcal{L}_{1} & =\mathcal{L}_{1}^{N}+\mathcal{L}_{1}^{e}+\mathcal{L}_{1}^{\bar{e}}, \quad \mathcal{L}_{2}=\mathcal{L}_{2}^{N}+\mathcal{L}_{2}^{e}+\mathcal{L}_{2}^{\bar{e}}, \quad \mathcal{L}_{3}=\mathcal{L}_{3}^{N}+\mathcal{L}_{3}^{e}+\mathcal{L}_{3}^{\bar{e}}, \\
\mathcal{L}_{1}^{N} & =\eta+\bar{c}_{\omega}^{N} \omega-\left(\bar{c}_{l} x+\overline{\mathbf{u}}^{N}\right) \cdot \nabla \omega+c_{\omega}^{N} \bar{\omega}-\mathbf{u}^{N} \cdot \nabla \bar{\omega}, \\
\mathcal{L}_{1}^{e} & =c_{\omega}^{e} \bar{\omega}-\mathbf{u}^{e} \cdot \nabla \bar{\omega}, \quad \mathcal{L}_{1}^{\bar{e}}=\bar{c}_{\omega}^{e} \omega-\overline{\mathbf{u}}^{e} \cdot \nabla \omega \\
\mathcal{L}_{2}^{N} & =-\left(\bar{c}_{l} x+\overline{\mathbf{u}}^{N}\right) \cdot \nabla \eta+\left(2 \bar{c}_{\omega}^{N}-\bar{u}_{x}^{N}\right) \eta-\bar{v}_{x}^{N} \xi-\mathbf{u}_{x}^{N} \cdot \nabla \bar{\theta}-\mathbf{u}^{N} \cdot \nabla \bar{\theta}_{x}+2 c_{\omega}^{N} \bar{\theta}_{x}, \\
\mathcal{L}_{2}^{e} & =-\mathbf{u}_{x}^{e} \cdot \nabla \bar{\theta}-\mathbf{u}^{e} \cdot \nabla \bar{\theta}_{x}+2 c_{\omega}^{e} \bar{\theta}_{x}, \quad \mathcal{L}_{2}^{\bar{e}}=-\overline{\mathbf{u}}^{e} \cdot \nabla \eta+\left(2 \bar{c}_{\omega}^{e}-\bar{u}_{x}^{e}\right) \eta-\bar{v}_{x}^{e} \xi \\
\mathcal{L}_{3}^{N} & =-\left(\bar{c}_{l} x+\overline{\mathbf{u}}^{N}\right) \cdot \nabla \xi+\left(2 \bar{c}_{\omega}^{N}-\bar{v}_{y}^{N}\right) \xi-\bar{u}_{y}^{N} \eta-\mathbf{u}_{y}^{N} \cdot \nabla \bar{\theta}-\mathbf{u}^{N} \cdot \nabla \bar{\theta}_{y}+2 c_{\omega}^{N} \bar{\theta}_{y}, \\
\mathcal{L}_{3}^{e} & =-\mathbf{u}_{y}^{e} \cdot \nabla \bar{\theta}-\mathbf{u}^{e} \cdot \nabla \bar{\theta}_{y}+2 c_{\omega}^{e} \bar{\theta}_{y}, \quad \mathcal{L}_{3}^{\bar{e}}=-\overline{\mathbf{u}}^{e} \cdot \nabla \xi+\left(2 \bar{c}_{\omega}^{e}-\bar{v}_{y}^{e}\right) \xi-\bar{u}_{y}^{e} \eta
\end{aligned}
$$

where $\mathcal{L}_{i}^{e}, \mathcal{L}_{i}^{\bar{e}}$ denote the errors from $\psi^{e}, \bar{\psi}^{e}$, respectively. These operators depend on $\omega, \eta, \xi$, and we drop the dependence in (3.12) to simplify the notations.
3.2. First correction and the construction of $\phi^{N}$. According to the normalization condition and (3.3), the solution to (3.1) satisfies $\omega_{x}(0, t)=\eta_{x}(0, t)=0$. To obtain an approximate solution with this condition, we make the first correction

$$
\begin{equation*}
\omega_{k} \rightarrow \omega_{k}-\omega_{k, x}(0,0) \chi_{11}, \quad \eta_{k} \rightarrow \eta_{k}-\eta_{k, x}(0,0) \chi_{21} \tag{3.13}
\end{equation*}
$$

where $\chi_{i j}$ are cutoff functions defined in (3.17) with $\chi_{i j}=x+O\left(|x|^{4}\right)$ near 0 . We do not modify $\xi_{k}$ since $\xi_{k}$ already vanishes quadratically near $(0,0)$. We remark that the first correction does not change the second order derivatives of the solution near 0 and $c_{\omega}$ since

$$
\partial_{x y} \chi_{11}(0)=\partial_{x y} \chi_{21}(0)=0, \quad c_{\omega}\left(\chi_{11}\right)=-\partial_{x y} \phi_{1}(0)=0
$$

where $\phi_{1}$ is defined below

$$
\begin{equation*}
\phi_{1}=-\frac{x y^{2}}{2} \kappa_{*}(x) \kappa_{*}(y) \tag{3.14}
\end{equation*}
$$

where $\kappa_{*}(x)$ is the cutoff function chosen in (D.5) in Appendix D.2 satisfying $\kappa_{*}(x)=1+O\left(|x|^{4}\right)$ near $x=0$, and $\phi_{1}$ satisfies $-\Delta \phi_{1}=x+O\left(|x|^{4}\right)$. For the numeric stream function $\phi_{k, 1}^{N}$ constructed at the beginning of this Section 3, we correct it as follows

$$
\phi_{k, 1}^{N} \rightarrow \phi_{k, 1}^{N}+\partial_{x} \Delta \phi_{k, 1}^{N}(0) \phi_{1} \triangleq \phi_{k}^{N}
$$

Since $\partial_{x} \Delta \phi_{1}(0)=-1$, this allows us to obtain

$$
\begin{align*}
& \partial_{x}(-\Delta) \phi_{k}^{N}(0)=-\partial_{x} \Delta \phi_{k, 1}^{N}(0)+\partial_{x} \Delta \phi_{k, 1}^{N}(0)=0 \\
& \Delta \phi_{k}^{N}=O\left(|x|^{2}\right), \quad \omega_{k}-(-\Delta) \phi_{k}^{N}=O\left(|x|^{2}\right) \tag{3.15}
\end{align*}
$$

We further extend it to Lipschitz continuous solutions $\widehat{W} \triangleq(\hat{\omega}(t), \hat{\eta}(t), \hat{\xi}(t))$ in time using a cubic polynomial interpolation in $t$. See section 3.4 for more details.
3.3. The second correction. The error

$$
\left(\partial_{t}-\mathcal{L}_{i}\right)(\hat{\omega}(t), \hat{\eta}(t), \hat{\xi}(t))
$$

may not vanish to the order $O\left(|x|^{3}\right)$, which is a property that we require in the energy estimate. Then we add the second correction

$$
\hat{\omega}(t) \rightarrow \hat{\omega}(t)+a_{1}(t) \chi_{12}, \quad \hat{\eta} \rightarrow \hat{\eta}+a_{2}(t) \chi_{22}, \quad \hat{\xi}(t) \rightarrow \hat{\xi}(t)+a_{3}(t) \chi_{32}
$$

so that the error satisfies

$$
\begin{equation*}
\varepsilon_{i}^{(2)} \triangleq\left(\partial_{t}-\mathcal{L}_{i}\right)\left(\hat{\omega}(t)+a_{1}(t) \chi_{12}, \hat{\eta}(t)+a_{2}(t) \chi_{22}, \hat{\xi}(t)+a_{3}(t) \chi_{32}\right)=O\left(|x|^{3}\right) \tag{3.16}
\end{equation*}
$$

near $x=0$. We use the following functions for these two corrections

$$
\begin{array}{lll}
\chi_{11}=-\Delta \phi_{1}, & \phi_{1}=-\frac{x y^{2}}{2} \kappa_{*}(x) \kappa_{*}(y), & \chi_{21}=x \kappa_{*}(x) \kappa_{*}(y) \\
\chi_{12}=-\Delta \phi_{2}, & \phi_{2}=-\frac{x y^{3}}{6} \kappa_{*}(x) \kappa_{*}(y), & \chi_{22}=x y \kappa_{*}(x) \kappa_{*}(y), \quad \chi_{32}=\frac{x^{2}}{2} \kappa_{*}(x) \kappa_{*}(y) \tag{3.17}
\end{array}
$$

where $\kappa_{*}(x)$ is chosen in (D.5), $\chi_{\cdot, 1}$ is used for the first correction, and $\chi_{\cdot, 2}$ for the second correction. We do not have $\chi_{31}$ since we do need the first correction for $\xi$ (3.13). Since $\kappa_{*}(x)$ satisfies $\kappa_{*}(x)=1+O\left(|x|^{4}\right)$ near $x=0$, the behaviors of the above functions near $x=0$ are given by

$$
\chi_{11}=y+\text { l.o.t., } \chi_{21}=x+\text { l.o.t., } \chi_{12}=x y+\text { l.o.t., } \chi_{22}=x y+\text { l.o.t., } \chi_{32}=x^{2} / 2+\text { l.o.t. }
$$

We choose $\chi_{1 j}=-\Delta \phi_{j}$ for the correction of $\omega$ so that its associated velocity $\nabla^{\perp}(-\Delta)^{-1} \chi_{1 j}$ can be obtained explicitly. We do not need such form for the correction of $\eta, \xi$ since we do not compute the velocity of $\eta, \xi$.

For cutoff functions $\chi_{1}, \chi_{2}, \chi_{3}$ with

$$
\begin{equation*}
c_{\omega}\left(\chi_{1}\right)=-\partial_{x y}(-\Delta)^{-1} \chi_{1}=0 \tag{3.18}
\end{equation*}
$$

e.g. $\chi_{i}=\chi_{i 2}$ chosen above, we have the following formulas of $\mathcal{L}_{i}\left(a_{1}(t) \chi_{1}, a_{2}(t) \chi_{2}, a_{3}(t) \chi_{3}\right)$ (3.1)
$\mathcal{L}_{1}\left(a_{1} \chi_{1}, a_{2} \chi_{2}, a_{3} \chi_{3}\right)=a_{1}(t)\left(-\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \chi_{1}+\bar{c}_{\omega} \chi_{1}-\mathbf{u}\left(\chi_{1}\right) \cdot \nabla \bar{\omega}\right)+a_{2}(t) \chi_{2}$,
$\mathcal{L}_{2}\left(a_{1} \chi_{1}, a_{2} \chi_{2}, a_{3} \chi_{3}\right)=a_{2}(t)\left(-\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \chi_{2}+\left(2 \bar{c}_{\omega}-\bar{u}_{x}\right) \chi_{2}\right)-a_{3}(t) \bar{v}_{x} \chi_{3}-a_{1}(t)\left(\mathbf{u}\left(\chi_{1}\right) \cdot \nabla \bar{\theta}\right)_{x}$,
$\mathcal{L}_{3}\left(a_{1} \chi_{1}, a_{2} \chi_{2}, a_{3} \chi_{3}\right)=a_{3}(t)\left(-\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \chi_{3}+\left(2 \bar{c}_{\omega}+\bar{u}_{x}\right) \chi_{3}\right)-a_{2}(t) \bar{u}_{y} \chi_{2}-a_{1}(t)\left(\mathbf{u}\left(\chi_{1}\right) \cdot \nabla \bar{\theta}\right)_{y}$,
where $\mathbf{u}\left(\chi_{1}\right)$ is the velocity associated with $\chi_{1}$. We want to apply the above formulas to the second corrections $\chi_{i 2}, i=1,2,3$ in (3.17). We use the Hadamard product

$$
\begin{equation*}
(A \circ B)_{i}=A_{i} B_{i} \tag{3.19}
\end{equation*}
$$

and (3.12) to simplify the notation as follows

$$
\begin{align*}
& \mathcal{L}_{i}(a \circ \chi)=\operatorname{Cor}_{i j}(x ; \chi) a_{j}(t), \quad \operatorname{Cor}_{i j}(x ; \chi)=\operatorname{Cor}_{i j}^{N}(x ; \chi)+\operatorname{Cor}_{i j}^{\bar{e}}(x ; \chi), \\
& \mathcal{L}_{i}^{N}(a \circ \chi) \triangleq \operatorname{Cor}_{i j}^{N}(x ; \chi) a_{j}(t), \quad \mathcal{L}_{i}^{\bar{e}}(a \circ \chi) \triangleq \operatorname{Cor}_{i j}^{\bar{e}}(x ; \chi) a_{j}(t) \tag{3.20}
\end{align*}
$$

Note that $\mathcal{L}_{i}^{e}(a \circ \chi)=0$ since we can obtain $\mathbf{u}\left(\chi_{1}\right)$ explicitly for $\chi_{1}=\chi_{11}, \chi_{12}$ (3.17).
Next, we derive the equations for $a_{i}(t), i=1,2,3$. Using (3.1) and the condition

$$
\partial_{x y} \varepsilon_{1}^{(2)}(0)=\partial_{x y} \varepsilon_{2}^{(2)}(0)=\partial_{x x} \varepsilon_{3}^{(2)}(0)=0
$$

from (3.16), we obtain the following ODEs for $a(t), b(t), c(t)$

$$
\begin{align*}
\dot{a}_{1}(t) & =\left(-2 \bar{c}_{l}+\bar{c}_{\omega}\right) a_{1}(t)+a_{2}(t)-F_{1}(t), \\
\dot{a}_{2}(t) & =\left(-2 \bar{c}_{l}+2 \bar{c}_{\omega}-\bar{u}_{x}(0)\right) a_{2}(t)-F_{2}(t),  \tag{3.21}\\
\dot{a}_{3}(t) & =\left(-2 \bar{c}_{l}+2 \bar{c}_{\omega}-\bar{u}_{x}(0)\right) a_{3}(t)-F_{3}(t),
\end{align*}
$$

where $F(t)=\left(F_{1}(t), F_{2}(t), F_{3}(t)\right)^{T}$ is the error associated to the second order derivatives of $\left(\partial_{t}-\mathcal{L}\right) \hat{W}$ near 0 . More precisely, we have

$$
\begin{align*}
& F_{1}(t)=\partial_{x y}\left(\partial_{t}-\mathcal{L}_{1}\right) \widehat{W}(0)=\frac{d}{d t} \hat{\omega}_{x y}(t, 0)-\left(-2 \bar{c}_{l}+\bar{c}_{\omega}\right) \hat{\omega}_{x y}(t, 0)-\hat{\eta}_{x y}(t, 0)-c_{\omega}(t) \bar{\omega}_{x y}(0)  \tag{3.22}\\
& F_{2}(t)=\partial_{x y}\left(\partial_{t}-\mathcal{L}_{2}\right) \widehat{W}(0)=\frac{d}{d t} \hat{\eta}_{x y}(t, 0)-\left(-2 \bar{c}_{l}+2 \bar{c}_{\omega}-\bar{u}_{x}(0)\right) \hat{\eta}_{x y}(t, 0)-c_{\omega}(t) \bar{\theta}_{x x y}(0) \\
& F_{3}(t)=\partial_{x}^{2}\left(\partial_{t}-\mathcal{L}_{3}\right) \widehat{W}(0)=\frac{d}{d t} \hat{\xi}_{x x}(t, 0)-\left(-2 \bar{c}_{l}+2 \bar{c}_{\omega}-\bar{u}_{x}(0)\right) \hat{\xi}_{x x}(t, 0)-c_{\omega}(t) \bar{\theta}_{x x y}(0)
\end{align*}
$$

Denote $D^{2}=\left(\partial_{x y}, \partial_{x y}, \partial_{x}^{2}\right)^{T}$. Then we can simplify (3.22) as

$$
\begin{equation*}
F_{i}=D_{i}^{2}\left(\partial_{t}-\mathcal{L}_{i}\right) \hat{W}(0)=D_{i}^{2}\left(\partial_{t}-\mathcal{L}_{i}^{N}-\mathcal{L}_{i}^{e}-\mathcal{L}_{i}^{\bar{e}}\right) \hat{W}(0) \tag{3.23}
\end{equation*}
$$

Denote by $M$ the coefficients in (3.21)

$$
M=\left(\begin{array}{ccc}
-2 \bar{c}_{l}+\bar{c}_{\omega} & 1 & 0  \tag{3.24}\\
0 & -2 \bar{c}_{l}+2 \bar{c}_{\omega}-\bar{u}_{x}(0) & 0 \\
0 & 0 & -2 \bar{c}_{l}+2 \bar{c}_{\omega}-\bar{u}_{x}(0)
\end{array}\right) \triangleq M^{N}+M^{\bar{e}}
$$

where the last identity is based on the decomposition $\bar{c}_{\omega}=\bar{c}_{\omega}^{N}+\bar{c}_{\omega}^{e}, \bar{u}_{x}(0)=\bar{u}_{x}^{N}(0)+\bar{u}_{x}^{e}(0)$, and $M^{\bar{e}}$ only contains the contribution from $\bar{c}_{\omega}^{e}, \bar{u}_{x}^{\bar{\varepsilon}}(0)$. According to the normalization condition (3.2), we have $\bar{u}_{x}(0)^{e}=\bar{c}_{\omega}^{e}$. It follows

$$
\begin{equation*}
M^{\bar{e}}=\bar{c}_{\omega}^{e} I_{3} \tag{3.25}
\end{equation*}
$$

We simplify the ODE for $a=\left(a_{1}, a_{2}, a_{3}\right)^{T}$ as

$$
\begin{equation*}
\dot{a}_{i}(t)=M_{i j} a_{j}(t)-F_{i}(t), \quad \dot{a}(t)=M a-F=M a-e_{i} D_{i}^{2}\left(\partial_{t}-\mathcal{L}_{i}\right) \hat{W}(0) \tag{3.26}
\end{equation*}
$$

Recall $\chi_{\cdot 2}=\left(\chi_{12}, \chi_{22}, \chi_{32}\right)$ from (3.17). In the $i-t h$ equation, the overall error for the approximate solution $\widehat{W}+a(t) \circ \chi \cdot 2^{2}$ is

$$
\begin{align*}
& \left(\partial_{t}-\mathcal{L}_{i}\right)\left(\widehat{W}+a(t) \circ \chi_{\cdot 2}\right)=\left(\partial_{t}-\mathcal{L}_{i}^{N}\right)\left(a(t) \circ \chi_{\cdot 2}\right)+\left(\left(\partial_{t}-\mathcal{L}_{i}^{N}\right) \widehat{W}\right.  \tag{3.27}\\
& \left.-\mathcal{L}_{i}^{e}\left(\widehat{W}+a(t) \circ \chi_{\cdot 2}\right)-\mathcal{L}_{i}^{\bar{e}}\left(\widehat{W}+a(t) \circ \chi_{\cdot 2}\right)\right) \triangleq J+I
\end{align*}
$$

Note that in the above notation, $\partial_{t}$ acts on $a_{i}(t) \chi_{i, 2}$. For $J$, using the ODE for $a(t)$ (3.26), (3.20), (3.23), and (3.24), we get

$$
\begin{aligned}
J & =\left(M_{i j} a_{j}-F_{i}\right) \chi_{i 2}-\operatorname{Cor}_{i j}^{N}\left(x ; \chi_{\cdot 2}\right) a_{j} \\
& =\left(M_{i j}^{N} \chi_{i 2}-\operatorname{Cor}_{i j}^{N}\left(x ; \chi_{\cdot 2}\right)\right) a_{j}+M_{i j}^{\bar{e}} a_{j} \chi_{i 2}-D_{i}^{2}\left(\partial_{t}-\mathcal{L}_{i}^{N}-\mathcal{L}_{i}^{e}-\mathcal{L}_{i}^{\bar{e}}\right) \widehat{W}(0) \chi_{i 2} \triangleq J_{1}+J_{2}+J_{3}
\end{aligned}
$$

where we have summation over $j=1,2,3$. Since $\mathcal{L}^{e}\left(a(t) \circ \chi_{\cdot 2}\right)=0$, using the above decomposition and combining $I, J_{2}, J_{3}$, we yield

$$
\begin{align*}
I+J_{2}+J_{3}= & \left(\left(\partial_{t}-\mathcal{L}_{i}^{N}\right) \widehat{W}-D_{i}^{2}\left(\partial_{t}-\mathcal{L}_{i}^{N}\right) \widehat{W}(0) \chi_{i 2}\right)-\left(\mathcal{L}_{i}^{e} \widehat{W}-D_{i}^{2} \mathcal{L}_{i}^{e} \widehat{W}(0) \chi_{i 2}\right)  \tag{3.28}\\
& -\left(\mathcal{L}_{i}^{\bar{e}}\left(\widehat{W}+a(t) \circ \chi_{\cdot 2}\right)-D_{i}^{2} \mathcal{L}_{i}^{e} \widehat{W}(0) \chi_{i 2}-M_{i j}^{\bar{e}} a_{j} \chi_{i 2}\right) \triangleq I_{i, N}+I_{i, e}+I_{i, \bar{e}}
\end{align*}
$$

Next, we check that $J_{1}, I_{i, N}, I_{i, e}, I_{i, \bar{e}}$ have a vanishing order $O\left(|x|^{3}\right)$. This is clear for $I_{i, N}, I_{i, e}$. Since we correct the second order derivatives and $\hat{\omega}, \hat{\eta}, \hat{\zeta}$ are odd with $\hat{\xi}=x \hat{\zeta}$, we get $\partial_{x}^{i} \partial_{y}^{j} I_{i, N}$, $\partial_{x}^{i} \partial_{y}^{j} I_{i, e}=0, i+j \leq 2$ at the origin. For $J_{1}$, we note that it is a linear combination of $a_{j}$ with given coefficients $M_{i j}^{N}-\operatorname{Cor}_{i j}^{N}$. Its cubic vanishing order follows from the definition. For example, when $i=j=1$, we have
$S=a_{1}(t) \cdot\left(\operatorname{Cor}_{11}^{\bar{e}}(x)-M_{11}^{\bar{e}} \chi_{12}\right)=a_{1}(t)\left(-\overline{\mathbf{u}}^{e} \cdot \nabla \chi_{12}+\bar{c}_{\omega}^{e} \chi_{12}-\bar{c}_{\omega}^{e} \chi_{12}\right)=a_{1}(t)\left(-\overline{\mathbf{u}}^{e} \cdot \nabla \chi_{12}\right)$.
Since $\chi_{12}=x y+O\left(|x|^{4}\right)$ (3.17), $\bar{u}^{e}=\bar{u}_{x}^{e}(0) x+O\left(|x|^{2}\right), \bar{v}^{e}=-\bar{u}_{x}^{e}(0) y$ near 0 , we have $S=O\left(|x|^{3}\right)$ near 0 . The vanishing order of other terms in $J_{1}$ can be obtained similarly. Then for $J_{1}$, we estimate the weighted norm for $\operatorname{Cor}_{i j}^{\bar{e}}(x)-M_{i j}^{\bar{e}} \chi_{i 2}$ and then apply the triangle inequality to further bound $J_{1}$. Similarly, for a fixed $i$, we have the following vanishing order

$$
\mathcal{L}_{i}^{\bar{e}}\left(a(t) \circ \chi_{\cdot 2}\right)-M_{i j}^{\bar{e}} a_{j} \chi_{i 2}=\operatorname{Cor}_{i j}^{\bar{e}} a_{j}(t)-M_{i j}^{\bar{e}} a_{j} \chi_{i 2}=O\left(|x|^{3}\right), \quad D_{i}^{2} \mathcal{L}_{i}^{\bar{e}}\left(a(t) \circ \chi_{\cdot 2}\right)(0)=M_{i j}^{\bar{e}} a_{j}
$$

Thus, we can rewrite $I_{i, \bar{e}}$ as follows

$$
\begin{equation*}
I_{i, \bar{e}}=-\left(\mathcal{L}_{i}^{\bar{e}}\left(\widehat{W}+a(t) \circ \chi_{\cdot 2}\right)-D_{i}^{2} \mathcal{L}_{i}^{\bar{e}}\left(\widehat{W}+a(t) \circ \chi_{\cdot 2}\right)(0) \chi_{i 2}\right) \tag{3.29}
\end{equation*}
$$

which clearly has a cubic vanishing order. Note that $\widehat{W}+a(t) \circ \chi_{\cdot 2}$ is our final approximate solution for solving (3.1).

In summary, to estimate the error $\left(\partial_{t}-\mathcal{L}\right)\left(\widehat{W}+a \circ \chi_{\cdot 2}\right)$, we will estimate $J_{1}, I_{i, N}, I_{i, e}, I_{i, \bar{e}}$ separately. The term $I_{i, N}$ is the local error of solving (3.1) numerically, $I_{i, e}, I_{i, \bar{e}}$ are due to the error of solving the Poisson equations for $\omega$ and $\widehat{\omega}$. Since we use a cubic polynomial interpolation to obtain the continuous function $\hat{W}(t)$, the errors $I_{i, N}, I_{i, e}$ are piecewise cubic polynomials in time, and we track the coefficients of these polynomials to verify that they are small. We discuss the estimate of nonlocal error in Section 3.7.
3.4. Cubic interpolation in time. Given the numerical solution with the first correction $\widehat{W}_{n}=\left(\hat{\omega}_{n}, \hat{\eta}_{n}, \hat{\xi}_{n}\right)$, we use a piecewise cubic interpolation to construct $\widehat{W}(t, x)$ over $(t, x) \in$ $[0, T] \times \mathbb{R}_{2}^{+}$. We partition the whole time interval $[0, T]$ into small subintervals $[3 m k, 3(m+1) k]$ with length $3 k$. For $s \in[-3 k / 2,3 k / 2]$ and $t_{m}=3 m k$, we construct

$$
\begin{aligned}
W\left(s+t_{m}+\frac{3 k}{2}\right)= & \frac{1}{16}\left(-W_{0}+9 W_{1}+9 W_{2}-W_{3}\right)+\frac{1}{24}\left(W_{0}-27 W_{1}+27 W_{2}-W_{3}\right) \frac{s}{k} \\
& +\frac{1}{4}\left(W_{0}-W_{1}-W_{2}+W_{3}\right)\left(\frac{s}{k}\right)^{2}+\frac{1}{6}\left(-W_{0}+3 W_{1}-3 W_{2}+W_{3}\right)\left(\frac{s}{k}\right)^{3} \\
\triangleq & \sum_{i \leq 3} C_{i} \cdot V \frac{1}{i!}\left(\frac{s}{k}\right)^{i}, \quad V=\left(W_{0}, W_{1}, W_{2}, W_{3}\right)
\end{aligned}
$$

where $k$ is the time step, $W_{i}=\hat{W}_{n+i}$ for $t_{n}=n k$, and $C_{i} \in \mathbb{R}^{4}$ is the coefficient determined by the interpolation formula. A direct calculation yields

$$
\begin{aligned}
\partial_{t} \widehat{W}-\mathcal{L} \widehat{W} & =\sum_{1 \leq i \leq 3} \frac{C_{i} \cdot V}{k} \frac{1}{(i-1)!}\left(\frac{s}{k}\right)^{i-1}-\sum_{i \leq 3} \mathcal{L}\left(C_{i} \cdot V\right) \frac{1}{i!}\left(\frac{s}{k}\right)^{i} \\
& =\sum_{i \leq 2}\left(\frac{C_{i+1} \cdot V}{k}-\mathcal{L}\left(C_{i} \cdot V\right)\right) \frac{1}{i!}\left(\frac{s}{k}\right)^{i}-\mathcal{L}\left(C_{4} \cdot V\right) \frac{s^{3}}{6 k^{3}} .
\end{aligned}
$$

To estimate $\partial_{t} \widehat{W}-\mathcal{L} \widehat{W}$, we will use the triangle inequality and estimate $\frac{C_{i+1} \cdot V}{k}-\mathcal{L}\left(C_{i} \cdot V\right), \mathcal{L}\left(C_{4}\right.$. $W$ ) rigorously using the methods in Section 3.6, 3.7.

Applying the triangle inequality and integrating the error over $s \in\left[-\frac{3 k}{2}, \frac{3 k}{2}\right]$ yield

$$
\begin{align*}
& \int_{|s| \leq 3 k / 2}\left|\partial_{t} W-\mathcal{L} W\right| d s \leq \sum_{i \leq 2}\left|\frac{C_{i+1} \cdot V}{k}-\mathcal{L}\left(C_{i} \cdot V\right)\right| \int_{|s| \leq 3 k / 2} \frac{1}{i!}\left|\frac{s}{k}\right|^{i} \\
& +\left|\mathcal{L}\left(C_{4} \cdot V\right)\right| \int_{|s| \leq \frac{3 k}{2}} \frac{1}{6}\left|\frac{s}{k}\right|^{3}=k\left(\sum_{i \leq 2}\left|\frac{C_{i+1} \cdot V}{k}-\mathcal{L}\left(C_{i} \cdot V\right)\right| C_{I}(i)+\left|\mathcal{L}\left(C_{4} \cdot V\right)\right| C_{I}(3)\right), \tag{3.30}
\end{align*}
$$

where

$$
C_{I}=\left[3, \frac{9}{4}, \frac{9}{8}, \frac{27}{64}\right]
$$

3.4.1. Decomposing the time interval for parallel computing. To verify that the posteriori error is small, we need to estimate the error rigorously at each time step, which takes a significant amount of time. Consider a partition of the time interval $0=T_{0}<T_{1}<. .<T_{n}=T$, where $T$ is the final time of the computation. To reduce the computational time, we first solve the equations on $[0, T]$ without any rigorously verification and save the solution $\left(\omega_{k}, \eta_{k}, \xi_{k}, \phi_{k, 1}^{N}\right)$ at $t_{k}=T_{i}$. Since we do not need to perform verification at this step, the running time for each time step is short. Then we solve the equations on a smaller time interval $\left[T_{i}, T_{i+1}\right], i=0,1,2 \ldots, n-1$ from the initial data $W\left(T_{i}\right)$ and then perform the verification in each time interval in parallel.
3.5. Compactly supported in time. To construct an approximate solution, we do not need to solve the linearized equations (3.1) for all time. In fact, since the solution decays in certain norm as $t$ increases, we stop the computation at time $T$ if $\hat{W}-D^{2} \hat{W} \circ \chi$ is small in the energy norm. Then we extend $\hat{W}(t, \cdot)$ trivially for $t>T$

$$
\widehat{W}(t, \cdot)=0, \quad t>T
$$

As a result, the error satisfies

$$
\mathcal{R}_{i}=\left(\partial_{t}-\mathcal{L}_{i}\right) \widehat{W}=\left(\partial_{t}-\mathcal{L}_{i}\right) \widehat{W} \mathbf{1}_{t \leq T}-\delta_{T} \widehat{W}_{i}(T)
$$

Let $F=\left(F_{1}, F_{2}, F_{3}\right), F_{i}=\left.D_{i}^{2}\left(\partial_{t}-\mathcal{L}_{i}\right) \widehat{W}\right|_{x=0}$ for $t \leq T$, where $D^{2}=\left(D_{x y}, D_{x y}, D_{x}^{2}\right)$. Then similarly, we get

$$
\left.F_{e x t} \triangleq D^{2}\left(\partial_{t}-\mathcal{L}\right) \widehat{W}\right|_{x=0} \cdot \mathbf{1}_{t \leq T}-D^{2} \widehat{W}(T, 0) \delta_{T}=F(t) \mathbf{1}_{t \leq T}-F(T) \delta_{T}
$$

We will test the above formulas with some Lipschitz function in time and the above formulas are well defined. Recall that the coefficients of the second correction a satisfy (3.26). Although $\hat{W}$ only has finite support in time, to achieve the vanishing order (3.16) for all time, we need to solve the ODE exactly for all time. If we stop solving the ODE at time $T$, we cannot achieve (3.16) at time $T$. Moreover, we cannot solve the ODE using a numerical method, e.g. the RungeKutta method, since it leads to an error. Instead, we solve the ODE exactly by diagonalizing the system. We introduce the following notations

$$
\begin{aligned}
& \lambda_{1}=-2 \bar{c}_{l}+\bar{c}_{\omega}, \quad \lambda_{2}=\lambda_{3}=-2 \bar{c}_{l}+2 \bar{c}_{\omega}-\bar{u}_{x}(0) \\
& \tilde{a}_{1}=a_{1}+\frac{a_{2}}{\lambda_{1}-\lambda_{2}}, \quad \tilde{F}_{1}=F_{1}+\frac{F_{2}}{\lambda_{1}-\lambda_{2}}, \quad \tilde{a}_{i}=a_{i}, \quad \tilde{F}_{i}=F_{i}, i=2,3
\end{aligned}
$$

and similar notations for $\tilde{F}_{\text {ext }}$. The coefficients satisfy $\lambda_{1} \approx-7, \lambda_{2}=\lambda_{3} \approx-5.5$. We diagonalize (3.21) as follows

$$
\frac{d}{d t} \tilde{a}_{i}=\lambda_{i} \tilde{a}_{i}-\tilde{F}_{e x t, i}
$$

Using Duhamel's formula and the definition of $\tilde{F}_{\text {ext }, i}$, we yield

$$
\begin{align*}
\tilde{a}_{j}(t) & =e^{\lambda_{j} t} \tilde{a}_{j}(0)-\int_{0}^{t} e^{\lambda_{j}(t-s)} \tilde{F}_{e x t, j}(s) d s  \tag{3.31}\\
& =e^{\lambda_{j} t} \tilde{a}_{j}(0)-\int_{0}^{t \wedge T} e^{\lambda_{j}(t-s)} \tilde{F}_{j}(s) d s+\tilde{F}(T) e^{\lambda_{j}(t-T)} \mathbf{1}_{t \geq T} \triangleq S_{1}+S_{2}+S_{3}
\end{align*}
$$

With the second correction, the above extension and the decomposition of error (3.27)-(3.28), the residual error for rank-one perturbation (3.7) with $n=1$ is given by

$$
\begin{align*}
& \mathcal{R}=c(t)\left(\widehat{W}_{0}+a_{0} \circ \chi_{\cdot 2}-\bar{W}_{0}\right)+\int_{0}^{t} c(t-s)\left(\partial_{t}-\mathcal{L}\right)\left(\widehat{W}+a \circ \chi_{\cdot 2}\right) d s=\mathcal{R}_{l o c}+\mathcal{R}_{n l o c} \\
& \mathcal{R}_{l o c, 0, \cdot}=c(t)\left(\widehat{W}_{0}+a_{0} \circ \chi_{\cdot 2}-\bar{W}_{0}\right)-\left(\widehat{W}(T)-D^{2} \widehat{W}(T) \circ \chi_{\cdot 2}\right) c(t-T) \\
& \left.\left.+\int_{0}^{t \wedge T} c(t-s) \sum_{i \leq 3} e_{i} I_{i, N}(s)\right) d s+\int_{0}^{t} c(t-s) \sum_{i \leq 3} e_{i} J_{1, i}(s)\right) d s=\int_{0}^{t} c(t-s) \mathcal{R}_{n u m}(s) d s,  \tag{3.32}\\
& \mathcal{R}_{n l o c}=\int_{0}^{t \wedge T} c(t-s) \sum_{1 \leq i \leq 3} e_{i} I_{i, e}(s) d s+\int_{0}^{t} c(t-s) \sum_{i \leq 3} e_{i} I_{i, \bar{e}}(s) d s
\end{align*}
$$

where $I_{i, N}, I_{i, e}, I_{i, \bar{e}}$ are given in (3.28), $J_{1, i}$ means $J_{1}$ (3.28) in $i$-th equation, and $\mathcal{R}_{n u m}$ is (3.33)

$$
\mathcal{R}_{n u m}(s) \triangleq \delta_{0} \cdot\left(\widehat{W}_{0}+a_{0} \circ \chi_{\cdot 2}-\bar{W}_{0}\right)-\delta_{T} \cdot\left(\widehat{W}(T)-D^{2} \widehat{W}(T) \circ \chi_{\cdot 2}\right)+e_{j}\left(\mathbf{1}_{t \leq T} I_{j, N}+J_{1, j}\right)
$$

We obtain the local part in (3.8) when $n=1$. The first term is the initial interpolation error for $\bar{W}_{0}$, and we choose $a_{0} \in \mathbb{R}^{3}$ to achieve vanishing order $\widehat{W}_{0}+a_{0} \circ \chi-\bar{W}_{0}=O\left(|x|^{3}\right)$. We use $\mathcal{R}_{l o c, 0, .}, \mathcal{R}_{n l o c}$ to denote the error that depends on the solution locally and nonlocally. We use the bootstrap assumption to obtain uniform control of $c(t)$ in $t$. See Section 5.7 in Part I [13]. The error estimate of the local part $\mathcal{R}_{\text {loc,0, }}$. follows Section 3.6. Moreover, we extract the essentially local part from $\mathcal{R}_{n l o c}$ and can estimate it with $\mathcal{R}_{l o c, 0, j}$ together (3.35). We decompose the nonlocal part $\mathcal{R}_{n l o c}$ in Section 3.7. To control the terms involving $a_{i}$, e.g. $J_{1, i}$ above (3.28), we can estimate the weighted norm of the functions $\operatorname{Cor}_{i j}^{N}(x)-M_{i j}^{N} \chi_{i 2}$ and then only need to estimate the integral of $\tilde{a}_{j}$.

Denote $x \wedge y \triangleq \min (x, y)$. Since the factor $\lambda_{j}<0$, using the formula of $\tilde{a}_{j}$ (3.31), we obtain

$$
\begin{aligned}
& \int_{0}^{\infty}\left|S_{1}\right| d t=\frac{1}{\left|\lambda_{j}\right|}\left|\tilde{a}_{j}(0)\right|, \quad \int_{0}^{\infty}\left|S_{3}(t)\right| d t=\int_{T}^{\infty}\left|\tilde{F}_{j}(T)\right| e^{\lambda_{j}(t-T)} d t=\frac{1}{\left|\lambda_{j}\right|}\left|\tilde{F}_{j}(T)\right| \\
& \int_{0}^{\infty}\left|S_{2}(t)\right| d t \leq \int_{0}^{\infty}\left(\int_{0}^{t \wedge T} e^{\lambda_{j}(t-s)}\left|\tilde{F}_{j}(s)\right| d s\right) d t=\int_{0}^{T}\left|\tilde{F}_{j}(s)\right|\left(\int_{s}^{\infty} e^{\lambda_{j}(t-s)} d t\right) d s=\frac{1}{\left|\lambda_{j}\right|} \int_{0}^{T}\left|\tilde{F}_{j}(s)\right| .
\end{aligned}
$$

It follows

$$
\int_{0}^{\infty}\left|\tilde{a}_{j}(t)\right| d t \leq \frac{1}{\left|\lambda_{j}\right|}\left(\left|\tilde{a}_{j}(0)\right|+\int_{0}^{T}\left|\tilde{F}_{j}(s)\right| d s+\left|\tilde{F}_{j}(T)\right|\right)
$$

Since $\hat{W}, \hat{F}, \tilde{F}(3.22)$ are cubic in time, we can estimate the above integrals of $\tilde{F}_{j}$ following (3.30). Using the linear relation between $a_{j}, \tilde{a}_{j}$, we can estimate $a_{j}$.

Using the above estimates, we can represent the rank-one solution and estimate it as follows

$$
\begin{align*}
\hat{G}(t, x) & =\int_{0}^{t} c(t-s)\left(\hat{W}+a \circ \chi_{l 2}\right)(s) d s \\
\left|\partial_{x}^{i} \partial_{y}^{j} G_{l}(t, x)\right| & \leq \sup _{t>0}|c(t)|\left(\int_{0}^{T}\left|\partial_{x}^{i} \partial_{y}^{j} \hat{W}_{l}(t)\right| d t+\left|\partial_{x}^{i} \partial_{y}^{j} \chi_{j 2}\right| \int_{0}^{\infty}\left|a_{j}(t)\right| d t\right) \tag{3.34}
\end{align*}
$$

Similarly, we can bound other quantities for $\hat{G}$ and complete the estimates in (3.6).
We generalize the above formula and estimate directly to the finite rank perturbation operator using linearity. For different initial data $\bar{W}_{0}$ related to the finite rank perturbation, we choose a different stopping time $T\left(\widehat{W}_{0}\right)$ to save computation cost. In practice, we construct the numerical solution up to time $T\left(\widehat{W}_{0}\right) \leq T=12$. At that time, the solution $\widehat{W}(T)$ is very small, which can be treated as a small perturbation. See figures in Section 4.3 in Part I [13].
Remark 3.1. Using linearity and the triangle inequality, we can assemble the estimates for $\mathcal{R}$ (3.7) from the estimates of each mode $\hat{W}_{i}$ in (3.6), (3.7). In practice, this means that we can implement the above estimate for each individual mode completely in parallel.

Finite support of the $c_{\omega}\left(\hat{W}_{2}\right)$ term. In Section 5 of Part I [13], we need to use $c_{\omega}(f)$, where $c_{\omega}(f)=u_{x}(f)(0)=-\partial_{x y}(-\Delta)^{-1} f(0)$. Since we choose the cutoff function $\chi_{12}$ for the second correction of $\hat{\omega}$ with properties (3.17), (3.18), we get

$$
c_{\omega}\left(\widehat{W}_{1}+a_{1}(t) \chi_{12}\right)=c_{\omega}\left(\widehat{W}_{1}\right)
$$

and it is supported in $[0, T]$.
3.6. Ideas of estimating the norm of the error. In this section, we discuss how to estimate the error derived in the previous section, e.g. $I_{i, N}$ (3.28), a-posteriori. The general idea is to first evaluate $f$ on some grid points and estimate the higher order derivatives of $f$ in a domain $D$. Then we can construct an approximation $\hat{f}$ of $f$ by interpolating the values of $f$ at different points. The approximation error $f-\hat{f}$ can be bounded by $C_{k}\|f\|_{C^{k}} h^{k}$, where $h$ measures the size of the domain. If the mesh $h$ is sufficiently small, the error term is small. See a simple second order error estimate in (C.11).

To develop an efficient method for rigorous estimates, we have the following considerations. Firstly, we should evaluate as a small number of points as possible so that the method is efficient. Secondly, most functions $f$ in the verification are complicated, e.g. $I_{i, N}$ (3.28), and it is difficult to obtain the sharp bound of the higher derivatives. Instead, we first estimate the piecewise derivatives of some simple functions, e.g. piecewise polynomials $(\hat{\omega}, \hat{\eta})$ or semi-analytic solutions following Appendix C, D. Then we use the triangle inequality and the Leibniz rule to estimate the products of these simple functions, and their linear combinations. Yet, in general, this approach overestimates the derivatives significantly. To compensate the overestimates, we use higher order interpolations and estimates with error bounds $C h^{k}, k=3,4,5$, which provide the small factor $h^{k}$. We develop three estimates based on different interpolations: the Newton interpolation, the Lagrangian interpolation, and the Hermite interpolation in Section 8 in the supplementary material II [11 (contained in this paper). The 1D interpolating polynomials are standard, and we generalize them to construct 2D interpolating polynomials.

We want to estimate the constant $C$ in the error bound $C h^{k}$ as sharp as possible to reduce the computational cost and improve the efficiency. In fact, when $k=4$, if we can obtain an interpolation method and reduce the constant $C$ to $\frac{C}{16}$, to achieve the same level of error, we can increase $h$ to $2 h$. In this verification step, since the domain is 2D, it means that we can evaluate only $\frac{1}{4}$ of the grid point values of $f$, which can reduce the computational cost by $75 \%$.

Using the above method, we can obtain a sharp estimate of the derivatives of $f$. Using the method in Section 8 in the supplementary material II [11] and Taylor expansion, we can further estimate the weighted norm of $f$ with a singular weight near 0 . We discuss the estimate of the nonlocal error in Section 3.7. Using these $L^{\infty}$ estimates of $f$ and its derivatives, we can further develop Hölder estimate for $f$. See Section E.1. We remark that the numerical solutions are regular, e.g. the approximate steady state and the solutions to the linearized equations are $C^{4,1}$. We use these methods to estimate piecewise $L^{\infty}\left(\varphi_{\text {evo, } i}\right)$ norm of the local residual error $\mathcal{R}_{n u m, i}$ (3.33) and the $C_{x_{i}}^{1 / 2}$ partial Hölder seminorm of $\mathcal{R}_{n u m, i} \psi_{i}$, where $\varphi_{\text {evo }, i}, \psi_{i}$ are defined in (A.3).

We remark that the weights $\varphi_{\text {evo }, i}$ and $\varphi_{i}, i=2,3$ in the $L^{\infty}$ energy estimate (see Section 5 in [13]) for $\eta, \xi$ are similar but with different coefficient $p_{5, .}, p_{6, .}$. Since $\varphi_{i}$ and $\varphi_{\text {evo }, i}$ are equivalent, we can obtain piecewise weighted $L^{\infty}\left(\varphi_{i}\right)$ estimate of the error by estimating the ratio $\varphi_{i} / \varphi_{\text {evo }, i}$. Similarly, we can obtain weighted $L^{\infty}\left(\varphi_{g, i}\right)$ estimate of the error, where $\varphi_{g, i}$ is another weight in the energy estimate in Section 5 in [13].

Estimate the local part of the residual error. Using the above methods, we can estimate the local part of the residual error $\overline{\mathcal{F}}_{i}$ for the approximate steady state and discuss the estimate in Appendix C.4 We further extract the local part of $\mathcal{R}_{n l o c}$ (3.32), which has the form (3.8) obtained in Section 3.7 and combine it with $\mathcal{R}_{l o c, 0, j}$ to get the essentially local residual error

$$
\begin{align*}
\mathcal{R}_{l o c, i} & =\mathcal{R}_{l o c, 0, i}+\mathcal{R}_{d i f, i}+M, \quad \mathcal{R}_{d i f, i} \triangleq D_{i}^{2} \mathcal{B}_{o p, i}(\mathbf{u}(\bar{\varepsilon}), \hat{G})(0) \cdot\left(\chi_{i 2}-f_{\chi, i}\right)  \tag{3.35}\\
M & \triangleq \mathcal{B}_{o p, i}(\mathbf{u}(\hat{\varepsilon}), \bar{W})-D_{i}^{2} \mathcal{B}_{o p, i}(\mathbf{u}(\hat{\varepsilon}), \bar{W})(0) \chi_{i 2}-\mathcal{B}_{o p, i}\left(\mathbf{u}_{A}\left(\hat{\varepsilon}_{1}\right),(\nabla \mathbf{u})_{A}\left(\hat{\varepsilon}_{1}\right), \bar{W}\right)
\end{align*}
$$

where $\chi_{i 2}$ is defined in (D.6). By definition (3.36) and following derivation of (3.23), we get

$$
D_{i}^{2} \mathcal{B}_{o p, i}(\mathbf{u}(\bar{\varepsilon}), \hat{G})(0)=u_{x}(\bar{\varepsilon})(0) V_{i}, \quad V=\left(\hat{G}_{1, x y}(0), \hat{G}_{2, x y}(0), \hat{G}_{3, x x}(0)\right)
$$

To estimate each term, we follow Section 3.6 and Appendix C.4. The term $I_{i, N}$ in $\mathcal{R}_{\text {loc }, 0, i}$ (3.28), (3.32) is similar to $I I_{i}^{N}$, and $M$ has the same form as $I I_{i}\left(\bar{\varepsilon}_{1}\right)+I I_{i}\left(\bar{\varepsilon}_{2}\right)$ in Appendix C.4

$$
M=I I_{i}\left(\hat{\varepsilon}_{1}\right)+I I_{i}\left(\hat{\varepsilon}_{2}\right), I I_{i}\left(\bar{\varepsilon}_{1}\right)=\mathcal{B}_{o p, i}\left(\hat{\mathbf{u}}\left(\hat{\varepsilon}_{1}\right), \widehat{\nabla \mathbf{u}}\left(\hat{\varepsilon}_{1}\right), \bar{W}\right), I I_{i}\left(\hat{\varepsilon}_{2}\right)=\mathcal{B}_{o p, i}\left(\mathbf{u}\left(\hat{\varepsilon}_{2}\right), \nabla \mathbf{u}\left(\hat{\varepsilon}_{2}\right), \bar{W}\right)
$$

Here, we perform the decomposition (C.17) with ( $\bar{\varepsilon}, \chi_{\bar{\varepsilon}}$ ) replacing by ( $\hat{\varepsilon}, \chi_{\hat{\varepsilon}}$ ), where $\chi_{\hat{e}}$ is defined in (D.6). Then the estimate of $\mathcal{R}_{l o c, j}$ is similar to that in Appendix C.4. See Section 5.8 in [13] for more discussion of the above forms.

Error for the initial data and at stopping time. The error $\widehat{W}(T)-D^{2} \widehat{W}(t) \circ \chi_{\cdot 2}$ at the stopping time has compact support and its estimate follows the methods in Section 3.6. To bound the initial interpolation error $e r r_{i n} \triangleq \widehat{W}_{0}+a_{0} \circ \chi_{\cdot 2}-\bar{W}_{0}$ (3.32) in a large domain, we follow similar methods. The error involves $\bar{\omega}, \bar{\theta}$ which are supported globally. To bound $\mathrm{err}_{i n}$ in the middle and far-field, since $\hat{W}_{0}+a_{0} \circ \chi_{i, 2}=0$, combining all the initial data from the finite rank perturbation (see Section C.2.1 of Part I [13]), we need to estimate

$$
\begin{aligned}
& I_{1}=c_{\omega}\left(\omega_{1}\right) \bar{\omega}-\hat{\mathbf{u}}\left(\omega_{1}\right) \cdot \nabla \bar{\omega}, \quad I_{2}=2 c_{\omega}\left(\omega_{1}\right) \bar{\theta}_{x}-\hat{\mathbf{u}} \cdot \nabla \bar{\theta}_{x}-\hat{\mathbf{u}}_{x} \cdot \nabla \bar{\theta} \\
& I_{3}=2 c_{\omega}\left(\omega_{1}\right) \bar{\theta}_{y}-\hat{\mathbf{u}} \cdot \nabla \bar{\theta}_{y}-\hat{\mathbf{u}}_{y} \cdot \nabla \bar{\theta}
\end{aligned}
$$

for large $|x|$. The approximation terms near 0 defined in Section 4.2.1 of Part I 13 are supported near 0 and decay to zero as $|x| \rightarrow \infty$. In the far-field, $\hat{\mathbf{u}}\left(\omega_{1}\right)$ is only a rank-one term. We estimate the above terms using (C.20), (C.21) with $a=c_{\omega}\left(\omega_{1}\right)$ and the estimates in Section C.4.
3.7. Posteriori error estimates of the velocity. In this section, we show that the nonlocal error in (3.32) has the desired forms in (3.8). Then we combine the estimate of such terms with the nonlinear energy estimate in Section 5.8 in [13. Using (3.5) and the definition of $\mathcal{L}^{\bar{\varepsilon}}, \mathcal{L}^{e}$ (3.12), we have

$$
\begin{equation*}
\mathcal{L}_{j}^{\bar{\varepsilon}}(G)=\mathcal{B}_{o p, j}(\mathbf{u}(\bar{\varepsilon}), G), \quad \mathcal{L}_{j}^{\varepsilon}(G)=\mathcal{B}_{o p, j}(\mathbf{u}(\hat{\varepsilon}), G) \tag{3.36}
\end{equation*}
$$

Given $c_{i}(t)$ Lipschitz in $t$ and $\bar{W}_{i}(0), i=1,2 . ., n$, we construct $\hat{W}_{i}(t)$ following previous sections and $G$ using (3.6). Using the derivations in (3.32), (3.28), (3.29) and the above relation, the contribution from the error type $I_{j, \bar{e}}$ term to the error (3.7) in the $j$ th equation is the following

$$
\mathcal{R}_{j}^{\bar{e}} \triangleq \mathcal{R}_{j 0}^{\bar{e}}-D_{j}^{2} \mathcal{R}_{j 0}^{\bar{e}}(0) \chi_{j 2}, \quad \mathcal{R}_{j 0}^{\bar{e}} \triangleq \sum_{i \leq n} \int_{0}^{t} c_{i}(s) \mathcal{B}_{o p, j}\left(\mathbf{u}(\bar{\varepsilon}), \hat{W}_{i}(t-s)\right) d t
$$

Since $\mathcal{B}_{o p, j}$ is bilinear and $c_{i}(t)$ is spatial-independent and Lipschitz in $t$, we get

$$
\mathcal{R}_{j 0}^{\bar{e}}=\mathcal{B}_{o p, j}\left(\mathbf{u}(\bar{\varepsilon}),\left(\sum_{i \leq n} \int_{0}^{t} c_{i}(t-s) \hat{W}_{i}(s) d s\right)=\mathcal{B}_{o p, j}(\mathbf{u}(\bar{\varepsilon}), \widehat{G}(t))\right.
$$

Denote by $\widehat{G}^{(l)}, \hat{W}_{j}^{(l)}$ the approximate solution with extension in $t$ in Section 3.4, and the first correction $l=1$ in Section 3.2 or two corrections $l=2$ in Section 3.2, 3.3. In particular, the full solution is given by $\hat{G}^{(2)}=\hat{G}, \hat{W}_{i}=\hat{W}_{i}^{(2)}$. Let $\hat{\phi}_{i}^{(l)}$ be the stream function associated with $\hat{W}_{i}^{(1)}$ constructed numerically with the first correction for $l=1$ and both corrections for $l=2$. We construct the stream function $\hat{\phi}^{N,(l)}$ associated with $\hat{G}^{(l)}$ and error $\hat{\varepsilon}$ as follows

$$
\hat{\phi}^{N,(l)} \triangleq \sum_{i \leq n} \int_{0}^{t} c_{i}(s) \hat{\phi}_{i}^{(l)}(t-s) d s, \quad \hat{\varepsilon}=\hat{W}^{(1)}+\Delta \hat{\phi}^{N,(1)}=\sum_{i \leq n} \int_{0}^{t} c_{i}(s)\left(\hat{W}_{i}^{(1)}+\Delta \hat{\phi}_{i}^{(l)}\right)(t-s) d s
$$

Since we can obtain $\mathbf{u}\left(a(t) \chi_{12}\right)$ exactly for the second correction (see Section 3.3), we have

$$
\hat{\varepsilon}(t)=\hat{W}^{(1)}-(-\Delta) \hat{\phi}^{N,(1)}=\hat{W}^{(2)}-(-\Delta) \hat{\phi}^{N,(2)}
$$

In practice, we estimate $\hat{\varepsilon}$ using the first identity since it does not involve $a_{i}(t)$ and the integrand $\hat{W}_{i}^{1}+\Delta \phi_{i}^{(l)}$ is piecewise cubic in time. We decompose $\hat{\varepsilon}$ as follows

$$
\begin{equation*}
\hat{\varepsilon}_{2}=\hat{\varepsilon}_{x y}(0) \Delta\left(\frac{x^{3} y}{2} \chi_{\hat{\varepsilon}}\right), \quad \hat{\varepsilon}=\left(\hat{\varepsilon}-\hat{\varepsilon}_{2}\right)+\hat{\varepsilon}_{2} \triangleq \hat{\varepsilon}_{1}+\hat{\varepsilon}_{2} \tag{3.37}
\end{equation*}
$$

where $\chi_{\hat{\varepsilon}}$ is defined in (D.6). Since $\hat{\varepsilon}$ only vanishes $O\left(|x|^{2}\right)$ near 0 , we perform the above decomposition so that $\hat{\varepsilon}_{1}=O\left(|x|^{3}\right)$ near 0. See Appendix C.4 and Section 5.8 in [13] for motivations of (3.37). We estimate $\hat{\varepsilon}_{1}, \hat{\varepsilon}_{x y}(0), \hat{\phi}^{N}$ following (3.34). We establish (3.6).

Similarly, using linearity, we can rewrite the residual error in (3.32) from $I_{j, e}$ term in (3.28) related to $\mathcal{L}_{i}^{e}(3.12)$ as follows and establish (3.8)

$$
\mathcal{R}_{j}^{e} \triangleq \mathcal{R}_{j 0}^{e}-D_{j}^{2} \mathcal{R}_{j 0}^{e}(0) \chi_{j 2}, \mathcal{R}_{j 0}^{e}=\sum_{i \leq n} \int_{0}^{t} c_{i}(t-s) \mathcal{B}_{o p, j}\left(\mathbf{u}\left(\hat{W}_{i}+\Delta \phi_{i}^{N}\right)(s), \bar{W}\right) d s=\mathcal{B}_{o p, j}(\hat{\varepsilon}(t), \bar{W})
$$

## 4. Estimate the norm of the velocity in the Regular case

In this section, we derive the constants in the upper bound in Lemma 2.3. We have constructed the finite rank approximation $\hat{f}$ for $f$ in Lemma 2.3 in Section 4.3 in Part I 13 . The estimate of the most singular part, e.g. $u_{x, a, b}(\omega \psi)$, in the $C^{1 / 2}$ estimate in Lemma 2.3 can be obtained using the sharp Hölder estimates in Section 3 of Part I [13], where $u_{x, a, b}$ is defined via a localized kernel. In this section, we estimate other terms in Lemma 2.3, e.g. $I=\psi u_{x}(\omega)-u_{x, a, b}(\omega \psi)-\psi \hat{u}_{x}(\omega)$, involving the velocity with desingularized kernels, which are more regular.

In Section 4.1, we outline the strategies in the estimate and decompose the integrals from the nonlocal terms into several parts based on their regularities. In Section 4.2, we perform the $L^{\infty}$ estimates in Lemma 2.3 and derive the constants. In Section 4.3 we perform the Hölder estimate of different parts. In Section 4.6, we combine the Hölder estimate of different parts, which provide the constants in Lemma 2.3. In particular, we reduce the $L^{\infty}$ estimates and the $C^{1 / 2}$ estimates in Lemma 2.3 to bounding some explicit $L^{1}$ integrals depending on the weights, which can be estimated by a numerical quadrature with rigorously error control. We estimate these integrals with computer assistance. See discussions in Section 2.2,

We will apply the second estimates in Lemma 2.3 for the nonlocal error, e.g. $\mathbf{u}(\bar{\varepsilon})$ and $\bar{\varepsilon}$ is the error of solving the Poisson equations. Since we can estimate piecewise bounds of $\bar{\varepsilon}$ following Section [3.6, instead of using global norm, we improve the estimate using the localized norms, which are much smaller than the global norm. See Section 4.7 ,

The kernels associated with $\mathbf{u}, \nabla \mathbf{u}$ are given by

$$
\begin{align*}
& K_{1} \triangleq \frac{y_{1} y_{2}}{|y|^{4}}, \quad K_{2} \triangleq \frac{1}{2} \frac{y_{1}^{2}-y_{2}^{2}}{|y|^{4}}, \quad K_{u} \triangleq \frac{y_{2}}{2|y|^{2}}, \quad K_{v} \triangleq-\frac{y_{1}}{2|y|^{2}}  \tag{4.1}\\
& K_{u_{x}}=-K_{1}, \quad K_{u_{y}}=K_{v_{x}}=K_{2}
\end{align*}
$$

Here, we have dropped the constant $\frac{1}{\pi}$, e.g. $u_{x}(\omega)=-\partial_{x y}(-\Delta)^{-1}=\frac{1}{\pi} K_{u_{x}} * \omega$. One needs to multiply $\frac{1}{\pi}$ back to obtain the final estimate.
Difficulties in the computations. In addition to the difficulties discussed in Section 5.1 of Part I [13], e.g. singularities caused by the weights and kernels, the singular integral introduces several technical difficulties in our estimates. To address these difficulties, we need to consider different scenarios and decompose the domain of the integrals carefully in our computer assisted estimates. Given $\omega \varphi \in L^{\infty}$, the velocity $\mathbf{u}$ and the commutator $\psi \cdot(\nabla \mathbf{u})(\omega)-(\nabla \mathbf{u})(\omega \psi)$ are only log-Lipschitz. The logarithm singularity introduces several difficulties. For example, if u is Lipschitz, a natural approach to estimate its Hölder norm in terms of $\|\omega \varphi\|_{\infty}$ is to estimate the piecewise bound of $\mathbf{u}$ and $\partial \mathbf{u}$, which are local in $\mathbf{u}$, and then use the method in Section E.1. However, since $\mathbf{u}$ is only log-Lipschitz, we need to perform a decomposition of $\mathbf{u}$ into the regular part and the singular part carefully. For different parts, we will apply different estimates. See Section 4.1.11 for the ideas. For $\nabla \mathbf{u}$, the estimates are more involved since it is more singular.
4.1. Several strategies. We outline several strategies to estimate the nonlocal terms.
4.1.1. Integral with approximation. In our computation of $\mathbf{u}_{A}=\mathbf{u}-\hat{\mathbf{u}}, \nabla \mathbf{u}_{A}=\nabla \mathbf{u}-\widehat{\nabla \mathbf{u}}_{A}$, where the approximation terms are defined in Section 4.3 of Part I [13], the rescaling argument still applies. We consider one approximation term $c(x) \int \mathbf{1}_{y \notin S} K\left(x_{a}, y\right) \omega(y) d y$ for $\int K(x, y) \omega(y)$ to illustrate the ideas, where $S$ is the singular region associated with $x_{a}$. Suppose that $K$ is $-d$-homogeneous. We want to estimate

$$
I=\rho(x) \int_{\mathbb{R}^{2}}\left(K(x, y)-c(x) K\left(x_{a}, y\right) \mathbf{1}_{y \notin S}\right) W(y) d y
$$

where $W$ is the odd extension of $\omega$ from $\mathbb{R}_{+}^{2}$ to $\mathbb{R}^{2}$ (see (4.23)). Denote

$$
\begin{equation*}
f_{\lambda}(x) \triangleq f(\lambda x) \tag{4.2}
\end{equation*}
$$

We choose $\lambda \asymp|x|$ and denote $x=\lambda \hat{x}, y=\lambda \hat{y}, x_{a}=\lambda \hat{x}_{a}$. Since $K(\lambda z)=\lambda^{-d} K(z)$, we have

$$
\begin{align*}
I & =\rho_{\lambda}(\hat{x}) \int_{\mathbb{R}^{2}}\left(K(\lambda \hat{x}, \lambda \hat{y})-c(\lambda \hat{x}) \mathbf{1}_{\lambda \hat{y} \notin S} K\left(\lambda \hat{x}_{a}, \lambda \hat{y}\right)\right) W(\lambda \hat{y}) \lambda^{2} d \hat{y} \\
& =\lambda^{2-d} \rho_{\lambda}(\hat{x}) \int_{\mathbb{R}^{2}}\left(K(\hat{x}, \hat{y})-c(\lambda \hat{x}) \mathbf{1}_{\hat{y} \notin S / \lambda} K\left(\hat{x}_{a}, \hat{y}\right)\right) W_{\lambda}(\hat{y}) d \hat{y} . \tag{4.3}
\end{align*}
$$

The singular region becomes $S / \lambda$ and close to $x_{a} / \lambda=\hat{x}_{a}$. For example, if $S=\left\{y: \max _{i} \mid y_{i}-\right.$ $\left.x_{a, i} \mid \leq a\right\}$, we have $S / \lambda=\left\{y: \max _{i}\left|y_{i}-\hat{x}_{a, i}\right| \leq a / \lambda\right\}$. For the above integral, we will symmetrize the kernel and then estimate it using the norms $\|W \varphi\|_{\infty}$ and $[\omega \psi]_{C_{x_{i}}^{1 / 2}}, i=1,2$.

The bulk and approximation. To take advantage of the scaling symmetry and overcome the singularity, in our computation for $x$ away from the origin and not too large, we choose several dyadic rescaling parameters $\lambda=2^{i}, i \in I$, e.g. $I=\{-4,-3, . ., 10\}$. Then for any $x$ with $\max \left(x_{1}, x_{2}\right) \in\left[2^{i} x_{c}, 2^{i+1} x_{c}\right]$, we can choose $\lambda=2^{i}$ so that the rescaled $\hat{x}=\frac{x}{\lambda}$ satisfies

$$
\hat{x} \in\left\{\begin{array}{l}
{\left[x_{c}, 2 x_{c}\right] \times\left[0,2 x_{c}\right] \triangleq \Omega_{1}, \quad \text { if } x_{2} \leq x_{1}}  \tag{4.4}\\
{\left[0, x_{c}\right] \times\left[x_{c}, 2 x_{c}\right] \triangleq \Omega_{2}, \quad \text { if } x_{2}>x_{1}}
\end{array}\right.
$$

We also choose $x_{i}\left(\left(x_{i}, 0\right)\right.$ is singularity) and the size of the singular region $t_{i}$ for the approximation term defined in Section 4.3.2 of Part I [13] such that $x_{i} / \lambda$ is on the grid point of the mesh and the boundary of the singular region $\left\{y:\left|x_{i}-y_{1}\right| \vee\left|y_{2}\right| \geq t_{i} / \lambda\right\}$, which aligns with one of the edges of a mesh cell. For example, this can be done by choosing the following $y$ mesh in the near-field to discretize the $y$-integral, $x_{i}$, and $t_{i}$

$$
y_{1, i}=i h, \quad y_{2, i}=i h, \quad x_{i}=2^{n_{i}} h, \quad t_{i}=2^{m_{i}} h
$$

Then when we discretize the rescaled integral in $y$, e.g. (4.3), the singular region is the union of several mesh cells. For large $y$, it is away from the singularity $\hat{x}$. Then we can use an adaptive mesh in $y_{1}, y_{2}$ to discretize the integral.

We remark that in (4.3), if $x_{a} \neq 0$ and $x_{a} / \lambda$ is too large or too small, since $c(x)$ is supported near $x_{a}, c(\lambda \hat{x})$ will be 0 . This means that when we compute $\mathbf{u}_{A}(x), \widehat{\nabla u}_{A}$, if the coefficient of an approximation term with center $x_{i}$ and parameter $t_{i}$ is nonzero, e.g., $c(x) \neq 0$, then $\lambda$ is comparable to $x_{i}$ when we rescale the integral by $\lambda$. Thus $\hat{x}_{i}=x_{i} / \lambda$ is on the grid. We also choose $t_{i}$ such that $t_{i} / \lambda$ is a multiple of mesh size $h$ for $\lambda$ comparable to $x_{i}$.

Remark 4.1. Using the scaling symmetry and rescaling the integral by dyadic scales, we can compute the integral for $x \in[0, D]^{2} \backslash[0, d]^{2}$ with roughly $O(\log (D / d))$ computational cost.

The near-field and the far-field. Recall the notations from Section 4.3 in Part I [13]

$$
\begin{align*}
& C_{u 0}=x, \quad C_{v 0}=-y, \quad C_{u_{x} 0}=1, \quad C_{u_{y} 0}=C_{v_{x} 0}=0 \\
& C_{u_{x}}=-\left(x^{2}-y^{2}\right), \quad C_{v_{x}}=2 x y, \quad C_{u_{y}}=2 x y, \quad C_{u}=-\left(\frac{1}{3} x^{3}-x y^{2}\right), C_{v}=x^{2} y-\frac{1}{3} y^{3},  \tag{4.5}\\
& K_{u x 0}=-\frac{4 y_{1} y_{2}}{|y|^{4}}, \quad K_{00}=\frac{24 y_{1} y_{2}\left(y_{1}^{2}-y_{2}^{2}\right)}{|y|^{8}}, \quad \mathcal{K}_{00}(\omega) \triangleq \frac{1}{\pi} \int_{\mathbb{R}_{++}^{2}} K_{00}(y) \omega(y) d y .
\end{align*}
$$

If $x$ is sufficiently small, i.e. $\max \left(x_{1}, x_{2}\right)<\min _{i \in I} 2^{i} x_{c}$, we choose $\lambda=\max \left(x_{1}, x_{2}\right) / x_{c}$ so that the rescaled $\hat{x}=\frac{x}{\lambda}$ is on the line $x_{1}=x_{c}$ or $x_{2}=x_{c}$. Assuming $\varphi(x) \geq|x|^{-\beta_{1}}|x|_{1}^{-\beta_{2}}$, $\rho \sim|x|^{-\alpha}$ near $x=0$, and $K$ is $-d$-homogeneous, then we get

$$
\begin{align*}
\left|\rho(x) \int_{\mathbb{R}_{2}^{++}} K(x, y) \omega(y) d y\right| & \leq\left\|\omega_{\lambda} \varphi_{\lambda}\right\|_{L^{\infty}} \rho_{\lambda}(\hat{x}) \int_{\mathbb{R}_{2}^{++}}|K(\hat{x}, \hat{y})| \varphi_{\lambda}(\hat{y})^{-1} \lambda^{2-d} d \hat{y} \\
& \leq\left\|\omega_{\lambda} \varphi_{\lambda}\right\|_{L^{\infty}} \lambda^{\beta_{1}+\beta_{2}+2-d} \rho_{\lambda}(\hat{x}) \int_{\mathbb{R}_{2}^{++}}|K(\hat{x}, \hat{y}) \| \hat{y}|^{\beta_{1}} \hat{y}_{1}^{\beta_{2}} d \hat{y} \tag{4.6}
\end{align*}
$$

As $x \rightarrow 0, \lambda \rightarrow 0$. The factor $\lambda^{\beta_{1}+\beta_{2}+2-d}$ absorbs the large factor $\lambda^{-\alpha}$ in $\rho_{\lambda}(\hat{x})$. In our estimate of $\mathbf{u}_{A}, \nabla \mathbf{u}_{A}$, we have $\beta_{1}+\beta_{2}=2.9$ for $\varphi_{1}, \varphi_{g, 1}, 2.5$ for $\varphi_{\text {elli }}$ A.2), $(\alpha, d)=(2,2)$ for $\left(\psi_{1}, \nabla \mathbf{u}_{A}\right)$ A.1), $(\alpha, d)=(3,1)$ for $\left(\rho_{10}, \mathbf{u}_{A}\right)$ A.2). We have $\beta_{1}+\beta_{2}+2-d-2>0$.

In general, the above integral may not be integrable due to the growing weight $|y|^{\beta_{1}} y_{1}^{\beta_{2}}$. For $\mathbf{u}_{A}, \nabla \mathbf{u}_{A}$ with small $x$, it takes the form (see Section 4.3 of Part I [13])

$$
\begin{equation*}
f(x)-C_{f 0}(x) u_{x}(0)-C_{f}(x) \mathcal{K}_{00}=\int_{\mathbb{R}_{++}^{2}}\left(K_{f}^{s y m}+C_{f 0}(x) \frac{4}{\pi} \frac{y_{1} y_{2}}{|y|^{4}}-C_{f}(x) K_{00}(y)\right) \omega(y) d y \tag{4.7}
\end{equation*}
$$

where $C_{f 0}, C_{f}$ and $K_{00}$ are defined in (4.5), and $f=u, v, u_{x}, v_{x}, u_{y}, v_{y}$. In particular, the associated kernel has a much faster decay rate $|y|^{-6}$, which will be shown in Appendix B.1.1. Thus, the integral is integrable.

Since $\lambda=\max \left(x_{1}, x_{2}\right) / x_{c}$ is very small, $\rho_{\lambda}(\hat{x})$ can be well approximated by the most singular power $c \lambda^{-\alpha}|x|^{-\alpha}$ for some $c>0$, which can be estimated effectively after factorizing out $\lambda^{-\alpha}$.

Similarly, if $x$ is sufficiently large, i.e. $\max \left(x_{1}, x_{2}\right)>\max _{i \in I} 2^{i+1} x_{c}$, we choose $\lambda=\frac{\max \left(x_{1}, x_{2}\right)}{x_{c}}$ so that the rescaled $\hat{x}=x / \lambda$ is on the line $x_{1}=x_{c}$ or $x_{2}=x_{c}$. Since $\lambda$ is sufficiently large, we can estimate the weight $\rho_{\lambda}, \varphi_{\lambda}$ based on their asymptotic behavior.

Integral near 0 . We have an approximation $I=-C_{f 0}(x) K_{u x 0}(y)-C_{2}(x) K_{00}(y)$ (4.5) for $K_{f 0}^{s y m}(x, y)$ with some smooth coefficients $C_{2}\left(C_{2}\right.$ may not be $\left.C_{f}\right)$. The term $C_{f 0}(x) K_{u x 0}(y)$ and $K_{f}$ are both $-d$ homogeneous, $d=1$ or 2 . Since $K_{u x 0}, K_{00}$ are singular near 0 , after rescale the integral following (4.3), we decompose the symmetrized integral for $y$ near 0 as follows

$$
\begin{align*}
I I & =\int_{\mathbb{R}_{2}^{++}}\left(K_{f}^{s y m}(\hat{x}, \hat{y}) \lambda^{2-d}-\lambda^{2-d} C_{f}(\hat{x}) K_{u x 0}(\hat{y})-C_{2}(\lambda \hat{x}) K_{00}(\hat{y}) \lambda^{-2}\right) \omega(\lambda \hat{y}) d \hat{y}  \tag{4.8}\\
= & \lambda^{2-d}\left(\int_{\mathbb{R}_{2}^{++}}\left(K_{f}^{s y m}(\hat{x}, \hat{y})-C_{f}(\hat{x}) \mathbf{1}_{|\hat{y}|_{\infty} \geq k_{01} h} K_{u x 0}(\hat{y})-\lambda^{-4+d} C_{2}(\lambda \hat{x}) \mathbf{1}_{|\hat{y}|_{\infty} \geq k_{02} h} K_{00}(\hat{y})\right) \omega(\lambda \hat{y}) d \hat{y}\right. \\
& \left.-\int_{\mathbb{R}_{2}^{++}}\left(C_{f}(\hat{x}) \mathbf{1}_{|\hat{y}|_{\infty} \leq k_{01} h} K_{u x 0}(\hat{y})-\lambda^{-4+d} C_{2}(\lambda \hat{x}) \mathbf{1}_{|\hat{y}|_{\infty} \leq k_{02} h} K_{00}(\hat{y})\right) \omega(\lambda \hat{y}) d \hat{y}\right)
\end{align*}
$$

for some small integers $k_{0 i}$ with $k_{0 i} h<|\hat{x}|_{\infty} / 2$, e.g. $k_{01}=4, k_{02}=20$, where $|a|_{\infty}=\max \left(a_{1}, a_{2}\right)$ and $h$ is chosen in (4.14). We will estimate the first integral using the method in Section 4.1.3, and the last two integrals for $|\hat{y}|_{\infty} \leq k_{01} h,|\hat{y}|_{\infty} \leq k_{02} h$ analytically in Section 4.4.1.

We apply the above decompositions to the integrals in both $L^{\infty}$ and $C^{1 / 2}$ estimates. We also apply the above decompositions to the approximation terms and estimate integral of $K_{u x 0}$ separately near $y=0$.
4.1.2. The scaling relations. We discuss several scaling relations, which will be useful in later computation. For a $-d$-homogeneous kernel $K$, i.e., $K(\lambda x)=\lambda^{-d} K(x)$, we have

$$
I(x)=\rho(x) \int K(x, y) \omega(y) d y=\rho_{\lambda}(\hat{x}) \int K(\hat{x}, \hat{y}) \omega_{\lambda}(\hat{y}) \lambda^{2-d} d \hat{y} \triangleq \lambda^{2-d} I_{\lambda}(\hat{x})
$$

where $x=\lambda \hat{x}, y=\lambda \hat{y}$. To compute the derivative of $I(x)$, using the chain rule, we have

$$
\partial_{x_{i}} I(x)=\lambda^{2-d} \frac{d \hat{x}_{i}}{d x_{i}} \partial_{\hat{x}_{i}} I_{\lambda}(\hat{x})=\lambda^{1-d} \partial_{\hat{x}_{i}} I_{\lambda}(\hat{x})
$$

For the $L^{\infty}$ part, clearly, we get $|I(x)|=\left|I_{\lambda}(\hat{x})\right|$. To compute the Hölder norm, we use the following relation $|x-z|=\lambda|\hat{x}-\hat{z}|$ and

$$
\frac{|I(x)-I(z)|}{|x-z|^{1 / 2}}=\lambda^{-1 / 2} \frac{\left|I_{\lambda}(\hat{x})-I_{\lambda}(\hat{z})\right|}{|\hat{x}-\hat{z}|^{1 / 2}}
$$

In particular, we have

$$
\begin{equation*}
\left\|\omega_{\lambda} \varphi_{\lambda}\right\|_{\infty}=\|\omega \varphi\|_{\infty}, \quad\left[\omega_{\lambda} \psi_{\lambda}\right]_{C_{x_{i}}^{1 / 2}}=\lambda^{1 / 2}[\omega \psi]_{C_{x_{i}}^{1 / 2}}, \quad i=1,2 \tag{4.9}
\end{equation*}
$$

Using these scaling relations, we can perform the estimate in a rescaled domain with any $\lambda>0$.
4.1.3. Mesh and the Trapezoidal rule. After rescaling the integral with suitable scaling factor $\lambda$, we can restrict the rescaled singularity $\hat{x} \in\left[0,2 x_{c}\right]^{2} \backslash\left[0, x_{c}\right]^{2}$ (see (4.3), (4.4)).

If a domain $Q$ is away from the singularity $\hat{x}$ of the kernel, applying (4.9), we get

$$
\begin{equation*}
\int_{Q}|K(\hat{x}, y)|\left|\omega_{\lambda}(y)\right| d y \leq\left\|\omega_{\lambda} \varphi_{\lambda}\right\|_{\infty} \int_{Q}|K(\hat{x}, y)| \varphi_{\lambda}^{-1}(y) d y=\|\omega \varphi\|_{\infty} \int_{Q}|K(\hat{x}, y)| \varphi_{\lambda}^{-1}(y) d y \tag{4.10}
\end{equation*}
$$

Then, it suffices to estimate the integral of an explicit function $|K(\hat{x}, y)| \varphi_{\lambda}^{-1}(y)$. If in addition, the region $Q$ is small, e.g. $Q$ is the grid $\left[y_{i}, y_{i+1}\right] \times\left[y_{j}, y_{j+1}\right]$ introduced below, we further apply

$$
\int_{Q}\left|K ( \hat { x } , y ) \left\|\omega_{\lambda}(y)\left|d y \leq\|\omega \varphi\|_{\infty}\left\|\varphi_{\lambda}^{-1}\right\|_{L^{\infty}(Q)} \int_{Q}\right| K(\hat{x}, y) \mid d y\right.\right.
$$

Since the domain $Q$ is small, the estimate is sharp. We use the following method to estimate $\int\left|K\left(\hat{x}_{i}, y\right)\right| d y$ for a suitable kernel $K$ and $\hat{x}_{i}$ on the grid points.

We consider the estimate of the $L^{1}$ norm of some function $f$ in $\mathbb{R}_{2}^{++}$, e.g. $f=K\left(\hat{x}_{i}, y\right)$ mentioned above. To discretize the integral, we design uniform mesh in the domain $[0, b]^{2}$ covering $\Omega_{1}$ and $\Omega_{2}$ with mesh size $h$ and adaptive mesh in the larger domain $[0, D]^{2}$

$$
\begin{equation*}
0=y_{0}<y_{1}<. .<y_{n}=D, \quad y_{i}=i h, i \leq b / h \tag{4.11}
\end{equation*}
$$

The finer mesh in the near field $[0, b]^{2}$ allows us to estimate the integral with higher accuracy. We choose sparser mesh in the far-field since $y$ is away from the singularity $\hat{x}$ and the kernel decays in $y$. We partition the integral as follows

$$
\begin{equation*}
\int_{\mathbb{R}_{2}^{++}}|f(y)| d y=\sum_{0 \leq i, j \leq n-1} \int_{\left[y_{i}, y_{i+1}\right] \times\left[y_{j}, y_{j+1}\right]}|f(y)| d y+\int_{y \notin D}|f(y)| d y \tag{4.12}
\end{equation*}
$$

We focus on how to estimate the first part for nonsingular $f$. In Section 4.4, we estimate the integral beyond $[0, D]^{2}$ using the decay of the integral. We will discuss how to estimate the integral near the singularity of the kernel in a later subsection.

Denote $Q=[a, b] \times[d, c], h_{1}=b-a, h_{2}=d-c$. We use the Trapezoidal rule

$$
\int_{[a, b] \times[c, d]}|f(y)| d y \leq T(|f|, Q)+\operatorname{Err}(f)
$$

where

$$
T(f, Q) \triangleq \frac{(b-a)(d-c)}{4}(f(a, c)+f(a, d)+f(b, c)+f(b, d))
$$

The error estimate of the above Trapezoidal rule is not obvious due to the absolute sign. In fact, even if $f$ is smooth, $|f|$ is only Lipschitz near the zeros of $f$. Since the set of zeros is hard to characterize and that $|f|$ can have low regularity, we do not pursue higher order quadrature rule. We have the following error estimate.
Lemma 4.2 (Trapezoidal rule for the $L^{1}$ integral). For $f \in C^{2}(Q)$, we have

$$
\int_{Q}|f(y)| d y \leq T(|f|, Q)+\frac{|Q|}{12}\left(h_{1}^{2}| | f_{x x}\left\|_{L^{\infty}(Q)}+h_{2}^{2} \mid f_{y y}\right\|_{L^{\infty}(Q)}\right)
$$

Remark 4.3. The above estimate shows that the Trapezoidal rule remains second order accurate from the above. In particular, this error estimate is comparable to the case without taking the absolute value.

Proof. Define the linear interpolation of $f$ in $Q$

$$
L(f)=\sum_{i=1}^{4} \lambda_{i}(x) f_{i}, \quad E(f)=f-L(f)
$$

where $\lambda_{i}(x)$ is linear and satisfies $\sum \lambda_{i}(x)=1$ and $\lambda_{i}(x) \geq 0$ for $x \in Q$. Using the triangle inequality, we obtain

$$
\int_{Q}|f| d y \leq \int_{Q}|E(f)| d y+\int_{Q} \lambda_{i}(x)\left|f_{i}\right| d y=T(|f|, Q)+\int_{Q}|E(f)| d y
$$

We have the standard error bound for linear interpolation $E(f)$

$$
\begin{equation*}
|E(f)| \leq \frac{\left\|f_{x x}\right\|_{L^{\infty}(Q)}}{2}|(x-a)(x-b)|+\frac{\left\|f_{y y}\right\|_{L^{\infty}(Q)}}{2}|(y-c)(y-d)| \tag{4.13}
\end{equation*}
$$

which can be obtained by first applying interpolation in $x$ and then in $y$. It can also be established using the error estimate for the 2D Lagrangian interpolation with $k=2$ in Section 8 in the supplementary material II [11]. Integrating the above estimate in $x, y$ and using $\frac{1}{2} \int_{0}^{1} t(1-t) d t=$ $\frac{1}{12}$ conclude the proof.

To estimate the integral $\int|K(x, y)|$ for all $\hat{x} \in \Omega_{1}, \Omega_{2}$ (4.4), we discretize $[0,2 a]^{2}$ using uniform mesh with mesh size $h_{x}=h / 2$. We use the above method to estimate $\int\left|K\left(\hat{x}_{i}, y\right)\right| d y$ for $x_{i}$ on the grid points. After we estimate the derivatives of the kernel, we use the following Lemma to estimate the integral for any $x$ in a domain.

Lemma 4.4. Suppose that $K(x, y) \in C^{2}(P \times Q), P=\left[a_{1}, b_{1}\right] \times\left[a_{2}, b_{2}\right], h_{i}=b_{i}-a_{i}, i=1,2$, and $Q=[a, b] \times[c, d]$. Let $L(K)(x, y)=\sum_{i, j=1,2} \lambda_{i j}(x) K\left(\left(a_{i}, b_{j}\right), y\right)$ be the linear interpolation of $K(x, y)$ in $x$ using $K\left(\left(a_{i}, b_{j}\right), y\right), i, j=1,2$. Then for any $x \in P$, we have
$\int_{Q}|K(x, y)| d y \leq \sum_{i, j=1,2} \lambda_{i j}(x) \int_{Q}\left|K\left(\left(a_{i}, b_{j}\right), y\right)\right| d y+\left(\frac{h_{1}^{2}}{8}\left\|K_{x x}\right\|_{L^{\infty}(P \times Q)}+\frac{h_{2}^{2}}{8}\left\|K_{y y}\right\|_{L^{\infty}(P \times Q)}\right)|Q|$.
The proof follows from (4.13), the triangle inequality and $\frac{1}{2}|t(1-t)| \leq \frac{1}{8}$ for $t \in[0,1]$. We will apply the above Lemma and sum $Q$ over all the near-field domains $Q_{i j}=\left[y_{i}, y_{i+1}\right] \times\left[y_{j}, y_{j+1}\right]$ (4.11). Since $\sum_{i j} \lambda_{i j}(x)=1$, we can simplify the first term as follows

$$
\sum_{i, j=1,2} \lambda_{i j}(x) \sum_{k, l \leq n} \int_{Q_{k l}}\left|K\left(\left(a_{i}, b_{j}\right), y\right)\right| d y \leq \max _{1 \leq i, j \leq 2} \sum_{k, l \leq n} \int_{Q_{k l}}\left|K\left(\left(a_{i}, b_{j}\right), y\right)\right| d y
$$

Therefore, it suffices to estimate the integral for $x$ on the grid points and the piecewise derivative bounds of the kernel.

We apply Lemmas 4.2, 4.4 to estimate the weighted integral related to the velocity. The integrands take the form (4.28), (4.29), (4.24). To estimate the error in the above integrals, we need to obtain piecewise $L^{\infty}$ estimate of the derivatives of the integrands in $P, Q$. We estimate the derivatives of the weights in Appendix A. 1 and the kernel in Appendix B

Parameters for the integrals. In our computation, we choose

$$
\begin{equation*}
h_{x}=13 \cdot 2^{-12}, \quad h=13 \cdot 2^{-11}, \quad x_{c}=13 \cdot 2^{-5} \tag{4.14}
\end{equation*}
$$

which can be represented exactly in binary system, to reduce the round off error. The approximate values of the above parameters are $h_{x} \approx 0.0032, h \approx 0.0064, x_{c} \approx 0.4$. For $x \in$ $\left[0,2 x_{c}\right]^{2} \backslash\left[0, x_{c}\right]^{2}$ (4.4), we have

$$
\begin{equation*}
\max \left(x_{1}, x_{2}\right) \geq x_{c}=64 h=128 h_{x} \tag{4.15}
\end{equation*}
$$

In our decomposition of the integral, e.g. (4.24), (4.45), (4.49), we impose a constraint on the size of the singular region to satisfy $(k+1) h<x_{c}$ such that the region does not cover the origin.
4.1.4. Decomposition, commutators and the Lipschitz norm. The most difficult part of the computation is to estimate the Hölder norm of $\nabla \mathbf{u}$, and we discuss several strategies. In this computation, we cannot first estimate the local Lipschitz norm of $\nabla u$ and then obtain the local Hölder norm due to the difficulties discussed at the beginning of Section 4 . We need to decompose the integral related to $\nabla u$ into several parts according to the distance between $y$ and the singularity and use different estimates for different parts.

We focus on the integral related to $u_{x}$ without subtracting any approximation term and assume that $x \in\left[0,2 x_{c}\right]^{2} \backslash\left[0, x_{c}\right]^{2}$. The approximation term $\widehat{\nabla u}_{A}$ is nonsingular and can be estimated using the method in Section 4.1.3. Let $h$ be the mesh size in the discretization of the integral in $y$. Suppose that

$$
\begin{equation*}
x \in \mathbb{R}_{2}^{++}, \quad x_{2} \leq x_{1}, \quad x \in B_{i_{1}, j_{1}}\left(h_{x}\right) \subset B_{i j}(h), \quad j \leq i \tag{4.16}
\end{equation*}
$$

where $h_{x}=h / 2$ and $B_{l m}(r)$ is defined as

$$
\begin{equation*}
B_{l m}(r)=[l r,(l+1) r] \times[m r,(m+1) r] . \tag{4.17}
\end{equation*}
$$

Denote by $R(x, k)$ the rectangle covering $x$

$$
\begin{equation*}
R(x, k) \triangleq[(i-k) h,(i+1+k) h] \times[(j-k) h,(j+1+k) h] \tag{4.18}
\end{equation*}
$$

for any $k>0$. If $k \in Z^{+}$, the boundary of $R(x, k)$ is along with the mesh grid and is at least $k h$ away from $x$. Denote by $R_{s}, R_{s, 1}, R_{s, 2}$ different symmetric rectangles with respect to $x$

$$
\begin{align*}
R_{s}(x, k) & \triangleq\left[x_{1}-k h, x_{1}+k h\right] \times\left[x_{2}-k h, x_{2}+k h\right] \\
R_{s, 1}(x, k) & \triangleq\left[x_{1}-k h, x_{1}+k h\right] \times[(j-k) h,(j+1+k) h]  \tag{4.19}\\
R_{s, 2}(x, k) & \triangleq[(i-k) h,(i+1+k) h] \times\left[x_{2}-k h, x_{2}+k h\right] .
\end{align*}
$$

Clearly, we have $R_{s}(x, k) \subset R_{s, 1}(x, k), R_{s, 2}(x, k) \subset R(x, k)$. We introduce the upper and lower parts of the rectangle

$$
\begin{equation*}
R^{+}(x, k) \triangleq R(x, k) \cap\left\{y: y_{2} \geq x_{2}\right\}, \quad R^{-}(x, k) \triangleq R(x, k) \cap\left\{y: y_{2} \leq x_{2}\right\} \tag{4.20}
\end{equation*}
$$

We use similar notations for $R_{s}(x, k), R_{s, 1}(x, k), R_{s, 2}(x, k)$. We further introduce the intersection of the rectangle and four half planes with reflection

$$
\begin{align*}
R(x, k, N)=R(x, k) \cap\left\{y: y_{2} \geq 0\right\}, & R(x, k, S)=\mathcal{R}_{2}\left(R(x, k) \cap\left\{y: y_{2} \leq 0\right\}\right) \\
R(x, k, E)=R(x, k) \cap\left\{y: y_{1} \geq 0\right\}, & R(x, k, W)=\mathcal{R}_{1}\left(R(x, k) \cap\left\{y: y_{1} \leq 0\right\}\right) \tag{4.21}
\end{align*}
$$

where $N, E, S, W$ are short for north, east, south, west, respectively and the reflection operators $\mathcal{R}_{1}, \mathcal{R}_{2}$ are given by

$$
\mathcal{R}_{1}\left(y_{1}, y_{2}\right)=\left(-y_{1}, y_{2}\right), \quad \mathcal{R}_{2}\left(y_{1}, y_{2}\right)=\left(y_{1},-y_{2}\right)
$$

It is clear that $R(x, k, S) \subset \mathbb{R}_{2}^{+}, R(x, k, W) \subset\left\{y: y_{1} \geq 0\right\}$. An illustration of these domains is given in Figure 1. If $x, y \in \mathbb{R}_{2}^{++}$, we have the equivalence

$$
\begin{equation*}
\left(y_{1},-y_{2}\right) \notin R(x, k) \Longleftrightarrow\left(y_{1},-y_{2}\right) \notin R(x, k) \cap\left\{y: y_{2} \leq 0\right\} \Longleftrightarrow y \notin R(x, k, S) \tag{4.22}
\end{equation*}
$$

The above notations will be very useful in our later decomposition of the symmetrized kernel.
Define the odd extension of $\omega$ in $y$ from $\mathbb{R}_{2}^{+}$to $\mathbb{R}_{2}$

$$
\begin{equation*}
W(y)=\omega(y) \text { for } y_{2} \geq 0, \quad W(y)=-\omega\left(y_{1},-y_{2}\right) \text { for } y_{2}<0 \tag{4.23}
\end{equation*}
$$

$W$ is odd in both $y_{1}$ and $y_{2}$ variables. For simplicity, we drop the $x$ variable in the $R$ notation. For $k>k_{2}, k, k_{2} \in Z^{+}$, we decompose the weighted $u_{x}(x)$ integral as follows

$$
\begin{align*}
& \psi(x) \int K_{1}(x-y) W(y) d y=\psi(x) \int_{R(k)^{c}} K_{1}(x-y) W(y) d y \\
& +\int_{R_{s, 1}(k)} K_{1}(x-y) \psi(y) W(y) d y+\int_{R(k) \backslash R_{s, 1}(k)} K_{1}(x-y) \psi(y) W(y) d y  \tag{4.24}\\
& +\int_{R(k) \backslash R\left(k_{2}\right)} K_{1}(x-y)(\psi(x)-\psi(y) W(y) d y)+\int_{R\left(k_{2}\right)} K_{1}(x-y)(\psi(x)-\psi(y)) W(y) d y \\
& \triangleq I_{1}(x, k)+I_{2}(x, k)+I_{3}(x, k)+I_{4}\left(x, k, k_{2}\right)+I_{5}\left(x, k_{2}\right)
\end{align*}
$$



Figure 1. Left: The large box is $R(x, k)$ and the red box is $R_{s, 1}(x, k)$. The small box containing $x$ has size $h \times h$. Right: The upper box is $R(x, k, N)$, and the shaded box is $R(x, k, S)$, the reflection of the region below the $y$-axis.
where

$$
K_{1}(s) \equiv \frac{s_{1} s_{2}}{|s|^{4}}
$$

We drop $-\frac{1}{\pi}$ in the integrand $-\frac{1}{\pi} K_{1}(s)$ for $u_{x}(x)$ at this moment to simplify the notation. We will estimate different parts in Section 4.3
4.1.5. Symmetrization. After we obtain the decomposition, we use the odd symmetry of $W$ in $y_{1}, y_{2}$ to symmetrize the integral and reduce the integral over $\mathbb{R}_{2}$ to the first quadrant $\mathbb{R}_{2}^{++}$. This enables us to exploit the cancellation in the integral and obtain a sharper estimate. In our computation, we symmetrize the integrals in $I_{1}(x, k)$ and $I_{4}\left(x, k, k_{2}\right)$, which are more regular. For a given kernel $K(x, y)$, we denote by $K^{\text {sym }}$ the symmetrization of $K$

$$
\begin{equation*}
K^{s y m}(x, y) \triangleq K(x, y)-K\left(x,-y_{1}, y_{2}\right)-K\left(x, y_{1},-y_{2}\right)+K(x,-y) \tag{4.25}
\end{equation*}
$$

We show how to symmetrize $I_{1}(x, k)$ as an example. Recall the notations in (4.21), (4.16). We assume $x_{1} \geq x_{2}$. We choose $k<i$ so that $R(x, k) \subset\left\{y: y_{1}>0\right\}$ and $R(x, k, W)=\emptyset$. By definition (4.18), the domains $R(x, k), R(x, k, N), R^{+}(x, k)$ etc are the same for all $x \in B_{i_{1}, j_{1}}\left(h_{x}\right)$. Yet, $R(x, k)$ may cross the boundary $y_{2}=0$, i.e. $R(x, k, S) \neq \emptyset$. See the right figure in Figure 1 for a possible configuration. Using the equivalence (4.22) and the property that $W$ is odd in $y_{1}$ and $y_{2}$, for general $x \in \mathbb{R}_{2}^{++}$(without $x_{1} \geq x_{2}$ ), we can symmetrize $I_{1}(x, k)$ as follows

$$
\begin{align*}
I_{1}(x, k)=\psi(x) \int_{\mathbb{R}_{2}^{++}} & \left(K_{1}(x-y) \mathbf{1}_{y \in R(k)^{c}}-K_{1}\left(x_{1}-y_{1}, x_{2}+y_{2}\right) \mathbf{1}_{y \notin R(k, S)}\right.  \tag{4.26}\\
& \left.-K_{1}\left(x_{1}+y_{1}, x_{2}-y_{2}\right) \mathbf{1}_{y \notin R(k, W)}+K_{1}(x+y)\right) \omega(y) d y
\end{align*}
$$

For $I_{4}(x)$ (4.24), we choose the weight $\psi(y)$ (A.1), (A.2) even in $y_{1}, y_{2}$. Then the symmetrization of $I_{4}$ is

$$
\begin{align*}
I_{4}\left(x, k, k_{2}\right)= & \int_{\mathbb{R}_{2}^{++}}\left(K_{1}(x-y) \mathbf{1}_{y \in R(k) \backslash R\left(k_{2}\right)}-K_{1}\left(x_{1}-y_{1}, x_{2}+y_{2}\right) \mathbf{1}_{y \in R(k, S) \backslash R\left(k_{2}, S\right)}\right.  \tag{4.27}\\
& \left.-K_{1}\left(x_{1}+y_{1}, x_{2}-y_{2}\right) \mathbf{1}_{y \in R(k, W) \backslash R\left(k_{2}, W\right)}\right)(\psi(x)-\psi(y)) W(y) d y
\end{align*}
$$

In (4.27), we do not have the term $K_{1}(x+y)$ since for $y \in \mathbb{R}_{2}^{++}, x+y \geq x_{c}>(k+1) h$ and $-y \notin R(k)$. See the discussion below (4.15). Thus after symmetrizing the kernel in $I_{4}$, we do not have such a term.

Though the symmetrized kernel is complicated, since these regions $R(l), R(l, \alpha), \alpha=N, E, l=$ $k, k_{2}$ (4.18), (4.21) can be decomposed into the union of the mesh girds [y, $\left.y_{i+1}\right] \times\left[y_{j}, y_{j+1}\right]$, in each grid, the indicator functions are constants. See also Remark 4.6. In each grid $y \in$ $\left[y_{i}, y_{i+1}\right] \times\left[y_{j}, y_{j+1}\right]$, we can write the integrand in $I_{1}+I_{4}$ as

$$
\begin{align*}
J & =K^{N C}(x, y) \cdot \psi(x)+K^{C}(x, y) \cdot(\psi(x)-\psi(y)), \\
\partial_{x_{i}} J & =\left(K^{N C}+K^{C}\right) \partial_{x_{i}} \psi(x)+\partial_{x_{i}} K^{N C} \cdot \psi(x)+\partial_{x_{i}} K^{C}(x, y) \cdot(\psi(x)-\psi(y)), \tag{4.28}
\end{align*}
$$

where $N C, C$ are short for non-commutator, commutator, respectively. For $y$ away from $x$, e.g. $\left|y_{1}\right| \vee\left|y_{2}\right| \geq 4 x_{c}$ in our computation, we have

$$
\begin{equation*}
J=K^{s y m}(x, y) \psi(x) \tag{4.29}
\end{equation*}
$$

In practice, we assemble the symmetrized integrand in $I_{1}+I_{4}$ in $\mathbb{R}_{2}^{++}$together. Using (4.28), we only need to assemble $K^{N C}, K^{C}$. We first initialize the integrand with $\left(K^{N C}, K^{C}\right)=$ $\left(K^{s y m}, 0\right)$. To assemble the integrand in the singular regions, we perform two replacements. In the first replacement, we pretend that $R\left(k_{2}\right)=\emptyset$ and replace the integrand in $R(k) \cap \mathbb{R}_{2}^{++}$. Based on $x \in B_{i j}(h)$ (4.16), we determine the regions $R(x, k), R(x, k, S)$ (4.18), (4.21). Since $x_{1} \geq x_{2}$, we get $R(x, k, W)=\emptyset$. See Figure (11). We partition $R(k) \cap \mathbb{R}_{2}^{++}$as follows

$$
\begin{equation*}
R(k) \cap \mathbb{R}_{2}^{++}=R(k, N)=(R(k, N) \backslash R(k, S)) \cup R(k, S) \triangleq D_{1} \cup D_{2} \tag{4.30}
\end{equation*}
$$

According to (4.26), (4.27) $\left(R\left(k_{2}\right)=\emptyset\right)$, for $i=1$, 2 , we first replace $\left(K^{N C}, K^{C}\right)$ in $D_{i}$ by
$\left(K^{N C}, K^{C}\right)=\left(K^{s y m}-K_{i}^{C}, K_{i}^{C}\right), K_{1}^{C}=K_{1}(x-y), K_{2}^{C}=K_{1}(x-y)-K_{1}\left(x_{1}-y_{1}, x_{2}+y_{2}\right)$,
respectively, where $K^{C}$ is from the integrand in (4.27). We have $i$ singular terms in $D_{i}$ in (4.27).
In the second replacement, we replace the integrand in the smaller singular region $R\left(k_{2}\right) \cap$ $\mathbb{R}_{2}^{++} \subset R(k) \backslash \mathbb{R}_{2}^{++}$. Outside this region, we have obtained the symmetrized integrand using (4.31). Since we assume $x_{1} \geq x_{2}$, similar to $R(k) \cap \mathbb{R}_{2}^{++}$(see Figure 1), we can decompose

$$
R\left(k_{2}\right) \cap \mathbb{R}_{2}^{++}=\left(R\left(k_{2}, N\right) \backslash R\left(k_{2}, S\right)\right) \cap R\left(k_{2}, S\right) \triangleq D_{3} \cup D_{4}
$$

In $D_{4}=R\left(k_{2}, S\right) \subset R\left(k_{2}\right), R(k, S)$, from (4.26), 4.27), we completely remove the $K_{1}(x-$ $y), K_{1}\left(x_{1}-y_{1}, x_{2}+y_{2}\right)$ terms in the integrand and have

$$
\left(K^{N C}, K^{C}\right)=\left(K_{1}(x+y)-K_{1}\left(x_{1}+y_{1}, x_{2}-y_{2}\right), 0\right)
$$

In $D_{3}$, since $D_{3} \subset R(k, N)=D_{1} \cup D_{2}$ (4.30), there are two cases. In $D_{3} \cap D_{1}, D_{1}=$ $R(k, N) \backslash R(k, S)$, we have three non-singular terms from (4.26) and 0 term from (4.27) and get

$$
\left(K^{N C}, K^{C}\right)=\left(K_{1}(x+y)-K_{1}\left(x_{1}+y_{1}, x_{2}-y_{2}\right)-K_{1}\left(x_{1}-y_{1}, x_{2}+y_{2}\right), 0\right)
$$

In $D_{3} \cap D_{2}, D_{2}=R(k, S)$, we have two terms from (4.26) and one term from (4.27). We get

$$
\left(K^{N C}, K^{C}\right)=\left(K_{1}(x+y)-K_{1}\left(x_{1}+y_{1}, x_{2}-y_{2}\right),-K_{1}\left(x_{1}-y_{1}, x_{2}+y_{2}\right)\right)
$$

For $x_{1}<x_{2}$, we assemble the integrand similarly. Using (4.28), we obtain the integrand $\partial_{x_{i}} J$ for the Hölder estimate.
$C_{y}^{1 / 2}$ estimate of $u_{y}, v_{x}$. In the $C_{y}^{1 / 2}$ estimate of $u_{y}, v_{x}$ with kernel $K_{2}$ (4.1), we symmetrize the integrand $K(x-y)(\psi(x)-\psi(y)$, see (4.64) in Section 4.3.9. In this case, the symmetrized integrand $W(y) T$ is similar to (4.26) with $\psi(x)$ replaced by $\psi(x)-\psi(y)$ with $T$
$T=(\psi(x)-\psi(y))\left(K_{2}(x-y) \mathbf{1}_{y \in R(k)^{c}}-K_{2}\left(x_{1}-y_{1}, x_{2}+y_{2}\right) \mathbf{1}_{y \notin R(k, S)}-K_{2}\left(x_{1}+y_{1}, x_{2}-y_{2}\right) \mathbf{1}_{y \notin R(k, W)}+K_{1}(x+y)\right)$.
Due to the weight $(\psi(x)-\psi(y))$, we always have $K^{N C}=0$. We initialize the $T$ using (4.28) with $\left.K^{C}=K_{2}^{\text {sym }} 4.25\right)$. In the singular region $R(x, k) \cap \mathbb{R}_{2}^{++}$, we only need to perform one replacement. Similar to (4.31), we use (4.30) and replace the integrand as follows
$K^{C}=K_{2}^{s y m}-K_{2}(x-y), y \in R(k, N) \backslash R(k, S), K^{C}=K_{2}^{s y m}-K_{2}(x-y)-K_{2}\left(x_{1}-y_{1}, x_{2}+y_{2}\right), y \in R(k, S)$.
For $L^{\infty}$ estimate, we do not multiply the integrand by the weight or the commutator. We decompose the integral as (4.45), and symmetrize the nonsingular part in $I_{1}$ using (4.26) without the weight $\psi(x)$. Symmetrizing $I_{4}(4.45)$ is similar. We initialize the symmetrized integrand as $K^{\text {sym }}(4.25)$, and then replace it in $R(k) \cap \mathbb{R}_{2}^{++}$. Without loss of generality, we assume $x_{1} \geq x_{2}$ and have the decomposition (4.30). Similar to (4.31), we replace the integrand as follows
$K^{\text {sym }}-K_{1}(x-y), y \in R(k, N) \backslash R(k, S), \quad K^{\text {sym }}-\left(K_{1}(x-y)-K_{1}\left(x_{1}-y_{1}, x_{2}+y_{2}\right), y \in R(k, S)\right.$.
That is, we remove one or two singular terms in $R(k, N) \backslash R(k, S), R(k, S)$.
4.1.6. Integral in domains depending on $x$. In the computation, we need to estimate several integrals in the domains depending on $x$, e.g. $I_{3}$ in (4.24). We use the $L^{\infty}$ estimate of $I_{3}$ to illustrate the ideas. A direct estimate yields

$$
\left|I_{3}(x)\right| \leq\|W \varphi\|_{\infty} \int_{R(k) \backslash R_{s, 1}(k)}\left|K_{1}(x-y)\right| \psi(y) \varphi^{-1}(y) d y
$$

We cannot apply the method in Section 4.1.3 to first estimate $I_{3}(x)$ for $x$ on the grid points and then estimate $\partial^{2} I_{3}(x)$ for the error since the kernel is singular and the error part associated with $\partial^{2} I_{3}(x)$ is more singular (see Lemma 4.4).

Denote $f=\psi \varphi^{-1}$. We consider a change of variable $y=x+s$ to center our analysis around the singularity $x$. The domain for $s$ is

$$
\begin{equation*}
\left\{y \in R(k) \backslash R_{s, 1}(k)\right\}=\{s \in R(k)-x\} \cap\left\{\left|s_{1}\right| \geq k h\right\} \triangleq D(x, k) \tag{4.32}
\end{equation*}
$$

It suffices to estimate

$$
\begin{equation*}
J=\int_{s \in D(x, k)}\left|K_{1}(-s)\right| f(x+s) d y, \quad f \geq 0 \tag{4.33}
\end{equation*}
$$

for all $x \in B_{i_{1}, j_{1}}\left(h_{x}\right)$ 4.16). We want to further simplify the above domain so that it does not depend on $x$. Recall the location of $x$ (4.16). To obtain a sharp estimate, we further partition the location of $x \in B_{i_{1}, j_{1}}\left(h_{x}\right)$ as follows

$$
\begin{equation*}
A_{a}=\left[i_{1} h_{x}+a h_{x} / m, i_{1} h_{x}+(a+1) h_{x} / m\right], B_{b} \triangleq\left[j_{1} h_{x}+b h_{x} / m, j_{1} h_{x}+(b+1) h_{x} / m\right] \tag{4.34}
\end{equation*}
$$

for some $m \in Z^{+}$and $0 \leq a, b \leq m-1$. Clearly, $A_{a} \times B_{b}$ is a partition of $B_{i_{1} j_{1}}\left(h_{x}\right)$. Recall (4.16) and (4.18). We have

$$
R(x, k)=[(i-k) h,(i+1+k) h] \times[(j-k) h,(j+1+k) h] .
$$

Now, for $x \in A_{a} \times B_{b}$, since $\left|s_{1}\right| \geq k h$, we have

$$
\begin{align*}
s_{1}=y_{1}-x_{1} & \in\left[(i-k) h-i_{1} h_{x}-(a+1) h_{x} / m,-k h\right] \cup\left[k h,(i+1+k) h-i_{1} h_{x}-a h_{x} / m\right]  \tag{4.35}\\
& \triangleq X_{l, a} \cup X_{r, a}
\end{align*}
$$

where the subscripts 1 , r are short for left, right, respectively. Similarly, for $s_{2}$, we have (4.36)

$$
\begin{aligned}
s_{2} & =y_{2}-x_{2} \in\left[(j-k) h-j_{1} h_{x}-(b+1) h_{x} / m,(j+k+1) h-j_{1} h_{x}-b h_{x} / m\right] \\
& \triangleq\left[(j-k) h-j_{1} h_{x}-(b+1) h_{x} / m,-k h\right] \cup[-k h, k h] \cup\left[k h,(j+1+k) h-j_{1} h_{x}-b h_{x} / m\right] \\
& \triangleq Y_{d, b} \cup Y_{m, b} \cup Y_{u, b}
\end{aligned}
$$

where the subscripts $\mathrm{d}, \mathrm{m}, \mathrm{u}$ are short for down, middle, upper, respectively. Note that the intervals $X, Y$ do not depend on $x$. We have

$$
\begin{equation*}
D(x, k) \subset\left(X_{l, a} \cup X_{r, a}\right) \times\left(Y_{d, b} \cup Y_{m, b} \cup Y_{u, b}\right) \tag{4.37}
\end{equation*}
$$

Now, we can decompose $J$ (4.33) as follows

$$
J \leq \sum_{\alpha=l, r, \beta=d, m, u} J_{\alpha, \beta}, \quad J_{\alpha, \beta} \triangleq \int_{X_{\alpha, a} \times Y_{\beta, b}}\left|K_{1}(-s)\right| f(s+x) d y, \alpha=l, r, \beta=d, m, u
$$

See the left figure in Figure 2 for different domains in the above decomposition. From the definitions of $X, Y$, the total width of the left and the right domains $X_{\alpha, a} \times\left(Y_{d, b} \cup Y_{m, b} \cup Y_{u, b}\right), \alpha=$ $l, u$ is

$$
\left|X_{l, a}\right|+\left|X_{r, a}\right|=h+h_{x} / m
$$

For a fixed $x$, from the definition (4.18), the width of $R(k) \backslash R_{s, 1}(k)$ is $h$. We choose a large $m$ and further partition the location of $x$ so that we do not overestimate the region too much.

For a small domain $Q=[a, b] \times[c, d]$, we can estimate the integral as follows

$$
\begin{equation*}
\int_{Q}\left|K_{1}(-s)\right| f(x+s) d s \leq \int_{Q}\left|K_{1}(-s)\right| d s| | f \|_{L^{\infty}\left(B_{i_{1} j_{1}}\left(h_{x}\right)+Q\right)} \tag{4.38}
\end{equation*}
$$



Figure 2. The largest box in the left and middle figure is $R(x, k)$. Left: The left and right blue regions are $X_{l, a} \times Y_{m, b}, X_{r, a} \times Y_{m, b}$. The four red regions correspond to $X_{\alpha, a} \times Y_{\beta, b}, \alpha=l, u, \beta=d, u$. Middle: Illustration of $R(x, k) \backslash R_{s}(x, k)$ and $R_{s}\left(x, k_{2}\right) . R(x, k) \backslash R_{s}(x, k)$ consists of the blue and the red regions. Right: different regions near the singularity for $u / x_{1}$. Blue, red, and white regions represent $S_{i n, 1}, S_{i n, 2}, S_{o u t}$, respectively.

Since $Q$ is given, $K_{1}(s)$ is explicit and has scaling symmetries, we can estimate the integral of $\left|K_{1}(s)\right|$ easily. For example, if $Q=[a h, b h]^{2}$, we can use the scaling symmetries of $K_{1}(s)$ to obtain $\int_{Q}\left|K_{1}(-s)\right|=h^{\beta} \int_{[a, b]^{2}}\left|K_{1}(-s)\right|$ for some $\beta$. Moreover, for many kernels in our computations, e.g. $K(s)=\frac{s_{1} s_{2}}{|s|^{4}}$, we have explicit formulas for the integral. See Section 5.1 in the supplementary material II [11].

We apply the above method to estimate the integral in $X_{\alpha, a} \times Y_{\beta, b}, \alpha=l, r, \beta=d, u$ (red region in Figure (2). Since $Y_{m, b}=[-k h, k h]$, for the integral in $X_{\alpha, a} \times Y_{m, b}$ (blue region), we further decompose it

$$
\begin{equation*}
J_{\alpha, m}=\sum_{-k \leq t \leq k-1} \int_{X_{\alpha, a} \times[t h,(t+1) h]}\left|K_{1}(-s)\right| f(s+x) d y \tag{4.39}
\end{equation*}
$$

and then apply the above method to estimate it.
Next, we further simplify $\|f\|_{L^{\infty}\left(B_{i_{1} j_{1}}\left(h_{x}\right)+Q\right)}$ in the above estimate. From (4.16), we get

$$
i h \leq i_{1} h_{x}<\left(i_{1}+1\right) h_{x} \leq(i+1) h, \quad j h \leq j_{1} h_{x}<\left(j_{1}+1\right) h_{x} \leq(j+1) h
$$

For $X_{l, a}$ (4.35) with $0 \leq a \leq m-1$, we have the lower bound for the endpoint
$(i-k) h-i_{1} h_{x}-(a+1) h_{x} / m \geq(i-k) h-i_{1} h_{x}-h_{x} \geq(i-k) h-\left((i+1) h-h_{x}\right)-h_{x}=-k h-h$.
See the left figure in Figure 2. The width of blue region is less than $h$. Similarly, we can cover the intervals of $X, Y$ (4.35), 4.36) uniformly for $0 \leq a, b \leq m-1$ and obtain

$$
\begin{aligned}
X_{l, a} & \subset\left[(i-k) h-i_{1} h_{x}-h_{x},-k h\right] \subset[-(k+1) h,-k h] \\
X_{r, a} & \subset\left[k h,(i+1+k) h-i_{1} h_{x}\right] \subset[k h,(k+1) h] \\
Y_{d, b} & \subset[-(k+1) h,-k h], \quad Y_{u, b} \subset[k h,(k+1) h]
\end{aligned}
$$

Thus, we only need to estimate the $L^{\infty}$ norm of $f$ in

$$
Q_{i_{1} j_{1}}\left(h_{x}\right)+[\alpha h,(\alpha+1) h] \times[\beta h,(\beta+1) h], \quad \alpha=-k-1, k, \quad \beta=-(k+1),-k, . ., k
$$

These estimates are independent of the choice of $m, a, b$. Since the size of each domain is at most $2 h \times 2 h$, the above estimates based on (4.38) are sharp. We estimate the piecewise bound of the weights $\psi, \varphi$ in Appendixes A.11A.2A.3.

Using the above decomposition and estimates, we obtain the estimate of $J$ (4.33) for $x \in$ $A_{a} \times B_{b}$ (4.34). Similarly, we can estimate $J$ for any $0 \leq a, b \leq m-1$. Taking the maximum of these $m^{2}$ estimates, we obtain the estimate of $J$ and $I_{3}(x)$ for all $x \in B_{i_{1} j_{1}}\left(h_{x}\right)$.
4.1.7. First generalization: integral in a ring. We generalize the above ideas to estimate the integrals in domain $D=R(x, k) \backslash R\left(x, k_{2}\right)=R(k) \backslash R\left(k_{2}\right)$
$J=\int_{R(k) \backslash R\left(k_{2}\right)}|K(y-x)||f(y)| d y=\int_{s \in D(x, k)}|K(s)||f(x+s) d y|, \quad D(x, k) \triangleq R(k) \backslash R\left(k_{2}\right)-x$
with $2 \leq k_{2}=k-\frac{i}{2}<k$ for some integer $i \geq 1$ and some kernel $K(z)$. Note that the inner region $R\left(k_{2}\right)$ is different from (4.32). See $I_{4}$ in (4.24) for an example of this integral region. Suppose $x \in B_{i j}(h)$ (4.16). We partition location of $x$ similar to (4.34) and introduce $p_{l}, q_{l}$ (4.40)

$$
\begin{aligned}
& A_{a}=[i h+a h / m, i h+(a+1) h / m], B_{b}=[j h+b h / m, j h+(b+1) h / m], 0 \leq a, b \leq m-1 \\
& p_{1}=-k_{2}-a / m, p_{2}=k_{2}+(m-a-1) / m, p_{3}=-k_{2}-b / m, p_{4}=k_{2}+(m-b-1) / m \\
& q_{1}=-k-(a+1) / m, q_{2}=k+(m-a) / m, q_{3}=-k-(b+1) / m, q_{4}=k+(m-b) / m
\end{aligned}
$$

For a fixed $x \in A_{a} \times B_{b}$, by comparing the boundaries of the following four rectangles, we get

$$
D_{i n} \triangleq\left[p_{1} h, p_{2} h\right] \times\left[p_{3} h, p_{4} h\right] \subset R\left(k_{2}\right)-x \subset R(k)-x \subset\left[q_{1} h, q_{2} h\right] \times\left[q_{3} h, q_{4} h\right] \triangleq D_{o u t}
$$

To obtain the above inclusions, for example, for $s=y-x, y \in R\left(k_{2}\right)$, we use

$$
\min _{y \in R\left(k_{2}\right)} y_{1}-x_{1}=i h-k_{2} h-x \leq i h-k_{2} h-(i h+a h / m)=-k_{2} h-a h / m=p_{1} h
$$

uniformly for $x \in A_{a} \times B_{b}$. For $R(k)-x \subset D_{\text {out }}$, we have $q_{1} h \leq \min _{y \in R(k)} y_{1}-x_{1}$. Other bounds for the inclusions are obtained similarly. We yield $D(x, k) \subset D_{\text {ring }}$. where

$$
\begin{equation*}
D_{\text {ring }} \triangleq D_{\text {out }} \backslash D_{i n} \tag{4.41}
\end{equation*}
$$

It suffices to estimate the integral $J$ in $D_{\text {ring }}$. We partition $s \in D_{\text {ring }}$ using mesh

$$
\begin{equation*}
Z_{1}=\{-k \leq i \leq k, i \in \mathbb{Z}\} \cup\left\{p_{1}, p_{2}, q_{1}, q_{2}\right\}, \quad Z_{2}=\{-k \leq i \leq k, i \in \mathbb{Z}\} \cup\left\{p_{3}, p_{4}, q_{3}, q_{4}\right\} \tag{4.42}
\end{equation*}
$$

and then order them in an increasing order $z_{l, 1}<z_{l, 2}<. .<z_{l, 2 k+5} \in Z_{l}, l=1,2$. Note that we do not multiply $z_{l, 1}$ by $h$ here. We estimate the integral $J$ in each grid $Q=\left[z_{1, c} h, z_{1, c+1} h\right] \times$ $\left[z_{2, d} h, z_{2, d+1} h\right]$ following (4.38) and using the norm $\|f\|_{L^{\infty}(x+Q)}$. We turn off the integral in region $Q$ if $Q \subset D_{\text {in }}$ since it is not in $D_{\text {ring }}$, where $D_{\text {ring }}$ is defined in (4.41).

Finally, we cover $x+Q$ uniformly for $a, b$ (the sub-partition of $x$ ) to bound $\|f\|_{L^{\infty}(x+Q)}$. Since we add 4 extra points in $Z_{1}$ and $Z_{2}$, and order them in an increasing order, the region $Q_{c, d}$ can change for fixed $c, d$ but with different $a, b$. We show that the $2 k+4$ intervals $\left[z_{1, c}, z_{1, c+1}\right], 1 \leq$ $c \leq 2 k+4$ can be covered by $\left[\alpha_{l}, \beta_{l}\right]$ uniformly for $a, b$

$$
\begin{align*}
& {\left[\alpha_{l}, \beta_{l}\right], \quad \alpha_{l} \in Z_{1}^{l}, \beta_{l} \in Z_{1}^{u}, Z_{1}^{l} \triangleq\{-(k+1) \leq i \leq k, i \in \mathbb{Z}\} \cup\left\{-s_{0}-2, s_{0}\right\}}  \tag{4.43}\\
& Z_{1}^{u} \triangleq\{-k \leq i \leq k+1, i \in \mathbb{Z}\} \cup\left\{-s_{0}, s_{0}+2\right\}, \quad s_{0}=\left\lfloor k_{2}\right\rfloor
\end{align*}
$$

with $\alpha_{l}, \beta_{l}$ increasing. From (4.40) and the definition of $s_{0}$, we get

$$
\begin{equation*}
p_{1} \in\left[-s_{0}-2,-s_{0}\right], p_{2} \in\left[s_{0}, s_{0}+2\right], q_{1} \in[-k-1,-k], q_{2} \in[k, k+1] \tag{4.44}
\end{equation*}
$$

The uniform covering is based on the following observations. Suppose that $u_{i} \leq v_{i}, i=$ $1,2, . ., n\left(u_{i}, v_{i}\right.$ may not be increasing). Let us denote by $\left\{U_{i}\right\}$ the re-ordering of $\left\{u_{i}\right\}$ in an increasing order and denote by $\left\{V_{i}\right\}$ the re-ordering of $\left\{v_{i}\right\}$ in an increasing order. Then we have $U_{i} \leq V_{i}$. In fact, for any $k \leq n$, from $u_{i} \leq v_{i}, V_{k}$ is larger than $u_{j}$ with at least $k$ different indexes $j$. Since $U_{k}$ is the $k$-smallnest value in $\left\{u_{i}\right\}_{i}$, we get $V_{k} \geq U_{k}$.

From (4.42), (4.44), since $q_{2}=\max _{c} z_{1, c}, q_{1}=\min _{c} z_{1, c}$, we get

$$
\begin{aligned}
& \left\{z_{1, c}, c \leq 2 k+4\right\}=\{-k \leq i \leq k, i \in \mathbb{Z}\} \cup\left\{p_{1}, p_{2}, q_{1}\right\},-k-1 \leq q_{1},-s_{0}-2 \leq p_{1}, s_{0} \leq p_{2} \\
& \left\{z_{1, c+1}, c \leq 2 k+4\right\}=\{-k \leq i \leq k, i \in \mathbb{Z}\} \cup\left\{p_{1}, p_{2}, q_{2}\right\}, p_{1} \leq-s_{0}, p_{2} \leq s_{0}+2, q_{2} \leq k+1
\end{aligned}
$$

We can bound each component in $Z_{1}^{l}$ (4.43) by a component in the above list. Using the above observations, after reordering two sequences in an increasing order, which gives $\left\{\alpha_{c}\right\},\left\{z_{1, c}\right\}_{c \leq 2 k+4}$, we get $\alpha_{c} \leq z_{1, c}, c \leq 2 k+4$. Similarly, we obtain $z_{1, c+1} \leq \beta_{c}$, and yield $\left[z_{1, c}, z_{1, c+1}\right] \in\left[\alpha_{c}, \beta_{c}\right], c \leq$ $2 k+4$.

Similarly, we obtain $\left[z_{2, d}, z_{2, d+1}\right] \subset\left[\alpha_{l}, \beta_{l}\right]$. Thus, we get $\left[z_{1, c}, z_{1, c+1}\right] \times\left[z_{2, d}, z_{2, d+1}\right] \in$ $\left[\alpha_{c}, \alpha_{c+1}\right] \times\left[\beta_{d}, \beta_{d+1}\right]$ uniformly for the sub-partition of $x \in A_{a} \times B_{b}$ with $0 \leq a, b \leq m-1$, and can cover $x+Q$ by $B_{i_{1} j_{1}}\left(h_{x}\right)+\left[\alpha_{c} h, \alpha_{c+1} h\right] \times\left[\beta_{d} h, \beta_{d+1} h\right]$ (4.16).
4.1.8. Second generalization: the boundary terms. We generalize the method to estimate some boundary terms. We estimate the $x_{1}$-derivative of $I_{3}(x)(4.24)$ to illustrate the ideas. In $\partial_{1} I_{3}$, we have an extra boundary term $I_{32}$
$\partial_{1} I_{3}(x)=\int_{R(k) \backslash R_{s, 1}(k)} \partial_{x_{1}} K_{1}(x-y)(W \psi)(y) d y-\left.\int_{(j-k) h}^{(j+1+k) h} K_{1}(x-y)(W \psi)(y)\right|_{y_{1}=x_{1}-k h} ^{x_{1}+k h} d y_{2} \triangleq I_{31}+I_{32}$,
where we have used the domain for $R(x, k)$ (4.18).
For $I_{31}$, we apply the previous method to estimate it. Denote $\left.\Gamma_{k} \triangleq[j-k) h,(j+1+k) h\right]$. Using a change of variable $y=x+s$, we can rewrite $I_{32}$ as follows
$I_{32}=-\int_{s_{2} \in \Gamma_{k}-x_{2}}\left(K_{1}\left(-k h,-s_{2}\right)(W \psi)\left(x_{1}+k h, x_{2}+s_{2}\right)-K_{1}\left(k h,-s_{2}\right)(W \psi)\left(x_{1}-k h, x_{2}+s_{2}\right)\right) d s_{2}$.
We partition the location of $x$ and assume $x \in A_{a} \times B_{b} \subset B_{i_{1}, j_{1}}\left(h_{x}\right)$ (4.34). From (4.36), we have

$$
s_{2} \in \Gamma_{k}-x_{2} \subset Y_{d, b} \cup Y_{m, b} \cup Y_{u, b} .
$$

Using the above decomposition and $|W \psi(x)| \leq\|W \varphi\|_{\infty} f(x), f=\psi \varphi^{-1}$, we obtain

$$
\left|I_{32}\right| \leq\|W \varphi\|_{\infty} \sum_{\alpha= \pm, \beta=d, m, u} M_{\alpha, \beta}, \quad M_{\alpha, \beta} \triangleq \int_{Y_{\beta, b}}\left|K_{1}\left(-\alpha k h,-s_{2}\right)\right| \cdot\left|f\left(x_{1}+\alpha k h, x_{2}+s_{2}\right)\right| d s_{2}
$$

for $\alpha= \pm, \beta=u, m, d$. For $\beta=u, d$, the domain $Y_{\beta, b}$ is small $\left|Y_{\beta, b}\right| \leq h$. We apply the method in (4.38) to estimate $M_{\alpha, \beta}$. The only difference is that we need consider a 1D integral here

$$
\int_{Q}\left|K_{1}\left(-\alpha k h,-s_{2}\right)\right| d s_{2}
$$

for some interval $Q$, rather than a 2D integral in (4.38). For $M_{\alpha, m}$, we decompose the domain $Y_{m, b}$ into small intervals with length $h$ similar to (4.39) and then apply the method in (4.38).

We combine these estimates to bound $I_{32}$ for $x \in A_{a} \times B_{b}$. Then, we maximize the estimates over $0 \leq a, b \leq m-1$ to bound $I_{32}$ for $x \in B_{i_{1}, j_{1}}\left(h_{x}\right)$.
4.1.9. Third generalization. In some of the computations, we need to estimate

$$
J=\int_{R(k) \backslash R_{s}\left(k_{2}\right)}|K(x-y)| f(y) d y
$$

for some $k_{2}<k$ with $2 k_{2}, k \in Z^{+}$, where $R_{s}(k)$ is defined in (4.19). Similarly, we use

$$
R_{s}\left(k_{2}\right) \subset R_{s}(k) \subset R(k), \quad R(k) \backslash R_{s}\left(k_{2}\right)=R(k) \backslash R_{s}(k) \cup R_{s}(k) \backslash R_{s}\left(k_{2}\right)
$$

and a change of variable $y=x+s$ to obtain

$$
J=\left(\int_{s \in R(k)-x,\left|s_{1}\right| \vee\left|s_{2}\right| \geq k h}+\int_{k_{2} h \leq\left|s_{1}\right| \vee\left|s_{2}\right| \leq k h}\right) K(-s) f(x+s) d y \triangleq J_{1}+J_{2}
$$

Compared to $R(k) \backslash R_{s, 1}(k)$, the domain $R(k) \backslash R_{s}(k)$ contains two more parts

$$
X_{m, a} \triangleq[-k h, k h], \quad X_{m, a} \times Y_{u, b}, \quad X_{m, a} \times Y_{d, b}
$$

i.e., the upper and lower blue regions in the right figure in Figure2, The integral in these regions is estimated similar to that in $X_{\alpha, a} \times Y_{m, b}$ (4.37), and the estimate of $J_{1}$ is similar to $J$ in (4.33).

For $J_{2}$, the domain is simpler. Since $2 k_{2} \in Z^{+}$, we partition the domain into $h_{x} \times h_{x}$ grids
$J_{2}=\sum_{(c, d) \in S_{k} \backslash S_{k_{2}}} \int_{\left[c h_{x},(c+1) h_{x}\right] \times\left[d h_{x},(d+1) h_{x}\right]}|K(-s)| f(s+x) d s, \quad S_{l} \triangleq\{-k \leq c<k,-k \leq d<k\}$.
For each integral, we estimate it using the method in (4.38). The remaining steps are the same as those of $J$ in (4.33) studied previously.

Remark 4.5. In the estimates in Section 4.1.6-4.1.9, we use the important property that the weights are locally smooth to move them outside the integral. Moreover, we use the fact that the singular region depend on $x$ monotonously to cover it effectively. Since the integral $\int_{Q}\left|K_{1}(s)\right| d y$ for different $Q, a, b$ in the above estimates does not depend on $x$, we first compute these integrals once and store them, and then use them in later estimate of different $x$.
4.1.10. Taylor expansion near the singularity. We need to estimate the integral

$$
J(x) \triangleq \int_{D} \partial_{x_{i}}(K(x-y)(\psi(x)-\psi(y)) W(y)) d y
$$

for $k_{2}<k$ in some region $D$ close to the singularity $x$. For example, $D=R\left(x, k_{2}\right) \backslash R\left(x, k_{3}\right)$, $R\left(x, k_{3}\right) \backslash R_{s 1}\left(x, k_{3}\right)$ in $\partial_{x_{i}} I_{5,0}, \partial_{x_{i}} I_{5,1}$ (4.51), To obtain a sharp estimate, we perform Taylor expansion on $\psi(x)$. We focus on $\partial_{x_{1}}$. Denote $z=x-y, x_{m}=\frac{x+y}{2}$. A direct computation yields

$$
I=\partial_{x_{1}}(K(x-y) \psi(x)-\psi(y))=\left(\partial_{1} K\right)(x-y)(\psi(x)-\psi(y))+K(x-y) \partial_{1} \psi(x)
$$

Using Taylor expansion of $\psi$ at $x_{m}$ and following (B.26), we get
$\psi(x)-\psi(y)=(x-y) \cdot \nabla \psi\left(x_{m}\right)+\varepsilon_{1}, \quad \psi_{x}(x)=\psi_{x}\left(x_{m}\right)+\varepsilon_{2}$
$\left|\varepsilon_{1}\right| \leq \sum_{i+j=2} c_{i j}| | \partial_{x}^{i} \partial_{y}^{j} \psi \|_{L^{\infty}(Q(y))}\left|z_{1}\right|^{i}\left|z_{2}\right|^{j}, \quad\left|\varepsilon_{2}\right| \leq \frac{1}{2}\left(\left\|\partial_{x x} \psi\right\|_{L^{\infty}(Q(y))}\left|z_{1}\right|+\left\|\partial_{x x} \psi\right\|_{L^{\infty}(Q(y))}\left|z_{2}\right|\right)$,
where $c_{20}=\frac{1}{4}, c_{11}=\frac{1}{2}, c_{02}=\frac{1}{4}$, and we have written $z_{i}=x_{i}-y_{i}$ and $Q(y)$ is one of the four quadrants $D \cap\left\{y: \operatorname{sgn}\left(y_{i}-x_{i}\right)= \pm 1\right\}$ covering both $x, y$. Combining the term with the same derivative of $\psi$, we need to estimate the following integrals
$\left|\int_{D} \psi_{x}\left(x_{m}\right)\left(\partial_{1} K(z) z_{1}+K(z)\right) W(y) d y\right|, \quad\left|\int_{D} \psi_{y}\left(x_{m}\right) \partial_{1} K(z) z_{2} W(y) d y\right|$
$\int_{D}\left|\partial_{x}^{i} \partial_{y}^{j} \psi\right|_{L^{\infty}(Q(y))}\left|\partial_{1} K(z) z_{1}^{i} z_{2}^{j} W(y)\right| d y, i+j=2, \quad \int_{D}\left|\partial_{x}^{i+1} \partial_{y}^{j} \psi\right|_{L^{\infty}(Q(y))}\left|K(z) z_{1}^{i} z_{2}^{j} W(y)\right| d y, i+j=1$,
We partition the region of $z=x-y \in x-D$, e.g. $D=R\left(k_{2}\right) \backslash R\left(k_{3}\right)$ (4.51) into small mesh, and estimate the piecewise bounds of weights and each integral following Sections 4.1.6 4.1.9,

We estimate the integral of $\left|\partial_{1}^{i} \partial_{2}^{j} K(z) z_{1}^{k} z_{2}^{l}\right|$ in Section 5.1 in the supplementary material II [11].
4.1.11. Hölder estimate of log-Lipschitz function. In some computation, we need to perform $C^{1 / 2}$ estimate of some log-Lipschitz function. We consider an example to illustrate the ideas

$$
F(x)=\int_{\max _{i}\left|x_{i}-y_{i}\right| \leq b} K(x, y) f(y) d y, \quad|K(x, y)| \leq C_{1}|x-y|^{-1}, \quad|\partial K(x, y)| \leq C_{2}|x-y|^{-2}
$$

for some constant $C_{1}, C_{2}$. Given $f \in L^{\infty}, F$ is log-Lipschitz. To estimate $[f]_{C_{x}^{1 / 2}}$, we cannot first estimate the piecewise values of $f$ and $\partial_{x} f$ and then combine them to obtain the $C_{x}^{1 / 2}$ estimate. Instead, given $x, z$, for $a$ to be determined, we decompose $F$ into the smooth part and the singular part

$$
F_{1}(x) \triangleq \int_{a \leq \max _{i}\left|x_{i}-y_{i}\right| \leq b} K(x, y) f(y) d y, \quad F_{2}(x) \triangleq \int_{\max _{i}\left|x_{i}-y_{i}\right| \leq a} K(x, y) f(y) d y
$$

Using the assumptions of the kernel, we have

$$
\left|\partial_{x_{1}} F_{1}(x)\right| \leq C_{3} \log \frac{b}{a}\|f\|_{\infty}, \quad\left|F_{2}(x)\right| \leq C_{4}|a| \cdot\|f\|_{\infty}
$$

where the constants $C_{3}, C_{4}$ depend on $b, C_{1}, C_{2}$. Applying the above estimates, we obtain
$\frac{|F(x)-F(z)|}{\left|x_{1}-z_{1}\right|^{1 / 2}} \leq \frac{\left|F_{1}(x)-F_{1}(z)\right|+\left|F_{2}(x)-F_{2}(z)\right|}{\left|x_{1}-z_{1}\right|^{1 / 2}} \leq\left(C_{3} \log \frac{b}{a} \cdot\left|x_{1}-z_{1}\right|^{1 / 2}+2 C_{4}|a|\left|x_{1}-z_{1}\right|^{-1 / 2}\right)\|f\|_{\infty}$.
We optimize the estimates by choosing $a=C_{5}\left|x_{1}-z_{1}\right|$ for some constant $C_{5}$ depending on $C_{3}, C_{4}$. Then we establish the estimate. The above simple estimates show that the choice of $a$ depends on $|x-z|$. Thus, in our later Hölder estimates, we perform decomposition guided by the above estimates and optimize the choice of size of the singular region $[-a, a]^{2}$. On the other
hand, since for different $|x-z|$ we need to choose different $a$, it increases the technicality of the computer-assisted estimates.
4.2. $L^{\infty}$ estimate. Let $\hat{u}_{x, A}$ be the approximation term of $u_{x}$ (see Section 4.3 of Part I [13]). We focus on the estimate of the piecewise $L^{\infty}$ norm of $u_{x, A}=u_{x}-\hat{u}_{x, A}$, which is a representative case. For simplicity, we assume the rescaling factor $\lambda=1$. We assume that $x$ satisfies (4.16) without loss of generality. We want to estimate $u_{x, A}$ for all $x \in B_{i_{1} j_{1}}\left(h_{x}\right)$.

We can write $u_{x, A}=u_{x}-\hat{u}_{x}$ as follows

$$
u_{x, A}=\int(K(x-y)-\hat{K}(x, y)) W(y) d y, \quad K_{A} \triangleq K(x-y)-\hat{K}(x, y)
$$

where $\hat{K}(x, y)$ is the kernel for the approximation term and $W$ is the odd extension of $\omega$ (see (4.23)). From Sections 4.3 .2 and 4.3 .3 of Part I [13], we remove the singular part in $\hat{K}$, and then $\hat{K}$ is nonsingular. Given $x$ with (4.16), similar to (4.24), for $k \geq k_{2}$, we perform the following decomposition

$$
\begin{align*}
u_{x, A} & =\left(\int_{R(k)^{c}}+\int_{R(k) \backslash R_{s}\left(k_{2}\right)}+\int_{R_{s}\left(k_{2}\right)}\right) K(x-y) W(y) d y-\int \hat{K}(x, y) W(y) d y  \tag{4.45}\\
& \triangleq I_{1}+I_{2}+I_{3}+I_{4}
\end{align*}
$$

where $R_{s}(k)$ is the symmetric singular region (4.19). See Section 4.2.3 for the choice of $k$.
Since $I_{1}+I_{4}$ is nonsingular, we use the ideas in Section 4.1.5 to symmetrize the kernels in $I_{1}+I_{4}$. Then we use the method in Section 4.1.3 to estimate it.
Remark 4.6. In our computation, the domain $[0, D]^{2} \cap R(k)^{c}$ can be decomposed into the union of small grids $\left[y_{i}, y_{i+1}\right] \times\left[y_{j}, y_{j+1}\right]$ (4.11) since the boundary of $R(x, k)$ aligns with the mesh (4.18). In particular, in each grid, the indicator function is constant, and the integrand is smooth in $y$.

Next we consider $I_{2}$. The domain of the integral is close to the singularity. If we use the method in Section 4.1.3 to estimate it, the error will be quite large since $\partial^{2} K(x-y)$ is very singular. We want to estimate $I_{2}$ using $\|W \varphi\|_{\infty}$ and the singular part $I_{3}$ using $\left[W \psi_{1}\right]_{C^{1 / 2}}$. Since $K(z)$ is singular of order -2 , we expect an estimate

$$
\left|I_{2}\right|+\left|I_{3}\right| \lesssim \log \frac{k}{k_{2}} \varphi^{-1}(x)| | W \varphi \|_{L^{\infty}[R(k)]}+\psi^{-1}(x) k_{2}^{1 / 2}[W \psi]_{C_{x}^{1 / 2}}
$$

Note that the weights $\varphi, \psi$ have a different order of singularity for small $x$ and a different rate of decay. Moreover, we need to control the right hand side using the energy, which assigns different weights to two norms (seminorms). Thus, to obtain a sharp estimate, we need to optimize the choice of $k_{2}$.

Firstly, we consider $k_{2}=2,2+\frac{1}{2}, . ., k$, we use the method in Section 4.1.9 to estimate $I_{2}$. We also consider very small $k_{2}<2$. In this case, we further decompose $I_{2}$ as follows

$$
I_{2}=\left(\int_{R(k) \backslash R_{s}(2)}+\int_{R_{s}(2) \backslash R_{s}\left(k_{2}\right)}\right) K(x-y) W(y) d y \triangleq I_{21}+I_{22}
$$

For $I_{21}$, we apply the method in Section4.1.9 For $I_{22}$, we use a change of variables $y=x+s h$

$$
\left|I_{22}\right|=\left|\int_{k_{2} \leq\left|s_{1}\right| \vee\left|s_{2}\right| \leq 2} K(-s h) W(x+s h) h^{2} d s\right|
$$

Since the region is very small, $x+s h \in B_{i_{1} j_{1}}\left(h_{x}\right)+[-2 h, 2 h]$, and $K_{1}(h s)=h^{-2} K_{1}(s)$, we get

$$
\left|I_{22}\right| \leq\|W \varphi\|_{\infty}\left\|\varphi^{-1}\right\|_{L^{\infty}\left(B_{i_{1} j_{1}}\left(h_{x}\right)+[-2 h, 2 h]\right)} \int_{k_{2} \leq\left|s_{1}\right| \vee\left|s_{2}\right| \leq 2}|K(s)| d s
$$

The integral can be computed explicitly and has the order $\log \frac{2}{k_{2}}$.
It remains to estimate the most singular part $I_{3}$ for different $k_{2}$. Using a change of variables $y=x+s h$, the scaling symmetries, and the above derivations, we get

$$
I_{3}=\int_{\left[-k_{2}, k_{2}\right]^{2}} K(-s) W(x+s h) d s
$$

To use the Hölder norm of $W \psi$, we decompose it as follows

$$
\begin{equation*}
I_{3}=\int_{\left[-k_{2}, k_{2}\right]^{2}} K(-s)(W \psi)(x+s h)\left(\frac{1}{\psi(x+s h)}-\frac{1}{\psi(x)}\right)+K(-s) \frac{(W \psi)(x+s h)}{\psi(x)} d s \triangleq I_{31}+I_{32} \tag{4.46}
\end{equation*}
$$

For $I_{32}$, using the Hölder seminorm, the odd symmetry of $K(s)=c \frac{s_{1} s_{2}}{|s|^{4}}$ in $s_{1}$, and $\mid(W \psi)(x+$ $s h)-(W \psi)(x-s h) \mid \leq \sqrt{2 s_{1} h}$, we get

$$
\left|I_{32}\right| \leq \frac{h^{1 / 2}}{\psi(x)}[W \psi]_{C_{x}^{1 / 2}} \int_{\left[0, k_{2}\right] \times\left[-k_{2}, k_{2}\right]}|K(s)| \sqrt{2 s_{1}} d s=\frac{2 k_{2}^{1 / 2} h^{1 / 2}}{\psi(x)}[W \psi]_{C_{x}^{1 / 2}} \int_{[0,1]^{2}}|K(s)| \sqrt{2 s_{1}} d s
$$

where we used the scaling symmetry of $K$ and a change of variables $s \rightarrow k_{2} s$ in the last equality.
4.2.1. The commutator. For $I_{31}$, we apply the simple Taylor expansion to $f=\psi^{-1}$

$$
\begin{equation*}
|f(x+s h)-f(x)| \leq\left|f_{x}(x) h s_{1}+f_{y}(x) h s_{2}\right|+h^{2}\left(\frac{m_{20} s_{1}^{2}}{2}+m_{11} s_{1} s_{2}+\frac{m_{02} s_{2}^{2}}{2}\right) \tag{4.47}
\end{equation*}
$$

where $m_{i j}$ is the bound for the second derivatives of $\psi^{-1}$

$$
m_{i j}(s)=\max _{B_{i_{1} j_{1}}(h)+I\left(\operatorname{sgn}\left(s_{1}\right)\right) \times I\left(\operatorname{sgn}\left(s_{2}\right)\right)}\left\|\partial_{x}^{i} \partial_{y}^{j}\left(\psi^{-1}\right)\right\|_{L^{\infty}}, \quad I_{+}=\left[0, k_{2} h\right], \quad I_{-}=\left[-k_{2} h, 0\right] .
$$

Note that $m_{i j}$ is constant in each quadrant of $\left[-k_{2}, k_{2}\right]$. We plug in the expansion (4.47) to estimate $I_{31}$. We only discuss a typical term $m_{20} s_{1}^{2} h^{2}$

$$
I_{31,20} \triangleq h^{2} \int_{\left[-k_{2}, k_{2}\right]^{2}}|K(-s)(W \psi)(x+s h)| m_{20}(s) \frac{s_{1}^{2}}{2} d s
$$

If $k_{2} \geq 2$, we can further partition $\left[-k_{2}, k_{2}\right]^{2}$ into $B_{2 p, 2 q}(1 / 2)=[p, p+1 / 2] \times[q, q+1 / 2],-k_{2} \leq$ $p, q \leq k_{2}-1 / 2$, where we use the notation (4.17). For each grid $B_{2 p, 2 q}(1 / 2)$, the sign of $s$ and $m_{20}(s)$ are fixed, and we have

$$
\int_{B_{2 p, 2 q}\left(\frac{1}{2}\right)}|K(-s)|(W \psi)(x+s h) m_{20}(s) \frac{s_{1}^{2}}{2} d s \leq m_{20}\|W \varphi\|_{\infty} \int_{B_{2 p, 2 q}\left(\frac{1}{2}\right)} \frac{|K(s)| s_{1}^{2}}{2}\left(\frac{\psi}{\varphi}\right)(x+s h) d s
$$

The last integral can be estimated using the method in (4.38). Combining the estimate of integral in different regions $B_{2 p, 2 q}(1 / 2)$, we obtain the estimate of $I_{31,02}$. Similarly, we can estimate the contributions of other terms in (4.47) to $I_{31}$.

For small $k_{2} \leq 2$, we do not partition the domain. We denote $D\left(k_{2}\right)=B_{i_{1}, j_{1}}\left(h_{x}\right)+$ $\left[-k_{2} h, k_{2} h\right]^{2}$. For $s \in\left[-k_{2}, k_{2}\right]$, we use $x+s h \subset D\left(k_{2}\right) \subset D(2)$ to get
$|f(x+s h)-f(x)| \leq\left\|f_{x}\right\|_{L^{\infty}\left(D\left(k_{2}\right)\right)} s_{1} h+\left\|f_{y}\right\|_{L^{\infty}\left(D\left(k_{2}\right)\right)} s_{2} h . \quad|W \psi(x+s h)| \leq\|W \varphi\|_{\infty}\left\|\frac{\psi}{\varphi}\right\|_{L^{\infty}(D(2))}$.
Plugging the above estimate into $I_{31}$, we get

$$
I_{31} \leq \sum_{(i, j)=(1,0),(0,1)} h\left\|\partial_{x}^{i} \partial_{y}^{j}\left(\psi^{-1}\right)\right\|_{L^{\infty}\left(D\left(k_{2}\right)\right)}\|W \varphi\|_{\infty}\left\|\frac{\psi}{\varphi}\right\|_{L^{\infty}(D(2))} \int_{\left[-k_{2}, k_{2}\right]^{2}}\left|K(s) s_{1}^{i} s_{2}^{j}\right| d s
$$

Using the scaling symmetry, we can reduce the last integral to $k_{2}^{i+j} \int_{[-1,1]^{2}}\left|K(s) s_{1}^{i} s_{2}^{j}\right| d s$.
We apply the above estimates to a list of $k_{2}$, and bound different norms using $\max \left(\|\omega \varphi\|_{\infty}\right.$, $\max _{i} \gamma_{i}\left[\omega \psi_{1}\right]_{C_{x_{i}}^{1 / 2}\left(\mathbb{R}_{2}^{+}\right)}$. Then by optimizing the $k_{2}$, we obtain the sharp estimate of $u_{x, A}$.

In (4.47), we do not bound $f(x+s h)-f(x)$ directly using the estimate 4.48) since $s$ is large. Instead, we perform a higher order expansion.

Estimate of $u_{y}, v_{x}$. The estimates of $u_{y}, v_{x}$ follow similar strategies and estimates. The only difference is the estimate of the most singular term similar to $I_{32}$ (4.46) for $u_{y}, v_{x}$ due to different symmetry property of the kernel. We estimate it using a combination of norms $\|\omega \varphi\|_{\infty}$, and semi-norms $[\omega \psi]_{C_{x_{i}}^{1 / 2}}$, and refer it to Section 6.1 in the supplementary material II [11].
4.2.2. Estimate of $\mathbf{u}_{A}$. The estimate of $\mathbf{u}_{A}$ is much simpler since it is more regular. Let $K$ and $\hat{K}$ be the kernel of $u, v$ and its approximation term, respectively. For $f=u$ or $v$, we perform a decomposition similar to (4.45)

$$
\begin{equation*}
f_{A}=\left(\int_{R(k)^{c}}+\int_{R(k) \backslash R_{s}(k)}+\int_{R_{s}(k)}\right) K(x-y) W(y) d y-\int \hat{K}(x, y) W(y) d y \triangleq I_{1}+I_{2}+I_{3}+I_{4} \tag{4.49}
\end{equation*}
$$

The estimates of $I_{1}+I_{4}$ follow the method for $u_{x, A}$. For $I_{2}$, we use the method in Section 4.1.6, For $I_{3}$, since $K$ has a singularity of order $|x|^{-1}$, which is locally integrable, we use a change of variable $y=x+s h$ to obtain

$$
I_{3}=h \int_{[-k, k]^{2}} K(-s) W(x+s h) d s
$$

Then we partition $[-k, k]^{2}$ into small grids, and use the method in (4.38) to estimate the integral in each grid. Here, we get a factor $h$ in the change of variables since $K(\lambda s)=\lambda^{-1} K(s)$.
4.2.3. Choice of parameters. Recall the choice of several parameters $a, h, h_{x}$ from (4.14). We choose $3 \leq k \leq 10$. We choose $k$ for the size of the singular region $k h$ (4.45), (4.49) not so small such that the error $h^{2} \partial^{2} K$ in Lemma 4.2, which has the order $h^{2}|x-y|^{-\alpha-2}$ near the singularity, is smaller than the main term $K$, which has the order $|x-y|^{-\alpha}, \alpha=1,2$. Since we will estimate $I_{1}+I_{4}, I_{2}, I_{3}$ in the decomposition separately using the triangle inequality, we do not choose $k$ to be too large so that we can exploit the cancellation in $I_{1}+I_{4}$.
4.3. Hölder estimates. We want to estimate $\frac{|f(x)-f(z)|}{|x-z|^{1 / 2}}$ for any $x, z \in \mathbb{R}_{2}^{++}$with $x_{1}=z_{1}$ or $x_{2}=z_{2}$ and some function $f$, e.g. $f=u_{x, A}$. Without loss of generality, we assume $|z|>$ $|x|$. Then in the $C_{x}^{1 / 2}$ estimate, we have $x_{1}<z_{1}, x_{2}=z_{2}$; in the $C_{y}^{1 / 2}$ estimate, we have $x_{1}=z_{1}, x_{2}<z_{2}$. Applying the rescaling argument in Section 4.1] we can restrict $\hat{x}=\frac{x}{\lambda}$ to $\hat{x} \in\left[0,2 x_{c}\right]^{2} \backslash\left[0, x_{c}\right]^{2}$. For this reason, we assume $\lambda=1$ for simplicity. We will only estimate the Hölder difference for comparable $x, z:|x| \asymp|z|$. If $|z| \gg|x|$, we simply apply the $L^{\infty}$ estimate to $f(x), f(z)$ and use the triangle inequality.

We focus on the Hölder estimate of $u_{x, A}$, which is a representative and the most important nonlocal term to estimate in our energy estimate.
4.3.1. $C_{x}^{1 / 2}$ estimate. Recall $I_{i}$ from the decomposition (4.24) and $K_{1}(s)=\frac{s_{1} s_{2}}{|s|^{4}}$. We apply the same decomposition to $u_{x, A}(z)$. We assume that the approximation term $\hat{u}_{x}$ (see Section 4.3.3 of Part I [13]) takes the following form

$$
\begin{equation*}
\hat{u}_{x}(x)=\int \hat{K}_{1}(x, y) W(y) d y, \quad I_{6}(x) \triangleq \psi(x) \hat{u}_{x}(x)=\psi(x) \int \hat{K}_{1}(x, y) W(y) d y \tag{4.50}
\end{equation*}
$$

with a nonsingular kernel $\hat{K}_{1}$. We first discuss how to estimate the regular part $I_{1}, I_{3}, I_{4}$ in (4.24) and $I_{6}$, which are Lipschitz. We will apply the sharp Hölder estimate in Lemmas 3.1-3.5 in Section 3 of Part I 13 to estimate the most singular part $I_{2}$. The most technical part is to estimate $I_{5}$, which is log-Lipschitz since the kernel $K_{1}(x-y)(\psi(x)-\psi(y))$ has a singularity of order -1 . We assemble the estimates of different parts to estimate $\left[u_{x, A} \psi\right]_{C_{x}^{1 / 2}}$ in Section 4.6,
4.3.2. Estimates of the regular terms $I_{1}, I_{3}, I_{4}, I_{6}$. Recall $I_{1}, I_{3}, I_{4}$ from (4.24) and $I_{6}$ from (4.50). Since the integrands in $I_{1}, I_{3}, I_{4}$ are supported at least $k_{2} h$ away from the singularity $x$, if $W$ is in some suitable weighted $L^{\infty}$ space, $I_{1}, I_{3}, I_{4}$ are Lipschitz and their derivatives can be bounded by $\|W \varphi\|_{\infty\left(\mathbb{R}_{2}^{++}\right)}=\|\omega \varphi\|_{\infty}$. In fact, $I_{1}$ and $I_{4}$ are piecewise smooth. Their derivatives jump when $R(x, k), R\left(x, k_{2}\right)$ change, or equivalently, $x$ moves from one grid to another. For $x \in B_{i_{1}, j_{1}}\left(h_{x}\right)$ (4.16), these rectangle domains are the same, and these functions are smooth. The approximation term $I_{6}(4.50)$ is locally smooth in $x$. To exploit the cancellation, we combine the estimates of $I_{1}, I_{4}, I_{6}$ together. We symmetrize the kernel in $I_{1}(x)+I_{4}(x)-I_{6}(x)$ following Section 4.1 .5 and use the method in Section 4.1.3 to estimate the derivatives of $I_{1}(x)+I_{4}(x)-I_{6}(x)$. See also (4.28), (4.29) for the form of the symmetrized integrands in these integrals.

We estimate both the $L^{\infty}$ and Lipschitz norm of $I_{3}$ using the method in Sections 4.1.6, 4.1.8, We will optimize two estimates to obtain a sharper Hölder norm of $I_{3}$.

We choose integer $k, k_{2}$ in the decomposition (4.24) . Then in each grid $\left[y_{i}, y_{i+1}\right] \times\left[y_{j}, y_{j+1}\right]$, the indicator functions in $I_{1}+I_{4}-I_{6}$, e.g. $\mathbf{1}_{R(k)^{c}}, \mathbf{1}_{R(k) \backslash R\left(k_{2}\right)}$, are constant. See Remark 4.6.
4.3.3. $C_{x}^{1 / 2}$ estimate of $I_{2}$. We first estimate the second term $I_{2}$ in (4.24). Recall $R(x, k), R_{s 1}(x, k), R_{s}(x, k)$ from (4.18), (4.19) and the location of $x$ (4.16). We have
$x_{2}-(j-k) h \leq(j+1) h-(j-k) h=(k+1) h, \quad(j+1+k h)-x_{2} \leq(j+1+k h)-j h=(k+1) h$.
Since $x_{2}=z_{2}$, using Lemma 3.1 from Section 3 of Part I 13 with $\left(a, b_{1}, b_{2}\right)=\left(k h, x_{2}-(j-\right.$ $\left.k) h,(j+1+k) h-x_{2}\right)$ and $\left|b_{1}\right|,\left|b_{2}\right| \leq(k+1) h$, we obtain

$$
\frac{1}{|x-z|^{1 / 2}}\left|I_{2}(x, k)-I_{2}(z, k)\right| \leq C_{1}\left(\frac{(k+1) h}{|x-z|}\right)[W \psi]_{C_{x}^{1 / 2}}=C_{1}\left(\frac{(k+1) h}{|x-z|}\right)[\omega \psi]_{C_{x}^{1 / 2}} .
$$

We only apply the Hölder estimate to $|x-z| \leq \frac{k h}{2}$ (rescaled $x, z$ ) and the assumption $a \geq \frac{1}{2}\left|x_{1}-z_{1}\right|$ in Lemma 3.1 in Part I [13] is satisfied. For $I_{2}(x, k)$ associated with other terms $u, v, u_{y}, v_{x}$, we can estimate it using similar ideas and Lemmas 3.1-3.5 in Part I [13. The $C_{y}^{1 / 2}$ estimate of $I_{2}(x, k)$ is completely similar. See Section 4.3.8 for more details.
4.3.4. $C_{x}^{1 / 2}$ estimate of $I_{5}$. For $I_{5}$ (4.24), $K_{1}(x-y)(\psi(x)-\psi(y))$ is singular of order -1 near $y=x$. Given $W \in L^{\infty}(\varphi), I_{5}$ is log-Lipschitz. There are several approaches to estimate its Hölder norm, see e.g., Section 4.1.11. We use part of the $C_{x}^{1 / 2}$ seminorm of $\omega$ to get a better estimate. We choose $k_{3}=k_{2}-\frac{i}{2} \geq 2, i=0,1,2, . ., 2 k_{2}-4$ and further decompose $I_{5}$ as follows

$$
\begin{align*}
I_{5}\left(x, k_{2}\right) & =\left(\int_{R\left(k_{2}\right) \backslash R\left(k_{3}\right)}+\int_{R\left(k_{3}\right) \backslash R_{s, 1}\left(k_{3}\right)}+\int_{R_{s, 1}\left(k_{3}\right)}\right) K_{1}(x-y)(\psi(x)-\psi(y)) W(y) d y  \tag{4.51}\\
& \triangleq I_{5,0}\left(x, k_{2}, k_{3}\right)+I_{5,1}\left(x, k_{3}\right)+I_{5,2}\left(x, k_{3}\right)
\end{align*}
$$

The domain in $I_{5,0}$ depends on $x$. For $x$ in a grid cell, it does not change with $x$. We estimate $\partial_{x_{1}} I_{5,0}$ using Taylor expansion in Section 4.1 .10 and following the method in Section 4.1.7 We estimate the $L^{\infty}$ and $x$-derivative of $I_{5,1}$ using the method in Sections 4.1.6, 4.1.8. For $\partial_{x_{1}} I_{5,1}$, we have
$\partial_{x_{1}} I_{5,1}=\int_{R\left(k_{3}\right) \backslash R_{s, 1}\left(k_{3}\right)} \partial_{x_{1}}\left(K_{1}(x-y)(\psi(x)-\psi(y))\right) W(y) d y-\left.\int_{\left(j-k_{3}\right) h}^{\left(j+1+k_{3}\right) h} K_{1}(x-y)(W \psi)(y)\right|_{y_{1}=x_{1}-k_{3} h} ^{x_{1}+k_{3} h} d y_{2}$.
We estimate the first part following Section 4.1.10, and the second part following Section 4.1.8.
For $I_{5,2}$, we will estimate it using a method similar to that of $I_{2}$. See the left figure in Figure 3 for the domains of the integrals in $I_{5,2}(x), I_{5,2}(z)$. The integrand satisfies

$$
\begin{aligned}
K_{1}(x-y)(\psi(x)-\psi(y)) W(y) & =\psi(x) K_{1}(x-y)\left(\psi^{-1}(y)-\psi^{-1}(x)(W \psi)(y)\right. \\
& \approx \psi(x) \partial_{i}\left(\psi^{-1}(x)\right) \cdot K_{1}(x-y)\left(y_{i}-x_{i}\right)(W \psi)(y)
\end{aligned}
$$

Thus, $I_{5,2}(x)$ can be seen as a weighted version of $I_{2}$ (4.24) with a weight $\psi(x) \partial_{i}\left(\psi^{-1}(x)\right)$, a more regular kernel $K_{1}(x-y)\left(y_{i}-x_{i}\right)$, and a smaller domain $R_{s, 1}\left(k_{3}\right)$. Since the kernel is more regular and the domain is smaller, our estimate for $I_{5,2}$ is much smaller than that of $I_{2}$.

Now, we justify this approach. Using a change of variables $y=x+s, s \in R_{s, 1}\left(k_{3}\right)-x$ and the above identity, we yield

$$
I_{5,2}\left(x, k_{3}\right)=\psi(x) \int_{R_{s, 1}\left(k_{3}\right)-x} K_{1}(-s)\left(\psi^{-1}(x+s)-\psi^{-1}(x)\right)(W \psi)(x+s) d s
$$

Using Newton's formula $f(1)=f(0)+f^{\prime}(0)+\int_{0}^{1}(1-t) f^{\prime \prime}(t) d t$ for $f(t)=\psi^{-1}(x+t s)$, we get

$$
\begin{aligned}
\psi^{-1}(x+s)-\psi^{-1}(x) & =s \cdot \nabla \psi^{-1}(x)+\int_{0}^{1}(1-t)\left(s \cdot\left(\nabla^{2} \psi^{-1}\right)(x+t s) \cdot s\right) d t \\
& =\sum_{i=1,2} s_{i} \partial_{i}\left(\psi^{-1}\right)(x)+\sum_{0 \leq i \leq 2}\binom{2}{i} s_{1}^{i} s_{2}^{2-i} \int_{0}^{1}(1-t) \partial_{1}^{i} \partial_{2}^{2-i}\left(\psi^{-1}\right)(x+t s) d t
\end{aligned}
$$

Denote

$$
\begin{aligned}
& Q_{i j}(x)=\psi(x) \int_{0}^{1}(1-t) \partial_{1}^{i} \partial_{2}^{j}\left(\psi^{-1}\right)(x+t s) d t, i+j=2, \quad D(x)=R_{s, 1}\left(x, k_{3}\right)-x \\
& Q_{i j}(x)=\psi(x) \cdot \partial_{1}^{i} \partial_{2}^{j}\left(\psi^{-1}\right)(x)=-\frac{\partial_{1}^{i} \partial_{2}^{j} \psi(x)}{\psi(x)}, i+j=1, \quad P_{i j}(x)=\int_{D(x)} K_{1}(-s) s_{1}^{i} s_{2}^{j}(W \psi)(x+s) d s
\end{aligned}
$$

Using the above expansion and notations, we get

$$
I_{5,2}\left(x, k_{3}\right)=\sum_{i+j=1} P_{i j} Q_{i j}+\sum_{i+j=2}\binom{2}{i} P_{i j} Q_{i j}
$$

Next, we use the above decomposition to estimate $I_{5,2}\left(x, k_{3}\right)-I_{5,2}\left(z, k_{3}\right)$. The leading order terms are $P_{i j} Q_{i j}$ with $i+j=1$. By definition of $R_{s, 1}$ (4.19), we observe that if $x_{2}=z_{2}$, we have

$$
D(x)=R_{s, 1}\left(x, k_{3}\right)-x=R_{s, 1}\left(z, k_{3}\right)-z=D(z) .
$$

Suppose that $x_{1}<z_{1}$. We perform a decomposition

$$
\begin{align*}
& \left|P_{i j}(x) Q_{i j}(x)-P_{i j}(z) Q_{i j}(z)\right| \leq J_{1}+J_{2} \\
& J_{1} \triangleq\left|Q_{i j}(z)\left(P_{i j}(x)-P_{i j}(z)\right)\right|, \quad J_{2} \triangleq \mid P_{i j}(x)\left(Q_{i j}(x)-Q_{i j}(z)\right) \tag{4.53}
\end{align*}
$$

Using $D(x)=D(z)$, we bound $J_{1}$ as follows

$$
\begin{aligned}
\left|J_{1}\right| & \leq\left|Q_{i j}(z)\right|\left|\int_{D(x)} K_{1}(-s) s_{1}^{i} s_{2}^{j}((W \psi)(x+s)-(W \psi)(z+s)) d s\right| \\
& \leq\left|Q_{i j}(z)\right| \cdot|x-z|^{1 / 2}| | \omega \psi \|_{C_{x}^{1 / 2}} \int_{s \in D(x)}\left|K_{1}(s) s_{1}^{i} s_{2}^{j}\right| d s .
\end{aligned}
$$

The term $Q_{i j}$ only depends on the weight and is smoother than $P_{i j}$. We can estimate $Q_{i j}(x)-Q_{i j}(z)$ by bounding $\partial_{1} Q_{i j}$ since $Q_{i j}$ is locally smooth. For $P_{i j}$ in $J_{2}$, we use the method in (4.38) to bound it by $C\|\omega \varphi\|_{\infty}$ with some constant $C$. Then we obtain the estimate

$$
\left|J_{2}\right| \leq C_{2}|x-z| \cdot\|\omega \varphi\|_{L^{\infty}}
$$

for some constant $C_{2}$. Note that the second order term $P_{i j} Q_{i j}, i+j=2$ is much smaller than the leading order terms. For $|x-z|$ not too small, we can estimate its contribution trivially

$$
\frac{1}{|x-z|^{1 / 2}}\left|P_{i j}(x) Q_{i j}(x)-P_{i j}(z) Q_{i j}(z)\right| \leq \frac{1}{|x-z|^{1 / 2}}\left(\left|P_{i j}(x) Q_{i j}(x)\right|+\left|P_{i j}(z) Q_{i j}(z)\right|\right)
$$

We optimize the above two estimates.
In summary, to obtain the above estimates, we estimate piecewise bounds for $\left|Q_{i j}(x)\right|$, $P_{i j}(x),\left|\partial_{k} Q_{i j}(x)\right|$, and the integrals $\int_{D(x)}\left|K_{1}(s) s_{1}^{i} s_{2}^{j}\right| d s, i+j=1,2$.

The above estimate of $I_{5}\left(x, k_{2}\right)$ can be generalized to the $C_{x}^{1 / 2}$ estimate of $u, v, v_{x}, u_{y}$. Yet, it does not apply to the $C_{y}^{1 / 2}$ estimate of $\mathbf{u}, \nabla \mathbf{u}$ since it requires the estimate of $(W \psi)(x+s)-$ $(W \psi)(z+s)$ for $s$ in some rectangle $R=D(x)=D(z)$. However, since $W$ is discontinuous across the boundary $y=0, W \psi \notin C_{y}^{1 / 2}(R)$ if $x+s, z+s$ are not in the same half plane. If $x_{1}<x_{2}$, then the rectangles $R\left(x, k_{2}\right), R\left(z, k_{2}\right)$ will not intersect the boundary and the previous estimate holds true. If $x_{1}>x_{2}$, we consider two modifications for different kernels in the following subsections.


Figure 3. Left: $R_{s, 1}\left(x, k_{3}\right)$ and $R_{s, 1}\left(z, k_{3}\right)$ with $x_{2}=z_{2}$. The small square is a mesh grid containing $x$ or $z . x, z$ can have different locations relative to the grids. Right: The large rectangle is $R\left(k_{2}\right)$, the upper part is $R^{+}\left(k_{2}\right)$, and the lower part is $R^{-}\left(k_{2}\right)$. The blue region is $R^{-}\left(k_{2}\right) \backslash R^{-}\left(k_{3}\right)$. $\Gamma$ is part of its boundary.
4.3.5. Ideas of the $C_{y}^{1 / 2}$ estimates of $I_{5}$. The main idea in the following $C_{y}^{1 / 2}$ estimates is to use a combination of the estimates for the log-Lipschitz function in Section 4.1.11 and the estimate in Section 4.3.4. The latter provides better estimates, and we try to use this method as much as possible. Following the ideas in Section4.1.11, we decompose $I_{5}(x)$ into the singular part and nonsingular part with different size $k_{3}$ of the singular region

$$
I_{5}(x)=I_{5, S}\left(x, k_{3}\right)+I_{5, N S}\left(x, k_{3}\right)
$$

Although we cannot apply the second method to the whole $I_{5}(x)$, we can apply it to the integrals in the upper part of the regions, e.g. $R^{+}\left(k_{2}\right), R^{+}\left(k_{3}\right)$ (4.20), since these integrals only involve $W \psi$ in $\mathbb{R}_{2}^{+}$and we have $W \psi \in C^{1 / 2}$. Thus, we will further decompose some of the regions into the upper part and the lower part, and then apply the first method to the lower part, and the second method to the upper part.
4.3.6. $C_{y}^{1 / 2}$ estimate of the velocity with a kernel of the first type. The kernels

$$
\begin{equation*}
K=\frac{y_{1} y_{2}}{|y|^{4}}, \quad \frac{y_{2}}{|y|^{2}} \tag{4.54}
\end{equation*}
$$

associated with $u_{x}=-\partial_{x y}(-\Delta)^{-1} \omega, u=-\partial_{y}(-\Delta)^{-1} \omega$ vanish when $y_{2}=0$. We call them the first type kernel. Let $K$ be a kernel of the first type. We use the following decomposition

$$
I_{5}\left(x, k_{2}\right)=\left(\int_{R^{+}\left(k_{2}\right)}+\int_{R^{-}\left(k_{2}\right)}\right) K(x-y)(\psi(x)-\psi(y)) W(y) d y \triangleq I_{5}^{+}\left(x, k_{2}\right)+I_{5}^{-}\left(x, k_{2}\right)
$$

See the right figure in Figure 3 for $R^{ \pm}\left(k_{2}\right)$. Since $R^{+}\left(x, k_{2}\right), R^{+}\left(z, k_{2}\right) \subset \mathbb{R}_{2}^{+}$, we can apply the same argument as that for $I_{5,1}\left(x, k_{3}\right), I_{5,2}\left(x, k_{3}\right)$ in Section 4.3.4 to obtain the desired estimates by restricting all the derivations in $R^{+}\left(x, k_{2}\right), R^{+}\left(z, k_{2}\right)$. Note that here, we do not further choose smaller window $R^{+}\left(x, k_{3}\right)$ to decompose $I_{5}^{+}\left(x, k_{2}\right)$, i.e. $k_{3}=k_{2}$ and $I_{5,0}=0$ in (4.51). For $I_{5,1}^{+}$, similar to (4.52), we get a boundary term from $\partial_{2}\left(R^{+}\left(k_{2}\right) \backslash R_{s 2}^{+}\left(k_{2}\right)\right)=\left[\left(i-k_{2}\right) h,(i+\right.$ $\left.\left.1+k_{2} h\right)\right] \times\left\{x_{2}+k_{2} h\right\}$. See (4.19), (4.18) for $R^{+}(k), R_{s 2}^{+}(k)$.

For the lower part $I_{5}^{-}\left(x, k_{2}\right)$, it is log-Lipschitz if $W \in L^{\infty}(\varphi)$. We cannot bound its derivative using $\|W \varphi\|_{\infty}$. We face the difficulty discussed at the beginning of Section 4 .

Alternatively, we follow the ideas in Section 4.1.11. We decompose it into the smooth part and rough part. We introduce $0<k_{3}<k_{2}$ and consider the following decomposition

$$
\begin{align*}
I_{5}^{-}\left(x, k_{2}\right) & =\int_{R^{-}\left(k_{2}\right) \backslash R^{-}\left(k_{3}\right)} K(x-y)(\psi(x)-\psi(y)) W(y) d y \\
& +\int_{R^{-}\left(k_{3}\right)} K(x-y)(\psi(x)-\psi(y)) W(y) d y \triangleq I_{5,1}^{-}\left(x, k_{2}\right)+I_{5,2}^{-}\left(x, k_{2}\right) . \tag{4.55}
\end{align*}
$$

See the right figure in Figure 3 for an illustration of different domains. Recall that $k_{2} \in Z_{+}$. We choose $k_{3}=k_{2}-\frac{i}{2} \geq 2, i=0,1,2 . ., 2 k_{2}-4$. Since the integrand in $I_{5,1}^{-}$supports at least
$k_{3} h$ away from the singularity, $I_{5,1}^{-}\left(x, k_{2}\right)$ is Lipschitz. We can estimate $\partial_{x_{2}} I_{5,1}^{-}(x, k)$ following Sections 4.1.7, 4.1.10. The domain $R^{-}\left(k_{2}\right) \backslash R^{-}\left(k_{3}\right)$ is not piecewise constant since the upper part of its boundary, i.e.

$$
\Gamma=\left\{\left(y_{1}, x_{2}\right): y_{1} \in\left[\left(i-k_{2}\right) h,\left(i+1+k_{2}\right) h\right] \backslash\left[\left(i-k_{3}\right) h,\left(i+1+k_{3}\right) h\right]\right\}
$$

depends on $x_{2}$. See Figure 3 for an illustration of $\Gamma$. Taking $x_{2}$ derivative on $I_{5,1}^{-}$, we get

$$
\begin{align*}
\left|\partial_{x_{2}} I_{5,1}^{-}\left(x, k_{2}\right)\right| & \leq\left|\int_{R^{-}\left(k_{2}\right) \backslash R^{-}\left(k_{3}\right)} \partial_{x_{2}}(K(x-y)(\psi(x)-\psi(y))) W(y) d y\right|  \tag{4.56}\\
& \left.+\mid \int_{y \in \Gamma} K(x-y)(\psi(x)-\psi(y))\right) W(y) d y_{1} \mid
\end{align*}
$$

Since $y \in \Gamma \subset\left\{y: y_{2}=x_{2}\right\}$ and that $K\left(y_{1}, 0\right) \equiv 0$, the second term vanishes. The first term can be estimated using a change of variables $y=x+s$ and the method in Section4.1.10. Section 4.1.7 since its support is at least $k_{3} h$ away from the singularity.

For $I_{5,2}^{-}$, the kernel satisfies $K(x-y)(\psi(x)-\psi(y)) \sim|x-y|^{-1}$ for small $|x-y|$ and is locally integrable. We estimate its piecewise $L^{\infty}$ bound using the method in Section 4.2.1 for the commutator.

The above decomposition can be applied to estimate

$$
\frac{\left|I_{5}^{-}\left(x, k_{2}\right)-I_{5}^{-}\left(z, k_{2}\right)\right|}{|x-z|^{1 / 2}} \leq \min _{k_{3}=k_{2}-\frac{i}{2}} \frac{\left|I_{5,1}^{-}\left(x, k_{3}\right)-I_{5,1}^{-}\left(z, k_{3}\right)\right|}{|x-z|^{1 / 2}}+\frac{\left|I_{5,2}^{-}\left(x, k_{3}\right)\right|+\left|I_{5,2}^{-}\left(z, k_{3}\right)\right|}{|x-z|^{1 / 2}}
$$

for $|x-z|$ not too small, e.g. $|x-z| \geq d_{s}=\frac{h}{10}$. When $|x-z|$ is sufficiently small, the second term in the above estimate can be very large.

According to the analysis in Section 4.1.11, for $|x-z|$ very small, we need to choose $k_{3} h \sim$ $|x-z|$ to get the sharp estimate. Thus, we consider one more decomposition for $a \leq 1$

$$
\begin{align*}
I_{5}^{-}\left(x, k_{2}\right) & =\int_{R^{-}\left(k_{2}\right) \backslash R_{s}^{-}(a)} K(x-y)(\psi(x)-\psi(y)) W(y) d y \\
& +\int_{R_{s}^{-}(a)} K(x-y)(\psi(x)-\psi(y)) W(y) d y \triangleq I_{5,3}^{-}(x, a)+I_{5,4}^{-}(x, a) \tag{4.57}
\end{align*}
$$

The above decomposition is slightly different from (4.55). We choose $R_{s}^{-}(a)$ rather than $R^{-}(a)$, since we need to choose the singular region with size going to 0 as $|x-z| \rightarrow 0$. Yet, $R^{-}(a)$ (4.18) does not satisfy this requirement for $a \rightarrow 0$. We can estimate the derivative of $I_{5,3}^{-}(x, a)$ following Sections 4.1.6 4.1.8, and the $L^{\infty}$ norm of $I_{5,4}^{-}(x, a)$ following Section 4.2.1. Again, in the computation of $\partial_{x_{2}} I_{5,3}^{-}(x, a)$, the boundary term vanishes due to $K\left(y_{1}, 0\right) \equiv 0$. In summary, we can obtain the following estimate

$$
\begin{equation*}
\left|\partial_{x_{2}} I_{5,3}^{-}(x, a)\right| \leq A(x)+B(x) \log (1 / a), \quad\left|I_{5,4}^{-}(x, a)\right| \leq C(x) a h \tag{4.58}
\end{equation*}
$$

for any $a \leq 1$, where $A(x), B(x)$ can be estimated following the method in Appendix B.5.1 and the estimate of $C(x)$ follows the method in Section 4.2.1. Using the above estimates and the ideas in Section 4.1.11 we can estimate $d_{y}\left(I_{5}^{-}\left(\cdot, k_{2}\right), x, z\right)$ for small $|x-z|$ by optimizing $a$, where $d_{y}$ is defined below

$$
\begin{equation*}
d_{y}(f, x, z)=|f(x)-f(z)||x-z|^{-1 / 2} \tag{4.59}
\end{equation*}
$$

We will assemble these estimates in Section 4.6.
4.3.7. $C_{y}^{1 / 2}$ estimate of the velocity with a kernel of the second type. For the kernels $K_{2}=\frac{y_{1}^{2}-y_{2}^{2}}{|y|^{4}}$ and $\frac{y_{1}}{|y|^{2}}$, they do not vanish on $y_{2}=0$ in general. We call them the second type kernel.

If we use the strategies in the previous subsection, the boundary term in the computation of $\partial_{x_{2}} I_{5,1}^{-}\left(x, k_{3}\right)$ or $\partial_{x_{2}} I_{5,3}^{-}\left(x, k_{3}\right)$ does not vanish on $\Gamma$ and can be large. To avoid picking up
a boundary term on $\Gamma$ and apply the ideas in Section 4.3.5, we consider another estimate on $I_{5}\left(x, k_{2}\right)$. For $k_{3}=k_{2}-\frac{i}{2}, i=0,1, . ., 2 k_{2}-4$, we perform the following decomposition

$$
\begin{aligned}
I_{5}\left(x, k_{2}\right) & =\int_{R\left(k_{2}\right) \backslash R\left(k_{3}\right)} K(x-y)\left(\psi(x)-\psi(y) W(y) d y+\int_{R^{+}\left(k_{3}\right)} K(x-y)(\psi(x)-\psi(y)) W(y) d y\right. \\
& +\int_{R^{-}\left(k_{3}\right)} K(x-y)(\psi(x)-\psi(y)) W(y) d y \triangleq I_{5,1}+I_{5,2}+I_{5,3}
\end{aligned}
$$

Following the ideas in Section 4.1.11 we estimate the derivative of the regular part and then the $L^{\infty}$ norm of the singular part. Indeed, we can estimate the $y$-derivative of $I_{5,1}$ following Sections 4.1.10, 4.1.7, and the $L^{\infty}$ norm of $I_{5,3}$ following Section 4.2.1. The estimate of $I_{5,1}$ is similar to that of $I_{4}$ in Section 4.3.2. For $I_{5,2}$, since $R^{+}\left(k_{3}\right)$ is in $\mathbb{R}_{2}^{+}$, we can obtain a better estimate following the method in the estimate of $I_{5,1}, I_{5,2}$ in Section 4.3.4.

After we estimate these quantities, we can estimate $d_{y}\left(I_{5}, x, z\right)$ (4.59) for $|x-z|$ not too small by optimizing $k_{3}$. To estimate $d_{y}\left(I_{5}, x, z\right)$ (4.59) for sufficiently small $|x-z|$, following (4.57), we use the following decomposition

$$
\begin{align*}
I_{5}\left(x, k_{2}\right) & =\int_{R\left(k_{2}\right) \backslash R_{s}(a)} K(x-y)\left(\psi(x)-\psi(y) W(y) d y+\int_{R_{s}^{+}(a)} K(x-y)(\psi(x)-\psi(y)) W(y) d y\right.  \tag{4.60}\\
& +\int_{R_{s}^{-}(a)} K(x-y)(\psi(x)-\psi(y)) W(y) d y \triangleq I_{5,4}+I_{5,5}+I_{5,6}
\end{align*}
$$

Then we estimate the derivative of $I_{5,4}$ and the $L^{\infty}$ norm of $I_{5,6}$ as follows

$$
\begin{equation*}
\left|\partial_{x_{2}} I_{5,4}\right| \leq A(x)+B(x) \log (1 / a), \quad\left|I_{5,6}\right| \leq C(x) a h \tag{4.61}
\end{equation*}
$$

where the estimates of $A, B$ are given in Appendix B.5.1, and the estimate of $C$ follows the method in Section 4.2.1. The Hölder estimate of $I_{5,5}$ follows the method in the estimate of $I_{5,2}$ in Section 4.3.4. With these estimates, we can further bound $d_{y}\left(I_{5}, x, z\right)$

$$
d_{x}(f, x, z) \triangleq \frac{|f(x)-f(z)|}{\left|x_{1}-z_{1}\right|^{1 / 2}}, \quad d_{y}(f, x, z) \triangleq \frac{|f(x)-f(z)|}{\left|x_{2}-z_{2}\right|^{1 / 2}}
$$

for sufficiently small $|x-z|$ by optimizing $a$. See Section 4.6.
Remark 4.7. We do not use the later decomposition on $I_{5}$, i.e. $I_{5}=I_{5,4}+I_{5,5}+I_{5,6}$, to estimate $d_{y}(f, x, z)$ when $|x-z|$ is not too small since the domain of the integral in $I_{5,4}$ is not piecewise constant. As a result, we need to bound the boundary term in the computation of $\partial_{x_{2}} I_{5,4}$. The resulting estimate is worse than the estimate using the decomposition $I_{5}=I_{5,1}+I_{5,2}+I_{5,3}$.

We do not apply the above computation with smaller window $[-a h, a h]^{2}$ in the $C_{x}^{1 / 2}$ estimate, since it leads to a worse estimate. See also the discussions in Section 4.3.5.
4.3.8. Hölder estimate of $u, v, u_{y}, v_{x}$. The ideas of the Hölder estimate for other terms are similar. For a kernel $K$ associated with $\mathbf{u}, \nabla \mathbf{u}$, we perform another decomposition similar to (4.24)

$$
\begin{gather*}
\psi(x) \int K(x-y) W(y) d y=\int\left(\psi(x) \mathbf{1}_{R(k)^{c}}+\mathbf{1}_{R_{s}(k)} \psi(y)+\mathbf{1}_{R(k) \backslash R_{s}(k)} \psi(y)\right. \\
\left.\quad+\mathbf{1}_{R(k) \backslash R\left(k_{2}\right)}(\psi(x)-\psi(y))+\mathbf{1}_{R\left(k_{2}\right)}(\psi(x)-\psi(y))\right) K(x-y) W(y) d y  \tag{4.62}\\
\triangleq I_{1}(x, k)+I_{2}(x, k)+I_{3}(x, k)+I_{4}\left(x, k, k_{2}\right)+I_{5}\left(x, k_{2}\right)
\end{gather*}
$$

Here, we use $R_{s}(x, k)$ 4.19), which is symmetric with respect to both $x_{1}$ and $x_{2}$, rather than $R_{s, 1}(x, k)$, since the singular region in the sharp Hölder estimate of $\left[u_{y}\right]_{C_{x_{i}}}^{1 / 2},\left[v_{x}\right]_{C_{x_{i}}^{1 / 2}},\left[u_{x}\right]_{C_{y}^{1 / 2}}$ in Lemma 3.3-3.5 in Part I [13] needs to be symmetric in both $x_{1}, x_{2}$. Denote by $I_{f 6}\left(x, k_{2}\right)$ the approximation term for $f=u_{x}, u_{y}, v_{x}, u, v$. It takes the form similar to (4.50).

We consider two cases of $\hat{x} \in\left[0,2 x_{c}\right]^{2} \backslash\left[0, x_{c}\right]^{2}$ (4.4). In the first case, we consider $\hat{x} \in$ $\left[x_{c}, 2 x_{c}\right] \times\left[0,2 x_{c}\right] \triangleq D_{X 1}$, where we have $\hat{x}_{1} \geq c \hat{x}_{2}$ for some constant $c>0$. In the second case, we consider $\hat{x} \in\left[0, x_{c}\right] \times\left[x_{c}, 2 x_{c}\right] \triangleq D_{X 2}$, where we have $\hat{x}_{1} \leq c \hat{x}_{2}$. We distinguish these two
cases since in the second case, the singular region does not touch the boundary, we can apply the method in Section 4.3.4.
$C_{x}^{1 / 2}$ estimate of $u_{y}, v_{x}$. In the $C_{x}^{1 / 2}$ estimate of $u_{y}, v_{x}$, we follow Section 4.3.2 to estimate the regular part $I_{1}+I_{4}-I_{6}$ and $I_{3}$. We follow Section 4.3.3 and use Lemma 3.4 in Section 3 of Part I [13] to estimate $I_{2}$. For $I_{5}$, we follow Section 4.3.4.
$C_{y}^{1 / 2}$ estimate of $u_{x}$. We perform the decomposition (4.62) rather than (4.24). The estimates of $I_{1}+I_{4}-I_{6}, I_{3}$ follow Section 4.3.2, For $I_{2}$, we use Lemma 3.3 in Section 3 of Part I [13]. We follow Section 4.3.6 to estimate $I_{5}$ if $\hat{x} \in D_{X 1}$, and Section 4.3.4 if $\hat{x} \in D_{X 2}$.

We remark that we use the decomposition (4.62) rather than (4.24) since in Lemma 3.3 in Section 3 of Part I [13, we need to assume that the singular region around $x$ is symmetric in both $x_{1}$ and $x_{2}$. The same reasoning applies to $C_{x_{i}}^{1 / 2}$ estimate of $u_{y}, v_{x}$.
$C_{x}^{1 / 2}$ and $C_{y}^{1 / 2}$ estimate of $u, v$. The Hölder estimates of $u, v$ are substantially easier since $u, v$ are more regular. We perform $C_{x}^{1 / 2}, C_{y}^{1 / 2}$ of $\rho \mathbf{u}_{A}$ for another weight $\rho=\psi_{u}$ (A.1). Below, we only use the weighted $L^{\infty}$ norm $\|\omega \varphi\|_{\infty}$. We decompose the integral as follows

$$
\begin{align*}
\rho(x) \int K(x-y) W(y) d y & =\int\left(\mathbf{1}_{R(k)^{c}} \rho(x)+\mathbf{1}_{R(k)} \rho(x)\right) K(x-y) W(y) d y  \tag{4.63}\\
& \triangleq I_{1}(x, k)+I_{2}(x, k)
\end{align*}
$$

We choose $k$ smaller than that in (4.24) for $\nabla \mathbf{u}$ since the kernel for $\mathbf{u}$ is more regular. We follow Section 4.3.2 to estimate $I_{1}-I_{6}$. For $I_{2}$, we follow the ideas in Sections 4.1.11, 4.3.6, 4.3.7 to estimate the log-Lipschitz function. We choose a list of $k_{2}$ and associated region $S\left(k_{2}\right)$ and decompose $I_{2}$ as follows
$I_{2}(x, k) \triangleq \int_{R(k) \backslash S\left(k_{2}\right)} \rho(x) K(x-y) W(y) d y+\int_{S\left(k_{2}\right)} \rho(x) K(x-y) W(y) d y \triangleq I_{21}\left(x, k_{2}\right)+I_{22}\left(x, k_{2}\right)$.
For large $k_{2}=k, k-1 / 2, . ., 2$, we choose $S\left(k_{2}\right)=R\left(k_{2}\right)$. For $k_{2}<2$, we choose $S\left(k_{2}\right)=$ $R_{s}\left(k_{2}\right)$. For $I_{21}\left(x, k_{2}\right)$, we estimate its derivatives following the estimate of $I_{50}$ (4.51) or Section 4.1.7 when $k_{2} \geq 2$, and the estimate of $I_{54}$ when $k_{2}<2$ in Section 4.3.7. For $I_{22}\left(x, k_{2}\right)$, we estimate its $L^{\infty}$ norm following the estimate of $I_{53}$ when $k_{2} \geq 2$, and the estimate of $I_{56}$ when $k_{2}<2$ in Section 4.3.7. The estimate is simpler since the above kernel is much simpler than $K(x-y)(\psi(x)-\psi(y))$ in Section 4.3.7
4.3.9. Special case: $C_{y}^{1 / 2}$ estimate of $u_{y}, v_{x}$. In this case, we apply Lemma 3.5 from Section 3 of Part I [13] to estimate the most singular part. Since in Lemma 3.5 from Section 3 of Part I, we do not localize the integral, we perform the following decomposition

$$
\begin{align*}
\psi(x) \int K(x-y) W(y) d y & =\int\left(\psi(y)+\mathbf{1}_{R\left(k_{2}\right)^{c}}(\psi(x)-\psi(y))+\mathbf{1}_{R\left(k_{2}\right)}(\psi(x)-\psi(y))\right) K(x-y) W(y) d y  \tag{4.64}\\
& \triangleq I_{1}(x, k)+I_{2}(x, k)+I_{3}(x, k)
\end{align*}
$$

For $I_{1}$, we apply Lemma 3.5 from Part I [13. We follow Section 4.3.7 to estimate $I_{3}$ if $\hat{x} \in D_{X 1}$, and Section 4.3 .4 if $\hat{x} \in D_{X 2}$. We follow Section 4.3 .2 to estimate $I_{2}-I_{6}$, where $I_{6}$ is the approximation terms for $u_{y}, v_{x}$ similar to (4.50). The symmetrized integrand is discussed in the paragraph " $C^{1 / 2}$ estimate of $u_{y}, v_{x}$ " in Section 4.1.5 There are additional difficulties since the weight $\psi(y)$ and the symmetrized integrand $I=K(x, y)(\psi(x)-\psi(y)$ ) (see similar derivations in (4.28), (4.29) are singular near 0 .

Estimate the integral near 0 . To estimate the $D_{1}=\partial_{x_{2}}$ derivative, we use

$$
\left|D_{1} I\right|=\left|D_{1} K(\psi(x)-\psi(y))+K \cdot D_{1} \psi(x)\right| \leq\left|D_{1} K \cdot \psi(x)+K \cdot D_{1} \psi(x)\right|+\left|D_{1} K \cdot \psi(y)\right|
$$

For $y$ close to 0 , since $\psi$ is singular, $\psi(y)$ is much larger than $\psi(x)$, and $K(x, y)$ is not singular. The main term in $D_{1} I$ is given by $D_{1} K \psi(y)$. It follows
$\int_{Q}\left|D_{1} I \cdot W(y)\right| d y \leq\|W \varphi\|_{\infty}\left(\left\|\varphi^{-1}\right\|_{L^{\infty}(Q)} \int_{Q}\left|D_{1} K \psi(x)+K \cdot D_{1} \psi(x)\right| d y+\left\|\frac{\psi}{\varphi}\right\|_{L^{\infty}(Q)} \int_{Q}\left|D_{1} K\right| d y\right)$,
where $Q$ is some grid near the origin. The integrands in both integrals do not involve the singular weight, and we can estimate them for each grid point $x$ using the previous methods.

To estimate the $X$ - discretization error, we need to estimate the integral of $\partial_{x i}^{2} \partial_{x_{2}} J$. Since $\psi(y)$ is independent of $x$, we get
$I=K(x, y)\left(\frac{\psi(x)}{\psi(y)}-1\right) \psi(y), \int_{Q}\left|\partial_{x_{i}}^{2} \partial_{x_{2}} I \cdot W(y)\right| d y \leq\|W \varphi\|_{\infty}\left\|\frac{\psi}{\varphi}\right\|_{L^{\infty}(Q)} \int_{Q}\left|\partial_{x_{i}}^{2} \partial_{x_{2}} K(x, y)\left(\frac{\psi(x)}{\psi(y)}-1\right)\right| d y$.
The last integrand is not singular in $y$ near $y=0$, and we estimate it using the previous method, e.g. Section 4.1.3

For $u_{y}, v_{x}$, we have a rank-one approximation $K_{a p p}(x, y)$ from $C_{u_{y}} \chi_{0} K_{00}$ (4.5) (see Section 4.3.2 from Part I [13]). The full integrand with approximation term and weight is given by
$I_{a p p}=K(x, y)(\psi(x)-\psi(y))-K_{a p p}(x, y) \psi(x)=\left(K(x, y)-K_{a p p}(x, y)\right) \psi(x)-K(x, y) \psi(y)=I_{a p p, 1}+I_{a p p, 2}$.
For $y$ away from the singularity $x$ and $0, I_{a p p, 1}$ has the same form as the previous case, e.g. the $C_{x}^{1 / 2}$ estimate. We improve the error estimate $\partial_{i}^{2} \partial_{x_{2}} I_{a p p}$ using the cancellation between the full symmetrized kernel $K(x, y)$ and $K_{a p p}$ from Lemma B. 2 and the estimate in (B.15) in Appendix B.1.1 and the property that $\psi(y)$ is much smaller than $\psi(x)$ for $|y|$ much larger than $|x|$.

Estimate in the far-field. For the tail part in this case, we have an improvement for small $|x|$ where $\chi_{0}(x)=1$ due to the approximation term near 0

$$
\hat{f}=C_{f 0}(x, y) u_{x}(0)+C_{f}(x, y) \mathcal{K}_{00}=C_{f}(x, y) \mathcal{K}_{00}
$$

where $f=u_{y}, v_{x}$ and $\mathcal{K}_{00}$ is defined in (4.5), and we have used $C_{f 0}(x, y)=0$. Its associated integrand is given by

$$
K_{a p p} \triangleq \pi^{-1} C_{f}(x, y) K_{00}(y)
$$

where $K_{00}$ is defined in (4.5). To estimate it, we use the following decomposition

$$
D_{1}\left(J-\psi(x) K_{a p p}\right)=D_{1}\left(\left(K-K_{a p p}\right) \cdot \psi(x)\right)-D_{1} K \cdot \psi(y) \triangleq P_{1}+P_{2}
$$

We estimate $P_{1}$ using the method in Section 4.4. Due to the approximation, $\left(K-K_{\text {app }}\right)$ has a much faster decay for large $y$ beyond $[0, D]^{2}$. See B.15) and Appendix B.1.1 For $P_{2}$, we have

$$
\int_{\Omega^{c}}\left|P _ { 2 } \left\|W(y)\left|d y \leq\|W \varphi\|_{\infty} \int_{\Omega^{c}}\right| D_{1} K \left\lvert\, \frac{\psi}{\varphi}(y) d y\right.\right.\right.
$$

where $\Omega=[0, D]^{2}$ with large $D$. The last integral is computed using the method in Section 4.4.
4.4. Estimate the integrals near 0 and in the far field. We use a combination of uniform mesh and adaptive mesh to compute the integral in a finite domain $[0, D]^{2}$, e.g. $D=1000$. See Section 4.1.3. Since the kernel decays and the singularity is in the near-field, the integral beyond this domain is small, and we estimate it directly. In addition, for $y$ near 0 , we estimate the integrals (the last two integrals in (4.8) from the approximations $u_{x}(0), K_{00}$ (4.7), which is singular of order $|y|^{-2}$ or $|y|^{-4}$. For simplicity, we consider $\lambda=1$. The estimates can be generalized to other scaling parameter $\lambda$. To estimate $\int_{D} k(y) \omega(y) d y$ for $D$ near 0 or $D$ in the far-field, following (4.10), we only need to estimate $\int_{D}|k(y)| \varphi^{-1}(y) d y$. Since $|y|$ is either very small or very large, we can use the asymptotics of $\varphi$ in these estimates.
4.4.1. Near-field estimate. Firstly, we estimate $\int_{\left[0, R_{1}\right]^{2}}|k(y)| \varphi^{-1}(y) d y$ for $k(y)=\frac{y_{1} y_{2}}{|y|^{4}}, \frac{y_{1} y_{2}\left(y_{1}^{2}-y_{2}^{2}\right)}{|y|^{8}}$ related to $u_{x}(0), K_{00}$ (4.7). We partition $\left[0, R_{1}\right]$ into

$$
0=z_{0}<z_{1}<\ldots<z_{n}=R_{1}
$$

with $z_{1}$ much smaller than $R_{1}$. Denote $Q_{i j}=\left[z_{i-1}, z_{i}\right] \times\left[z_{j-1}, z_{j}\right]$. Clearly, we have

$$
\int_{\left[0, R_{1}\right]^{2}}|k(y)| \varphi^{-1}(y) d y \leq \sum_{1 \leq i, j \leq n} I_{i j}, \quad I_{i j} \triangleq \int_{Q_{i j}}|k(y)| \varphi^{-1}(y) d y
$$

For $I_{i j},(i, j) \neq(1,1)$, we apply a trivial bound

$$
\begin{equation*}
I_{i j} \leq\left\|\varphi^{-1}\right\|_{L^{\infty}\left(Q_{i j}\right)} \int_{Q_{i j}}|k(y)| d y \leq\left|Q_{i j}\right| \cdot\|k\|_{L^{\infty}\left(Q_{i j}\right)}\left\|\varphi^{-1}\right\|_{L^{\infty}\left(Q_{i j}\right)} \tag{4.65}
\end{equation*}
$$

For $k(y)=\frac{y_{1} y_{2}}{|y|^{4}}, \frac{y_{1} y_{2}\left(y_{1}^{2}-y_{2}^{2}\right)}{|y|^{8}}$, the estimate of $\|k\|_{L^{\infty}\left(Q_{i j}\right)}$ is established in Appendix B. It remains to estimate the first term $I_{11}$. Denote $r=y_{1}$. Suppose that

$$
\varphi(x) \geq q|x|^{a}(\cos \beta)^{b}, \quad b \leq 0
$$

See (A.2). If $k(y)=\frac{y_{1} y_{2}}{|y|^{4}}$ and $a<0$, we yield

$$
\begin{aligned}
I_{11} & \leq q^{-1} \int_{0}^{\sqrt{2} r} \int_{0}^{\pi / 2} \frac{\sin \beta \cos \beta}{r^{2}} r^{-a}(\cos \beta)^{-b} r d r d \beta=q^{-1} \int_{0}^{\sqrt{2} r} r^{-a-1} d r \int_{0}^{\pi / 2} \sin \beta(\cos \beta)^{-b+1} d \beta \\
& =q^{-1} \frac{(\sqrt{2} r)^{-a}}{-a} \int_{0}^{1} t^{-b+1} d t=q^{-1} \frac{(\sqrt{2} r)^{-a}}{-a} \frac{1}{2-b}
\end{aligned}
$$

If $k(y)=\frac{y_{1} y_{2}\left(y_{1}^{2}-y_{2}^{2}\right)}{|y|^{8}}$, we yield $|k(y)| \leq \frac{1}{4} \frac{\sin 4 \beta}{r^{4}}$. Since $b \leq 0$, if $a<-2$, we get $\varphi \geq q r^{a}$ and

$$
\begin{aligned}
I_{11} & \leq q^{-1} \int_{0}^{\sqrt{2} r} \int_{0}^{\pi / 2} \frac{1}{4} \frac{|\sin 4 \beta|}{s^{4}} s^{-a} s d s d \beta=\frac{1}{4 q} \int_{0}^{\sqrt{2} r} s^{-a-3} d s \frac{1}{4} \int_{0}^{2 \pi}|\sin \beta| d \beta \\
& =\frac{1}{4 q} \frac{(\sqrt{2} r)^{-a-2}}{-2-a} \int_{0}^{\pi / 2} \sin \beta d \beta=\frac{1}{4 q} \frac{(\sqrt{2} r)^{-a-2}}{-2-a}
\end{aligned}
$$

4.4.2. Far-field estimate. Denote $a \vee b=\max (a, b)$. To estimate the far field integral $I \triangleq$ $\int_{y_{1} \vee y_{2} \geq R_{0}}|k(y)| \varphi^{-1}(y) d y$, we first pick sufficient large $R$, and then partition the domain

$$
0=z_{0}<z_{1}<. .<z_{m}=R_{0}<z_{m+1}<\ldots<z_{n}=R_{1}<+\infty
$$

Denote $Q_{i j}=\left[z_{i-1}, z_{i}\right] \times\left[z_{j-1}, z_{j}\right]$. Clearly, we have

$$
I=\sum_{m+1 \leq \max (i, j) \leq n} I_{i j}+J, \quad I_{i j} \triangleq \int_{Q_{i j}}|k(y)| \varphi^{-1}(y) d y, \quad J=\int_{y_{1} \vee y_{2} \geq R_{1}}|k(y)| \varphi^{-1}(y) d y
$$

For $I_{i j}$, we apply the trivial estimate (4.65). Suppose that

$$
\varphi \geq q r^{a}(\cos \beta)^{b}, \quad|k(y)| \leq|y|^{-p}, \quad b \in[-1,0], \quad p+a>2
$$

We get

$$
J \leq \frac{1}{q} \int_{R_{1}}^{\infty} \int_{0}^{\pi / 2} r^{-p-a}(\cos \beta)^{-b} r d r d \beta=\frac{1}{q} \frac{R_{1}^{-p-a+2}}{|p+a-2|} \int_{0}^{\pi / 2}(\cos \beta)^{-b} d \beta
$$

Using Hölder's inequality and $b \in[-1,0]$, we get

$$
\int_{0}^{\pi / 2}(\cos \beta)^{-b} d \beta \leq\left(\int_{0}^{\pi / 2} \cos \beta d \beta\right)^{-b}\left(\int_{0}^{\pi / 2} 1\right)^{1+b}=(\pi / 2)^{1+b}
$$

It follows

$$
J \leq \frac{1}{q} \frac{R_{1}^{-p-a+2}}{|p+a-2|}(\pi / 2)^{1+b}
$$

Application. We apply the above calculations to estimate the integral and its derivatives beyond the mesh $[0, D]^{2}(4.12)$. Since the domain is far away from the singularity, the integrand is the symmetrized kernel, e.g., 4.29). From Appendix B.1.1 and Lemma B. 2 in Appendix B for $\mathbf{u}_{A}, \nabla \mathbf{u}_{A}, \partial_{i}\left(\rho \mathbf{u}_{A}\right), \partial_{i}\left(\psi \nabla \mathbf{u}_{A}\right)$, the integrand in the far-field ( $y$ is large) satisfies

$$
|K(x, y)| \leq C(x) \operatorname{Den}^{-k}
$$

with some $k \geq 2$ and coefficients $C(x)$, where Den is defined in (B.20).
In our computation, we rescale $x$ to $\hat{x}$ and restrict it to the near-field $[0, b]^{2}$ with $b<2$. Note that $y \notin[0, D]^{2}$ and $|y| \geq D \gg b$. From ( (B.20), we get

$$
\text { Den } \geq \min _{\left|z_{1}\right| \leq x_{1},\left|z_{2}\right| \leq x_{2}}|y-z|^{2} \geq \min _{\left|z_{1}\right| \leq x_{1},\left|z_{2}\right| \leq x_{2}}(|y|-|z|)^{2} \geq(|y|-|x|)^{2}=|y|^{2}\left(1-\frac{|x|}{|y|}\right)^{2}
$$

Since $\frac{|x|}{|y|} \leq \sqrt{2} b / D$, we yield

$$
\text { Den } \geq\left(1-C_{s}\right)^{2}|y|^{2}, \quad C_{s}=\sqrt{2} b / D
$$

It follows

$$
\int_{y \notin[0, D]^{2}}|K(x, y)| \varphi^{-1}(y) d y \leq\left(1-C_{s}\right)^{-2 k} C(x) \int_{y \notin[0, D]^{2}}|y|^{-2 k} \varphi^{-1}(y) d y .
$$

Using the method in Section 4.4.2, we can estimate the above integral.
4.5. Estimate for very small or large $x$. The rescaling argument and the methods in the previous subsections apply to the estimate of $\mathbf{u}_{A}(x), \nabla \mathbf{u}_{A}(x)$ for $x \in\left[0, x_{M}\right]^{2} \backslash\left[0, x_{m}\right]^{2}, 0<x_{m}<$ $x_{M}$. For very small or large $x$, we cannot use a finite number of dyadic scales $\lambda=2^{i}$ to rescale $x$ such that $x / \lambda \in\left[0,2 x_{c}\right]^{2} \backslash\left[0, x_{x}\right]^{2}$. Instead, we choose $\lambda=\frac{\max \left(x_{1}, x_{2}\right)}{x_{c}}$. We want to estimate the rescaled integral with a $-d$-homogeneous kernel $K$

$$
p(x) \int K(x-y) W(y) d y=p_{\lambda}(x) \int K(\hat{x}-\hat{y}) \lambda^{2-d} W_{\lambda}(\hat{y}) d y
$$

uniformly for all small $\lambda \ll 1$ or large $\lambda \gg 1$, where $p$ is some weight and $p_{\lambda}$ is defined in (4.2). The rescaled singularity $\hat{x}=x / \lambda$ satisfies $\max _{i} \hat{x}_{i}=x_{c}$. We simplify $\hat{x}, \hat{y}$ as $x, y$.

We can use the asymptotic of the weights to estimate the integral, see e.g. (4.6). The new difficulty is that the estimate involves the rescaled weight $p_{\lambda}(y)$. Since $\lambda$ is not fixed and depends on $x$ that tends to 0 or $\infty$, we cannot evaluate $p_{\lambda}(y)$ and the integrand directly. In the following derivation, $\lambda$ is comparable to $|x|$, which is either very small or very large.

For $y$ away from the singular region, the integrand of the regular part is given by $J=K(x, y)$. $p_{\lambda}(x)$ 4.29). We choose a radial weight $p$ defined in Appendix A. $1 p(x)=\sum_{1 \leq i \leq n} q_{i}|x|^{a_{i}}$. See $\psi_{1}, \psi_{u}, \psi_{d u}$ A.1). We introduce the asymptotics of these weights

$$
R_{\lim } \triangleq \lim _{x \rightarrow A} \frac{D_{1} p_{\lambda}(x)}{p_{\lambda}(x)}, \quad p_{l i m}=q_{i}|x|^{a_{i}}
$$

with $(A, i)=(0,1)$ or $(A, i)=(\infty, n)$, where $\left(q_{n}, a_{n}\right)$ denotes the last power in the weight. We use the following decomposition to compute $D_{1} J$ with $D_{1}=\partial_{x_{i}}$

$$
\begin{aligned}
\left|D_{1} J\right| & =\left|D_{1}\left(K(x, y) \cdot p_{\lambda}(x)\right)\right|=\left|D_{1} K(x, y) \cdot p_{\lambda}(x)+K(x, y) \cdot D_{1} p_{\lambda}(x)\right| \\
& =\left|p_{\lambda}(x)\left\{D_{1} K(x, y)+R_{\text {lim }} K(x, y)+\left(\frac{D_{1} p_{\lambda}(x)}{p_{\lambda}(x)}-R_{\text {lim }}\right) K(x, y)\right\}\right|
\end{aligned}
$$

Since we consider very small $\lambda$ or very large $\lambda$, the error term $\frac{D_{1} p_{\lambda}(x)}{p_{\lambda}(x)}-R_{l i m}$ is small. Hence, we use a triangle inequality to bound $D_{1} J$

$$
\left|D_{1} J\right| \leq p_{\lambda}(x)\left|D_{1} K(x, y)+R_{\text {lim }} K(x, y)\right|+p_{\lambda}(x)\left|\left(\frac{D_{1} p_{\lambda}(x)}{p_{\lambda}(x)}-R_{\text {lim }}\right) K(x, y)\right|
$$

The advantage of the above decomposition is that the main term $D_{1} K(x, y)+R_{\text {lim }} K(x, y)$ does not depend on $\lambda$ so that we can estimate it using previous methods.

Since the estimate of derivative of $u, v$ does not involve the commutator, see, e.g. (4.63), we can apply the above method to compute the integral of $D_{1} u$ for small $x$ or large $x$.

For $y$ near the singular region, from (4.28), the symmetrized integrand is given by

$$
J=K^{C}\left(p_{\lambda}(x)-p_{\lambda}(y)\right)+K^{N C} p_{\lambda}(x)
$$

where we use $p$ for the weight. Firstly, we have

$$
\left|D_{1} J\right|=\left|D_{1} K^{C}\left(p_{\lambda}(x)-p_{\lambda}(y)\right)+D_{1} K^{N C} p_{\lambda}(x)+\left(K^{C}+K^{N C}\right) D_{1} p_{\lambda}(x)\right|
$$

Denote $K=K^{C}+K^{N C}$. We use the following method to bound $D_{1} J$

$$
\begin{aligned}
\left|D_{1} J\right| \leq & p_{\lambda}(x)\left|D_{1} K^{C} \cdot\left(1-\frac{p_{\lambda}(y)}{p_{\lambda}(x)}\right)+D_{1} K^{N C}+K \cdot \frac{D_{1} p_{\lambda}}{p_{\lambda}}\right| \\
\leq & p_{\lambda}(x)\left\{\left|D_{1} K^{C} \cdot\left(1-\frac{p_{\text {lim }}(y)}{p_{\text {lim }}(x)}\right)+D_{1} K^{N C}+K \cdot \frac{D_{1} p_{\text {lim }}}{p_{\text {lim }}}\right|\right. \\
& \left.\quad+\left|D_{1} K^{C}\left(\frac{p_{\lambda}(y)}{p_{\lambda}(x)}-\frac{p_{\text {lim }}(y)}{p_{\text {lim }}(x)}\right)\right|+K\left|\frac{D_{1} p_{\text {lim }}}{p_{\text {lim }}}-\frac{D_{1} p_{\lambda}}{p_{\lambda}}\right|\right\} .
\end{aligned}
$$

The second and the third term on the right hand side can be seen as an error term. The main term $\left|D_{1} K^{C} \cdot\left(1-\frac{p_{l i m}(y)}{p_{\text {lim }}(x)}\right)+D_{1} K^{N C}+K \cdot \frac{D_{1} p_{\text {lim }}}{p_{\text {lim }}}\right|$ does not depend on $\lambda$, and the singularity $x$ is in the near-field and away from 0 . We can apply all the delicate decompositions developed in previous sections to estimate $D_{1} J$.

In the Hölder estimates, we need various bounds for the weights $p_{\lambda}$. Using the asymptotics of $p(x)$, we can estimate the derivatives of $p_{\lambda}$ for very small $\lambda$ or very large $\lambda$ uniformly. See Appendix A.1, A.2 Once we obtain the estimates of $\psi_{\lambda}$, and the weight $\varphi_{\lambda}$ in the $L^{\infty}$ norm $\left\|\omega_{\lambda} \varphi_{\lambda}\right\|_{\infty}$, we can use the methods in the previous subsections and the scaling relations in Section 4.1.2 to perform the Hölder estimates.

The $L^{\infty}$ estimate follows similar ideas and is much easier. We refer more details to Section 7 in the supplementary material II [11].

We remark that since we have much larger damping coefficients in the energy estimates (see Section 5 in Part I [13) near $x=0$ and in the far-field, the estimates of the nonlocal terms in these regions, though technical, only have minor effects on the nonlinear stability estimates.
4.6. Assemble the Hölder estimates. In Section 4.3, we decompose the velocity in several parts and estimate them separately using the norms $\|\omega \varphi\|_{\infty},[\omega \psi]_{C_{x_{i}}^{1 / 2}}$. In this section, we assemble these estimates and estimate

$$
\delta(f, x, z) \triangleq \frac{|f(x)-f(z)|}{|x-z|^{1 / 2}}
$$

for $f=\psi_{u} \mathbf{u}_{A}, \psi \nabla \mathbf{u}_{A}$ with weights in A.1). To obtain better estimates, we combine some of the estimates.

In the proof of the first inequality in Lemma 2.3, we combine and bound different norms using $\max \left(\|\omega \varphi\|_{\infty}, \max _{j=1,2} \gamma_{j}\left[\omega \psi_{1}\right]_{C_{x_{j}^{\prime}}^{1 / 2}\left(\mathbb{R}_{2}^{+}\right)}\right)$. We apply the second inequality to the error $\varepsilon=$ $\omega-(-\Delta) \phi^{N}$ (3.10) and can evaluate the localized norm using piecewise bounds of the error.

To illustrate the ideas, we focus on the $C_{x}^{1 / 2}$ estimate, $x \in\left[x_{c}, 2 x_{c}\right] \times\left[0,2 x_{c}\right]$, i.e. $x_{1}$ is large relative to $x_{2}, z_{1} \geq x_{1}$, and $x_{2}=z_{2}$. For general pairs $(x, z)$, we can rescale $(x, z)$ to $(\lambda x, \lambda z)$ such that $\lambda x \in\left[0,2 x_{c}\right]^{2} \backslash\left[0, x_{c}\right]^{2}$. Using the scaling relations in (4.1.2), we can estimate the rescaled version of $\delta(f, x, z)$. See also the discussion at the beginning of Section 4.3.

We assume that $z_{1} \in\left[x_{c}, 2(1+\nu) x_{c}\right]$ with $\nu<1$. For $z_{1} \geq 2(1+\nu) x_{c}$, we have $z_{1}>(1+\nu) x_{1}$. Since $z_{1}, x_{1}$ are large relative to $z_{2}, x_{2}$, respectively, we have

$$
|x-z|=\left|z_{1}-x_{1}\right| \asymp\left|z_{1}\right| \gtrsim|x|,|z|
$$

Then, we can use the $L^{\infty}$ estimate and triangle inequality to estimate $\delta(f, x, z)$. Note that we can estimate the piecewise $L^{\infty}$ norm of $|x|^{-1 / 2} \rho(x) \mathbf{u}_{A}(x)$ and $|x|^{-1 / 2} \psi \nabla \mathbf{u}_{A}$ following Section 4.2. where $\rho, \psi$ are the weights in the Hölder estimate of $\rho \mathbf{u}_{A}, \psi \nabla \mathbf{u}_{A}$. See Section 7.4 in the supplementary material II [11] for more details.

We focus on $f=\psi u_{x, A}$. We partition the domain $D_{\nu}=\left[x_{c}, 2(1+\nu) x_{c}\right] \times\left[0,2 x_{c}\right]$ into $h_{x} \times h_{x}$ grids $D_{i j}, 1 \leq i \leq 2(1+\nu) x_{c} / h_{x}, 1 \leq j \leq 2 x_{c} / h_{x}$. We apply the decomposition (4.63) with the
same parameters $k, k_{2}$ to $x$ in different grids $D_{i j}$. For $x \in D_{i j}$, using the method in Section 4.3, we obtain the estimate

$$
\begin{align*}
& f(x)=I_{1}(x)+I_{2}(x)+I_{3}(x)+I_{4}(x)+I_{5}(x)-I_{6}(x), \quad I_{5}=I_{5,0}+I_{5,1}+I_{5,2} \\
& \left|\partial_{x}\left(I_{1}+I_{4}+I_{5,0}-I_{6}\right)\right| \leq a_{i j, 1}| | \omega \varphi\left\|_{\infty}, \quad\left|\partial_{x} I_{3}\right| \leq a_{i j, 2}\right\| \omega \varphi\left\|_{\infty},\left|I_{3}\right| \leq b_{i j, 2}\right\| \omega \varphi \|_{\infty}  \tag{4.66}\\
& \left|\partial_{x} I_{5,1}\right| \leq a_{i j, 3}\|\omega \varphi\|_{\infty}, \quad\left|I_{5,1}\right| \leq b_{i j, 3}\|\omega \varphi\|_{\infty}
\end{align*}
$$

for some constants $a_{i j, l}, b_{i j} \geq 0$, where $I_{5,1}, I_{5,2}$ are defined and estimated in Section 4.3.4.
For $x, z \in D_{\nu}$ with $x_{2}=z_{2}, z_{1} \leq z_{1}$, we have $x \in D_{i_{1}, j}, z \in D_{i_{2}, j}$ for some $i_{1} \leq i_{2}$. We apply the method in Section 4.3.3 to estimate $\delta\left(I_{2}, x, z\right)$ and the method in Section 4.3.4 to estimate $J_{1}$ related to $\delta\left(I_{52}, x, z\right)$ 4.53). These estimates contribute to the bound $C_{h o l}[\omega \psi]_{C_{x}^{1 / 2}}$ for some $C_{h o l}>0$, which can be computed.

By averaging the piecewise derivative bounds and using the estimates in Appendix E.2, for $x \in D_{i_{1}, j}, z \in D_{i_{2}, j}$, we can obtain

$$
\left|\left(I_{1}+I_{4}-I_{6}\right)(x)-\left(I_{1}+I_{4}-I_{6}\right)(z)\right| \leq C_{l i p}\left|x_{1}-z_{1}\right| \cdot\|\omega \varphi\|_{\infty}
$$

for constant $C_{l i p}$ depending only on $\left\{a_{k l, 1}\right\}_{k, l \geq 1}$ and the mesh $h_{x}$ explicitly. Similar estimates hold for $I_{5,0}, I_{3}, I_{5,1}$. Hence, for the remaining terms in $f$ not estimated using the seminorm $[\omega \psi]_{C_{x}^{1 / 2}}$, e.g. $I_{1}+I_{4}-I_{6}, I_{3}, I_{5,0}, I_{5,1}$ and $J_{2}$ related to $I_{5,2}$ (4.53), they satisfy

$$
f_{R}(x)=\sum_{1 \leq l \leq N} f_{l}(x), \quad\left|f_{l}(x)-f_{l}(z)\right| \leq \min \left(p_{l}\left|x_{1}-z_{1}\right|, q_{l}\right) \cdot\|\omega \varphi\|_{\infty}
$$

for some $N$, where we can choose $q_{l}=\infty$ if we do not have $L^{\infty}$ estimate for $f_{l}(x)$. Similar consideration applies to $p_{l}$. In our problem, there are only a few terms and $N<10$. Now, for $x \in D_{i_{1}, j}, z \in D_{i_{2}, j}$, we have

$$
\begin{align*}
\frac{\mid f_{R}(x)-f_{R}(z)}{\left|z_{1}-x_{1}\right|^{1 / 2}} & \leq \sum_{1 \leq l \leq N} \min \left(p_{l} \delta^{1 / 2}, q_{l} \delta^{-1 / 2}\right)\|\omega \varphi\|_{\infty}  \tag{4.67}\\
\delta & =z_{1}-x_{1} \in\left[\max \left(i_{2}-i_{1}-1,0\right) h_{x},\left(i_{2}-i_{1}+1\right) h_{x}\right]
\end{align*}
$$

The upper bound can be obtained explicitly by partitioning the range of $z_{1}-x_{1}$ into finite many subintervals $M_{l}$ according to the threshold $\delta_{l}=q_{l} / p_{l}$. In each $M_{l}$, the bound reduces to

$$
P \delta^{1 / 2}+Q \delta^{-1 / 2}
$$

for some constants $P, Q$. It is convex in $\delta^{1 / 2}$ and can be optimized easily and explicitly in any interval $\left[\delta_{l}, \delta_{u}\right], \delta_{l}>0$.

Remark 4.8. We combine the estimates of different parts in (4.66) using (4.67) to obtain a sharp estimate. If one estimate different parts separately, the distance $\delta=z_{1}-x_{1}$ for the optimizer may not be achieved for the same value, which leads to an overestimate. We remark that for small distance $\left|z_{1}-x_{1}\right|$, such an overestimate can be significant since the ratio between the endpoints $\left|i_{2}-i_{1}+1\right| / \max \left(i_{2}-i_{1}-1,0\right)$ varies a lot.

In some estimates, e.g. the $C_{y}^{1 / 2}$ estimate of $u_{x}$ in Section 4.3.6, we need to decompose $I_{5}$ using different size of small singular region $k_{3}$. In such a case, we have a list of estimates associated to different $k_{3}$ for the part $f_{R}$ not estimated by $[\omega \psi]_{C_{x}^{1 / 2}}$ or $[\omega \psi]_{C_{y}^{1 / 2}}$ :

$$
\frac{\left|f_{R}(x)-f_{R}(z)\right|}{\left|z_{1}-x_{1}\right|^{1 / 2}} \leq \sum_{1 \leq l \leq N} \min \left(p_{l, k_{3}} \delta^{1 / 2}, q_{l, k_{3}} \delta^{-1 / 2}\right)\|\omega \varphi\|_{\infty}
$$

For $\left|x_{1}-z_{1}\right|$ bounded away from 0 , e.g. $\left|x_{1}-z_{1}\right| \geq \frac{1}{10} h_{x}$, we can still partition the range of $\left|x_{1}-z_{1}\right|$ and optimizing the above estimates first over $\delta$ and then $k_{3}$.
4.6.1. Hölder estimate for small distance. In some Hölder estimates, e.g. the $C_{y}^{1 / 2}$ estimate in Sections 4.3.6, 4.3.7, when $|x-z|$ is very small, e.g. $|x-z| \leq c h_{x}$ with $c<1$, we need to choose a singular region with size $a$ to be arbitrary small. See also Section 4.1.11for the estimates of a log-Lipschitz function. In these estimates, we can decompose $f_{R}(x)$ that is not estimated using the Hölder norm of $\omega \psi$ as follows

$$
f_{R}(x)=f_{1}(x, a, b)+f_{2}(x, a)
$$

for $a<b$ and $b$ is fixed. We can estimate the derivative of $f_{1}$, and the $L^{\infty}$ norm for $f_{2}$

$$
\left|\partial_{x} f_{1}(x, a, b)\right| \leq\left(A_{i}+B_{i} \log \frac{b}{a}\right)\|\omega \varphi\|_{\infty}, \quad\left|f_{2}\right| \leq \frac{C_{i} a}{2}\|\omega \varphi\|_{\infty}
$$

in each grid $D_{i j}$ for any $a \leq b$, see e.g., (4.58) and (4.66). We drop $j$ since we consider $x, z$ with $x_{2}=z_{2}$. For $t=|x-z| \leq h_{x}$, we get

$$
\begin{equation*}
\frac{|f(x)-f(z)|}{|x-z|^{1 / 2}} \leq\left(A+B \log \frac{b}{a}\right) \sqrt{t}+\frac{C a}{\sqrt{t}} \triangleq F(a, t) \tag{4.68}
\end{equation*}
$$

where $A=\max \left(A_{i}, A_{i+1}\right), B=\max \left(B_{i}, B_{i+1}\right), C=\max \left(C_{i}, C_{i+1}\right)$. For each $t \leq c h_{x}$, we can optimize the above estimate over $a \leq b$ explicitly. Then we maximize the estimate over $t \leq c h_{x}$ to obtain uniform estimate for small $|x-z| \leq c h_{x}$. We refer the derivations to Appendix B.5.2,
4.7. Improved estimate for the nonlocal error. In Section 3.7 we discuss the estimates of the nonlocal error $\mathbf{u}(\bar{\varepsilon})$ based on the functional inequalities established in this section. Since the weight is singular $\varphi \sim|x|^{-2}\left|x_{1}\right|^{-1 / 2}, \varphi=\varphi_{\text {elli }}$ A.2) near the origin, $\bar{\varepsilon}_{1} \varphi$ is much larger near $x=0$. Due to the anisotropic mesh for large $x$ and small $y$, or small $x$ and large $y$, and the round off error, $\bar{\varepsilon}_{1}$ is not very small in these far-field regions. On the other hand, these regions are small since either $|(x, y)|$ is very small or the ratio $x / y, y / x$ is very small, and the error is very small in the bulk, e.g. $x=O(1)$. See Figure 4 for the rigorous weighted bound of the error in the adaptive mesh. The weighted error of $\bar{\varepsilon}_{1}$ is larger near 0 , while the error for $\hat{\varepsilon}_{1}$ is larger in the far-field. If we simply use the global norm $\|\omega \varphi\|_{\infty}, \omega=\bar{\varepsilon}, \hat{\varepsilon}$, and then apply the previous estimates to bound $\mathbf{u}(\bar{\varepsilon})$, we overestimate the nonlocal error significantly. For $x=O(1)$, where we have the smallest damping for the energy estimate, due to the decay of kernel and the smallness of these regions, the integral $\int K(x, y) \bar{\varepsilon}(y) d y$ near $y=0$ or in the far-field is very small.


Figure 4. Piecewise $L^{\infty}\left(\varphi_{\text {elli }}\right)$ bound of the error $\bar{\varepsilon}_{1}, \hat{\varepsilon}_{1}$ in solving the Poisson equations. Left: error for the approximate steady steate. Right: error for the approximate space-time solution $\hat{W}_{2}$

Note that we can obtain the piecewise derivative bounds for the error $\bar{\varepsilon}_{1}, \hat{\varepsilon}_{1}$ and we partition the domain of the integral into different regions (4.45). Instead of using the global norm to bound the integral, we use the localized norms $\left\|W \varphi_{\text {elli }}\right\|_{l^{\infty}(D)},\left[W \psi_{1}\right]_{C_{x_{i}}^{1 / 2}(D)}$ (A.2), (A.1) to exploit the smallness of the error in most part of the domains and improve the error estimate.

Recall the regions of rescaled $\hat{x}$ (4.4) and the mesh $y_{i}$ partitioning the domain (4.11). We fix a scale $\lambda$ and assume $\hat{\epsilon}\left[x_{c}, 2 x_{c}\right] \times\left[0,2 x_{c}\right]$. By definition, the singular region $R(\hat{x}, k)$ (4.18) satisfies

$$
-R(\hat{x}, k) \cap \mathbb{R}_{2}^{+}, \quad R(\hat{x}, k) \cap \mathbb{R}_{2}^{+} \subset\left[x_{c}-k h, 2 x_{c}+k h\right] \times\left[0,2 x_{c}+k h\right] \triangleq S_{k h}
$$

Thus, in the estimates of $I_{2}, I_{3}, I_{4}$ in (4.45), instead of using the global norm $\|W \varphi\|_{L^{\infty}}$, we use $\left\|\omega_{\lambda} \varphi_{\lambda}\right\|_{L^{\infty}\left(S_{k h}\right)}=\|\omega \varphi\|_{L^{\infty}\left(\lambda S_{k h}\right)}$. For the error $\omega=\bar{\varepsilon}, \hat{\varepsilon}$, we can bound $\|\omega \varphi\|_{L^{\infty}\left(\lambda S_{k h}\right)}$ by using the piecewise estimates of $\bar{\varepsilon}, \hat{\varepsilon}$ and covering the region $\lambda S_{k h}$. Similarly, we use the localized bound $\left[\omega_{\lambda} \psi_{\lambda}\right]_{C_{x_{i}}^{1 / 2}\left(S_{k h}\right)}=\lambda^{1 / 2}[\omega \psi]_{C_{x_{i}}^{1 / 2}\left(\lambda S_{k h}\right)}$ for the Hölder seminorm in the estimate of $I_{2}, I_{3}, I_{4}$, and similar localized norms for $I_{5}$.

For the regular part $I_{1}$, we partition $[0, D]^{2}, \mathbb{R}_{2}^{++}$into disjoint domains: near-field $D_{n, i}$ the bulk $D_{B}$ and the far-field $D_{f, i}$, e.g.

$$
D_{n, 1}=[8 h, 16 h], D_{B}=[0,2]^{2} \backslash D_{n, 1}, D_{f, 1}=[0, D]^{2} \backslash[0,2]^{2}, D_{f, 2}=\mathbb{R}_{2}^{++} \backslash[0, D]^{2}
$$

where $h$ is the mesh size in (4.11). Then we use the norm $\left\|\omega_{\lambda} \varphi_{\lambda}\right\|_{L^{\infty}(D)}=\|\omega \varphi\|_{L^{\infty}(\lambda D)}$ for the estimate of the integral in region $D$.

In (4.8), we estimate the integral of $K_{00}(y)$ (4.5) for $|\hat{y}|_{\infty} \leq k_{02} h$ and $|\hat{y}|_{\infty} \geq k_{02} h$ separately. Since the kernel is very singular near 0 , the $L^{1}$ estimate of the integral in $|\hat{y}|_{\infty} \leq k_{02} h$ in Section 4.4.1 is not very small. Since we can evaluate $\omega=\bar{\varepsilon}, \hat{\varepsilon}$, we we change the rescaling from $\hat{y}$ back to $y$ by using $y=\lambda \hat{y}$ in (4.8)

$$
J=\int_{|\hat{y}|_{\infty} \leq k_{02} h} K_{00}(\hat{y}) \omega(\lambda \hat{y}) d \hat{y}=\lambda^{2} \int_{|y|_{\infty} \leq \lambda k_{02} h} K_{00}(y) \omega(y) d y
$$

where we get $\lambda^{2}$ since $K_{00}$ is -4 homogeneous. For a list of dyadic scales $\lambda=2^{k}$, we estimate the integral using Simpson's rule with very small mesh. This allows us to exploit the cancellation in the integral. For $|y|$ very close to 0 , we use Taylor expansion. See Section 6.4.1 in supplementary material II 11 (contained in this paper) for more details.

In the estimate of the integral for very small $x$ or large $x$ in Section 4.5 (see more details in Section 7 in the Supplementary Material II [11), we estimate the rescaled integral for $\lambda \leq \lambda_{1}$ and $\lambda \geq \lambda_{n}$ with small $\lambda_{1}$ and large $\lambda_{n}$ uniformly. In the case of $\lambda \leq \lambda_{1}$, we bound $\left\|\omega_{\lambda} \varphi_{\lambda}\right\|_{L^{\infty}([a, b] \times[c, d])} \leq\|\omega \varphi\|_{L^{\infty}\left(\lambda_{1}[0, b] \times[0, d]\right)}$. Other norms in different cases are estimated similarly.

We do not track the bound $\left\|\omega_{\lambda} \varphi_{\lambda}\right\|_{L^{\infty}\left(Q_{i j}\right)}$ in each small grid $Q_{i j}$ for computational efficiency.

## Appendix A. Weights and parameters

A.1. Estimate of the weights. Recall the following weights for the Hölder estimate of $\omega, \eta, \xi$ and $\mathbf{u}$

$$
\begin{align*}
& \psi_{1}=|x|^{-2}+0.5|x|^{-1}+0.2|x|^{-1 / 6}, \quad \psi_{d u}=\psi_{1}, \quad \psi_{u}=|x|^{5 / 2}+0.2|x|^{-7 / 6} \\
& \psi_{2}=p_{2,1}|x|^{-5 / 2}+p_{2,2}|x|^{-1}+p_{2,3}|x|^{-1 / 2}+p_{2,4}|x|^{1 / 6}  \tag{A.1}\\
& \psi_{3}=\psi_{2}, \quad \vec{p}_{2, .}=(0.46,0.245,0.3,0.112)
\end{align*}
$$

and the following weights for $\omega, \rho_{i}$ for $\mathbf{u}$ and the error

$$
\begin{align*}
\varphi_{1} & =x^{-1 / 2}\left(|x|^{-2.4}+0.6|x|^{-1 / 2}\right)+0.3|x|^{-1 / 6}, \quad \varphi_{g 1}=\varphi_{1}+|x|^{1 / 16} \\
\varphi_{\text {elli }} & =\left|x_{1}\right|^{-1 / 2}\left(|x|^{-2}+0.6|x|^{-1 / 2}\right)+0.3|x|^{-1 / 6}, \quad \rho_{10}=|x|^{-3}+|x|^{-7 / 6}, \quad \rho_{20}=\psi_{1}  \tag{A.2}\\
\rho_{3} & =|x|^{-1}+|x|^{-1 / 6}, \quad \rho_{4}=x^{-1 / 2}\left(|x|^{-2.5}+0.6|x|^{-1 / 2}\right)+0.3|x|^{-1 / 6}
\end{align*}
$$

To estimate the weighted $L^{\infty}$ norm of the residual error in Section 3, we use $\psi_{i}, \varphi_{\text {evo }, i}$

$$
\begin{align*}
& \varphi_{\text {evo }, 1}=\varphi_{1}, \quad \varphi_{\text {evo }, 2}=x^{-1 / 2}\left(\tilde{p}_{5,1}|x|^{-5 / 2}+\tilde{p}_{5,2}|x|^{-3 / 2}+\tilde{p}_{5,3}|x|^{-1 / 6}\right)+\tilde{p}_{5,4}|x|^{-1 / 4}+\tilde{p}_{5,5}|x|^{1 / 7},  \tag{A.3}\\
& \varphi_{\text {evo }, 3}=x^{-1 / 2}\left(\tilde{p}_{6,1}|x|^{-5 / 2}+\tilde{p}_{6,2}|x|^{-3 / 2}+\tilde{p}_{6,3}|x|^{-1 / 6}\right)+\tilde{p}_{6,4}|x|^{-1 / 4}+\tilde{p}_{6,5}|x|^{1 / 7} \\
& \tilde{p}_{5, \cdot}=(0.42,0.135,0.216,0.182,0.0349) \cdot \mu_{0}, \quad \mu_{0}=0.917 \\
& \tilde{p}_{6, \cdot}=\left(2.5 \cdot \tilde{p}_{5,1}, 2.9 \cdot \tilde{p}_{5,2}, 3.115 \cdot \tilde{p}_{5,3}, 1.82 \cdot \tilde{p}_{5,4}, 2.72 \cdot \tilde{p}_{5,5}\right)
\end{align*}
$$

where $\varphi_{1}$ is defined in A.2).
In our energy estimates and the estimates of the nonlocal terms, we need various estimates of the weights and their derivatives. From Appendix C. 1 of Part I [13] and (A.2), (A.1), we have two types of weights. The first one is the radial weight

$$
\rho(x, y)=\sum_{i} p_{i} r^{a_{i}}, \quad r=\left(x^{2}+y^{2}\right)^{1 / 2}
$$

where $a_{i}$ is increasing and $p_{i} \geq 0$. We use these weights for the Hölder estimates. See e.g. (A.1).
The second type of weights is the following

$$
\rho(x, y)=\rho_{1}(r)|x|^{-\alpha}+\rho_{2}(r),
$$

where $\rho_{1}, \rho_{2}$ are the radial weights.
We use $f_{l}, f_{u}$ to denote the lower and upper bound of $f$. We have the following simple inequalities

$$
\begin{align*}
& (f-g)_{l}=f_{l}-g_{u}, \quad(f-g)_{u}=f_{u}-g_{l}, \quad(f+g)_{\gamma}=f_{\gamma}+g_{\gamma} \\
& (f g)_{l}=\min \left(f_{l} g_{l}, f_{u} g_{l}, f_{l} g_{u}, f_{u} g_{u}\right), \quad(f g)_{u}=\max \left(f_{l} g_{l}, f_{u} g_{l}, f_{l} g_{u}, f_{u} g_{u}\right) \tag{A.4}
\end{align*}
$$

where $\gamma=l$, $u$. If $g \geq 0$, we can simplify the formula for the product

$$
\begin{equation*}
(f g)_{l}=\min \left(f_{l} g_{l}, f_{l} g_{u}\right), \quad(f g)_{u}=\max \left(f_{u} g_{l}, f_{u} g_{u}\right) \tag{A.5}
\end{equation*}
$$

Given the piecewise bounds of $\partial^{j} f, \partial^{j} g, j \leq k$, we can estimate $\partial^{k}(f g)$ using the Leibniz rule

$$
\begin{equation*}
\left|\partial_{x}^{i} \partial_{y}^{j}(f g)\right| \leq \sum_{k \leq i, l \leq j}\binom{i}{k}\left|\partial_{x}^{k} \partial_{y}^{l} f\right| \cdot\left|\partial_{x}^{i-k} \partial_{y}^{j-l} g\right| \tag{A.6}
\end{equation*}
$$

A.2. Radial weights. The advantage of radial weights is that we can estimate them easily.
A.2.1. Bounds for the derivatives. We can easily derive the derivatives and their upper and lower bound as follows. Firstly, we have

$$
\begin{equation*}
\left(\partial_{x}^{i} \partial_{y}^{j} \rho(x, y)\right)_{\gamma}=\sum_{1 \leq k \leq n} p_{k}\left(\partial_{x}^{i} \partial_{y}^{j} r^{a_{k}}\right)_{\gamma} \tag{A.7}
\end{equation*}
$$

where $\gamma=l, u$. Using induction, for any $\alpha, i, j$, we can obtain
$\partial_{x}^{i} \partial_{y}^{j} r^{\alpha}=\sum_{k \leq i+j, l \leq \min (j, 1)} C_{i, j, k, l}(\alpha) x^{k} y^{l} r^{\alpha-i-j-k-l}=\sum_{k \leq i+j, l \leq \min (j, 1)}\left(C_{i, j, k, l}^{+}(\alpha)-C_{i, j, k, l}^{-}(\alpha)\right) x^{k} y^{l} r^{\alpha-i-j-k-l}$,
with $C_{i, j, k, l}^{ \pm}(\alpha) \triangleq \max \left(0, C_{i, j, k, l}(\alpha)\right)$. The bounds for $C_{i, j, k, l}^{ \pm}(\alpha) x^{k} y^{l} r^{\alpha-i-j-k-l}$ are simple:

$$
\begin{equation*}
\left(C_{i, j, k, l}^{ \pm}(\alpha) x^{k} y^{l} r^{\alpha-i-j-k-l}\right)_{\gamma}=C_{i, j, k, l}^{ \pm}(\alpha) x_{\gamma}^{k} y_{\gamma}^{l} r_{\gamma}^{\alpha-i-j-k-l} \tag{A.8}
\end{equation*}
$$

In particular, we use the derivatives bound for $i+j \leq 4$ and we have
$\partial_{x} r^{a}=a x r^{a-2}, \quad \partial_{x}^{2} r^{a}=a r^{a-2}+a(a-2) x^{2} r^{a-4}, \quad \partial_{x y} r^{a}=a(a-2) x y r^{a-4}$,
$\partial_{x}^{3} r^{a}=a(a-2)(a-4) x^{3} r^{a-6}+3 a(a-2) x r^{a-4}, \partial_{x}^{2} \partial_{y} r^{a}=a(a-2) y r^{a-4}+a(a-2)(a-4) x^{2} y r^{a-6}$,
$\partial_{x}^{4} r^{a}=3 a(a-2) r^{a-4}+6 a(a-2)(a-4) x^{2} r^{a-6}+a(a-2)(a-4)(a-6) x^{4} r^{a-8}$,
$\partial_{x}^{3} \partial_{y} r^{a}=a(a-2)(a-4) x y r^{a-6}+2 a(a-2)(a-4) x y r^{a-6}+a(a-2)(a-4)(a-6) x^{3} y r^{a-8}$,
$\partial_{x}^{2} \partial_{y}^{2} r^{a}=a(a-2)(a-3) r^{a-4}+a(a-2)(a-4)(a-6) x^{2} r^{a-6}-x^{4} a(a-2)(a-4)(a-6) r^{a-8}$.
Using (A.4), the above identities, and linearity, we can obtain the upper and lower bounds for $\partial_{x}^{i} \partial_{y}^{j} \rho$. Since $\rho(x, y)$ is symmetric in $x, y$, we have $\partial_{1}^{i} \partial_{2}^{j} \rho(x, y)=\left(\partial_{1}^{j} \partial_{2}^{i} \rho\right)(y, x)$ and can obtain piecewise bounds of $\partial_{1}^{i} \partial_{2}^{j} \rho$ from that of $\partial_{1}^{j} \partial_{2}^{i} \rho$.

For the estimate in Section 4.5, we need to use the estimates of $\partial_{x}^{i} \partial_{y}^{j} \rho(\lambda x)$ for very small $\lambda \leq \lambda_{*}$ or very large $\lambda \geq \lambda_{*}$ uniformly. Obviously, the bounds are mainly determined by the leading order power of $p(\lambda x)$, i.e. $p_{1}|\lambda r|^{a_{1}}$ for small $\lambda$ and $p_{n}|\lambda r|^{a_{n}}$ for large $\lambda$. We would like
to estimate $\left(\partial_{x}^{i} \partial_{y}^{j} \rho(\lambda x)\right)_{\gamma} \lambda^{-\beta}$ for $\lambda \leq \lambda_{*}, \beta=a_{1}$ and $\lambda \geq \lambda_{*}, \beta=a_{n}, \gamma=l$,u. Using the above derivations (A.7), we have

$$
\lambda^{-\beta}\left(\partial_{x}^{i} \partial_{y}^{j} \rho(x, y)\right)_{\gamma}=\sum_{1 \leq k \leq n} p_{k}\left(\partial_{x}^{i} \partial_{y}^{j} \lambda^{a_{k}-\beta} r^{a_{k}}\right)_{\gamma}, \quad \gamma=l, u,
$$

and we only need to derive the upper and the lower bounds for $C_{i, j, k, l}^{ \pm}\left(a_{m}\right) x^{k} y^{l} r^{\alpha-i-j-k-l} \lambda^{a_{m}-\beta}$ uniformly for $\lambda \leq \lambda_{*}, \beta=a_{1}$ or $\lambda \geq \lambda_{*}, \beta=a_{n}$. Since $a_{i}$ is increasing, in the first case, we have

$$
\lambda^{a_{1}-a_{1}}=1, \quad a_{m}-a_{1}>0, \quad\left(\lambda^{a_{m}-a_{1}}\right)_{l}=0, \quad\left(\lambda^{a_{m}-a_{1}}\right)_{u}=\lambda_{*}^{a_{m}-a_{1}}, m>1 .
$$

In the second case, we get

$$
\lambda^{a_{n}-a_{n}}=1, \quad a_{m}-a_{n}<0, \quad\left(\lambda^{a_{m}-a_{n}}\right)_{l}=0,\left(\lambda^{a_{m}-a_{n}}\right)_{u}=\lambda_{*}^{a_{m}-a_{n}}, m>1 .
$$

In both cases, if $a_{m}=\beta$, we get a trivial bound 1 for $\lambda^{a_{m}-\beta}$; if $a_{m} \neq \beta$, we get $0 \leq$ $\lambda^{a_{m}-\beta} \leq \lambda_{*}^{a_{m}-\beta}$. Using these bounds for $\lambda^{a_{m}-\beta}$, (A.8), (A.4), A.5), we obtain the bounds for $\lambda^{-\beta} \partial_{x}^{i} \partial_{y}^{j} \psi(\lambda x)$ uniformly for small $\lambda, \beta=a_{1}$ and large $\lambda, \beta=a_{n}$.

We also need to bound $M=\lambda^{-\beta} \rho_{\lambda}(x)\left|\frac{\rho_{\lambda}(y)}{\rho_{\lambda}(x)}-\frac{\rho_{\text {lim }}(y)}{\rho_{\text {lim }}(x)}\right|$ used in Section 4.5 uniformly for $\lambda \leq \lambda_{*}, \beta=a_{1}, \rho_{\text {lim }}(y)=p_{1}|y|^{a_{1}}$ or $\lambda \geq \lambda_{*}, \beta=a_{n}, \rho_{\text {lim }}(y)=p_{n}|y|^{a_{n}}$. Using the formula of $\rho$ and a direct computation yield

$$
\frac{\rho_{l i m}(y)}{\rho_{\text {lim }}(x)}=\frac{|y|^{\beta}}{|x|^{\beta}}, \left.\quad M \leq\left.\sum_{i \leq n} p_{i} \lambda^{a_{i}-\beta}| | y\right|^{a_{i}}-\left.\left.|x|^{a_{i}} \frac{|y|^{\beta}}{|x|^{\beta}}\left|\leq \sum_{i \leq n} p_{i} \lambda_{*}^{a_{i}-\beta}\right| y\right|^{\beta}| | y\right|^{a_{i}-\beta}-|x|^{a_{i}-\beta} \right\rvert\, .
$$

We remark that the leading power $\lambda_{*}^{a_{i}-\beta}$ for $a_{i}=\beta$ is cancelled due to $|y|^{0}=|x|^{0}=1$ in the above estimate and we gain the small factor $\lambda_{*}^{a_{i}-\beta}$ for $a_{i} \neq \beta$.
A.2.2. Leading order behavior of $\partial \rho / \rho$. In our verification, we need to bound $\partial \rho(\lambda x) / \rho(\lambda x)$ as $\lambda \rightarrow 0$ or $\lambda \rightarrow \infty$ uniformly. A direct calculation yields

$$
\frac{\partial_{x_{i}} \rho}{\rho}=\frac{x_{i}}{|x|^{2}} \frac{\sum_{i} p_{i} a_{i} r^{a_{i}}}{\sum_{i} p_{i} r^{a_{i}}} \triangleq \frac{x_{i}}{|x|^{2}} S(x), \quad S(x) \triangleq \frac{\sum_{i} p_{i} a_{i} r^{a_{i}}}{\sum_{i} p_{i} r^{a_{i}}} .
$$

For $x$ close to 0 , we introduce $b=a-a_{1}$. Clearly, we get $b_{i} \geq 0$ and

$$
S(x)=a_{1}+\frac{\sum_{i} p_{i} b_{i} r^{a_{i}}}{\sum_{i} p_{i} r^{a_{i}}}=a_{1}+\frac{\sum_{i} p_{i} b_{i} r^{b_{i}}}{\sum_{i} p_{i} r^{b_{i}}} \triangleq a_{1}+\frac{A(r)}{B(r)} .
$$

Using $b_{i} \geq 0$ and the Cauchy-Schwarz inequalities, we yield
$A^{\prime} B-A B^{\prime}=r^{-1}\left(\left(\sum p_{i} b_{i}^{2} r^{b_{i}}\right)\left(\sum p_{i} r^{b_{i}}\right)-\left(\sum p_{i} b_{i} r^{b_{i}}\right)^{2}\right)=r^{-1} \frac{1}{2} \sum_{i j} p_{i} p_{j}\left(b_{i}-b_{j}\right)^{2} r^{b_{i}+b_{j}} \geq 0$,
and thus $A / B$ is increasing. For $\lambda \leq \lambda_{*}, r \in\left[r_{l}, r_{u}\right]$, we get the uniform bound for $S(\lambda x)$

$$
a_{1} \leq S(\lambda x) \leq a_{1}+\frac{A\left(\lambda_{*} r_{u}\right)}{B\left(\lambda_{*} r_{u}\right)}
$$

For $\lambda=1$, we simply obtain

$$
a_{1}+\frac{A\left(r_{l}\right)}{B\left(r_{l}\right)} \leq S(x) \leq a_{1}+\frac{A\left(r_{u}\right)}{B\left(r_{u}\right)} .
$$

Similarly, for $\lambda \geq \lambda_{*}, r \in\left[r_{l}, r_{u}\right]$, we get

$$
a_{n}+\frac{A\left(\lambda_{*} r_{l}\right)}{B\left(\lambda_{*} r_{l}\right)} \leq S(\lambda x) \leq a_{n}, \quad \frac{A(r)}{B(r)}=\frac{\sum_{i} p_{i} b_{i} r^{b_{i}}}{\sum_{i} p_{i} r^{b_{i}}},
$$

where $b=a-a_{n} \leq 0$. Here, we have used that $A(r) / B(r)$ is increasing. Thought $b_{i}$ is negative, we still have $(A / B)^{\prime}=\frac{A^{\prime} B-A B^{\prime}}{B^{2}}>0$. From the above estimates, we yield

$$
\begin{aligned}
& \lim _{\lambda \rightarrow 0} \frac{\partial_{x_{i}} \rho}{\rho}=\frac{x_{i}}{|x|^{2}} a_{1} \triangleq R_{0}(x), \quad\left|\frac{\partial_{x_{i}} \rho}{\rho}(\lambda x)-R_{0}(\lambda x)\right| \leq \lambda^{-1} \frac{x_{i}}{|x|^{2}} \frac{\left|A\left(\lambda_{*} x\right)\right|}{\left|B\left(\lambda_{*} x\right)\right|}, \lambda \leq \lambda_{*}, \\
& \lim _{\lambda \rightarrow \infty} \frac{\partial_{x_{i}} \rho}{\rho}=\frac{x_{i}}{|x|^{2}} a_{n} \triangleq R_{\infty}(x), \quad\left|\frac{\partial_{x_{i}} \rho}{\rho}(\lambda x)-R_{\infty}(\lambda x)\right| \leq \lambda^{-1} \frac{x_{i} i}{|x|^{2}} \frac{\left|A\left(\lambda_{*} x\right)\right|}{\left|B\left(\lambda_{*} x\right)\right|}, \lambda \geq \lambda_{*} .
\end{aligned}
$$

A.2.3. Bounds for the derivatives of $1 / \rho$. The bounds for $d_{x}^{i} d_{y}^{j} \rho^{-1}$ is more complicated since $\rho^{-1}$ is not linear in the summand $p_{i} r^{a_{i}}$. We need such estimates in the estimate of the velocity. Firstly, using the bounds in Section A.2.1 and A.5), we can obtain the upper and the lower bounds for $R_{i j}$

$$
R_{i j}=\frac{\partial_{x}^{i} \partial_{y}^{j} \rho}{\rho}
$$

For $i+j=1$ and $k=2,3$, we use the estimate in Section A.2.1 to obtain the bounds for

$$
R_{10}=\frac{x}{|x|^{2}} S(x), \quad R_{0,1}=\frac{y}{|x|^{2}} S(x), \quad\left(R_{i j}\right)^{k}
$$

In our estimate, we need $\partial_{x}^{i} \partial_{y}^{j} \rho^{-1}$ for $i+j \leq 3$. A direct calculation yields
$\partial_{x} \rho^{-1}=-\frac{\rho_{x}}{\rho^{2}}=-\frac{R_{10}}{\rho}, \quad \partial_{x x} \rho^{-1}=-\frac{\rho_{x x}}{\rho^{2}}+2 \frac{\rho_{x}^{2}}{\rho^{3}}=\rho^{-1}\left(-R_{20}+2 R_{10}^{2}\right)$,
$\partial_{x y} \rho^{-1}=-\frac{\rho_{x y}}{\rho}+\frac{2 \rho_{x} \rho_{y}}{\rho^{3}}=\rho^{-1}\left(-R_{11}+2 R_{10} R_{01}\right)$,
$\partial_{x x x} \rho^{-1}=-\frac{\rho_{x x x}}{\rho^{2}}+\frac{6 \rho_{x x} \rho_{x}}{\rho^{3}}-\frac{6 \rho_{x}^{3}}{\rho^{4}}=\rho^{-1}\left(-R_{30}+6 R_{20} R_{10}-6 R_{10}^{3}\right)$,
$\partial_{x x y} \rho^{-1}=-\frac{\rho_{x x y}}{\rho^{2}}+\frac{2 \rho_{x x} \rho_{y}}{\rho^{3}}+\frac{4 \rho_{x} \rho_{x y}}{\rho^{3}}-6 \frac{\rho_{x}^{2} \rho_{y}}{\rho^{4}}=\rho^{-1}\left(-R_{21}+2 R_{20} R_{01}+4 R_{10} R_{11}-6 R_{10}^{2} R_{01}\right)$.
Next, we estimate $\partial_{x}^{i} \partial_{y}^{j}\left(\partial_{x_{l}} \rho / \rho\right)$ for $i \leq 2, j=0$ or $i=0, j \leq 2$. Denote $f=\partial_{x_{l}} \rho$. Using a direct computation, for $D_{2}=\partial_{x}^{i_{2}} \partial_{y}^{j_{2}}$ with $i_{2}+j_{2}=1$, we yield

$$
D_{2} \frac{f}{\rho}=\frac{D_{2} f}{\rho}-\frac{f D_{2} \rho}{\rho^{2}}=\rho^{-1}\left(D_{2} f-f R_{i_{2}, j_{2}}\right)
$$

For $\left(i_{2}, j_{2}\right)=(2,0),(0,2)$, denote $i_{3}=i_{2} / 2, j_{3}=j_{2} / 2, D_{3}=\partial_{x}^{i_{3}} \partial_{y}^{j_{3}}$. We yield

$$
\begin{aligned}
D_{3}^{2} \frac{f}{\rho} & =\frac{D_{3}^{2} f}{\rho}-\frac{2 D_{3} f \cdot D_{3} \rho}{\rho^{2}}+f D_{3}^{2}\left(\frac{1}{\rho}\right)=\frac{D_{3}^{2} f}{\rho}-\frac{2 D_{3} f \cdot D_{3} \rho}{\rho^{2}}+f\left(-\frac{D_{3}^{2} \rho}{\rho^{2}}+\frac{2\left(D_{3} \rho\right)^{2}}{\rho^{3}}\right) \\
& =\rho^{-1}\left(D_{3}^{2} f-2 D_{3} f R_{i_{3}, j_{3}}-f R_{i_{2}, j_{2}}+2 f R_{i_{3}, j_{3}}^{2}\right)
\end{aligned}
$$

where we have used $D_{3}^{2} \frac{1}{\rho}=D_{3}\left(-\frac{D_{3} \rho}{\rho^{2}}\right)=-\frac{D_{3}^{2} \rho}{\rho^{2}}+\frac{2\left(D_{3} \rho\right)^{2}}{\rho^{3}}$.
Since we have estimated $\partial_{x}^{i} \partial_{y}^{j} \rho$ and $R_{i j}$, we can bound these derivatives of $D_{1} \rho / \rho$ using (A.4).
We also need to obtain the uniform estimates of $\lambda^{\beta} \partial_{x}^{i} \partial_{y}^{j}\left(\rho^{-1}(\lambda x)\right)$ for $\lambda \leq \lambda_{*}, \beta=a_{1}$ and $\lambda \geq \lambda_{*}, \beta=a_{n}$. Denote $\rho_{\lambda}(x)=\rho(\lambda x)$. For example, for $D_{1}=\partial_{x_{i}}$, we have

$$
\lambda^{\beta} D_{1}\left(\rho_{\lambda}^{-1}(x)\right)=-\lambda^{1+\beta} \frac{\left(D_{1} \rho\right)(\lambda x)}{\rho_{\lambda}^{2}(x)}=-\lambda^{1+\beta} \rho_{\lambda}^{-1}(x) \lambda^{-1} \frac{x_{i}}{|x|^{2}} S(\lambda x)=-\lambda^{\beta} \rho_{\lambda}^{-1}(x) \frac{x_{i}}{|x|^{2}} S(\lambda x)
$$

which can be estimated using the estimates in Sections A.2.1 A.2.2 The power $\lambda^{\beta}$ and the leading power $\lambda^{-\beta}$ in $\rho_{\lambda}^{-1}(x)$ cancel each other. The estimates of $\lambda^{\beta} \partial_{x}^{i} \partial_{y}^{j}\left(\rho^{-1}(\lambda x)\right)$ with $i+j \geq 2$ and $\partial_{x}^{i} \partial_{y}^{j} \frac{\partial_{x}^{i}\left(\rho_{\lambda}\right)}{\rho_{\lambda}}$ are similar, and follow from the above estimates for $\partial_{x}^{i} \partial_{y}^{j} \rho^{-1}, \partial_{x}^{i} \partial_{y}^{j}(\partial \rho / \rho)$, the uniform estimates for $\partial_{x}^{i} \partial_{y}^{j} p(\lambda x)$ in Section A.2.1 and $\frac{\partial \rho}{\rho}$ in Section A.2.2 We remark that in all of these estimates for $\rho_{\lambda}(x)$, taking derivatives in $x$ does not change the asymptotic power in $\lambda$.
A.2.4. Improved estimates for $\rho^{-1}$ near $x=0$. For the special case $a_{1}=-2$, we can write

$$
\rho(x)=r^{-2} \sum_{i} p_{i} r^{a_{i}+2}=r^{-2} \tilde{\rho}(x), \quad \rho^{-1}=\left(x^{2}+y^{2}\right) \tilde{\rho}(x)^{-1}
$$

To obtain a better estimate of $\rho^{-1}$, we use the fact that $x^{2}+y^{2}$ is a polynomial. Firstly, we can obtain the bounds for $\partial_{x}^{i} \partial_{y}^{j} \tilde{\rho}^{-1}$. The bound for $S_{0}=x^{2}+y^{2}$ is trivial, e.g.,

$$
\left(\partial_{x} S_{0}\right)_{\gamma}=2 x_{\gamma},\left(\partial_{y} S_{0}\right)_{\gamma}=2 y_{\gamma}, \gamma=u, l, \quad \partial_{x y} S_{0}=0, \quad \partial_{x x} S_{0}=\partial_{y y} S_{0}=2
$$

Then using (A.4)-(A.5), we can bound $\rho^{-1}$.
A.3. The mixed weight. For the second type of weights $W=\rho_{1}(r)|x|^{-1 / 2}+\rho_{2}(r)$, we can compute its derivatives and its upper and lower bounds using linearity and the Leibniz rule (A.6). We consider $x, y \geq 0$. For example, we have

$$
W_{l}=\rho_{1, l} x_{u}^{-1 / 2}+\rho_{2, l}, \quad\left(W^{-1}\right)_{u}=\left(W_{l}\right)^{-1}, \quad W_{x}=\partial_{x} \rho_{1} x^{-1 / 2}-\frac{1}{2} \rho_{1} x^{-3 / 2}+\partial_{x} \rho_{2}
$$

To obtain the upper bound for $\partial_{x}^{i} \partial_{y}^{j} W$, we use the Leibniz rule (A.6):

$$
\left|\partial_{x}^{i} \partial_{y}^{j} W\right| \leq \sum_{k \leq i}\binom{i}{k}\left|\partial_{x}^{i-k} \partial_{y}^{j} \rho_{1}\right| \frac{(2 k-1)!!}{2^{k}} x^{-1 / 2-k}+\left|\partial_{x}^{i} \partial_{y}^{j} \rho_{2}\right|
$$

We need to bound $\rho(r) / W(x, y)$ in the estimate of the integrals. Suppose that the leading and the last powers of $\rho$ is $a_{1}, a_{n}$. The leading and the last terms of $W$ are given by $p_{i} r^{b_{i}} \cos (\beta)^{-\alpha_{i}}, \alpha_{i} \geq 0$.

$$
W \geq p_{1} r^{b_{1}}, \quad W \geq p_{n} r^{b_{n}}
$$

We estimate

$$
\frac{\rho}{W} \leq C_{1} r^{a_{1}-b_{1}}, \quad \frac{\rho}{W} \leq C_{2} r^{a_{n}-b_{n}}
$$

for all $x, y \in \mathbb{R}_{2}^{+}$. We apply the above estimates for $x$ near 0 or $x$ sufficiently large.
Using $W(\lambda x) \geq \rho_{1}(\lambda x) \lambda^{-1 / 2}\left|x_{1}\right|^{-1 / 2}, W(\lambda x) \geq \rho_{2}(\lambda x)$, the uniform estimates of $\rho_{i}(\lambda x)$ in $\lambda$ in Section A.2.1 we can obtain the lower bound of $W(\lambda x)$ and the upper bound of $\frac{\rho(\lambda x)}{W(\lambda x)}$ uniformly in $\lambda$.

## Appendix B. Estimate the derivatives of the velocity kernel and integrands

In this appendix, we estimate the derivatives of the kernel $-\frac{1}{2 \pi} \log |x|$ associated to the velocity $\mathbf{u}=\nabla^{\perp}(-\Delta)^{-1} \omega$ and its symmetrization (4.25). These estimates are used to estimate the error terms in Lemmas 4.2, 4.4. We will perform an additional estimate for $u$ with weight $\varphi(x)$ singular along $x_{1}=0$ in Section B.4. Some additional derivations related to the estimate of the velocity are given in Appendix B. 5
B.1. Estimate the symmetrized kernel. In this section, we estimate the symmetrized kernel. We develop several symmetrized estimates for harmonic functions. Before we introduce the estimates, we have a simple 1D estimate, which is useful for later estimates.

Lemma B.1. We have
$|f(x)+f(-x)-2 f(0)| \leq x^{2}\left|\left\|f_{x x}\right\|_{L^{\infty}[-x, x]}, \quad\right| f(x)+f(-x)-2 f(0)-x^{2} f_{x x}(0) \left\lvert\, \leq \frac{x^{4}}{12}\left\|\partial_{x}^{4} f\right\|_{L^{\infty}[-x, x]}\right.$.
Proof. Denote $G(x)=f(x)+f(-x)$. Clearly, $G$ is even and

$$
\begin{equation*}
G(0)=2 f(0), \quad G^{\prime}(0)=0, \quad \partial_{x}^{2} G(0)=2 f_{x x}(0), \quad \partial_{x}^{3} G(0)=0 \tag{B.1}
\end{equation*}
$$

Using the Taylor expansion, we obtain

$$
G(x)=G(0)+G^{\prime}(0) x+\frac{\partial_{x}^{2} G(0) x^{2}}{2}+\frac{\left.\partial_{x}^{3} G^{( } 0\right) x^{3}}{6}+\frac{\partial_{x}^{4} G^{(\xi)} x^{4}}{24}
$$

for some $\xi \in[0, x]$. Using (B.1), we get

$$
\left|G(x)-G(0)-G^{\prime \prime}(x) \frac{x^{2}}{2}\right| \leq\left\|\partial_{x}^{4} G\right\|_{L^{\infty}[0, x]} \frac{x^{4}}{24} \leq\left\|\partial_{x}^{2} f\right\|_{L^{\infty}[-x, x]} \frac{x^{4}}{12}
$$

Plugging the identity (B.1) into the above estimate proves the second estimate in Lemma B. 1 The first estimate is simpler.

The following lemma is useful for estimating the symmetrized kernel (4.25) and its derivatives.
Lemma B.2. Suppose that $Q_{x}=\left[-x_{1}, x_{1}\right] \times\left[-x_{2}, x_{2}\right]$ and $f \in C^{4}\left(Q_{x}\right)$ is harmonic. Denote

$$
\begin{align*}
G_{1}(1, x) & \triangleq f\left(x_{1}, x_{2}\right)+f\left(-x_{1}, x_{2}\right)+f\left(x_{1},-x_{2}\right)+f\left(-x_{1},-x_{2}\right)-4 f(0,0) \\
G_{2}(1, x) & \triangleq f\left(x_{1}, x_{2}\right)-f\left(-x_{1}, x_{2}\right)-f\left(x_{1},-x_{2}\right)+f\left(-x_{1},-x_{2}\right)  \tag{B.2}\\
\hat{G}_{1}(x) & \triangleq 2 x_{1}^{2} f_{x x}(0,0)+2 x_{2}^{2} f_{y y}(0,0), \quad \hat{G}_{2}(x) \triangleq 4 x_{1} x_{2} f_{x y}(0,0)
\end{align*}
$$

We have

$$
\begin{align*}
& \left|G_{1}(1, x)\right| \leq 2|x|^{2}\left\|f_{x x}\right\|_{L^{\infty}\left(Q_{x}\right)}, \quad\left|\partial_{x_{i}} G_{1}(1, x)\right| \leq 4\left|x_{i}\right| \cdot\left\|f_{x x}\right\|_{L^{\infty}\left(Q_{x}\right)}  \tag{B.3}\\
& \left|G_{1}(1, x)-\hat{G}_{1}(x)\right| \leq \frac{\left(x_{1}^{4}+6 x_{1}^{2} x_{2}^{2}+x_{2}^{4}\right)}{6}\left\|\partial^{4} f\right\|_{L^{\infty}\left(Q_{x}\right)} \leq \frac{|x|^{4}}{3}\left\|\partial^{4} f\right\|_{L^{\infty}\left(Q_{x}\right)}  \tag{B.4}\\
& \left|G_{1}\left(1, x_{1}, 0\right)-\hat{G}_{1}\left(x_{1}, 0\right)\right| \leq \frac{1}{6} x_{1}^{4}\left\|\partial^{4} f\right\|_{L^{\infty}\left(Q_{x}\right)}  \tag{B.5}\\
& \left|\partial_{x_{i}}\left(G_{1}(1, x)-\hat{G}_{1}(x)\right)\right| \leq \frac{2}{3}\left(3 x_{3-i}^{2} x_{i}+x_{i}^{3}\right)\left\|\partial^{4} f\right\|_{L^{\infty}\left(Q_{x}\right)} \leq \frac{2 \sqrt{2}}{3}|x|^{3}\left\|\partial^{4} f\right\|_{L^{\infty}\left(Q_{x}\right)} \tag{B.6}
\end{align*}
$$

where $\left\|\partial^{4} f\right\|_{L^{\infty}}=\max _{0 \leq i \leq 4}\left\|\partial_{x}^{i} \partial_{y}^{j} f\right\|_{L^{\infty}\left(Q_{x}\right)}$. For $G_{2}$, we have the following estimate

$$
\begin{align*}
& \left|G_{2}(1, x)\right| \leq 4 x_{1} x_{2}\left\|f_{x y}\right\|_{L^{\infty}\left(Q_{x}\right)}, \quad\left|\partial_{x_{i}} G_{2}(1, x)\right| \leq 4\left|x_{3-i}\right| \cdot\left\|f_{x y}\right\|_{L^{\infty}\left(Q_{x}\right)}  \tag{B.7}\\
& \left|G_{2}(1, x)-\hat{G}_{2}(x)\right| \leq \frac{2 x_{1} x_{2}|x|^{2}}{3}\left\|\partial^{4} f\right\|_{L^{\infty}\left(Q_{x}\right)}  \tag{B.8}\\
& \left|\partial_{x_{i}}\left(G_{2}(1, x)-\hat{G}_{2}(x)\right)\right| \leq \frac{2}{3}\left(3 x_{i}^{2} x_{3-i}+x_{3-i}^{3}\right)\left\|\partial^{4} f\right\|_{L^{\infty}\left(Q_{x}\right)} \leq \frac{2 \sqrt{2}}{3}|x|^{3}\left\|\partial^{4} f\right\|_{L^{\infty}\left(Q_{x}\right)} \tag{B.9}
\end{align*}
$$

Note that $G_{1}(\cdot, x)$ is even in $x_{i}$, and $G_{2}(\cdot, x)$ is odd in $x_{i}$. The polynomials of $x_{i}$ in the upper bounds (without absolute value) have the same symmetries. Similar properties hold for $\partial G_{1}, \partial G_{2}$. Moreover the above bound satisfies the differential relation. These properties are useful for tracking different bounds for $G_{1}, G_{2}$.

Proof. Recall $Q_{x}=\left[-x_{1}, x_{1}\right] \times\left[-x_{2}, x_{2}\right]$. Denote

$$
A_{i j}(x)=\left\|\partial_{x}^{i} \partial_{y}^{j} f\right\|_{L^{\infty}\left(Q_{x}\right)}
$$

Using Lemma B.1 for any $t \in[0,1]$, we obtain

$$
\left|f\left(t x_{1}, x_{2}\right)+f\left(t x_{1},-x_{2}\right)-2 f\left(t x_{1}, 0\right)\right| \leq A_{02} x_{2}^{2}, \quad\left|f\left(x_{1}, 0\right)+f\left(-x_{1}, 0\right)-2 f(0,0)\right| \leq A_{20} x_{1}^{2}
$$

Since $f$ is harmonic function, we have $\partial_{x}^{i+2} \partial_{y}^{j} f=-\partial_{x}^{i} \partial_{y}^{j+2} f$ and obtain $A_{i+2, j}=A_{i, j+2}$. Taking $t= \pm 1$ in the above estimate and using the triangle inequality, we prove

$$
|G(1, x)| \leq 2 A_{20} x_{1}^{2}+2 A_{02} x_{2}^{2}=2 A_{20}\left(x_{1}^{2}+x_{2}^{2}\right)=2 A_{20}|x|^{2}
$$

which is the first estimate in (B.3).
The second estimate in ( $\overline{\mathrm{B} .3})$ is simple. We consider $i=1$ without loss of generality. We get $\left.\left|\partial_{x_{1}} G_{1}(1, x)\right|=\mid\left(\partial_{1} f\right)\left(x_{1}, x_{2}\right)-\left(\partial_{1} f\right)\left(-x_{1}, x_{2}\right)+\left(\partial_{1} f\right)\left(x_{1},-x_{2}\right)-\left(\partial_{1} f\right)\left(-x_{1},-x_{2}\right)\right) \mid \leq 4 x_{1} A_{20}(x)$.

For (B.4), using Lemma B.1, we yield

$$
\begin{align*}
\left|f\left(t x_{1}, x_{2}\right)+f\left(t x_{1},-x_{2}\right)-2 f\left(t x_{1}, 0\right)-x_{2}^{2}\left(\partial_{2}^{2} f\right)\left(t x_{1}, 0\right)\right| & \leq A_{04}(x) \frac{x_{2}^{4}}{12} \\
\left|\partial_{2}^{2} f\left(x_{1}, 0\right)+\partial_{2}^{2} f\left(-x_{1}, 0\right)-2 \partial_{2}^{2} f(0,0)\right| & \leq x_{1}^{2} A_{2,2}(x)  \tag{B.10}\\
\left|f\left(x_{1}, 0\right)+f\left(-x_{1}, 0\right)-2 f(0)-x_{1}^{2} \partial_{1}^{2} f(0)\right| & \leq A_{40} \frac{x_{1}^{4}}{12}
\end{align*}
$$

for $t= \pm 1$. Combining the above estimates and using the triangle inequality and $A_{40}=A_{22}=$ $A_{04}$, we prove the first estimate in ( $\overline{\mathrm{B} .4}$ ). The second estimate follows from $2|x|^{4}-x_{1}^{4}-6 x_{1}^{2} x_{2}^{2}-$ $x_{2}^{4}=\left(x_{1}^{2}-x_{2}^{2}\right)^{2} \geq 0$.

Estimate ( (B.5) follows from ( (B.4) by taking $x_{2}=0$.

For (B.6), we consider the estimate of $\partial_{x_{1}}$. The other case is similar. Using

$$
\partial_{1} f\left(x_{1}, s\right)-\left(\partial_{1} f\right)\left(-x_{1}, s\right)=\int_{0}^{x_{1}}\left(\partial_{1}^{2}\right) f(t, s)+\left(\partial_{1}^{2}\right) f(-t, s) d t
$$

we obtain

$$
\begin{aligned}
\partial_{1}(G(1, x) & \left.-\hat{G}_{1}(x)\right)=\left(\partial_{1}\right) f\left(x_{1}, x_{2}\right)-\left(\partial_{1} f\right)\left(-x_{1}, x_{2}\right)+\left(\partial_{1}\right) f\left(x_{1},-x_{2}\right)-\left(\partial_{1} f\right)\left(-x_{1},-x_{2}\right)-4 x_{1} \partial_{1}^{2} f(0) \\
& =\int_{0}^{x_{1}}\left(\left(\partial_{1}^{2} f\right)\left(z, x_{2}\right)+\left(\partial_{1}^{2} f\right)\left(-z, x_{2}\right)+\left(\partial_{1}^{2}\right) f\left(z,-x_{2}\right)+\left(\partial_{1}^{2} f\right)\left(-z, x_{2}\right)-4 \partial_{1}^{2} f(0)\right) d z
\end{aligned}
$$

Applying (B.3), we yield

$$
\left|\partial_{1}\left(G(1, x)-\hat{G}_{1}(x)\right)\right| \leq \int_{0}^{x_{1}} 2\left(z^{2}+x_{2}^{2}\right) d z A_{4,0}(x)=\left(\frac{2}{3} x_{1}^{3}+2 x_{1} x_{2}^{2}\right) A_{4,0}(x)
$$

and complete the proof of the first estimate in (B.6). For the second estimate, we use the AM-GM inequality to yield

$$
\begin{equation*}
\left(3 x_{2}^{2} x_{1}+x_{1}^{3}\right)^{2}=\left(3 x_{2}^{2}+x_{1}^{2}\right)^{2} x_{1}^{2}=\frac{1}{4}\left(3 x_{2}^{2}+x_{1}^{2}\right)^{2} 4 x_{1}^{2} \leq \frac{1}{4}\left(\frac{2\left(3 x_{2}^{2}+x_{1}^{2}\right)+4 x_{1}^{2}}{3}\right)^{3}=2|x|^{6} \tag{B.11}
\end{equation*}
$$

Taking a square root completes the estimate.
To estimate $G_{2}$ in ( $\overline{\mathrm{B} .2}$ ), we rewrite it as follows

$$
\begin{align*}
G_{2}(1, x)-c \hat{G}_{2}(x)= & \int_{-x_{1}}^{x_{1}} \int_{-x_{2}}^{x_{2}} \partial_{12} f\left(z_{1}, z_{2}\right)-c \partial_{12} f(0) d z \\
= & \int_{0}^{x_{1}} \int_{0}^{x_{2}}\left(\partial_{12} f\right)\left(z_{1}, z_{2}\right)+\left(\partial_{12} f\right)\left(-z_{1}, z_{2}\right)+\left(\partial_{12} f\right)\left(z_{1},-z_{2}\right)  \tag{B.12}\\
& +\left(\partial_{12} f\right)\left(-z_{1}, z_{2}\right)-4 c\left(\partial_{12} f\right)(0) d z
\end{align*}
$$

for $c=0,1$. The integrand has the same form as $G_{1}$ in (B.2). For $c=0$, using the above decomposition, we prove

$$
\left|G_{2}(1, x)\right| \leq 4 x_{1} x_{2} A_{11}
$$

When $c=1$, using ( $(\overline{\mathrm{B} .6})$, we yield

$$
\left|G_{2}(1, x)-\hat{G}_{2}(x)\right| \leq A_{40} 2 \int_{0}^{x_{1}} \int_{0}^{x_{2}}|y|^{2} d y=A_{40} \frac{2}{3}\left(x_{1}^{3} x_{2}+x_{1} x_{2}^{3}\right)=A_{40} \frac{2}{3} x_{1} x_{2}|x|^{2}
$$

To estimate the derivatives, we focus on $\partial_{x_{1}}$. Using the above representation, we obtain

$$
\begin{aligned}
\partial_{x_{1}}\left(G_{2}(1, x)-c \hat{G}_{2}(x)\right)= & \int_{0}^{x_{2}}\left(\left(\partial_{12} f\right)\left(x_{1}, y_{2}\right)+\left(\partial_{12} f\right)\left(-x_{1}, y_{2}\right)\right) \\
& +\left(\left(\partial_{12} f\right)\left(x_{1},-y_{2}\right)+\left(\partial_{12} f\right)\left(-x_{1},-y_{2}\right)-4 c\left(\partial_{12} f\right)(0)\right) d y
\end{aligned}
$$

We apply the same estimates to the integrands with $c=0,1$ and yield
$\left|\partial_{x_{1}} G_{2}(1, x)\right| \leq 4 x_{2} A_{11}, \quad\left|\partial_{x_{1}}\left(G_{2}(1, x)-\hat{G}_{2}(x)\right)\right| \leq A_{31} 2 \int_{0}^{x_{2}}\left(x_{1}^{2}+y_{2}^{2}\right) d y_{2}=A_{31}\left(2 x_{1}^{2} x_{2}+\frac{2}{3} x_{2}^{3}\right)$.
The second inequality in (B.9) follows from (B.11). The above estimates imply (B.7)-(B.9).
Recall the kernels associated with $\nabla \mathbf{u}, \mathbf{u}$ in (4.1). These kernels are the derivatives of the Green function $-\frac{1}{2 \pi} \log |x|$ and are harmonic away from 0 . We have the following estimates for their derivatives.

Lemma B.3. Denote $r=\left(x^{2}+y^{2}\right)^{\frac{1}{2}}$ and $f(x, y)=\log r$. For any $i, j \geq 0$ with $i+j \geq 1$, we have

$$
\left|\partial_{x}^{i} \partial_{y}^{j} f(x, y)\right| \leq(i+j-1)!\cdot r^{-i-j}
$$

As a result, for $K_{1}(y)=-\frac{1}{2} \partial_{12} f(y), K_{2}(y)=-\frac{1}{2} \partial_{1}^{2} f(y)$, we have
$\left|K_{i}\right| \leq \frac{1}{2|y|^{2}}, \quad\left|\partial_{y_{1}}^{j} \partial_{y_{2}}^{2-j} K_{i}\right| \leq \frac{3}{|y|^{4}}, \quad\left|\partial_{y_{1}}^{j} \partial_{y_{2}}^{4-j} K_{i}\right| \leq \frac{60}{|y|^{6}}, \quad\left|\partial_{y_{1}}^{j} \partial_{y_{2}}^{6-j} K_{i}\right| \leq \frac{2520}{|y|^{8}}, \quad i=1,2$.

Proof. Consider the polar coordinate $\beta=\arctan (y / x), r=\left(x^{2}+y^{2}\right)^{1 / 2}$. We use induction on $n=i+j$ to prove

$$
\begin{equation*}
\partial_{x}^{i} \partial_{y}^{j} f=(n-1)!\cos \left(n \beta-\beta_{i j}\right) r^{-n} \tag{B.13}
\end{equation*}
$$

for some constant $\beta_{i j}$. We have the formula

$$
\begin{equation*}
\partial_{x} g=\left(\cos \beta \partial_{r}-\frac{\sin \beta}{r} \partial_{\beta}\right) g, \quad \partial_{y} g=\left(\sin \beta \partial_{r}+\frac{\cos \beta}{r} \partial_{\beta}\right) g \tag{B.14}
\end{equation*}
$$

Firstly, for $n=1$, a direct calculation yields

$$
\partial_{x} f=\frac{x}{r^{2}}=\frac{\cos \beta}{r}, \quad \partial_{y} f=\frac{y}{r^{2}}=\frac{\sin \beta}{r}=\frac{\cos (\beta-\pi / 2)}{r} .
$$

Suppose that (B.13) holds for any $i, j$ with $i+j=n$ and $n \geq 1$. Now, since

$$
\begin{aligned}
\partial_{x} \partial_{x}^{i} \partial_{y}^{j} f & =(n-1)!\partial_{x}\left(\cos \left(n \beta-\beta_{i j}\right) r^{-n}\right) \\
& =(n-1)!\left(-n \cos \beta \cos \left(n \beta-\beta_{i j}\right) r^{-n-1}+n \sin \beta \sin \left(n \beta-\beta_{i j}\right) r^{-n-1}\right) \\
& =n!\left(-\cos \left(n \beta-\beta_{i j}+\beta\right) r^{-n-1}\right)=n!\cos \left((n+1) \beta-\beta_{i j}-\pi\right) r^{-n-1}
\end{aligned}
$$

using a similar computation and $\sin (x)=\cos (x-\pi / 2)$, we can obtain that $\partial_{y} \partial_{x}^{i} \partial_{y}^{j} f$ has the form (B.13). Using induction, we prove ( $\overline{\mathrm{B} .13)}$. The desired estimate follows from (B.13).

Using the above two Lemmas, we can estimate the error in the discretization of the kernels $K(x, y)$ in both $x$ and $y$ directions.
B.1.1. Estimate the kernels in the far field. We apply Lemma B. 2 to estimate the decay of $F_{1}, F_{2}$ (B.15)
$F_{0} \triangleq G(y-x)-G\left(y_{1}-x_{1}, y_{2}+x_{2}\right)-G\left(y_{1}+x_{1}, y_{2}-x_{2}\right)+G(y+x), G(y)=-\log |y| / 2$,
$F_{1} \triangleq F_{0}-4 x_{1} x_{2} \partial_{12} G(y), F_{2} \triangleq F_{1}-\frac{2\left(x_{1}^{2}-x_{2}^{2}\right) x_{1} x_{2}}{3} \partial_{1}^{3} \partial_{2} G(y), I_{i j k l}(P) \triangleq \partial_{x_{1}}^{i} \partial_{x_{2}}^{j} \partial_{y_{1}}^{k} \partial_{y_{2}}^{l} P(x, y)$.
Note that for stream function $\phi=(-\Delta)^{-1} \omega(y)=C \cdot G * W$, where $W$ is the odd extension of $\omega$ from $\mathbb{R}_{2}^{+}$to $\mathbb{R}_{2}^{++}$, since $G(z)$ is even in $z_{i}$, after symmetrization, we have

$$
\tilde{\phi}(x)=\phi(x)-x_{1} x_{2} \phi_{12} \phi(0)=C \int_{\mathbb{R}^{2}} G(y-x) W(y) d y=C \int F_{1}(x, y) W(y) d y
$$

where $\phi_{12} \phi(0)$ is related to $C_{f 0} K_{u x 0}$ in (4.5). In the estimate of $\mathbf{u}, \nabla \mathbf{u}$ related to $\partial_{x_{1}}^{i} \partial_{x_{2}}^{j} \tilde{\phi}$, e.g. $(1,1)$ for $u_{x}=-\partial_{x_{1} x_{2}} \phi$, for $y \in Q$ away from the singularity, we get the symmetrized integrand

$$
\partial_{x_{1}}^{i} \partial_{x_{2}}^{j} \int_{Q} F_{1}(x, y) W(y) d y=\int_{Q} \partial_{x_{1}}^{i} \partial_{x_{2}}^{j} F_{1}(x, y) W(y) d y
$$

In the error estimate of the Trapezoidal rule Lemma4.2 we estimate $\partial_{x_{1}}^{i} \partial_{x_{2}}^{j} \partial_{y_{i}}^{2} F_{1}(x, y)$, which is $I_{i j 20}\left(F_{1}\right)$ or $I_{i j 02}\left(F_{1}\right)$ in (B.15). We apply the estimate of $F_{2}$ to $K_{f}-C_{f 0} K_{u x 0}-C_{f} K_{00}$ (4.5). Below, we show that $I_{i j k l}\left(F_{i}\right), i=1,2$ has faster decay in $|y|$ than $\partial_{x_{1}}^{i} \partial_{x_{2}}^{j} \partial_{y_{1}}^{k} \partial_{y_{2}}^{l} G(y+x)$.

By definition, we get $i_{1}, j_{1} \leq 1$. Next, we fix $y$ and introduce
$g_{p q}(z) \triangleq \partial_{y_{1}}^{p} \partial_{y_{2}}^{q} G(y+z), \quad M_{G, k} \triangleq \max _{a+b \leq k}\left\|\left(\partial_{y_{1}}^{a} \partial_{y_{2}}^{b} G\right)(y+\cdot)\right\|_{L^{\infty}\left(Q_{x}\right)}, Q_{x}=\left[-x_{1}, x_{1}\right] \times\left[-x_{2}, x_{2}\right]$.
We have

$$
\begin{align*}
& \partial_{x_{i}}^{k} G\left(y_{1}+s_{1} x_{1}, y_{2}+s_{2} x_{2}\right)=s_{i}^{k} \partial_{y_{i}}^{k} G\left(x_{1}+s_{1} y_{1}, x_{2}+s_{2} y_{2}\right), \quad s_{l} \in\{ \pm 1\} \\
& \partial_{1}^{2} G(y)=-\partial_{2}^{2} G(y),\left.\quad \partial_{x_{1} x_{2}} g_{p q}(x)\right|_{x=0}=\partial_{y_{1}}^{p+1} \partial_{y_{2}}^{q+1} G(y), \quad \partial_{22} g_{r s}(0)=-\partial_{11} g_{r s}(0) \tag{B.17}
\end{align*}
$$

Second approximation $F_{2}$. Note that taking $\partial_{y_{i}}$ in $F_{i}$ does not change the sign of coefficient of $G$ term in ( (B.15). Applying (B.12) with $c=1$ and $f=g$ in $G_{2}$, we yield

$$
\begin{aligned}
& I_{p q r s}\left(F_{2}\right)=\partial_{x_{1}}^{p} \partial_{x_{2}}^{q} \int_{0}^{x_{1}} \int_{0}^{x_{2}} g_{r s, a l l}(z) d z \\
& g_{r s, a l l}=g_{r s}(z)+g_{r s}(-z)+g_{r s}\left(z_{1},-z_{2}\right)+g_{r s}\left(-z_{1}, z_{2}\right)-4 g_{r s}(0)-2\left(z_{1}^{2}-z_{2}^{2}\right) \partial_{11} g_{r s}(0)
\end{aligned}
$$

If $\max (i, j) \leq 1$, using the above notation to $I_{i j k l}\left(F_{2}\right)$ and the estimate of $G_{1}-\hat{G}_{1}$ in Lemma B. 2 with $f=g_{k l}$, and then integrating the bounds in $z_{2}$, we get
$\left|I_{10 k l}\left(F_{2}\right)\right|=\left|\int_{0}^{x_{2}} g_{k l, a l l}\left(x_{1}, z_{2}\right) d z_{2}\right| \leq M_{G, d_{2}} \int_{0}^{x_{2}} \frac{x_{1}^{4}+6 x_{1}^{2} z_{2}^{2}+z_{2}^{4}}{6} d z=\left(\frac{x_{1}^{4} x_{2}}{6}+\frac{x_{1}^{2} x_{2}^{3}}{3}+\frac{x_{2}^{5}}{30}\right) M_{G, d_{2}}$,
where $d_{2}=k+l+6$. Similarly, we get

$$
\left|I_{01 k l}\left(F_{2}\right)\right| \leq\left(\frac{x_{1}^{5}}{30}+\frac{x_{1}^{3} x_{2}^{2}}{3}+\frac{x_{1}^{4} x_{2}}{6}\right) A_{G, d_{2}}, \quad I_{11 k l}\left(F_{2}\right) \leq \frac{x_{1}^{4}+6 x_{1}^{2} x_{2}^{2}+x_{2}^{4}}{6} A_{G, d_{2}}
$$

If $\max (i, j) \geq 2, i+j \leq 3$, without loss of generality, we consider $i \geq 2$. We choose $\left(i_{1}, j_{1}, k_{1}, l_{1}\right)=(i-2, j, k+2, l)$. From (B.17), we get

$$
\begin{aligned}
& \partial_{x_{1}}^{2}\left(x_{1} x_{2} \partial_{12} G(y)\right)=0 \\
& \left.\partial_{x_{1}}^{2} \partial_{y_{1}}^{k} \partial_{y_{2}}^{l}\left(\frac{2\left(x_{1}^{2}-x_{2}^{2}\right) x_{1} x_{2}}{3} \partial_{1}^{3} \partial_{2} G(y)\right)=4 x_{1} x_{2} \partial_{y_{1}}^{k_{1}+1} \partial_{y_{2}}^{l_{1}+1} G\right)(y)=4 x_{1} x_{2} \partial_{12} g_{k_{1} l_{1}}(0)
\end{aligned}
$$

Using (B.17) again, we rewrite $\partial_{x_{1}}^{i} \partial_{y_{1}}^{k} G(x+y)=\partial_{x_{1}}^{i_{1}} \partial_{y_{1}}^{k_{1}} G(x+y)$ and get (B.18)

$$
I_{i j k l}\left(F_{2}\right)=\partial_{x_{1}}^{i_{1}} \partial_{x_{2}}^{j_{1}}\left(g_{k_{1} l_{1}}(x)-g_{k_{1} l_{1}}\left(x_{1},-x_{2}\right)-g_{k_{1} l_{1}}\left(-x_{1}, x_{2}\right)+g_{k_{1} l_{1}}(-x)-4 x_{1} x_{2} \partial_{12} g_{k_{1} l_{1}}(0)\right)
$$

The same derivativation applies to the case of $j \geq 2$, where we choose $\left(i_{1}, j_{1}, k_{1}, l_{1}\right)=(i, j-$ $2, k, l+2)$. Since $i_{1}, j_{1} \leq 1$, using the estimate of $G_{2}-\hat{G}_{2}$ in Lemma B.2 with $f=g_{k_{1} l_{1}}$, we get

$$
\begin{aligned}
& \left|I_{20 k l}\left(F_{2}\right)\right|,\left|I_{02 k l}\left(F_{2}\right)\right| \leq \frac{2 x_{1} x_{2}|x|^{2}}{3} M_{G, d_{2}}, \quad\left(i_{1}, j_{1}\right)=(0,0) \\
& \left|I_{30 k l}\left(F_{2}\right)\right|,\left|I_{12 k l}\left(F_{2}\right)\right| \leq \frac{2}{3}\left(3 x_{1}^{2} x_{2}+x_{2}^{3}\right) M_{G, d_{2}},\left(i_{1}, j_{1}\right)=(1,0) \\
& \left|I_{21 k l}\left(F_{2}\right)\right|,\left|I_{03 k l}\left(F_{2}\right)\right| \leq \frac{2}{3}\left(x_{1}^{3}+3 x_{1} x_{2}^{2}\right) M_{G, d_{2}},\left(i_{1}, j_{1}\right)=(0,1), d_{2}=k_{1}+l_{1}+4=k+l+6
\end{aligned}
$$

Note that the form (B.18) can be seen as the $\partial_{x_{1}}^{i_{1}} \partial_{x_{2}}^{i_{2}} F_{1}$. If $4 \leq i+j \leq 5$, we still first perform (B.18) by choosing $\left(i_{1}, j_{1}, k_{1}, l_{1}\right)=(i-2, j, k+2, l)$ or $(i, j-2, k, l+2)$ and get

$$
I_{i j k l}\left(F_{2}\right)=I_{i_{1} j_{1} k_{1} l_{1}}\left(\tilde{F}_{1}\right)
$$

where $\tilde{F}_{1}$ is similar to $F_{1}$ in (B.15) with $G$ replaced by $g_{i-i_{1}, j-j_{1}}=\partial_{y_{1}}^{i-i_{1}} \partial_{y_{2}}^{j-j_{1}} G(y)$. Then we apply the estimate for the first approximation below with $i_{1}+j_{1} \leq 3$.

First approximation. The estimate of $I_{i j k l}\left(F_{1}\right)$ is similar. Denote

$$
i_{2}=i-2\left\lfloor\frac{i}{2}\right\rfloor, j_{2}=j-2\left\lfloor\frac{j}{2}\right\rfloor, k_{2}=k+2\left\lfloor\frac{i}{2}\right\rfloor, \quad l_{2}=l+2\left\lfloor\frac{j}{2}\right\rfloor .
$$

If $\max (i, j) \leq 1$, we get $(i, j, k, l)=\left(i_{2}, j_{2}, k_{2}, l_{2}\right)$. Applying the estimate $G_{2}-\hat{G}_{2}$ in Lemma (B.2) with $f=g_{k_{2} l_{2}}$, we get

$$
\begin{aligned}
& I_{10 k l}\left(F_{1}\right) \leq \frac{2}{3} x_{2}\left(x_{2}^{2}+3 x_{1}^{2}\right)\left\|\partial^{d} G(y+\cdot)\right\|_{L^{\infty}\left(Q_{x}\right)}=\frac{2}{3} x_{2}\left(x_{2}^{2}+3 x_{1}^{2}\right) M_{G, d} \\
& I_{01 k l}\left(F_{1}\right) \leq \frac{2}{3} x_{1}\left(x_{1}^{2}+3 x_{2}^{2}\right) M_{G, d}, \quad I_{00 k l}\left(F_{1}\right) \leq \frac{2 x_{1} x_{2}|x|^{2}}{3} M_{G, d}, \quad d=k_{2}+l_{2}+4=k+l+4
\end{aligned}
$$

If $(i, j)=(1,1)$, we apply the estimate of $G_{1}$ in Lemma (B.2) with $f=\partial_{x_{1} x_{2}} g_{k l}(x)(k, l$ are number of derivatives on $G(y+z))$ to get

$$
\left|I_{11 k l}\left(F_{1}\right)\right| \leq 2|x|^{2} M_{G, d}, \quad d=k_{2}+l_{2}+4=k+l+4
$$

If $\max (i, j) \geq 2, x_{1} x_{2} \partial_{12} G(0)$ vanishes in $I_{i j k l}$. We apply derivation similar to (B.18) without $4 x_{1} x_{2} \partial_{12} g_{k_{2} l_{2}}(0)$ and then the estimate of $G_{2}$ in Lemma B. 2 with $f=g_{k_{2} l_{2}}$ to get

$$
\mid I_{i j k l}\left(F_{1}\right) \leq 4 x_{1}^{1-i_{2}} x_{2}^{1-j_{2}}\left\|\partial^{2} g_{k_{1} l_{1}}\right\|_{L^{\infty}\left(Q_{x}\right)} \leq 4 x_{1}^{1-i_{2}} x_{2}^{1-j_{2}} M_{G, d}, \quad d=k_{2}+l_{2}+2=k+l+4
$$

To bound $M_{G, k}$, we apply Lemma B. 3 to get

$$
\begin{equation*}
M_{G, k}=\max _{a+b \leq k}\left\|\left(\partial_{y_{1}}^{a} \partial_{y_{2}}^{b} G\right)(y+\cdot)\right\|_{L^{\infty}\left(Q_{x}\right)} \leq \frac{(k-1)!}{2 \cdot \operatorname{Den}(x, y)^{k / 2}}, \quad \operatorname{Den}(x, y)=\min _{z \in Q_{x}}|y-z|^{2} \tag{B.19}
\end{equation*}
$$

It is not difficult to obtain that for $x, y \in \mathbb{R}_{2}^{++}$, we have

$$
\begin{equation*}
\operatorname{Den}(x, y)=\sum_{i=1,2} \min _{\left|z_{i}\right| \leq x_{i}}\left|y_{i}-z_{i}\right|^{2}=\sum_{i=1,2}\left(\max \left(y_{i}-x_{i}, 0\right)\right)^{2} \tag{B.20}
\end{equation*}
$$

Using the above estimates, for $|y| \gg|x|$, we get Den $\sim|y|^{2}$ and the decay estimate for $I_{i j k l}\left(F_{1}\right)\left(\right.$ B.15) with a rate $|y|^{-k-l-4}$ and $I_{i j k l}\left(F_{2}\right)$ with a rate $|y|^{-k-l-6}$.
B.2. Piecewise $L^{\infty}$ estimate of derivatives of the Green function. In this section, we develop sharp $L^{\infty}$ estimates of the derivatives of the Green function $G(x)=-\frac{1}{2 \pi} \log |x|$ and their linear combinations in a small domain $[a, b] \times[c, d]$. They will be used in Lemmas 4.2, 4.4 to estimate the error, especially near the singularity of the kernel. We remark that the linear combinations of $\partial_{1}^{i} \partial_{2}^{j} G$ can be quite complicated. If we simply use the triangle inequality to estimate it, we can overestimate some terms with cancellation significantly, especially near the singularity of $G$. These sharp estimates are useful for reducing the estimate of the error term in Lemmas 4.2, 4.4 without choosing very small mesh, which can lead to large computational cost.
B.2.1. Coefficients of the derivatives of the Green function. To simplify the notation, we drop $\frac{1}{\pi}$ from $G$ and denote $f_{p}=-\frac{1}{2} \log |x|$. Firstly, we derive the formulas of $\partial_{1}^{i} \partial_{2}^{j} f_{p}$. Due to homogeneity, we assume

$$
\begin{equation*}
\partial_{x_{1}}^{k} \partial_{x_{2}}^{l} f_{p}=\frac{\sum_{i+j=k+l} c_{i j} x_{1}^{i} x_{2}^{j}}{|x|^{2(k+l)}} \tag{B.21}
\end{equation*}
$$

Next, we derive the recursive formula for $c_{i j}$. Using induction, we can obtain

$$
\begin{aligned}
\partial_{x_{1}}^{k+1} \partial_{x_{2}}^{l} f_{p} & =\frac{\sum_{i+j=k+l} c_{i j} i x_{1}^{i-1} x_{2}^{j}}{|x|^{2(k+l)}}-\frac{2(k+l) x_{1}}{|x|^{2(k+l+1)}} \sum_{i+j=k+l} c_{i j} x_{1}^{i} x_{2}^{j} \\
& =\frac{1}{|x|^{2(k+l+1)}}\left(\sum_{i+j=k+l} c_{i j} i x_{1}^{i+1} x_{2}^{j}+c_{i j} i x_{1}^{i-1} x_{2}^{j+2}-2(k+l) c_{i j} x_{1}^{i+1} x_{2}^{j}\right) \\
& =\frac{1}{|x|^{2(k+l+1)}}\left(\sum_{i+j=k+l}\left(c_{i j} i+c_{i+2, j-2}(i+2)-2(k+l) c_{i j}\right) x_{1}^{i+1} x_{2}^{j}\right)
\end{aligned}
$$

Therefore, we obtain the recursive formula

$$
c_{i+1, j}=i c_{i j}+(i+2) c_{i+2, j-2}-2(k+l) c_{i j}
$$

for all $i+j=k+l$, or equivalently,

$$
c_{i, j}=(i-1) c_{i-1, j}-2(k+l) c_{i-1, j}+(i+1) c_{i+1, j-2}
$$

for all $i+j=k+l+1$. Similarly, for $\partial_{x_{2}}$, we yield

$$
c_{i, j}=(j-1) c_{i, j-1}-2(k+l) c_{i, j-1}+(j+1) c_{i-2, j+1}
$$

for all $i+j=k+l+1$.
B.2.2. Estimates of rational functions. We use the above formulas to develop sharp estimates of the derivatives of $f_{p}$ and their linear combinations in a small grid cell $\left[y_{1 l}, y_{1 u}\right] \times\left[y_{2 l}, y_{2 u}\right]$. For $k<k_{2}$ and $S \subset\{(i, j): i+j=k\}$, we estimate

$$
\begin{equation*}
I_{S} \triangleq \frac{\sum_{(i, j) \in S} c_{i j} y_{1}^{i} y_{2}^{j}}{|y|^{k_{2}}} \tag{B.22}
\end{equation*}
$$

We assume that $I_{S}(x)$ is either odd in $x_{i}$ or even in $x_{i}$ for $i=1,2$. Clearly, this properties hold for $\partial_{x_{1}}^{k} \partial_{x_{2}}^{l} f_{p}\left(\overline{\mathrm{~B} .21)}\right.$. Denote $i_{1}=\min _{i \in S} i, j_{1}=\min _{j \in S} j$. We yield

$$
I_{S}=\frac{y_{1}^{i_{1}} y_{2}^{j_{1}}}{|y|^{i_{1}+j_{1}}} \frac{\sum_{(i, j) \in S} c_{i j} y_{1}^{i-i_{1}} y_{2}^{j-j_{1}}}{|y|^{k_{2}-i_{1}-j_{1}}}
$$

We further introduce

$$
P \triangleq \sum_{(i, j) \in S} c_{i j}^{+} y_{1}^{i-i_{1}} y_{2}^{j-j_{1}}, \quad Q \triangleq \sum_{(i, j) \in S} c_{i j}^{-} y_{1}^{i-i_{1}} y_{2}^{j-j_{1}}
$$

We claim that $i-i_{1}, j-j_{1}$ are even for all $(i, j) \in S$. Since $I_{S}$ is either odd or even in $x_{i}, i=1,2$, the numerator $\sum c_{i j} x_{1}^{i} x_{2}^{j}$ in (B.22) have the same symmetries in $x_{1}, x_{2}$. In particular, each monomial $c_{i j} x_{1}^{i} x_{2}^{j}$ in (B.22) also enjoys the same symmetries in $x_{1}, x_{2}$ as $I_{S}$. If $i-i_{1}$ is odd for some $i$, then $c_{i j} x_{1}^{i-i_{1}} x_{2}^{j-j_{1}}$ must be odd in $x_{1}$. It implies $i-i_{1} \geq 1$ for any $(i, j) \in S$ and contradicts the minimality of $i_{1}$. The same argument applies to $j_{1}$.

As a result, $P$ and $Q$ are monotone increasing in $\left|y_{1}\right|,\left|y_{2}\right| \geq 0$. For $\left|y_{i}\right|_{l} \leq\left|y_{i}\right| \leq\left|y_{i}\right|_{u}, i=1,2$, we can derive the upper and lower bounds for $P, Q$ and yield

$$
\begin{aligned}
|I| & \leq \frac{\max \left(P_{u}-Q_{l}, Q_{u}-P_{l}\right)}{|y|_{l}^{k_{2}-i_{1}-j_{1}}} \max _{y \in \Omega} \frac{\left|y_{1}\right|^{i_{1}}\left|y_{2}\right|^{j_{1}}}{|y|^{i_{1}+j_{1}}} \\
& \leq \frac{\max \left(P_{u}-Q_{l}, Q_{u}-P_{l}\right)}{|y|_{l}^{k_{2}-i_{1}-j_{1}}}\left(\frac{\left|y_{1}\right|_{u}}{\left(\left|y_{1}\right|_{u}^{2}+\left|y_{2}\right|_{l}^{2}\right)^{1 / 2}}\right)^{i_{1}}\left(\frac{\left|y_{2}\right|_{u}}{\left(\left|y_{1}\right|_{l}^{2}+\left|y_{2}\right|_{u}^{2}\right)^{1 / 2}}\right)^{j_{1}}
\end{aligned}
$$

where $|y|_{l}$ is the lower bound of $|y|$ and we have used the fact that $z_{i} /|z|$ is increasing in $z_{i}$ for $z_{i} \geq 0$ to obtain its upper bound. Now, for $y_{i} \in\left[y_{i l}, y_{i u}\right]$, we estimate $\left|y_{i}\right|_{l},\left|y_{i}\right|_{u}$ as follows

$$
\begin{align*}
& \left|y_{i}\right| \geq \max \left(0,\left|y_{i l}+y_{i u}\right| / 2-\left(y_{i u}-y_{i l}\right) / 2\right) \triangleq\left|y_{i}\right|_{l} \\
& \left|y_{i}\right| \leq \max \left(\left|y_{i l}\right|,\left|y_{i u}\right|\right) \triangleq\left|y_{i}\right|_{u}, \mid y_{l} \triangleq\left(\left|y_{1}\right|_{l}^{2}+\left|y_{2}\right|_{l}^{2}\right)^{1 / 2} \tag{B.23}
\end{align*}
$$

Note that for $y_{i} \in\left[y_{i l}, y_{i u}\right], y_{i}$ can change sign.
B.3. Improved estimate of the higher order derivatives of the integrands. In the Hölder estimate, we need to estimate the derivatives of the integrands (4.28), (4.29), (4.24), which take the form

$$
K^{C}(x, y)(p(x)-p(y))+K^{N C} p(x)
$$

for some weight $p$ and kernels $K^{C}, K^{N C}$. Using the estimates of the kernels in Appendix B.1. B. 2 and the weights in Section A.1, the Leibniz rule (A.6), and the triangle inequality, we can estimate the derivative of the integrands. Howover, such an estimate can lead to significant overestimates near the singularity of the integrand. We use the estimates in Appendix B. 2 to handle the cancellations among different terms and obtain improved estimates for the integrand and its derivatives near the singularity:

$$
\begin{equation*}
T_{00}(x, y) \triangleq K(y-x)(p(x)-p(y)), \quad \partial_{x_{i}} T_{00}(x, y) \tag{B.24}
\end{equation*}
$$

We choose weight $p(x)$ that is even in $x$ and $y$. The basic idea is to perform a Taylor expansion on $p(x)-p(y)$ and obtain the factor $|x-y|$, which cancels one order of singularity from $K(x, y)$. We use the formulas in Appendix B. 2 to collect the terms with the same singularity and exploit the cancellation.
B.3.1. $Y$-discretization. In the Y-discretization of the integral, we need to estimate the $y$-derivatives of the integrand ( $\overline{\mathrm{B} .24})$. For $a, b=1,2$, denote

$$
\begin{equation*}
D_{1}=\partial_{a}, \quad D_{2}=\partial_{b}, \quad x_{m}=\frac{x+y}{2} . \tag{B.25}
\end{equation*}
$$

Next, we compute $\partial_{y_{b}}^{j} \partial_{x_{a}}^{i} T_{00}$. The reader should be careful about the sign. Note that

$$
\partial_{x_{a}}(K(y-x))=-\left(\partial_{a} K\right)(y-x)=-D_{1} K(y-x)
$$

Using the Leibniz rule, we get

$$
\begin{aligned}
\partial_{y_{b}}^{2} \partial_{x_{a}} T_{00} & =\partial_{y_{b}}^{2}\left(-D_{1} K(p(x)-p(y))+K \cdot D_{1} p(x)\right)=\partial_{y_{b}}^{2}\left(D_{1} K \cdot(p(y)-p(x))+K \cdot D_{1} p(x)\right) \\
& =D_{2}^{2} D_{1} K \cdot(p(y)-p(x))+2 D_{2} D_{1} K \cdot D_{2} p(y)+D_{1} K \cdot D_{2}^{2} p(y)+D_{2}^{2} K \cdot D_{1} p(x)
\end{aligned}
$$

We use Taylor expansion at $x=x_{m}$ and write

$$
\begin{equation*}
p(y)-p(x)=(y-x) \cdot \nabla p\left(x_{m}\right)+p_{m, 2, e r r}, \partial_{i} p(z)=\partial_{i} p\left(x_{m}\right)+\left(\partial_{i} p(z)-\partial_{i} p\left(x_{m}\right)\right), z=x, y \tag{B.26}
\end{equation*}
$$

$\left|f(z)-f\left(x_{m}\right)-\left(z-x_{m}\right) \cdot \nabla f\left(x_{m}\right)\right| \leq \frac{1}{2} \frac{d_{1}^{2}}{4}\left\|f_{x x}\right\|_{L^{\infty}(Q)}+\frac{d_{1} d_{2}}{4}\left\|f_{x y}\right\|_{L^{\infty}(Q)}+\frac{1}{2} \frac{d_{1}^{2}}{4}\left\|f_{x x}\right\|_{L^{\infty}(Q)} \triangleq I_{f}$.
for $d=y-x, z=x, y$ and any $f$, where $Q$ is the rectangle covering $x, y$. Then $p_{m, 2, e r r}$ is bounded by $2 I_{p}$. Combining the terms involving $\nabla p$, we get

$$
\begin{align*}
\partial_{y_{b}}^{2} \partial_{x_{a}} T_{00}= & \sum_{i=1,2}\left(D_{2}^{2} D_{1} K \cdot\left(y_{i}-x_{i}\right)+\mathbf{1}_{D_{2}=\partial_{i}} 2 D_{2} D_{1} K+\mathbf{1}_{D_{1}=\partial_{i}} D_{2}^{2} K\right) \cdot \partial_{x_{i}} p\left(x_{m}\right)+D_{2}^{2} D_{1} K \cdot p_{m, 2, \text { err }}  \tag{B.27}\\
& +2 D_{2} D_{1} K \cdot\left(D_{2} p(y)-D_{2} p\left(x_{m}\right)\right)+D_{2}^{2} K \cdot\left(D_{1} p(x)-D_{1} p\left(x_{m}\right)\right)+D_{1} K \cdot D_{2}^{2} p(y) \\
\triangleq & \left(\sum_{i=1,2} I_{i} \cdot \partial_{x_{i}} p\left(x_{m}\right)\right)+I I_{1}+I I_{2}+I I_{3}+I I_{4}, \\
I_{i} \triangleq & D_{2}^{2} D_{1} K \cdot\left(y_{i}-x_{i}\right)+\mathbf{1}_{D_{2}=\partial_{i}} 2 D_{2} D_{1} K+\mathbf{1}_{D_{1}=\partial_{i}} D_{2}^{2} K,
\end{align*}
$$

where $\partial_{1}^{i} \partial_{2}^{j} K$ is evaluated at $y-x$, and $I I_{i}$ denotes the last four terms in the second equation. The first term is the most singular term. We combine the most singular terms to exploit the cancellation and improve the estimates. We estimate the kernels

$$
\begin{equation*}
K_{\text {mix }}\left(D_{1}, D_{2}, i, s\right)\left(z_{1}, z_{2}\right) \triangleq D_{2}^{2} D_{1} K(z) z_{i}+\mathbf{1}_{D_{2}=\partial_{i}} 2 D_{2} D_{1} K(z)+s \mathbf{1}_{D_{1}=\partial_{i}} D_{2}^{2} K(z) \tag{B.28}
\end{equation*}
$$

with $s= \pm 1$ and $D_{1}, D_{2} \in\left\{\partial_{1}, \partial_{2}\right\}$. Then we can bound $\partial_{y_{b}}^{2} \partial_{x_{a}} T_{00}$ using the triangle inequality. When $D_{1}=D_{2}$, we have an improved estimate for $I I_{2}, I I_{3}$

$$
\begin{equation*}
I I_{2}+I I_{3}=D_{2}^{2} K\left(D_{2} p(y)-D_{2} p\left(x_{m}\right)+\left(D_{2} p(y)+D_{2} p(x)-2 D_{2} p\left(x_{m}\right)\right)\right) \tag{B.29}
\end{equation*}
$$

We estimate $D_{2} p(y)+D_{2} p(x)-2 D_{2} p\left(x_{m}\right)$ using (B.27) with $f=D_{2} p$ and $z=x, y$.
B.3.2. The second singular term. For $x=\left(x_{1}, x_{2}\right)$ close to the $y$-axis or the $x$-axis, since we have symmetrized the integral, we have another singular term in the integrand

$$
T_{01} \triangleq K\left(y_{1}-x_{1}, y_{2}+x_{2}\right)(p(x)-p(y)), \text { or } T_{10} \triangleq K\left(y_{1}+x_{1}, y_{2}-x_{2}\right)(p(x)-p(y))
$$

We have the first term if $x_{2}<x_{1}$ and $x_{2}$ close to 0 , and the second term if $x_{1}<x_{2}$ and $x_{1}$ close to 0 . We label the former case with side $=1$ and the latter side $=2$. See the right figure in Figure 1 for an illustration of the first case. The $T_{01}$ term is supported in the blue region $R(x, k, S)$. Denote

$$
\begin{equation*}
\left(s_{1}, s_{2}\right)=(1,-1) \text { if side }=1, \quad\left(s_{1}, s_{2}\right)=(-1,1) \text { if side }=2 \tag{B.30}
\end{equation*}
$$

Case I. If $\left(D_{1}\right.$, side $)=\left(\partial_{1}, 1\right)$ or $\left(\partial_{2}, 2\right)$, we obtain

$$
\partial_{x_{j}} K\left(y_{1}-s_{1} x_{1}, y_{2}-s_{2} x_{2}\right)=-\partial_{y_{j}} K\left(y_{1}-s_{1} x_{1}, y_{2}-s_{2} x_{2}\right)
$$

for $\left(j, s_{1}, s_{2}\right)=(1,1,-1)$ or $(2,-1,1)$. The computations for $\partial_{y_{b}}^{2} \partial_{x_{1}} T_{01}, \partial_{y_{b}}^{2} \partial_{x_{2}} T_{10}$ are the same as (B.27) with $K$ and its derivatives evaluating at $z=\left(y_{1}-s_{1} x_{1}, y_{2}-s_{2} x_{2}\right)$.

We estimate $I I_{i}$ in (B.27) directly using the triangle inequality and the bounds for $\partial_{1}^{i} \partial_{2}^{j} K$ in Section B. 1 B. 2 and $p$ in Section A.1. For $I_{i}$ in B.27) in the most singular term, if $i=$ side, from definition ( $\overline{\mathrm{B} .30}$ ), we get

$$
s_{i}=1, \quad s_{3-i}=-1, \quad z_{i}=y_{i}-s_{i} x_{i}=y_{i}-x_{i}, \quad z_{3-i}=y_{3-i}+x_{3-i}
$$

Therefore, it follows

$$
I_{i}=D_{2}^{2} D_{1} K(z) \cdot\left(y_{i}-x_{i}\right)+\mathbf{1}_{D_{2}=\partial_{i}} 2 D_{2} D_{1} K(z)+\mathbf{1}_{D_{1}=\partial_{i}} D_{2}^{2} K(z)=K_{\operatorname{mix}}\left(D_{1}, D_{2}, i, 1\right)(z)
$$

where $K_{\text {mix }}$ is defined in (B.28). If $i \neq$ side, we have $z_{i}=y_{i}+x_{i} \geq\left|y_{i}-x_{i}\right|, z_{3-i}=y_{3-i}-x_{3-i}$. We simply bound the summand using the triangle inequality

$$
\left|I_{i}\right| \leq\left|D_{2}^{2} D_{1} K(z)\right| \cdot\left|y_{i}-x_{i}\right|+\mathbf{1}_{D_{2}=\partial_{i}} 2\left|D_{2} D_{1} K(z)\right|+\mathbf{1}_{D_{1}=\partial_{i}}\left|D_{2}^{2} K(z)\right|
$$

Case II. If $\left(D_{1}\right.$, side $)=\left(\partial_{1}, 2\right)$ or $\left(\partial_{2}, 1\right)$, we obtain

$$
\partial_{x_{j}} K\left(y_{1}-s_{1} x_{1}, y_{2}-s_{2} x_{2}\right)=\left(\partial_{y_{j}} K\right)\left(y_{1}-s_{1} x_{1}, y_{2}-s_{2} x_{2}\right)
$$

for $\left(j, s_{1}, s_{2}\right)=(1,-1,1)$ or $(2,1,-1)$. Recall the definitions of $D_{1}, D_{2}$ (B.25). Using the above identity, we yield

$$
\partial_{y_{b}}^{2} \partial_{x_{a}} T=\partial_{y_{b}}^{2}\left(D_{1} K \cdot(p(x)-p(y))+K \cdot D_{1} p\right)=-\left(\partial_{y_{b}}^{2}\left(D_{1} K \cdot(p(y)-p(x))-K \cdot D_{1} p\right)\right)
$$

for $T=T_{01}$ or $T_{10}$. Using an expansion similar to that in (B.27), (B.26), we get (B.31)

$$
\begin{aligned}
-\partial_{y_{b}}^{2} \partial_{x_{a}} T= & \sum_{i=1,2}\left(D_{2}^{2} D_{1} K \cdot\left(y_{i}-x_{i}\right)+\mathbf{1}_{D_{2}=\partial_{i}} 2 D_{2} D_{1} K-\mathbf{1}_{D_{1}=\partial_{i}} D_{2}^{2} K\right) \cdot \partial_{x_{i}} p\left(x_{m}\right)+D_{2}^{2} D_{1} K \cdot p_{m, 2, e r r} \\
& +2 D_{2} D_{1} K \cdot\left(D_{2} p(y)-D_{2} p\left(x_{m}\right)\right)-D_{2}^{2} K \cdot\left(D_{1} p(x)-D_{1} p\left(x_{m}\right)\right)+D_{1} K \cdot D_{2}^{2} p(y) \\
\triangleq & \left(\sum_{i=1,2} I_{i} \cdot \partial_{x_{i}} p\left(x_{m}\right)\right)+I I_{1}+I I_{2}+I I_{3}+I I_{4}, \\
I_{i} \triangleq & D_{2}^{2} D_{1} K \cdot\left(y_{i}-x_{i}\right)+\mathbf{1}_{D_{2}=\partial_{i}} 2 D_{2} D_{1} K-\mathbf{1}_{D_{1}=\partial_{i}} D_{2}^{2} K
\end{aligned}
$$

where $\partial_{1}^{i} \partial_{2}^{j} K$ is evaluated at $z=\left(y_{1}-s_{1} x_{1}, y_{2}-s_{2} x_{2}\right)$. We bound $I I_{i}$ using triangle inequality, the estimate (B.29), and the bounds for $K$, its derivatives, and $p$ in Sections B.1 B.2, and A.1.

For $I_{i}$, if $i=$ side, from ( $\overline{\mathrm{B} .30}$ ), we get $s_{i}=1$ and $z_{i}=y_{i}-s_{i} x_{i}=y_{i}-x_{i}$. Hence, we get

$$
I_{i}=D_{2}^{2} D_{1} K \cdot\left(y_{i}-x_{i}\right)+\mathbf{1}_{D_{2}=\partial_{i}} 2 D_{2} D_{1} K-\mathbf{1}_{D_{1}=\partial_{i}} D_{2}^{2} K=K_{m i x}\left(D_{1}, D_{2}, i,-1\right)(z)
$$

where $K_{\text {mix }}$ is defined in (B.28).
If $i \neq$ side and $D_{1}=D_{2}=\partial_{i}$, we have $z_{i}=y_{i}-s_{i} x_{i}=y_{i}+x_{i}$ and get a cancellation between $D_{2} D_{1} K$ and $D_{2}^{2} K$ and yield

$$
\left|I_{i}\right|=\left|D_{2}^{2} D_{1} K \cdot\left(y_{i}-x_{i}\right)+D_{2} D_{1} K\right| \leq\left|D_{2}^{2} D_{1} K\right| \cdot\left|y_{i}-x_{i}\right|+\left|D_{2} D_{1} K\right|
$$

Otherwise, we simply bound each term in $I_{i}$ using the triangle inequality.
B.3.3. $X$-discretization. For $K(s)=\frac{s_{1} s_{2}}{|s|^{4}}, \frac{1}{2} \frac{s_{1}^{2}-s_{2}^{2}}{|s|^{4}}$, we have $K(s)=K(-s)$. Denote

$$
T=K(y-x)(p(x)-p(y))=K(x-y)(p(x)-p(y))
$$

In this section, we compute $\partial_{x_{b}}^{i} \partial_{x_{a}}^{j} T$. Using the Taylor expansion at $x$

$$
p(x)-p(y)=(x-y) \cdot \nabla p(x)+p_{x, 2, e r r}
$$

and calculations similar to those in Section B.3.1 we get
(B.32)

$$
\begin{aligned}
\partial_{x_{b}}^{2} \partial_{x_{a}} T= & \partial_{x_{b}}^{2}\left(D_{1} K \cdot(p(x)-p(y))+K D_{1} p(x)\right)=D_{2}^{2} D_{1} K \cdot(p(x)-p(y))+2 D_{1} D_{2} K \cdot D_{2} p(x) \\
& +D_{1} K \cdot D_{2}^{2} p(x)+D_{2}^{2} K \cdot D_{1} p(x)+2 D_{2} K \cdot D_{1} D_{2} p(x)+K \cdot D_{1} D_{2}^{2} p(x) \\
= & \sum_{i=1,2}\left(D_{2}^{2} D_{1} K \cdot\left(x_{i}-y_{i}\right)+\mathbf{1}_{D_{2}=\partial_{i}} 2 D_{1} D_{2} K+\mathbf{1}_{D_{1}=\partial_{i}} D_{2}^{2} K\right) \partial_{i} p(x)+D_{2}^{2} D_{1} K \cdot p_{x, 2, e r r} \\
& +D_{1} K \cdot D_{2}^{2} p(x)+2 D_{2} K \cdot D_{1} D_{2} p(x)+K \cdot D_{1} D_{2}^{2} p(x) \triangleq\left(\sum_{i=1,2} I_{i} \cdot \partial_{i} p(x)\right)+I I, \\
I_{i} \triangleq & D_{2}^{2} D_{1} K \cdot\left(x_{i}-y_{i}\right)+\mathbf{1}_{D_{2}=\partial_{i}} 2 D_{1} D_{2} K+\mathbf{1}_{D_{1}=\partial_{i}} D_{2}^{2} K,
\end{aligned}
$$

where $I I$ consists of the last four terms in the third equation, $K$ and its derivatives are evaluated at $x-y$. Since $D_{1}, D_{2}=\partial_{x_{i}}$, we get

$$
I_{i}=D_{2}^{2} D_{1} K \cdot\left(x_{i}-y_{i}\right)+\mathbf{1}_{D_{2}=\partial_{i}} 2 D_{1} D_{2} K+\mathbf{1}_{D_{1}=\partial_{i}} D_{2}^{2} K=K_{m i x}\left(D_{1}, D_{2}, i, 1\right)(x-y),
$$

where $K_{m i x}$ is defined in (B.28). We use the bound for $K_{m i x}, \partial_{1}^{i} \partial_{2}^{j} K$ and $p$ to estimate $D_{2}^{2} D_{1} T$.
B.3.4. The second singular term. Similar to Section B.3.2 we have the second singular term for $x$ close to the $x$-axis or $y$-axis

$$
T_{01} \triangleq K\left(x_{1}-y_{1}, x_{2}+y_{2}\right)(p(x)-p(y)), \quad T_{10} \triangleq K\left(x_{1}+y_{1}, x_{2}-y_{2}\right)(p(x)-p(y)) .
$$

We have the former if $x_{2}<x_{1}$ and $x_{2}$ close to 0 , and the latter if $x_{1}<x_{2}$ and $x_{1}$ close to 0 . Using the definition of side, $s_{1}, s_{2}$ from Section B.3.2 and (B.30), we get

$$
\partial_{x_{a}} K\left(x_{1}-y_{1} s_{1}, x_{2}-y_{2} s_{2}\right)=\left(D_{1} K\right)\left(x_{1}-y_{1} s_{1}, x_{2}-y_{2} s_{2}\right) .
$$

Then the computations of $D_{2}^{2} D_{1} T$ are the same as those in (B.32) with $\partial_{1}^{i} \partial_{2}^{j} K$ evaluated at $z=\left(x_{1}-s_{1} y_{1}, x_{2}-s_{2} y_{2}\right)$. We bound $I I$ in (B.32) directly using the triangle inequality and the bounds for $\partial_{1}^{i} \partial_{2}^{j} K$ and $p$. For $I_{i}$ in (B.32), if $i=s i d e$, from (B.30), we get $s_{i}$ and $z_{i}=$ $x_{i}-s_{i} y_{i}=x_{i}-y_{i}$. It follows

$$
I_{i}=D_{2}^{2} D_{1} K \cdot z_{i}+\mathbf{1}_{D_{2}=\partial_{i}} 2 D_{1} D_{2} K+\mathbf{1}_{D_{1}=\partial_{i}} D_{2}^{2} K=K_{m i x}\left(D_{1}, D_{2}, i, 1\right)(z) .
$$

If $i \neq$ side, we have $z_{i}=x_{i}+y_{i}>\left|x_{i}-y_{i}\right|$. We bound each term in $I_{i}$ separately by following the previous argument.
B.4. Estimate of $u(x)$ for small $x_{1}$. In the energy estimate, we need to estimate $(u(x)-$ $\hat{u}(x)) \varphi(x)$ with weight $\varphi$ singular along the line $x_{1}=0$, where $\hat{u}(x)$ is a finite rank approximation of $u(x)$. We use the property that $u$ vanishes on $x_{1}=0$ to establish such an estimate.

By definition and symmetrizing the kernel using the odd symmetry of $\omega$, we have

$$
u(x, y)=\frac{1}{2 \pi} \int_{y_{1} \geq 0}\left(\frac{x_{2}-y_{2}}{|x-y|^{2}}-\frac{x_{2}-y_{2}}{\left(x_{1}+y_{1}\right)^{2}+\left(x_{2}-y_{2}\right)^{2}}\right) \omega(y) d y=\frac{1}{\pi} \int_{y_{1} \geq 0} K(x, y) W(y) d y,
$$

where
(B.33)

$$
\begin{aligned}
K & =\frac{1}{2}\left(\frac{x_{2}-y_{2}}{|x-y|^{2}}-\frac{x_{2}-y_{2}}{\left(x_{1}+y_{1}\right)^{2}+\left(x_{2}-y_{2}\right)^{2}}\right)=x_{1} \cdot \frac{2\left(x_{2}-y_{2}\right) y_{1}}{|x-y|^{2}\left(\left(x_{1}+y_{1}\right)^{2}+\left(x_{2}-y_{2}\right)^{2}\right)} \\
& \triangleq x_{1} K_{d u}(x, y)=x_{1} \tilde{K}_{d u}\left(x_{1}, y_{1}, x_{2}-y_{2}\right), \quad \tilde{K}_{d u}(x, y, z)=\frac{2 y z}{\left((x-y)^{2}+z^{2}\right)\left((x+y)^{2}+z^{2}\right)} .
\end{aligned}
$$

We define $K_{\text {app }}$ as the symmetrized kernel in $\mathbb{R}_{2}^{++}$for $\hat{u}$ similar to that in Section 4.2. Since $W$ is odd in $y_{2}$, we can symmetrize the integral in $y_{2}$ and obtain the full symmetrized integrand

$$
x_{1} K_{d u}(x, y)-x_{1} K_{d u}\left(x_{1}, x_{2}, y_{1},-y_{2}\right)=x_{1}\left(\tilde{K}_{d u}\left(x_{1}, y_{1}, x_{2}-y_{2}\right)-\tilde{K}_{d u}\left(x_{1}, y_{1}, x_{2}+y_{2}\right)\right) .
$$

Since $K$ is -1 homogeneous, using a rescaling argument, for $x=\lambda \hat{x}, y=\lambda \hat{y}$, we have

$$
\begin{equation*}
u=\frac{\lambda}{\pi} \int_{\hat{y}_{1} \geq 0}\left(\mathbf{1}_{S^{c}}(\hat{y}) K(\hat{x}, \hat{y})-K_{a p p, \lambda}(\hat{x}, \hat{y})\right) \omega_{\lambda}(\hat{y})+\mathbf{1}_{S}(\hat{y}) K(\hat{x}, \hat{y}) \omega_{\lambda}(\hat{y}) d \hat{y} \triangleq I+I I, \tag{B.34}
\end{equation*}
$$

for some rescaled kernel $K_{a p p, \lambda}(\hat{x}, \hat{y})$, where $S=R(\hat{x}, k)$ is the singular region (4.18) adapted to $\hat{x}$. For $I$, we further rewrite it and estimate it as follows

$$
\begin{align*}
|I| & =\frac{\lambda}{\pi} \hat{x}_{1}\left|\int_{\hat{y}_{1} \geq 0, \hat{y} \notin S}\left(\mathbf{1}_{S^{c}}(\hat{y}) K_{d u}(\hat{x}, \hat{y})-\frac{1}{\hat{x}_{1}} K_{a p p, \lambda}(\hat{x}, \hat{y})\right) \omega_{\lambda}(\hat{y}) d y\right|  \tag{B.35}\\
& \leq \frac{\lambda}{\pi} \hat{x}_{1}\|\omega \varphi\|_{L^{\infty}} \int_{\hat{y}_{1} \geq 0}\left|\mathbf{1}_{S^{c}}(\hat{y}) K_{d u}(\hat{x}, \hat{y})-\frac{1}{\hat{x}_{1}} K_{a p p, \lambda}(\hat{x}, \hat{y})\right| \varphi_{\lambda}^{-1}(\hat{y}) d \hat{y}
\end{align*}
$$

Since the integral is not singular, we can use the previous method to discretize the integral and obtain its tight bound.

Derivative bounds. To estimate the error in the Trapezoidal rule in Lemma 4.2, we need to bound $\partial_{x_{i}}^{2} K_{d u}(x, y), \partial_{y_{i}}^{2} K_{d u}(x, y)$. Since $\frac{1}{x} C_{u 0}(x, y), \frac{1}{x} C_{u}(x, y)$ (4.5) are smooth, from the construction in Section 4.3, the kernel $\frac{1}{x_{1}} K_{\text {app }}(x, y)$ and its rescaled version are regular in $\hat{x}$. We estimate its derivatives following Section 4.1. Since $K_{d u}(x, y)=\frac{1}{x_{1}} K(x, y)$ (B.33), $K(x, y)$ is harmonic in $y$, and $\left|\partial_{x_{2}}^{2} K(x, y)\right|=\left|\partial_{y_{2}}^{2} K(x, y)\right|$, we get

$$
\partial_{y_{1}}^{2} K_{d u}(x, y)=-\partial_{y_{2}}^{2} K_{d u}(x, y), \quad\left|\partial_{y_{2}}^{2} K_{d u}(x, y)\right|=\left|\partial_{x_{2}}^{2} K_{d u}(x, y)\right|
$$

Thus, we only need to bound $\left|\partial_{x_{1}}^{2} K_{d u}\right|$ and $\left|\partial_{y_{1}}^{2} K_{d u}\right|$, or $\partial_{x}^{2} \tilde{K}_{d u}$ and $\partial_{y}^{2} \tilde{K}_{d u}$ using the relation (B.33). We derive the formulas of $\partial_{x}^{2} \tilde{K}_{d u}$ and $\partial_{y}^{2} \tilde{K}_{d u}$ and then estimate them using methods similar to that in Appendix B.2. We have an improved estimate for $\partial_{y} \tilde{K}_{d u}$ in $\{x\} \times\left[y_{l}, y_{u}\right] \times\left[z_{l}, z_{u}\right]$ near the singularity. A direct computation yields

$$
\begin{aligned}
& \partial_{y}^{2} \tilde{K}_{d u}(x, y, z)=24 y z \frac{\left(z^{4}-\left(x^{2}-y^{2}\right)^{2}\right)\left(x^{2}+y^{2}+z^{2}\right)}{T_{-}^{3} T_{+}^{3}}+64 \frac{x^{2} y^{3} z^{3}}{T_{-}^{3} T_{+}^{3}} \\
& =\frac{y z}{T_{-}^{2} T_{+}^{2}}\left(12\left(\frac{1}{T_{-}}+\frac{1}{T_{+}}\right)\left(z^{4}-(x-y)^{2}(x+y)^{2}\right)+64 x^{2} \frac{y^{2} z^{2}}{T_{-} T_{+}}\right), \quad T_{ \pm}=(x \pm y)^{2}+z^{2}
\end{aligned}
$$

where we have used $\frac{1}{T_{-}}+\frac{1}{T_{+}}=2 \frac{x^{2}+y^{2}+z^{2}}{T_{-} T_{+}}$. We apply the estimate of $K_{d u}$ to $x, y \geq 0$. Since $\left|\partial_{y}^{2} \tilde{K}_{d u}\right|$ is even in $z$, without loss of generality, we consider $z \geq 0$. Then for $P_{2}$, we have $z / T_{-}^{1 / 2}, y / T_{+}^{1 / 2}$ are increasing in $z, y$, respectively. To bound other terms, we simply use the monotonicity of the polynomials, (B.22), interval operation (A.4), (A.5), and follow Section B.2.1. For example, we use (B.23) to bound $(x-y)^{2},(x+y)^{2}$ and

$$
0 \leq \frac{y}{T_{+}^{1 / 2}} \leq \frac{y_{u}}{\left(\left(x+y_{u}\right)^{2}+z_{l}^{2}\right)^{1 / 2}}, \quad 0 \leq \frac{z}{T_{-}^{1 / 2}} \leq \frac{z_{u}}{\left(|x-y|_{l}^{2}+z_{u}^{2}\right)^{1 / 2}}
$$

$\hat{x}_{1}$ not small. For $I I$ in (B.34), if $\hat{x}_{1} \geq x_{l}=2 h>0$ away from 0 , we have $\left|K_{d u}(\hat{x}, \hat{y})\right| \lesssim \frac{1}{x_{l}} \frac{1}{\hat{x}-y \mid}$, which is integrable near the singularity $\hat{x}$. We estimate $I I$ using

$$
|I I| \leq \frac{\lambda}{\pi} \hat{x}_{1} \int_{\hat{y}_{1} \geq 0, \hat{y} \in S}\left|K_{d u}(\hat{x}, \hat{y})\right| \varphi_{\lambda}^{-1}(\hat{y}) d \hat{y}\|\omega \varphi\|_{\infty}, \quad S=R(\hat{x}, k)
$$

We follow Section 4.1.6 by introducing $\hat{y}=\hat{x}+s, s \in S-\hat{x}$, decomposing $S-x$ into the symmetric part $D_{\text {sym }}$ and non-symmetric part $D_{n s}$ and estimating the piecewise integral of $K_{d u}(\hat{x}, \hat{y})$

$$
\begin{aligned}
& D_{s y m}=R_{s}(\hat{x}, k)-\hat{x}, D_{n s}=\left(R(\hat{x}, k) \backslash R_{s}(\hat{x}, k)\right)-\hat{x} \\
& \left|K_{d u}(\hat{x}, \hat{y})\right| \mathbf{1}_{\hat{y}_{1} \geq 0}=|F| \mathbf{1}_{\hat{x}_{1}+s_{1} \geq 0}, F=\frac{\left(\hat{x}_{1}+s_{1}\right) s_{2}}{|s|^{2}\left(\left(s_{1}+2 \hat{x}_{1}\right)^{2}+s_{2}^{2}\right)}
\end{aligned}
$$

and piecewise bounds of $\varphi_{\lambda}^{-1}(y)$, where we have used $(\bar{B} .33)$ to obtain the above formula. We observe that $|F|$ is even in $s_{2}$ and $F \geq 0$ for $s \in Q=[a, b] \times[c, d]$ with $c, d \geq 0$. We estimate the piecewise integrals of $F$ in $Q$ in Section 6.2 in the supplementary material II [11]. Denote $X_{1}^{+} \triangleq\left\{y: y_{1} \geq 0\right\}$. If $\hat{x}_{1} \geq k h$, we get $S \cap X_{1}^{+}=R(\hat{x}, k)$ and the regions $D_{\text {sym }}, D_{n s}$ are the same as those in Section 4.1.6. If $\hat{x}_{1} \in[i h,(i+1) h), i<k$, the region $S$ touches $\left\{y: y_{1}=0\right\}$ and we get

$$
S \cap X_{1}^{+}=[0,(i+k+1) h] \times[(j-k) h,(j+1+k) h], \quad \text { for } x_{2} \in[j h,(j+1) h]
$$

In this case, the symmetric and non-symmetric region becomes smaller. We do not have the left edge in the middle figure in Figure 2, part of the upper and the lower edge due to the restriction $\hat{y}_{1}=s_{1}+\hat{x}_{1} \geq 0$. The estimate of the integrals for $s \in S \cap X_{1}^{+}-\hat{x}_{1}$ follows similar argument.
Small $\hat{x}_{1}$. The difficulty is to estimate $I I$ for small $\hat{x}_{1} \leq 2 h$. It is not difficult to obtain that

$$
\begin{equation*}
|I I| \lesssim \frac{\lambda}{\pi}\left\|\left|\omega_{\lambda} \|_{L^{\infty}(S)} \hat{x}_{1}\right| \log \left(\hat{x}_{1}\right) \mid .\right. \tag{B.36}
\end{equation*}
$$

Thus we cannot bound $I I$ by $C \hat{x}_{1}$ for some constant $C$ uniformly for small $\hat{x}_{1}$. Denote by

$$
\begin{align*}
S_{\text {sym }} & =\left[0, \hat{x}_{1}+k h\right] \times\left[\hat{x}_{2}-k h, \hat{x}_{2}+k h\right], \quad S_{i n, 1}=\left[0, \hat{x}_{1}\right] \times\left[\hat{x}_{2}-k h, \hat{x}_{2}+k h\right] \\
S_{\text {in }, 2} & =\left[\hat{x}_{1}, \hat{x}_{1}+h\right] \times\left[\hat{x}_{2}-h, \hat{x}_{2}+h\right], \quad S_{\text {in }}=S_{i n, 1} \cup S_{i n, 2}  \tag{B.37}\\
S_{\text {out }} & =\left[\hat{x}_{1}, \hat{x}_{1}+h k\right] \times\left[\hat{x}_{2}-k h, \hat{x}_{2}+k h\right] \backslash S_{i n, 2}, \quad \hat{y}=\hat{x}+\hat{x}_{1} s
\end{align*}
$$

See the right figure in Figure 2 for an illustration of different regions. By definition, we have $S_{\text {sym }}=S_{\text {out }} \cup S_{i n, 1} \cup S_{i n, 2}$. Here $S_{\text {in }}$ captures the most singular region. Then $\hat{y} \in S_{\text {in }}$ is equivalent to

$$
\begin{align*}
& s \in \hat{x}_{1}^{-1}\left(S_{i n}-\hat{x}\right)=x_{1}^{-1}\left(\left[-\hat{x}_{1}, 0\right] \times[-k h, k h] \cup[0, h] \times[-h, h]\right) \triangleq R_{1}\left(B_{1}\right) \cup R_{2}\left(B_{2}\right) \\
& R_{1}\left(B_{1}\right)=[-1,0] \times\left[-\frac{1}{B_{1}}, \frac{1}{B_{1}}\right], R_{2}\left(B_{2}\right)=\left[0, \frac{1}{B_{2}}\right] \times\left[-\frac{1}{B_{2}}, \frac{1}{B_{2}}\right], B_{1}=\frac{\hat{x}_{1}}{k h}, B_{2}=\frac{\hat{x}_{1}}{h} \tag{B.38}
\end{align*}
$$

We further decompose $I I$ as follows
$I I=\frac{\lambda}{\pi} \hat{x}_{1} \int_{y_{1} \geq 0}\left(\mathbf{1}_{S \backslash S_{s y m}}(\hat{y})+\mathbf{1}_{S_{\text {out }}}(\hat{y})+\mathbf{1}_{S_{\text {in }, 1}}(\hat{y})+\mathbf{1}_{S_{i n, 2}}(\hat{y})\right) K_{d u}(\hat{x}, \hat{y}) \omega_{\lambda}(\hat{y}) d \hat{y}=\frac{\lambda \hat{x}_{1}}{\pi}\left(I I_{1}+I I_{2}+I I_{i n, 1}+I I_{\text {in }, 2}\right)$.
The integrals $I I_{1}, I I_{2}$ capture the non-symmetric part and the symmetric part away from the singularity. We apply $L^{\infty}$ estimate and the method in Sections 4.1.6, 4.1.9, For $I I_{i n, i}$, using a change of variables (B.37), (B.38), we derive

$$
I I_{i n, i}=\int_{s \in R_{i}\left(B_{i}\right)} K_{d u}\left(\hat{x}, \hat{x}+\hat{x}_{1} s\right) \hat{x}_{1}^{2} \omega_{\lambda}\left(\hat{x}+\hat{x}_{1} s\right) d s
$$

Note that $\hat{y}-\hat{x}=\hat{x}_{1} s, \hat{y}_{1}+\hat{x}_{1}=\hat{x}_{1}\left(2+s_{1}\right), \hat{y}_{2}-\hat{x}_{2}=\hat{x}_{1} s_{2}$. By definition (B.33), we get

$$
\begin{aligned}
K_{d u}\left(\hat{x}, \hat{x}+\hat{x}_{1} s\right) \hat{x}_{1}^{2} & =-\frac{2 \hat{x}_{1} s_{2} \cdot\left(\hat{x}_{1}+\hat{x}_{1} s_{1}\right)}{\hat{x}_{1}^{2}|s|^{2} \cdot \hat{x}_{1}^{2}\left(\left(s_{1}+2\right)^{2}+s_{2}^{2}\right)} \hat{x}_{1}^{2}=-\frac{2\left(s_{1}+1\right) s_{2}}{|s|^{2}\left(\left(s_{1}+2\right)^{2}+s_{2}^{2}\right)} \triangleq-K_{s}(s) \\
I I_{i n, i} & =-\int_{R_{i}\left(B_{i}\right)} K_{s}(s) \omega_{\lambda}\left(\hat{x}+\hat{x}_{1} s\right) d s
\end{aligned}
$$

Since $K_{s}(s)$ is symmetric in $s_{2}$, we derive

$$
\begin{aligned}
& \left|I I_{i n, 1}\right| \leq\|\omega \varphi\|_{\infty}\left(\max _{z \in\left[-\hat{x}_{1}, 0\right] \times[0, k h]} \varphi_{\lambda}^{-1}(\hat{x}+z)+\max _{z \in\left[-\hat{x}_{1}, 0\right] \times[-k h, 0]} \varphi_{\lambda}^{-1}\right) J_{1}\left(B_{1}\right) \\
& \left|I I_{i n, 2}\right| \leq\|\omega \varphi\|_{\infty}\left(\max _{z \in[0, h] \times[0, h]} \varphi_{\lambda}^{-1}+\max _{z \in[0, h] \times[-h, 0]} \varphi_{\lambda}^{-1}\right) J_{2}\left(B_{2}\right)
\end{aligned}
$$

where $B_{i}$ is given in (B.38) and

$$
J_{1}\left(B_{1}\right)=\left|\int_{[-1,0] \times\left[0,1 / B_{1}\right]} K_{s}(s) d s\right|=\int_{[-1,0] \times\left[0,1 / B_{1}\right]} K_{s}(s) d s, \quad J_{2}\left(B_{2}\right)=\int_{\left[0,1 / B_{2}\right]^{2}} K_{s}(s) d s
$$

The formula of $J_{i}$ can be obtained using the analytic integral formula for $K_{s}$, and obviously $J_{i}$ is decreasing in $B$. Note that $J_{1}(B)$ is bounded, but $J_{2}(B) \lesssim 1+\log (B) \lesssim 1+\left|\log \hat{x}_{1}\right|$, which relates to the estimate (B.36). We refer the formulas of $J_{i}$ to Section 6.2 in the supplementary material II 11.

## B.5. Additional derivations.

B.5.1. Estimate of the log-Lipschitz integral. In this section, we derive the coefficient in the estimate of $\partial_{x_{2}} I_{5,4}(x)(4.60)$, (4.61). For $I_{5,4}$, we further decompose it as follows
$I_{5,4}=\left(\int_{R\left(k_{2}\right) \backslash R_{s}\left(k_{2}\right)}+\int_{R_{s}\left(k_{2}\right) \backslash R_{s}(b)}+\int_{R_{s}(b) \backslash R_{s}(a)}\right) K(x-y)\left(\psi(x)-\psi(y) W(y) d y \triangleq I_{5,4,1}+I_{5,4,2}+I_{5,4,3}\right.$.
In practice, we choose $b=2$. The first two terms are nonsingular and their derivatives can be estimated using the method in Sections 4.1.6-4.1.9. For $I_{5,4,3}$, using the second order Taylor expansion to $\psi(x)-\psi(y)$ centered at $x$, we have

$$
\begin{aligned}
& \partial_{x_{2}}(K(x-y)(\psi(x)-\psi(y)))=\left(\partial_{2} K\right)(x-y)(\psi(x)-\psi(y))+K(x-y) \partial_{2} \psi(x) \\
= & \left(\partial_{2} K(x-y)\left(x_{2}-y_{2}\right)+K(x-y)\right) \partial_{2} \psi(x)+\partial_{2} K(x-y)\left(x_{1}-y_{1}\right) \partial_{1} \psi(x)+\mathcal{R}_{K}
\end{aligned}
$$

where the remainder $\mathcal{R}_{K}$ coming from the higher order term in the Taylor expansion satisfies

$$
\left|\mathcal{R}_{K}\right| \leq \sum_{i+j=2}| | \partial_{x}^{i} \partial_{y}^{j} \psi \|_{L^{\infty}(Q)}\left|x_{1}-y_{1}\right|^{i}\left|x_{2}-y_{2}\right|^{j} c_{i j}
$$

where $Q=B_{i_{1} j_{1}}\left(h_{x}\right)+[-b h, b h]^{2}$ and $c_{20}=c_{02}=\frac{1}{2}, c_{11}=1$. It follows

$$
\left|\partial_{x_{2}} I_{5,4,3}\right| \leq\|\omega \varphi\|_{\infty} \sum_{0 \leq i \leq 1,0 \leq j \leq i+1} S c o e_{i j}(x) \cdot f_{i j}(a, b),
$$

where the coefficients $S c o e_{i j}(x)$ depend on the weight $\psi, \varphi$, and $f_{i j}(a, b)$ is the upper bound of the integral

$$
\begin{equation*}
\int_{[-b, b]^{2} \backslash[-a, a]^{2}}\left|\partial_{2} K(y) \cdot y_{1}^{i} y_{2}^{j}+\mathbf{1}_{(i, j)=(0,1)} K(y)\right| d y \leq f_{i j}(a, b) \tag{B.39}
\end{equation*}
$$

For example, $S c o e_{01}$ comes from the following estimate for $I_{5,4,3}$

$$
\begin{aligned}
& \int_{R_{s}(b) \backslash R_{s}(a)}\left|\left(\partial_{2} K(x-y)\left(x_{2}-y_{2}\right)+K(x-y)\right) \partial_{2} \psi(x)\right| \omega(y) d y \\
\leq & \|\omega \varphi\|_{\infty}\left\|\varphi^{-1}\right\|_{L^{\infty}(Q)} \cdot\left|\partial_{2} \psi(x)\right| \int_{[-b, b]^{2} \backslash[-a, a]^{2}}\left|\partial_{2} K(s) s_{2}+K(s)\right| d s
\end{aligned}
$$

The function $f_{i j}(a, b)$ satisfies the following estimates

$$
f_{1 j}(a, b) \leq B_{1 j} \log (b / a), \quad j=1,2
$$

with some constants $B_{1 j}$. We refer the derivations to Section 5.1.5 in the supplementary material II [11].
B.5.2. Optimization in the Hölder estimate. Consider

$$
\max _{t \leq t_{u}} \min _{a \leq b} F(a, t), \quad F(a, t)=\left(A+B \log \frac{b}{a}\right) \sqrt{t}+\frac{C a}{\sqrt{t}}
$$

in the upper bound in (4.68). For each $t \leq t_{u}$, we first optimize $F(a, t)$ over $a \leq b$. We assume that $A, B, C, b, c, h, h_{x}$ are given. Denote

$$
t_{u}=c h_{x}, \quad t_{1}=\frac{C b}{B}
$$

For a fixed $t$, since $\partial_{a}^{2} F>0, \partial_{a} F(0, t)<0$ and $\partial_{a} F(a, t)=0$ if $a=\frac{B t}{C}$, we choose $a=$ $\min \left(b, \frac{B t}{C}\right)$. For $t \leq \frac{C b}{B}=t_{1}$, we get

$$
\min _{a \leq b} F(a, t) \leq F\left(\frac{B t}{C}, t\right)=\left(A+B \log \frac{b C}{B}+B\right) \sqrt{t}-B \sqrt{t} \log t
$$

The right hand side can be further estimated by studying the concave function on $s=t^{1 / 2} \leq s_{u}$

$$
f(p, q, s)=(p-q \log s) s \leq f\left(p, q, \min \left(s_{u}, s_{*}\right)\right), \quad s_{*}=\exp \left(\frac{p-q}{q}\right)
$$

with $p=A+B \log \left(\frac{b C}{B}\right)+B, q=2 B, s_{u}=\min \left(t_{u}^{1 / 2}, t_{1}^{1 / 2}\right)$. We get the above inequality since $f(p, q, s)$ is increasing for $s \leq s_{*}$ and is decreasing for $s \geq s_{*}$.

If $\frac{C b}{B} \leq t \leq t_{u}$, we choose $a=b$ and get

$$
\min _{a \leq b} F(a, t) \leq F(b, t)=A \sqrt{t}+\frac{C b}{\sqrt{t}}
$$

which is convex in $t^{1 / 2}$. Thus its maximum is achieved at the endpoints.

## Appendix C. Representations and estimates of the solutions

In Section 7 of Part I [13, we represent the approximate steady state as follows

$$
\begin{align*}
& \bar{\omega}=\bar{\omega}_{1}+\bar{\omega}_{2}, \quad \bar{\theta}=\bar{\theta}_{1}+\bar{\theta}_{2}, \quad \bar{\omega}_{1}=\chi(r) r^{-\alpha} g_{1}(\beta), \quad \bar{\theta}_{1}=\chi(r) r^{1-2 \alpha} g_{2}(\beta) \\
& \bar{\phi}^{N}=\bar{\phi}_{1}^{N}+\bar{\phi}_{2}^{N}+\bar{\phi}_{3}^{N}+\bar{\phi}_{c o r}^{N}, \quad \bar{\phi}_{3}^{N}=\bar{a} \chi_{\phi, 2 D}, \quad \chi_{\phi, 2 D}=-x y \chi_{\phi}(x) \chi_{\phi}(y)  \tag{C.1}\\
& \bar{\phi}_{c o r}^{N}=-c \cdot \frac{x y^{2}}{2} \kappa_{*}(x) \kappa_{*}(y)=c \phi_{1}, \quad c=\partial_{x}\left(\bar{\omega}+\Delta\left(\bar{\phi}_{1}^{N}+\bar{\phi}_{2}^{N}+\bar{\phi}_{3}^{N}\right)\right), \quad \alpha=-\frac{\bar{c}_{\omega}}{\bar{c}_{l}} \approx \frac{1}{3},
\end{align*}
$$

where $\bar{\omega}_{2}, \bar{\theta}_{2}, \bar{\phi}_{2}^{N}$ have compact supports and are represented as piecewise polynomials, $\bar{a} \in \mathbb{R}$ is some coefficient, $\kappa_{*}$ is given in (D.5), $\phi_{1}$ is the same as (3.14), $\chi_{\phi}$ is given in (D.7). We choose a small correction $\bar{\phi}_{\text {cor }}$ similar to that in Section 3.2 so that $\bar{\omega}+\Delta \bar{\phi}^{N}=O\left(|x|^{2}\right)$ near 0 . We use upper script $N$ to distinguish the numerical approximation $\bar{\phi}^{N}$ for the exact stream function $\bar{\phi}=(-\Delta)^{-1} \bar{\omega}$. We have discussed how to find the semi-analytic part in Section 7 of Part I [13]. We will discuss how to estimate the semi-analytic part in Section C.3. In the following sections, we discuss more details about the representations and establish rigorous estimate of the derivatives of $\bar{\omega}_{2}, \bar{\theta}_{2}$.

Note that we we do not need an approximation term $\bar{\phi}_{3}$ for the stream function in solving the linearized equation in Section 3 since we can allow a larger residual error in Section 3 ,
C.1. Representations. In a large domain $[0, L]^{2}$, we use piecewise polynomials to represent the solution. Firstly, we choose a large $L$ of order $10^{15}$ and then design the adaptive mesh $y_{-5}<. .<y_{0}=0<y_{1}<. .<y_{N-1}=L, N=748$ to partition $[0, L]$.

Adaptive mesh. We design three parts of the mesh $y_{i}, i \in I_{j} \triangleq\left[a_{j}, b_{j}\right], a_{0}=0$ as follows

$$
\begin{aligned}
& y_{i}=\frac{i}{256}, i=-5,1, . ., b_{1}, \quad y_{a_{2}+i}=y_{a_{2}}+F\left(i h_{3}\right), i=1, . ., b_{2}-a_{2} \\
& y_{a_{3}+i}=y_{a_{3}} \exp \left(i r_{1}\right), i=1, . ., b_{3}-a_{3}, \quad r_{0}=1.025, r_{1}=1.15 \\
& F(z)=\frac{h_{2}}{h_{3}} z \exp \left(r z^{2}\right), r=\log \left(\frac{r_{0}}{1+h_{3}}\right) \frac{1}{\left(1+h_{3}\right)^{2}-1}, \quad h_{2}=\frac{1}{128}, h_{3}=\frac{1}{b_{2}-a_{2}} .
\end{aligned}
$$

Since we need to estimate the weighted $L^{\infty}$ norm of the residual error with a singular weight of order $|x|^{-\beta}, \beta \approx 3$ near $x=0$, we use uniformly dense mesh near 0 so that we have a very small residual error. In the far-field, we use a mesh that grows exponentially fast in space. Note that the error estimate $f-I(f)$ for the $k$-th order interpolation of $f$ on $\left[y_{i}, y_{i+1}\right]$ reads

$$
|f-I(f)| \leq C\left(y_{i+1}-y_{i}\right)^{k}\left|\partial_{x}^{k} f\right|
$$

For large $x$, we expect that $\partial_{x}^{k} f$ has a decay rate $|y|^{-k-\alpha}$ if $|f| \lesssim|y|^{-\alpha}$ for $\alpha>0$. Thus, to get a uniformly small error in the far-field, we just require $\frac{y_{i+1}-y_{i}}{y_{i}} \leq \varepsilon$ with $\varepsilon<1$. This allows us to choose an exponentially growing mesh in the far-field and cover a very large domain without using too many points. We use the second part of the mesh to glue the first part of the mesh, which grows linearly, and the third part of the mesh. The functions $F(z)$ behaves linearly for $z$ close to 0 , and it grows exponentially fast with rate $r_{1}$ for $z$ close to 1 :

$$
F\left(1+h_{3}\right) / F(1)=\left(1+h_{3}\right) \exp \left(r\left(\left(1+h_{3}\right)^{2}-1\right)\right)=\left(1+h_{3}\right) \exp \left(\log \left(r_{0} /\left(1+h_{3}\right)\right)\right)=r_{0}
$$

Parameters $h_{2}, h_{3}$ control the mesh size $y_{a_{2}+1}-y_{a_{2}}=F\left(h_{3}\right)=h_{2} \exp \left(r h_{3}^{2}\right) \approx h_{2}$. We further glue $y_{i}, i \in\left[b_{j}, a_{j+1}\right], j=1,2$ using the Lagrangian interpolation for $j=1$. For $j=2$, we interpolate the growth rate using $\exp \left(\log \left(r_{0}\right) l(i)+(1-l(i)) \log \left(r_{1}\right)\right)$ with $l(i)$ linear in $i \in\left[b_{2}, a_{3}\right]$.

In our numerical computation, we compute the derivatives of the solution using the B-spline basis, see e.g., (C.5), and do not use the Jacobian related to the adaptive mesh. In particular, we do not use derivatives of the map $f(i)=y_{i}$, and have more flexibility to design the mesh.

Let $n_{1}=720<N$. We solve the dynamic rescaling equation (2.10)-(2.11) on first $n_{1} \times$ $n_{1},\left(y_{i}, y_{j}\right), i, j \leq n_{1}-1$ grids. We construct

$$
\begin{equation*}
\bar{\omega}_{2}(x, y)=\sum_{0 \leq i \leq n_{1}+11,-2 \leq j \leq n_{1}+1} a_{i j} B_{1, i}(x) B_{j}(y) \tag{C.2}
\end{equation*}
$$

where $a_{i j} \in \mathbb{R}$ is the coefficient, $B_{i}(x), B_{j}(y)$ are constructed from the 6 -th order B-spline

$$
\begin{equation*}
B_{i}(x)=C_{i} B_{i 0}(x), \quad B_{i 0}(x)=\sum_{0 \leq j \leq k} k \frac{\left(s_{i j}-x\right)_{+}^{k-1}}{d_{j}}, \quad d_{j}=\prod_{0 \leq l \leq k, l \neq j}\left(s_{i j}-s_{i l}\right) \tag{C.3}
\end{equation*}
$$

with $k=6$. The constant $C_{i}$ will be chosen in (C.9), (C.10) so that the stiffness matrix associated to these Bspine basis has a better condition number. The points $s_{i j}$ are chosen as follows

$$
s_{i j}=y_{i+j-3}, \quad 0 \leq j \leq k=6
$$

Then the B-spline $B_{i}$ is supported in $\left[y_{i-3}, y_{i+3}\right]$ and is centered around $y_{i}$. Since $\omega$ is odd in $x$, to impose this symmetry in the representation, we modify the first few basis

$$
\begin{equation*}
B_{1, i}(x)=B_{i}(x)-B_{i}(-x), \quad i \leq 2 \tag{C.4}
\end{equation*}
$$

Then $B_{i}$ is odd. We remark that $B_{1,0}(x) \equiv 0$.
Extrapolation. Near the boundary $y=0$, we need 2 extra basis functions $a_{i,-j} B_{-j}(y), j=1,2$ that are not zeros in $y_{1} \geq 0$. Without these 2 basis functions, the representation (C.2) does not approximate $\bar{\omega}$ with a $6-$ th order error. We use a 7 -th order extrapolation [41,42] to determine $a_{i,-j}$
$a_{i,-j}=\sum_{0 \leq l \leq 6} C_{3-j, l+1} a_{i, l}, C_{1, \cdot}=(28,-112,210,-224,140,-48,7), C_{2, \cdot}=(7,-21,35,-35,21,-7,1)$.
We choose $C_{j, l}$ such that the 7 -th difference of $a_{i, j},-2 \leq j \leq 6$ is 0 . Since $a_{i,-j}$ depends on $a_{i, l}$ linearly, we can combine $a_{i,-j} B_{i,-j}, j=1,2$ with $a_{i, l} B_{i, l}$ and modify (C.2) as follows

$$
\begin{align*}
\bar{\omega}_{2}(x, y) & =\sum_{0 \leq i, j \leq n_{1}+1} a_{i j} B_{1, i}(x) B_{2, j}(y)  \tag{C.5}\\
B_{2, j}(y) & =B_{j}(y)+C_{2, j+1} B_{-1}(y)+C_{1, j+1} B_{-2}(y), 0 \leq j \leq 6, \quad B_{2, j}(y)=B_{j}(y), j \geq 7
\end{align*}
$$

The modified basis functions $B_{1, i}, B_{2, j}$ are still piecewise polynomials in [ $y_{l}, y_{l+1}$ ].
Far-field extension. In (C.2), (C.5), we use Bspline $B_{1, i}(x), B_{j}(y)$ up to $i, j \leq n_{1}+1$ rather than $n_{1}-1$ since the support of $B_{1, i}, B_{j}$ intersects $\left[0, y_{n-1}\right]^{2}$ for $i, j \leq n_{1}-1$. To determine the extra coefficients, we first extend the grid point values of $\omega_{2}(x, y)$ from $\left(y_{i}, y_{j}\right)$ with $i, j \leq$ $n_{1}-1$ to $i, j \leq n_{1}+l_{0}-1$ by $\omega_{2}\left(y_{n_{1}+l}, y_{j}\right)=P\left(y_{n_{1}+l} ; y_{j}\right), l=0,1, . ., l_{0}-3$, where $P$ is the Lagrangian interpolation polynomials on $\left.\left(y_{n_{1}-1}, \omega\left(y_{n_{1}-1}, y_{j}\right)\right),\left(y_{n_{1}+l_{0}-3}, 0\right),\left(y_{n_{1}+l_{0}-2}\right), 0\right)$. We impose $\omega_{2}\left(y_{n_{1}+l}, y_{j}\right)=0, l=l_{0}-3, l_{0}-2, l_{0}-1$. Similarly, we extend $\omega\left(y_{i}, y_{n+l}\right)$. Note that $\omega_{2}$ is odd and $B_{1,0}=0$. We solve the coefficents $a_{k l}, 1 \leq k \leq M, 0 \leq l \leq M$ from

$$
\omega_{2}\left(y_{p}, y_{q}\right)=\sum_{1 \leq i \leq M, 0 \leq j \leq M} a_{i j} B_{1, i}(x) B_{2, j}(y), 1 \leq p \leq M, 0 \leq q \leq M, M=n_{1}+l_{0}-1
$$

The value $a_{0 j}$ is not used since $B_{1,0} \equiv 0$. To simplify the notation, we keep it. We only keep $a_{i j}, i, j \leq n_{1}+1$ and obtain (C.5). In practice, we choose $l_{0}=8$ and the above construction provides a solution with tail decaying smoothly to 0 for $|y|_{\infty} \geq y_{n_{1}+l_{0}-1}$.

To solve the dynamic rescaling equations numerically (2.10)-(2.12) (see Section 7 Part I), we update the grid point value of $\omega_{n+1}$ at time $t_{n+1}$, and then use the above method to obtain $a_{i j}$.

For the density $\bar{\theta}_{2}$, the representation is similar

$$
\begin{equation*}
\bar{\theta}_{2}=x \sum_{0 \leq i, j \leq n_{1}+1} a_{i j} B_{1, i}(x) B_{2, j}(y) \tag{C.6}
\end{equation*}
$$

Here, we multiply $x$ since $\bar{\theta}$ is even and vanishes $O\left(x^{2}\right)$ near $x=0$.

For the stream function $\bar{\phi}_{2}^{N}$ (C.1), we choose $n_{2}>n_{1}$ and represent it as follows

$$
\begin{equation*}
\bar{\phi}_{2}^{N}=\sum_{0 \leq i, j \leq n_{2}-1} a_{i j} \tilde{B}_{1, i}(x) \tilde{B}_{2, j}(y) \rho_{p}(y) \tag{C.7}
\end{equation*}
$$

Instead of using the above extension to determine the extra coefficients, we perform an additional extrapolation for the basis in the far-field similar to (C.5)

$$
\tilde{B}_{l, j}(z)=B_{l, j}(z), \quad j \leq n_{2}-8, \quad \tilde{B}_{l, j}(z)=B_{j}(z)+C_{2, n_{2}-j} B_{n_{2}}(z)+C_{1, n_{2}-j} B_{n_{2}+1}(z)
$$

We multiply $\rho_{p}(y)$ given below to impose the Dirichlet boundary condition

$$
\begin{equation*}
\rho_{p}(y)=\arctan (1+y)-1 \tag{C.8}
\end{equation*}
$$

We can obtain the exact formulas of $\partial_{x}^{i} \rho_{p}$ using a symbolic computation. We use induction to obtain rigorous estimate of $\partial_{x}^{i} \rho_{p}$. See Section D.3.

We choose $C_{i}$ in (C.3) of order $s_{i, j+1}-s_{i, j}$ as follows

$$
\begin{equation*}
C_{i}=y_{1}, i \leq 9, \quad C_{i}=\left(s_{i, 4}-s_{i, 2}\right) / 2, i>9 \tag{C.9}
\end{equation*}
$$

so that the summand in (C.3) has order 1 for $x$ in the support $\left[y_{i-3}, y_{i+3}\right]$. When we need to perform extrapolation for $a_{n} B_{n}, a_{n+1} B_{n+1}$ from $a_{i} B_{i}, i \leq n-1$, e.g. (C.7), we modify the last few terms as follows

$$
\begin{equation*}
C_{i}=\left(y_{n}-y_{n-1}\right) / 100, n-9 \leq i \tag{C.10}
\end{equation*}
$$

We choose $C_{i}$ to be constant for $i$ close to 0 or $i$ close to $n_{1}$ since we need to perform extrapolation, and the choice of the constant does not affect the extrapolation formula for $a_{i j}$.
Far-field angular profile. To represent the far-field angular profile of $\bar{\omega}_{1}, \bar{\theta}_{1}, \bar{\phi}_{1}^{N}$ (C.1), we design adaptive mesh $0=\beta_{0}<\beta_{1}<.<\beta_{m}=\pi / 2$, and use 8-th order Bspline to represent $\bar{\omega}, \bar{\zeta}=\frac{\bar{\theta}}{x_{1}}$

$$
g(\pi / 2-\beta)=\sum_{i \geq 0} b_{i} B_{1, i}^{(8)}(\beta), \quad g_{\phi}(\pi / 2-\beta)=\left((\pi / 2)^{2}-\beta^{2}\right) \sum_{i} b_{i} \tilde{B}_{i}^{(8)}(\beta), \beta \in[0, \pi / 2]
$$

where $B_{1, i}^{(8)}$ is 8 -th order Bspline (C.3) $k=8$ with odd modification (C.4). Since $\bar{\omega}, \bar{\zeta}$ are odd in $x$, in the angular direction, this symmetry becomes odd in $\beta=\pi / 2$. To impose it, we write $g$ in terms of $\pi / 2-\beta$ and modify the first few B-spline $B_{i}\left(\mathbf{C . 3 )}\right.$ following (C.4) so that $\tilde{B}_{1, i}$ is odd at $\beta=0$. Then $g$ is odd in $\beta=\pi / 2$. The stream function $\bar{\phi}^{N}$ satisfies the boundary condition $\bar{\phi}^{N}(x, 0)=0$. For the angular profile, we need $g_{\phi}(0)=0$, and use the weight $\pi / 2-\beta$ to impose this condition. We further modify a few $\operatorname{Bspline} B_{1, i}(\beta)$ supported near $\beta=\pi / 2$ using 9 -th order extrapolation similar to (C.5) near $\beta=\pi / 2$ and get $\tilde{B}_{1, i}(\beta)$. We choose the mesh $\beta_{i}$ to be equi-spaced near $\beta=\pi / 2$ and determine the coefficients for extrapolation similar to (C.5). We remark that to evaluate the derivative $\partial_{\beta}^{i} g$ at $\pi / 2-\beta$, we have the $\operatorname{sign}(-1)^{k}$

$$
\left(\partial_{\beta}^{k} g\right)(\pi / 2-\beta)=(-1)^{k} \partial_{\beta}^{k} g(\pi / 2-\beta)=\sum b_{i} \partial_{\beta}^{k} B_{1, i}^{(8)}(\beta)
$$

We discuss how to obtain these angular profiles in Section 7 in 13 .
C.2. Estimate of the derivatives of piecewise polynomials. Our approximate steady state in a very large domain is represented as piecewise polynomials. We discuss how to estimate its derivatives. Suppose that we can evaluate a function $f$ on finite many points. For example, $f$ is an explicit function or a polynomial. To obtain a piecewise sharp bound of $f$ on $I=\left[x_{l}, x_{u}\right]$, we use the following standard error estimate

$$
\begin{equation*}
\max _{x \in I}|f(x)| \leq \max \left(\left|f\left(x_{l}\right)\right|,\left|f\left(x_{u}\right)\right|\right)+\frac{h^{2}}{8}\left\|f_{x x}\right\|_{L^{\infty}(I)}, \quad h=x_{u}-x_{l} \tag{C.11}
\end{equation*}
$$

If we can obtain a rough bound for $f_{x x}$, as long as the interval $I$ is small, i.e., $h$ is small, the error part is small. Similarly, if we can obtain a rough bound for $\partial_{x}^{k+2} f$, using induction and the above estimate recursively,

$$
\max _{x \in I}\left|\partial_{x}^{i} f(x)\right| \leq \max \left(\left|\partial_{x}^{i} f\left(x_{l}\right)\right|,\left|\partial_{x}^{i} f\left(x_{u}\right)\right|\right)+\frac{h^{2}}{8}\left\|\partial_{x}^{i+2} f\right\|_{L^{\infty}(I)}
$$

for $i=k, k-1, \ldots, 0$, we can obtain the sharp bound for $\partial_{x}^{i} f$ on $I$. We call the above method the second order method since the error term is second order in $h$.
C.2.1. Estimate a piecewise polynomial in $1 D$. Suppose that $p(x)$ is a piecewise polynomials on $x_{0}<x_{1}<. .<x_{n}$ with degree $d$, e.g. Hermite spline. Denote $I_{i}=\left[x_{i}, x_{i+1}\right]$. Then $p(x)$ is a polynomial in each $I_{i}$ with degree $\leq d$. Our goal is to estimate $\partial_{x}^{k} p(x)$ in $I_{i}$ for all $k$ by only finite many evaluations of $p(x)$ and its derivatives.

Firstly, we have

$$
\partial_{x}^{k} p(x)=0, \quad k>d, \quad \partial_{x}^{d} p(x)=c_{p}
$$

for some constant $c_{p}$ in $I_{i}$.
Now, using induction from $k=d-1, d-2, . ., 0$, we have

$$
\max _{x \in I_{i}}\left|\partial_{x}^{k} p(x)\right| \leq \max \left(\left|\partial_{x}^{k} p\left(x_{i}\right)\right|,\left|\partial_{x}^{k} p\left(x_{i+1}\right)\right|\right)+\frac{h_{i}^{2}}{8}\left\|\partial_{x}^{k+2} p\right\|_{L^{\infty}\left(I_{i}\right)}, \quad h_{i}=x_{i+1}-x_{i}
$$

Since we know $\partial_{x}^{d+1} p(x)=0$ on $I_{i}$, using the above method, we can obtain the sharp piecewise bounds for all derivatives of $p(x)$ on $I_{i}$.

Using the above approach, we can estimate the derivatives of the angular profile defined Section 7.1 of Part I [13] rigorously.
C.2.2. Estimate a piecewise polynomial in 2D. Now, we generalize the above ideas to 2 D so that we can estimate the approximate steady state (C.5). We assume that $p(x, y)$ is a piecewise polynomials in the mesh $Q_{i j}=\left[x_{i}, x_{i+1}\right] \times\left[y_{j}, y_{j+1}\right]$ with degree $d$. That is, in $Q_{i j}, p(x, y)$ can be written as a linear combination of

$$
x^{k} y^{l}, \quad \max (k, l) \leq d
$$

e.g. (C.5). For (C.5), we have $d=5$. Similar to the 1D case, we have

$$
\partial_{x}^{k} \partial_{y}^{l} p(x, y)=0, \quad \max (k, l)>d
$$

Moreover, we know $\partial_{x}^{d-1} \partial_{y}^{d-1}$ is linear in $x, y$.
We use the following direct generalization of (C.11) to 2 d

$$
\begin{align*}
\max _{(x, y) \in Q} \mid f(x, y \mid & \leq \max _{\alpha, \beta=l, u}\left|f\left(x_{\alpha}, y_{\beta}\right)\right|+\frac{\left\|f_{x x}\right\|_{L^{\infty}(Q)}\left(x_{u}-x_{l}\right)^{2}}{8}+\frac{\left\|f_{y y}\right\|_{L^{\infty}(Q)}\left(y_{u}-y_{l}\right)^{2}}{8}  \tag{C.12}\\
Q & =\left[x_{l}, x_{u}\right] \times\left[y_{l}, y_{u}\right]
\end{align*}
$$

Denote

$$
A_{k l} \triangleq \max _{Q_{i j}}\left\|\partial_{x}^{k} \partial_{y}^{l} p\right\|_{L^{\infty}\left(Q_{i j}\right)}, B_{k l} \triangleq \max _{\alpha, \beta=l, u}\left|\partial_{x}^{k} \partial_{y}^{l} p\left(x_{\alpha}, y_{\beta}\right)\right|, \quad h_{1}=x_{i+1}-x_{i}, \quad h_{2}=y_{j+1}-y_{j}
$$

Since $p$ is given, we can evaluate $B_{k l}$. Clearly, we have $A_{k l}=0$ for $\max (k, l)>d$. For $k=d-1, d$, using (C.12) and induction in the order $l=d, d-1, d-2, . ., 0$, we can obtain

$$
A_{k l} \leq B_{k l}+\frac{1}{8}\left(h_{1}^{2} A_{k+2, l}+h_{2}^{2} A_{k, l+2}\right)
$$

This allows us to bound $A_{k l}$ for $k=d, d-1$ and all $l$. Similarly, we can bound $A_{k l}$ for $l=d, d-1$ and all $k$.

For the remaining cases, we can use induction on $n=\max (k, l)=d-2, d-1, . ., 0$ to estimate

$$
A_{k l} \leq B_{k l}+\frac{1}{8}\left(h_{1}^{2} A_{k+2, l}+h_{2}^{2} A_{k, l+2}\right)
$$

This allows us to estimate all derivatives of $p(x, y)$ in $Q_{i j}$.
C.2.3. Estimate a piecewise polynomial in 2D with weights. We consider how to estimate the derivatives of $f=\rho(y) p(x, y)$, where $\rho$ is a given weight in $y$ and $p(x, y)$ is the piecewise polynomials in 2D. For example, our construction of the stream function (C.7) has such a form. Firstly, we can estimate the derivatives of $p(x, y)$ using the method in Appendix C.2.2, For the weight $\rho$, we estimate its derivatives in Section D.3. Then, using the Leibniz rule (A.6) and the triangle inequality, we can estimate the derivatives $f$

$$
\left|\partial_{x}^{i} \partial_{y}^{j} f\right| \leq \sum_{l \leq j}\binom{j}{l}\left|\partial_{x}^{i} \partial_{y}^{l} p(x, y)\right|\left|\partial_{y}^{j-l} \rho(y)\right|
$$

for high enough derivatives.
Now, we plug the above bounds for $\partial_{x}^{i+2} \partial_{j}^{y}, \partial_{x}^{i} \partial_{y}^{j+2} f$ in (C.12) and evaluate $\partial_{x}^{i} \partial_{y}^{j} f$ on the grid points to obtain the sharp estimate of $\partial_{x}^{i} \partial_{y}^{j} f$.
C.3. Estimate of the far-field approximation. We estimate the derivatives of

$$
g(x, y)=g(r, \beta)=A(r) B(\beta), \quad r=\left(x^{2}+y^{2}\right)^{1 / 2}, \quad \beta=\arctan (y / x)
$$

where $(r, \beta)$ is the polar coordinate. The semi-analytic parts of $\bar{\omega}, \bar{\theta}$ have the above forms.
C.3.1. Formulas of the derivatives of $g$. Firstly, we use induction to establish

$$
\begin{equation*}
F_{i, j} \triangleq \partial_{x}^{i} \partial_{y}^{j} g(r, \beta)=\sum_{k+l \leq i+j} C_{i, j, k, l}(\beta) r^{-i-j+k} \partial_{r}^{k} A \partial_{\beta}^{l} B \tag{C.13}
\end{equation*}
$$

with $C_{i, j, k, l}=0$, for $k<0, l<0$, or $k+l>i+j$. Let us motivate the above ansatz. Recall from (B.14) that

$$
\partial_{x}=\cos \beta \partial_{r}-\frac{\sin \beta}{r} \partial_{\beta}, \quad \partial_{y}=\sin \beta \partial_{r}+\frac{\cos \beta}{r} \partial_{\beta}
$$

For each derivative $\partial_{x}$ or $\partial_{y}$, we get the factor $\frac{1}{r}$ or a derivative $\partial_{r}$, which leads to the form $r^{-i-j+k} \partial_{r}^{k} A$. Moreover, we get a derivative $\partial_{\beta}$ and some functions depending on $\beta$, which leads to the form $C_{i, j, k, l}(\beta) \partial_{\beta}^{l} B$.

For $D=\partial_{x}$ or $\partial_{y}$, a direct calculation yields
(C.14) $D F_{i, j}=\sum_{k+l \leq i+j} D\left(C_{i, j, k, l} r^{-i-j+k}\right) \cdot \partial_{r}^{k} A \partial_{\beta}^{l} B+C_{i, j, k, l} r^{-i-j+k}\left(D \partial_{r}^{k} A \cdot \partial_{\beta}^{l} B+\partial_{r}^{k} A \cdot D \partial_{\beta}^{l} B\right)$.

Using the formula of $\partial_{x}, \partial_{y}$, we get

$$
\begin{aligned}
& \partial_{x}\left(C_{i, j, k, l}(\beta) r^{-i-j+k}\right)=-\sin \beta \partial_{\beta} C_{i, j, k, l} r^{-i-j-1+k}+(k-i-j) \cos \beta C_{i, j, k, l} r^{-i-j-1+k} \\
& \partial_{x} \partial_{r}^{k} A=\cos \beta \partial_{r}^{k+1} A, \quad \partial_{x} \partial_{\beta}^{l} B=-\frac{\sin \beta}{r} \partial_{\beta}^{l+1} B
\end{aligned}
$$

Using $\partial_{x} F_{i, j}=F_{i+1, j}$ and comparing the above formulas and the ansatz (C.13), we yield
(C.15) $\quad C_{i+1, j, k, l}=(k-i-j) \cos \beta C_{i, j, k, l}-\sin \beta \partial_{\beta} C_{i, j, k, l}+\cos \beta C_{i, j, k-1, l}-\sin \beta C_{i, j, k, l-1}$, for $k \leq i+j$. Similarly, for $D=\partial_{y}$, plugging the following identities

$$
\begin{aligned}
& \partial_{y}\left(C_{i, j, k, l}(\beta) r^{-i-j+k}\right)=\cos \beta \partial_{\beta} C_{i, j, k, l} r^{-i-j-1+k}+(k-i-j) \sin (\beta) C_{i, j, k, l} r^{-i-j-1+k} \\
& \partial_{y} \partial_{r}^{k} A=\sin \beta \partial_{r}^{k+1} A, \quad \partial_{y} \partial_{\beta}^{l} B=\frac{\cos \beta}{r} \partial_{\beta}^{l+1} B
\end{aligned}
$$

into (С.14) and then comparing (С.13) and (С.14), we yield

$$
\begin{equation*}
C_{i, j+1, k, l}=(k-i-j) \sin \beta C_{i, j, k, l}+\cos \beta \partial_{\beta} C_{i, j, k, l}+\sin \beta C_{i, j, k-1, l}+\cos \beta C_{i, j, k, l-1} \tag{C.16}
\end{equation*}
$$

The based case is given by

$$
F_{0,0}=A(r) g(\beta), \quad C_{0,0,0,0}=1
$$

Using induction and the above recursive formulas, we can derive $C_{i, j, k, l}(\beta)$ in (C.13).
C.3.2. Estimates of $F_{i, j}$. To estimate $F_{i, j}$, using (C.13) and triangle inequality, we only need to estimate $\partial_{r}^{k} A, \partial_{\beta}^{l} B(\beta)$, and $C_{i, j, k, l}(\beta)$. In our case, $B(\beta)$ is piecewise polynomials, whose estimates follow the method in Appendix C.2.1). Function $A(r)$ is some explicit function, which will be constructed and estimated in Section D.1. To estimate $C_{i, j, k, l}(\beta)$ on $\beta \in\left[\beta_{1}, \beta_{2}\right]$, we use the second order estimate in (C.11) and the induction ideas in Section C.2.1. We can evaluate $C_{i, j, k, l}$ using its exact formula. It remains to bound $\partial_{\beta}^{2} C_{i, j, k, l}$.

An important observation from (C.15), (C.14) is that $C_{i, j, k, l}$ is a polynomial on $\sin \beta$ and $\cos \beta$ with degree less than $i+j$, which can be proved easily using induction. In particular, we can write $C_{i, j, k, l}$ as follows
$C_{i, j, k, l}=\sum_{0 \leq k \leq n} a_{k} \sin (k \beta)+b_{k} \cos (k \beta), \quad f \triangleq \partial_{\beta}^{2} C_{i, j, k, l}=\sum_{1 \leq k \leq n} c_{k} \sin (k \beta)+d_{k} \cos (k \beta), \quad n=i+j$
for some $a_{k}, b_{k}, c_{k}, d_{k} \in \mathbb{R}$. It is easy to see that $C_{i, j, k, l}$ is either odd or even in $\beta$ depending on $j-l$, which implies $c_{k} \equiv 0$ or $d_{k} \equiv 0$. Using Cauchy-Schwarz's inequality, we get

$$
\|f\|_{\infty} \leq \sum_{1 \leq k \leq n}\left(\left|c_{k}\right|+\left|d_{k}\right|\right) \leq\left(n \sum_{k \leq n}\left(c_{k}^{2}+d_{k}^{2}\right)\right)^{1 / 2}=\left(\frac{n}{\pi} \int_{0}^{2 \pi} f^{2}\right)^{1 / 2}
$$

where we have used orthogonality of $\sin k x, \cos k x$ and $\|f\|_{L^{2}}^{2}=\pi \sum_{k \leq n}\left(c_{k}^{2}+d_{k}^{2}\right)$ in the last equality. It is easy to see that $f^{2}$ is again a polynomial in $\sin \beta, \cos \beta$ with degree $\leq 2 n$. We fix $M>2 n$. For any $0 \leq k<M$, it is easy to obtain

$$
\frac{1}{2 \pi} \int_{0}^{2 \pi} e^{i k x}=\frac{1}{M} \sum_{j=1}^{M} \exp \left(i \frac{2 k j}{M} \pi\right)=\delta_{k 0}
$$

Using the above identity, we establish

$$
\|g\|_{L^{2}}^{2}=\frac{2 \pi}{M} \sum_{j=1}^{M}\left|g\left(\frac{2 j \pi}{M}\right)\right|^{2}
$$

for any polynomial $g$ in $\sin \beta, \cos \beta$ with degree $<M / 2$. Hence, we prove

$$
\|f\|_{\infty} \leq\left(\frac{2 n}{M} \sum_{k=1}^{M} f^{2}\left(\frac{2 j \pi}{M}\right)\right)^{1 / 2}
$$

The advantage of the above estimate is that to obtain the sharp bound of $C_{i, j, k, l}$, we only need to evaluate $C_{i, j, k, l}, f=\partial_{\beta}^{2} C_{i, j, k, l}$ on finite many points.
C.3.3. From polar coordinates to the Cartesian coordinate. We want to obtain the piecewise estimate of $F_{p, q}=\partial_{x}^{p} \partial_{y}^{q}(A(r) g(\beta))$ on $Q_{i j}=\left[x_{i}, x_{i+1}\right] \times\left[y_{j}, y_{j+1}\right], 1 \leq i, j \leq n$. Firstly, we partition the $(r, \beta)$ coordinate into $r_{1}<r_{2}<\ldots<r_{n_{1}}, 0=\beta_{0}<b_{1}<\ldots<\beta_{n_{2}}=\frac{\pi}{2}$. Then we apply the methods in Section C. 3 to bound $F_{p, q}$ on $S_{i j} \triangleq\left[r_{i}, r_{i+1}\right] \times\left[\beta_{j}, \beta_{j+1}\right]$. We cover $Q_{i j}$ by $S_{k, l}$ and transfer the bound from $(r, \beta)$ coordinate to $(x, y)$ coordinate

$$
\max _{x \in Q_{i j}}\left|F_{p, q}(x)\right| \leq \max _{S_{k, l} \cap Q_{i j} \neq \emptyset}\left\|F_{p, q}(r, \beta)\right\|_{L^{\infty}\left(S_{k, l}\right)}
$$

For $(r, \beta) \in Q_{i, j}$, we get

$$
r \in\left[\left(x_{i}^{2}+y_{j}^{2}\right)^{1 / 2},\left(x_{i+1}^{2}+y_{j+1}^{2}\right)^{1 / 2}\right], \quad \beta \in\left[\arctan \frac{y_{j}}{x_{i+1}}, \arctan \frac{y_{j+1}}{x_{i}}\right]
$$

Therefore, we yield the necessary conditions for $Q_{i, j} \cap S_{k, l} \neq \emptyset$ :

$$
x_{i+1}^{2}+y_{j+1}^{2} \geq r_{k}^{2}, \quad x_{i}^{2}+y_{i}^{2} \leq r_{u}^{2}, \quad \arctan \frac{y_{j+1}}{x_{i}} \geq \beta_{l}, \quad \arctan \frac{y_{j}}{x_{i+1}} \leq \beta_{l+1}
$$

Given $Q_{i, j}$, we maximize $\left\|F_{p, q}\right\|_{L^{\infty}\left(S_{k, l}\right)}$ over $(k, l)$ satisfying the above bounds to control $\left\|F_{p, q}\right\|_{L^{\infty}\left(Q_{i, j}\right)}$.
C.4. Estimates of the residual error. Let $\chi_{\bar{\varepsilon}}=1+O\left(|x|^{4}\right)$ be the cutoff function in (D.6). Firstly, we decompose the error of solving the Poisson equations $\bar{\varepsilon}=\bar{\omega}-(-\Delta) \bar{\phi}^{N}$ as follows

$$
\begin{align*}
& \bar{\varepsilon}=\bar{\varepsilon}_{1}+\bar{\varepsilon}_{2}, \quad \bar{\varepsilon}_{2}=\bar{\varepsilon}_{x y}(0) \Delta\left(\frac{x^{3} y}{2} \chi_{\bar{\varepsilon}}\right), \quad \mathbf{u}\left(\bar{\varepsilon}_{2}\right)=\nabla^{\perp}(-\Delta)^{-1} \bar{\varepsilon}_{2}=\frac{1}{2} \bar{\varepsilon}_{x y}(0) \nabla^{\perp}\left(x^{3} y \chi_{\bar{\varepsilon}}\right)  \tag{C.17}\\
& \mathbf{u}(\bar{\varepsilon})=\mathbf{u}\left(\bar{\varepsilon}_{1}\right)+\mathbf{u}\left(\bar{\varepsilon}_{2}\right)=\mathbf{u}_{A}\left(\bar{\varepsilon}_{1}\right)+\left(\hat{\mathbf{u}}\left(\bar{\varepsilon}_{1}\right)+\mathbf{u}\left(\bar{\varepsilon}_{2}\right)\right) \triangleq \mathbf{u}_{A}\left(\bar{\varepsilon}_{1}\right)+\mathbf{u}_{l o c}(\bar{\varepsilon})
\end{align*}
$$

where $\hat{\mathbf{u}}$ is the approximation term for $\mathbf{u}$ defined in Section 4.3 in Part I [13]. We perform the above correction near 0 so that $\bar{\varepsilon}_{1}=O\left(|x|^{3}\right)$ near 0 . We perform a similar decomposition for $(\nabla \mathbf{u})_{A}$. Note that we do not have $\partial_{x_{i}} \mathbf{u}_{A}=\left(\partial_{x_{i}} \mathbf{u}\right)_{A}$. Using the above decomposition and the notation (3.4), we can rewrite the residual error (2.14) with rank-one correction as follows (C.18)

$$
\begin{aligned}
\overline{\mathcal{F}}_{l o c, i} & =I I_{i}-D_{i}^{2} I I_{i}(0) f_{\chi, i}=I I_{i}-D_{i}^{2} \overline{\mathcal{F}}_{i}(0) f_{\chi, i}, I I_{i}=\overline{\mathcal{F}}_{i}-\mathcal{B}_{o p, i}\left(\left(\mathbf{u}_{A}\left(\bar{\varepsilon}_{1}\right),(\nabla \mathbf{u})_{A}\left(\bar{\varepsilon}_{1}\right)\right), \bar{W}\right) \\
\mathbf{u}(\bar{\omega}) & =\overline{\mathbf{u}}=\overline{\mathbf{u}}^{N}+\mathbf{u}_{l o c}(\bar{\varepsilon})+\mathbf{u}_{A}\left(\bar{\varepsilon}_{1}\right), \bar{c}_{\omega}=\bar{c}_{\omega}^{N}+u_{x}\left(\bar{\varepsilon}_{1}\right)(0), \bar{c}_{\omega}^{N} \triangleq \frac{\bar{c}_{l}}{2}+\bar{u}_{x}^{N}(0), c_{\omega}\left(\bar{\varepsilon}_{1}\right) \triangleq u_{x}\left(\bar{\varepsilon}_{1}\right)(0) \\
I I_{1} & =-\left(\bar{c}_{l} x+\overline{\mathbf{u}}^{N}+\mathbf{u}_{l o c}(\bar{\varepsilon})\right) \cdot \nabla \bar{\omega}+\bar{\theta}_{x}+\left(\bar{c}_{\omega}^{N}+\bar{c}_{\omega}\left(\bar{\varepsilon}_{1}\right)\right) \bar{\omega} \\
I I_{2} & =-\left(\bar{c}_{l} x+\overline{\mathbf{u}}^{N}+\mathbf{u}_{l o c}(\bar{\varepsilon})\right) \cdot \nabla \bar{\theta}_{x}+2\left(\bar{c}_{\omega}^{N}+\bar{c}_{\omega}\left(\bar{\varepsilon}_{1}\right)\right) \bar{\theta}_{x}-\left(\bar{u}_{x}^{N}+u_{x, l o c}\left(\bar{\varepsilon}^{\prime}\right)\right) \bar{\theta}_{x}-\left(\bar{v}_{x}^{N}+v_{x, l o c}(\bar{\varepsilon})\right) \bar{\theta}_{y}, \\
I I_{2} & =-\left(\bar{c}_{l} x+\overline{\mathbf{u}}^{N}+\mathbf{u}_{l o c}(\bar{\varepsilon})\right) \cdot \nabla \bar{\theta}_{y}+2\left(\bar{c}_{\omega}^{N}+c_{\omega}\left(\bar{\varepsilon}_{1}\right)\right) \bar{\theta}_{y}-\left(\bar{u}_{y}^{N}+u_{y, l o c}(\bar{\varepsilon})\right) \bar{\theta}_{x}-\left(\bar{v}_{y}^{N}+v_{y, l o c}(\bar{\varepsilon})\right) \bar{\theta}_{y}
\end{aligned}
$$

where $f_{\chi, i}$ is defined in (D.6), and we have used (2.11) for $\bar{c}_{\omega}, u_{x}\left(\bar{\varepsilon}_{2}\right)=0, u_{x, A}\left(\bar{\varepsilon}_{1}\right)=0\left(\mathbf{u}_{A}=\right.$ $\left.O\left(|x|^{3}\right),(\nabla \mathbf{u})_{A}=O\left(|x|^{2}\right)\right)$ from the definition. The above decomposition is essentially the same as (3.12). We apply the functional inequalities in Section 4 to estimate $\mathbf{u}_{A}\left(\bar{\varepsilon}_{1}\right),(\nabla \mathbf{u})_{A}\left(\bar{\varepsilon}_{1}\right)$, and combine the estimate $\mathcal{B}_{o p, i}\left(\left(\mathbf{u}_{A},(\nabla \mathbf{u})_{A}\right), \bar{W}\right)$ with the energy estimate. See Section 5.8 in Part I [13] for more details about the decompositions and estimates. Using the decomposition (C.17), we can further decompose the above $I I_{i}$ as follow
$I I_{i}=I I_{i}^{N}+I I_{i}\left(\bar{\varepsilon}_{1}\right)+I I_{i}\left(\bar{\varepsilon}_{2}\right), I I_{i}\left(\bar{\varepsilon}_{1}\right)=\mathcal{B}_{o p, i}\left(\hat{\mathbf{u}}\left(\bar{\varepsilon}_{1}\right), \widehat{\nabla \mathbf{u}}\left(\bar{\varepsilon}_{1}\right), \bar{W}\right), I I_{i}\left(\bar{\varepsilon}_{2}\right)=\mathcal{B}_{o p, i}\left(\mathbf{u}\left(\bar{\varepsilon}_{2}\right), \nabla \mathbf{u}\left(\bar{\varepsilon}_{2}\right), \bar{W}\right)$,
where $I I_{i}^{N}$ contain the terms in $I I_{i}$ except the $u_{l o c}, u\left(\bar{\varepsilon}_{1}\right)$ terms.
For $\hat{\mathbf{u}}\left(\bar{\varepsilon}_{1}\right)$, it is a finite rank operator on $\bar{\varepsilon}_{1}$, and we can write it as

$$
\hat{\mathbf{u}}\left(\bar{\varepsilon}_{1}\right)=\sum_{i=1}^{n} a_{i}\left(\bar{\varepsilon}_{1}\right) \bar{g}_{i}(x) \triangleq C_{\mathbf{u} 0}(x) u_{x}\left(\bar{\varepsilon}_{1}\right)(0)+\tilde{\hat{\mathbf{u}}}\left(\bar{\varepsilon}_{1}\right), \quad a_{i}\left(\bar{\varepsilon}_{1}\right)=\int_{\mathbb{R}_{2}^{++}} \bar{\varepsilon}_{1}(y) q_{i}(y) d y
$$

for some functions $\bar{g}_{i}(x)$ and $q_{i}(y)$, where $C_{\mathbf{u} 0}(x)$ is given in (4.5), and $\tilde{\hat{\mathbf{u}}}\left(\bar{\varepsilon}_{1}\right)$ denotes other modes with $O\left(|x|^{3}\right)$ vanishing order near 0 . See Section 4.3 in 13 for definition. We can obtain more regular estimates, e.g. $C^{3}$ estimates, of $\hat{\mathbf{u}}\left(\varepsilon_{1}\right)$ since $\bar{g}_{1}(x)$ is smooth. Similarly, we decompose $\widehat{\nabla \mathbf{u}}\left(\bar{\varepsilon}_{1}\right)$. We obtain piecewise estimates of $\partial_{x}^{i} \partial_{y}^{j} \bar{\varepsilon}_{1}, i+j \leq 1$ following the methods in Section 3.6 and Section 8 in the supplementary material II [11] (contained in [10]) and then the above integrals on $\bar{\varepsilon}_{1}$. The main term in $\hat{\mathbf{u}}\left(\bar{\varepsilon}_{1}\right)$ is $C_{u 0} u_{x}(0)$ with

$$
\begin{align*}
& u_{x}\left(\bar{\varepsilon}_{1}\right)(0)=u_{x}(\bar{\varepsilon})(0)=-\frac{4}{\pi} \int_{\mathbb{R}_{2}^{++}} \bar{\varepsilon}(y) \frac{y_{1} y_{2}}{|y|^{4}} d y  \tag{C.19}\\
& u_{x}\left(\bar{\varepsilon}_{2}\right)(0)=-\varepsilon_{x y}(0) /\left.2 \cdot \partial_{y}\left(x^{3} y \chi_{\bar{\varepsilon}}\right)\right|_{(0,0)}=0
\end{align*}
$$

Since the kernel $\frac{y_{1} y_{2}}{|y|^{4}}$ has a slow decay for large $|y|$ (not in $L^{1}$ ), we need to estimate $u_{x}(\bar{\varepsilon})(0)$ carefully, using Simpson's rule. See Section 6.4.2 in supplementary material II for Part II 11.

Using the above decomposition, we further decompose $\hat{\mathbf{u}}\left(\hat{\varepsilon}_{1}\right)$

$$
I I_{i}\left(\bar{\varepsilon}_{1}\right)=u_{x}(\bar{\varepsilon})(0) \mathcal{B}_{o p, i}\left(\left(C_{\mathbf{u} 0}(x), C_{\nabla \mathbf{u} 0}(x), \bar{W}\right)+\mathcal{B}_{o p, i}(\widetilde{\widetilde{\mathbf{u}}}, \widetilde{\widehat{\nabla \mathbf{u}}}, \bar{W}) \triangleq I I_{i, M}\left(\bar{\varepsilon}_{1}\right)+I I_{i, R}\left(\bar{\varepsilon}_{1}\right)\right.
$$

Since $D_{i}^{2}$ is linear, we estimate each term $g_{i}-D_{i}^{2} g_{i} f_{\chi, i}$ for $g_{i}=I I_{i, M}\left(\bar{\varepsilon}_{1}\right), I I_{i, R}\left(\bar{\varepsilon}_{1}\right), I I_{i}^{N}, I I_{i}\left(\bar{\varepsilon}_{2}\right)$ to bound $\mathcal{F}_{l o c, i}$. To estimate $I I_{i, R}$, since $\tilde{\hat{\mathbf{u}}}\left(\bar{\varepsilon}_{1}\right)=O\left(|x|^{3}\right)$ near 0 , (see Section 4.3 in [13]), we get $D_{i}^{2} I I_{i, R}\left(\bar{\varepsilon}_{1}\right)=O\left(|x|^{3}\right)$ and estimate

$$
\tilde{\tilde{\mathbf{u}}}\left(\bar{\varepsilon}_{1}\right) \rho_{10}, \partial_{i} \tilde{\tilde{\mathbf{u}}}\left(\bar{\varepsilon}_{1}\right) \rho_{20}, \widetilde{\widehat{\nabla \mathbf{u}}}\left(\bar{\varepsilon}_{1}\right) \rho_{20}, \partial_{i} \widetilde{\widehat{\nabla \mathbf{u}}}\left(\bar{\varepsilon}_{1}\right) \rho_{3}, \rho_{4} \hat{\mathbf{u}}\left(\bar{\varepsilon}_{1}\right)
$$

for $\rho_{i 0}\left(\underline{\text { A.2 })}\right.$ with $\rho_{i 0} \sim|x|^{-4+i}, i \leq 3$ near 0 using the $C^{3}$ bounds of $\tilde{\hat{\mathbf{u}}}, \widetilde{\widehat{\nabla \mathbf{u}}}$. Note that $\partial_{i} \tilde{\tilde{\mathbf{u}}} \neq \widetilde{\overline{\partial_{i} \mathbf{u}}}$. The former is the derivative of $\tilde{\hat{\mathbf{u}}}$, and the later is the approximation term for $\partial_{i} \mathbf{u}$. With the above weighted estimate, we can bound a typical terms, e.g. $\widetilde{u_{x}} \bar{\theta}_{x} \varphi_{2}$ in $I I_{i, R}\left(\bar{\varepsilon}_{1}\right) \varphi_{2}$ as follows

$$
\widetilde{\widetilde{u_{x}}} \bar{\theta}_{x} \varphi_{2}=\widetilde{\widehat{u_{x}}} \rho_{20} \cdot\left(\bar{\theta}_{x} \frac{\varphi_{2}}{\rho_{20}}\right), \partial_{x}\left(\widetilde{\widetilde{u_{x}}} \bar{\theta}_{x}\right) \rho=\left(\partial_{x} \widetilde{\widetilde{u_{x}}} \bar{\theta}_{x}+\widetilde{\widetilde{u_{x}}} \partial_{x} \bar{\theta}_{x}\right) \rho=\partial_{x} \widetilde{\widetilde{u_{x}}} \rho_{3} \cdot \frac{\bar{\theta}_{x} \rho}{\rho_{3}}+\widetilde{\widetilde{u_{x}}} \rho_{20} \cdot \frac{\partial_{x} \bar{\theta}_{x} \rho}{\rho_{20}}
$$

where $\varphi_{2}$ is given in (A.2). Each term $A, B$ in the above products $A \cdot B$ is regular and we estimate each term and then the product to bound weighted $L^{\infty}$ and $C^{1}$ norm of $I I_{i, R}\left(\bar{\varepsilon}_{1}\right)$.

The remaining part in $I I_{i}^{N}, I I_{i, M}\left(\bar{\varepsilon}_{1}\right), I I_{i}\left(\bar{\varepsilon}_{2}\right)$ depends on $\left(\bar{\phi}^{N}, \bar{\omega}, \bar{\theta}\right)$ locally and are given functions. To estimate the weighted $L^{\infty}$ and $C^{1 / 2}$ norms of $g_{i}-D_{i}^{2} g_{i}(0) f_{\chi, i-}=O\left(|x|^{3}\right)$ with $g=I I_{i, M}\left(\bar{\varepsilon}_{1}\right), I I_{i}(\bar{\varepsilon})$, we follow the methods in Sections 3.6, 3.7 with $\partial_{t} \bar{\omega}=\partial_{t} \bar{\theta}=0$.

Estimate in the far-field. Since $\bar{\omega}, \bar{\theta}$ are supported globally, we need to estimate the error in the far-field. We consider $|x|_{\infty} \geq R_{1} \geq 10^{12}>10 a_{2}$ beyond the support of $\bar{\omega}_{2}, \bar{\theta}_{2}, \bar{\psi}_{2}^{N}$ so that $\chi(r)=1$ (D.4) and

$$
\bar{\omega}=\bar{\omega}_{1}=\bar{g}_{1}(\beta) r^{\bar{\alpha}_{1}}, \quad \bar{\theta}=\bar{\theta}_{1}=r^{1+2 \bar{\alpha}_{1}} \bar{g}_{2}(\beta), \quad \bar{\phi}^{N}=\bar{\phi}_{1}^{N}=r^{2+\bar{\alpha}_{1}} \bar{f}(\beta)
$$

We estimate the angular derivatives of $f(\beta), g_{i}(\beta)$ using the methods in Section C.2.1 Using the above representation, $x \cdot \nabla r^{\beta}=r \partial_{r} r^{\beta}=\beta r^{\beta}, \bar{c}_{\omega}=\bar{c}_{\omega}^{N}+\bar{c}_{\omega}^{\bar{\varepsilon}}$ (3.11), and separating $\mathbf{u}^{N}$ and $\mathbf{u}_{l o c}$, we obtain
$\bar{F}_{l o c, 1}=\left(\left(\bar{c}_{\omega}^{N}-\bar{c}_{l} \bar{\alpha}_{1}\right) \bar{\omega}_{1}-\overline{\mathbf{u}}^{N} \cdot \nabla \bar{\omega}_{1}+\bar{\theta}_{1, x}\right)+\bar{c}_{\omega}^{\varepsilon} \bar{\omega}_{1}-\mathbf{u}_{l o c} \cdot \nabla \bar{\omega}_{1} \triangleq I_{11}+I_{12}$,
$\bar{F}_{l o c, 2}=\left(\left(2 \bar{c}_{\omega}^{N}+2 \bar{c}_{l} \bar{\alpha}_{1}\right) \bar{\theta}_{1, x}-\partial_{x}\left(\overline{\mathbf{u}}^{N} \cdot \nabla \bar{\theta}_{1}\right)\right)+2 \bar{c}_{\omega}^{\varepsilon} \bar{\theta}_{1, x}-\mathbf{u}_{l o c} \cdot \nabla \bar{\theta}_{1, x}-u_{x, l o c} \bar{\theta}_{x}-v_{x, l o c} \bar{\theta}_{y} \triangleq I_{21}+I_{22}$,
$\bar{F}_{l o c, 3}=\left(\left(2 \bar{c}_{\omega}^{N}+2 \bar{c}_{l} \bar{\alpha}_{1}\right) \bar{\theta}_{1, y}-\partial_{x}\left(\overline{\mathbf{u}}^{N} \cdot \nabla \bar{\theta}_{1}\right)\right)+2 \bar{c}_{\omega}^{\varepsilon} \bar{\theta}_{1, y}-\mathbf{u}_{l o c} \cdot \nabla \bar{\theta}_{1, y}-u_{y, l o c} \bar{\theta}_{x}-v_{y, l o c} \bar{\theta}_{y} \triangleq I_{31}+I_{32}$,
where we simplify $\mathbf{u}_{l o c}(\bar{\varepsilon})$ as $\mathbf{u}_{l o c}$, and we have used $\bar{c}_{\theta}=\bar{c}_{l}+2 \bar{c}_{\omega}$, and $f_{\chi, j}$ is supported near 0 to get $f_{\chi, i}=0$. The terms $I_{11}, I_{21}, I_{31}$ are local and have the form $r^{\gamma} q(\beta)$ for some angular function $q$ and decay rate $\gamma$. We estimate its piecewise bound and derivative bounds using the formula (B.14). From our choice of $\bar{\alpha}_{1}, \bar{c}_{\omega}-\bar{c}_{l} \bar{\alpha}_{1}, \bar{c}_{l}+2 \bar{c}_{\omega}^{N}-\bar{c}_{l}\left(1+2 \bar{\alpha}_{1}\right)=2\left(\bar{c}_{\omega}-\bar{c}_{l} \bar{\alpha}_{1}\right)$ is very small, and thus the first terms in $I_{11}, I_{21}$ are small. The second terms in $I_{11}, I_{21}, I_{31}$ have decay rates $r^{2 \alpha_{1}}, r^{3 \alpha_{1}}$ and are also very small.

Estimate of the velocity approximation. Since $\bar{\varepsilon}_{2}$ is supported near 0, we get $\mathbf{u}_{l o c}=\hat{\mathbf{u}}\left(\varepsilon_{1}\right)$. It remains to estimate

$$
\begin{equation*}
\bar{c}_{\omega}^{e} \bar{\omega}-\hat{\mathbf{u}}\left(\bar{\varepsilon}_{1}\right) \cdot \nabla \bar{\omega}, 2 \bar{c}_{\omega}^{e} \bar{\theta}_{x}-\hat{\mathbf{u}}_{x}\left(\bar{\varepsilon}_{1}\right) \cdot \nabla \bar{\theta}-\hat{\mathbf{u}}\left(\bar{\varepsilon}_{1}\right) \cdot \nabla \bar{\theta}_{x}, 2 \bar{c}_{\omega}^{e} \bar{\theta}_{y}-\hat{\mathbf{u}}_{y}\left(\bar{\varepsilon}_{1}\right) \cdot \nabla \bar{\theta}-\hat{\mathbf{u}}\left(\bar{\varepsilon}_{1}\right) \cdot \nabla \bar{\theta}_{x} \tag{C.20}
\end{equation*}
$$

Note that $c_{\omega}\left(\bar{\varepsilon}_{1}\right)=c_{\omega}(\bar{\varepsilon})\left(\overline{C .19)}\right.$ and $c_{\omega}(\bar{\varepsilon})=\bar{c}_{\omega}^{e}$ in our notation. For any $a \in \mathbb{R}$, we estimate

$$
A(f, g)=a g-\hat{\mathbf{u}}(f) \cdot \nabla g, \quad B_{i}(f, g)=2 a \partial_{i} g-\hat{\mathbf{u}}(f) \cdot \nabla \partial_{i} g-\widehat{\partial_{i} \mathbf{u}}(f) \cdot \nabla g, i=1,2
$$

for $|x|_{\infty} \geq R_{1}$. From Sections 4.3.2-4.3.3 in Part I [13], for $|x|_{\infty} \geq R_{1}$, we yield

$$
\hat{u}(f)=x_{1} I_{\text {far }}(f), \quad \hat{v}(f)=-x_{2} I_{\text {far }}(f), \quad I_{\text {far }}(f) \triangleq-\frac{4}{\pi} \int_{\max \left(y_{1}, y_{2}\right) \geq R_{n}} \frac{y_{1} y_{2}}{|y|^{4}} \omega(y) d y
$$

where $R_{n}=1024 \cdot 64 h_{x}$ is the largest threshold. Denote $b=I_{f a r}(f)$. A direct calculation yields (C.21)

$$
\begin{aligned}
& A(f, g)=(a-b) g+b\left(g-x_{1} \partial_{1} g+x_{2} \partial_{2} g\right) \\
& B_{1}(f, g)=2 a \partial_{1} g-b \partial_{1} g-b x_{1} \partial_{11} g+b x_{2} \partial_{12} g=(2 a-2 b) \partial_{1} g+b\left(\partial_{1} g-x_{1} \partial_{11} g+x_{2} \partial_{12} g\right) \\
& B_{2}(f, g)=2 a \partial_{2} g+b \partial_{2} g-b x_{1} \partial_{12} g+b x_{2} \partial_{22} g=(2 a-2 b) \partial_{2} g+b\left(3 \partial_{1} g-x_{1} \partial_{11} g+x_{2} \partial_{12} g\right)
\end{aligned}
$$

Therefore, we only need to bound the functions following Section C.2, e.g. $g-x_{1} \partial_{1} g+x_{2} \partial_{2} g$ and $g$, and the functional $b(f)$ and $a$. We apply these estimates for (C.20) with $a=\bar{c}_{\omega}^{e}, f=$ $\bar{\varepsilon}_{1}, g=\bar{\omega}, \bar{\theta}$.

## Appendix D. Estimate of explicit functions

In this section, we estimate the derivatives of several explicit or semi-explicit functions using induction, including several cutoff functions used in the estimates and the weight in the stream function (C.7).

## D.1. Estimate of the radial functions.

D.1.1. Estimate of the cutoff function. We estimate the derivatives of the cutoff function

$$
\begin{equation*}
\chi_{e}(x)=\left(1+\exp \left(\frac{1}{x}+\frac{1}{x-1}\right)\right)^{-1} \tag{D.1}
\end{equation*}
$$

where $e$ is short for exponential. In our verification, it involves high order derivatives of $\chi_{e}$. Although $\chi_{e}$ is explicit, its formula is complicated and is difficult to estimate. Instead, we use the structure of $\partial_{x}^{i} \chi_{e}$ and induction to estimate $\partial_{x}^{i} \chi_{e}$. Denote

$$
p(x)=\frac{1}{x}+\frac{1}{x-1}, \quad f=\frac{1}{1+x}, \quad \chi_{e}=f\left(e^{p}\right)
$$

Firstly, we use induction to derive

$$
d_{x}^{k} \chi_{e}=\sum_{i=1}^{k}\left(\partial^{i} f\right)\left(e^{p}\right) e^{i p} Q_{k, i}(x)
$$

where $Q_{k, i}=0$ for $i>k, i<0$. A direct calculation yields

$$
\begin{aligned}
\partial \sum_{i=1}^{k} \partial^{i} f e^{i p} Q_{k, i}(x) & =\sum_{i=1}^{k}\left(\partial^{i+1} f\right)\left(e^{p}\right) \cdot p^{\prime} e^{p} e^{i p} Q_{k, i}+\left(\partial^{i} f\right) \partial_{x}\left(e^{i p} Q_{k, i}\right) \\
& =\sum_{i=1}^{k}\left(\partial^{i+1} f\right)\left(e^{p}\right) \cdot e^{(i+1) p} p^{\prime} Q_{k, i}+\left(\partial^{i} f\right) e^{i p}\left(i p^{\prime} Q_{k, i}+Q_{k, i}^{\prime}\right) .
\end{aligned}
$$

Comparing the above two equations, we derive

$$
Q_{k+1, i}=p^{\prime} Q_{k, i-1}+i p^{\prime} Q_{k, i}+Q_{k, i}^{\prime}
$$

The first few terms in $Q_{k, i}$ are given by

$$
Q_{0,0}=1, \quad Q_{1,1}=p^{\prime}, \quad Q_{1,0}=0
$$

It is not difficult to see that $Q_{k, i}$ is a polynomial of $\partial_{x}^{j} p, j \leq k$. Thus, using triangle inequality, we only need to bound $\partial_{x}^{j} p$. We have

$$
\left|\partial_{x}^{n} p(x)\right|=n!\left|x^{-n-1}+(x-1)^{-n-1}\right| \leq n!\left(|z|^{-n-1}+2^{n+1}\right), \quad z=\min (|x|,|1-x|)
$$

If $n$ is even, $x^{-n-1}$ and $(x-1)^{-n-1}$ have different sign, and we get better estimate

$$
\left|\partial_{x}^{n} p(x)\right| \leq n!\max \left(|x|^{-n-1},|x-1|^{-n-1}\right)=n!\cdot z^{-n-1}
$$

Substituting the above bounds into the formula of $Q_{k, i}$, we can obtain the upper bound $Q_{k, i}^{u}(x)$ for $Q_{k, i}(x)$, which is a polynomial of $z^{-1}$ with positive coefficient. Since each term in $Q_{k, i}$ is given by $c_{i_{1}, i_{2}, . ., i_{m}} \prod_{j=1}^{m} \partial_{x}^{i_{j}} p$ with $\sum i_{j}=k$, the above estimate implies

$$
\left|c_{i_{1}, i_{2}, . ., i_{m}} \prod_{j=1}^{m} \partial_{x}^{i_{j}} p\right| \leq c_{i_{1}, i_{2}, . ., i_{m}} \prod_{j=1}^{m} i_{j}!\left(|z|^{-i_{j}-1}+2^{i_{j}+1}\right)
$$

Since $m \leq k$, the highest order of $z^{-1}$ in the upper bound is bounded by $2 k$. Thus, we obtain that $Q_{k, i}^{u}$ is a polynomial in $z^{-1}$ with $\operatorname{deg} Q_{k, i}^{u} \leq 2 k$. Next, we bound

$$
\left|e^{i p} Q_{k, i}\right| \leq e^{i p} Q_{k, i}^{u}
$$

For $k \leq 20, x \geq 1-\frac{1}{2 k} \geq \frac{1}{2}, z^{-1}=|x-1|^{-1} \geq 2 k$, a direct calculation implies that $e^{i p(x)} Q_{k, i}^{u}(x)$ is decreasing. In fact, for $l \leq 2 k$, we have $z=|x-1|=1-x$ and

$$
\begin{aligned}
& \partial_{x}\left(\exp (i p(x))(1-x)^{-l}\right)=\exp (i p(x))\left(i p^{\prime}(1-x)^{-l}+l(1-x)^{-l-1}\right) \\
= & \exp (i p(x))\left(-\frac{i}{x^{2}}-\frac{i}{(x-1)^{2}}+l(1-x)^{-1}\right)(1-x)^{-l} \leq 0
\end{aligned}
$$

In the last inequality, we have used $-\frac{i}{1-x}+l \leq-2 k i+2 k \leq 0$.
Note that $\left|\left(\partial_{x}^{i} f\right)\left(e^{p}\right)\right|=i!\left|\left(1+e^{p}\right)^{-i-1}\right| \leq i!$. Thus, for $x \in\left[x_{l}, x_{u}\right]$ with $x_{l}$ close to 1 , we get

$$
\left|\partial_{x}^{k} \chi_{e}(x)\right| \leq \sum_{i=1}^{k}\left|\left(\partial^{i} f\right)\left(e^{p}\right)\right| e^{i p(x)} Q_{k, i}^{u}(x) \leq \sum_{i=1}^{k} i!\frac{e^{i p(x)}}{\left(1+e^{p}\right)^{i+1}} Q_{k, i}^{u}(x) \leq \sum_{i=1}^{k} i!e^{i p\left(x_{l}\right)} Q_{k, i}^{u}\left(x_{l}\right)
$$

For $x$ away from 1, we use monotonicities of $p, Q^{u}$ and the above estimate to estimate piecewise bounds of $\partial_{x}^{k} \chi_{e}(x)$. Using the above derivatives bound, the symbolic formula of $\partial_{x}^{k} \chi_{e}$, and the refined second order estimate in Section C.2.1 we can obtain sharp bounds for $\partial_{x}^{k} \chi_{e}$. Remark that we only apply the above estimate to $k \leq 15$.
D.1.2. Estimate of polynomial decay functions. For cutoff function $\chi_{e}\left(\frac{|x|-a}{b}\right)$ based on the exponential cutoff function (D.1), it has rapid change from $|x| \leq a$ to $|x| \geq a+b$, which is not very smooth in the computational domain if there are not enough mesh for $x$ with $a \leq|x| \leq b$. We apply these cutoff functions to the far-field, e.g. $|x| \geq 10$, where the mesh is relatively sparse. Thus, we need another function similar to a cutoff function that has a slower change than the exponential cutoff function. We consider

$$
\begin{equation*}
\chi(x)=\frac{x^{7}}{\left(1+x^{2}\right)^{7 / 2}}, \quad x \in \mathbb{R}_{+} \tag{D.2}
\end{equation*}
$$

and will use its rescaled version, e.g., $\chi\left(\frac{x-a}{b}\right)$, in our verification.
Firstly, we use induction to derive

$$
\partial_{x}^{k} \chi=\frac{p_{k}(x)}{\left(1+x^{2}\right)^{7 / 2+k}}, \quad p_{0}=x^{7}
$$

where $p_{k}(x)$ is a polynomial. A direct calculation yields

$$
\partial_{x}^{k+1} \chi=\frac{p_{k}^{\prime}(x)\left(1+x^{2}\right)-\left(\frac{7}{2}+k\right) \cdot 2 x p_{k}(x)}{\left(1+x^{2}\right)^{7 / 2+k+1}}
$$

Comparing the above two formulas, we yield

$$
p_{k+1}=p_{k}^{\prime}\left(1+x^{2}\right)-(7+2 k) x p_{k}(x)
$$

The first few terms are given by $p_{0}=x^{7}, p_{1}=7 x^{6}$. Using the recursive formula and $\operatorname{deg} p_{1}=$ 6 , we yield

$$
\begin{equation*}
\operatorname{deg} p_{k+1} \leq \operatorname{deg} p_{k}+1, \quad \operatorname{deg} p_{k} \leq k+5, \quad k \geq 1 \tag{D.3}
\end{equation*}
$$

Since $p_{k}$ is a polynomial, the above recursive formula shows that $p_{k+1}$ is also a polynomial.
To estimate $\partial_{x}^{k} \chi$, we decompose $p_{k}$ into the positive and the negative parts. Suppose that $p_{k}=\sum_{i} a_{i} x^{i}$. We have

$$
p_{k}=p_{k}^{+}-p_{k}^{-}, \quad p_{k}^{+}=\sum a_{i}^{+} x^{i}, \quad p_{k}^{-}=\sum a_{i}^{-} x^{i}
$$

For $x \geq 0, p_{k}^{+}, p_{k}^{-}$are increasing. Thus, for $x \in\left[x_{l}, x_{u}\right]$, we get

$$
\left|\partial_{x}^{k} \chi\right| \leq \frac{\max \left(p_{k}^{+}\left(x_{u}\right)-p_{k}^{-}\left(x_{l}\right), p_{k}^{-}\left(x_{u}\right)-p_{k}^{+}\left(x_{l}\right)\right)}{\left(1+x_{l}^{2}\right)^{7 / 2+k}}
$$

Next, we estimate $\partial_{x}^{k} \chi$ for large $x$. For $x \geq 2, k \geq 1$ and any polynomial $q(x)$ with nonnegative coefficients and $\operatorname{deg} q \leq k+5$, we yield

$$
x q^{\prime} \leq(k+5) q, \quad \frac{q^{\prime}\left(1+x^{2}\right)}{(7+2 k) x q} \leq \frac{\left(1+x^{2}\right)(k+5)}{(7+2 k) x^{2}} \leq \frac{5(k+5)}{4(7+2 k)}<1
$$

The first inequality follows by comparing the coefficients of $x q^{\prime}$ and $(k+5) q$, which are nonnegative. It follows

$$
\partial_{x} \frac{q}{\left(1+x^{2}\right)^{7 / 2+k}}=\frac{q^{\prime}\left(1+x^{2}\right)-(7 / 2+k) 2 x q}{\left(1+x^{2}\right)^{7 / 2+k+1}} \leq 0, \quad k \geq 1, x \geq 2
$$

Thus $\frac{q}{\left(1+x^{2}\right)^{7 / 2+k}}$ is decreasing. For $k \geq 1$ and $x \geq x_{l} \geq 2$, using (D.3) and the monotonicity, we yield

$$
\left|\partial_{x}^{k}(x)\right| \leq \frac{p_{k}^{+}(x)+p_{k}^{-}(x)}{\left(1+x^{2}\right)^{7 / 2+k}} \leq \frac{p_{k}^{+}\left(x_{l}\right)+p_{k}^{-}\left(x_{l}\right)}{\left(1+x_{l}^{2}\right)^{7 / 2+k}}
$$

For $k=0$, the estimate is trivial: $\chi(x) \leq 1$. Using these higher order derivative bounds, we can use the discrete values of $\partial_{x}^{k} \chi$ and the bound for $\partial_{x}^{k+2} \chi$ to obtain sharp bounds of $\partial_{x}^{k} \chi$.

Note that $\chi_{1}(x-a)=\frac{(x-a)_{+}^{7}}{\left(1+(x-a)^{2}\right)^{7 / 2}}$ is only $C^{6,1}$. Suppose that $a \in\left[x_{l}, x_{u}\right]$. Since $\chi_{1}$ is smooth on $x \leq a$ and on $x \geq a$, we can still use first order estimate to estimate $\partial_{x}^{k} \chi_{1}$ as follows

$$
\left|\partial_{x}^{k} \chi_{1}(x)\right| \leq \max _{\alpha \in\{l, u\}}\left|\partial_{x}^{k} \chi_{1}\left(x_{\alpha}\right)\right|+\max \left(\left\|\partial_{x}^{k+1} \chi_{1}\right\|_{\left.L^{\infty}\right]\left[x_{l}, a\right]}\left\|\partial_{x}^{k+1} \chi_{1}\right\|_{\left.L^{\infty}\right]\left[a, x_{u}\right]}\right)\left|x_{u}-x_{l}\right|
$$

D.1.3. Radial cutoff function. Now, we construct the radial cutoff functions for the far-field approximation terms of $\omega$ and $\phi$ as follows

$$
\begin{align*}
\chi(r) & =\chi_{1}\left(1-\chi_{2}\right)+\chi_{2}, \quad \chi_{1}(r)=\chi_{r a t i}\left(\frac{r-a_{1}}{l_{1}^{1 / 2}}\right), \quad \chi_{2}(r)=\chi_{\exp }\left(\frac{r-a_{2}}{9 a_{2}}\right)  \tag{D.4}\\
a_{1} & =10, \quad l_{1}=50000, \quad a_{2}=10^{5}
\end{align*}
$$

where $\chi_{\text {exp }}$ and $\chi_{\text {rati }}$ are defined in (D.1) and (D.2), respectively. Using the estimates of $\chi_{\text {rati }}, \chi_{\text {exp }}$ established in the last two sections, the Leibniz rule (A.6), and (C.11), we can evaluate $\chi$ on the grid points and estimate its derivative bounds.
D.2. Cutoff function near the origin. For the cutoff function $\kappa(x)$ used in Section 3, we choose it as follows

$$
\begin{equation*}
\kappa(x ; a, b)=\kappa_{1}\left(\frac{x}{a}\right)\left(1-\chi_{e}\left(\frac{x}{b}\right)\right), \quad \kappa_{1}(x)=\frac{1}{1+x^{4}}, \quad \kappa_{*}(x)=\kappa\left(x ; \frac{1}{3}, \frac{3}{2}\right), \tag{D.5}
\end{equation*}
$$

where $\chi_{e}$ is the cutoff function chosen in (D.1). We mostly use the cutoff $\kappa_{*}$. Since $\chi_{e}(y)=1$ for $y \geq 1$ and $\chi_{e}(y)=0$ for $y \leq 0$. The above cutoff function is supported in $x \leq a_{2}$. Using Taylor expansion, we have the following properties for $\kappa$

$$
\kappa_{1}\left(x / a_{1}\right)=1+O\left(x^{4}\right), \quad \kappa(x)=1+O\left(x^{4}\right)
$$

For the cutoff functions $\chi_{N F}$ in Section 4.2.1 in Part I [13] and $\chi_{\bar{\varepsilon}}$ in (C.17), we choose

$$
\begin{align*}
& \chi_{\bar{\varepsilon}}(x, y)=\kappa\left(x ; \nu_{\bar{\varepsilon}, 1}, \nu_{\bar{\varepsilon}, 2}\right) \kappa\left(y ; \nu_{\bar{\varepsilon}, 1}, \nu_{\bar{\varepsilon}, 2}\right), \quad \nu_{\bar{\varepsilon}, 1}=1 / 192, \quad \nu_{\bar{\varepsilon}, 2}=3 / 2 \\
& \chi_{\hat{\varepsilon}}(x, y)=\kappa_{*}(x) \kappa_{*}(y), \quad \chi_{N F}(x, y)=\kappa(x ; 2,10) \kappa(y ; 2,10)  \tag{D.6}\\
& f_{\chi, 1}=\Delta\left(\frac{x y^{3}}{6} \chi_{N F}(x, y)\right), \quad f_{\chi, 2}=x y \chi_{N F}(x, y), \quad f_{\chi, 3}=\frac{x^{2}}{2} \chi_{N F}(x, y)
\end{align*}
$$

For the cutoff function for the stream function (C.1), we choose

$$
\begin{equation*}
\chi_{\phi}=\kappa_{2}\left(\frac{x}{\nu_{4,1}}\right)\left(1-\chi_{e}\left(\frac{x}{\nu_{4,2}}\right)\right), \quad \kappa_{2}(x)=\frac{1}{1+x^{2}}, \quad \nu_{4,1}=2, \quad \nu_{4,2}=128 \tag{D.7}
\end{equation*}
$$

For $\kappa_{1}(x), \kappa_{2}(x)$, we use induction to obtain

$$
\partial_{x}^{k} \kappa_{1}(x)=\frac{P_{k}^{+}(x)-P_{k}^{-}(x)}{\left(1+x^{4}\right)^{k+1}}, \quad \partial_{x}^{k} \kappa_{2}(x)=\frac{R_{k}^{+}(x)-R_{k}^{-}(x)}{\left(1+x^{2}\right)^{k+1}}
$$

for some polynomials $P_{k}^{ \pm}, R_{k}^{ \pm}$with non-negative coefficients, and the same method as that in Section D.1.2 to estimate the derivatives of $\partial_{x}^{i} \kappa_{1}(x)$. The estimate of $\kappa_{1}$ is simpler since $\kappa_{1}$ has a simpler form. Using the Leibniz rule (A.6) and the triangle inequality, we can obtain estimate $\partial_{x}^{l} \kappa_{1}(x)$ in $[a, b]$. Then we use these derivative estimates for $\partial_{x}^{l+2} \kappa_{1}(x)$, evaluate $\kappa\left(x ; a_{1}, a_{2}\right)$ on the grid points, and then use (C.11) to obtain a sharp estimate of $\partial_{x}^{l} \kappa_{1}(x)$ on $[a, b]$. The same method applies to estimate $\kappa_{2}, \chi_{\phi}$.

For large $x$, e.g. $x \geq 100$, the above estimates can lead to a very large round off error. Instead, for $a \geq 2, a \in \mathbb{Z}_{+}$, we use the Taylor expansion

$$
F_{a}=\frac{1}{1+x^{a}}=\sum_{k \geq 0}(-1)^{k} x^{-a(k+1)}, \quad \partial_{x}^{i} F_{a}=\sum_{k \geq 0}(-1)^{k+i} C_{i, k} x^{-a(k+1)-i}, \quad C_{i, k}=\prod_{0 \leq j \leq i-1}(a(k+1)+j)
$$

We want to bound $\left|\partial_{x}^{i} F_{a}\right| \leq C_{i, 0}\left(1+C_{\varepsilon}\right) x^{-a-i}$ for $x \geq x_{l}=100, i \leq 20$. For $k \leq 20$, we bound

$$
C_{i, k} x^{-a(k+1)-i} \leq C_{i, k} x_{l}^{-(a-1) k} x^{-a-i-k} \leq C_{i, 0} \varepsilon_{1}^{-a-i-k}, \quad \varepsilon_{1} \triangleq \max _{i \leq 20, k \leq 20} x_{l}^{-(a-1) k} C_{i, k} C_{i, 0}^{-1}
$$

For the tail part $k>20$, we consider $G(k)=k \log x-i \log (1+k)$. Since $x>21, i \leq 20$, we get

$$
\partial_{k} G=\log x-\frac{i}{1+k} \geq \log x-1>\log 4-1>0, \quad G(k) \geq G(21)=21 \log x-i \log 21>0
$$

It follows $x^{k}>(1+k)^{i}$. Using $\frac{a(k+1)+j}{a+j} \leq 1+k, C_{i, k} \leq C_{i, 0}(1+k)^{i}$, and $a \geq 2$, we further get

$$
C_{i, k} x^{-a(k+1)-i} \leq x^{-k-a-i} C_{i, k} x^{-k} \leq x^{-k-a-i} C_{i, 0}(1+k)^{i} x^{-k} \leq C_{i, 0} x^{-k-a-i}, k>20
$$

Combining the above estimates and $x \geq x_{l}>10$, we obtain

$$
\left|\partial_{x}^{i} F_{a}\right| \leq C_{i, 0} x^{-a-i} C_{a}, \quad C_{a} \leq 1+\varepsilon_{1} \sum_{k=1}^{20} x^{-k}+\sum_{k \geq 21} x^{-k} \leq 1+\frac{\varepsilon_{1} x^{-1}}{1-x^{-1}}+\frac{x^{-21}}{1-x^{-1}} \leq 1+\frac{\varepsilon_{1}}{x_{l}-1}+x_{l}^{-20}
$$

D.3. Estimate of $\rho_{p}(y)$. We estimate the weight $\rho_{p}(y)$ C.8) in the representation of the stream function. Using symbolic computation, e.g., Matlab or Mathematica, we yield

$$
\begin{aligned}
& \partial_{x}^{9} \rho_{p}(y)=\frac{f_{2}(y)-f_{1}(y)}{(g(y))^{9}}, \quad g(y)=2+2 y+y^{2} \\
& f_{1}=288 y^{2}+672 y^{3}+504 y^{4}, \quad f_{2}=16+168 y^{6}+72 y^{7}+9 y^{8}
\end{aligned}
$$

Since $f_{1}, f_{2}, g \geq 0$ are increasing in $y \geq 0$, for $y \in\left[y_{l}, y_{u}\right]$, we yield

$$
\left|\partial_{x}^{9} \rho_{p}(y)\right| \leq \frac{\max \left(f_{2}\left(y_{u}\right)-f_{1}\left(y_{l}\right), f_{1}\left(y_{u}\right)-f_{2}\left(y_{l}\right)\right.}{\left(g\left(y_{l}\right)\right)^{9}}
$$

We have a trivial estimate similar to (C.11)

$$
\begin{equation*}
\max _{x \in I}|f(x)| \leq \max \left(\left|f\left(x_{l}\right)\right|,\left|f\left(x_{u}\right)\right|\right)+\frac{h}{2}\left\|f_{x}\right\|_{L^{\infty}(I)} \tag{D.8}
\end{equation*}
$$

which is useful if we do not have bound for $f_{x x}$.
Based on the above estimates, using the estimates (C.11), (D.8), ideas in Section C.2.1, and evaluating $\rho_{p}$ on some grid points, we can obtain piecewise sharp bounds for $\partial_{x}^{i} \rho_{p}$ for $i \leq 8$.

## Appendix E. Piecewise $C^{1 / 2}$ and Lipschitz estimates

In this section, we estimate the piecewise $C^{1 / 2}$ bound and Lipschitz bound for a function.
E.1. Hölder estimate of the functions. In the following two sections, we estimate the Hölder seminorms $[f]_{C_{x}^{1 / 2}}$ or $[f]_{C_{y}^{1 / 2}}$ of some function $f$, e.g. $f=\left(\partial_{t}-\mathcal{L}\right) \widehat{W}$ in (3.27), based on the previous $L^{\infty}$ estimates. We will develop two approaches.

Suppose that we have bounds for $\partial_{x} f, \partial_{y} f$ and $f$. Firstly, we consider the $C_{x}^{1 / 2}$ estimate. For $x_{1}<y_{1}$ and $x_{2}=y_{2}$, we have

$$
I=\frac{|f(x)-f(y)|}{|x-y|^{1 / 2}} \leq|x-y|^{1 / 2} \frac{1}{|x-y|} \int_{x_{1}}^{y_{1}}\left|f_{x}\left(z_{1}, x_{2}\right)\right| d z_{1}
$$

We further bound the average of $f_{x}$ piecewisely using the method in Appendix E. 2 to obtain the first estimate. We have a second estimate

$$
\begin{aligned}
|I| & =\left|\int_{x_{1}}^{y_{1}} f_{x}\left(z_{1}, x_{2}\right) d z\right| \cdot \frac{1}{|x-y|^{1 / 2}} \leq\left\|f_{x} x^{1 / 2}\right\|_{\infty} \int_{x_{1}}^{y_{1}} z_{1}^{-1 / 2} d z_{1} \cdot \frac{1}{|x-y|^{1 / 2}} \\
& \leq\left\|f_{x} x^{1 / 2}\right\|_{\infty} 2 \frac{y_{1}^{1 / 2}-x_{1}^{1 / 2}}{|x-y|^{1 / 2}}=\left\|f_{x} x^{1 / 2}\right\|_{\infty} \frac{2 \sqrt{y_{1}-x_{1}}}{\sqrt{x_{1}}+\sqrt{y_{1}}}
\end{aligned}
$$

We also have a trivial $L^{\infty}$ estimate

$$
|I| \leq\left\|f x_{1}^{-1 / 2}\right\|_{\infty} \frac{x_{1}^{1 / 2}+y_{1}^{1 / 2}}{|x-y|^{1 / 2}}, \quad|I| \leq\|f\|_{\infty} \frac{2}{|x-y|^{1 / 2}}
$$

Similar $L^{\infty}$ and Lipschitz estimates apply to $\|f\|_{C_{y}^{1 / 2}}$.
Near the origin, optimizing the above estimates, for $x_{2}=y_{2}$, we obtain

$$
\left|\frac{f(x)-f(y)}{|x-y|^{1 / 2}}\right| \leq \min \left(\left\|f_{x} x^{1 / 2}\right\|_{\infty} 2 t, \quad\left\|f x_{1}^{-1 / 2}\right\|_{\infty} t^{-1}\right), \quad t=\frac{\sqrt{y_{1}-x_{1}}}{\sqrt{x_{1}}+\sqrt{y_{1}}}
$$

In the $Y$-direction, $x_{1}=y_{1}, x_{2} \leq y_{2}$, we use
$I_{Y}=\left|\frac{f(x)-f(y)}{|x-y|^{1 / 2}}\right| \leq \frac{1}{\left|x_{2}-y_{2}\right|^{1 / 2}} \int_{x_{2}}^{y_{2}}\left|f_{y}\left(x_{1}, z_{2}\right)\right||z|^{1 / 2} \cdot|z|^{-1 / 2} d z_{2} \leq\left\|f_{y}|x|^{1 / 2}\right\|_{\infty} \frac{\left|x_{2}-y_{2}\right|^{1 / 2}}{|x|^{1 / 2}}$,
$I_{Y} \leq\left\|f x^{-1 / 2}\right\|_{\infty} \frac{2 x_{1}^{1 / 2}}{\left|x_{2}-y_{2}\right|^{1 / 2}}$.
Since $x_{1} \leq|x|$, the minimum of these two estimates are not singular near $x=0$. In particular, we optimize two estimates to estimate $I_{Y}$.

From the above estimates, to obtain sharp Hölder estimate of $f$, we estimate the piecewise bounds of $f, f x_{1}^{-1 / 2}, f|x|^{-1 / 2}, f_{x}, f_{y}, f_{x}\left|x_{1}\right|^{1 / 2}, f_{y}|x|^{1 / 2}$, which are local quantities. These estimates can be established using the piecewise bounds of $\partial_{x}^{i} \partial_{y}^{j} f$ and the methods in Section 8 in the supplementary material II [11].
E.1.1. The second approach of Hölder estimate. We develop an additional approach to estimate $I(f)=\frac{|f(x)-f(z)|}{|x-z|^{1 / 2}}$ that is sharper if $|x-z|$ is not small and $f$ is smooth. We need the grid point values and derivative bounds of $f$.

We estimate $I(f)=\frac{|f(x)-f(z)|}{|x-z|^{1 / 2}}$ for $x \in\left[x_{l}, x_{u}\right], z \in\left[z_{l}, z_{u}\right]$. Denote by $\hat{f}$ the linear approximation of $f$ with $\hat{f}\left(x_{i}\right)=f\left(x_{i}\right)$ on the grid point $x_{i}$. We have the following Lemma.

Lemma E.1. Suppose that $f$ is linear on $\left[x_{l}, x_{u}\right],\left[z_{l}, z_{u}\right]$ and $x_{l} \leq x_{u} \leq z_{l} \leq z_{u}$. Then we have

$$
\max _{x \in\left[x_{l}, x_{u}\right], z \in\left[z_{l}, z_{u}\right]} \frac{|f(x)-f(z)|}{|x-z|^{1 / 2}}=\max _{\alpha, \beta \in\{l, u\}} \frac{\left|f\left(x_{\alpha}\right)-f\left(z_{\beta}\right)\right|}{\left|x_{\alpha}-z_{\beta}\right|^{1 / 2}} .
$$

The above Lemma shows that for the linear interpolation of $f$, the maximum of the Holder norm is achieved at the grid point.

Proof. Denote by $M$ the right hand side in the Lemma. Clearly, it suffices to prove that the left hand side is bounded by $M$. We fix $x \in\left[x_{l}, x_{u}\right], z \in\left[z_{l}, z_{u}\right]$. Suppose that

$$
x=a_{l} x_{l}+a_{u} x_{u}, \quad z=b_{l} z_{l}+b_{u} z_{u}, a_{u}+a_{l}=1, \quad b_{l}+b_{u}=1
$$

for $a_{l}, b_{l} \in[0,1]$. Denote

$$
m_{\alpha \beta}=a_{\alpha} b_{\beta}, \quad \alpha, \beta \in\{l, u\}
$$

Since $f(x)$ is linear on $\left[x_{l}, x_{u}\right]$ and $\left[z_{l}, z_{u}\right]$, we get

$$
f(x)=a_{l} f\left(x_{l}\right)+a_{u} f\left(x_{u}\right), \quad f(z)=b_{l} f\left(z_{l}\right)+b_{u} f\left(z_{u}\right)
$$

For any function $g$ linear on $\left[x_{l}, x_{u}\right]$, $\left[z_{l}, z_{u}\right]$, e.g., $g(x)=1, g(x)=x, g(x)=f(x)$, we have

$$
\begin{equation*}
g(x)=\sum_{\alpha, \beta \in\{l, u\}} m_{\alpha \beta} g\left(x_{\alpha}\right), \quad g(z)-g(x)=\sum_{\alpha, \beta \in\{l, u\}} m_{\alpha \beta}\left(g\left(z_{\beta}\right)-g\left(x_{\alpha}\right)\right), \tag{E.1}
\end{equation*}
$$

Using the above identities and the triangle inequality and the definition of $M$, we yield

$$
|f(x)-f(z)|=\left|\sum_{\alpha, \beta \in\{l, u\}} m_{\alpha \beta}\left(f\left(x_{\alpha}\right)-f\left(z_{\beta}\right)\right)\right| \leq \sum_{\alpha, \beta \in\{l, u\}} m_{\alpha \beta} M\left|x_{\alpha}-z_{\beta}\right|^{1 / 2}
$$

Using the Cauchy-Schwarz inequality, $\left|x_{\alpha}-z_{\beta}\right|=z_{\beta}-x_{\alpha}$ and (E.1), we establish

$$
\begin{aligned}
|f(x)-f(z)| & \leq \sum_{\alpha, \beta \in\{l, u\}} m_{\alpha \beta} \sum_{\alpha, \beta \in\{l, u\}} m_{\alpha \beta} M\left|x_{\alpha}-z_{\beta}\right|^{1 / 2}=\sum_{\alpha, \beta \in\{l, u\}} m_{\alpha \beta} M\left|x_{\alpha}-z_{\beta}\right|^{1 / 2} \\
& =M\left(\sum_{\alpha, \beta \in\{l, u\}} m_{\alpha \beta}\left(z_{\beta}-x_{\alpha}\right)\right)^{1 / 2}=M(z-x)^{1 / 2}
\end{aligned}
$$

The desired result follows.
We generalize Lemma E. 1 to 2D as follows.
Lemma E.2. Let $I_{x}=\left[x_{l}, x_{u}\right], I_{z}=\left[z_{l}, z_{u}\right], I_{y}=\left[y_{l}, y_{u}\right]$ with $x_{l} \leq x_{u} \leq z_{l} \leq z_{u}$. Suppose that $f$ is linear on $I_{x} \times I_{y}$ and $I_{z} \times I_{y}$. Then we have

$$
\max _{x \in I_{x}, z \in I_{z}, y \in I_{y}} \frac{|f(x, y)-f(z, y)|}{|x-z|^{1 / 2}}=\max _{\alpha, \beta, \gamma \in\{l, u\}} \frac{\left|f\left(x_{\alpha}, y_{\gamma}\right)-f\left(z_{\beta}, y_{\gamma}\right)\right|}{\left|x_{\alpha}-z_{\beta}\right|^{1 / 2}} .
$$

Proof. Note that the function $I(x, z, y)=\frac{f(x, y)-f(z, y)}{|x-z|^{1 / 2}}$ is linear in $y$. We get

$$
\mid I\left(x, z, y \mid=\max \left(\left|I\left(x, z, y_{l}\right)\right|,\left|I\left(x, z, y_{u}\right)\right|\right)\right.
$$

Applying Lemma E. 1 completes the proof.
Let $\hat{f}$ be the linear interpolation of $f$. Suppose that $x \in I_{x}, z \in I_{z}, y \in I_{y}$ with $x_{u} \leq z_{l}$. Using the above estimates and notations, we can bound $I(f)$ as follows

$$
\begin{aligned}
I(f) & =\frac{|f(z, y)-f(x, y)|}{|x-z|^{1 / 2}} \leq \frac{|\hat{f}(x, y)-f(x, y)|+|\hat{f}(z, y)-f(z, y)|}{|x-z|^{1 / 2}}+\max _{\alpha, \beta, \gamma \in\{l, u\}} \frac{\left|f\left(x_{\alpha}, y_{\gamma}\right)-f\left(z_{\beta}, y_{\gamma}\right)\right|}{\left|x_{\alpha}-z_{\beta}\right|^{1 / 2}} \\
& \leq\left(\frac{h_{x}^{2}}{8}\left\|f_{x x}\right\|_{I_{x} \times I_{y}}+\frac{h_{y}^{2}}{8}\left(\left\|f_{y y}\right\|_{I_{x} \times I_{y}}+| | f_{y y} \|_{I_{z} \times I_{y}}\right)+\frac{h_{z}^{2}}{8}\left\|f_{x x}\right\|_{I_{z} \times I_{y}}\right)|x-z|^{-1 / 2}+M .
\end{aligned}
$$

E.2. Piecewise derivative bounds. In this section, we discuss how to obtain the sharp bound of $\frac{p(b)-p(a)}{b-a}$ using piecewise derivative bounds of $p$.

Suppose that $\left|p^{\prime}(y)\right| \leq C_{i}, y \in I_{i}=\left[y_{i}, y_{i+1}\right]$. For any $a \in I_{k}, b \in I_{l}, a<b$, we have the bound

$$
\begin{aligned}
& |p(b)-p(a)| \leq \int_{a}^{b}\left|p^{\prime}(y)\right| d y \leq\left|y_{k+1}-a\right| C_{k}+\left|b-y_{l}\right| C_{l}+\sum_{k+1 \leq m \leq l-1} C_{m}\left(y_{m+1}-y_{m}\right) \\
= & \left(y_{k+1}-a\right) C_{k}+\left(b-y_{l}\right) C_{l}+M_{k l}\left(y_{l}-y_{k+1}\right) \mathbf{1}_{l \geq k+1},
\end{aligned}
$$

where $M_{k l}$ is defined below:

$$
M_{k l}=\left|y_{l}-y_{k+1}\right|^{-1}\left(\sum_{k+1 \leq m \leq l-1} C_{m}\left|y_{m+1}-y_{m}\right|\right)
$$

Next, we want to bound $\frac{|p(b)-p(a)|}{|b-a|}$. If $l-k \leq 1$, we get

$$
|p(b)-p(a)| \leq(b-a) \max \left(C_{k}, C_{l}\right)
$$

Otherwise, if $l \geq k+2$, we have

$$
|p(b)-p(a)| \leq\left(y_{k+1}-a\right)\left(C_{k}-M_{k l}\right)+\left(b-y_{l}\right)\left(C_{l}-M_{k l}\right)+M_{k l}(b-a)
$$

Since $\frac{y_{k+1}-a}{b-a}$ is decreasing in $a$ and $b, \frac{b-y_{l}}{b-a}$ is increasing in $b$ and $a$, we get

$$
0 \leq \frac{y_{k+1}-a}{b-a} \leq \frac{y_{k+1}-y_{k}}{y_{l}-y_{k}}, \quad 0 \leq \frac{b-y_{l}}{b-a} \leq \frac{y_{l+1}-y_{l}}{y_{l+1}-y_{k+1}}
$$

Using the above estimates, for $a \in I_{k}, b \in I_{l}$, we obtain

$$
\frac{|p(b)-p(a)|}{|b-a|} \leq \max \left(C_{k}-M_{k l}, 0\right) \frac{y_{k+1}-y_{k}}{y_{l}-y_{k}}+\max \left(C_{l}-M_{k l}, 0\right) \frac{y_{l+1}-y_{l}}{y_{l+1}-y_{k+1}}+M_{k l} .
$$

For uniform mesh, i.e. $y_{i+1}-y_{i}=h$, we can simplify the above estimate as follows

$$
\frac{|p(b)-p(a)|}{|b-a|} \leq \frac{\left(\max \left(C_{k}-M_{k l}, 0\right)+\max \left(C_{l}-M_{k l}, 0\right)\right)}{l-k}+M_{k l}, \quad M_{k l}=\frac{1}{l-k-1} \sum_{k+1 \leq m \leq l-1} C_{m} .
$$

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# SUPPLEMENTARY MATERIAL FOR "STABLE NEARLY SELF-SIMILAR BLOWUP OF THE 2D BOUSSINESQ AND 3D EULER EQUATIONS WITH SMOOTH DATA II: RIGOROUS NUMERICS" 

JIAJIE CHEN AND THOMAS Y. HOU


#### Abstract

In this supplementary material, we provide all the remaining details in the estimate of integrals related to the nonlocal terms $\mathbf{u}, \nabla \mathbf{u}$. In Section 5 we estimate the integrals in the upper bounds of the sharp Hölder estimate established in Section 3 in 3. In Section 6 we provide detailed formulas in the estimate of $\nabla \mathbf{u}, u /\left|x_{1}\right|$ in the singular regions and some explicit integrals. In Section 7 we generalize the estimate of integrals of $\mathbf{u}(x), \nabla \mathbf{u}(x)$ in Section 4 in Part II 2 from $x=O(1)$ to $x$ either close to 0 or $x$ very large. In Section 8, we use grid points and piecewise derivative bounds estimated in Appendix in Part II 2] to construct three interpolating polynomials in 2D with rigorous error estimates. Then we use these interpolating polynomials to obtain sharp piecewise estimates of functions, e.g. the residual error.


## 5. Computation of the sharp constants for in the Hölder estimates

In this section, we estimate the integrals in the upper bounds of the sharp Hölder estimate established in Section 3 in [3].
5.1. Integrals related to the kernels. Firstly, we derive some analytic integral formulas. The kernels associated with $\nabla u$ are given by

$$
\begin{equation*}
G(y)=-\frac{1}{2} \log |y|, \quad K_{1}(y) \triangleq G_{x y}=\frac{y_{1} y_{2}}{|y|^{4}}, \quad K_{2}(y) \triangleq G_{x x}=-G_{y y}=\frac{1}{2} \frac{y_{1}^{2}-y_{2}^{2}}{|y|^{4}} \tag{5.1}
\end{equation*}
$$

and we drop the factor $\frac{1}{\pi}$ for simplicity.
5.1.1. Basic Lemmas. We use the follwing Lemma from Appendix B. 1 in Part I 3 to estimate $K_{2}$.

Lemma 5.1. Suppose that $f \in L^{\infty}$, is Hölder continuous near 0. For $0<a, b<\infty$ and $Q=[0, a] \times[0, b],[0, a] \times[-b, 0],[-a, 0] \times[0, b]$, or $[-a, 0] \times[-b, 0]$, we have
P.V. $\int_{Q} K_{2}(y) f(y) d y=\lim _{\varepsilon \rightarrow 0} \int_{Q \cap\left|y_{1}\right| \geq \varepsilon} K_{2}(y) f(y) d y-\frac{\pi}{8} f(0)=\lim _{\varepsilon \rightarrow 0} \int_{Q \cap\left|y_{2}\right| \geq \varepsilon} K_{2}(y) f(y) d y+\frac{\pi}{8} f(0)$.

We have the following estimate for the Green function from Appendix B. 1 in Part II [2].
Lemma 5.2. Denote $r=\left(x^{2}+y^{2}\right)^{\frac{1}{2}}$ and $G(x, y)=-\frac{1}{2} \log r$. For any $i, j \geq 0$ with $i+j \geq 1$, we have

$$
\left|\partial_{x}^{i} \partial_{y}^{j} G(x, y)\right| \leq \frac{1}{2}(i+j-1)!\cdot r^{-i-j}
$$

Consider the odd extension of $\omega$ in $y$ from $\mathbb{R}_{2}^{+}$to $\mathbb{R}_{2}$

$$
\begin{equation*}
W(y)=\omega(y) \text { for } y_{2} \geq 0, \quad W(y)=-\omega\left(y_{1},-y_{2}\right) \text { for } y_{2}<0 \tag{5.2}
\end{equation*}
$$

$W$ is odd in both $y_{1}$ and $y_{2}$ variables.

We have the following indefinite integral formulas for $K_{1}, K_{2}$ (5.1)

$$
\begin{aligned}
& \int K_{1}(y) d y_{1} d y_{2}=-\frac{1}{2} \log |y|+C, \quad \int K_{2}(y) d y=\frac{1}{2} \arctan \frac{y_{1}}{y_{2}}+C=-\frac{1}{2} \arctan \frac{y_{2}}{y_{1}}+C, \\
& \int K_{1}(y) d y_{1}=-\frac{1}{2} \frac{y_{2}}{|y|^{2}}+C, \quad \int K_{1}(y) d y_{2}=-\frac{1}{2} \frac{y_{1}}{|y|^{2}}+C \\
& \int K_{2}(y) d y_{1}=-\frac{1}{2} \frac{y_{1}}{|y|^{2}}+C, \quad \int K_{2}(y) d y_{2}=\frac{1}{2} \frac{y_{2}}{|y|^{2}}+C
\end{aligned}
$$

For $K(y)=-\frac{y_{i}}{2|y|^{2}}=\partial_{y_{i}} G$ (5.1), we have

$$
\begin{align*}
\int \partial_{y_{2}} G d y & =\int G d y_{1}
\end{align*}=\frac{1}{4}\left(2 y_{1}-2 y_{2} \arctan \frac{y_{1}}{y_{2}}-y_{1} \log \left(y_{1}^{2}+y_{2}^{2}\right)\right)+C,
$$

5.1.2. Formulas for integrals with a half power. We introduce

$$
\begin{equation*}
f_{s}(t) \triangleq \int_{0}^{t} \frac{s^{2}}{1+s^{4}} d s=\frac{1}{2} \int_{0}^{t^{1 / 2}} \frac{1}{1+s^{2}} s^{1 / 2} d s \tag{5.5}
\end{equation*}
$$

Using changes of variables $s=p s$ and $s=t^{2}$, we yield

$$
\begin{align*}
\int_{a}^{b} \frac{p}{p^{2}+s^{2}} s^{1 / 2} d s & =p^{1 / 2} \int_{a / p}^{b / p} \frac{1}{1+s^{2}} s^{1 / 2} d s=2 p^{1 / 2} \int_{\sqrt{a / p}}^{\sqrt{b / p}} \frac{t^{2}}{1+t^{4}} d t  \tag{5.6}\\
& =2 p^{1 / 2}\left(f_{s}(\sqrt{b / p})-f_{s}(\sqrt{a / p})\right)
\end{align*}
$$

Clearly $f_{s}$ is increasing. Using the Beta function $B(x, 1-x)=\frac{\pi}{\sin (\pi x)}$ and $t=s^{1 / 4}$ we have

$$
f_{s}(\infty)=\frac{1}{4} \int_{0}^{\infty} \frac{t^{1 / 2}}{1+t} t^{-3 / 4} d t=\frac{1}{4} \int_{0}^{\infty} \frac{t^{-1 / 4}}{1+t} d t=\frac{1}{4} B\left(\frac{3}{4}, \frac{1}{4}\right)=\frac{1}{4} \frac{\pi}{\sin \frac{\pi}{4}}=\frac{\pi}{2 \sqrt{2}}
$$

For $a, b, c, d \geq 0$, we introduce

$$
\begin{equation*}
F_{K_{2}, h}(a, b, c, d)=\int_{[a, b] \times[c, d]}\left|K_{2}(y)\right| y_{2}^{1 / 2} d y \tag{5.7}
\end{equation*}
$$

Clearly, it suffices to study $F_{K_{2}, h}(0, a, 0, b)$ for any $a, b>0$. Then we have

$$
\begin{equation*}
F_{K_{2}, h}(a, b, c, d)=\left|F_{K_{2}, h}(a, c)+F_{K_{2}, h}(b, d)-F_{K_{2}, h}(a, d)-F_{K_{2}, h}(b, c)\right| \tag{5.8}
\end{equation*}
$$

Denote $m=\min (a, b)$. Using (5.3) and (5.6), we yield

$$
\begin{align*}
& \int_{0}^{m} \int_{0}^{m}\left|K_{2}(y)\right| y_{2}^{1 / 2} d y=\int_{0}^{m} y_{2}^{1 / 2} d y_{2}\left(\int_{y_{2}}^{m} K_{2}(y)-\int_{0}^{y_{2}} K_{2}(y)\right) d y_{1}  \tag{5.9}\\
= & \frac{1}{2} \int_{0}^{m}\left(-\left.\frac{y_{1}}{|y|^{2}}\right|_{y_{2}} ^{m}+\left.\frac{y_{1}}{|y|^{2}}\right|_{0} ^{y_{2}}\right) y_{2}^{1 / 2} d y_{2}=\frac{1}{2} \int_{0}^{m}\left(\frac{2 y_{2}}{2 y_{2}^{2}}-\frac{m}{m^{2}+y_{2}^{2}}\right) y_{2}^{1 / 2} d y_{2}=\sqrt{m}-\sqrt{m} f_{s}(1) .
\end{align*}
$$

Next, we consider the integral in $R=[0, a] \times[0, b] \backslash[0, m]^{2}$. If $a>b$, we get $y_{1}>y_{2}, K_{2}(y)>0$ in $R$ and

$$
\begin{aligned}
\left|\int_{b}^{a} \int_{0}^{b} K_{2}(y) y_{2}^{1 / 2} d y\right| & =\left|\int_{0}^{b}\left(-\left.\frac{1}{2} \frac{y_{1}}{|y|^{2}}\right|_{b} ^{a}\right) y_{2}^{1 / 2} d y_{2}\right|=\left|\frac{1}{2} \int_{0}^{b}\left(\frac{b}{b^{2}+y_{2}^{2}}-\frac{a}{a^{2}+y_{2}^{2}}\right) y_{2}^{1 / 2} d y_{2}\right| \\
& =\left|b^{1 / 2} f_{s}(1)-a^{1 / 2} f_{s}(\sqrt{b / a})\right|
\end{aligned}
$$

If $a<b$, we yield $y_{1}<y_{2}, K_{2}(y)<0$ in $R$ and

$$
\left.\left|\int_{0}^{a} \int_{a}^{b}-K_{2}(y) y_{2}^{1 / 2} d y\right|=\left|\frac{1}{2} \int_{a}^{b} \frac{y_{1}}{|y|^{2}}\right|_{0}^{a} y_{2}^{1 / 2} d y_{2}\left|=\left|\frac{1}{2} \int_{a}^{b} \frac{a}{a^{2}+y_{2}^{2}} y_{2}^{1 / 2} d y_{2}\right|=a^{1 / 2}\right| f_{s}(\sqrt{b / a})-f_{s}(1) \right\rvert\,
$$

For $0<a<b$, we introduce

$$
\begin{equation*}
F_{d i a g}(a, b) \triangleq \int_{[a, b]^{2}}\left|K_{2}(y)\right| d y=\frac{1}{2} \log \left(\frac{b}{a}\right)-\left(\frac{\pi}{4}-\arctan \frac{a}{b}\right) \tag{5.10}
\end{equation*}
$$

To prove the identity, since $K_{2}(y)$ is symmetric in $y_{1}, y_{2}$, using (5.3), we yield

$$
\begin{aligned}
& \int_{[a, b]^{2}}\left|K_{2}(y)\right| d y=2 \int_{a}^{b} \int_{a}^{y_{1}} K_{2}(y) d y=\left.\int_{a}^{b} \frac{y_{2}}{|y|^{2}}\right|_{a} ^{y_{1}} d y_{1}=\int_{a}^{b} \frac{1}{2 y_{1}}-\frac{a}{y_{1}^{2}+a^{2}} d y_{1} \\
= & \frac{1}{2} \log \frac{b}{a}-\left.\arctan \frac{y_{1}}{a}\right|_{a} ^{b}=\frac{1}{2} \log \frac{b}{a}-\left(\arctan \frac{b}{a}-\frac{\pi}{4}\right)=\frac{1}{2} \log \frac{b}{a}-\left(\frac{\pi}{4}-\arctan \frac{a}{b}\right) .
\end{aligned}
$$

Denote some constants

$$
\begin{align*}
C_{K_{2}, u p} & \triangleq \int_{0}^{1} \int_{y_{1}}^{1}\left|K_{2}(y)\right| \cdot\left|\frac{y_{1}^{2}}{y_{2}}-y_{2}\right|^{1 / 2} d y, \quad C_{K_{2}, l o w} \triangleq \int_{0}^{1} \int_{0}^{y_{1}^{2}}\left|K_{2}(y)\right| \cdot\left|y_{2}\right|^{1 / 2} d y_{2}  \tag{5.11}\\
C_{K_{2}} & \triangleq C_{K_{2}, u p}+C_{K_{2}, \text { low }}
\end{align*}
$$

Using (5.3), (5.5), and $1 \geq y_{1} \geq y_{2}^{1 / 2} \geq y_{2}$, we can simplify the integral in $C_{K_{2}, \text { low }}$ as follows

$$
\begin{align*}
C_{K_{2}, \text { low }} & =\int_{0}^{1} y_{2}^{1 / 2} d y_{2} \int_{y_{2}^{1 / 2}}^{1} K_{2}(y) d y_{1}=\int_{0}^{1}\left(-\left.\frac{1}{2} \frac{y_{1}}{|y|^{2}}\right|_{y_{2}^{1 / 2}} ^{1}\right) y_{2}^{1 / 2} d y_{2} \\
& =\frac{1}{2} \int_{0}^{1}\left(\frac{y_{2}^{1 / 2}}{y_{2}^{2}+y_{2}}-\frac{1}{y_{2}^{2}+1}\right) y_{2}^{1 / 2} d y_{2}=\frac{1}{2}\left(\log (2)-2 f_{s}(1)\right) \tag{5.12}
\end{align*}
$$

For $C_{K_{2}, u p}$, we follow the strategies in Section [5.2, In a small region $[0, \varepsilon]^{2}$ near the singularity $y=0$, since $0 \leq y_{2}-\frac{y_{1}^{2}}{y_{2}} \leq y_{2}$, we bound the integrand by $\left|K_{2}(y) \| y_{2}\right|^{1 / 2}$ and follow (5.9) to get $\int_{0}^{\varepsilon} \int_{y_{1}}^{\varepsilon}\left|K_{2}(y)\right|\left|y_{2}\right|^{1 / 2} d y=\left|\int_{0}^{\varepsilon} y_{2}^{1 / 2} d y_{2} \int_{0}^{y_{2}} K_{2}(y) d y_{1}\right|=\left.\frac{1}{2} \int_{0}^{\varepsilon} y_{2}^{1 / 2} \frac{y_{1}}{|y|^{2}}\right|_{0} ^{y_{2}} d y_{2}=\frac{1}{2} \int_{0}^{\varepsilon} \frac{y_{2}^{3 / 2}}{2 y_{2}^{2}} d y_{2}=\frac{1}{2} \varepsilon^{1 / 2}$.

We introduce $f_{h}$ similar to $f_{s}$

$$
\begin{equation*}
f_{h}(a) \triangleq \int_{0}^{a} \frac{s^{3 / 2}}{1+s^{2}} d s=2 \int_{0}^{a^{1 / 2}} \frac{s^{4}}{1+s^{4}} d s \tag{5.13}
\end{equation*}
$$

Using a change of variables, we obtain

$$
\begin{equation*}
\int_{0}^{a} \frac{s^{3 / 2}}{p^{2}+s^{2}} d s=p^{1 / 2} \int_{0}^{a / p} \frac{s^{3 / 2}}{1+s^{2}} d s=p^{1 / 2} f_{h}(a / p) \tag{5.14}
\end{equation*}
$$

5.1.3. Integral formulas for $K_{1}, K_{2}$. We discuss how to compute the integrals

$$
J_{i j}(a, b, c, d)=\int_{[a, b] \times[c, d]}\left|K(y) y_{1}^{i} y_{2}^{j}\right| d y, \quad 1 \leq i+j, \quad K=K_{l}, \quad K=\partial_{k} K_{l}
$$

Due to symmetry of $\left|K(y) y_{1}^{i} y_{2}^{j}\right|$, we can assume that $0 \leq a \leq b, 0 \leq c \leq d$. For $K=K_{1}=$ $\frac{y_{1} y_{2}}{|y|^{4}}$, the computation is simple since $K_{1}$ has a fixed sign in each quardrant and $\mid K_{1}(y) y_{1}^{i} y_{2}^{\mid}=$ $K_{1}(y) y_{1}^{i} y_{2}^{j}$. For $K=K_{2}, \nabla K$, we have

$$
K_{2}=\frac{1}{2} \frac{y_{1}^{2}-y_{2}^{2}}{|y|^{4}}, \quad \partial_{1}^{3} G=\frac{y_{1}\left(-y_{1}^{2}+3 y_{2}^{2}\right)}{|y|^{6}}, \quad \partial_{1}^{2} \partial_{2} G=\frac{y_{2}\left(-3 y_{1}^{2}+y_{2}^{2}\right)}{|y|^{6}},
$$

where $G$ is the Green function (5.1) and is harmonic. The above kernels can change sign in $[a, b] \times[c, d]$, and the sign is determined by $S(y)=m^{2} y_{1}^{2}-y_{2}^{2}$ for some $m>0$. We fix a kernel $K=K_{2}$ or $\partial_{m} K_{n}$ for some $m, n=1,2$ and $i, j$. We have

$$
F(y) \triangleq K(y) y_{1}^{i} y_{2}^{j}, \quad|F|=\left|K(y) y_{1}^{i} y_{2}^{j}\right|=\left|m^{2} y_{1}^{2}-y_{2}^{2}\right| P(y)
$$

for some functions $P(y)$ having a fixed sign in $\mathbb{R}_{2}^{++}$. For example, $K_{2}(y) y_{1} y_{2}=\left(y_{1}^{2}-y_{2}^{2}\right) \frac{y_{1} y_{2}}{2|y|^{4}}, P=$ $\frac{y_{1} y_{2}}{2|y|^{4}}$. Denote the indefinite integral

$$
\begin{equation*}
I_{y}\left(y_{1}, y_{2}\right) \triangleq \int F(y) d y_{2}, \quad I_{x y} \triangleq \int F(y) d y_{1} d y_{2}, \quad I_{\text {diag }} \triangleq \int I_{y}\left(y_{1}, m y_{1}\right) d y_{1} \tag{5.15}
\end{equation*}
$$

which can be derived using symbolic computation. The constant in the indefinite integral formula will be cancelled when we apply it to evaluate an integral in a specific domain. Next, we evaluate

$$
S_{d i a g}^{l}(p, q) \triangleq \int_{p}^{q} \int_{m p}^{m y_{1}}|F(y)| d y, \quad S_{d i a g}^{u}(p, q) \triangleq \int_{p}^{q} \int_{m y_{1}}^{m q}|F(y)| d y, \quad S_{d i a g}=S_{d i a g}^{l}+S_{d i a g}^{u}
$$

The domain of the integrals $S_{d i a g}^{\alpha}, \alpha=l, u$ is a triangle, and $S_{d i a g}^{l}, S_{\text {diag }}^{u}$ denote the integral in the lower and the upper triangle $[p, q] \times[m p, m q]$, respectively. We first evaluate $S_{d i a g}^{l}$. Since $y_{2} \leq m y_{1}$ in the domain, $F(y)$ has a fixed sign in the domain of $S_{\text {diag }}^{l}$ and
$S_{\text {diag }}^{l}(p, q)=\left|\int_{p}^{q} I_{y}\left(y_{1}, \cdot\right)\right|_{m p}^{m y_{1}} d y_{1}\left|=\left|\int_{p}^{q} I_{y}\left(y_{1}, m y_{1}\right)-I_{y}\left(y_{1}, m p\right) d y_{1}\right|=\left|I_{\text {diag }}(\cdot)\right|_{p}^{q}-I_{x y}\left(y_{1}, m p\right)\right|_{p}^{q} \mid$.
Similarly, we get

$$
\begin{equation*}
S^{u} d i a g=\left|\int_{p}^{q} I_{y}\left(y_{1}, m q\right)-I_{y}\left(y_{1}, m y_{1}\right) d y_{1}\right|=\left|I_{x y}\left(y_{1}, m q\right)\right|_{p}^{q}-\left.I_{\operatorname{diag}}(\cdot)\right|_{p} ^{q} \mid \tag{5.17}
\end{equation*}
$$

and then yield the formula for $S_{d i a g}$.
Next, we consider the integral of $|F(y)|$ in a general domain $Q=[a, b] \times[c, d] \subset \mathbb{R}_{2}^{++}$. We can decompose the regions into several parts according to the intersection between $Q$ and the line $y_{2}=m y_{1}$, and then apply the integral formulas $S_{d i a g}$ and $I_{x y}$ (5.15).

Fixed sign: (1) $c \geq m b$, (6) $d \leq m a$. In these two cases, we have $y_{2} \geq m y_{1}$ for any $y \in Q$ or $y_{2} \leq m y_{1}$ for any $y \in Q$. Thus, $F(y)$ has a fixed $\operatorname{sign}$ in $Q$ and we can use the analytic formula $I_{x y}$ (5.15) to evaluate $S$.
(2) $m a \leq c \leq m b, m b \leq d$. We decompose $Q$ as follows

$$
[a, b] \times[c, d]=[a, c / m] \times[c, d] \cup[c / m, b] \times[m b, d] \cup[c / m, b] \times[c, m b] \triangleq Q_{1} \cup Q_{2} \cup Q_{3}
$$

(3) $m a \leq c \leq m b, d \leq m b$. We decompose $Q$ as follows

$$
[a, b] \times[c, d]=[a, c / m] \times[c, d] \cup[d / m, b] \times[c, d] \cup[c / m, d / m] \times[c, d] \triangleq Q_{1} \cup Q_{2} \cup Q_{3}
$$

(4) $c \leq m a, m b \leq d$. We decompose $Q$ as follows

$$
Q=[a, b] \times[c, m a] \cup[a, b] \times[m b, d] \cup[a, b] \times[m a, m b]=Q_{1} \cup Q_{2} \cup Q_{3}
$$

(5) $c \leq m a, m a \leq d \leq m b$. We decompose $Q$ as follows

$$
Q=[a, d / m] \times[c, m a] \cup[d / m, b] \times[c, d] \cup[a, d / m] \times[m a, d] \triangleq Q_{1} \cup Q_{2} \cup Q_{3}
$$

In each case (2)-(5), for $Q_{1}, Q_{2},\left(m y_{1}\right)^{2}-y_{2}^{2}$ and $F$ have fixed signs and we use (5.15) to evaluate the integral. For $Q_{3}$, we apply (5.16), (5.17). See Figure 1 for an illustration of decompositions in different cases.

We need to estimate the integral

$$
\int_{Q}|F| d y, \quad|F|=\left|\partial_{2}\left(K_{2}(y) y_{2}\right)\right|=\left|\frac{y_{1}^{4}-6 y_{1}^{2} y_{2}^{2}+y_{2}^{4}}{2|y|^{6}}\right|
$$

in $Q=[a, b] \times[c, d]$. Without loss of generality, we assume $Q \subset \mathbb{R}_{2}^{++}$. We define

$$
p_{1}=y_{1}^{4}+y_{2}^{4}, \quad p_{2}=6 y_{1}^{2} y_{2}^{2}, \quad b(Q)=\max \left(p_{1}^{u}-p_{2}^{l}, p_{2}^{u}-p_{1}^{l}\right), \quad f(Q)=1-\left(p_{1}^{l} \leq p_{2}^{u}\right)\left(p_{2}^{l} \leq p_{1}^{u}\right)
$$

Since $p_{i}$ is increasing in $y_{1}, y_{2}$, it is easy to obtain the piecewise upper and lower bounds $p_{i}^{l}, p_{i}^{u}$. If $f=1$, we get $p_{1}(y) \geq p_{2}(y)$ or $p_{2}(y) \geq p_{1}(y)$ for any $y \in Q$, and $F(y)$ has a fixed sign in $Q$. Then we can use the analytic integral formula for $\partial_{2}\left(K_{2}(y) y_{2}\right)$ to evaluate the integral. If $f=0$, we bound the integral as follows

$$
\int_{Q}|F(y)| d y \leq b(Q) \int_{Q} \frac{1}{2|y|^{6}} d y
$$



Figure 1. Decompositions in different cases. First figure for case (1), (6), the last four figures for case (2)-(5). The blue region denotes $Q_{3}$.
and further evaluate the integral using the analytic formula that can be obtained by symbolic computation.
5.1.4. Symmetric bounds for $K(y)$. The kernel $K_{l}(y)$ (5.1) enjoys the symmetry in $y_{1}, y_{2}$

$$
K_{l}\left(y_{2}, y_{1}\right)=s_{l} K_{l}\left(y_{1}, y_{2}\right), \quad s_{1}=1, s_{2}=-1
$$

Therefore, we have

$$
\partial_{1}^{m} \partial_{2}^{n}\left(K_{l}\left(y_{1}, y_{2}\right)\right) \cdot y_{1}^{i} y_{2}^{j}=s_{l} \partial_{1}^{m} \partial_{2}^{n}\left(K_{l}\left(y_{2}, y_{1}\right)\right) \cdot y_{1}^{i} y_{2}^{j}=\left.s_{l}\left(\partial_{1}^{n} \partial_{2}^{m} K_{l}(z) z_{1}^{j} z_{2}^{i}\right)\right|_{\left(z_{1}, z_{2}\right)=\left(y_{2}, y_{1}\right)}
$$

We need to derive several estimates on $\partial_{1} K_{l}\left(y_{1}, y_{2}\right) y_{1}^{i} y_{2}^{j}$. Using the above relations, once we obtain the estimates related to $\partial_{1} K_{l}$, we can obtain the estimate for $\partial_{2} K_{l}$. Moreover, since

$$
K_{1}=\partial_{12} G, \quad K_{2}=\partial_{1}^{2} G, \quad \partial_{1} K_{1}=\partial_{2} K_{2}, \quad \partial_{2} K_{1}=\partial_{122} G=-\partial_{1}^{3} G=-\partial_{1} K_{2}
$$

once we derive the estimate related to $\partial_{i} K_{1}$, we can derive the estimate for $\partial_{3-i} K_{2}$ using the above relation.
5.1.5. Asymptotic integral formulas. In the estimate of the derivatives of the regular part in $\nabla \mathbf{u}$, we need to estimate several integrals
$I_{i, j, m, n} \triangleq \int_{Q_{a, b}}\left|\partial_{j} K_{i}(y) y_{1}^{m} y_{2}^{n}\right| d y, m+n=2, \quad J_{i, j, l} \triangleq \int_{Q_{a, b}}\left|\partial_{j}\left(K_{i}(y) y_{l}\right)\right| d y, \quad Q_{a, b} \triangleq[-b, b]^{2} \backslash[-a, a]^{2}$
for $i, j, l=1,2$, where $K_{i}$ is given in (5.1). When $m+n=2$, the integrand has a singularity of order -1 and it is locally integrable. Thus, we can apply the estimates and the integral formulas developed in Section 5.1 .3 to estimate $I_{i, j, m, n}$. Since the integrand is symmetric in $y_{1}, y_{2}$, we only need to estimate the integral in $[0, b]^{2} \backslash[0, a]^{2}$.

In the second integral, the integrand has a singularity of order -2 and is not integrable near 0. Denote $F_{i, j, l}(y)=\partial_{j} K_{i}(y) y_{l}$. Using the scaling symmetry of $K_{i}$, we get $F(\lambda y)=\lambda^{-2} F(y)$ and

$$
\begin{aligned}
\partial_{a} J_{i, j, l}(a, b) & =-\int_{0}^{a}\left(\left|F_{i, j, l}(a, s)\right|+\left|F_{i, j, l}(s, a)\right|\right) d s=-\int_{0}^{1}\left(\left|F_{i, j, l}(a, a s)\right|+\left|F_{i, j, l}(a s, a)\right|\right) a d s \\
& =-\frac{1}{a} \int_{0}^{1}\left(\left|F_{i, j, l}(1, s)\right|+\left|F_{i, j, l}(s, 1)\right|\right) d s
\end{aligned}
$$

where we have used a change of variables $s \rightarrow a s$ in the last inequality. We remark that the last integral is independent of $a, b$ and is a constant. Since $J_{i, j, l}(b, b)=0$, we derive
$J_{i, j, l}(a, b)=J_{i, j, l}(b, b)-\int_{a}^{b} \partial_{x} J_{i, j, l}(s) d s=C_{i, j, l} \log \frac{b}{a}, \quad C_{i, j, l} \triangleq \int_{0}^{1}\left(\left|F_{i, j, l}(1, s)\right|+\left|F_{i, j, l}(s, 1)\right|\right) d s$.
It remains to estimate $C_{i, j, l}$. This can be done by using the above identity with $(a, b)=(1,2)$,

$$
C_{i, j, l}=\frac{J_{i, j, l}(1,2)}{\log 2}
$$

and applying the estimates in Section 5.1.3 to $J_{i, j, l}$ in the domain $Q_{1,2}$. In particular, using $\left|K_{i}\left(y_{1}, y_{2}\right)\right|=\left|K_{i}\left(y_{2}, y_{1}\right)\right|$ and (5.3)

$$
\begin{aligned}
& \int_{0}^{1}\left(\left|K_{1}(s, 1)\right|+\left|K_{1}(1, s)\right|\right) d s=2 \int_{0}^{1} K_{1}(s, 1) d s=\left.\frac{1}{1+s^{2}}\right|_{0} ^{1}=\frac{1}{2} \\
& \int_{0}^{1}\left|K_{2}(s, 1)\right|+\left|K_{2}(1, s)\right| d s=2 \int_{0}^{1} K_{2}(1, s) d s=\left.\frac{s}{s^{2}+1}\right|_{0} ^{1}=\frac{1}{2}
\end{aligned}
$$

we get

$$
\begin{equation*}
\int_{[-b, b]^{2} \backslash[-a, a]^{2}}\left|K_{i}(y)\right| d y=2 \log (b / a) \tag{5.18}
\end{equation*}
$$

5.2. Strategy of estimating the explicit integrals. In the sharp Hölder estimates, we need to estimate several explicit integrals, e.g. (5.34)

$$
\begin{equation*}
\left|\int_{0}^{\infty} \int_{1 / 2}^{\infty} \mathbf{1}_{s_{2} \leq b} \mathbf{1}_{s_{1} \geq f\left(s_{2}\right)}\right| T\left(s_{1}, s_{2}\right)-\left.s_{1}\right|^{1 / 2} \Delta(s) d s \mid, \quad \Delta(s)=K\left(s-s_{*}\right)-K\left(s-s_{r}\right) \tag{5.19}
\end{equation*}
$$

in the Hölder estimate of $u_{x}$, where $\Delta$ is some kernel singular at $s_{*}, T$ is the map, and $s_{r}$ is another fixed point, e.g. $(-1 / 2,0)$, outside the domain of the integral. In all cases, the integrand can be written as

$$
I\left(s_{1}, s_{2}\right)=\mathbf{1}_{A\left(s_{1}, s_{2}\right) \geq 0}\left|T(s)-s_{j}\right|^{1 / 2}|\Delta(s)|
$$

for $j=1$ or 2 , where $\mathbf{1}_{A\left(s_{1}, s_{2}\right) \geq 0}$ is some indicator function. To obtain a sharp estimate, we decompose the domain of the integral into the bulk, and the far-field. In the far-field, $s$ is very large. Using the cancellation in the kernel $\Delta$, we have

$$
|\Delta(s)| \leq C\left|s_{*}-s_{r}\right|| | \partial_{j} K(s) \mid
$$

for some constant $C$ and $j=1$ or 2 . In the domain $A(s) \geq 0$, we have $0 \leq T(s) \leq s_{j}$, and apply the trivial bound $\left|T(s)-s_{j}\right| \leq s_{j}$. This allows us to estimate the far-field of the integral easily. Due to the decay of the integrand, the contribution of the integral in the far-field is very small.

In the bulk, we partition the integral using dense mesh and estimate the integral in each small region $Q=\left[x_{l}, x_{u}\right] \times\left[y_{l}, y_{u}\right]$. We study the monotonicity of the function $A(s)$ and $T(s)$ and obtain their piecewise bounds in $Q$

$$
A^{l}(Q) \leq A(s) \leq A^{u}(Q), \quad T^{l}(Q) \leq T(s) \leq T^{u}(Q, \quad s \in Q
$$

The upper and lower bounds of $s_{j}$ are given by the endpoints of the domain. It follows

$$
\begin{align*}
& \mathbf{1}_{A^{l} \geq 0} \leq \mathbf{1}_{A(s) \geq 0} \leq \mathbf{1}_{A^{u} \geq 0}, \quad\left|T(s)-s_{j}\right| \leq \max _{\alpha, \beta=l, u}\left|T^{\alpha}-s^{\beta}\right| \\
& S=\int_{Q} I(s) d s \leq \mathbf{1}_{A^{u} \geq 0} \max _{\alpha, \beta=l, u}\left|T^{\alpha}-s^{\beta}\right|^{1 / 2} \int_{Q}|\Delta(s)| d s \tag{5.20}
\end{align*}
$$

In particular, the integral is 0 if $A^{u}<0$. The set $A(s)=0$ has zero measure in our integrals. The integral for $|\Delta(s)|$ can be estimated using the Trapezoidal rule in Section 5.1 The same method applies to estimate more general integrals

$$
\prod_{1 \leq i \leq n} \mathbf{1}_{A_{i}(s) \geq 0} f(s)|\Delta(s)|
$$

in $Q$, for some function $f \geq 0$ and several indicator functions $\mathbf{1}_{A_{i}(s) \geq 0}$. There are three improvements.
5.2.1. $\Delta(s)$ has a fixed sign. In the support of the integrand $I(s), \Delta(s)$ has a fixed sign. Yet, it may not have a fixed sign in $Q$ since we estimate the indicator function $\mathbf{1}_{A \geq 0} \leq \mathbf{1}_{A^{u} \geq 0}$. If $A^{l} \geq 0$, then we have $\mathbf{1}_{A(s) \geq 0} \geq \mathbf{1}_{A^{l} \geq 0}=1, \Delta(s)$ has a fixed $\operatorname{sign}$ in $Q$ and
$S=\int_{Q}\left|T(s)-s_{j}\right|^{1 / 2}|\Delta(s)| d s \leq \max _{s \in Q}\left|T(s)-s_{j}\right|^{1 / 2} \int_{Q}|\Delta(s)| d s=\max _{Q}\left|T(s)-s_{j}\right|^{1 / 2}\left|\int_{Q} \Delta(s) d s\right|$.
For the last integral, we can use the analytic integral formula for $\Delta(s)$.
5.2.2. Estimate of $|T(s)-s|$ near the singularity. Near the singularity of $\Delta(s)$ at $s_{*}$, we need to exploit the smallness of $\left|T(s)-s_{j}\right| \lesssim\left|s-s_{*}\right|$ to cancel the singularity of $\Delta(s)$, which has order of $\left|s-s_{*}\right|^{-2}$. The previous piecewise bounds for $T(s)-s_{j}$ in (5.20) do not provide such an order. We estimate

$$
\begin{equation*}
\mathbf{1}_{A(s) \geq 0}\left|T(s)-s_{j}\right| \leq c_{1}(Q)\left|s_{1}-s_{*, 1}\right|+c_{2}\left(Q\left|s_{2}-s_{*, 2}\right|\right. \tag{5.21}
\end{equation*}
$$

The piecewise constant $c_{i}(Q)$ can be obtained using the piecewise bounds of $\nabla T$. We will study the piecewise bounds of $T, \nabla T$ in the following section.
5.2.3. Estimate of integral near the singularity. Recall from (5.19) that the kernel $\Delta$ can be written as $K\left(s-s_{*}\right)-K\left(s-s_{r}\right)$. Near the singularity, $K\left(s-s_{r}\right)$ is regular and its related integral is much smaller. Using (5.21) and the triangle inequality, we get

$$
S \leq \max _{Q}\left|T(s)-s_{j}\right|^{1 / 2} \int_{Q}\left|K\left(s-s_{r}\right)\right| d s+\int_{Q}\left|T(s)-s_{j}\right|^{1 / 2}\left|K\left(s-s_{*}\right)\right| d s \triangleq S_{1}+S_{2}
$$

For $S_{1}$, we estimate it using the previous method. For $S_{2}$, we further bound $\left|T(s)-s_{j}\right|$ as follows

$$
\left|T(s)-s_{j}\right| \leq c_{3}(Q)\left|s_{j}-s_{*, j}\right|
$$

for some piecewise constant $c_{3}(Q)$ depending on $T$ and $\nabla T$. Then we estimate $S_{2}$ as follows

$$
S_{2} \leq c_{3}(Q) \int_{Q}\left|K\left(s-s_{*}\right)\right|\left|s_{j}-s_{*, j}\right|^{1 / 2} d s=c_{3}(Q) \int_{Q-a}\left|K(s) \| s_{j}\right|^{1 / 2} d s
$$

for $a=\left(s_{*, 1}, 0\right)$ or $a=\left(0, s_{*, 2}\right)$, and evaluate the integral using the analytic integral formula for $K(s)\left|s_{j}\right|^{1 / 2}$.
5.2.4. Partition the domain of the integrals and the piecewise bounds. To estimate the 2D integral, we will partition the domain $[0, \infty)^{2}$ using adaptive mesh $x_{1}<x_{2}<. .,<x_{n_{1}+1}=\infty, y_{1}<$ $y_{2}<. .,<y_{n_{2}+1}=\infty$. The integrals we will estimate depend on 1 or 2 parameters. For example, the integral in (5.34) depends on $b$, and the integrals in (5.43), (5.49) depend on $[A, B]$. These parameters impose a constraint on the domain of the integral, e.g. $\mathbf{1}_{y_{2} \leq b}$ in (5.34), $\mathbf{1}_{y_{2} \leq A}$ in (5.49). Using the monotonicity of the integrals (or different parts of the integrals) on these parameters, we only need to estimate the integrals with parameters on the mesh, e.g. $b=y_{i}$ for (5.34) or $A=y_{i}, B=x_{j}$ for (5.49),

$$
\int_{0}^{b}\left|F\left(s_{1}, s_{2}\right)\right| d s_{2} \leq \int_{0}^{b^{u}}\left|F\left(s_{1}, s_{2}\right)\right| d s_{2}, \quad b \in\left[b^{l}, b^{u}\right] .
$$

For $b, A, B$ on the grid point, the support of the integrand is the union of the small grid $\left[x^{l}, x^{u}\right] \times\left[y^{l}, y^{u}\right]$. Thus we can take the sum of the integrals in these grids to estimate the integrals with parameter $b, A, B$. For example, after we estimate

$$
Q_{i}=\int_{b_{i}}^{b_{i+1}}|I(s)| d s
$$

the cumulative sum of $\sum_{j \leq i} Q_{j}$ provides a piecewise bound for $\int_{0}^{b}|I(s)| d s$. This strategy allows us to obtain piecewise bounds for parameters $b, A, B$ and the uniform bound.
5.3. Estimate of the sharp constants for $u_{x}$. We first study some properties of the transport map $T\left(s_{1}, s_{2}\right)$ for $u_{x}$. Recall from Sectin 3.3 [3] that $T$ solves the cubic equation

$$
\begin{equation*}
0=-1-8 T s_{1}+16 T s_{1}\left(T^{2}+T s_{1}+s_{1}^{2}\right)-8 s_{2}^{2}\left(1-4 s_{1} T+2 s_{2}^{2}\right) \tag{5.22}
\end{equation*}
$$

Solving

$$
\Delta\left(y_{1}, y_{2}\right)=0, \quad \Delta(y) \triangleq K_{1}\left(y_{1}+1 / 2, y_{2}\right)-K_{1}\left(y_{1}-1 / 2, y_{2}\right)=0
$$

for $y_{1} \geq 0$, we yield the threshold $f\left(s_{2}\right)$ determining the sign of the kernel

$$
\begin{equation*}
f\left(s_{2}\right)=\left(\frac{\frac{1}{2}-2 s_{2}^{2}+\sqrt{16 s_{2}^{4}+4 s_{2}^{2}+1}}{6}\right)^{1 / 2} \tag{5.23}
\end{equation*}
$$

For $s_{1}, s_{2}>0$, we establish in Section 3.3 [3]

$$
\begin{equation*}
\Delta(s) \leq 0, s_{1} \in\left[f\left(s_{2}\right), \infty\right), \quad \Delta(s) \geq 0, s_{1} \in\left(0, f\left(s_{2}\right)\right] \tag{5.24}
\end{equation*}
$$

Firstly, for a fixed $s_{2} \neq 0$ and $s_{1}>0$, we show that it has a unique solution on $[0, \infty)$. Dividing $s_{1}$ on both side of (5.22) yields

$$
\begin{equation*}
g(T) \triangleq T^{3}+T^{2} s_{1}+T\left(s_{1}^{2}-\frac{1}{2}+2 s_{2}^{2}\right)-\frac{\left(4 s_{2}^{2}+1\right)^{2}}{16 s_{1}}=0 \tag{5.25}
\end{equation*}
$$

Since $g(0)<0$ and $g(\infty)>0$, the above equation has at least one real root on $\mathbb{R}_{+}$. We introduce $Z=T+\frac{s_{1}}{3}$ and can rewrite the above equation in terms of $Z$

$$
\begin{align*}
0 & =\left(T+\frac{s_{1}}{3}\right)^{3}-\frac{s_{1}^{3}}{27}+T\left(\frac{2}{3} s_{1}^{2}-\frac{1}{2}+2 s_{2}^{2}\right)-\frac{\left(4 s_{2}^{2}+1\right)^{2}}{16 s_{1}} \\
& \left.=Z^{3}+Z\left(\frac{2}{3} s_{1}^{2}-\frac{1}{2}+2 s_{2}^{2}\right)\right)-\left(\frac{\left(4 s_{2}^{2}+1\right)^{2}}{16 s_{1}}+\frac{7}{27} s_{1}^{3}+\frac{s_{1}}{3}\left(2 s_{2}^{2}-\frac{1}{2}\right)\right)  \tag{5.26}\\
& \triangleq Z^{3}+p\left(s_{1}, s_{2}\right) Z+q\left(s_{1}, s_{2}\right)
\end{align*}
$$

The discriminant is given by

$$
\begin{equation*}
\Delta_{Z}\left(s_{1}, s_{2}\right)=-\left(27 q\left(s_{1}, s_{2}\right)^{2}+4 p\left(s_{1}, s_{2}\right)^{3}\right) \tag{5.27}
\end{equation*}
$$

Note that

$$
-q \geq \frac{7}{27} s_{1}^{3}-\frac{s_{1}}{6}+\frac{1}{16 s_{1}} \geq\left(2 \sqrt{\frac{7}{27} \cdot \frac{1}{16}}-\frac{1}{6}\right) s_{1} \geq 0
$$

and $-q, p$ are increasing in $\left|s_{2}\right|$. We yield

$$
-\Delta_{Z}\left(s_{1}, s_{2}\right) \geq-D_{Z}\left(s_{1}, 0\right)=\frac{\left(1-4 s_{1}^{2}\right)^{2}\left(27-56 s_{1}^{2}+48 s_{1}^{4}\right)}{256 s_{1}^{2}} \geq 0
$$

When $s \neq\left(\frac{1}{2}, 0\right)$, the above inequality is strict. Using the solution formula for a cubic equation, we obtain that the cubic equation for $T$ or $Z$ has a unique real root for $s \neq\left( \pm \frac{1}{2}, 0\right)$ given by

$$
\begin{equation*}
Z=r_{1}-\frac{p}{3 r_{1}}, \quad r_{1}=\left(\frac{-q+\sqrt{q^{2}+\frac{4}{27} p^{3}}}{2}\right)^{1 / 3}, \quad T=Z-\frac{s_{1}}{3} \tag{5.28}
\end{equation*}
$$

where $p, q$ are defined in (5.26). For $s=\left(\frac{1}{2}, 0\right)$, it is easy to get that $T=\frac{1}{2}=s_{1}$ is the unique solution in $(0, \infty)$, which is also given by the above formula.

We have the following basic properties for the map $T$ and the threshold $f\left(s_{2}\right)$ (5.23).
Lemma 5.3. The map $T\left(s_{1}, s_{2}\right)$ is increasing in $s_{2}$ and decreasing in $s_{1}$. Moreover, we have $f\left(s_{2}\right) \geq \frac{1}{2}, f\left(s_{2}\right)$ is increasing in $s_{2}$ for $s_{2}>0$, and

$$
\begin{align*}
& 4 s_{2}^{2}+1 \geq \max \left(4 s_{1} T\left(s_{1}, s_{2}\right), 1\right), \quad T^{2}+T s_{1}+s_{1}^{2}+2 s_{2}^{2} \geq \frac{3}{4}, \quad \text { for } s_{1} \geq 0 \\
& \left|T\left(s_{1}, s_{2}\right)-s_{1}\right| \leq \max _{x \in\left[f\left(s_{2}\right), s_{1}\right]}\left(\left|T_{x}\left(x, s_{2}\right)\right|+1\right)\left|s_{1}-\frac{1}{2}\right|, \quad s_{1} \geq f\left(s_{2}\right) \tag{5.29}
\end{align*}
$$

Proof. The estimate $f\left(s_{2}\right) \geq \frac{1}{2}$ follows from (5.23) and $1+4 s_{2}^{2}+16 s_{2}^{4} \geq\left(1+2 s_{2}^{2}\right)^{2}$. From (5.23), for $s_{2}>0$, since

$$
\frac{d}{d s_{2}}\left(6 f^{2}\left(s_{2}\right)\right)=-4 s_{2}+\frac{64 s_{2}^{3}+8 s_{2}}{2 \sqrt{16 s_{2}^{4}+4 s_{2}^{2}+1}}, \quad\left(8 s_{2}^{2}+1\right)^{2}>16 s_{2}^{4}+4 s_{2}^{2}+1
$$

$f\left(s_{2}\right)$ is increasing in $s_{2}>0$. Denote $P=4 s_{1} T, Q=4 s_{2}^{2}+1$. Using (5.22), we get

$$
\begin{aligned}
Q^{2} & =\left(4 s_{2}^{2}+1\right)^{2}=16 T s_{1}\left(T^{2}+T s_{1}+s_{1}^{2}\right)+32 s_{2}^{2} s_{1} T-8 T s_{1} \\
& \geq 16 T s_{1} \cdot 3 T s_{1}+2(Q-1) P-2 P=3 P^{2}+2 P Q-4 P
\end{aligned}
$$

If $P \leq 1$, we derive $Q \geq 1 \geq P$. If $P>1$, solving the above quadratic inequality in $Q$, we yield

$$
Q \geq P+\sqrt{4 P^{2}-4 P}, \quad \text { or } \quad Q \leq P-\sqrt{4 P^{2}-4 P}
$$

Note that for $P>1$, we have $P-\sqrt{4 P^{2}-4 P}<1$. Thus, we must have

$$
Q \geq P+\sqrt{4 P^{2}-4 P} \geq P
$$

This proves the first inequality in (5.29). Using (5.22) again, we derive

$$
T^{2}+T s_{1}+s_{1}^{2}+2 s_{2}^{2}=\frac{1}{16 T s_{1}}\left(8 T s_{1}+\left(4 s_{2}^{2}+1\right)^{2}\right) \geq \frac{1}{16 T s_{1}}\left(8 T s_{1}+4 T s_{1}\right)=\frac{3}{4}
$$

where we have used the first inequality in (5.29) that we just proved. The last inequality in (5.29) follows directly from $f\left(s_{2}\right) \geq \frac{1}{2}$ and the mean value theorem.

Since (5.22) is symmetric in $T$ and $s_{1}$, and its has a unique positive real root for any $s_{1}>0$, we get $T\left(T\left(s_{1}, s_{2}\right), s_{2}\right)=s_{1}$, or $T \circ T=I d$. Using (5.90) and (5.24), we get

$$
\Delta\left(s_{1}, s_{2}\right)=T_{s_{1}}\left(s_{1}, s_{2}\right) \Delta\left(T\left(s_{1}, s_{2}\right), s_{2}\right)
$$

Since $\Delta\left(s_{1}, s_{2}\right)$ and $\Delta\left(T\left(s_{1}, s_{2}\right), s_{2}\right)$ have opposite sign, it follows $T_{s_{1}} \leq 0$.
Taking $s_{2}$ derivative on both sides of (5.25), we yield

$$
\begin{equation*}
\frac{d T}{d s_{2}}\left(3 T^{2}+2 T s_{1}+s_{1}^{2}-\frac{1}{2}+2 s_{2}^{2}\right)+s_{2}\left(4 T-\frac{4 s_{2}^{2}+1}{s_{1}}\right)=0 \tag{5.30}
\end{equation*}
$$

Using the first and the second inequality in (5.29), we prove $\frac{d T}{d s_{2}} \geq 0$.
5.3.1. Piecewise bounds for $T, \nabla T$. Using the monotonicity of $T$ in Lemma 5.3 and its analytic formula, we can derive the piecewise upper and lower bounds for the map $T(s)$ in $\left[s_{1}^{l}, s_{1}^{u}\right] \times$ $\left[s_{2}^{l}, s_{2}^{u}\right] \subset \mathbb{R}_{+}^{2}$,

$$
\begin{equation*}
T^{l}=T\left(s_{1}^{u}, s_{2}^{l}\right), \quad T^{u}=T\left(s_{1}^{l}, s_{2}^{u}\right) \tag{5.31}
\end{equation*}
$$

Such a formula can lead to a large round off error when $s_{1} / s_{2}$ and $s_{1}$ are large. In such a case, we derive another bound. Firstly, we write (5.22), (5.25) as follows

$$
T^{3}+T^{2} s_{1}+(T-\hat{T})\left(s_{1}^{2}-\frac{1}{2}+2 s_{2}^{2}\right)=0, \quad \hat{T}=\frac{\left(4 s_{2}^{2}+1\right)^{2}}{16 s_{1}\left(s_{1}^{2}-\frac{1}{2}+2 s_{2}^{2}\right)}
$$

The function $\hat{T}$ can be seen as the approximation of $T$, and we can estimate the piecewise bounds for $\hat{T}$ easily. For $s_{1}>1, s_{2}>0$, since $T>0, s_{1}^{2}-\frac{1}{2}+2 s_{2}^{2}>0$, we yield $T \leq \hat{T}$ and

$$
\begin{equation*}
T=\hat{T}-\frac{T^{3}+T^{2} s_{1}}{s_{1}^{2}-\frac{1}{2}+2 s_{2}^{2}} \geq \hat{T}-\frac{\hat{T}^{3}+\hat{T}^{2} s_{1}}{s_{1}^{2}-\frac{1}{2}+2 s_{2}^{2}} \tag{5.32}
\end{equation*}
$$

Using the piecewise bounds for $\hat{T}$ and the above estimates, we can obtain another piecewise bounds for $T$. Taking $s_{1}$ derivative of (5.25), we yield

$$
\frac{d T}{d s_{1}}\left(3 T^{2}+2 T s_{1}+s_{1}^{2}-\frac{1}{2}+2 s_{2}^{2}\right)+T^{2}+2 s_{1} T+\frac{\left(4 s_{2}^{2}+1\right)^{2}}{16 s_{1}^{2}}=0
$$

From (5.30) and the above formula, we obtain $\partial_{s_{i}} T=\frac{P_{i}\left(T, s_{1}, s_{2}\right)}{Q_{i}\left(T, s_{1}, s_{2}\right)}$ for some polynomials $P_{i}, Q_{i}$. Using the above piecewise bounds for $T$, we can further obtain piecewise bounds for $\nabla T$.

Near the singularity of the integral, we need to obtain a sharp estimate of $T(s)-s_{1}$. Since $T\left(f\left(s_{2}\right), s_{2}\right)=f\left(s_{2}\right)$, and $f\left(s_{2}\right) \geq 1 / 2$, for $s_{1} \geq f\left(s_{2}\right)$, we yield (5.33)

$$
\begin{aligned}
\left|T(s)-s_{1}\right| & =\left|T(s)-s_{1}-\left(T\left(f\left(s_{2}\right), s_{2}\right)-f\left(s_{2}\right)\right)\right| \leq\left|s_{1}-f\left(s_{2}\right)\right|\left(\max _{\xi \in\left[f\left(s_{2}\right), s_{1}\right]}\left|T_{s_{1}}\left(\xi, s_{2}\right)\right|+1\right) \\
& \leq\left|s_{1}-1 / 2\right|\left(\max _{\xi \in\left[1 / 2, s_{1}\right]}\left|T_{s_{1}}\left(\xi, s_{2}\right)\right|+1\right)
\end{aligned}
$$

We can estimate the upper bound using the piecewise bounds for $\nabla T$.
5.3.2. Estimate of the explicit integrals for $\left[u_{x}\right]_{C_{x}^{1 / 2}}$. In this section, we estimate the integral
$S(b)=\left|\int_{0}^{b} d s_{2} \int_{f\left(s_{2}\right)}^{\infty}\right| T\left(s_{1}, s_{2}\right)-\left.s_{1}\right|^{1 / 2} \Delta(s) d s_{1}\left|=\left|\int_{0}^{\infty} \int_{1 / 2}^{\infty} \mathbf{1}_{s_{2} \leq b} \mathbf{1}_{s_{1} \geq f\left(s_{2}\right)}\right| T\left(s_{1}, s_{2}\right)-s_{1}\right|^{1 / 2} \Delta(s) d s \mid$,
$\Delta(s)=K_{1}\left(s_{1}+\frac{1}{2}, s_{2}\right)-K_{1}\left(s_{1}-\frac{1}{2}, s_{2}\right)$,
in the upper bound in Lemma 3.1 [3], and obtain its piecewise bounds for $b \in[1, \infty)$, where $f\left(s_{2}\right)$ and $T(s)$ are determined by (5.23), (5.22). Note that one needs to multiply $S(b)$ by a factor $\frac{4}{\pi}$ to get the constant in Lemma 3.1 [3]. In the second identity, we have used $f\left(s_{2}\right) \geq \frac{1}{2}$ from Lemma 5.3. The kernel satisfies $\Delta(s) \leq 0$ in the domain of the integral.

For very large $R_{0}, R_{1}$, e.g. $R_{0}=R_{1}=10^{8}$, and small $m_{1}>0$, we decompose the domain integral into four parts

$$
\begin{align*}
D_{1} & =\left[1 / 2, R_{1}\right] \times\left[m_{1}, R_{0}\right], D_{2}=\left[1 / 2, R_{1}\right] \times\left[0, m_{1}\right] \\
D_{3} & =\left[R_{1}, \infty\right] \times\left[0, R_{0}\right], D_{4}=[1 / 2, \infty) \times\left[R_{0}, \infty\right) \tag{5.35}
\end{align*}
$$

The first part captures the bulk part of the integrals, $D_{2}$ captures the integral near the singularity, and $D_{3}$ and $D_{4}$ capture the far-field of the integral. We follow the strategy in Section 5.2 to estimate the integrals.

Integral in the finite domain. To estimate the integrals in $D_{1}, D_{2}$, we discretize the domain $\left[1 / 2, R_{1}\right] \times\left[0, R_{0}\right]$ using

$$
1 / 2=x_{0}<x_{2}<\ldots<x_{n_{1}}=R_{1}, \quad 0=z_{0}<z_{1}<\ldots<z_{n_{2}}=R_{0}
$$

Since $S(b)$ is increasing in $b$, it suffices to estimate $S(b)$ for $b=z_{i}$ and $S(\infty)$. For a fixed domain $s \in\left[x_{i-1}, x_{i}\right] \times\left[z_{j-1}, z_{j}\right] \triangleq X_{i} \times Z_{j} \triangleq Q_{i j}$, since $f\left(s_{2}\right)$ is increasing (Lemma 5.3), we have

$$
S_{i j}=\int_{Q_{i j}} \mathbf{1}_{s_{1} \geq f\left(s_{2}\right)}\left|T(s)-s_{1}\right|^{1 / 2}|\Delta(s)| d s \leq \mathbf{1}_{x_{i} \geq f\left(z_{j-1}\right)} \int_{Q_{i j}}\left|T(s)-s_{1}\right|^{1 / 2}|\Delta(s)| d s
$$

Using the piecewise bounds of $T(s)$ in $Q_{i j}$ (5.31), (5.33), we get

$$
\left|T\left(s_{1}, s_{2}\right)-s_{1}\right| \leq \min \left(\max \left(T^{u}-s_{1}^{l}, s_{1}^{u}-T^{l}\right),\left|s_{1}-1 / 2\right|\left(\max _{\xi \in\left[1 / 2, x_{i}\right] \times Z_{j}}\left|T_{s_{1}}\left(\xi, s_{2}\right)\right|+1\right)\right)
$$

For the integral $\int_{Q}|\Delta(s)|$, we use the two estimates in Section 5.1 based on the Trapezoidal rule and the analytic integral formulas. If $x_{i-1} \geq f\left(z_{j}\right)$, from (5.24), we get

$$
s_{1} \geq x_{i-1} \geq f\left(z_{j}\right) \geq f\left(s_{2}\right), \quad|\Delta(s)|=-\Delta(s)
$$

for any $s \in Q_{i j}$. Hence, we can use the analytic integral formula for $\Delta(s)$ in this case.
Near the singularity $s=(1 / 2,0)$, we use the triangle inequality to obtain another estimate

$$
S_{i j} \leq \int_{Q_{i j}}\left|T(s)-s_{1}\right|^{1 / 2}\left(K_{1}\left(s_{1}+1 / 2, s_{2}\right)+K_{1}\left(s_{1}-1 / 2, s_{2}\right) \mid\right) d s \triangleq I_{1}+I_{2}
$$

where we have used $K_{1}\left(s_{1}, s_{2}\right)>0$ for $s_{1}, s_{2}>0$. The regular part $I_{1}$ is estimated using the previous method. Using (5.33), for the singular part $I_{2}$, we have

$$
I_{2} \leq\left(\max _{\xi \in\left[1 / 2, x_{i}\right] \times Z_{j}}\left|T_{s_{1}}\left(\xi, s_{2}\right)\right|+1\right) \int_{Q_{i j}}\left(s_{1}-1 / 2\right)^{1 / 2} K_{1}\left(s-1 / 2, s_{2}\right) d s
$$

and obtain the integral using the analytic integral formula for $K_{1}(s) s_{1}^{1 / 2}$.

Integral in the far-field. The integrals in the far-field $D_{3}, D_{4}$ (5.35) are very small. Next, we estimate the integral in $D_{3}$, for $s_{1} \geq f\left(s_{2}\right)$, we derive

$$
\begin{equation*}
s_{1} \geq f\left(s_{2}\right) \geq T(s), \quad\left|s_{1}-T(s)\right| \leq s_{1} . \tag{5.36}
\end{equation*}
$$

Denote $R_{m}=\min \left(R_{0}, R_{1}\right)$. In $D_{3} \cup D_{4}$, we have $\max \left(s_{1}, s_{2}\right) \geq R_{m}$. Applying Lemma 5.2 with $i+j=3$ and the Mean Value Theorem yields

$$
\left|\Delta_{i n}(s)\right|=\left|\partial_{1} K_{1}\left(\xi, s_{2}\right)\right| \leq \frac{1}{\left(\xi^{2}+s_{2}^{2}\right)^{3 / 2}} \leq C_{R}|s|^{-3}, \quad C_{R}=\left(\frac{R_{m}}{R_{m}-1 / 2}\right)^{3}
$$

for some $\xi \in\left[s_{1}-\frac{1}{2}, s_{1}+\frac{1}{2}\right]$, where we have used $\left|\left(\xi, s_{2}\right)\right| \geq|s|-1 / 2 \geq|s|\left(1-\frac{1}{2 R_{m}}\right)=|s| \frac{R_{m}}{R_{m}-1 / 2}$. For the integrals in $D=D_{3}$ or $D=D_{3} \cup D_{4}$ (5.35), using the above estimates, we yield

$$
\begin{aligned}
S_{D} & \triangleq \int_{D} \mathbf{1}_{s_{2} \leq b} \mathbf{1}_{s_{1} \geq f\left(s_{2}\right)}\left|T-s_{1}\right|^{\frac{1}{2}}|\Delta(s)| d s \leq C_{R} \int_{D} \frac{s_{1}^{1 / 2}}{|s|^{3}} d s, \\
S_{D_{3}} & \leq C_{R} \int_{R_{1}}^{\infty} d s_{1} \int_{0}^{R_{0}} \frac{s_{1}^{1 / 2}}{|s|^{3}} d s \leq C_{R} \int_{R_{1}}^{\infty} s_{1}^{-5 / 2} R_{0} d s_{1}=C_{R} \frac{2}{3} R_{1}^{-3 / 2} R_{0}=C_{R} \frac{2}{3} R_{1}^{-3 / 2} R_{0}, \\
S_{D_{3}, D_{4}} & \leq C_{R} \int_{|s| \geq R_{m}, s \in \mathbb{R}_{2}^{++}} \frac{s_{1}^{1 / 2}}{|s|^{3}} d s \leq C_{R} \int_{|s| \geq R_{m}, s \in \mathbb{R}_{2}^{++}}|s|^{-\frac{5}{2}} d s=C_{R} \int_{R_{m}}^{\infty} r^{-\frac{3}{2}} d r \int_{0}^{\frac{\pi}{2}} d \beta=C_{R} \pi R_{m}^{-\frac{1}{2}} .
\end{aligned}
$$

Note that if $b \leq R_{0}$, due to the restriction $\mathbf{1}_{s_{2} \leq b}$, we only need to estimate the integral in $D_{3}$.
Combining the estimate of the integrals $S_{i j}$ and $S_{D_{3}}$, we can estimate $S(b)$ for $b=z_{i}, i=$ $1,2, . ., n_{2}$. Adding the contribution from $S_{D_{4}}$, we obtain the estimate of $S(\infty)$. Since $S(b)$ is increasing, we obtain piecewise bounds and the uniform bound for $S(b)$ (5.34). We remark that to obtain the constant in Lemma 3.1s [3, one needs to further multiply the constant $\frac{4}{\pi}$.
5.3.3. Estimate the integrals for $\left[u_{x}\right]_{C_{y}}^{1 / 2}$. In this section, we estimate the integral

$$
\begin{equation*}
S(a)=\int_{0}^{a} \int_{0}^{\infty} y_{1}^{1 / 2}\left|\frac{y_{1}\left(1 / 2-y_{2}\right)}{\left(y_{1}^{2}+\left(1 / 2-y_{2}\right)^{2}\right)^{2}}+\frac{y_{1}\left(1 / 2+y_{2}\right)}{\left(y_{1}^{2}+\left(1 / 2+y_{2}\right)^{2}\right)^{2}}\right| d y \tag{5.37}
\end{equation*}
$$

in Lemma 3.2 [3] for the estimate of $\left[u_{x}\right]_{C_{y}^{1 / 2}}$. Note that one needs to multiply $S(a)$ by a factor $\frac{\sqrt{2}}{\pi}$ to get the constant in Lemma 3.2. Clearly, $S$ is increasing in $a$.

Swapping the dummy variables $y_{1}, y_{2}$ and writing $\left(1 / 2-y_{1}\right)=-\left(y_{1}-1 / 2\right)$, we yield
$S(a)=\int_{0}^{\infty} d y_{1} \int_{0}^{a} y_{2}^{1 / 2}\left|\frac{y_{2}\left(y_{1}-1 / 2\right)}{\left(y_{2}^{2}+\left(1 / 2-y_{1}\right)^{2}\right)^{2}}-\frac{y_{2}\left(1 / 2+y_{1}\right)}{\left(y_{2}^{2}+\left(1 / 2+y_{1}\right)^{2}\right)^{2}}\right| d y_{2}=\int_{0}^{a} y_{2}^{1 / 2} d y_{2} \int_{0}^{\infty}|\Delta(y)| d y_{1}$,
where $\Delta(s)$ is defined in (5.24).
Using the sign property (5.24), we can first compute the integral in $y_{1}$

$$
\begin{align*}
\int_{0}^{\infty}|\Delta(y)| d y_{1} & =\left(\int_{0}^{f\left(y_{2}\right)}-\int_{f\left(y_{2}\right)}^{\infty}\right) \Delta(y) d y_{1}=\frac{1}{2}\left(-\frac{y_{2}}{y_{2}^{2}+\left(y_{1}+\frac{1}{2}\right)^{2}}+\frac{y_{2}}{y_{2}^{2}+\left(y_{1}-\frac{1}{2}\right)^{2}}\right)\left(\left.\right|_{0} ^{f\left(y_{2}\right)}-\left.\right|_{f\left(y_{2}\right)} ^{\infty}\right)  \tag{5.39}\\
& =g\left(f\left(y_{2}\right)-1 / 2, y_{2}\right)-g\left(f\left(y_{2}\right)+1 / 2, y_{2}\right), \quad g(y) \triangleq \frac{y_{2}}{y_{1}^{2}+y_{2}^{2}}
\end{align*}
$$

where we have used $A\left(\left.\right|_{a} ^{b}-\left.\right|_{c} ^{d}\right)$ to denote $(A(b)-A(a))-(A(d)-A(c))$. The boundary term vanishes at 0 and $\infty$. It follows

$$
S(a)=\int_{0}^{a} y_{2}^{1 / 2}\left(g\left(f\left(y_{2}\right)-1 / 2, y_{2}\right)-g\left(f\left(y_{2}\right)+1 / 2, y_{2}\right)\right) d y_{2}, \quad g(y) \triangleq \frac{y_{2}}{y_{1}^{2}+y_{2}^{2}}
$$

We partition $[0, \infty)$ using mesh $0=z_{0}<z_{1}<\ldots<z_{n_{2}}=R_{0}$ and $\infty$. Since $S(a)$ is increasing in $a$, it suffices to estimate $S(a)$ for $a=z_{i}$ and $a=\infty$.

Integral in a finite domain. Clearly, $g(y)$ is decreasing in $y_{1}$ for $y_{1}>0$. For the integral in $Z_{j}=\left[z_{j-1}, z_{j}\right]$, since $f\left(y_{2}\right)$ is increasing in $y_{2}$ and $f\left(y_{2}\right) \geq 1 / 2$ (see Lemma 5.3), we get

$$
1 / 2 \leq f^{l}=f\left(z_{j-1}\right) \leq f\left(y_{2}\right) \leq f\left(z_{j}\right)=f^{u}, \quad y_{2} \in Z_{j}
$$

Using the above estimate and $d_{y_{2}} \log |y|=g$, we derive

$$
\begin{aligned}
S_{j} & \triangleq \int_{z_{j-1}}^{z_{j}} y_{2}^{1 / 2}\left(g\left(f\left(y_{2}\right)-\frac{1}{2}, y_{2}\right)-g\left(f\left(y_{2}\right)+\frac{1}{2}, y_{2}\right)\right) d y_{2} \leq z_{j}^{1 / 2} \int_{Z_{j}}\left(g\left(f^{l}-\frac{1}{2}, y_{2}\right)-g\left(f^{u}+\frac{1}{2}, y_{2}\right)\right) d y_{2} \\
& =\left.z_{j}^{1 / 2} \frac{1}{2}\left(\log \left(y_{2}^{2}+\left(f^{l}-\frac{1}{2}\right)^{2}\right)-\log \left(y_{2}^{2}+\left(f^{u}+\frac{1}{2}\right)^{2}\right)\right)\right|_{z_{j-1}} ^{z_{j}}
\end{aligned}
$$

We remark that $f^{l}, f^{u}$ do not depend on $y_{2}$. Using

$$
\begin{aligned}
& g\left(f\left(y_{2}\right)-\frac{1}{2}, y_{2}\right)-g\left(f\left(y_{2}\right)+\frac{1}{2}, y_{2}\right)=\frac{y_{2}}{\left(f-\frac{1}{2}\right)^{2}+y_{2}^{2}}-\frac{y_{2}}{\left(f+\frac{1}{2}\right)^{2}+y_{2}^{2}}=\frac{2 y_{2} f}{\left(\left(f-\frac{1}{2}\right)^{2}+y_{2}^{2}\right)\left(\left(f+\frac{1}{2}\right)^{2}+y_{2}^{2}\right)} \\
\leq & \frac{f_{u}}{f_{l}} \frac{2 y_{2} f_{l}}{\left(\left(f_{l}-1 / 2\right)^{2}+y_{2}^{2}\right)\left(\left(f_{l}+1 / 2\right)^{2}+y_{2}^{2}\right)}=\frac{f_{u}}{f_{l}}\left(g\left(f^{l}-\frac{1}{2}, y_{2}\right)-g\left(f^{l}+\frac{1}{2}, y_{2}\right)\right),
\end{aligned}
$$

and computation similar to the above, we obtain another estimate

$$
S_{j} \leq\left. z_{j}^{1 / 2} \frac{f^{u}}{f^{l}} \frac{1}{2}\left(\log \left(y_{2}^{2}+\left(f^{l}-\frac{1}{2}\right)^{2}\right)-\log \left(y_{2}^{2}+\left(f^{l}+\frac{1}{2}\right)^{2}\right)\right)\right|_{z_{j-1}} ^{z_{j}}=\left.z_{j}^{1 / 2} \frac{f^{u}}{2 f^{l}} \log \left(1-\frac{2 f_{l}}{y_{2}^{2}+\left(f^{l}+1 / 2\right)^{2}}\right)\right|_{z_{j-1}} ^{z_{j}}
$$

For the integral near the singularity, e.g. when $z_{j}$ is small, we need to exploit the cancellation between the kernel and $y_{2}^{1 / 2}$. Since the more regular part $-g\left(f\left(y_{2}\right)+1 / 2, y_{2}\right) \leq 0$, we derive an improved estimate near the singularity

$$
S_{j} \leq \int_{z_{j-1}}^{z_{j}} y_{2}^{1 / 2} g\left(f\left(y_{2}\right)-1 / 2, y_{2}\right) d y_{2} \leq \int_{z_{j-1}}^{z_{j}} y_{2}^{1 / 2} \frac{1}{y_{2}} d y_{2}=2\left(z_{j}^{1 / 2}-z_{j-1}^{1 / 2}\right)
$$

Integral in the far-field. For the integral in the region $y_{2} \geq R_{0}$, we treat it as an error. Using $\xi^{2}+y_{2}^{2} \geq 2\left|\xi y_{2}\right|$ and the Mean Value Theorem, we get

$$
\left|g\left(f\left(y_{2}\right)-1 / 2, y_{2}\right)-g\left(f\left(y_{2}\right)+1 / 2, y_{2}\right)\right|=\left|\partial_{y_{1}} g\left(\xi, y_{2}\right)\right|=\frac{2\left|\xi y_{2}\right|}{\left(\xi^{2}+y_{2}^{2}\right)^{2}} \leq \frac{1}{y_{2}^{2}}
$$

for some $\xi \in[f-1 / 2, f+1 / 2]$. Using (5.39) and the above estimate, we yield
$\int_{R_{0}}^{\infty} y_{2}^{\frac{1}{2}} \int_{0}^{\infty}|\Delta(y)| d y_{1}=\int_{R_{0}}^{\infty} y_{2}^{1 / 2}\left(g\left(f\left(y_{2}\right)-\frac{1}{2}, y_{2}\right)-g\left(f\left(y_{2}\right)+\frac{1}{2}, y_{2}\right)\right) d y_{2} \leq \int_{R_{0}}^{\infty} y_{2}^{\frac{1}{2}} y_{2}^{-2} d y_{2}=2 R_{0}^{-\frac{1}{2}}$.
5.4. Estimate the sharp constants for $u_{y}, v_{x}$. In this section, we estimate the integrals in the sharp Hölder estimate of $u_{y}, v_{x}$. Recall the monotonicity lemma proved in Section 3.4 [3].

Lemma 5.4. Suppose $f, f g \in L^{1}$ and $g \geq 0$ is monotone increasing on $[0, \infty]$. For any $0 \leq k \leq$ $b \leq c$, we have

$$
\int_{b-k}^{b+k}|f(x-k)| g(x) d x \leq \int_{b-k}^{c-k}|f(x-k)-f(x+k)| g(x) d x+\int_{c-k}^{c+k}|f(x-k)| g(x) d x .
$$

We have the following basic Lemma for the map $T$ (5.92). See the left figure in Figure 2 for an illustration of the sign of the kernel and the map.

Lemma 5.5. For $y_{1}, y_{2}>0$, equation (5.92) has a unique solution $T$ in $(0, \infty)$. For $y_{1}, y_{2}>0$, $T(y)$ is decreasing in $y_{2}$, and $\left|\frac{d}{d y_{2}} T(y)\right| \leq 1$ if $T(y) \leq y_{2} ; T\left(y_{1}, y_{2}\right)$ is decreasing in $y_{1}$ for $y_{1} \leq 1$, and $T$ is increasing in $y_{1}$ for $y_{1}>1$.

Proof. We fix $y_{1}, y_{2}>0$. Recall the equation (5.92)

$$
\begin{equation*}
g(T) \triangleq T^{3}+T^{2} y_{2}+T\left(y_{2}^{2}+2+2 y_{1}^{2}\right)-\frac{\left(y_{1}^{2}-1\right)^{2}}{y_{2}}=0 \tag{5.40}
\end{equation*}
$$

Since $g(0)<0$ and $g(\infty)>0$, there exists at least one real root in $(0, \infty)$. The uniqueness is discussed below (5.92) and follows from the discriminant. Taking $y_{2}$ derivative, we yield

$$
\frac{d T}{d y_{2}}\left(3 T^{2}+2 T y_{2}+y_{2}^{2}+2+2 y_{1}^{2}\right)+T^{2}+2 T y_{2}+\frac{\left(y_{1}^{2}-1\right)^{2}}{y_{2}^{2}}=0
$$

which implies $\frac{d T}{d y_{2}}<0$. For $T(y) \leq y_{2}$, using (5.40), we get
$T^{2}+2 T y_{2}+\frac{\left(y_{1}^{2}-1\right)^{2}}{y_{2}^{2}}=T^{2}+2 T y_{2}+\frac{T^{3}+T^{2} y_{2}+T\left(y_{2}^{2}+2+2 y_{1}^{2}\right)}{y_{2}} \leq T^{2}+2 T y_{2}+2 T^{2}+\left(y_{2}^{2}+2+2 y_{1}^{2}\right)$,
which along with the formula of $\frac{d T}{d y_{2}}$ implies $\left|\frac{d T}{d y_{2}}\right| \leq 1$.
Taking $y_{1}$ derivative, we yield

$$
\frac{d T}{d y_{1}}\left(3 T^{2}+2 T y_{2}+y_{2}^{2}+2+2 y_{1}^{2}\right)+4 T y_{1}-\frac{4 y_{1}\left(y_{1}^{2}-1\right)}{y_{2}}=0 .
$$

From the above equation for $T$, we get

$$
\begin{aligned}
& \left(y_{1}^{2}-1\right)^{2}=y_{2}\left(T^{3}+T^{2} y_{2}+T\left(y_{2}^{2}+2+2 y_{1}^{2}\right)\right)>\left(T y_{2}\right)^{2} \\
& y_{1}^{2}-1>T y_{2}, \quad \text { for } y_{1}>1, \quad y_{1}^{2}-1<-T y_{2}, \quad \text { for } y_{1}<1
\end{aligned}
$$

Hence, we prove the monotonicity properties of $T$.
The above derivations provide the formula of $\nabla T$ in terms of $T, y_{1}, y_{2}$. We can use the monotonicity of $T$ to obtain its piecewise bounds. For large $y_{1}, y_{2}$, to reduce the round off error, following the argument in Section 5.3.1 and (5.32) and using (5.40), we obtain another piecewise bounds for $T$

$$
\begin{equation*}
\hat{T} \geq T=\hat{T}-\frac{T^{3}+T^{2} y_{2}}{y_{2}^{2}+2+2 y_{1}^{2}} \geq \hat{T}-\frac{\hat{T}^{3}+\hat{T}^{2} y_{2}}{y_{2}^{2}+2+2 y_{1}^{2}}, \quad \hat{T} \triangleq \frac{\left(y_{1}^{2}-1\right)^{2}}{y_{2}\left(y_{2}^{2}+2+2 y_{1}^{2}\right)} \tag{5.41}
\end{equation*}
$$

Using the formulas of $\nabla T$ and the bounds for $T$, we can further obtain the piecewise bounds for $\nabla T$. Near the singularity of the kernel $( \pm 1,0)$, we need a sharp bound for $T(y)-y_{2}$ by $C|y-1|$ so that it can cancel the singularity of the kernel with order -2 . Note that the bound based on the piecewise bounds for $T(y)-y_{2}$ does not provide the order $|y-1|$ when $y$ is sufficiently close to 1. Recall $s_{c}\left(y_{1}\right)$ from (5.89). It satisfies $T\left(y_{1}, s_{c}\left(y_{1}\right)\right)=s_{c}\left(y_{1}\right)$. Following (5.36), for $y_{2} \geq s_{c}\left(y_{1}\right) \geq T(y)$ we have
$\left|y_{2}-T\left(y_{1}, y_{2}\right)\right|=\left|y_{2}-T\left(y_{1}, y_{2}\right)-s_{c}\left(y_{1}\right)-T\left(y_{1}, s_{c}\left(y_{1}\right)\right)\right| \leq\left|y_{1}-s_{c}\left(y_{1}\right)\right|\left(\max _{\xi \in\left[s_{c}\left(y_{1}\right), y_{2}\right]}\left|\partial_{y_{2}} T\left(y_{1}, \xi\right)\right|+1\right)$.
Using the formula (5.89) and

$$
\left(2 y_{1}^{2}-3 y_{1}+2\right)^{2}-\left(y_{1}^{4}-y_{1}^{2}+1\right)=3 y_{1}^{4}-12 y_{1}^{3}+(8+9+1) y_{1}^{2}-12 y_{1}+3=3\left(y_{1}-1\right)^{4} \geq 0
$$

we yield

$$
\begin{aligned}
\left(s_{c}\left(y_{1}\right)\right)^{2} & =\frac{-\left(y_{1}^{2}+1\right)+2\left(y_{1}^{4}-y_{1}^{2}+1\right)^{1 / 2}}{3}=\frac{-\left(y_{1}^{2}+1\right)^{2}+4\left(y_{1}^{4}-y_{1}^{2}+1\right)}{3\left(y_{1}^{2}+1+2\left(y_{1}^{4}-y_{1}^{2}+1\right)^{1 / 2}\right)} \\
& \geq \frac{3\left(y_{1}^{2}-1\right)^{2}}{3\left(y_{1}^{2}+1+2\left(2 y_{1}^{2}-3 y_{1}+2\right)\right)}=\frac{\left(y_{1}-1\right)^{2}\left(y_{1}+1\right)^{2}}{5 y_{1}^{2}-6 y_{1}+5}
\end{aligned}
$$

Thus, we get

$$
y_{2}-s_{c}\left(y_{1}\right) \leq y_{2}-\left|y_{1}-1\right| f\left(y_{1}\right) \leq y_{2}, \quad f\left(y_{1}\right)=\frac{y_{1}+1}{\left(5 y_{1}^{2}-6 y_{1}+5\right)^{1 / 2}}
$$

Near $y=1, f\left(y_{1}\right) \approx 1$ and the upper bound is $O(|y-1|)$. Moreover, from the above formula, since $\left(y_{1}^{4}-y_{1}^{2}+1\right)^{1 / 2} \geq\left(y_{1}^{2}\right)^{1 / 2}=y_{1}$, we have
$\left(s_{c}\left(y_{1}\right)\right)^{2}=\frac{3\left(y_{1}-1\right)^{2}\left(y_{1}+1\right)^{2}}{3\left(y_{1}^{2}+1+2\left(y_{1}^{4}-y_{1}^{2}+1\right)^{1 / 2}\right)} \leq \frac{3\left(y_{1}-1\right)^{2}\left(y_{1}+1\right)^{2}}{3\left(y_{1}^{2}+1+2 y_{1}\right)}=\left(y_{1}-1\right)^{2}, \quad s_{c}\left(y_{1}\right) \leq\left|y_{1}-1\right|$.
5.4.1. Estimate the explicit integrals for $u_{y}, v_{x}$. We follow the strategy in Section5.2 to estimate the integrals in the $C_{x}^{1 / 2}$ estimate of $v_{x}, u_{y}$. Recall from Appendix B. 1 [3] the estimate of $\left[v_{x}\right]_{C_{x}^{1 / 2}}$

$$
\begin{align*}
& 2^{-1 / 2}\left|v_{x}\left(-1, x_{2}\right)-v_{x}\left(1, x_{2}\right)\right| \leq 2^{-1 / 2} \frac{1}{\pi}\left(\left(S_{i n, x}+S_{1 D}\right)[\omega]_{C_{x}^{1 / 2}}+4\left(S_{i n, y}+S_{o u t}\right)[\omega]_{C_{y}^{1 / 2}}\right), \\
& S_{i n, x}=\int_{y_{1} \notin J_{1}, y_{1} \geq 0} \sqrt{2 y_{1}}\left|\Delta_{1 D}\left(y_{1}\right)\right| d y_{1}, \\
& S_{i n, y}=S_{u p}+S_{l o w}, S_{i n, y, n s}=S_{u p, n s}(\varepsilon)+S_{l o w}, S_{u p}=S_{u p, n s}(\varepsilon)+S_{i n, y, \varepsilon}, \\
& S_{u p, n s}(\varepsilon)=\int_{y_{1} \in \mathbb{R}_{+} \backslash[1-\varepsilon, 1+\varepsilon]} \int_{s_{c}\left(y_{1}\right)}^{A}\left|T(y)-y_{2}\right|^{\frac{1}{2}}|\Delta(y)| d y_{2} d y_{1},  \tag{5.43}\\
& S_{l o w}=\int_{y_{1} \in J_{1}^{+} \backslash[1-\varepsilon, 1+\varepsilon]} \int_{0}^{T\left(y_{1}, A\right)}\left|y_{2}\right|^{\frac{1}{2}}|\Delta(y)| d y_{2} d y_{1}, \\
& S_{o u t}=\frac{1}{4} \sqrt{2 x_{2}} \int_{\mathbb{R}} \int_{x_{2}}^{B}|\Delta(y)| d y=\frac{1}{2} \sqrt{2 x_{2}} \int_{\mathbb{R}^{+}} \int_{x_{2}}^{B}|\Delta(y)| d y, \\
& S_{1 D}=2 \int_{\frac{1}{9}}^{9}\left|\Delta_{1 D}\right| y_{1}-\left.1\right|^{1 / 2} d y_{1}+\int_{0}^{\frac{1}{9}}\left|\Delta_{1 D}\right| \sqrt{2 y_{1}} d y_{1}+\left(\pi+P\left(\frac{1}{9}\right)\right) \sqrt{2},
\end{align*}
$$

and the estimate of $\left[u_{y}\right]_{C_{x}^{1 / 2}}$

$$
\begin{align*}
& 2^{-1 / 2}\left|u_{y}\left(-1, x_{2}\right)-u_{y}\left(1, x_{2}\right)\right| \leq 2^{-1 / 2} \frac{1}{\pi}\left(\tilde{S}_{1 D}[\omega]_{C_{x}^{1 / 2}}+4\left(S_{u p}+S_{o u t}\right)[\omega]_{C_{y}^{1 / 2}}\right) \\
& \tilde{S}_{1 D}=\int_{\mathbb{R}_{+}}\left|\Delta_{1 D}\left(y_{1}\right)\right| \sqrt{2 y_{1}} d y_{1} \tag{5.44}
\end{align*}
$$

where $S_{o u t}$ and $S_{u p}$ are given above,

$$
\begin{align*}
& A=\min \left(x_{2}, B\right), \quad P(k) \triangleq-\int_{k}^{9}\left|\Delta_{1 D}\left(y_{1}\right)\right| d y_{1}, \quad J_{1}=[-9,9], \quad J_{1}^{+}=[0,9]  \tag{5.45}\\
& \Delta_{1 D}\left(y_{1}\right)=g_{A}\left(y_{1}+1\right) \mathbf{1}_{\left|y_{1}+1\right| \leq B}-g_{A}\left(y_{1}-1\right) \mathbf{1}_{\left|y_{1}-1\right| \leq B}, \quad g_{b}(y)=\frac{b}{y^{2}+b^{2}}
\end{align*}
$$

$s_{c}\left(y_{1}\right)$ is given below, and $S_{i n, y, \varepsilon, \text {. estimates the following integrals near the singularity }}$

$$
\begin{align*}
& I_{ \pm}(\varepsilon) \triangleq \int_{J_{\varepsilon}} \int_{0}^{A} K_{2}\left(y_{1} \pm 1, y_{2}\right) \tilde{W}(y) d y, \quad I(\varepsilon) \triangleq \int_{J_{\varepsilon}} \int_{0}^{A} \Delta(y) \tilde{W}(y) d y=I_{+}(\varepsilon)-I_{-}(\varepsilon) \\
& \left|I_{ \pm}(\varepsilon)\right| \leq S_{i n, y, \varepsilon, \pm} \cdot[\omega]_{C_{y}^{1 / 2}}, \quad|I(\varepsilon)| \leq S_{i n, y, \varepsilon}, \quad \tilde{W}(y)=\omega\left(y_{1}, x_{2}-y_{2}\right)-\omega\left(y_{1}, x_{2}\right)  \tag{5.46}\\
& J_{\varepsilon} \triangleq[1-\varepsilon, 1+\varepsilon]
\end{align*}
$$

We will estimate $S_{i n, y, \delta, \pm}$ in Section 5.4.3. Note that the above variables, e.g. $S_{o u t}, S_{i n, x}$, are different from [3] by a constant. Here, we have restricted the domain of the integral to $\mathbb{R}_{2}^{++}$due to symmetry. The upper bounds for $v_{x}, u_{y}$ are the same as [3]. The subscript $n s$ in (5.43) is short for non-singular. For $s_{1}, s_{2}>0$, the map $T$ and $s_{c}\left(s_{1}\right)$ are obtained from

$$
\begin{aligned}
& \Delta(s)=K_{2, B}\left(s_{1}+1, s_{2}\right)-K_{2, B}\left(s_{1}-1, s_{2}\right), \quad \Delta\left(s_{1}, s_{c}\left(s_{1}\right)\right)=0 \\
& \int_{T}^{y_{2}} \Delta\left(y_{1}, s_{2}\right) d s_{2}=0, \quad K_{2, B}(s)=\frac{1}{2} \frac{s_{1}^{2}-s_{2}^{2}}{|s|^{4}} \mathbf{1}_{\left|s_{1}\right|,\left|s_{2}\right| \leq B}
\end{aligned}
$$

When $0 \leq s_{1} \leq B-1$, we have

$$
\begin{equation*}
\Delta(s)=\Delta_{i n}(s)=K_{2}\left(s_{1}+1, s_{2}\right)-K_{2}\left(s_{1}-1, s_{2}\right), \quad s_{c}\left(s_{1}\right)=s_{c, \text { in }}\left(s_{1}\right) \tag{5.47}
\end{equation*}
$$

$s_{c, i n}$ is given by (5.89), and the map $T$ is given by (5.92), which is denoted as $T_{i n}$. See the left figure in Figure 2 for an illustrations of the sign and the map in the inner regions. For $s_{1} \in[B-1, B+1]$, we get

$$
\begin{equation*}
T_{o u t}=\frac{\left(y_{1}-1\right)^{2}}{y_{2}}, \quad \Delta(s)=\Delta_{\text {out }}(s)=-K_{2}\left(s_{1}-1, s_{2}\right), \quad s_{c}\left(s_{1}\right)=s_{1}-1 \tag{5.48}
\end{equation*}
$$



Figure 2. Illustration of the sign of the kernel $\Delta(y)$ and transportation plan. The sign of $\Delta(y)$ in different regions is indicated by $\pm$. The blue arrows indicate the direction of 1D transportation plan. Left for $C_{x}^{1 / 2}$ estimate: The black curve and the red curve represents $y_{2}= \pm s_{c, i n}\left(y_{1}\right), y_{2}= \pm T_{i n}\left(y_{1}, A\right)$ for $y_{1} \geq 0$, respectively. Right for $C_{y}^{1 / 2}$ estimate: The black curve is for $y_{2}= \pm s_{c}\left(y_{1}\right)$, or equivalently $y_{1}=h_{c}^{-}\left(\left|y_{2}\right|\right)$ (two left black curves) and $y_{1}=h_{c}^{+}\left(y_{2}\right)$ (two right black curves). The red curve represents $y_{1}=T\left(m,\left|y_{2}\right|\right)$. Note that these curves do not agree with the actual functions.

For $y_{1} \geq 1$ or $y_{1} \leq 1$ and $y_{2}>0$, the piecewise bounds for $T_{\text {out }}(y)$ are trivial, and $T_{\text {out }}$ is decreasing in $y_{2}$.

The above integral depends on two parameters $x_{2}, B$. Denote $A=\min \left(x_{2}, B\right)$. Our goal is to obtain a uniform bound for all $0 \leq A \leq B \leq \infty, B \geq 2$. We partition these two parameters $0=A_{1}<. .<A_{n_{1}}<A_{n_{1}+1}=\infty, 2 \leq B_{1}<B_{2}<. .<B_{n_{2}+1}=\infty$, and estimate the bound for $A \in\left[A^{l}, A^{u}\right], B \in\left[B^{l}, B^{u}\right]$. We discuss how to estimate each part below.
5.4.2. Bulk part $S_{i n, y}$. Using the above notation and the localization of the kernel, we have

$$
\begin{align*}
S_{i n, y, n s}(\varepsilon)= & \int_{[0, B-1] \backslash J_{\varepsilon}} \int_{0}^{A}\left(\mathbf{1}_{y_{2} \geq s_{c}\left(y_{1}\right)}\left|T_{i n}(y)-y_{2}\right|^{1 / 2}+\mathbf{1}_{y_{1} \leq 9} \mathbf{1}_{y_{2} \leq T_{\text {in }}\left(y_{1}, A\right)} y_{2}^{1 / 2}\right)\left|\Delta_{\text {in }}(y)\right| d y  \tag{5.49}\\
& +\int_{B-1}^{B+1} \int_{0}^{A}\left(\mathbf{1}_{y_{2} \geq s_{c}\left(y_{1}\right)}\left|T_{\text {out }}(y)-y_{2}\right|^{1 / 2}+\mathbf{1}_{y_{1} \leq 9} \mathbf{1}_{y_{2} \leq T_{\text {out }}\left(y_{1}, A\right)} y_{2}^{1 / 2}\right)\left|\Delta_{\text {out }}(y)\right| d y \\
\triangleq & S_{\text {in }, y 1, n s}+S_{\text {in }, y 2}
\end{align*}
$$

where we have combined the integrals

$$
\mathbf{1}_{y_{1} \in J_{1}} \mathbf{1}_{y_{2} \in\left[s_{c}\left(y_{1}\right), A\right]}+\mathbf{1}_{y_{1} \notin J_{1}} \mathbf{1}_{y_{2} \in\left[s_{c}\left(y_{1}\right), A\right]}
$$

related to $\left|T-y_{2}\right|^{1 / 2}$ in $S_{i n, y}$ in (5.43). Since $T\left(y_{1}, y_{2}\right)$ is decreasing in $y_{2}$ (see Lemma 5.5, for $A \in\left[A_{l}, A_{u}\right], B \in\left[B_{l}, B_{u}\right]$, clearly, we have

$$
\begin{aligned}
S_{i n, y 1, n s} & \leq \int_{\left[0, B^{u}-1\right] \backslash[1-\varepsilon, 1+\varepsilon]} \int_{0}^{A^{u}}\left(\mathbf{1}_{y_{2} \geq s_{c}\left(y_{1}\right)}\left|T_{\text {in }}(y)-y_{2}\right|^{1 / 2}+\mathbf{1}_{y_{1} \leq 9} \mathbf{1}_{y_{2} \leq T_{i n}\left(y_{1}, A^{l}\right)} y_{2}^{1 / 2}\right)\left|\Delta_{i n}(y)\right| d y \\
S_{i n, y 2} & \leq \int_{B^{l}-1}^{B^{u}+1} \int_{0}^{A^{u}}\left(\mathbf{1}_{y_{2} \geq s_{c}\left(y_{1}\right)}\left|T_{\text {out }}(y)-y_{2}\right|^{1 / 2}+\mathbf{1}_{y_{1} \leq 9} \mathbf{1}_{y_{2} \leq T_{\text {out }}\left(y_{1}, A^{l}\right)} y_{2}^{1 / 2}\right)\left|\Delta_{\text {out }}(y)\right| d y
\end{aligned}
$$

We choose our mesh aligning with $y_{1}=9, y_{1}=1 \pm \varepsilon, y_{2}=A_{u}$ to partition the integrals so that for each small domain $Q=\left[y_{1}^{l}, y_{1}^{u}\right] \times\left[y_{2}^{l}, y_{2}^{u}\right]$, the restrictions are automatically imposed i.e.

$$
\mathbf{1}_{y_{1} \leq 9} \mathbf{1}_{y_{2} \leq A_{u}} \mathbf{1}_{Q} \mathbf{1}_{[1-\varepsilon, 1+\varepsilon]}\left(y_{1}\right)=\mathbf{1}_{Q}, \text { or } 0
$$

We follow the strategy in Section 5.2 to handle the indicators, the integral $\int_{Q}\left|\Delta_{\alpha}(s)\right|$, and $\left|T_{\alpha}-y_{2}\right|^{1 / 2}, \alpha=$ in, out. The analytic integral formula for $K_{2}(s)$ and $K_{2}(s) s_{2}^{1 / 2}$ is given in Section 5.1, and the estimate of $\left|T_{i n}-y_{2}\right|^{1 / 2}$ is given in Section 5.4 after Lemma 5.5. The
estimate of the second part $S_{i n, y 2}$ is much easier since the integrand is supported away from the singularity $(1,0)$ since $B \geq 3$ and the map $T_{\text {out }}$ (5.48) and $s_{c}\left(y_{1}\right)$ (5.89) are simple.

For $S_{i n, y 1, n s}$, we apply the above strategy to estimate the integrals in $[0,1-\varepsilon] \times[0, A],[1+$ $\varepsilon, \infty)$. For the first integrand in $S_{i n, y 2}$, if $B^{u}-B^{l}$ is large, which is the case for large $B$ since we partition the domain of $B$ using adaptive mesh, the above estimate for $S_{i n, y 2}$ is not sufficient. Note that the second integrand in $S_{i n, y 2}$ vanishes for large $B$ since it is supported in $y_{1} \leq 9$.
Additional estimate for $S_{i n, y 2}$. We consider another estimate for $S_{i n, y 2}$ by exploiting the boundedness of the interval $y_{1} \in[B-1, B+1]$

$$
\begin{aligned}
I & =\int_{B-1}^{B+1} \int_{0}^{A} \mathbf{1}_{y_{2} \geq s_{c}\left(y_{1}\right)}\left|T_{\text {out }}(y)-y_{2}\right|^{1 / 2}\left|\Delta_{\text {out }}(y)\right| d y \\
& =\frac{1}{2} \int_{B-1}^{B+1} \int_{y_{1}-1}^{A}\left|\frac{\left(y_{1}-1\right)^{2}}{y_{2}}-y_{2}\right|^{1 / 2}\left|K_{2}\left(y_{1}-1, y_{2}\right)\right| d y=\frac{1}{2} \int_{B-2}^{B} \int_{y_{1}}^{A}\left|\frac{y_{1}^{2}}{y_{2}}-y_{2}\right|^{1 / 2}\left|K_{2}(y)\right| d y
\end{aligned}
$$

in $S_{i n, y 2}$ in (5.49), where we have used $s_{c}\left(y_{1}\right)=y_{1}-1$ (5.48), and a change of varialbes $y_{1} \rightarrow y_{1}+1$. Since $A \leq B$ (5.45) and

$$
B-2 \leq y_{1} \leq y_{2} \leq A \leq B \leq y_{1}+2, \quad y_{1}+y_{2} \leq \sqrt{2}\left(y_{1}^{2}+y_{2}^{2}\right)^{1 / 2}, \quad|y| \geq \sqrt{2}(B-2)
$$

in the support, we get

$$
\begin{aligned}
\left|\frac{y_{1}^{2}}{y_{2}}-y_{2}\right|^{1 / 2}\left|K_{2}(y)\right| & =\left|\frac{\left(y_{2}-y_{1}\right)\left(y_{2}+y_{1}\right)}{y_{2}}\right|^{1 / 2} \frac{\left(y_{2}-y_{1}\right)\left(y_{2}+y_{1}\right)}{|y|^{4}} \leq(2 \cdot 2)^{1 / 2} 2 \sqrt{2}|y|^{-3} \\
& \leq 4 \sqrt{2}(\sqrt{2})^{-3}(B-2)^{-3}=2(B-2)^{-3}
\end{aligned}
$$

Since $A \leq B, B_{l} \leq B$, we yield

$$
I \leq \frac{1}{2} \int_{B-2}^{B} \int_{y_{1}}^{A} 2(B-2)^{-3} d y \leq(B-2)^{-3} \int_{B-2}^{B}\left(B-y_{1}\right) d y_{1}=2(B-2)^{-3} \leq 2\left(B_{l}-2\right)^{-3}
$$

5.4.3. Near the singularity. Near $s_{*}=(1,0)$, the integrand in $S_{i n, y}$ (5.43), (5.46) is singular of order $|x|^{-3 / 2}$ and quite complicated. To ease our computation, we use another estimate and separate the estimate of two kernels in $\Delta(s)=K_{2}\left(s_{1}+1, s_{2}\right)-K_{2}\left(s_{1}-1, s_{2}\right)$. Below, we estimate $I_{ \pm}(\varepsilon)$ from (5.46) and derive the bound $S_{i n, y, \varepsilon, \pm}$. We fix $\varepsilon$ and then drop $\varepsilon$ for simplicity.
(a) Regular part. Since the kernel $K_{2}\left(y_{1}+1, y_{2}\right)$ is regular, using $K_{2}(y)=\frac{1}{2} \frac{y_{1}^{2}-y_{2}^{2}}{|y|^{4}}$, (5.46),

$$
\begin{equation*}
|\tilde{W}| \leq y_{2}^{1 / 2}[\omega]_{C_{y}^{1 / 2}}, \quad \int_{1-\varepsilon}^{1+\varepsilon}\left|K_{2}\left(y_{1}+1, y_{2}\right)\right| d y_{1} \leq \int_{1-\varepsilon}^{1+\varepsilon} \frac{1}{2}|y|^{-2} d y_{1} \leq \frac{\varepsilon}{(2-\varepsilon)^{2}+y_{2}^{2}} \tag{5.50}
\end{equation*}
$$

we get

$$
\begin{equation*}
\left|I_{+}(\varepsilon)\right| \leq \int_{1-\varepsilon}^{1+\varepsilon} \int_{0}^{A}\left|K_{2}\left(y_{1}+1, y_{2}\right)\right| y_{2}^{1 / 2} d y[\omega]_{C_{y}^{1 / 2}} \leq[\omega]_{C_{y}^{1 / 2}} \int_{0}^{A^{u}} \frac{\varepsilon y_{2}^{1 / 2}}{(2-\varepsilon)^{2}+y_{2}^{2}} d y_{2} \tag{5.51}
\end{equation*}
$$

where we have used the fact that the integral is increasing in $A$ and $A \in\left[A^{l}, A^{u}\right]$ in the last inequality. Following the strategy (d) in Section 5.2, we partition the domain of the integral using the same mesh as that for $S_{i n, y}$. In each interval $\left[y_{2}^{l}, y_{2}^{u}\right]$, we have

$$
\begin{equation*}
\int_{y^{l}}^{y^{u}} \frac{\varepsilon y_{2}^{1 / 2}}{(2-\varepsilon)^{2}+y_{2}^{2}} d y \leq\left.\sqrt{y^{u}} \frac{\varepsilon}{2-\varepsilon} \arctan \left(\frac{y_{2}}{2-\varepsilon}\right)\right|_{y^{l}} ^{y^{u}} \tag{5.52}
\end{equation*}
$$

(b) Singular part. Denote

$$
a=\min (A, \varepsilon), \quad Q_{a} \triangleq[-a, a] \times[0, a], \quad Q_{a}^{+} \triangleq[0, a]^{2} .
$$

Recall $I_{-}(\varepsilon)$ from (5.46). We have

$$
\begin{aligned}
I_{-}(\varepsilon) & \triangleq\left|\int_{1-\varepsilon}^{1+\varepsilon} \int_{0}^{A} K_{2}\left(y_{1}-1, y_{2}\right) \tilde{W}(y) d y\right|=\left|\int_{-\varepsilon}^{\varepsilon} \int_{0}^{A} K_{2}(y) \tilde{W}\left(y_{1}+1, y_{2}\right) d y\right| \\
& \leq\left|\int_{Q_{a}} K_{2}\left(y_{1}, y_{2}\right) \tilde{W}\left(y_{1}+1, y_{2}\right) d y\right|+\left|\int_{[-\varepsilon, \varepsilon] \times[0, A] \backslash Q_{a}} K_{2}\left(y_{1}, y_{2}\right) \tilde{W}\left(y_{1}+1, y_{2}\right) d y\right| \triangleq I_{-, 1}+I_{-, 2} .
\end{aligned}
$$

For the second part, using the bound for $\tilde{W}$ (5.50), we get

$$
] I_{-, 2}\left|\leq\left|\int_{[-\varepsilon, \varepsilon] \times[0, A] \backslash Q(a)}\right| K_{2}\left(y_{1}, y_{2}\right)\right| y_{2}^{1 / 2} d y=2 \int_{[0, \varepsilon] \times[0, A] \backslash Q_{a}^{+}}\left|K_{2}(y)\right| y_{2}^{1 / 2} d y .
$$

If $A \geq A^{l} \geq \varepsilon$, we yield $a=\varepsilon, A \leq A^{u}$. Using $K_{2}(y)=\frac{1}{2} \frac{y_{1}^{2}-y_{2}^{2}}{|y|^{4}} \leq 0$ for $y_{1} \leq \varepsilon \leq y_{2}$, we yield

$$
\left.I_{-, 2} \leq 2 \int_{0}^{\varepsilon} \int_{\varepsilon}^{A^{u}}\left|K_{2}(y)\right| y_{2}^{1 / 2} d y=\left|\int_{\varepsilon}^{A^{u}} \frac{y_{1}}{|y|^{2}}\right|_{y_{2}=0}^{\varepsilon} y_{2}^{1 / 2} d y_{2} \right\rvert\,=\int_{\varepsilon}^{A^{u}} \frac{\varepsilon}{\varepsilon^{2}+y_{2}^{2}} y_{2}^{1 / 2} d y_{2} .
$$

We use the same method as (5.52) and $\frac{\varepsilon}{\varepsilon^{2}+y_{2}^{2}}=\frac{d}{d y_{2}} \arctan \left(y_{2} / \varepsilon\right)$ to estimate the integral.
We choose $\varepsilon$ from the mesh for $A$. Then when $A^{l}<\varepsilon$, we get $A^{u} \leq \varepsilon$. (This is similar to $a<b, a, b \in \mathbb{Z}$ implies $a+1 \leq b$.) Since $A \leq A^{u} \leq \varepsilon$, we get $K_{2}(y)=\frac{1}{2} \frac{y_{1}^{2}-y_{2}^{2}}{|y|^{4}} \geq 0$ in $[A, \varepsilon] \times[0, A]$,

$$
\begin{aligned}
I_{-, 2} & \leq 2 \int_{A}^{\varepsilon} \int_{0}^{A} K_{2}(y) y_{2}^{1 / 2} d y=\int_{0}^{A}-\left.\frac{y_{1}}{|y|^{2}}\right|_{A} ^{\varepsilon} y_{2}^{1 / 2} d y_{2}=\int_{0}^{A}\left(\frac{A}{A^{2}+y_{2}^{2}}-\frac{\varepsilon}{\varepsilon^{2}+y_{2}^{2}}\right) y_{2}^{1 / 2} d y_{2} \\
& =2 A^{1 / 2} f_{s}(1)-2 \varepsilon^{1 / 2} f_{s}\left((A / \varepsilon)^{1 / 2}\right) \leq 2 A_{u}^{1 / 2} f_{s}(1)-2 \varepsilon^{1 / 2} f_{s}\left(\left(A_{l} / \varepsilon\right)^{1 / 2}\right),
\end{aligned}
$$

where we have used (5.5), (5.6).
For $I_{-, 1}$ and a fixed $x_{2}$, using the scaling relation, $(5.46)\left[\tilde{W}\left(a y_{1}, a y_{2}\right)\right]_{C_{y}^{1 / 2}\left(Q_{1}\right)}=a^{1 / 2}[\tilde{W}]_{C_{y}^{1 / 2}\left(Q_{a}\right)} \leq$ $a^{1 / 2}[\omega]_{C_{y}^{1 / 2}}$, and (5.86), we get

$$
\left|I_{-, 1}\right| \leq\left|\int_{Q_{1}} K_{2}\left(y_{1}, y_{2}\right) \tilde{W}\left(a y_{1}+1, a y_{2}\right) d y\right| \leq 2 C_{K_{2}}\left[\tilde{W}\left(a y_{1}+1, a y_{2}\right)\right]_{C_{y}\left(Q_{1}\right)} \leq 2 \min (\varepsilon, A)^{1 / 2} C_{K_{2}}[\omega]_{C_{y}^{1 / 2}} .
$$

5.4.4. Estimate of $S_{\text {out }}$. In this section, we estimate $S_{\text {out }}$ in (5.43), which is much easier than that of $S_{i n, y}$. Note that if $B<x_{2}$, we have $S_{\text {out }}=0$. Thus, we consider $x_{2}<B$, and get $A=\min \left(x_{2}, B\right)=x_{2}$. Using (5.47) and (5.48), we yield

$$
\begin{equation*}
S_{\text {out }}(A, B) \triangleq \frac{\sqrt{2 x_{2}}}{2} \int_{\mathbb{R}} \int_{x_{2}}^{B}|\Delta(y)| d y=\frac{\sqrt{2 A}}{2} \int_{A}^{B} d s_{2}\left(\int_{0}^{B-1}\left|\Delta_{\text {in }}(s)\right| d s_{1}+\int_{B-1}^{B+1} \Delta_{\text {out }}(s) d s_{1}\right) . \tag{5.53}
\end{equation*}
$$

For a fixed $y_{2}$, applying Lemma 5.4 with $k=1, f(x)=K\left(x, y_{2}\right), b=B, c=B^{u}$, and $B \leq B_{u}$ we obtain

$$
\begin{equation*}
\int_{B-1}^{B+1}\left|K_{2}\left(s-1, s_{2}\right)\right| d s_{1} \leq \int_{B-1}^{B^{u}-1}\left|K_{2}\left(s-1, s_{2}\right)-K_{2}\left(s+1, s_{2}\right)\right|+\int_{B^{u}-1}^{B^{u}+1}\left|K_{2}\left(s_{1}-1, s_{2}\right)\right| d s_{1} . \tag{5.54}
\end{equation*}
$$

As a result, $S_{\text {out }}(A, B)$ is increasing in $B$ and we get $S_{\text {out }}(A, B) \leq S_{\text {out }}\left(A, B^{u}\right)$. Denote $I_{\varepsilon}=[1-\varepsilon, 1+\varepsilon]$. Firstly, using $\Delta(s)=\Delta_{\text {in }}(s)$ for $y_{1} \in I_{\varepsilon}, y_{2} \in[A, B]$, we have

$$
S_{\text {out }} \leq \frac{\sqrt{2 A^{u}}}{2} \int_{A^{l}}^{B^{u}} \int_{y_{1} \in\left[0, B^{u}+1\right] \backslash I_{\varepsilon}}|\Delta(s)| d s+\frac{\sqrt{2 A}}{2} \int_{A}^{B} \int_{I_{\varepsilon}}\left|\Delta_{\text {in }}(s)\right| d s \triangleq S_{\text {out }, 1}+S_{\text {out }, 2} .
$$

We do not bound the integral region $[A, B]$ in $S_{\text {out }, 2}$ at this moment. The first part is away from the singularity. We partition the domain and apply the strategy in Section 5.2 to estimate it.

In particular, the integral in $S_{\text {out }, 1}$ with $y_{1} \in\left[B^{u}-1, B^{u}+1\right]$ is given by

$$
I=\int_{A^{l}}^{B^{u}} \int_{B^{u}-1}^{B^{u}+1}\left|K_{2}\left(s_{1}-1, s_{2}\right)\right| d s=\int_{A^{l}}^{B^{u}} \int_{B^{u}-2}^{B^{u}}\left|K_{2}\left(s_{1}, s_{2}\right)\right| d s
$$

We decompose the domain of the integral as follows
$\left[B^{u}-2, B^{u}\right] \times\left[A^{l}, B^{u}\right]=\left[B^{u}-2, B^{u}\right] \times\left[A^{l}, B^{u}-2\right] \cup\left[B^{u}-2, B^{u}\right]^{2} \triangleq Q_{1} \cup Q_{2}, \quad A^{l} \leq B^{u}-2$,
$\left[B^{u}-2, B^{u}\right] \times\left[A^{l}, B^{u}\right]=\left[B^{u}-2, A^{l}\right] \times\left[A^{l}, B^{u}\right] \cup\left[A^{l}, B^{u}\right]^{2} \triangleq Q_{1} \cup Q_{2}, \quad A^{l}>B^{u}-2$.
In $Q_{1}$, since $y_{1}^{2}-y_{2}^{2}$ has a fixed sign, $K_{2}(s)$ has a fixed sign. We apply the analytic formula for $K_{2}(s)$ to evaluate the integral in $Q_{1}$. In $Q_{2}$, we use the formula (5.10).

For $S_{\text {out }, 2}$, we denote $A_{\varepsilon, l}=\max \left(\varepsilon, A^{l}\right)$. We decompose $S_{\text {out }, 2}$ as follows

$$
S_{o u t, 2} \leq \frac{\sqrt{2 A}}{2} \int_{A_{\varepsilon, l}}^{B} \int_{1-\varepsilon}^{1+\varepsilon}\left|\Delta_{i n}(s)\right| d s+\frac{\sqrt{2 A}}{2} \int_{A}^{\varepsilon} \int_{1-\varepsilon}^{1+\varepsilon}\left|\Delta_{i n}(s)\right| d s \triangleq I_{1}+I_{2} .
$$

Note that $I_{2}=0$ if $A \geq \varepsilon$. From (5.42), in the support of the integral $I_{1}$, we have $y_{2} \geq \varepsilon \geq$ $\left|y_{1}-1\right| \geq s_{c}\left(y_{1}\right)$ (5.42) and yield that $\Delta(s)$ has a fixed sign. We estimate $I_{1}$ as follows

$$
I_{1} \leq \frac{\sqrt{2 A^{u}}}{2}\left|\int_{A_{\varepsilon, l}}^{B_{u}} \int_{1-\varepsilon}^{1+\varepsilon} \Delta(s) d s\right|
$$

and evaluate the integral using the analytic integral formula for $\Delta(s)$. For $I_{2}$, we use the fact that $\sqrt{A} \leq s_{2}^{1 / 2}$ in the support of the integral and triangle inequality to get

$$
I_{2} \leq \frac{\sqrt{2}}{2} \int_{A}^{\varepsilon} \int_{1-\varepsilon}^{1+\varepsilon}\left|K_{2}\left(s_{1}-1, s_{2}\right)\right| s_{2}^{1 / 2} d s+\frac{\sqrt{2 A^{u}}}{2} \int_{A^{l}}^{\varepsilon} \int_{1-\varepsilon}^{1+\varepsilon}\left|K_{2}\left(s_{1}+1, s_{2}\right)\right| d s=I_{21}+I_{22}
$$

For $I_{22}$, we choose $\varepsilon<\frac{1}{100}$. In the support of the integral, $K_{2}>0$ and we get

$$
I_{22}=\frac{\sqrt{2 A^{u}}}{2}\left|\int_{A^{l}}^{\varepsilon} \int_{1-\varepsilon}^{1+\varepsilon} K_{2}\left(s_{1}+1, s_{2}\right) d s\right|
$$

and can evaluate the integral using the analytic integral formula. For $I_{21}$, we have

$$
I_{21} \leq \sqrt{2} \int_{A^{l}}^{\varepsilon} \int_{0}^{\varepsilon}\left|K_{2}\left(s_{1}, s_{2}\right)\right| s_{2}^{1 / 2} d s \leq \sqrt{2} F_{K_{2}, h}\left(0, \varepsilon, A^{l}, \varepsilon\right)
$$

where $F_{K_{2}, h}$ is defined in (5.7).
5.4.5. Integral $S_{i n, y, \varepsilon}, S_{i n, y, n s}, S_{\text {out }}$ in the far-field. The previous method can handle the case that $A \leq B<\infty$. Next, we consider the remaining case $A \leq B, B^{l} \leq B \leq \infty$ with $B^{l}$ sufficiently large, e.g. $B^{l}=10^{8}$. Recall $S_{i n, y, n s}$ from (5.43), (5.46), (5.49)

$$
\begin{equation*}
S_{i n, y, n s}=\int_{0}^{A} d y_{2}\left(\int_{0}^{B-1} I_{i n, 1}\left|\Delta_{i n}(y)\right| d y_{1}+\int_{B-1}^{B+1} I_{\text {in,2 }}\left|\Delta_{o u t}(y)\right| d y_{1}\right) \triangleq \int_{0}^{A} I_{\text {in }}\left(y_{2}\right) d y_{2} \tag{5.55}
\end{equation*}
$$

where $I_{i n, 1}, I_{i n, 2}$ denotes the integrand,

$$
\begin{aligned}
& I_{\text {in }, 1}=\left(\mathbf{1}_{y_{2} \geq s_{c}\left(y_{1}\right)}\left|T_{\text {in }}(y)-y_{2}\right|^{1 / 2}+\mathbf{1}_{y_{1} \leq 9} \mathbf{1}_{y_{2} \leq T_{\text {in }}\left(y_{1}, A\right)} y_{2}^{1 / 2}\right) \mathbf{1}_{y_{1} \notin J_{\varepsilon}} \\
& I_{\text {in }, 2}=\left(\mathbf{1}_{y_{2} \geq s_{c}\left(y_{1}\right)}\left|T_{\text {out }}(y)-y_{2}\right|^{1 / 2}+\mathbf{1}_{y_{1} \leq 9} \mathbf{1}_{y_{2} \leq T_{\text {out }}\left(y_{1}, A\right)} y_{2}^{1 / 2}\right) \mathbf{1}_{y_{1} \notin J_{\varepsilon}}, \quad J_{\varepsilon}=[1-\varepsilon, 1+\varepsilon] .
\end{aligned}
$$

We decompose the integral in $y_{1}$ as follows

$$
\int_{0}^{B-1} I_{i n, 1}\left|\Delta_{i n}(y)\right| d y_{1}=\int_{0}^{B^{l}-1} I_{i n, 1}\left|\Delta_{i n}(y)\right| d y_{1}+\int_{B^{l}-1}^{B-1} I_{i n, 1}\left|\Delta_{i n}(y)\right| d y_{1} \triangleq I_{2}+I_{3}
$$

Since for $A \geq y_{2} \geq s_{c}\left(y_{1}\right)$, we have $y_{2} \geq s_{c}\left(y_{1}\right) \geq T\left(y_{1}, y_{2}\right) \geq T\left(y_{1}, A\right)$, clearly, we have

$$
\begin{equation*}
\left|I_{i n, 1}\right| \leq\left|y_{2}\right|^{1 / 2} \mathbf{1}_{y_{1} \notin J_{\varepsilon}}, \quad I_{i n, 2} \leq\left|y_{2}\right|^{1 / 2} \mathbf{1}_{y_{1} \notin J_{\varepsilon}} . \tag{5.56}
\end{equation*}
$$

Using (5.56) and combining $I_{3}$ and the integral of $I_{i n, 2}$ in (5.55), we further obtain

$$
I_{3}+\int_{B-1}^{B+1} I_{\text {in,2 }}\left|\Delta_{\text {out }}(y)\right| d y_{1} \leq y_{2}^{1 / 2}\left(\int_{B^{l}-1}^{B-1}\left|\Delta_{\text {in }}(y)\right| d y_{1}+\int_{B-1}^{B+1}\left|\Delta_{o u t}(y)\right| d y_{1}\right)
$$

Using the monotonicity Lemma 5.4, we get

$$
I_{3}+\int_{B-1}^{B+1} I_{i n, 2}\left|\Delta_{o u t}(y)\right| d y_{1} \leq y_{2}^{1 / 2} \int_{B^{l}-1}^{\infty}\left|\Delta_{i n}(y)\right| d y_{1}
$$

It follows

$$
\begin{equation*}
S_{i n, y, n s} \leq \int_{0}^{A} \int_{0}^{B^{l}-1} I_{i n, 1}\left|\Delta_{i n}\right| d y+\int_{0}^{A} y_{2}^{1 / 2} \int_{B^{l}-1}^{\infty}\left|\Delta_{i n}(y)\right| d y_{1} \tag{5.57}
\end{equation*}
$$

for any $B \geq B^{l}$ and a fixed $A$. The first part is the main part.
Applying a similar argument to $S_{\text {out }}$ in (5.53), we yield
$S_{\text {out }}(A, B)=\frac{\sqrt{2 A}}{2} \int_{A}^{B} d s_{2}\left(\int_{0}^{B-1}\left|\Delta_{\text {in }}(s)\right| d s_{1}+\int_{B-1}^{B+1}\left|\Delta_{\text {out }}(s)\right| d s_{1}\right) \leq \frac{\sqrt{2 A}}{2} \int_{A}^{B} \int_{0}^{\infty}\left|\Delta_{\text {in }}(s)\right| d s$,
where $a \vee b=\max (a, b)$. Using $\sqrt{2 A} \leq 2 s_{2}^{1 / 2}$ in the support of the integral in $S_{\text {out }}$ and $I_{i n, 1} \leq y_{2}^{1 / 2}$, we can bound the integrands in $S_{i n, y, n s}, S_{\text {out }}$ by $y_{2}^{1 / 2}|\Delta(y)|$. Denote

$$
R\left(B^{l}\right) \triangleq\left[0, B^{l}\right] \times\left[0, B^{l}-1\right], \quad A_{2} \triangleq \min \left(A, B^{l}\right)
$$

For $I(\varepsilon), S_{i n, y, \varepsilon}\left(\underline{(5.46)}\right.$, using $|\tilde{W}| \leq y_{2}^{1 / 2}[\omega]_{C_{y}^{1 / 2}}$ (5.50) we perform a similar decomposition

$$
|I(\varepsilon)| \leq\left|\int_{J_{\varepsilon}} \int_{0}^{A_{2}} \Delta(y) \tilde{W}(y) d y\right|+\left|\int_{J_{\varepsilon}} \int_{A_{2}}^{A} y_{2}^{1 / 2}\right| \Delta_{\text {in }}(y)|d y|[\omega]_{C_{y}^{1 / 2}} \triangleq I\left(\varepsilon, A_{2}\right)+S_{\text {in, }, \text {, }, \text { out }}[\omega]_{C_{y}^{1 / 2}}
$$

We treat the integrals in $S_{i n, y, n s}, S_{\text {out }}$ beyond $R\left(B^{l}\right)$ and $S_{i n, y, \varepsilon, o u t}$ as error $\mathcal{R}$ and yield

$$
\begin{aligned}
& S_{\text {in }, y, n s}+S_{\text {in }, y, \varepsilon, \text { out }}+S_{\text {out }} \leq\left(\int_{0}^{A}+\int_{A}^{B}\right) \int_{0}^{\infty} \mathbf{1}_{R\left(B^{l}\right)^{c}}(y) y_{2}^{1 / 2}\left|\Delta_{\text {in }}(y)\right| d y \\
& +\int_{0}^{A} \int_{0}^{B^{l}-1} \mathbf{1}_{R\left(B^{l}\right)}(y) I_{\text {in }, 1}\left|\Delta_{\text {in }}(y)\right| d y+\frac{\sqrt{2 A}}{2} \int_{A}^{B} \int_{0}^{\infty} \mathbf{1}_{R\left(B^{l}\right)}(y)\left|\Delta_{\text {in }}(y)\right| d y \triangleq \mathcal{R}+M_{\text {in }}+M_{\text {out }}
\end{aligned}
$$

Note that

$$
\begin{aligned}
& {[0, A] \cup[A, B]=[0, B],\left([0, A] \times\left[0, B^{l}-1\right]\right) \cap R\left(B^{l}\right)=\left[0, A_{2}\right] \times\left[0, B^{l}-1\right] \triangleq R_{2}\left(B^{l}\right)} \\
& ([A, B] \times[0, \infty]) \cap R\left(B^{l}\right)=\left([A, B] \cap\left[0, B^{l}\right]\right) \times\left[0, B^{l}-1\right] \subset\left[A_{2}, B^{l}\right] \times\left[0, B^{l}-1\right]
\end{aligned}
$$

where we have used $[A, B] \cap\left[0, B^{l}\right]=\emptyset$ if $A>B^{l}$ in the last inclusion. Since $M_{\text {out }}=0$ if $A>B^{l}$, we get
$S_{\text {in }, y, n s}+S_{\text {out }}+S_{\text {in }, y, \varepsilon, \text { out }} \leq \int_{R\left(B^{l}\right)^{c}} y_{2}^{1 / 2}\left|\Delta_{\text {in }}(y)\right|+\int_{0}^{A_{2}} \int_{0}^{B^{l}-1} I_{\text {in, } 1}\left|\Delta_{\text {in }}(y)\right|+\frac{\sqrt{2 A_{2}}}{2} \int_{A_{2}}^{B^{l}} \int_{0}^{B^{l}-1}\left|\Delta_{\text {in }}(y)\right|$.
The estimate of the last two integrals and $I\left(\varepsilon, A_{2}\right)$ defined above (see also (5.46)) reduce to the case $A_{2} \leq B^{l}$ studied in the previous section. The first term is very small and treated as an error term. Applying Lemma 5.2 with $i+j=3$ and mean value theorem yields

$$
\left|\Delta_{i n}(s)\right|=\left|2 \partial_{y_{1}} K_{2}\left(\xi, y_{2}\right)\right| \leq \frac{2}{\left(\xi^{2}+y^{2}\right)^{3 / 2}} \leq C_{B}|y|^{-3}, \quad C_{B}=2\left(\frac{B^{l}-1}{B^{l}-2}\right)^{3}
$$

for some $\xi \in\left[y_{1}-1, y_{1}+1\right]$, where we have used $\left|\left(\xi, y_{2}\right)\right| \geq|y|-1 \geq|y|\left(1-\frac{1}{B^{l}-1}\right)$. Using $y_{2}^{1 / 2} \leq|y|^{1 / 2}$, we estimate the remaining part as follows

$$
\begin{align*}
\int_{R\left(B^{l}\right)^{c}} y_{2}^{1 / 2}\left|\Delta_{i n}(y)\right| & \leq C_{B} \int_{|y| \geq B^{l}-1, y_{1}, y_{2} \geq 0}|y|^{-5 / 2} d y \leq C_{B} \int_{B^{l}-1} r^{-3 / 2} d r \int_{0}^{\pi / 2} 1 d \beta  \tag{5.58}\\
& =C_{B} 2\left(B^{l}-1\right)^{-\frac{1}{2}} \cdot \frac{\pi}{2}=C_{B} \pi\left(B^{l}-1\right)^{-\frac{1}{2}}
\end{align*}
$$

5.4.6. Estimate of $S_{i n, x}$ and $S_{1 D}$. Recall $S_{i n, x}, S_{1 D}$ from (5.43). Denote $J_{2}=[1 / 9,9]$. Clearly, we have
$F(A, B)=S_{i n, x}+S_{1 D}=\int_{y_{1} \notin J_{2}, y_{1} \geq 0} \sqrt{2 y_{1}}\left|\Delta_{1 D}\left(y_{1}\right)\right| d y_{1}+2 \int_{\frac{1}{9}}^{9}\left|\Delta_{1 D}\right| \cdot\left|y_{1}-1\right|^{1 / 2} d y_{1}+\left(\pi+P\left(\frac{1}{9}\right)\right) \sqrt{2}$,
where $\Delta_{1 D}\left(y_{1}\right), P\left(\frac{1}{9}\right)$ are given in (5.45). From the definition, we obtain $\Delta_{1 D}<0$ for $y_{1}>0$.
First, we show that $F$ is increasing in $B$. Using (5.45) and $B \geq 3$, we get

$$
\begin{align*}
F(A, B) & =\pi \sqrt{2}+\int_{0}^{\infty}\left|\Delta_{1 D}\left(y_{1}\right)\right| h\left(y_{1}\right) d y_{1} \\
& =\pi \sqrt{2}+\int_{0}^{B-1}\left|g_{A}\left(y_{1}+1\right)-g_{A}\left(y_{1}-1\right)\right| h\left(y_{1}\right)+\int_{B-1}^{B+1}\left|g_{A}\left(y_{1}-1\right)\right| h\left(y_{1}\right) d y_{1}  \tag{5.59}\\
h\left(y_{1}\right) & =\sqrt{2 y_{1}} \mathbf{1}_{J_{2}^{c}}\left(y_{1}\right)-\sqrt{2} \mathbf{1}_{J_{2}}\left(y_{1}\right)+2\left|y_{1}-1\right|^{1 / 2} \mathbf{1}_{J_{2}}\left(y_{1}\right)
\end{align*}
$$

where $J_{2}=[1 / 9,9]$, and the negative term comes from $P\left(\frac{1}{9}\right)$. The function $h\left(y_{1}\right)$ satisfies

$$
h\left(y_{1}\right)=2\left|y_{1}-1\right|^{1 / 2}-\sqrt{2}, y_{1} \in[1 / 9,9], \quad h\left(y_{1}\right)=\left|2 y_{1}\right|^{1 / 2}, y_{1}>9, y_{1} \in[0,1 / 9] .
$$

Since $h(9-)=2 \sqrt{8}-\sqrt{2}=3 \sqrt{2}=h(9+)$, and $h\left(y_{1}\right)$ is increasing on $[2,9]$ and $[9, \infty]$, we obtain that $h\left(y_{1}\right)$ is increasing. Using the monotonicity Lemma 5.4 we get

$$
F(A, B) \leq F(A, \infty)
$$

Thus, it suffices to consider the case $B=\infty$, where we do not have localization of $\Delta_{1 D}$ in (5.45).
Now, we fix $A \in\left[A^{l}, A^{u}\right]$. We partition the integral in (5.59) with $B=\infty$ into

$$
D_{1}=[0,1 / 9], D_{2}=[1 / 9,1-\varepsilon], D_{3}=[1-\varepsilon, 1+\varepsilon], D_{4}=[1+\varepsilon, 9], D_{5}=\left[9, R_{0}\right], D_{f}=\left[R_{0}, \infty\right]
$$

for some $\varepsilon<1 / 4$. In each $D_{i}$, we can simplify the function $h\left(y_{1}\right)$ and $h\left(y_{1}\right)$ is monotone. Thus, we can obtain its piecewise bounds easily. Next, we estimate the integral of $\Delta_{1 D}$ uniformly for $A \in\left[A_{l}, A_{u}\right]$. Using $\Delta_{1 D}\left(y_{1}\right)<0$ for $y_{1}>0$ and

$$
\begin{aligned}
\left|\Delta_{1 D}\left(y_{1}, A\right)\right| & =\frac{4 A y_{1}}{\left(\left(y_{1}+1\right)^{2}+A^{2}\right)\left(\left(y_{1}-1\right)^{2}+A^{2}\right)} \leq \frac{A_{u}}{A_{l}} \frac{4 A^{l} y_{1}}{\left(\left(y_{1}+1\right)^{2}+A_{l}^{2}\right)\left(\left(y_{1}-1\right)^{2}+A_{l}^{2}\right)} \\
& =-\frac{A_{u}}{A_{l}} \Delta_{1 D}\left(y_{1}, A_{l}\right)=\frac{A_{u}}{A_{l}} \frac{d}{d y_{1}}\left(\arctan \frac{y_{1}-1}{A_{l}}-\arctan \frac{y_{1}+1}{A_{l}}\right)
\end{aligned}
$$

we get

$$
\int_{a}^{b}\left|\Delta_{1 D}\left(y_{1}, A\right)\right| d y_{1} \leq\left.\frac{A_{u}}{A_{l}}\left(\arctan \frac{y_{1}-1}{A_{l}}-\arctan \frac{y_{1}+1}{A_{l}}\right)\right|_{a} ^{b}
$$

We apply the above estimate and the strategy in Section 5.2 to estimate the integral in the finite domains $D_{1}, D_{2}, D_{4}, D_{5}$ away from the singularity 1 . When $A$ is small and near the singularity 1 , the above formula can be $\infty$. Denote $P=\left(y_{1}-1\right)^{2}+A^{2}, Q=\left(y_{1}+1\right)^{2}+A^{2}$. Using the above formula for $\left|D_{1 D}\right|$, we have

$$
\frac{1}{4 y_{1}} \partial_{A}\left|\Delta_{1 D}\right|=\partial_{A} \frac{A}{P Q}=\frac{1}{P Q}\left(1-A \frac{P^{\prime}}{P}-A \frac{Q^{\prime}}{Q}\right)=\frac{1}{P Q}\left(1-2 A^{2}\left(\frac{1}{P}+\frac{1}{Q}\right)\right)
$$

For $\left|y_{1}-1\right| \geq \sqrt{3} A$, we get $Q \geq P \geq 4 A^{2}$ and yield

$$
2 A^{2}\left(P^{-1}+Q^{-1}\right) \leq 2 A^{2} \cdot 2\left(4 A^{2}\right)^{-1}=1, \quad \partial_{A}\left|\Delta_{1 D}\right| \geq 0
$$

Therefore, if $\min (|a-1|,|b-1|) \geq \sqrt{3} A$, we have an improved estimate

$$
\int_{a}^{b}\left|\Delta_{1 D}\left(y_{1}, A\right)\right| \leq \int_{a}^{b}\left|\Delta_{1 D}\left(y_{1}, A_{u}\right)\right|=-\int_{a}^{b} \Delta_{1 D}\left(y_{1}, A_{u}\right)
$$

which can be evaluated using the analytic integral formula.

For the term involving $P(1 / 9)$, using the formula of $\int \Delta_{1 D}(s)$, we get

$$
\begin{aligned}
P(1 / 9, A) & =\arctan \frac{8}{A}-\arctan \frac{10}{A}-\arctan \frac{-8}{9 A}+\arctan \frac{10}{9 A} \\
\partial_{A} P(1 / 9, A) & =-\frac{13120\left(6400+6724 A^{2}+243 A^{4}\right)}{\left(A^{2}+64\right)\left(A^{2}+100\right)\left(64+81 A^{2}\right)\left(100+81 A^{2}\right)}<0 .
\end{aligned}
$$

Thus $P(1 / 9, A)$ is monotone in $A$ and

$$
\pi+P(1 / 9, A) \leq \max \left(\pi+P\left(1 / 9, A_{l}\right), \pi+P\left(1 / 9, A_{u}\right)\right)
$$

In the domain $D_{3}$, for $A$ very small, we handle the singularity near $\left(A, y_{1}\right)=(0,0)$ as follows

$$
\begin{aligned}
I_{3} & =\int_{1-\varepsilon}^{1+\varepsilon}\left(g_{A}\left(1-y_{1}\right)-g_{A}\left(1+y_{1}\right)\left|y_{1}-1\right|^{1 / 2} d y_{1} \leq \int_{1-\varepsilon}^{1+\varepsilon}\left(g_{A}\left(1-y_{1}\right)\left|y_{1}-1\right|^{1 / 2} d y_{1}=2 \int_{0}^{\varepsilon} \frac{A}{A^{2}+y_{1}^{2}} y_{1}^{1 / 2} d y_{1}\right.\right. \\
& =2 A^{1 / 2} \int_{0}^{\varepsilon / A} \frac{1}{y_{1}^{2}+1} y_{1}^{1 / 2} d y_{1} \leq 2 A_{u}^{1 / 2} \int_{0}^{\varepsilon / A_{l}} \frac{1}{y_{1}^{2}+1} y_{1}^{1 / 2} d y_{1}=4 A_{u}^{1 / 2} f_{s}\left(\left(\varepsilon / A_{l}\right)^{1 / 2}\right)
\end{aligned}
$$

where $f_{s}$ is the function defined in (5.5).
In the far-field $y_{1} \geq R_{0}$, we have

$$
\left|\Delta_{1 D}\left(y_{1}\right)\right|=\left|2 \partial_{y_{1}} g_{A}(\xi)\right|=\left|\frac{4 A \xi}{\left(A^{2}+\xi^{2}\right)^{2}}\right| \leq \frac{2}{\xi^{2}+A^{2}} \leq 2\left(y_{1}-1\right)^{-2}
$$

for some $\xi \in\left[y_{1}-1, y_{1}+1\right]$. Since $y_{1} \leq\left(y_{1}-1\right) \frac{R_{0}}{R_{0}-1}$ for $y_{1} \geq R_{0}$, we yield
$I_{f}=\int_{R_{0}}^{\infty}\left|\Delta_{1 D}\left(y_{1}\right)\right| \sqrt{2 y_{1}} d y_{1} \leq 2\left(\frac{2 R_{0}}{R_{0}-1}\right)^{1 / 2} \int_{R_{0}}^{\infty}\left(y_{1}-1\right)^{-2+1 / 2} d y_{1}=4\left(\frac{2 R_{0}}{R_{0}-1}\right)^{1 / 2}\left(R_{0}-1\right)^{-1 / 2}$.
Finally, we handle the case $A$ sufficiently large and can be $\infty$. For $A \geq A_{l}$ with $A_{l}$ sufficiently large, e.g. $A^{l}=10^{8}$, we use $h\left(y_{1}\right) \leq\left|2 y_{1}\right|^{1 / 2}$ (5.59) and the identity (5.60) below to get

$$
\begin{aligned}
F(A, B) \leq F(A, \infty) & =\sqrt{2} \pi+\int_{0}^{\infty}\left|\Delta_{i n}\left(y_{1}\right)\right| \sqrt{2 y_{1}} d y_{1}=\sqrt{2} \pi+\pi \sqrt{2} \sqrt{\sqrt{A^{2}+1}-A} \\
& =\sqrt{2} \pi\left(1+\left(\sqrt{A^{2}+1}+A\right)^{-1 / 2}\right) \leq \sqrt{2} \pi\left(1+\left(\sqrt{A_{l}^{2}+1}+A_{l}\right)^{-1 / 2}\right)
\end{aligned}
$$

where we have used $\sqrt{A^{2}+1}-A=\frac{1}{\sqrt{A^{2}+1}+A}$.
Combining the above estimates, we complete the estimate of the integrals in (5.43) for $\left[v_{x}\right]_{C_{y}^{1 / 2}}$.
The estimate of $\left[u_{y}\right]_{C_{y}^{1 / 2}}$ in (5.44) is similar and simpler. We have estimated the terms in $S_{l o w}$ (5.44) in our estimate of $S_{i n, y}$ (5.43). The estimate of $\tilde{S}_{1 D}$ is similar. Using the monotonicity Lemma 5.4, the identity (5.60) below, we get

$$
\begin{aligned}
\tilde{S}_{1 D} & \leq \int_{0}^{\infty}\left(g_{A}\left(y_{1}-1\right)-g_{A}\left(y_{1}+1\right)\right)\left(2 y_{1}\right)^{1 / 2} d y_{1}=\sqrt{2} \pi \sqrt{\sqrt{1+A^{2}}-A} \\
& =\sqrt{2} \pi\left(\sqrt{1+A^{2}}+A\right)^{-1 / 2} \leq \sqrt{2} \pi\left(\sqrt{1+A_{l}^{2}}+A_{l}\right)^{-1 / 2}
\end{aligned}
$$

An integral identity. We prove the following identity using the residue formula

$$
\begin{equation*}
T=\int_{0}^{\infty}\left(g_{A}\left(y_{1}-1\right)-g_{A}\left(y_{1}+1\right)\right) y_{1}^{1 / 2} d y_{1}=\pi \sqrt{\sqrt{1+A^{2}}-A} \tag{5.60}
\end{equation*}
$$

Using a change of variable $y_{1}=s^{2}$ and writing the integral on $\mathbb{R}$ using the symmetry, we get

$$
T=\int_{-\infty}^{\infty} s^{2}\left(\frac{A}{\left(s^{2}-1\right)^{2}+A^{2}}-\frac{A}{\left(s^{2}+1\right)^{2}+A^{2}}\right) d s \triangleq T_{1}-T_{2}
$$

The integrand is analytic except a few poles. In the upper half plane, we have poles

$$
s_{1}=(1+i A)^{1 / 2}=r e^{i \theta}, \quad s_{2}=r e^{\pi-\theta}=-\bar{s}_{1}, s_{2}^{2}=\bar{s}_{1}^{2}=1-i A
$$

for some $\theta \in[0, \pi]$. Denote $f(s)=\left(s^{2}-1\right)^{2}+A^{2}$. We have $\partial_{s} f=4 s\left(s^{2}-1\right)$. Applying the residue formula, we get

$$
T_{1}=2 \pi i\left(\frac{A s_{1}^{2}}{f^{\prime}\left(s_{1}\right)}+\frac{A s_{1}^{2}}{f^{\prime}\left(s_{1}\right)}\right)=\frac{1}{2} \pi i A\left(\frac{s_{1}}{s_{1}^{2}-1}+\frac{s_{2}}{s_{2}^{2}-1}\right)=\frac{\pi i A}{2}\left(\frac{s_{1}}{i A}+\frac{-\bar{s}_{1}}{-i A}\right)=\frac{\pi}{2}\left(s_{1}+\bar{s}_{1}\right)
$$

Since $\left(s_{1}+\bar{s}_{1}\right)^{2}=s_{1}^{2}+\bar{s}_{1}^{2}+2\left|s_{1}\right|^{2}=2+2\left(1+A^{2}\right)^{1 / 2}$ and $T_{1}>0$, we get $T_{1}=\frac{\pi}{2} \sqrt{2+2 \sqrt{1+A^{2}}}$. For $T_{2}$, the computation is similar. Let $-1+i A=r_{2}^{2} e^{i \theta_{2}}, \theta_{2}>0$. We have poles in $\mathbb{R}_{2}^{+}$

$$
s_{3}=r_{2} e^{i \theta_{2} / 2}, s_{3}^{2}=-1+i A, \quad s_{4}=\bar{s}_{3}=-r_{2} e^{-i \theta_{2} / 2}, s_{4}^{2}=r_{2}^{2} e^{-i \theta_{2}}=-1-i A
$$

and apply the residue formula to get

$$
T_{2}=\frac{\pi}{2}\left(s_{3}+\bar{s}_{3}\right)=\frac{1}{2} \pi i A\left(\frac{s_{3}}{s_{3}^{2}+1}+\frac{s_{4}}{s_{4}^{2}+1}\right)=\frac{\pi}{2}\left(s_{3}+\bar{s}_{3}\right)=\frac{\pi}{2}\left(-2+2 \sqrt{1+A^{2}}\right)^{1 / 2}
$$

Denote $G=\left(1+A^{2}\right)^{1 / 2}$. Using the above identities and $G^{2}-1=A^{2}$, we prove

$$
\begin{aligned}
T & =T_{1}-T_{2}=\sqrt{2} \frac{\pi}{2}\left((G+1)^{1 / 2}-(G-1)^{1 / 2}\right)=\sqrt{2} \frac{\pi}{2}\left((G+1)+(G-1)-2\left(G^{2}-1\right)^{1 / 2}\right)^{1 / 2} \\
& =\sqrt{2} \frac{\pi}{2}(2 G-2 A)^{1 / 2}=\pi\left(\left(1+A^{2}\right)^{1 / 2}-A\right)^{1 / 2}
\end{aligned}
$$

5.5. Estimate of the constant for $\left[u_{y}\right]_{C_{y}^{1 / 2}},\left[v_{x}\right]_{C_{y}^{1 / 2}}$. Recall from Appendix B. 2 in [3] the estimate of $\left[u_{y}\right]_{C_{y}^{1 / 2}},\left[v_{x}\right]_{C_{y}^{1 / 2]}}$

$$
\begin{align*}
& \frac{\left|u_{y}(z)-u_{y}(x)\right|}{\sqrt{2}} \leq \frac{1}{\pi \sqrt{2}}\left(\left(\tilde{C}_{i n}(\varepsilon)+\min \left(\tilde{C}_{m i d, 1}(\varepsilon), C_{m i d, 3}\right)\right)[\eta]_{C_{x}^{1 / 2}}+C_{o u t}(m, \varepsilon)[\eta]_{C_{y}^{1 / 2}}\right) \\
& \frac{\left|v_{x}(z)-v_{x}(x)\right|}{\sqrt{2}} \leq \frac{1}{\pi \sqrt{2}}\left(\tilde{C}_{i n}(\varepsilon)[\eta]_{C_{x}^{1 / 2}}+C_{o u t}(1, \delta)[\eta]_{C_{y}^{1 / 2}}+\frac{\pi}{2} \sqrt{2}[\eta]_{C_{x}^{1 / 2}}\right) \tag{5.61}
\end{align*}
$$

where $\eta$ is a rotation of the original variable $\omega$ and the upper bounds are given by (5.62)

$$
\begin{aligned}
& \tilde{C}_{i n}(\varepsilon)=C_{i n, \varepsilon, n s}+C_{i n, \varepsilon}, \quad \tilde{C}_{m i d, 1}(m, \varepsilon)=C_{m i d, 1, \varepsilon, n s}+C_{m i d, 1, \varepsilon} \\
& C_{\text {in }, \varepsilon, n s}=4 \int_{\varepsilon}^{y_{c}} d y_{2} \int_{0}^{h_{c}^{-}\left(y_{2}\right)}|\Delta(y)|\left|y_{1}-T_{1}(y)\right|^{1 / 2} d y_{1}+2 \int_{R_{i n, \varepsilon}^{++}}|\Delta(y)| \sqrt{2 y_{1}} d y \\
& C_{m i d, 1, \varepsilon, n s}=4 \int_{\varepsilon}^{s_{c}(m)} d y_{2} \int_{h_{c}^{+}\left(y_{2}\right)}^{m}|\Delta(y)|\left|y_{1}-T_{1}(y)\right|^{1 / 2} d y_{1}+2 \int_{R_{m i d}, y_{2} \geq \varepsilon}|\Delta(y)| \sqrt{2 y_{1}} d y \\
& C_{m i d, 2}(\varepsilon)=4 \int_{1+\varepsilon}^{\infty} d y_{1} \int_{s_{c}\left(y_{1}\right)}^{\infty}|\Delta(y)|\left|y_{2}-T(y)\right|^{1 / 2} d y_{2}, \quad \varepsilon_{m}=\max (m, \varepsilon+1), \\
& C_{\text {out }}(m, \varepsilon)=C_{m i d, 2}\left(\varepsilon_{m}-1\right)+4\left(I_{K_{2}, \infty}\left(m+1, \varepsilon_{m}+1\right), I_{K_{2}, \infty}\left(m-1, \varepsilon_{m}-1\right)\right), \\
& C_{\text {out }}(1, \varepsilon)=C_{m i d, 2}(\varepsilon)+4\left(I_{K_{2}, \infty}(2,2+\varepsilon)+I_{K_{2}, \infty}(0, \varepsilon)\right)
\end{aligned}
$$

and we have replaced the dummy parameter $\delta$ used in [3] by $\varepsilon$. The value of $\varepsilon$ can be different for different variables. The subscript $n s$ means non-singular. The functions $C_{m i d, 3}, C_{i n, \varepsilon}, C_{m i d, \varepsilon}$ are upper bounds of the following integrals

$$
\begin{align*}
& S_{i n, \varepsilon} \triangleq \lim _{\delta \rightarrow 0} \int_{\delta \leq\left|y_{2}\right| \leq \varepsilon,\left|y_{1}\right| \leq 1} \Delta(y) \eta_{m}(y) d y, \quad\left|S_{i n}\right| \leq C_{i n, \varepsilon}[\eta]_{C_{x}^{1 / 2}} \\
& S_{m i d, 1, \varepsilon} \triangleq \lim _{\varepsilon \rightarrow 0} \int_{\varepsilon \leq\left|y_{2}\right| \leq \delta, 1 \leq\left|y_{1}\right| \leq m} \Delta(y) \eta_{m}(y) d y, \quad\left|S_{m i d, 1, \varepsilon}\right| \leq C_{m i d, 1, \varepsilon}[\eta]_{C_{x}^{1 / 2}}  \tag{5.63}\\
& S_{m i d, i n t} \triangleq \lim _{\delta \rightarrow 0} \int_{\delta \leq\left|y_{2}\right|,\left|y_{1}\right| \in[1, m]} \Delta(y) \eta_{m}(y) d y, \quad\left|S_{m i d, i n t}\right| \leq C_{m i d, 3}[\eta]_{C_{x}^{\frac{1}{2}}}
\end{align*}
$$

and will be established in Sections 5.5.2, 5.5.3, In (5.62), $I_{K_{2}, \infty}$ is defined in (5.88), and $T_{1}$ is the transport map in $x$ direction given in (5.93), $T$ is the previous transport map in $y$ direction
(5.92), $s_{c}$ is given in (5.89), $h_{c}^{ \pm}$are given in (5.89), the kernel $\Delta$ and $\Delta_{m}$ are given by

$$
\begin{align*}
& \Delta(y)=K_{2}\left(y_{1}+1, y_{2}\right)-K_{2}\left(y_{1}-1, y_{2}\right), \quad \Delta_{m}\left(y_{1}\right)=\frac{1}{2}\left(g_{m+1}\left(y_{2}\right)-g_{m-1}\left(y_{2}\right)\right)  \tag{5.64}\\
& y_{m}=\sqrt{m^{2}-1}, \quad y_{c}=3^{-1 / 2}
\end{align*}
$$

Different domains are given by

$$
\begin{align*}
& \Omega_{\text {in }} \triangleq\left\{y:\left|y_{1}\right| \leq 1\right\}, \quad \Omega_{\text {mid }} \triangleq\left\{y:\left|y_{1}\right| \in[1, m]\right\}, \quad \Omega_{\text {out }} \triangleq\left\{y:\left|y_{1}\right|>m\right\} \\
& R_{\text {in, }} \triangleq\left\{\left|y_{1}\right| \leq 1,\left|y_{2}\right| \geq y_{c}\right\} \cup\left\{T_{1}\left(0,\left|y_{2}\right|\right) \leq y_{1} \leq 1, \varepsilon \leq\left|y_{2}\right| \leq y_{c}\right\} \\
& R_{\text {in, }}^{+}=R_{\text {in }, \varepsilon} \cap[0,1] \times \mathbb{R}, \quad R_{\text {in, }}^{++}=R_{\text {in, }} \cap[0,1] \times \mathbb{R}^{+},  \tag{5.65}\\
& R_{\text {mid }} \triangleq\left\{\left|y_{1}\right| \in[1, m],\left|y_{2}\right| \geq s_{c}(m)\right\} \cup\left\{\left|y_{2}\right|<s_{c}(m), 1 \leq\left|y_{1}\right| \leq T\left(m,\left|y_{2}\right|\right)\right\}, \\
& R_{\text {mid }}^{+} \triangleq R_{\text {out }} \cap\left\{y_{1} \geq 0\right\}, \quad R_{\text {mid }}^{++} \triangleq R_{\text {mid }} \cap \mathbb{R}_{2}^{++}
\end{align*}
$$

To estimate these integrals, we follow the strategy in Section 5.2. We need to derive the piecewise bounds for the map $T_{1}$. For $y_{2}>0$, it is easy to see that $h_{c}^{-}\left(y_{2}\right)$ is decreasing and $h_{c}^{+}\left(y_{2}\right)$ is increasing by checking the sign of $\frac{d}{d y_{2}} h_{c}^{ \pm}\left(y_{2}\right)$. Thus the bounds for $h_{c}^{ \pm}$are trivial.

Remark 5.6. Although the bounds in (5.62) look quite complicated, we develop several estimates for the same integral so that for the integrals in different region, especially near the singularity, we can choose the easiest one to compute.
5.5.1. Basic properties of $T_{1}, y_{m}, h_{c}^{ \pm}$. Recall that $h_{c}^{ \pm}$determine the sign of $\Delta(y)$

$$
\begin{equation*}
\Delta(y)<0, \quad y_{1} \in\left(0, h_{c}^{-}\right), y_{1}>h_{c}^{+}, \quad \Delta(y)>0, \quad y_{1} \in\left(h_{c}^{-}, h_{c}^{+}\right) \tag{5.66}
\end{equation*}
$$

for $y_{2}>0$. See the right figure in Figure 2 for an illustrations of different regions in $\left\{y_{1} \geq 0\right\}$ and the sign of the kernels. We have the following basic property.

Lemma 5.7. Consider the equation and region $D_{i}$
$\int_{T}^{y_{1}} \Delta\left(s, y_{2}\right) d s=0, \quad T \geq 0, \quad D_{1}=\left\{y_{2}<3^{-\frac{1}{2}}, y_{1}<h_{c}^{-}\left(y_{2}\right)\right\} \cap \mathbb{R}_{2}^{++}, D_{2}=\left\{y_{1}>h_{c}^{+}\left(y_{2}\right)\right\} \cap \mathbb{R}_{2}^{++}$.
For $y \in D_{1}$, there is a unique solution $T(y) \in\left[h_{c}^{-}\left(y_{2}\right), 1\right)$, and $T$ is decreasing in $y_{1}$ and $y_{2}$. For $y \in D_{2}$, there is a unique solution $T(y) \in\left[1, h_{c}^{+}\right]$, and $T$ is decreasing in $y_{1}$ and increasing in $y_{2}$. In both cases, the solution is given by (5.93).

Proof. Clearly, using (5.93), we know that $T$ solves the equation. Next, we consider the bound of $T$. For a fixed $y_{2} \leq 3^{-1 / 2}$. Denote

$$
F(t)=\int_{0}^{t} \Delta\left(s, y_{2}\right) d s=\left.\frac{1}{2}\left(\frac{y_{1}-1}{\left|y_{1}-1\right|^{2}+y_{2}^{2}}-\frac{y_{1}+1}{\left|y_{1}+1\right|^{2}+y_{2}^{2}}\right)\right|_{0} ^{t}
$$

Using (5.66), we get $F(t)<0, t \in\left[0, h_{c}^{-}\right], F(t)$ is stricitly decreasing on $\left[h_{c}^{-}, 1\right]$, and $F(1)<0$. Thus, we have a unique solution $T \in\left[h_{c}^{-}, 1\right]$. Since $y_{1}<h_{c}^{-}\left(y_{1}\right)<T(y), \Delta(y)$ and $\Delta\left(T(y), y_{2}\right)$ have opposite signs. Taking $y_{1}$ derivatives yields $\partial_{y_{1}} T \leq 0$. The properties of $T$ with $y_{1}>1$ follow by a similar argument and studying $F(t)=\int_{t}^{\infty} \Delta\left(s, y_{2}\right) d s$.

From the formula of $T$ (5.93), we have $T^{2}=\frac{P(y)}{Q(y)}$ for some polynomials $P, Q$ with $P$ increasing in $y_{2}$ and $Q$ decreasing in $y_{2}$. Moreover, $P=Q T^{2}$ and $Q$ have the same sign. For $y_{2}>0$, using (5.89), if $y \in D_{1}$, we get $y_{1}<h_{c}^{-}\left(y_{2}\right), y_{1}^{2}<y_{2}^{2}+1, Q<0$ and $P=T^{2} Q<0$. Thus, $\partial_{y_{2}}(P / Q)=\frac{P^{\prime} Q-P Q^{\prime}}{Q^{2}}<0$ and $T$ is decreasing in $y_{2}$. As a result, we have

$$
\begin{equation*}
y_{1} \leq h_{c}^{-}\left(y_{2}\right) \leq T(y), \quad\left|T(y)-y_{1}\right|=T(y)-y_{1} \leq T\left(y_{1}^{l}, y_{2}^{l}\right)-y_{1}^{l} \tag{5.67}
\end{equation*}
$$

for $y \in D_{1} \cap\left[y_{1}^{l}, y_{1}^{u}\right] \times\left[y_{2}^{l}, y_{2}^{u}\right]$. Since $y_{1}^{l} \leq y_{1}<h_{c}^{-}\left(y_{2}\right)<h_{c}^{-}\left(y_{2}^{l}\right),\left(y_{1}^{l}, y_{2}^{l}\right)$ is still in $D_{1}$ and the upper bound is well-defined.

Similarly, if $y_{1} \in D_{2}$, we get $y_{1}>1, P>0, Q=P T^{2}>0$ and obtain that $T$ is decreasing in $y_{1}$, but increasing in $y_{2}$. Moreover, for $y \in D_{2}$, we have

$$
\begin{equation*}
y_{1} \geq h_{c}^{+}\left(y_{2}\right) \geq T(y), \quad\left|T(y)-y_{1}\right|=y_{1}-T(y) \leq y_{1}^{u}-T\left(y_{1}^{u}, y_{2}^{l}\right) \tag{5.68}
\end{equation*}
$$

Since $y_{1}^{u}>y_{1}>h_{c}^{+}\left(y_{2}\right)>h_{c}^{+}\left(y_{2}^{l}\right)$, we get $\left(y_{1}^{u}, y_{2}^{l}\right) \in D_{2}$ and the upper bound is well-defined. We conclude the proof.

We remark that since $\int_{0}^{1} \Delta\left(s, y_{2}\right) d s \neq 0$ and $\int_{1}^{\infty} \Delta\left(s, y_{2}\right) d s$, for a fixed $y_{2}$, the total mass of the positive part and the negative part of $\Delta\left(s, y_{2}\right) d s$ are not equal. As a result, we cannot construct the map $T_{1}$ for all $y_{1}>0$.
5.5.2. Estimate the integrals in the inner region. Recall the integrals from (5.62). We have one parameter $m>1$ and want to obtain a uniform estimate for all $m>1$. We follow the strategy (d) to partition the domains of the integrals and this parameter. The singularity of the integral is at $y=(1,0)$. We follow the strategy in Section 5.2 to modify the estimate of $S_{i n, \varepsilon}$ near the singularity, which gives the bound $C_{i n, \varepsilon}$ (5.63). For a fixed small $\varepsilon>0$, denote

$$
\begin{equation*}
Q_{1}(\varepsilon)=[0,1] \times[0, \varepsilon], \quad Q_{2}(\varepsilon)=[1, m] \times[0, \varepsilon], \quad Q_{3}(\varepsilon)=[m, \infty) \times[0, \varepsilon] \tag{5.69}
\end{equation*}
$$

Recall from (5.62) that

$$
\tilde{C}_{i n}=C_{i n, \varepsilon, n s}+C_{i n, \varepsilon}
$$

The integral in $C_{i n, \varepsilon, n s}$ is restricted to $\left|y_{2}\right| \geq \varepsilon$, while $C_{i n, \varepsilon}$ is in $\left|y_{2}\right| \leq \varepsilon$. We estimate $C_{i n, \varepsilon, n s}$ using previous method and the strategy in Section 5.2. To estimate $\left|T(y)-y_{1}\right|$, we use the piecewise bounds (5.67). For the integral in $C_{i n, \varepsilon, n s}$ in $Q_{1, \varepsilon}^{c} \cap R_{i n, 0}^{++}$, since the kernel $\Delta(y)$ has a fixed sign, it becomes

$$
\begin{equation*}
C_{i n, Q_{1, \varepsilon}^{c}} \triangleq 2 \int_{0}^{1}\left(\int_{y_{c}}^{\infty} \Delta(y) d y_{2}\right) \sqrt{2 y_{1}} d y_{1}+2 \int_{\varepsilon}^{y_{c}} d y_{2} \int_{T_{1}\left(0, y_{2}\right)}^{1}|\Delta(y)| \sqrt{2 y_{1}} d y_{1} \tag{5.70}
\end{equation*}
$$

For the first integral, we first apply the integral formula (5.3) for $K_{2}$, and then follow Section 5.2 by bounding the integral of $\Delta$ and $y_{1}^{1 / 2}$ piecewisely. See (5.51), (5.52) for an example. For the second integral, we estimate the piecewise integrals of $|\Delta(y)| \sqrt{2 y_{1}}$ and then estimate the indicator function of $y_{1} \leq T_{1}\left(0, y_{2}\right)=\max \left(1-3 y_{2}^{2}, 0\right)^{1 / 2}$ (see (5.93)).

Next, we estimate $S_{i n, \varepsilon}$ and the bound $C_{i n, \varepsilon}$ (5.62), (5.63). We follow Section5.2 and estimate two kernels in $\Delta(y)=K_{2}\left(y_{1}+1, y_{2}\right)-K_{2}\left(y_{1}-1, y_{2}\right)$ separately. Since $\Delta$ is odd in $y_{1}$, we have

$$
\begin{align*}
& S_{i n, \varepsilon}=\lim _{\delta \rightarrow 0} \int_{\delta \leq\left|y_{2}\right| \leq \varepsilon} \int_{0}^{1} \tilde{\eta}_{m}(y)\left(K_{2}\left(y_{1}+1, y_{2}\right)-K_{2}\left(y_{1}-1, y_{2}\right)\right) d y \\
& =\int_{\left|y_{2}\right| \leq \varepsilon} \int_{0}^{1} K_{2}\left(y_{1}+1, y_{2}\right) \tilde{\eta}_{m}(y) d y-\lim _{\delta \rightarrow 0} \int_{\delta \leq\left|y_{2}\right| \leq \varepsilon} \int_{-1}^{0} K_{2}\left(y_{1}, y_{2}\right) \tilde{\eta}_{m}\left(y_{1}+1, y_{2}\right) d y  \tag{5.71}\\
& \triangleq I_{1}+I_{2}, \quad \tilde{\eta}_{m}(y) \triangleq \eta_{m}(y)-\eta_{m}\left(-y_{1}, y_{2}\right)
\end{align*}
$$

where we have used a change of variable $y_{1} \rightarrow y_{1}+1$ for the second integral. Using

$$
\begin{equation*}
\left|\tilde{\eta}_{m}(y)\right| \leq \sqrt{2 y_{1}}\left[\eta_{m}\right]_{C_{x}^{1 / 2}}=\sqrt{2 y_{1}}[\eta]_{C_{x}^{1 / 2}}, \quad\left(y_{1}+1\right)>\left|y_{2}\right|, y \in[0,1] \times[-\varepsilon, \varepsilon] \tag{5.72}
\end{equation*}
$$

and the formula for $\int K_{2} d y_{2}$ (5.3), for $I_{1}$, we get

$$
\begin{equation*}
\left|I_{1}\right| \leq \int_{0}^{1}\left(\int_{\left|y_{2}\right| \leq \varepsilon} K_{2}\left(y_{1}+1, y_{2}\right) d y_{2}\right)\left|2 y_{1}\right|^{1 / 2} d y_{1}[\eta]_{C_{x}^{1 / 2}}=2 \sqrt{2} \cdot \frac{1}{2} \int_{0}^{1} \frac{\varepsilon y_{1}^{1 / 2}}{\left(y_{1}+1\right)^{2}+\varepsilon^{2}} d y_{1}[\eta]_{C_{x}^{1 / 2}} \tag{5.73}
\end{equation*}
$$

We can bound the integral using the previous method, see e.g. (5.52). For $I_{2}$, applying the estimate (5.85) in Section 5.5.4 in the domain $[-1,0] \times \pm[0, \varepsilon]$ and

$$
\begin{equation*}
\left|\tilde{\eta}_{m}\left(y_{1}+1, y_{2}\right)\right| \leq \sqrt{2\left|y_{1}+1\right|}[\eta]_{C_{x}^{1 / 2}} \leq \sqrt{2}[\eta]_{C_{x}^{1 / 2}}, \quad\left[\tilde{\eta}_{m}\right]_{C_{x}^{1 / 2}} \leq 2\left[\eta_{m}\right]_{C_{x}^{1 / 2}} \leq 2[\eta]_{C_{x}^{1 / 2}} \tag{5.74}
\end{equation*}
$$

for $y_{1} \in[-1,0]$ from (5.72), we get

$$
\begin{equation*}
\left|I_{2}\right| \leq 2\left(\sqrt{2} \frac{1}{2} \arctan (\varepsilon)+2 \varepsilon^{1 / 2}\left(f_{s}(\sqrt{1 / \varepsilon})-f_{s}(1)+C_{K_{2}, u p}\right)\right)[\eta]_{C_{x}^{1 / 2}} \tag{5.75}
\end{equation*}
$$

where $f_{s}$ is defined in (5.6). We get a factor 2 since we have two domains $y_{2} \in \pm[0, \varepsilon]$.
5.5.3. Estimate of the integrals in the middle parts and outer parts. Other integrals in (5.62) are in the region $\Omega_{\text {mid }}(m), \Omega_{\text {out }}(m)$ (5.65) and depend on the parameter $m>1$ and we want to obtain piecewise bound for $m \in\left[m^{l}, m^{u}\right]$. Recall the region $Q_{2, \varepsilon}=[1, m] \times[0, \varepsilon]$ (5.69).

The integrand in $C_{m i d, 1, \varepsilon, n s}(m)$ is restricted to $\left|y_{2}\right| \geq \varepsilon$ away from the singularity $(1,0)$. We estimate it using the strategy in Section 5.2 and the above argument for $C_{i n}$. To control $\left|T(y)-y_{1}\right|$, since $y_{1} \geq h_{c}^{+}\left(y_{2}\right)$ in the support of the integrand, we use the piecewise bound (5.68). Note that from (5.89), for $y_{1}>1, y_{2}>0, y_{2}<s_{c}\left(y_{1}\right)$ is equivalent to $h_{c}^{+}\left(y_{2}\right)<y_{1}$. See the right black curve in right Figure 2 for an illustration. The integral region in the first term satisfies

$$
\left\{y: h_{c}^{+}\left(y_{2}\right)<y_{1}<m, \varepsilon_{2} y_{2}<s_{c}(m)\right\}=\left\{y: h_{c}^{+}\left(y_{2}\right)<y_{1}<m, y_{2}>\varepsilon\right\}
$$

which is increasing in $m$. Thus, we ce can bound the first integral in $C_{m i d, 1, \varepsilon, n s}(m)$ for $m \in$ [ $m^{l}, m^{u}$ ] uniformly by the case of $m=m^{u}$.

Since $s_{c}(m)$ is increasing in $m$, the second integral of $C_{m i d, 1, \varepsilon, n s}$ in $Q_{2, \varepsilon}^{c} \cap R_{m i d}^{++}$(5.65) satisfies

$$
\begin{aligned}
C_{m i d, 1, Q_{2, \varepsilon}^{c}}(m) & \leq 2 \int_{\varepsilon}^{s_{c}\left(m^{u}\right)} \int \mathbf{1}_{1 \leq y_{1} \leq T\left(m, y_{2}\right)}|\Delta(y)|\left|2 y_{1}\right|^{1 / 2} d y+2 \int_{1}^{m^{u}} \int_{\max \left(s_{c}\left(m^{l}\right), \varepsilon\right)}^{\infty}|\Delta(y)| d y_{2}\left(2 y_{1}\right)^{1 / 2} d y_{1} \\
& \triangleq I_{1}+I_{2}
\end{aligned}
$$

For $m \in\left[m^{l}, m^{u}\right]$, we estimate the indicator function in $I_{1}$ using Lemma 5.7 for $T$, and then estimate $I_{1}$ following Section 5.2 and a method similar to the above. For the second term, we denote $y c_{\alpha}=\max \left(s_{c}\left(m^{l}\right), \varepsilon\right), \alpha=l, u$ and decompose the domain into three parts

$$
D_{1}=\left[1, m^{l}\right] \times\left[y c_{l}, \infty\right], \quad D_{2}=\left[m^{l}, m^{u}\right] \times\left[y c_{u}, \infty\right], \quad D_{3}=\left[m^{l}, m^{u}\right] \times\left[y c_{l}, y c_{u}\right] .
$$

For $y \in D_{1}, D_{2}$, since $s_{c}\left(y_{1}\right)$ (5.89) is increasing in $y_{1}$, by definition, we get $y_{2} \geq s_{c}\left(m_{\alpha}\right)>$ $s_{c}\left(y_{1}\right), \alpha=l$, $u$. Thus, the kernel $\Delta$ has a fixed sign in $D_{1}, D_{2}$. We estimate the integral $I_{2}$ in $D_{1}, D_{2}$ using an argument similar to that in (5.70). The kernel $\Delta$ can change sign on $D_{3}$. We estimate $I_{2}$ in $D_{3}$ using the piecewise integral bound for $|\Delta|$ and $y_{1}^{1 / 2} \leq\left(m^{u}\right)^{1 / 2}$.

For $S_{m i d, 1, \varepsilon}, C_{m i d, 1, \varepsilon}$ (5.63), following the estimates (5.72)-(5.75) for $S_{i n, \varepsilon}$ and applying the estimate in Section 5.5 .4 to the region $[0, m-1] \times[0, \varepsilon]$ when $m-1 \geq \varepsilon$, we obtain

$$
\begin{aligned}
& \left|S_{m i d, 1, \varepsilon}\right| \leq \int_{\left|y_{2}\right| \leq \varepsilon} \int_{1}^{m^{u}} K_{2}\left(y_{1}+1, y_{2}\right)\left|2 y_{1}\right|^{1 / 2} d y_{1}[\eta]_{C_{x}^{1 / 2}} \\
& \quad+2\left(\sqrt{2 m} \frac{1}{2} \arctan \left(\frac{\varepsilon}{m-1}\right)+2 \varepsilon^{1 / 2}\left(f_{s}(\sqrt{(m-1) / \varepsilon})-f_{s}(1)+C_{K_{2}, u p}\right)\right)[\eta]_{C_{x}^{1 / 2}}
\end{aligned}
$$

where $f_{s}$ is defined in (5.6) and is increasing in $m$. The bound $2 m^{1 / 2}$ follows from $\mid \eta_{m}\left(y_{1}, y_{2}\right)-$ $\eta_{m}\left(-y_{1}, y_{2}\right) \mid \leq \sqrt{2\left|y_{1}\right|}[\eta]_{C_{x}^{1 / 2}} \leq \sqrt{2 m}[\eta]_{C_{x}^{1 / 2}}$ for $y_{1} \in[1, m]$, which is used to bound the $L^{\infty}$ norm in (5.85). Note that $\arctan \frac{\varepsilon}{m-1}$ is decreasing in $m$ and other functions in the upper bound of $I I$ are increasing in $m$. Then we can obtain piecewise upper bounds for $m \in\left[m^{l}, m^{u}\right]$. We estimate the first integral following (5.70) using analytic formulas for piecewise integral of $K_{2}\left(y_{1}+1, y_{2}\right)$ and bounding $y_{1}^{1 / 2}$ piecewisely.
Small $m-1$. Different from the integral $C_{i n}$, for $C_{m i d, 1}$, if $m$ is very close to 1 , the above method does not work since we require $m-1 \geq 2 \varepsilon$, which relates to the condition $a>b$ in the estimate of $J_{a, b}$ (5.85). In this case, we follow the strategy in Section 5.2.3 to separate two kernels in $\Delta$ and estimate $S_{m i d, i n t}, C_{m i d, 3}$ (5.63). Recall from (5.47) that $\Delta$ is odd. To overcome the computational difficulties in this singular case, we symmetrize the integral following (5.71) and then decompose it as follows

$$
\begin{aligned}
S_{m i d, i n t} & =\int_{\left|y_{2}\right| \geq m-1} \int_{1}^{m} \Delta(y) \tilde{\eta}_{m}(y) d y+\int_{\left|y_{2}\right| \leq m-1} \int_{1}^{m} K_{2}\left(y_{1}+1, y_{2}\right) \tilde{\eta}_{m}(y) d y \\
& +\int_{\left|y_{2}\right| \leq m-1} \int_{1}^{m} K_{2}\left(y_{1}-1, y_{2}\right) \tilde{\eta}_{m}(y) d y=P_{1}+P_{2}+P_{3}
\end{aligned}
$$

In the domain of $P_{1}$, since $\left|y_{2}\right| \geq y_{1}-1 \geq s_{c}\left(y_{1}\right)$ (5.89), (5.42), we get $\Delta(s) \geq 0$. Denote $\gamma=m-1$. Using $\left|\tilde{\eta}_{m}\right| \leq \sqrt{2 m}[\eta]_{C_{x}^{1 / 2}}$ (5.74) and the formula (5.3), we get

$$
\begin{aligned}
\left|P_{1}\right| & \leq 2 \sqrt{2 m}[\eta]_{C_{x}^{1 / 2}} \int_{m-1}^{\infty} \int_{1}^{m} \Delta(y) d y=2 \sqrt{2 m}[\eta]_{C_{x}^{1 / 2}} \cdot \frac{1}{2} \int_{1}^{1+\gamma} \frac{\gamma^{2}}{\left(y_{1}-1\right)^{2}+\gamma^{2}}-\frac{\gamma}{\left(y_{1}+1\right)^{2}+\gamma^{2}} d y_{1} \\
& \leq \sqrt{2 m}[\eta]_{C_{x}^{1 / 2}} \int_{0}^{\gamma} \frac{\gamma}{y_{1}^{2}+\gamma^{2}} d y_{1}=\frac{\pi}{4} \sqrt{2 m}[\eta]_{C_{x}^{1 / 2}}
\end{aligned}
$$

For $P_{2}$, using $\left|K_{2}\left(y_{1}+1, y_{2}\right)\right| \leq \frac{1}{2} \frac{1}{\left(y_{1}+1\right)^{2}+y_{2}^{2}} \leq \frac{1}{8}$ for $y_{1} \in[1, m]$ and (5.74), we get

$$
P_{2} \leq \frac{1}{8} \cdot 2(m-1)^{2} \sqrt{2 m}[\eta]_{C_{x}^{1 / 2}}=\frac{1}{4}(m-1)^{2} \sqrt{2 m}[\eta]_{C_{x}^{1 / 2}}
$$

For $P_{3}$, applying (5.85) with $a=b=m-1$ and (5.74), we get

$$
\begin{aligned}
P_{3} & \leq 2\left(\sqrt{2 m} \frac{1}{2} \arctan (1)+2(m-1)^{1 / 2}\left(f_{s}(1)-f_{s}(1)+C_{K_{2}, u p}\right)\right)[\eta]_{C_{x}^{1 / 2}} \\
& =2\left(\sqrt{2 m} \frac{\pi}{8}+2(m-1)^{1 / 2} C_{K_{2}, u p}\right)[\eta]_{C_{x}^{1 / 2}}
\end{aligned}
$$

Combining the above estimates, we yield the upper bounds $C_{m i d, 3}$ for $S_{m i d, i n t}$ in (5.63). The above bounds are increasing in $m$ and we get $C_{m i d, 3}(m) \leq C_{m i d, 3}\left(m^{u}\right)$.
Estimate of $C_{\text {out }}$. Recall $C_{\text {out }}(m, \varepsilon), C_{\text {out }}(1, \varepsilon)$ from (5.62). For $m \in\left[m^{l}, m^{u}\right]$, since the region $\Omega_{m i d, 2}(m)$ and $I_{K_{2}}(0, m-1), I_{K_{2}}(2, m+1)$ (see (5.88)) are increasing in $m$, for a fixed $\varepsilon$, we have $C_{\text {out }}(m) \leq C_{\text {out }}\left(m^{l}\right)$. We fixe a small $\varepsilon>0$ and get $\max \left(m^{l}, 1+\varepsilon\right)$. We have estimated the integral of $\left|\Delta(y) \| y_{2}-T(y)\right|^{1 / 2}$ in $C_{m i d, 2}\left(\varepsilon_{m}\right)$ (5.62) in Section 5.4, e.g. $S_{u p}$, and $I_{K_{2}, \infty}$ in (5.88) and the paragraph therein.

Estimate of the integrals in the far-field. To estimate the case of $m \geq m_{f}$ with $m_{f}=R_{1}$ sufficient large, we want to reduce it to the case of $m=m_{f}$ with an integral in the far-field, which is small. Recall $R_{m i d}, R_{\text {mid }}^{+}$from (5.65)

$$
\begin{equation*}
R_{m i d}^{++}(m) \triangleq\left\{y_{1} \in[1, m], y_{2} \geq s_{c}(m)\right\} \cup\left\{y_{2}<s_{c}(m), 1 \leq y_{1} \leq T\left(m,\left|y_{2}\right|\right)\right\} \tag{5.76}
\end{equation*}
$$

Note that from definition of $s_{c}, h_{c}^{+}$(5.89), for $y_{1}>1, y_{2}>0$, we have

$$
\begin{equation*}
h_{c}^{+}\left(y_{2}\right)<y_{1}, \Longleftrightarrow y_{2}<s_{c}\left(y_{1}\right) \tag{5.77}
\end{equation*}
$$

See the right black curve in right Figure 2 for an illustration. For $m \geq m_{f}$, we decompose the domain of integrals in $C_{m i d}, S_{m i d, 1, \varepsilon}$ (5.62), (5.63) into $y_{1} \in\left[1, m_{f}\right]$ and $y_{1} \in\left[m_{f}, m\right]$ as follows

$$
\begin{aligned}
& R_{\text {mid,low }}(m) \triangleq\left\{y: h_{c}^{+}\left(y_{2}\right)<y_{1}<m, y_{2} \geq \varepsilon\right\}=\left\{y: h_{c}^{+}\left(y_{2}\right)<y_{1}<m_{f}, y_{2} \geq \varepsilon\right\} \\
& \cup\left\{y: h_{c}^{+}\left(y_{2}\right)<y_{1}, m_{f}<y_{1}<m, y_{2} \geq \varepsilon\right\} \triangleq \Omega_{1 M} \cup \Omega_{1 R} \\
& R_{\text {mid }}^{++}(m)=R_{m i d}^{++}(m) \cap\left\{y: y_{1} \in\left[1, m_{f}\right]\right\} \cup R_{m i d}^{++}(m) \cap\left\{y: y_{1} \in\left[m_{f}, m\right]\right\} \triangleq \Omega_{2 M} \cap \Omega_{2 R}, \\
& {[1, m] \times[0, \varepsilon]=\left[1, m_{f}\right] \times[0, \varepsilon] \cup\left[m_{f}, m\right] \times[0, \varepsilon] \triangleq \Omega_{3 M} \cup \Omega_{3 R}}
\end{aligned}
$$

In $\Omega_{1 M}$, due to (5.77) and $s_{c}\left(y_{1}\right)$ is increasing in $y_{1}>1$ (5.89), we get $y_{2}<s_{c}\left(y_{1}\right)<s_{c}\left(m_{f}\right)$ and $\Omega_{1 M}=R_{\text {mid,low }}\left(m_{f}\right)$. For $\Omega_{2 M}$, we want to show $\Omega_{2 M} \subset R_{m i d}^{++}\left(m_{f}\right)$. We fix $y \in \Omega_{2 M}$. If $y_{2} \geq s_{c}\left(m_{f}\right)$, since $y_{1} \in\left[1, m_{f}\right]$ for $y \in \Omega_{2 M}$, from (5.76), we get $y \in R_{m i d}^{++}\left(m_{f}\right)$. If $y_{2}<s_{c}\left(m_{f}\right)$, from (5.77), we get $y_{1}>h_{c}^{+}\left(m_{f}\right)$ and $y$ is in the definition of $T$. See Lemma 5.7. Using $T(y)$ is decrreasing in $y_{1}$ from Lemma 5.7 and $y_{1}<T\left(m, y_{2}\right)$ for $y \in \Omega_{2 M}$, we get $y_{1}<T\left(m, y_{2}\right)<$ $T\left(m_{f}, y_{2}\right)$. Thus $\Omega_{2 M} \subset R_{m i d}^{++}\left(m_{f}\right)$.

For the integral $S_{m i d, 1, \varepsilon}$ in $\left|y_{1}\right| \in\left[m_{f}, m\right]$, since $\Delta$ is odd, using $\left|\eta_{m}\left(y_{1}, y_{2}\right)-\eta_{m}\left(-y_{1}, y_{2}\right)\right| \leq$ $\sqrt{2 y_{1}}[\eta]_{C_{x}^{1 / 2}}$ and following (5.71), (5.72), we get

$$
\left|\int_{\left|y_{1}\right| \in\left[m_{f}, m\right],\left|y_{2}\right| \leq \varepsilon} \Delta(y) \eta_{m}(y) d y\right| \leq 2 \sqrt{2} \int_{\Omega_{3 R}}|\Delta(y)| \sqrt{y_{1}} d y[\eta]_{C_{x}^{1 / 2}}
$$

Thus, the integral in $\tilde{C}_{m i d, 1}(m, \varepsilon)$ (5.62) in $\Omega_{1 M}, \Omega_{2 M}, \Omega_{3 M}$ can be bounded by the case of $m=m_{f}$. In $\Omega_{1 R}$, since $0<T<y_{1}$, we get $\left|y_{1}-T_{1}(y)\right|^{1 / 2} \leq\left|y_{1}\right|^{1 / 2}$. Since $\Omega_{i R}, \Omega_{i M}$ are disjoint,
following (5.58), we bound the integral in $\tilde{C}_{m i d, 1}$ for $\left|y_{1}\right| \geq m_{f}=R_{1}$ or $\left|y_{2}\right| \geq R_{2}$, which cover $\Omega_{i R}$, by

$$
4 \int_{y_{1} \geq R_{1}, \text { or } y_{2} \geq R_{2}}|\Delta(y)| y_{1}^{1 / 2} d y \leq 4 C_{B} \pi(B-1)^{-1 / 2}, \quad B=\min \left(R_{1}, R_{2}\right)-1
$$

To estimate the integral in $C_{o u t}$ in the far-field, e.g. outside a domain $\left[0, R_{1}\right] \times\left[0, R_{2}\right]$ with large $R_{1}, R_{2} \geq 10^{8}$, we use $\left|y_{2}-T\right| \leq\left|y_{2}\right|$ in the support of the integral and the estimate (5.58).
5.5.4. Estimate in a strip. Near the singularity, we need to obtain a sharp estimate of the integral

$$
\begin{equation*}
I_{a, b}=\lim _{\varepsilon \rightarrow 0} \frac{1}{2} \int_{Q_{\varepsilon, a, b}} \frac{y_{1}^{2}-y_{2}^{2}}{|y|^{4}} f(y) d y, \quad Q_{\varepsilon, a, b}=( \pm[\varepsilon, a]) \times( \pm[0, b]), \quad a<b . \tag{5.78}
\end{equation*}
$$

Without loss of generality, we assume $Q_{\varepsilon, a, b}=[\varepsilon, a] \times[0, b]$. Firstly, for a fixed $y_{1}$, we can construct a map $T_{3}(y) \leq y_{1}$ by solving

$$
\int_{T}^{y_{2}} K_{2}(y) d y=0, \quad K_{2}(y)=\frac{y_{1}^{2}}{y_{2}}
$$

Since $K_{2}(y)>0$ for $y_{2}<y_{1}$ and $K_{2}(y)<0$ for $y_{2}>y_{1}$, applying the transportation lemma, Lemma 3.6 3], we get

$$
\begin{equation*}
I_{a, b} \leq \lim _{\varepsilon \rightarrow 0} \int_{\varepsilon}^{a} \int_{y_{1}}^{b}\left|K_{2}(y) \| T(y)-y_{2}\right|^{1 / 2} d y[f]_{C_{y}^{1 / 2}}+\left|\int_{0}^{a} \int_{0}^{y_{1}^{2} / b} K_{2}(y) f(y) d y\right| \triangleq I_{1}[f]_{C_{y}^{1 / 2}}+I_{2} \tag{5.79}
\end{equation*}
$$

Denote $Q=[0, a] \times[0, a]$. Since $y_{1}^{2} / b \leq a^{2} / b \leq a$, in $I_{2}$, we have $|f(y)| \leq\|f\|_{L^{\infty}(Q)}$.
For $I_{2}$, we have $K_{2}(y)>0$ and

$$
\begin{aligned}
I_{2} & \leq\|f\|_{L^{\infty}(Q)} \lim _{\varepsilon \rightarrow 0} \int_{\varepsilon}^{a} \int_{0}^{y_{1}^{2} / b} K_{2}(y) d y=\left.\|f\|_{L^{\infty}(Q)} \int_{0}^{a} \frac{1}{2} \frac{y_{2}}{|y|^{2}}\right|_{0} ^{y_{1}^{2} / b} d y_{1} \\
& =\|f\|_{L^{\infty}(Q)} \frac{1}{2} \int_{0}^{a} \frac{y_{1}^{2} / b}{y_{1}^{2}+\left(y_{1}^{2} / b\right)^{2}} d y_{1}=\|f\|_{L^{\infty}(Q)} \frac{1}{2} \int_{0}^{a} \frac{b}{y_{1}^{2}+b^{2}} d y_{1}=\|f\|_{L^{\infty}(Q)} \frac{1}{2} \arctan \left(\frac{a}{b}\right) .
\end{aligned}
$$

For $I_{1}$, since $y_{2} \geq y_{1} \geq T$, we have $\left|T-y_{2}\right| \leq\left|y_{2}\right|$, and

$$
\begin{equation*}
I_{1} \leq \int_{0}^{a} \int_{a}^{b}\left|K_{2}(y)\right| y_{2}^{1 / 2} d y_{2}+\int_{0}^{a} \int_{y_{1}}^{a} K_{2}(y)\left|y_{2}-\frac{y_{1}^{2}}{y_{2}}\right|^{1 / 2} d y_{2} \triangleq I_{11}+I_{12} \tag{5.80}
\end{equation*}
$$

Using the scaling symmetry and (5.11), we get

$$
\begin{equation*}
I_{12}=a^{1 / 2} C_{K_{2}, u p} \tag{5.81}
\end{equation*}
$$

In $I_{11}, K_{2}(y)$ has a fixed sign in the domain of the integral. Integrating $y_{1}$ first and then using (5.5), and we yield

$$
\begin{align*}
I_{11} & =\left.\frac{1}{2} \int_{a}^{b} \frac{y_{1}}{y_{1}^{2}+y_{2}^{2}}\right|_{0} ^{a} y_{2}^{1 / 2} d y_{2}=\frac{1}{2} \int_{a}^{b} \frac{a y_{2}^{1 / 2}}{a^{2}+y_{2}^{2}} d y_{2}=\frac{1}{2} \cdot 2 a^{1 / 2}\left(f_{s}(\sqrt{b / a})-f_{s}(1)\right)  \tag{5.82}\\
& =a^{1 / 2}\left(f_{s}(\sqrt{b / a})-f_{s}(1)\right)
\end{align*}
$$

In summary, we establish

$$
\begin{equation*}
\left|I_{a, b}\right| \leq|f|_{L^{\infty}(Q)} \frac{1}{2} \arctan \left(\frac{a}{b}\right)+a^{1 / 2}\left(f_{s}(\sqrt{b / a})-f_{s}(1)+C_{K_{2}, u p}\right)[f]_{C_{y}^{1 / 2}}, \quad Q=[0, a]^{2} \tag{5.83}
\end{equation*}
$$

Using the same argument and swapping the variable $y_{1}, y_{2}$, for $0<b<a$ and

$$
\begin{equation*}
J_{a, b}=\lim _{\varepsilon \rightarrow 0} \frac{1}{2} \int_{R_{\varepsilon, a, b}} \frac{y_{1}^{2}-y_{2}^{2}}{|y|^{4}} f(y) d y, \quad R_{\varepsilon, a, b}=( \pm[0, a]) \times( \pm[\varepsilon, b]), \quad a<b \tag{5.84}
\end{equation*}
$$

we also obtain
$\left|J_{a, b}\right| \leq|f|_{L^{\infty}(Q)} \frac{1}{2} \arctan \left(\frac{b}{a}\right)+b^{1 / 2}\left(f_{s}(\sqrt{a / b})-f_{s}(1)+C_{K_{2}, u p}\right)[f]_{C_{x}^{1 / 2}}, Q= \pm[0, b] \times \pm[0, b]$.

A special case. Consider a similar integral

$$
M(a) \triangleq \int_{[0, a]^{2}} \frac{y_{1}^{2}-y_{2}^{2}}{2|y|^{4}} \tilde{f}(y) d y, \quad \tilde{f}(y)=f(y)-f\left(y_{1}, 0\right), \quad Q_{a} \triangleq=[0, a]^{2}
$$

For $f \in C^{1 / 2}\left(Q_{a}\right)$, the integrand is locally integrable, and we do not need to take the principle value. Applying (5.78), (5.79) with $a=b,|\tilde{f}(y)| \leq y_{2}^{1 / 2}[f]_{C_{y}^{1 / 2}}\left(Q_{a}\right),[\tilde{f}]_{C_{y}^{1 / 2}\left(Q_{a}\right)} \leq[f]_{C_{y}^{1 / 2}\left(Q_{a}\right)}$, the scaling symmetry, and (5.11), we can estimate $M(a)$ as follows

$$
\begin{equation*}
|M(a)| \leq[f]_{C_{y}^{\frac{1}{2}}\left(Q_{a}\right)}\left(\int_{0}^{a} \int_{y_{1}}^{a}\left|K_{2}(y)\right|\left|\frac{y_{1}^{2}}{y_{2}}-y_{2}\right|^{\frac{1}{2}}+\int_{0}^{a} \int_{0}^{y_{1}^{2} / a}\left|K_{2}(y)\right| y_{2}^{\frac{1}{2}} d y\right) \leq a^{\frac{1}{2}} C_{K_{2}}[f]_{C_{y}^{\frac{1}{2}}\left(Q_{a}\right)} \tag{5.86}
\end{equation*}
$$

The infinite length case. We consider estimating

$$
\begin{equation*}
I=I_{b, \infty}-I_{a, \infty}=\lim _{\varepsilon \rightarrow 0} \int_{a_{\varepsilon}}^{b} \int_{0}^{\infty} K_{2}(y) f(y) d y \tag{5.87}
\end{equation*}
$$

Applying the above argument and estimate (5.79), we get

$$
\begin{equation*}
I \leq I_{K_{2}, \infty}(a, b)[f]_{C_{y}^{1 / 2}}, \quad I_{K_{2}, \infty}(a, b) \triangleq \int_{a}^{b} \int_{y_{1}}^{\infty} \frac{1}{2}\left|\frac{y_{1}^{2}-y_{2}^{2}}{|y|^{4}}\right|\left|\frac{y_{1}^{2}}{y_{2}}-y_{2}\right|^{1 / 2} d y \tag{5.88}
\end{equation*}
$$

The second term in (5.79) vanishes since $\frac{y_{1}^{2}}{\infty}=0$. Applying a change of variable $y_{2}=s y_{1}$ yields
$I \leq \frac{1}{2} \int_{a}^{b} y_{1}^{-1 / 2} d y_{1} \int_{1}^{\infty} \frac{s^{2}-1}{\left(s^{2}+1\right)^{2}}\left|s-\frac{1}{s}\right|^{1 / 2} d s[f]_{C_{y}^{1 / 2}}=\left(b^{1 / 2}-a^{1 / 2}\right) \int_{1}^{\infty} \frac{s^{2}-1}{\left(s^{2}+1\right)^{2}}\left|s-\frac{1}{s}\right|^{1 / 2} d s[f]_{C_{y}^{1 / 2}}$.
The integral can be estimated using the strategy in Section 5.2 by partitioning $[0, \infty)$ and the integral formula for $K_{2}(s, 1)$ (5.3). Note that in the far-field $s \geq R>1$, we have

$$
\int_{R}^{\infty} \frac{s^{2}-1}{\left(s^{2}+1\right)^{2}}\left|s-\frac{1}{s}\right|^{1 / 2} \leq \int_{R}^{\infty} s^{-2+1 / 2} d s=2 R^{-1 / 2}
$$

5.6. Functions and transportation maps. We present the formulas of the transportation maps and the functions related to the sign of the kernels in the sharp Hölder estimate. Recall

$$
K_{1}=\frac{y_{1} y_{2}}{|y|^{4}}, \quad K_{2}=\frac{1}{2} \frac{y_{1}^{2}-y_{2}^{2}}{|y|^{4}}
$$

5.6.1. Sign functions. Solving $K_{2}\left(y_{1}+1, y_{2}\right)-K_{2}\left(y_{1}-1, y_{2}\right)=0$ for $y_{2} \geq 0$, we yield

$$
\begin{align*}
& y_{1}=h_{c}^{ \pm}\left(y_{2}\right) \triangleq\left(y_{2}^{2}+1 \pm 2 y_{2} \sqrt{y_{2}^{2}+1}\right)^{1 / 2} \\
& y_{2}=s_{c, i n}\left(y_{1}\right) \triangleq\left(\frac{-\left(y_{1}^{2}+1\right)+2\left(y_{1}^{4}-y_{1}^{2}+1\right)^{1 / 2}}{3}\right)^{1 / 2} \tag{5.89}
\end{align*}
$$

5.6.2. Transportation maps.

Map for $u_{x}$. For a fixed $s_{2} \neq 0$ and $s_{1}>0$, solving

$$
\begin{equation*}
\int_{T(s)}^{s_{1}}\left(K_{1}\left(s_{1}+1 / 2, s_{2}\right)-K_{1}\left(s_{1}-1 / 2, s_{2}\right)\right) d s_{1}=0 \tag{5.90}
\end{equation*}
$$

yields the equation of the transportation map in the $x$ direction

$$
\begin{equation*}
T^{3}+T^{2} s_{1}+T\left(s_{1}^{2}-\frac{1}{2}+2 s_{2}^{2}\right)-\frac{\left(4 s_{2}^{2}+1\right)^{2}}{16 s_{1}}=0 \tag{5.91}
\end{equation*}
$$

Map for $\left[u_{y}\right]_{C_{x}^{1 / 2}}$. For a fixed $y_{1} \geq 0$, solving

$$
\int_{T(y)}^{y_{2}}\left(K_{2}\left(y_{1}+1, y_{2}\right)-K_{2}\left(y_{1}-1, y_{2}\right)\right) d y_{2}=0
$$

yields the equation of the transportation map in the $y$ direction

$$
\begin{equation*}
T^{3}+T^{2} y_{2}+T\left(y_{2}^{2}+2+2 y_{1}^{2}\right)-\frac{\left(y_{1}^{2}-1\right)^{2}}{y_{2}}=0 . \tag{5.92}
\end{equation*}
$$

We rewrite the above equation as an equation for $W=T+\frac{y_{2}}{3}$
$0=W^{3}+W\left(\frac{2 y_{2}^{2}}{3}+2+2 y_{1}^{2}\right)-\left(\frac{\left(y_{1}^{2}-1\right)^{2}}{y_{2}}+\frac{y_{2}^{3}}{27}+\frac{y_{2}}{3}\left(\frac{2 y_{2}^{2}}{3}+2+2 y_{1}\right)\right) \triangleq W^{3}+p_{2}(y) W+q_{2}(y)$.
Since $p_{2}>0$, using the discriminant (5.27), we obtain $-\Delta_{W}(y)>0$. Thus the cubic equation of $T$ or $W$ has a unique real root that can be obtained by the formula similar to (5.28).

Map for $\left[u_{y}\right]_{C_{y}^{1 / 2}}$. For a fixed $y_{1}, y_{2} \geq 0$, solving

$$
\int_{y_{1}}^{T_{1}(y)} \Delta\left(s, y_{2}\right) d s=0
$$

with $T \geq 0$ yields the equation of the transportation map in $x$ direction

$$
1-T^{2}-y_{1}^{2}+T^{2} y_{1}^{2}-2 y_{2}^{2}-T^{2} y_{2}^{2}-y_{1}^{2} y_{2}^{2}-3 y_{2}^{4}=0
$$

or equivalently

$$
\begin{equation*}
T^{2}=\frac{y_{1}^{2}+2 y_{2}^{2}+y_{1}^{2} y_{2}^{2}+3 y_{2}^{4}-1}{y_{1}^{2}-y_{2}^{2}-1} \tag{5.93}
\end{equation*}
$$

We apply the above map to the following two regions separately

$$
y_{1} \in[0,1], y_{1} \leq h_{c}^{-}\left(y_{2}\right), \quad y_{1} \in[1, \infty], y_{1} \geq h_{c}^{+}\left(y_{2}\right)
$$

## 6. Additional $L^{\infty}$ estimates of $\nabla \mathbf{u}$ and some explicit integrals

We have discussed the $L^{\infty}$ estimates of $\nabla \mathbf{u}$ in Section 4 in Part II [2]. In this section, we provide a few more detailed calculations in the singular region. Recall from Section 5.3, 5 in [3] the energy $E_{1}, E_{4}$ for $W_{1}=\left(\omega_{1}, \eta_{1}, \xi_{1}\right)$, which satisfy

$$
\begin{align*}
& \max \left(\left\|\omega_{1} \varphi_{1}\right\|_{\infty}, \sqrt{2} \tau_{1}^{-1}\left\|\omega_{1}\left|x_{1}\right|^{-1 / 2} \psi_{1}\right\|_{\infty}, \tau_{1}^{-1}\left\|\omega_{1} \psi_{1}\right\|_{C_{g_{1}}^{1 / 2}}\right) \leq E_{1}(t)  \tag{6.1}\\
& \max \left(E_{1}(t), \mu_{g, 1}\left\|\omega_{1} \varphi_{g, 1}\right\|_{\infty}\right) \leq E_{4}(t), \quad \mu_{g, 1}=\tau_{2} \mu_{4}
\end{align*}
$$

with weights and parameters given below. In Appendix C. 1 in Part I 3], we choose the following parameters for the energy $E_{1}$

$$
\begin{equation*}
\tau_{1}=5, \quad \mu_{4}=0.065, \quad \tau_{2}=0.23 \tag{6.2}
\end{equation*}
$$

the following weights in the estimate of nonlocal terms

$$
\begin{align*}
& \psi_{1}=|x|^{-2}+0.5|x|^{-1}+0.2|x|^{-1 / 6}, \quad \psi_{d u}=\psi_{1}, \quad \psi_{u}=|x|^{5 / 2}+0.2|x|^{-7 / 6} \\
& g_{1}(h)=g_{10}(h) g_{10}(1,0)^{-1}, g_{10}(h)=\left(\sqrt{h_{1}+q_{11} h_{2}}+q_{13} \sqrt{h_{2}+q_{12} h_{1}}\right)^{-1}  \tag{6.3}\\
& \vec{q}_{1}=(0.12,0.01,0.25)
\end{align*}
$$

and the following weights for $\omega$ and the error

$$
\begin{align*}
\varphi_{1} & =x^{-1 / 2}\left(|x|^{-2.4}+0.6|x|^{-1 / 2}\right)+0.3|x|^{-1 / 6}, \quad \varphi_{g 1}=\varphi_{1}+|x|^{1 / 16}, \\
\varphi_{\text {elli }} & =\left|x_{1}\right|^{-1 / 2}\left(|x|^{-2}+0.6|x|^{-1 / 2}\right)+0.3|x|^{-1 / 6} \tag{6.4}
\end{align*}
$$

We do not write down the full energy since we do not use other norms in $E_{i}$ in this supplementary material. Below, to simplify the notation, we simplify $\omega_{1}$ as $\omega$ in the energy.
6.1. $L^{\infty}$ estimate of $\nabla \mathbf{u}$ in the singular region. In this section, we estimate

$$
\begin{align*}
& S=P . V . \int_{Q_{\delta}} K(y)(W \psi)(x+y)+s(W \psi)(x), \quad Q_{\delta}=[-\delta, \delta]^{2}, \quad Q_{\delta}^{ \pm} \triangleq[-\delta, \delta] \times \pm[0, \delta] \\
& K=K_{1}=\frac{y_{1} y_{2}}{|y|^{4}}, \quad K_{2}=\frac{1}{2} \frac{y_{1}^{2}-y_{2}^{2}}{|y|^{4}}, \quad s=0, \frac{\pi}{2} \text { or }-\frac{\pi}{2} \tag{6.5}
\end{align*}
$$

related to the piecewise $L^{\infty}$ estimate of $\psi \nabla \mathbf{u}$ using the energy $E_{1}$ defined in (6.1) which satisfies

$$
\begin{equation*}
\|\omega \varphi\|_{L^{\infty}} \leq E_{1},[\omega \psi]_{C_{x}^{1 / 2}} \leq \gamma_{1} E_{1},[\omega \psi]_{C_{x}^{1 / 2}} \leq \gamma_{2} E_{1}, \gamma_{1}=\tau_{1}, \gamma_{2}=\tau_{1} g_{1}(0,1)^{-1}, \gamma_{1}>\gamma_{2} \tag{6.6}
\end{equation*}
$$

Our goal is to establish the following estimate

$$
\begin{equation*}
|S| \leq C_{1}| | \omega \varphi \|_{L^{\infty}}+C_{2}[\omega \psi]_{C_{x}^{1 / 2}}+C_{3}[\omega \psi]_{C_{y}^{1 / 2}} \leq\left(C_{1}+\gamma_{1} C_{2}+C_{3} \gamma_{2}\right) E_{1} \tag{6.7}
\end{equation*}
$$

with constant $C_{1}+\gamma_{1} C_{2}+C_{3} \gamma_{2}$ as small as possible. Below, we focus on the case using the norm $\|\omega \varphi\|_{\infty}$ with $\varphi=\varphi_{1}$. The estimate using other norms $\|\omega \varphi\|_{\infty}, \varphi=\varphi_{g, 1}, \varphi_{\text {elli }}$ is similar and is discussed in Section 6.3. We adopt the notations from Section 3 in [3, Section 4 in Part II [2]. Here, $W$ is the odd extension of $\omega$ in $y$ form $\mathbb{R}_{2}^{+}$to $\mathbb{R}_{2}, \tau_{1}=5$ and $g_{1}, \psi=\psi_{1}, \varphi$ are the weights for $\omega$ in the energy. We use

$$
(K, s)=\left(K_{1}, 0\right), \text { for } u_{x}, \quad\left(K_{2},-\frac{\pi}{2}\right), \text { for } v_{x}, \quad\left(K_{2}, \frac{\pi}{2}\right) \text { for } u_{y}
$$

Note that one needs to multiply $\frac{1}{\pi}$ to get the estimate for $\psi \nabla \mathbf{u}$. We have discussed the estimate for $u_{x}$ in Section 4.2 in Part II [2].

We assume that $x_{i} \in\left[x_{i}^{l}, x_{i}^{u}\right] \in \mathbb{R}_{+}$, and derive the piecewise bound for $S(x)$. Denote

$$
\begin{equation*}
F=W \psi, \quad x_{2, \delta}=\min \left(x_{2}, \delta\right), \quad \alpha=x_{2, \delta} / \delta, \quad \alpha^{l}=\min \left(x_{2}^{l} / \delta, 1\right), \quad \alpha^{u}=\min \left(x_{2}^{u} / \delta, 1\right) \tag{6.8}
\end{equation*}
$$

6.1.1. Estimate of $u_{x}$. In the case of $u_{x}$, we have $K=K_{1}, s=0$ and $K_{1}(s)$ is odd in $s_{1}, s_{2}$. Using $\left|W \psi(x+y)-W \psi\left(x+\left(-y_{1}, y_{2}\right)\right)\right| \leq \sqrt{2 y_{1}}[W \psi]_{C_{x}^{1 / 2}}$, we get

$$
|S| \leq[W \psi]_{C_{x}^{1 / 2}} \int_{[0, \delta] \times[-\delta, \delta]}\left|K_{1}(s)\right|\left|2 s_{1}\right|^{1 / 2} d s=2 \sqrt{2 \delta}[\omega \psi]_{C_{x}^{1 / 2}} \int_{[0,1]^{2}} K_{1}(s)\left|s_{1}\right|^{1 / 2} d s
$$

Using (5.3) and (5.14), we get

$$
\int_{[0,1]^{2}} K_{1}(s)\left|s_{1}\right|^{1 / 2} d s=\int_{0}^{1}-\left.\frac{1}{2} \frac{s_{1}^{3 / 2}}{|s|^{2}}\right|_{0} ^{1} d s_{1}=\frac{1}{2} \int_{0}^{1}-\frac{s_{1}^{3 / 2}}{s_{1}^{2}+1}+\frac{s_{1}^{3 / 2}}{s_{1}^{2}} d s_{1}=1-\frac{1}{2} f_{h}(1)
$$

The above estimate only involves $[\omega]_{C_{x}^{1 / 2}}$ and is used when $x$ is close to the boundary $x_{2}=0$. For $x$ away from the boundary, $W$ is also Hölder continuous in $y$ in $Q(x, r)$, since $\gamma_{2}<\gamma_{1}$ (6.6), we can use the seminorm $[\omega]_{C_{y}^{1 / 2}}$ to control $S$ and improve the estimate. We have

$$
S=\int_{-\delta}^{\delta} d y_{1}\left(\int_{-\delta}^{-x_{2, \delta}}+\int_{-x_{2, \delta}}^{x_{2, \delta}}+\int_{x_{2, \delta}}^{\delta}\right) F(x+y) K_{1}(y) d y_{2} \triangleq I_{1}+I_{2}+I_{3}
$$

The domain in $I_{1}$ is below the boundary, and the domain in $I_{2}, I_{3}$ is above the boundary, where $F \in C_{y}^{1 / 2}$. For $I_{1}, I_{3}$, we use $[F]_{C_{x}}^{1 / 2}$ to control it. For $I_{2}$, we use $[F]_{C_{y}}^{1 / 2}$ to control it. Using the fact that $K_{1}(y)$ is odd in $y_{1}, y_{2}$ (6.8) and (6.6), we get

$$
\begin{aligned}
S & \leq[F]_{C_{x}^{1 / 2}} \int_{0}^{\delta}\left(\int_{-\delta}^{-x_{2, \delta}}+\int_{x_{2, \delta}}^{\delta}\right)\left|K_{1}(y)\right|\left|2 y_{1}\right|^{1 / 2}+[F]_{C_{y}^{1 / 2}} \int_{-\delta}^{\delta} \int_{0}^{x_{2, \delta}}\left|K_{1}(y) \| 2 y_{2}\right|^{1 / 2} d y \\
& =E_{1} 2 \sqrt{2 \delta}\left(\gamma_{1} \int_{0}^{1} \int_{\alpha}^{1} K_{1}(y) y_{1}^{1 / 2} d y_{1}+\gamma_{2} \int_{0}^{1} \int_{0}^{\alpha} K_{1}(y) y_{2}^{1 / 2} d y_{1}\right) \triangleq E_{1} 2 \sqrt{2 \delta}\left(\gamma_{1} I_{1}+\gamma_{2} I_{2}\right)
\end{aligned}
$$

where $\alpha$ is defined in (6.8), and $I_{1}, I_{2}$ denote the first and the second integral and we have rescaled the integral by changing $y \rightarrow \delta y$. Using (5.3), (6.8), and (5.13), we yield

$$
\begin{aligned}
I_{2} & =\frac{1}{2} \int_{0}^{\alpha}-\left.\frac{y_{2}^{3 / 2}}{y_{1}^{2}+y_{2}^{2}}\right|_{y_{1}=0} ^{1} d y_{2}=\frac{1}{2} \int_{0}^{\alpha}\left(\frac{1}{y_{2}^{1 / 2}}-\frac{y_{2}^{3 / 2}}{1+y_{2}^{2}}\right) d y_{2} \leq \frac{1}{2} \int_{0}^{\alpha^{u}}\left(\frac{1}{y_{2}^{1 / 2}}-\frac{y_{2}^{3 / 2}}{1+y_{2}^{2}}\right) d y_{2} \\
& =\frac{1}{2}\left(2 \sqrt{\alpha^{u}}-f_{h}\left(\alpha^{u}\right)\right)
\end{aligned}
$$

where the inequality follows from $\frac{1}{y_{2}^{1 / 2}}-\frac{y_{2}^{3 / 2}}{1+y_{2}^{2}} \geq 0$. For $I_{1}$, using (5.3) and (5.14), we get

$$
\begin{aligned}
I_{1} & =\frac{1}{2} \int_{0}^{1}-\left.\frac{y_{1}^{3 / 2}}{|y|^{2}}\right|_{y_{2}=\alpha} ^{1} d y_{1}=\frac{1}{2} \int_{0}^{1} \frac{y_{1}^{3 / 2}}{\alpha^{2}+y_{1}^{2}}-\frac{y_{1}^{3 / 2}}{1+y_{1}^{2}} d y_{1} \leq \frac{1}{2} \int_{0}^{1} \frac{y_{1}^{3 / 2}}{\left(\alpha^{l}\right)^{2}+y_{1}^{2}}-\frac{y_{1}^{3 / 2}}{1+y_{1}^{2}} d y_{1} \\
& =\frac{1}{2}\left(\sqrt{\alpha^{l}} f_{h}\left(\frac{1}{\alpha^{l}}\right)-f_{h}(1)\right) .
\end{aligned}
$$

If $x_{2} \geq \delta$, we get $\alpha^{l}=1$ and the first part vanishes $I_{1}=0$.
Improvement for $x_{2} \geq \delta$. When $x_{2} \geq \delta$, we have $x+Q_{\delta} \subset \mathbb{R}_{2}^{+}$and $F \in C^{1 / 2}\left(x+Q_{\delta}\right)$. Denote

$$
M=\gamma_{2}^{2} / \gamma_{1}^{2}<1
$$

Symmetrizing the integrals, using (6.6),

$$
\begin{aligned}
& \left|F(x+y)+F(x-y)-F\left(x_{1}-y_{1}, x_{1}+y_{2}\right)-F\left(x_{1}+y_{1}, x_{2}-y_{2}\right)\right| \\
\leq & 2 \min \left([F]_{C_{x}^{1 / 2}}\left|2 y_{1}\right|^{1 / 2},[F]_{C_{y}^{1 / 2}}\left|2 y_{2}\right|^{1 / 2}\right) \leq 2 \sqrt{2} \min \left(\gamma_{1}\left|y_{1}\right|^{1 / 2}, \gamma_{2}\left|y_{2}\right|^{1 / 2}\right) E_{1}
\end{aligned}
$$

$\gamma_{1}>\gamma_{2}$, and an argument similar to the above, we derive

$$
|S| \leq 2 \sqrt{2 \delta} E_{1} \int_{[0,1]^{2}} K_{1}(y) \min \left(\gamma_{1}\left|y_{1}\right|^{1 / 2}, \gamma_{2}\left|y_{2}\right|^{1 / 2}\right) . \triangleq 2 \sqrt{2 \delta} E_{1} I
$$

The threshold between two estimates is $y_{2}=y_{1} / M$ or $y_{1}=M y_{2}$. We have

$$
\begin{aligned}
I & =\int_{[0,1]^{2}} K_{1}(y)\left(\mathbf{1}_{\frac{y_{1}}{M} \leq y_{2}}\left|y_{1}\right|^{1 / 2} \gamma_{1}+\mathbf{1}_{\frac{y_{1}}{M} \geq y_{2}}\left|y_{2}\right|^{1 / 2} \gamma_{2}\right) d y \\
& =\gamma_{1} \int_{0}^{M} d y_{1} \int_{y_{1} / M}^{1} \frac{y_{1} y_{2}}{|y|^{4}} y_{1}^{1 / 2} d y_{2}+\gamma_{2} \int_{0}^{1} d y_{2} \int_{M y_{2}}^{1} \frac{y_{1} y_{2}}{|y|^{4}} y_{2}^{1 / 2} d y_{1} \triangleq I_{1}+I_{2}
\end{aligned}
$$

For $I_{1}$, using (5.3) and (5.13), we get
$I_{1}=\frac{\gamma_{1}}{2} \int_{0}^{M}-\left.\frac{y_{1}^{3 / 2}}{y_{1}^{2}+y_{2}^{2}}\right|_{y_{1} / M} ^{1} d y_{1}=\frac{\gamma_{1}}{2} \int_{0}^{M}-\frac{y_{1}^{3 / 2}}{1+y_{1}^{2}}+y_{1}^{-1 / 2} \frac{1}{(1 / M)^{2}+1} d y_{1}=\gamma_{1} \frac{M^{5 / 2}}{M^{2}+1}-\frac{\gamma_{1}}{2} f_{h}(M)$.
For $I_{2}$, using (5.3) and (5.13), we yield

$$
I_{2}=\frac{\gamma_{2}}{2} \int_{0}^{1}-\left.\frac{y_{2}^{3 / 2}}{y_{1}^{2}+y_{2}^{2}}\right|_{M y_{2}} ^{1} d y_{2}=\frac{\gamma_{2}}{2} \int_{0}^{1} y_{2}^{3 / 2}\left(\frac{1}{\left(M^{2}+1\right) y_{2}^{2}}-\frac{1}{y_{2}^{2}+1}\right) d y_{2}=\frac{\gamma_{2}}{2}\left(\frac{2}{M^{2}+1}-f_{h}(1)\right)
$$

Since $M=\gamma_{2}^{2} / \gamma_{1}^{2}, \gamma_{1} M^{1 / 2}=\gamma_{2}$, we establish

$$
|S| \leq \sqrt{2 \delta} E_{1}\left(2 \gamma_{1} \frac{M^{5 / 2}}{1+M^{2}}-\gamma_{1} f_{h}(M)+2 \gamma_{2} \frac{1}{M^{2}+1}-\gamma_{2} f_{h}(1)\right)=\sqrt{2 \delta} E_{1}\left(2 \gamma_{2}-\gamma_{1} f_{h}\left(\gamma_{2}^{2} / \gamma_{1}^{2}\right)-\gamma_{2} f_{h}(1)\right)
$$

6.1.2. Estimate of $v_{x}$. In the case of $v_{x}, u_{y}$ (6.5), we have $K(y)=K_{2}(y)$ which is even in $y_{1}, y_{2}$, and $s=-\frac{\pi}{2}$ in the case of $v_{x}$, and $s=\frac{\pi}{2}$ for $u_{y}$. Firstly, if $x_{2} \geq \delta$, we get $x+Q_{\delta} \subset \mathbb{R}_{2}^{+}$and $\omega \psi \in C^{1 / 2}\left(x+Q_{\delta}\right)$. Using Lemma 5.1, we rewrite $S$ in the case of $v_{x}$ as follows

$$
\begin{aligned}
S & =P . V \cdot \int_{Q_{\delta}} K_{2}(y)(F(x+y)-F(x)) d x-\frac{\pi}{2} F(x)=\lim _{\varepsilon \rightarrow 0} \int_{Q_{\delta},\left|y_{1}\right| \geq \varepsilon} K_{2}(y)(F(x+y)-F(x)) d y-\frac{\pi}{2} F(x) \\
& \triangleq S_{1}-\frac{\pi}{2} F(x) .
\end{aligned}
$$

The first term $S_{1}$ has the form (5.78). Applying (5.79) to four regions $\pm[0, \delta] \times[0, \delta]$ and $[F(x+\cdot)-F(x)]_{C_{y}^{1 / 2}}=[F]_{C_{y}^{1 / 2}}$, we yield

$$
\begin{align*}
\left|S_{1}\right| \leq & {[F]_{C_{y}^{1 / 2}} \int_{-\delta}^{\delta} d y_{1} \int_{\left|y_{1}\right| \leq\left|y_{2}\right| \leq \delta}\left|\frac{y_{1}^{2}}{y_{2}}-y_{2}\right|^{\frac{1}{2}}\left|K_{2}(y)\right| d y_{2} } \\
& +\left|\int_{-\delta}^{\delta} d y_{1} \int_{\left|y_{2}\right| \leq \frac{y_{1}^{2}}{\delta}} K_{2}(y)(F(x+y)-F(x)) d y_{2}\right| \triangleq S_{11}+S_{12} \tag{6.9}
\end{align*}
$$

For $S_{11}$, using scaling symmetries, (5.11), and (6.6), we get

$$
S_{11}=4[F]_{C_{y}^{1 / 2}} \delta^{1 / 2} C_{K_{2}, u p} \leq 4 \gamma_{2} \delta^{1 / 2} C_{K_{2}, u p} E_{1}
$$

For $S_{12}$, using (6.6), $|F(x+y-F(x))| \leq[F]_{C_{x}}^{1 / 2}\left|y_{1}\right|^{1 / 2}+[F]_{C_{y}}^{1 / 2}\left|y_{2}\right|^{1 / 2} \leq E_{1}\left(\gamma_{1}\left|y_{1}\right|^{1 / 2}+\gamma_{2}\left|y_{2}\right|^{1 / 2}\right)$, the symmetries of $K_{2}$ in $y$, and $\left|y_{2}\right| \leq\left|y_{1}\right|$, we yield

$$
\left|S_{12}\right| \leq 4 E_{1} \int_{0}^{\delta} \int_{0}^{y_{1}^{2} / \delta}\left(\gamma_{1}\left|y_{1}\right|^{1 / 2}+\gamma_{2}\left|y_{2}\right|^{1 / 2}\right) K_{2}(y) d y=4 E_{1}\left(\gamma_{1} I_{1}+\gamma_{2} I_{2}\right)
$$

where $I_{i}$ denotes two integrals. Using the scaling symmetry of $K_{2}$, (5.3), (5.6) and (5.12), we get

$$
\begin{aligned}
& I_{2}=\delta^{1 / 2} \int_{0}^{1} \int_{0}^{y_{1}^{2}} K_{2}(y)\left|y_{2}\right|^{1 / 2} d y=\delta^{1 / 2} C_{K_{2}, \text { low }} \\
& I_{1}=\delta^{1 / 2} \int_{0}^{1} \int_{0}^{y_{1}^{2}} K_{2}(y) y_{1}^{1 / 2} d y=\left.\delta^{1 / 2} \int_{0}^{1} \frac{y_{2}}{2|y|^{2}}\right|_{0} ^{y_{1}^{2}} d y_{1}=\delta^{1 / 2} \int_{0}^{1} \frac{1}{2\left(1+y_{1}^{2}\right)} y_{1}^{1 / 2} d y_{1}=\delta^{1 / 2} f_{s}(1)
\end{aligned}
$$

Combining the above estimates of $S_{11}, S_{12}$ and further bounding $F(x) \leq \psi(x) / \varphi(x) E_{1}$ (6.6), we establish

$$
|S(x)| \leq\left|S_{11}\right|+\left|S_{12}\right|+\frac{\pi}{2}|F(x)| \leq 4 E_{1} \delta^{1 / 2}\left(\gamma_{1} f_{s}(1)+\gamma_{2}\left(C_{K_{2}, l o w}+C_{K_{2}, u p}\right)\right)+\frac{\pi}{2} \frac{\psi(x)}{\varphi(x)} E_{1}
$$

The above estimate does not hold for $x_{2}<\delta$, i.e. the singular region touches the boundary, since $W$ is discontinuous across the boundary. Before we estimate $u_{y}$ and the case $x_{2}<\delta$, we discuss another $L^{\infty}$ estimate for $S_{12}$.
6.1.3. $L^{\infty}$ estimate in a curved region. Since $K_{2}(y) F(x+y) \mathbf{1}_{\left|y_{2}\right| \leq y_{1}^{2} / \delta} \mathbf{1}_{\left|y_{1}\right| \leq \delta}$ is locally integrable, we can bound it using $\|\omega \varphi\|_{\infty}$. We focus on a specific quadrant $y_{1}, y_{2} \geq 0$. Then in the integral region $y_{2} \leq y_{1}^{2} / \delta \leq y_{1}$, we get $K_{2}(y) \geq 0$. We partition [ 0,1$]$ using mesh $0=z_{0}<\ldots<z_{m}=1$ and use a change of variables $y=\delta s, K_{2}(\delta s) \delta^{2}=K_{2}(s)$ to get

$$
\begin{aligned}
\left|S_{12}^{++}\right| & =\left|\int_{0}^{\delta} \int_{0}^{y_{2}^{2} / \delta} K_{2}(y)(W \psi)(x+y) d y\right|=\left|\sum_{0 \leq i \leq m-1} \int_{z_{i}}^{z_{i+1}} \int_{0}^{s_{1}^{2}} K_{2}(s)(W \psi)(x+\delta s) d s\right| \\
& \leq \sum_{0 \leq i \leq m-1}\|W \varphi\|_{L^{\infty}}\left\|\frac{\psi}{\varphi}\right\|_{L^{\infty}\left(x+R_{i}\right)}\left|\int_{z_{i}}^{z_{i+1}} \int_{0}^{s_{1}^{2}} K_{2}(s) d s\right|, \quad R_{i}=\left[z_{i} \delta, z_{i+1} \delta\right] \times\left[0, z_{i+1}^{2} \delta\right]
\end{aligned}
$$

The piecewise $L^{\infty}$ bound $\left\|\frac{\psi}{\varphi}\right\|_{L^{\infty}\left(x+R_{i}\right)}$ for $x \in\left[x_{1}^{l}, x_{1}^{u}\right] \times\left[x_{2}^{l}, x_{2}^{u}\right] \triangleq B_{x}$ can be obtained by covering the region $B_{x}+R_{i}$. See Section 4.1.6 in Part II [2]. Since $K_{2}(s)$ has a fixed sign in the domain, using (5.3), we get

$$
\begin{equation*}
\int_{z_{i}}^{z_{i+1}} \int_{0}^{s_{1}^{2}} K_{2}(s) d s=\int_{z_{i}}^{z_{i+1}} \frac{1}{2} \frac{s_{1}^{2}}{s_{1}^{2}+s_{1}^{4}} d s_{1}=\left.\frac{1}{2} \arctan s_{1}\right|_{z_{i}} ^{z_{i+1}} \tag{6.10}
\end{equation*}
$$

Combining the above estimates, we obtain a sharp $L^{\infty}$ estimate for $S_{12}^{++}$. When $\delta$ is small enough, we do not further partition the domain $\delta \cdot[0,1]$ and estimate $S_{12}^{++}$directly

$$
\left|S_{12}^{++}\right| \leq\|W \varphi\|_{\infty}\left\|\frac{\psi}{\varphi}\right\|_{L^{\infty}\left(x+[0, \delta]^{2}\right)} \cdot \frac{\pi}{8}
$$



Figure 3. Illustration of the estimates. The shaded region represents the curved region, where we estimate the integrals $S_{c u r, i}^{\alpha, \beta}$ using the estimate in Section 6.1.3 and $\|\omega \varphi\|_{\infty}$. For other regions, we estimate the integrals using $C_{x_{i}}^{1 / 2}$ seminorm. The red arrow represents the direction of the $C_{x_{i}}^{1 / 2}$ seminorm. Left: estimate of $u_{y}, v_{x}$ with $x_{2}<\delta$. Right: estimate of $u_{y}$ with $x_{2}>\delta$.
where $\pi / 4$ is the integral of $K_{2}$ in the whole region, which follows from (6.10) with $z_{i}=0, z_{i+1}=$ 1.

Similar estimates apply to the integrals in other curved regions

$$
\begin{align*}
S_{c u r, i}^{\alpha \beta}(x, \delta) & \triangleq \int_{R_{i} \cap \mathbb{R}^{\alpha} \times \mathbb{R}^{\beta}} \mathbf{1}_{s \in \mathbb{R}_{\alpha} \times \mathbb{R}_{\beta}} K_{2}(s)(W \psi)(x+s) d s, \quad \alpha, \beta= \pm  \tag{6.11}\\
R_{1} & =\left\{\left|s_{1}\right| \leq \delta,\left|s_{2}\right| \leq s_{1}^{2} / \delta\right\}, \quad R_{2}=\left\{\left|s_{2}\right| \leq \delta,\left|s_{1}\right| \leq s_{2}^{2} / \delta\right\}
\end{align*}
$$

6.1.4. Estimate of $v_{x}$ for $x_{2} \leq \delta$. For $x_{2} \leq \delta$, the singular region $x+Q_{\delta}$ touches the boundary $x_{2}=0$ and $F \notin C_{y}^{1 / 2}\left(x+Q_{\delta}\right)$. Using Lemma 5.1] we decompose the integral (6.5) as follows (6.12)
$S(x)=\lim _{\varepsilon \rightarrow 0}\left(\int_{Q_{\delta}^{+},\left|y_{1}\right| \geq \varepsilon}+\int_{Q_{\delta}^{-}, y_{2} \leq-\varepsilon}\right) K_{2}(y) F(x+y) d y-\frac{\pi}{2} F(x) \triangleq S_{1}+S_{2}+S_{3}, \quad Q_{\delta}^{ \pm} \triangleq Q_{\delta} \cap \mathbb{R}_{2}^{ \pm}$.
We get a factor $-\frac{\pi}{4} F(x)$ in the upper part $Q^{+}$and $\frac{\pi}{4} F(x)$ in $Q^{-}$from Lemma 5.1, and they are canceled. We first apply estimate (5.79) and then the scaling symmetries of $K_{2}$, (5.11) and (6.11) to get

$$
\begin{aligned}
S_{1} & \leq[F]_{C_{y}^{1 / 2}} \int_{-\delta}^{\delta} d y_{1} \int_{y_{1}}^{\delta} K_{2}(y)\left|\frac{y_{1}^{2}}{y_{2}}-y_{2}\right|^{1 / 2} d y_{2}+\int_{-\delta}^{\delta} \int_{y_{2} \leq y_{1}^{2} / \delta} K_{2}(y) F(x+y) d y \\
& =2[F]_{C_{y}^{1 / 2}} \delta^{1 / 2} C_{K_{2}, u p}+S_{c u r, 1}^{++}+S_{c u r, 1}^{-+}
\end{aligned}
$$

where we have a factor 2 since the domain contains 2 quardrants.
For $S_{2}, F \notin C_{y}^{1 / 2}\left(x+Q_{\delta}^{-}\right)$. Thus, we apply the estimate for (5.84) using $[F]_{C_{x}}^{1 / 2}$ instead. Using an estimate similar to (5.79), (5.11), and (6.11), we get

$$
\begin{aligned}
S_{2} & =[F]_{C_{x}}^{1 / 2} \int_{-\delta}^{0} d y_{2} \int_{\left|y_{2}\right| \leq\left|y_{1}\right| \leq \delta}\left|\frac{y_{2}^{2}}{y_{1}}-y_{1}\right|^{1 / 2}\left|K_{2}(y)\right| d y+\int_{-\delta}^{0} \int_{\left|y_{1}\right| \leq y_{2}^{2} / \delta} K_{2}(y) F(x+y) d y \\
& =2[F]_{C_{x}}^{1 / 2} \delta^{1 / 2} C_{K_{2}, u p}+S_{c u r, 2}^{+-}+S_{c u r, 2}^{--}
\end{aligned}
$$

The remaining terms $S_{c u r, i}^{\alpha, \beta}$ in the above are estimated using the method in Section 6.1.3, The last term $S_{3}=\frac{\pi}{2} F(x)$ is estimated directly using $\frac{\pi}{2} \frac{\psi(x)}{\varphi(x)} E_{1}$ (6.6), (6.8). See the left figure in Figure 3 for various regions. We estimate $S_{c u r, i}^{\alpha, \beta}$ in the shaded region.
6.1.5. Estimate for $u_{y}$. The estimate of $u_{y}$ is completely similar. In this case, (6.5) becomes

$$
S(x)=P . V . \int_{Q_{\delta}} K_{2}(y) F(x+y) d y+\frac{\pi}{2} F(x)
$$

If $x_{2} \geq \delta$, we have $x+Q_{\delta} \subset \mathbb{R}_{2}^{+}$and $F \in C^{1 / 2}\left(x+Q_{\delta}\right)$. Applying Lemma 5.1 to 4 quadrants and (5.79) yields

$$
\begin{aligned}
|S(x)| & =\left|\lim _{\varepsilon \rightarrow 0} \int_{Q_{\delta},\left|y_{1}\right| \geq \varepsilon} K_{2}(y) F(x+y) d y\right| \\
& \leq[F]_{C_{y}^{1 / 2}} \int_{-\delta}^{\delta} d y_{1} \int_{\left|y_{1}\right| \leq\left|y_{2}\right| \leq \delta}\left|\frac{y_{1}^{2}}{y_{2}}-y_{2}\right|^{1 / 2}\left|K_{2}(y)\right| d y_{2}+\left|\int_{-\delta}^{\delta} d y_{1} \int_{\left|y_{2}\right| \leq y_{1}^{2} / \delta} K_{2}(y) F(x+y) d y_{2}\right| \\
& =4 \delta^{1 / 2}[F]_{C_{y}^{1 / 2}} C_{K_{2}, u p}+S_{c u r, 1}^{++}+S_{c u r, 1}^{+-}+S_{c u r, 1}^{-+}+S_{c u r, 1}^{--} .
\end{aligned}
$$

For $x_{2}<\delta$, we do not have $F \in C_{y}^{1 / 2}(x+Q)$. Using Lemma 5.1] yields another decomposition

$$
S(x)=\lim _{\varepsilon \rightarrow 0}\left(\int_{Q_{\delta}^{+},\left|y_{1}\right| \geq \varepsilon}+\int_{Q_{\delta}^{-}, y_{2} \leq-\varepsilon}\right) K_{2}(y) F(x+y) d y d y+\frac{\pi}{2} F(x) \triangleq S_{1}+S_{2}+S_{3}
$$

The estimates for $S_{1}, S_{2}, S_{3}$ are completely the same as those in Section 6.1.4. See the left figure in Figure 3 for an illustration of the estimate in the case of $x_{2} \geq \delta$, and right figure for $x_{2}>\delta$.
6.2. Estimate of $u(x) / x_{1}^{1 / 2}$. In the weighted $L^{\infty}$ energy estimate in [3], we need to estimate $u(x) /\left|x_{1}\right|^{1 / 2}$. We have discussed how to estimate $u(x) /\left|x_{1}\right|^{1 / 2}$ in Appendix B. 4 in Part II [2]. In this section, we derive the piecewise bounds for $J_{i}(B)$

$$
J_{1}(B)=\int_{[-1,0] \times[0,1 / B]} K_{s}(s) d s, \quad J_{2}(B)=\int_{[0,1 / B]^{2}} K_{s}(s) d s, \quad K_{s}(s)=\frac{2\left(s_{1}+1\right) s_{2}}{|s|^{2}\left(\left(s_{1}+2\right)^{2}+s_{2}^{2}\right)}
$$

which captures the singular part in the estimate of $u(x) / x_{1}$. Clearly, $J_{i}(B)$ is decreasing in $B$, for $B \in\left[B^{l}, B^{u}\right]$, we get $J_{i}(B) \leq J_{i}\left(B^{l}\right)$. Since

$$
K_{s}(s)=\frac{1}{2}\left(\frac{s_{2}}{|s|^{2}}-\frac{s_{2}}{\left(s_{1}+2\right)^{2}+s_{2}^{2}}\right)
$$

we can derive its analytic integral formula using (5.4). In particular, we have

$$
\begin{aligned}
& J_{1}(B)=\frac{B\left(\log \left(\frac{1}{B^{2}}+1\right)-\log \left(\frac{1}{B^{2}}+4\right)+\log (4)\right)+2 \arctan (B)-\arctan (2 B)}{2 B} \\
& J_{2}(B)=J_{21}(B)-\log B
\end{aligned}
$$

where the formula for $J_{2}(B)$ is too lengthy and we refer it to the Mathematica code in [1]. $J_{21}(B)$ is the regular part as $B \rightarrow 0$ and we define it by $J_{2}(B)-\log B$. Since $J_{i}(B)$ is decreasing in $B$, using the above formulas and $J_{i}(B) \leq J_{i}\left(B^{l}\right)$, we get the piecewise bounds for $J_{i}(B)$ in $\left[B^{l}, B^{u}\right]$. For $B \in\left[B^{l}, B^{u}\right]=\left[0, B^{u}\right]$, we derive the asymptotic behavior of the formula as $B \rightarrow 0$. For $J_{1}(B)$, since $2 \arctan B-\arctan (2 B)=O\left(B^{3}\right)$, we get

$$
\lim _{B \rightarrow 0} J_{1}(B)=\lim _{B \rightarrow 0} \frac{1}{2} \log \left(\frac{1 / B^{2}+1}{1 / B^{2}+4} \cdot 4\right)=\lim _{B \rightarrow 0} \frac{1}{4} \log \left(\frac{4+4 B^{2}}{1+4 B^{2}}\right)=\frac{1}{2} \log 4=\log 2
$$

Next, we show that $J_{21}(B)$ is increasing for $B$ close to 0 , which allows us to estimate $J_{2}(B)$ near $B=0$. Using symbolic computation, we get
$\partial_{B} J_{21}(B)=-\frac{-2 \log \left(\frac{1}{B^{2}}+\frac{2}{B}+2\right)+2 \log \left(\frac{(2 B+1)^{2}}{B^{2}}\right)-8 B+4 \tan ^{-1}(2 B)-4 \tan ^{-1}(2 B+1)+\pi}{8 B^{2}} \triangleq-\frac{S}{8 B^{2}}$.
We can rewrite the numerator $S$ as follows

$$
S=2 \log \left((1+2 B)^{2}\right)-2 \log \left(2 B^{2}+2 B+1\right)-8 B+4 \tan ^{-1}(2 B)-4 \tan ^{-1}(2 B+1)+\pi
$$

Clearly, we have $S(0)=0$. Next, we show that $S^{\prime}(B) \leq 0$ :

$$
\begin{aligned}
S^{\prime}(B) & =-8+\frac{8}{1+(2 B)^{2}}-\frac{8}{1+(1+2 B)^{2}}+\frac{8}{1+2 B}-\frac{2(4 B+2)}{1+2 B+2 B^{2}} \\
& \leq-\frac{8}{2+4 B+4 B^{2}}+\frac{8}{1+2 B}-\frac{2(4 B+2)}{1+2 B+2 B^{2}}=\frac{8}{1+2 B}-\frac{8 B+8}{1+2 B+2 B^{2}} \leq 0
\end{aligned}
$$

where we have used $1+2 B+2 B^{2}-(1+B)(1+2 B)=-B \leq 0$. Thus $S(B) \leq S(0)=0$ and $J_{21}(B)$ is increasing for $B>0$. Recall from Appendix B. 4 in Part II [2] that $B_{2}=\frac{\hat{x}_{1}}{h}$,
where $h$ is the mesh size and $\hat{x}_{1}$ is the rescaled $x$ in the computation domain. Using the above monotonicity property, for $\hat{x}_{1} \in\left[z^{l}, z^{u}\right], z^{u} \leq \frac{h}{2}$ and $\alpha \in(0,1)$, we have $B_{2} \leq \frac{z^{u}}{h} \leq \frac{1}{2}$ and

$$
J_{2}\left(B_{2}\right) \hat{x}_{1}^{1-\alpha}=J_{21}\left(\frac{\hat{x}_{1}}{h}\right) \hat{x}_{1}^{1-\alpha}-\log \left(\frac{\hat{x}_{1}}{h}\right) \hat{x}_{1}^{1-\alpha} \leq J_{21}\left(\frac{z^{u}}{h}\right)\left(z^{u}\right)^{1-\alpha}+I\left(\hat{x}_{1}\right), \quad I(t) \triangleq \log \left(\frac{h}{t}\right) t^{1-\alpha} .
$$

To obtain the piecewise bound for $I$, taking derivative, we yields

$$
\partial_{t} I=(1-\alpha) t^{-\alpha} \log \frac{h}{t}-t^{-\alpha}=t^{-\alpha}\left((1-\alpha) \log \frac{h}{t}-1\right), \quad \partial_{t} I>0, \text { for } t<h e^{-1 /(1-\alpha)}
$$

Therefore, for $z^{u} \leq \min \left(h e^{-1 /(1-\alpha)}, h / 2\right), I\left(\hat{x}_{1}\right)$ is increasing and thus

$$
I\left(\hat{x}_{1}\right) \leq I\left(z^{u}\right), \quad J_{2}\left(B_{2}\right) \hat{x}_{1}^{1-\alpha} \leq J_{2}\left(\frac{z^{u}}{h}\right)\left(z^{u}\right)^{1-\alpha}
$$

The quantities $J_{2}(B) \hat{x}_{1}^{1-\alpha}$ appear in our bound for $\frac{u(x)}{\left|x_{1}\right| \alpha}$. In our application, we choose $\alpha=1 / 2$. The above estimate allows us to control $\frac{u(x)}{\left|x_{1}\right|^{\alpha}}$ for small $x_{1}$.
Integral close to the singularity. In addition to $J_{i}(B)$, we need to estimate the integral

$$
\int_{a}^{b} \int_{c}^{d} K_{d u}(x, s) d s, \quad K_{d u}(x, s)=\frac{2\left(s_{1}+x_{1}\right) s_{2}}{|s|^{2}\left(\left(s_{1}+2 x_{1}\right)^{2}+s_{2}^{2}\right)} \mathbf{1}_{x_{1}+s_{1} \geq 0}, \quad Q=[a, b] \times[c, d]
$$

for $x_{1} \in\left[x_{1}^{l}, x_{1}^{u}\right] \subset \mathbb{R}^{+}$, in the region close to the singularity, e.g. $s \in\left[-k h_{0}, k h_{0}\right]^{2} \backslash\left[-h_{0}, h_{0}\right]^{2}$. See $I I_{1}, I I_{2}$ in the Appendix B. 4 in Part II [2]. Denote $h=x_{1}^{u}-x_{1}^{l}$. Without loss of generality, we consider $c, d \geq 0, s_{1} \in[a, b], s_{2} \in[c, d]$. For $s_{1}+x_{1} \geq 0$,
$K_{d u} \geq 0, \quad\left|s_{1}+2 x_{1}\right|^{2} \geq s_{1}^{2}, \quad\left|s_{1}+2 x_{1}\right|^{2}-\left|s_{1}+2 x_{1}^{l}\right|^{2}=\left(x_{1}-x_{1}^{l}\right)\left(2 s_{1}+2 x_{1}+2 x_{1}^{l}\right) \geq 0, x_{1} \leq x_{1}^{l}+h$.
Estimate I: $a+x_{1}^{l} \geq 0$. In this cas, we have $s_{1}+x_{1} \geq a+x_{1}^{l} \geq 0$ uniformly for $s_{1} \in[a, b]$ and $x \in\left[x_{1}^{l}, x_{1}^{u}\right]$. Thus, the indicator function is 1 in $Q$, and we yield
$0 \leq K_{d u} \mathbf{1}_{x_{1}+s_{1} \geq 0} \leq \frac{2\left(s_{1}+x_{1}^{l}+h\right) s_{2}}{|s|^{2}\left(\left(s_{1}+2 x_{1}^{l}\right)^{2}+s_{2}^{2}\right)}=\frac{1}{2 x_{1}^{l}}\left(\frac{s_{2}}{|s|^{2}}-\frac{s_{2}}{\left(s_{1}+2 x_{1}^{l}\right)^{2}+s_{2}^{2}}\right)+\frac{2 h s_{2}}{|s|^{2}\left(\left(s_{1}+2 x_{1}^{l}\right)^{2}+s_{2}^{2}\right)} \triangleq I_{1}+I_{2}$.
For $I_{1}$, if $x_{1}^{l}>0$, we use the analytic formula (5.4) to evaluate the integral. For $I_{2}$, it is much smaller than the main term. Since $\left(s_{1}+2 x_{1}^{l}\right)^{2}+s_{2}^{2}$ is regular and $a+x_{1}^{l} \geq 0$, we bound it as follows

$$
\begin{align*}
& \int_{Q} I_{2} d s \leq \max _{Q} \frac{2 h}{\left(s_{1}+2 x_{1}^{l}\right)^{2}+s_{2}^{2}} \int_{Q} \frac{s_{2}}{|s|^{2}} d s  \tag{6.13}\\
& s_{1}+2 x_{1}^{l} \geq \max \left(a+2 x_{1}^{l}, x_{1}^{l}, 0\right) \triangleq d i s^{l}, \quad s_{2}^{2} \geq \min \left(c^{2}, d^{2}\right), c d \geq 0
\end{align*}
$$

and evaluate the integral using (5.4). If $x_{1}^{l}=0, I_{1}, I_{2}$ reduce to $I_{1}=\frac{2 s_{1} s_{2}}{|s|^{4}}=2 K_{1}(s), I_{2}=\frac{2 h s_{2}}{|s|^{4}}$. We evaluate their integrals using the analytic integral formula (5.3) and

$$
\int \frac{s_{2}}{|s|^{4}} d s=\frac{1}{2} s_{2}^{-1} \arctan \frac{s_{2}}{s_{1}}+C
$$

Note that the integrand is singular near 0 when $x_{1}^{l}=0$. We only apply it to the region away from the origin $s=0$.

If $a+x_{1}^{l} \leq 0$, since in the support of the integrand, we have

$$
0 \leq s_{1}+x_{1} \leq s_{1}+x_{1}-a-x_{1}^{l} \leq b-a+h, \quad s_{1}+2 x_{1} \geq \max \left(x_{1}^{l}, a+2 x_{1}^{l}, 0\right)=d i s^{l}
$$

We bound the integrand as follows

$$
0 \leq K_{d u} \mathbf{1}_{x_{1}+s_{1} \geq 0} \leq \frac{2(b-a+h) s_{2}}{|s|^{2}\left(\left(s_{1}+2 x_{1}\right)^{2}\right)+s_{2}^{2}} \mathbf{1}_{x_{1}+s_{1} \geq 0} \leq \frac{2(b-a+h) s_{2}}{|s|^{2}\left(\left(d i s^{l}\right)^{2}+s_{2}^{2}\right.}
$$

and then estimate the integral following (6.13).

Estimate II. If $a+x_{1}^{u} \geq 0$, using $x_{1}+s_{1} \leq s_{1}+x_{1}^{u}$, we have another estimate for $K_{d u}$

$$
0 \leq x_{1} K_{d u} \mathbf{1}_{x_{1}+s_{1} \geq 0}=\frac{1}{2}\left(\frac{s_{2}}{|s|^{2}}-\frac{s_{2}}{\left(s_{1}+2 x_{1}\right)^{2}+s_{2}^{2}}\right) \mathbf{1}_{x_{1}+s_{1} \geq 0} \leq \frac{1}{2}\left(\frac{s_{2}}{|s|^{2}}-\frac{s_{2}}{\left(s_{1}+2 x_{1}^{u}\right)^{2}+s_{2}^{2}}\right)
$$

Since $a+x_{1}^{u} \geq 0$, the right hand side is nonnegative for $s_{1} \in[a, b]$. We further use (5.4) to evaluate the integral of the upper bound.

If $a+x_{1}^{u} \leq 0$, in the support of the integrand, we yield
$0 \leq s_{1}+x_{1} \leq s_{1}-a+\left(x_{1}-x_{1}^{u}\right) \leq b-a, \quad 0 \leq x_{1} K_{d u} \mathbf{1}_{x_{1}+s_{1} \geq 0} \leq \frac{2(b-a) x_{1}^{u} s_{2}}{|s|^{2}\left(\left(s_{1}+2 x_{1}\right)^{2}+s_{2}^{2}\right)} \mathbf{1}_{x_{1}+s_{1} \geq 0}$.
Then we use $s_{1}+2 x_{1} \geq \max \left(a+2 x_{1}^{l}, x_{1}^{l}, 0\right)$ and follow (6.13) to estimate the integral. We also have a simple bound using $x_{1} K_{d u} \mathbf{1}_{x_{1}+s_{1}} \leq \frac{s_{2}}{2|s|^{2}}$ and then apply the integral formula (5.4).

We apply the above estimate to obtain sharp estimates of the integral of $u$ (not divided by $\frac{1}{x_{1}}$ ). Dividing both sides by $\frac{1}{x_{1}}$ and using $\frac{1}{x_{1}} \leq \frac{1}{x_{1}^{\tau}}$, we yield another estimate for the integral of $K_{d u}$. This estimate is better if $x_{1}^{u} / x_{1}^{l}$ is close to 1 .

We remark that if $b+x_{1}^{u} \leq 0$, since $s_{1}+x_{1} \leq b+x_{1}^{u} \leq 0$, the integral is 0 .
6.3. Estimate using other norms. The estimate using the weights $\left\|\omega \varphi_{g, 1}\right\|_{\infty}$ is similar. From (6.1), we have $\left\|\omega \varphi_{g, 1}\right\|_{\infty} \leq \mu_{g, 1}^{-1} E_{4}, E_{1} \leq E_{4}$. Using the norm $\left\|\omega \varphi_{g, 1}\right\|_{\infty},\left[\omega \psi_{1}\right]_{C_{x_{i}}^{1 / 2}}$, we develop another estimate for $\mathbf{u}_{A},(\nabla \mathbf{u})_{A}$. For the singular part $S$ (6.5), using $\mu_{g, 1} \leq 0.02$, we estimate

$$
\begin{align*}
|S| & \leq C_{1}\left\|\omega \varphi_{g, 1}\right\|_{L^{\infty}}+C_{2}[\omega \psi]_{C_{x}^{1 / 2}}+C_{3}[\omega \psi]_{C_{y}^{1 / 2}} \leq\left(C_{1} \mu_{g, 1}^{-1}+\gamma_{1} C_{2}+C_{3} \gamma_{2}\right) E_{4}  \tag{6.14}\\
& \leq \mu_{g, 1}^{-1}\left(C_{1}+\mu_{g, 1} \gamma_{1} C_{2}+\mu_{g, 1} \gamma_{2} C_{3}\right) E_{4} \leq \mu_{g, 1}^{-1}\left(C_{1}+0.02 \gamma_{1} C_{2}+0.02 \gamma_{2} C_{3}\right) E_{4}
\end{align*}
$$

similar to (6.7), with constant $C_{1}+\left(0.02 \gamma_{1}\right) C_{2}+\left(0.02 \gamma_{2}\right) C_{3} \gamma_{2}$ as small as possible. For this purpose, we can apply the estimates in the previous sections with $\left(\gamma_{1}, \gamma_{2}\right)$ replaced by $002\left(\gamma_{1}, \gamma_{2}\right)$. Note that this additional estimate for $\mathbf{u}_{A},(\nabla \mathbf{u})_{A}$ is used to close the nonlinear estimates. The overestimate $\mu_{g, 1} \leq 0.02$ only slightly increases the constant in the nonlinear estimates.

To estimate the nonlocal error $\mathbf{u}(\varepsilon), \varepsilon=\bar{\omega}-(-\Delta) \bar{\phi}^{N}, \varepsilon=\hat{\omega}-(-\Delta) \hat{\phi}^{N}$ for the approximate steady state or $\hat{W}_{2}$ (see Section 5.8 in [3]), we develop similar estimates for $\mathbf{u}_{A}(\varepsilon),(\nabla \mathbf{u})_{A}$ using the norm $\left\|\varepsilon \varphi_{\text {elli }}\right\|_{\infty},\left[\varepsilon \psi_{1}\right]_{C_{x_{i}}^{1 / 2}}$ (6.3), (6.4). For the singular part $S$, we estimate it using the norm localized to $D$ containing $x+Q_{\delta}$ in (6.5)

$$
\begin{align*}
|S(\varepsilon)| & \leq C_{1}(x)\left\|\varepsilon \varphi_{g, 1}\right\|_{L^{\infty}(D)}+C_{2}(x)[\varepsilon \psi]_{C_{x}^{1 / 2}(D)}+C_{3}(x)[\varepsilon \psi]_{C_{y}^{1 / 2}(D)} \\
& \leq \bar{B}_{0}\left(C_{1}+\bar{B}_{0}^{-1} \bar{B}_{1} C_{2}+\bar{B}_{0}^{-1} \bar{B}_{2} C_{3}\right), \quad\left\|\varepsilon \varphi_{g, 1}\right\|_{L^{\infty}(D)} \leq \bar{B}_{0},\left[\varepsilon \psi_{1}\right]_{C_{x_{i}}^{1 / 2}(D)} \leq \bar{B}_{i} \tag{6.15}
\end{align*}
$$

with constant $C_{1}(x)+\bar{B}_{0}^{-1} \bar{B}_{1} C_{2}+\bar{B}_{0}^{-1} \bar{B}_{2} C_{3}(x)$ as small as possible. Since $\varepsilon$ depends on the numerical solution, e.g. $\bar{\omega}, \bar{\phi}^{N}$, locally, we can bound $\bar{B}_{i}$ directly. To get a sharp estimate, we can apply the estimate in the previous sections with $\left(\gamma_{1}, \gamma_{2}\right)$ replaced by $\left(\bar{B}_{0}^{-1} \bar{B}_{1}, \bar{B}_{0}^{-1} \bar{B}_{2}\right)$. See Section 4.7 in Part II [2] for more discussions of the localized estimate.

We can generalize the above estimates of combining different norms in the estimates of $\mathbf{u}_{A},(\nabla \mathbf{u})_{A}$ for $x$ very small or very large. See Section 7.5,
6.4. Estimate of some integrals. We discuss the refined estimate of $u_{x}(0)(\varepsilon)$ and $K_{00}(\varepsilon)$ for the error $\varepsilon$ of solving the Poisson equation. The refined estimates of these terms are important for us to show that the error is small.
6.4.1. Estimate of $K_{00}$. In the estimate of the integrals, near the origin, we use the triangle inequality and we estimate the approximation term

$$
I=p_{\lambda}(\hat{x}) \lambda^{-2} C(\lambda \hat{x}) \int_{|\hat{y}|_{\infty} \leq k h} K_{00}(\hat{y}) W_{\lambda}(\hat{y}) d \hat{y}=p(\lambda x) C(x) \int_{|y| \leq \lambda k h} K_{00}(y) W(y) d y
$$

for a fixed $k$, e.g. $k=12$, separately since the kernel given below is singular with order $|y|^{-4}$ near $y=0$

$$
K_{00}(y)=\frac{24 y_{1} y_{2}\left(y_{1}^{2}-y_{2}^{2}\right)}{|y|^{8}}=\partial_{1}^{3} \partial_{2} f(y), \quad f(y)=-\log |y|
$$

We need to estimate the above term for finitely many $\lambda \leq 10$ and $\lambda \leq \lambda_{*}$ uniformly for $x$ near 0 . In the energy estimate for linear stability analysis, we bound it using $\|\omega \varphi\|_{\infty}$. In the error estimate, such an estimate is not sufficient. Since we can evaluate $\omega(y)$, we exploit the cancellation in the integral. We partition the domain using fine mesh. Denote

$$
\begin{equation*}
0=y_{1}<y_{2}<. .<y_{N}, \quad y_{i, 1 / 2}=\left(y_{i}+y_{i+1}\right) / 2, \quad Q_{i j}=\left[y_{i}, y_{i+1}\right] \times\left[y_{j}, y_{j+1}\right] . \tag{6.16}
\end{equation*}
$$

We evaluate the integral in $Q$ using Simpson's rule
$\int_{\left[a_{1}, a_{3}\right] \times\left[b_{1}, b_{3}\right]} g(y) d y=\sum_{i, j \leq 3} c_{i} c_{j} f\left(a_{i}, b_{j}\right)+$ err,$\quad|e r r| \leq \frac{1}{2880}\left(h_{1}^{4}\left\|\partial_{x}^{4} g\right\|_{L^{\infty}(Q)}+h_{2}^{4}\left\|\partial_{y}^{4} g\right\|_{L^{\infty}(Q)}\right)$,
$h_{1}=a_{3}-a_{1}, h_{2}=b_{3}-b_{1} \quad Q=\left[a_{1}, a_{3}\right] \times\left[b_{1}, b_{3}\right], \quad a_{2}=\frac{a_{1}+a_{3}}{2}, \quad b_{2}=\frac{b_{1}+b_{3}}{2}, c=\left[\frac{1}{6}, \frac{4}{6}, \frac{1}{6}\right]$.
We obtain the piecewise derivative bound of $K_{00}(y) W$ using the bound of $W$ established by the method in Appendix C in Part II [2] and (5.2) for $K_{00}$. The above error estimate is obtained by applying the error estimate of Simpson's rule first in $x$ and then in $y$.

For the integral very close to origin, e.g. in $Q=[0, D]^{2}=\left[0, y_{m}\right]^{2}$, the above method fails since the kernel is singular. We defer the estimate below. With these estimates, for a fixed $\lambda$, we pick $l$ such that $y_{l} \leq r<y_{l+1}, r=\lambda k h$ and decompose the integral into three regions
$\int_{[0, r]^{2}} g(y) d y=\left(\int_{\left[0, y_{m}\right]^{2}}+\int_{[0, y]^{2} \backslash\left[0, y_{m}\right]^{2}}+\int_{\left.[0, r]^{2} \backslash[0, y]\right]^{2}}\right) g(y) d y \triangleq S_{1}+S_{2}+S_{3},\left|S_{3}\right| \leq \int_{\left[0, y_{l+1}\right]^{2} \backslash[0, y]^{2}}|g(y)| d y$.
For $S_{1}$, we apply the estimate near 0 discussed below. For $S_{2}$, we use the above Simpson's rule. Since $[0, r]^{2} \backslash\left[0, y_{l}\right]^{2}$ is small, we treat $S_{3}$ as an error and use the piecewise bounds for $g(y)=\omega(y) K_{00}(y)$.

The above method also provide the piecewise bound of the integral for $\lambda k h \in\left[y_{l}, y_{l+1}\right]$. This allows us to obtain the uniform bound for $\lambda \in\left[y_{m} /(k h), \lambda_{*}\right]$ by covering the interval. To obtain the uniform estimate for all small $\lambda \leq \lambda_{*}$, we further apply the method discussed below to estimate the case of $\lambda \leq y_{m} /(k h)$ uniformly.

Estimate near $y=0$. Consider $Q=[0, D]^{2}$. In our case, $\omega$ is odd and satisfies $\omega=O\left(|x|^{3}\right)$ near $x=0$ and we have piecewise $C^{3}$ bounds for $\omega$. Using integration by parts, we get

$$
\begin{aligned}
& \int_{Q} \omega(y) f_{x x x y}(y) d y=-\int_{Q} \omega_{x}(y) f_{x x y} d y+\int_{0}^{D} \omega(D, y) f_{x x y}(D, y) d y \\
= & \int_{Q} \omega_{x x}(D, y) f_{x y}(D, y) d y-\int_{0}^{D} \omega_{x}(y) f_{x y} d y+\int_{0}^{D} \omega(D, y) f_{x x y}(D, y) d y \triangleq I+I I+I I I .
\end{aligned}
$$

The boundary term vanishes on $x=0$ since $\omega f_{x x y}, \omega_{x} f_{x y}$ is odd. Denote $M_{i j}=\left\|\partial_{x}^{i} \partial_{y}^{j} \omega\right\|_{L^{\infty}(Q)}$. Since $\omega$ is odd, for $y \in Q$, using $\omega=O\left(|x|^{3}\right)$, Taylor expansion

$$
\omega=\int_{0}^{y_{1}} \omega_{x}\left(z, y_{2}\right) d z=\omega_{x}\left(0, y_{2}\right) y_{1}+\int_{0}^{y_{1}} \omega_{x x x}\left(z, y_{2}\right) \frac{\left(y_{1}-z\right)^{2}}{2} d z, \omega_{x}\left(0, y_{2}\right)=\int_{0}^{y_{2}} \omega_{x y y}(0, z)\left(y_{2}-z\right) d z,
$$

taking derivatives on the above expansions, and using $\int_{0}^{y} z^{k} d z=\frac{y^{k+1}}{k+1}$, we get

$$
|\omega(y)| \leq \frac{y_{1}^{3}}{6} M_{30}+\frac{y_{1} y_{2}^{2}}{2} M_{12},\left|\omega_{x}\right| \leq \frac{y_{1}^{2}}{2} M_{30}+\frac{y_{2}^{2}}{2} M_{12},\left|\omega_{x x}\right| \leq y_{1} M_{30}, \quad M_{i j} \triangleq\left\|\partial_{x}^{i} \partial_{y}^{\omega}\right\|_{L^{\infty}(Q)} .
$$

Since

$$
f_{x y}=\frac{2 y_{1} y_{2}}{|y|^{4}}, \quad f_{x x y}=\frac{2 y_{2}\left(-3 y_{1}^{2}+y_{2}^{2}\right)}{|y|^{6}},
$$

$f_{x y}, f_{x x y}$ have fixed signs in $[0, D]^{2},\{D\} \times[0, D]$, respectively.
For $I$, since $\omega_{x x}$ is odd, using the scaling symmetry of the kernel, we get
$|I| \leq M_{30} \int_{Q}\left|f_{x y} y_{1}\right| d y=M_{30} \int_{Q} \frac{2 y_{1}^{2} y_{2}}{|y|^{4}} d y=M_{30} D \int_{[0,1]^{2}} \frac{2 y_{1}^{2} y_{2}}{|y|^{4}} d y=\left.M_{30} D \cdot y_{2} \arctan \frac{y_{1}}{y_{2}}\right|_{[0,1]^{2}}=M_{30} \frac{D \pi}{4}$,
where we use $\partial_{12}\left(y_{2} \arctan \frac{y_{1}}{y_{2}}\right)=\partial_{2}\left(y_{2} \frac{1 / y_{2}}{\left(y_{1} / y_{2}\right)^{2}+1}\right)=\partial_{2} \frac{y_{2}^{2}}{y_{1}^{2}+y_{2}^{2}}=-\partial_{2} \frac{y_{1}^{2}}{y_{1}^{2}+y_{2}^{2}}=\frac{2 y_{1}^{2} y_{2}}{|y|^{4}}$.

For $I I$ and $I I I$, we use the expansion at $(D, y)$. We get

$$
|I I| \leq \frac{M_{30} D^{2}}{2}\left|\int_{0}^{D} f_{x y}(D, y) d y\right|+\frac{M_{12}}{2}\left|\int_{0}^{D} f_{x y}(D, y) y^{2} d y\right| \triangleq \frac{M_{30}}{2} I I_{1}+\frac{M_{12}}{2} I I_{2}
$$

Using the scaling symmetry, we get

$$
\begin{aligned}
& I I_{1}=D\left|\int_{0}^{1} f_{x y}(1, y) d y\right|=D\left|f_{x}(1, y)\right|_{0}^{1}|=D| f_{x}(1,1)-f_{x}(1,0) \mid \\
& I I_{2}=D\left|\int_{0}^{1} \frac{2 y^{3}}{\left(1+y^{2}\right)^{2}} d y\right|=\left.D\left(\frac{1}{1+y^{2}}+\log \left(1+y^{2}\right)\right)\right|_{0} ^{1}=D(\log 2-1 / 2)
\end{aligned}
$$

For $I I I$, since $f_{x x y}(D, y)$ has a fixed sign in $[0, D]$, we get

$$
|I I I| \leq \frac{M_{30} D^{3}}{6}\left|\int_{0}^{D} f_{x x y}(D, y) d y\right|+\frac{M_{12}}{2} D\left|\int_{0}^{D} f_{x x y}(D, y) y^{2} d y\right|=\frac{M_{30}}{6} I I I_{1}+\frac{M_{12}}{2} I I I_{2}
$$

Since $f$ is harmonic, $f_{x x y}=-f_{y y y}$, using the scaling symmetry and integration by parts, we get

$$
\begin{aligned}
& I I I_{1}=D\left|\int_{0}^{1} f_{x x y}(1, y) d y\right|=D\left|f_{x x}(1,1)-f_{x x}(1,0)\right| \\
& I I I_{2}=D\left|\int_{0}^{1} f_{x x y}(1, y) y^{2} d y\right|=D\left|\int_{0}^{1} f_{y y y}(1, y) y^{2} d y\right|=\left.D| |\left(f_{y y}(1, y) y^{2}-2 f_{y}(1, y) y+2 f\right)\right|_{0} ^{1} \mid
\end{aligned}
$$

All the above terms are linear in $D$ since $y_{1}^{i} y_{2}^{j} K_{00}(y) d y=D \hat{y}_{1}^{i} \hat{y}_{2}^{j} K_{00}(\hat{y}) d \hat{y}$ for $y=D \hat{y}, i+j=3$.
6.4.2. Estimate of $c_{\omega}(\bar{\omega})$. Recall from Appendix C in Part II [2] that we represent $\bar{\omega}$ using piecewise polynomials $\bar{\omega}_{2}$ and the semi-analytic part $\bar{\omega}_{1}=\chi(r) r^{-\alpha} g(\beta)$. We discuss the computation of

$$
c_{\omega}(f)=u_{x}(f)(0)=-\frac{4}{\pi} \int_{\mathbb{R}_{2}^{++}} K(y) f(y) d y, \quad K(y)=\frac{y_{1} y_{2}}{|y|^{4}}
$$

for $f$ being a single-level B-spline and $f=\bar{\omega}_{1}$. If $f$ is a multi-level B-spline (see Section 7 in [3]), i.e. $f=\sum_{i \leq n} f_{i}$ with $f_{i}$ being a single level B-spline, we estimate $c_{\omega}\left(f_{i}\right)$ and then use linearity to estimate $c_{\omega}(f)$. For $f$ being B-spline with supporting points $y_{0}<y_{1}<\ldots,<y_{n}$, its support is contained in $D=\left[0, y_{n+7}\right]^{2}$. We partition the domain into

$$
D_{i}=\left[0, y_{k_{i}}\right]^{2}, \quad i=1,2, . ., l, \quad y_{k_{1}}<y_{k_{2}}<. .<y_{k_{l}}=y_{n+7}
$$

For each $i$, we refine each mesh $Q=\left[y_{i_{1}}, y_{i_{1}+1}\right] \times\left[y_{j_{1}}, y_{j_{1}+1}\right]$ in $D_{i+1} \backslash D_{i} 2 n_{i}$ times

$$
Q=\cup_{k, l \leq n_{i}} Q_{k l}, Q_{k l}=\left[y_{i_{1}}+(k-1) h_{1}, y_{i_{1}}+k h_{1}\right] \times\left[y_{j_{1}}+(j-1) h_{2}, y_{j_{1}}+j h_{2}\right], h_{1}=\frac{y_{i_{1}+1}-y_{i_{1}}}{n_{i}}, h_{2}=\frac{y_{j_{1}+1}-y_{j_{1}}}{n_{i}}
$$

and then apply Simpson's rule (6.16) to estimate the integral $\int_{Q_{k l}} f(y) K(y) d y$ by evaluating $f K$ on $y_{i_{1}}+\frac{p h_{1}}{2}, y_{j_{1}}+\frac{q h_{2}}{2}, 0 \leq p, q \leq 2 n_{i}$ and estimating the error. In $Q$, using (6.16), the error is given by

$$
\frac{1}{2880} \sum_{k, l \leq n_{i}} \int_{Q_{k l}} h_{1}^{4}\left\|\partial_{x}^{4}(f K)\right\|_{L^{\infty}}+h_{2}^{4}\left\|\partial_{y}^{4}(f K)\right\|_{L^{\infty}}
$$

Using $K(y)=-\frac{1}{2} \partial_{1} \partial_{2} \log (|y|)$ and $\left|\partial_{x}^{i} \partial_{y}^{j} K(y)\right| \leq \frac{(i+j+1)!}{2}|y|^{-i-j-2}$ from Lemma 5.2, we get

$$
\sum_{k, l} \int_{Q_{k l}}\left\|\partial_{x}^{4}(f K)\right\|_{L^{\infty}} \leq \sum_{0 \leq m \leq 4}\binom{4}{m}\left\|\partial_{x}^{4-m} f\right\|_{L^{\infty}(Q)} h_{1} h_{2} \frac{(m+1)!}{2} \sum_{k l} \max _{y \in Q_{k l}}|y|^{-m-2}
$$

The bound for $\partial_{y}^{4}(f K)$ is similar.

Near 0. Since the integrand is singular, near 0, we use Taylor expansion. We choose $D_{0} \subset$ $\left[0, y_{1}\right]^{2}$. For $f\left(x_{1}, x_{2}\right)$ odd in $x_{1}$, we get
$f(x)=\partial_{1} f(0) x_{1}+\partial_{12} f(0) x_{1} x_{2}+\frac{\partial_{111} f(0) x_{1}^{3}}{6}+\frac{\partial_{122} f(0) x_{1} x_{2}^{2}}{2}+\varepsilon,|\varepsilon| \leq \frac{1}{24} \sum_{0 \leq i \leq 4}\binom{4}{i} x_{1}^{i} x_{2}^{4-i}\left\|\partial_{x}^{i} \partial_{y}^{4-i} f\right\|_{L^{\infty}\left(D_{0}\right)}$.
We integrate $K(y) y_{1}^{i} y_{2}^{j}$ analytically using symbolic computation and then obtain the error estimate and the approximation of the integral by evaluating $\partial_{1}^{i} \partial_{2}^{j} f(0)$.
Integral for $\bar{\omega}_{1}$. For $f=\bar{\omega}_{1}=\chi(r) r^{-\alpha-1} g(\beta), \chi(r)$ is supported in $[A, \infty)$ and $\chi(r)=1$ for $r \geq B, A=5, B=2 \cdot 10^{6}$. Using $K(y)=\frac{\sin (2 \beta)}{2 r^{2}}$ and a direct calculation yields

$$
\begin{aligned}
-\frac{4}{\pi} \int_{\mathbb{R}_{++}^{2}} K(y) \bar{\omega}_{1} d y & =-\frac{2}{\pi} \int_{A}^{\infty} \chi(r) r^{-\alpha-1} d r \int_{0}^{\pi / 2} g(\beta) \sin (2 \beta) d \beta \\
& =-\frac{2}{\pi}\left(\int_{A}^{B} \chi(r) r^{-\alpha-1} d r+\alpha^{-1} B^{-\alpha}\right) \int_{0}^{\pi / 2} g(\beta) \sin (2 \beta) d \beta
\end{aligned}
$$

We apply 1D Simpson's rule to estimate two 1D integrals.

## 7. Estimate of the velocity near 0 and in the far-field

For $x$ very close to the origin or very large, we cannot rescale the integral by choosing finitely many rescaling factors $\lambda_{i}$. See Section 4.5 in Part II [2] for more discussions. Instead, we choose $\lambda=\frac{\max \left(x_{1}, x_{2}\right)}{x_{c}}$ and estimate the rescaled integral with a $-d$-homogeneous kernel $K$

$$
\begin{equation*}
p(x) \int K(x-y) W(y) d y=p_{\lambda}(x) \int K(\hat{x}-\hat{y}) \lambda^{2-d} W_{\lambda}(\hat{y}) d y, \quad f_{\lambda}(x) \triangleq f(\lambda x) \tag{7.1}
\end{equation*}
$$

uniformly for all small $\lambda \ll 1$ or large $\lambda \gg 1$. The rescaled singularity $\hat{x}=x / \lambda$ satisfies $\max _{i} \hat{x}_{i}=x_{c}$ and is in the bulk of our computation domain. The method is essentially the same as those described in Sections 4.1, 4.2 in Part II [2]. We have the following scaling relation

$$
\begin{align*}
& \left\|\omega_{\lambda} \varphi_{\lambda}\right\|_{\infty}=\|\omega \varphi\|_{\infty}, \quad\left[\omega_{\lambda} \psi_{\lambda}\right]_{C_{x_{i}}^{1 / 2}}=\lambda^{\frac{1}{2}}[\omega \psi]_{C_{x_{i}}^{1 / 2}} \\
& \partial_{x_{i}} f(x)=\frac{d \hat{x}_{i}}{d x_{i}} \partial_{\hat{x}_{i}} f_{\lambda}(\hat{x})=\frac{1}{\lambda} \partial_{\hat{x}_{i}} f_{\lambda}(\hat{x}), \quad \frac{|f(x)-f(z)|}{|x-z|^{1 / 2}}=\lambda^{-\frac{1}{2}} \frac{\left|f_{\lambda}(\hat{x})-f_{\lambda}(\hat{z})\right|}{|\hat{x}-\hat{z}|^{1 / 2}} . \tag{7.2}
\end{align*}
$$

Denote by $y_{i}$ the adaptive mesh for computing the approximate steady state designed in Appendix C. 1 in Part II [2]

$$
\begin{equation*}
y_{1}<y_{2}<. .<y_{N}, \quad Q_{i j}=\left[y_{i}, y_{i+1}\right] \times\left[y_{j}, y_{j+1}\right] \tag{7.3}
\end{equation*}
$$

We have $y_{N}>10^{15}$. In Part II [2], we introduce the following notations: $h_{x}=h / 2$,

$$
\begin{equation*}
|x|_{\infty}=\max \left(x_{1}, x_{2}\right), \quad x \in B_{i_{1}, j_{1}}\left(h_{x}\right) \subset B_{i j}(h), \quad B_{l m}(r)=[l r,(l+1) r] \times[m r,(m+1) r] \tag{7.4}
\end{equation*}
$$

and singular regions near $x$

$$
\begin{align*}
R(x, k) & =[(i-k) h,(i+1+k) h] \times[(j-k) h,(j+k+1] h]  \tag{7.5}\\
R_{s}(k) & =\left[x_{1}-k h, x_{1}+k h\right] \times\left[x_{2}-k h, x_{2}+k h\right] .
\end{align*}
$$

We introduce the following domain and points

$$
\begin{align*}
& M_{l i_{1} j_{1}}=\lambda_{l} B_{i_{1} j_{1}}\left(h_{x}\right), \quad \Gamma(a) \triangleq\left\{\hat{x}: \max _{i} \hat{x}_{i}=a\right\}=\cup_{i h_{x} \leq a} \Gamma_{1, i}(a) \cup \Gamma_{2, i}(a) \\
& \Gamma_{1, i}(a) \triangleq\{a\} \times\left[(i-1) h_{x}, i h_{x}\right], \Gamma_{2, i}(a) \triangleq\left[(i-1) h_{x}, i h_{x}\right] \times\{a\}  \tag{7.6}\\
& x_{c}=128 h_{x}, x_{c 2}=256 h_{x}, x_{c 3}=3 x_{c 2}, m_{3}=2 x_{c 2} h_{x}^{-1}=512
\end{align*}
$$

Domain $M_{l, i_{1}, j_{1}}$ is a dyadic mesh, and $x_{c}, x_{c i}$ can be viewed as the reference centers. We choose $a$ to be a multiple of $h_{x}$, and then $\Gamma(a)$ consists of two intervals. In the weighted $L^{\infty}$ estimate, we rescale $x=\lambda \hat{x}$ such that $\hat{x} \in \Gamma\left(x_{c}\right)$. In the Hölder estimate, we rescale $(x, z)$ with $|x| \leq|z|$ such that $x=\lambda \hat{x}$ with $\hat{x} \in \Gamma\left(2 x_{c}\right)$.

We use the $L^{\infty}$ norm $\|\omega \varphi\|_{\infty}$ and the Hölder semi-norm $[\omega \psi]_{C_{x_{i}}^{1 / 2}}$ to control the piecewise $L^{\infty}$ norm of $\mathbf{u}, \nabla \mathbf{u}$ and the Hölder semi-norm of the weighted quantities $\psi_{u} \mathbf{u}, \psi \nabla \mathbf{u}$. Denote by $a_{1}, a_{n}$ the power of $\varphi(x)$ near 0 and $\infty, b_{1}, b_{n}$ the power for $\psi$, and $c_{1}, c_{n}$ for $\psi_{u}$. From the definitions of these weights (6.3), (6.4) we have

$$
\begin{align*}
& \varphi(x) \geq q_{1} r^{a_{1}}(\cos \beta)^{-\frac{1}{2}}+q_{n} r^{a_{n}}, \quad\left(a_{1}, a_{n}\right)=(-2.9,-1 / 6),(-2.9,1 / 16),(-5 / 2,-1 / 6), \\
& \psi(x) \sim B_{1} r^{b_{1}}, r \ll 1, \quad \psi(x) \sim B_{n} r^{b_{n}}, r \gg 1, \quad\left(b_{1}, b_{n}\right)=(-2,-1 / 6)  \tag{7.7}\\
& \psi_{u} \sim C_{1} r^{c_{1}}, r \ll 1, \quad \psi_{u} \sim C_{n} r^{c_{n}}, r \gg 1, \quad\left(c_{1}, c_{n}\right)=(-5 / 2,-7 / 6)
\end{align*}
$$

for some constants $q_{i}, B_{j}, C_{j}$, where $r=|x|, \beta=\arctan \left(\frac{x_{2}}{x_{1}}\right)$. Here $\psi=\psi_{1}$ (6.3) and we drop 1 to simplify the notation. We use the norm $\|\omega \varphi\|_{\infty}$ with three different weights $\varphi$ in (6.4) and the above power $\left(a_{1}, a_{n}\right)$. We have

$$
\begin{equation*}
\varphi_{\lambda}(x) \geq \lambda^{a_{\alpha}} \varphi_{\infty, \alpha}(x), \alpha=1, n \tag{7.8}
\end{equation*}
$$

We choose $c_{n}=b_{n}-1$ to capture the fact that $\mathbf{u}$ decays one order slower than $\nabla \mathbf{u}$.
7.1. $L^{\infty}$ estimate. Denote by $\phi$ the stream function. Recall from Section 4.3 [3] that for $\lambda$ very small, i.e. $x$ near 0 , the velocity with approximation terms is given by

$$
\begin{equation*}
\left(\partial_{x}^{i} \partial_{y}^{j} \phi\right)_{A} \triangleq \partial_{x}^{i} \partial_{y}^{j}\left(\phi-\phi_{x y}(0) x y-\frac{1}{6} \phi_{x x x}(0)\left(x^{3} y-x y^{3}\right)\right) \tag{7.9}
\end{equation*}
$$

for $i+j=1$, 2. For example, we have $u_{x}=-\partial_{x y} \phi, v_{x}=\partial_{x x} \phi$. See Section 4.3 in 3]. The vorticity $\omega$ vanishes like $O\left(|x|^{2+\alpha}\right)$ for some $\alpha>0$ near $x=0$, and $\phi, \phi_{x y}(0), \phi_{x x x y}(0)$ are given by
$\phi(x)=-\frac{1}{\pi} \int_{\mathbb{R}_{2}} G(x-y) W(y) d y, \phi_{x y}(0)=\frac{4}{\pi} \int_{\mathbb{R}_{2}^{++}} \frac{y_{1} y_{2}}{|y|^{4}}, \phi_{x x x y}(0)=\frac{2}{\pi} \int_{\mathbb{R}_{2}^{++}} \omega(y) \frac{24 y_{1} y_{2}\left(y_{1}^{2}-y_{2}^{2}\right)}{|y|^{8}} d y$,
where $G$ is the Green function (5.1), and $W$ is given in (5.2).
For $\lambda$ large, $x$ near $\infty$, we estimate

$$
\begin{equation*}
\left(\partial_{x}^{i} \partial_{y}^{j} \phi\right)_{A}=\partial_{x}^{i} \partial_{y}^{j}\left(\phi+L_{12}(0) x y\right), \quad L_{12}=-\frac{4}{\pi} \int_{y \in \mathbb{R}_{++}^{2} \backslash\left[0, R_{n}\right]^{2}} \frac{y_{1} y_{2}}{|y|^{4}} \omega(y) d y \tag{7.11}
\end{equation*}
$$

for some parameter $R_{n}$ (the largest threshold) given in Appendix C. 2 in [3], where $L_{12}(0)$ approximates $u_{x}(0)=-\phi_{x y}(0)$. Note that for $|x| \geq R_{n},\left(\partial_{x}^{i} \partial_{y}^{j} \phi\right)_{A}$ satisfies the differential relation

$$
\begin{equation*}
\partial_{x}^{i} \partial_{y}^{j}\left(\partial_{x}^{i_{2}} \partial_{y}^{j_{2}} \phi\right)_{A}=\left(\partial_{x}^{i+i_{2}} \partial_{y}^{j+j_{2}} \phi\right)_{A}, \quad 1 \leq i+j+i_{1}+j_{1} \leq 2 \tag{7.12}
\end{equation*}
$$

Using the scaling symmetry of the kernels, we can rewrite the above quantities as integrals of $\omega$ and in the form (7.1). We estimate piecewise bounds for $\nabla \mathbf{u}(\lambda \hat{x}), \mathbf{u}(\lambda \hat{x})$ for $\hat{x} \in \Gamma_{i, j}$ and $\lambda \leq \lambda_{*}$ or $\lambda \geq \lambda_{*}$ uniformly.
7.1.1. Near field, far field, and the nonsingular part. For the integral in these regions, the integrand is not singular, and we follow the method in Sections 4.1, 4.2 in Part II 22 to estimate them. For example, for a region $Q \subset \mathbb{R}_{2}^{++}$away from the singularity, using (7.8), we estimate the integral (7.1) with symmetrized kernel

$$
\begin{equation*}
K^{s y m}(x, y) \triangleq K(x-y)+K(x+y)-K\left(x_{1}-y_{1}, x_{2}+y_{2}\right)-K\left(x_{1}+y_{1}, x_{2}-y_{2}\right) \tag{7.13}
\end{equation*}
$$

and approximation terms $K=K^{\text {sym }}-K_{a p p}$ as follows
$\left|\int_{D} K(\hat{x}, \hat{y}) W_{\lambda}(\hat{y}) d \hat{y}\right| \leq\left\|W_{\lambda} \varphi_{\lambda}\right\|_{\infty} \int_{D}|K(\hat{x}, \hat{y})| \varphi_{\lambda}^{-1}(\hat{y}) d \hat{y} \leq \lambda^{-a_{\alpha}}| | \omega \varphi \|_{\infty} \int_{D}|K(\hat{x}, \hat{y})| \varphi_{\infty, \alpha}^{-1}(\hat{y}) d \hat{y}$, for $\alpha=1, n$. Using the decay estimates of the symmetrized integrals of (7.10) with approximations in Appendix B. 1 in Part II [2] and (7.7), one can show that $|K(x, y)| \varphi_{\infty, \alpha}^{-1}(y)$ is integrable away from the singularity. The last integral does not depend on $\lambda$ and can be estimated using the method in [3]. We track the integral and the power $\lambda^{-a_{\alpha}}$. Thus, the estimates of the integrals in these regions are essentially the same as those for finite rescaling factor $\lambda_{i}$ discussed
in Sections 4.1, 4.2 in Part II [2], except that we use the asymptotic behavior of the weights $\varphi_{\infty, 1}, \varphi_{\infty, n}$ instead of $\varphi$.

For large $\lambda$, we need to estimate (7.11). Since $\left[0, R_{n}\right]^{2}$ does not enjoy the scaling symmetry, $y=\lambda \hat{y} \in\left[0, R_{n}\right]^{2}$ if and only if $\hat{y} \in\left[0, R_{n} / \lambda\right]^{2}$, we decompose (7.11) as follows

$$
\begin{aligned}
L_{12} & =-\frac{4}{\pi} \int_{\mathbb{R}_{2}^{++} \backslash\left[0, R_{n} / \lambda\right]^{2}} \frac{\hat{y}_{1} \hat{y}_{2}}{|\hat{y}|^{4}} \omega_{\lambda}(\hat{y}) d \hat{y}=-\frac{4}{\pi} \int_{y \in \mathbb{R}_{2}^{++}} \mathbf{1}_{D_{1}^{c}}+\left(\mathbf{1}_{\left(\left[0, R_{n} / \lambda\right]^{2}\right)^{c}}(\hat{y})-\mathbf{1}_{D_{1}^{c}}(\hat{y})\right) K_{0}(\hat{y}) \omega_{\lambda}(\hat{y}) d \hat{y} \\
& \triangleq \hat{L}_{12}(\lambda)+I, \quad K_{0}(y)=\frac{4 y_{1} y_{2}}{|y|^{4}}, \quad D_{1}=\left[0, k_{1} h\right]^{2} .
\end{aligned}
$$

The domain of the rescaled integral in $L_{12}(\lambda)$ does not depend on $\lambda$, and we estimate the integrals for $\left.\partial_{x}^{i} \partial_{y}^{j} \phi-\partial_{x}^{i} \partial_{y}^{j}(x y) \hat{L}_{12}\right)(\lambda)$ using the above method. The second part $I$ is treated as an error term. Using (7.7), (7.8) and $\left|\omega_{\lambda}(\hat{y})\right| \leq \lambda^{-a_{n}} q_{n}^{-1}|\hat{y}|^{-a_{n}}\|\omega \varphi\|_{\infty}$, we have

Next, we estimate $|I I|$ uniformly for $\lambda \geq \lambda_{*}$. For $0<c<d$, since $\frac{y_{1} y_{2}}{|y|^{\beta}}$ is symmetric in $y_{1}, y_{2}$, we get

$$
\begin{aligned}
J(c, d) & =\int_{[0, d]^{2} \backslash[0, c]^{2}} \frac{4 y_{1} y_{2}}{|y|^{4+a_{n}}} d y=8 \int_{c}^{d} \int_{0}^{y_{1}} \frac{y_{1} y_{2}}{|y|^{4+a_{n}}} d y=\frac{8}{2+a_{n}} \int_{c}^{d} \frac{-y_{1}}{\left.|y|^{2+a_{n}}\right|_{0} ^{y_{1}} d y_{1}} \\
& =\frac{8\left(1-2^{-\left(2+a_{n}\right) / 2}\right)}{2+a_{n}} \int_{c}^{d} y_{1}^{-1-a_{n}} d y_{1}=C_{a_{n}}\left|d^{-a_{n}}-c^{-a_{n}}\right|, \quad C_{\alpha}=\frac{8}{(2+\alpha)|\alpha|}\left(1-2^{-(2+\alpha) / 2}\right)
\end{aligned}
$$

Applying the above estimate to $I I$, we obtain

$$
|I I| \leq C_{a_{n}}\left|\left(\frac{R_{n}}{\lambda}\right)^{-a_{n}}-\left(k_{1} h\right)^{-a_{n}}\right| \lambda^{-a_{n}}=C_{a_{n}}\left|R_{n}^{-a_{n}}-\left(\lambda k_{1} h\right)^{-a_{n}}\right|
$$

If $R_{n} \leq \lambda_{*} k_{1} h$, for $\lambda \geq \lambda_{*}$, we have

$$
|I I| \leq C_{a_{n}}\left(R_{n}^{-a_{n}}-\left(\lambda k_{1} h\right)^{-a_{n}}\right) \leq C_{a_{n}} R_{n}^{-a_{n}}, a_{n}>0, \quad|I I| \leq C_{a_{n}}\left(\left(\lambda k_{1} h\right)^{-a_{n}}-R_{n}^{-a_{n}}\right), a_{n}<0
$$

If $R_{n}>\lambda_{*} k_{1} h$ and $a_{n}<0$, for $\lambda \geq \lambda_{*}$, we have two cases, $R_{n} \geq \lambda k_{1} h$ and $R_{n}<\lambda k_{1} h$. In the first case, since $\lambda \geq \lambda_{*}, a_{n}<0$, we get $\lambda^{a_{n}} \leq \lambda_{*}^{a_{n}}$ and

$$
|I I| \leq C_{a_{n}}\left(\left(\frac{R}{\lambda}\right)^{-a_{n}}-\left(k_{1} h\right)^{-a_{n}}\right) \lambda^{-a_{n}} \leq C_{a_{n}}\left(\left(\frac{R}{\lambda_{*}}\right)^{-a_{n}}-\left(k_{1} h\right)^{-a_{n}}\right) \lambda^{-a_{n}}
$$

In the second case, we get

$$
|I I| \leq C_{a_{n}}\left(\left(k_{1} h\right)^{-a_{n}}-(R / \lambda)^{-a_{n}}\right) \lambda^{-a_{n}} \leq C_{a_{n}}\left(k_{1} h\right)^{-a_{n}} \lambda^{-a_{n}}
$$

Combining two cases, we yield

$$
|I I| \leq C_{a_{n}} \lambda^{-a_{n}} \max \left(\left(k_{1} h\right)^{-a_{n}},\left(R / \lambda_{*}\right)^{-a_{n}}-\left(k_{1} h\right)^{-a_{n}}\right) .
$$

In our estimate, we do not have the case $R_{n}>\lambda_{*} k_{1} h$ and $a_{n}>0$. We can pick $\lambda_{*}$ large enough and the power $a_{n}$ in the weight to avoid such a case. In all cases, we can estimate

$$
|I I| \leq C_{1} \lambda^{-a_{n}}+C_{2}
$$

for some $C_{1}, C_{2}$ (can be negatively) independent of $\lambda$ uniformly for $\lambda \geq \lambda_{*}$.
7.1.2. Singular part. We focus on the estimate of $\nabla \mathbf{u}$. which is much more difficult. For the singular part, we follow Section 4.2 in Parr II [2] to decompose it as follows

$$
\begin{aligned}
(\nabla \mathbf{u})_{S} & \left.=\int_{R(x, k)} K(\hat{x}-\hat{y}) W_{\lambda}(\hat{y}) d \hat{y}=\int_{R(x, k) \backslash R_{s}(b)}+\int_{R_{s}(b) \backslash R_{s}(a)}+\int_{R_{s}(a)}\right) K(\hat{x}-\hat{y}) W_{\lambda}(\hat{y}) d \hat{y} \\
& \triangleq I+I I+I I I
\end{aligned}
$$

for fixed $k, b \geq 1$ and $a$ to be chosen, where $R(x, k), R_{s}(k)$ are singular regions defined in (7.5). The integrand in $I$ is nonsingular and we estimate it by $\|\omega \varphi\|_{\infty}$ using the same method as that
in Section 7.1.1] and Sections 4.1, 4.2 in Part II [2], For $I I$, using $L^{\infty}$ estimate, (7.8), (5.18), and $R_{s}(b) \subset R(b)$, we get

$$
|I I| \leq\|W \varphi\|_{L^{\infty}} \lambda^{-a_{\alpha}}\left\|\varphi_{\infty, \alpha}^{-1}\right\|_{L^{\infty}(R(b))} \int_{R_{s}(b) \backslash R_{s}(a)}|K(\hat{x}-\hat{y})| d \hat{y}=\|\omega \varphi\|_{L^{\infty}} \lambda^{-a_{\alpha}}\left\|\varphi_{\infty, \alpha}^{-1}\right\|_{L^{\infty}(R(b))} 2 \log \left(\frac{b}{a}\right) .
$$

Following the estimate of the singular part in Section 4.2 in Part II [2], we get

$$
I I I=\int_{R_{s}(a)} K(\hat{x}-\hat{y})\left(\psi_{\lambda}^{-1}(\hat{x})-\psi_{\lambda}^{-1}(\hat{y})\right) \omega_{\lambda} \psi_{\lambda}(\hat{y}) d y+\psi_{\lambda}^{-1}(\hat{x}) \int_{R_{s}(a)} K(\hat{x}-\hat{y}) \omega_{\lambda}(\hat{y}) \psi_{\lambda}(\hat{y}) d \hat{y}=I I I_{1}+I I I_{2}
$$

For $I I I_{1}$, we follow Section 4.2 in Part II [2] using Taylor expansion and the $L^{\infty}$ estimate

$$
\begin{aligned}
& \psi_{\lambda}^{-1}(\hat{y})-\psi_{\lambda}^{-1}(\hat{x})=\left(\nabla \psi_{\lambda}^{-1}\right)(\hat{x}) \cdot(\hat{y}-\hat{x})+P_{e}, \quad\left|P_{e}\right| \leq C\left|\nabla^{2} \psi_{\lambda}^{-1}\right||\hat{y}-\hat{x}|^{2}, \quad\left|W_{\lambda} \psi_{\lambda}(x)\right| \leq \frac{\psi_{\lambda}}{\varphi_{\lambda}}(\hat{x})\|\omega \varphi\|_{L^{\infty}}, \\
& \int_{R_{s}(a)}\left|K(\hat{x}-\hat{y})\left(\hat{x}_{1}-\hat{y}_{1}\right)^{i}\left(\hat{x}_{2}-\hat{y}_{2}\right)^{j}\right| d \hat{y}=4 \int_{[0, a]^{2}}\left|K(s) s_{1}^{i} s_{2}^{j}\right| d s=4 a^{i+j} \int_{[0,1]^{2}}\left|K(s) s_{1}^{i} s_{2}^{j}\right| d s, i+j \geq 1
\end{aligned}
$$

where the last integral can be evaluated using the methods in Section 5.1.3 In particular, we gain a small factor $|a|$ in the estimate of $I I I_{1}$. We refer detailed estimate of $I I I_{1}$ to Section 4.2 in Part II [2]. To factorize out the dependence of $\lambda$ in the above estimates, we need the following estimates for the weights

$$
\begin{equation*}
\left|\frac{\partial^{i}}{\partial \hat{x}_{1}} \frac{\partial^{j}}{\partial \hat{x}_{2}} \psi_{\lambda}^{-1}(\hat{x})\right| \leq \lambda^{a_{\alpha}}\left(\partial_{1}^{i} \partial_{2}^{j} \psi\right)_{\infty, \alpha}(\hat{x}), \quad \frac{\psi_{\lambda}(\hat{x})}{\varphi_{\lambda}(\hat{x})} \leq \lambda^{b_{\alpha}-a_{\alpha}}\left(\frac{\psi}{\varphi}\right)_{\infty, \alpha}(\hat{x}) \tag{7.15}
\end{equation*}
$$

uniformly for $\lambda \leq \lambda_{1}$ for $\alpha=1$, i.e. $x$ close to 0 , and $\lambda \geq \lambda_{n}$ for $\alpha=n$, i.e. $x$ in the far-field, which have been established in Appendix A.2, A. 3 in Part II [2]. Note that we do not pick up extra $\lambda$ power in the asymptotic analysis when we take derivatives $\partial_{\hat{x}_{i}}^{j}$ on $\psi_{\lambda}^{-1}$. For example, if $\psi(x)=|x|^{-2}, \partial_{\hat{x}_{1}} \psi_{\lambda}^{-1}(\hat{x})=\partial_{\hat{x}_{1}} \lambda^{2}|\hat{x}|^{2}=\lambda^{2} 2 \hat{x}_{1}$. Using these estimates, we derive

$$
I I I_{1} \leq \lambda^{-a_{\alpha}}|a| C(\hat{x})
$$

for some constant $C(\hat{x})$.
For $I I I_{2}$, using the estimates in Section 6.1 we yield

$$
\left|\psi_{\lambda}(\hat{x}) I I I_{2}\right| \leq C_{1}(\hat{x})| | \frac{\psi_{\lambda}}{\varphi_{\lambda}}\left\|_{L^{\infty}(R(b))}\right\| \omega_{\lambda} \varphi_{\lambda} \|_{L^{\infty}}+C_{2}(\hat{x})|a|^{1 / 2}\left[\omega_{\lambda} \psi_{\lambda}\right]_{C_{x}^{1 / 2}}+C_{3}(\hat{x})|a|^{1 / 2}\left[\omega_{\lambda} \psi_{\lambda}\right]_{C_{y}^{1 / 2}},
$$

for some $C_{i}(\hat{x})$ independent of the weights $\varphi$ and $\lambda$, which can be derived using the same method as that in Section 6.1. Using the scaling relation (7.2), we yield

$$
\left|\psi_{\lambda}(\hat{x}) I I I_{2}\right| \leq C_{1}(\hat{x}) \lambda^{a_{\alpha}-b_{\alpha}}| | \omega \varphi \|_{L^{\infty}}+C_{2}(\hat{x})|\lambda a|^{1 / 2}[\omega \psi]_{C_{x}^{1 / 2}}+C_{3}(\hat{x})|\lambda a|^{1 / 2}[\omega \psi]_{C_{y}^{1 / 2}}
$$

where we have changed $C_{1}(\hat{x})$ to track the constant $\left\|\frac{\psi_{\lambda}}{\varphi_{\lambda}}\right\|_{L^{\infty}(R(b))}$.
Using the above estimates and (6.6), we can bound the singular part as follows

$$
\begin{align*}
(\nabla \mathbf{u})_{S} & \leq \lambda^{-a_{\alpha}}\left(C_{1}(\hat{x})+C_{2}(\hat{x}) \log \frac{b}{a}+C_{3}(\hat{x})|a|\right)| | \omega \varphi \|_{\infty}+\lambda^{-b_{\alpha}+\frac{1}{2}}\left(C_{41}(\hat{x})[\omega \psi]_{C_{x}^{1 / 2}}+C_{42}(\hat{x})[\omega \psi]_{C_{y}^{1 / 2}}\right)  \tag{7.16}\\
& \leq E_{1}\left(\lambda^{-a_{\alpha}}\left(C_{1}(\hat{x})+C_{2}(\hat{x}) \log \frac{b}{a}+C_{3}(\hat{x})|a|\right)+\left(\gamma_{1} C_{41}(\hat{x})+\gamma_{2} C_{42}(\hat{x})\right) \lambda^{-b_{\alpha}+1 / 2}\right)
\end{align*}
$$

for $x=\lambda \hat{x}$ close to $0, \alpha=1$, and for $x=\lambda \hat{x}$ in the far-field, $\alpha=n$, with $\hat{x}$ in each $\Gamma_{1, i}, \Gamma_{2, i}$ (7.6).

We perform the above estimate for a list of $a, a_{1}<a_{2}<\ldots,<a_{N} \leq b$ and uniform estimate for $a h \leq h \leq b h$. We will optimize $a$ to obtain a sharp estimate in Section 7.3.1,
7.1.3. Summary of the estimates and scaling. Combining the estimates in Section 7.1.1 e.g. (7.14), and (7.16), we obtain the estimate for $\nabla \mathbf{u}$. From the above estimates, we can apply the same estimates as those in [3] and Section 6.1 except that we need to perform the estimates using the asymptotic properties of the weights (7.7), (7.8), (7.15) uniformly for small $\lambda$ or large $\lambda$, instead of the weights $\varphi^{-1}, \frac{\psi}{\varphi}, \partial_{1}^{i} \partial_{2}^{j}\left(\psi^{-1}\right)$ in the estimates for the finite $\lambda$ case. Moreover, we need to track the power in $\lambda$. In particular, we can obtain piecewise estimates for $\mathbf{u}, \nabla \mathbf{u}$ (7.1) for $x \in \Gamma_{i, j}\left(x_{c}\right)(7.6)$ and $\lambda \leq \lambda_{*}$ or $\lambda \geq \lambda_{*}$ uniformly. The estimates weighted by $\psi_{\lambda}(\hat{x})$ can be obtained similarly.

For $\mathbf{u}(\lambda \hat{x})$, the estimate is much easier since the kernel is locally integrable. Using the argument in Section 7.1.1, the scaling relation (7.1) with $d=1$, and the same method in [3] except that we use the weight $\varphi_{\infty, \alpha}\left(\begin{array}{l}(7.8)\end{array}\right.$ instead of $\varphi$, we can bound the integral and $\mathbf{u}(\lambda \hat{x})$ by

$$
C(\hat{x}) \lambda^{-a_{\alpha}+1}\|\omega \varphi\|_{\infty}
$$

We get the power 1 since the kernel in $\mathbf{u}$ is -1 homogeneous and we use the scaling relation (7.1) with $d=1$.

Remark 7.1. To track the $\lambda$ power in the estimates (7.14), (7.16), we have the power $\lambda^{-a_{\alpha}}$ in the coefficient of $\|\omega \varphi\|_{\infty}$ since $\varphi^{-1}(\lambda x)$ has asymptotic scaling property $\lambda^{-a_{\alpha}}(7.8)$, and $\lambda^{-b_{\alpha}+1 / 2}$ in the coefficient of $[\omega \psi]_{C_{x_{i}}^{1 / 2}}$ since $\psi^{-1}(\lambda x)$ has asymptotic scaling property $\lambda^{-b_{\alpha}}$ (7.7), (7.8) and we get an extra $\lambda^{1 / 2}$ from the scaling law (7.2). In the weighted estimate, e.g. $\psi_{\lambda} \nabla \mathbf{u}$, we will obtain an additional power $\lambda^{b_{\alpha}}$ in the upper bound since the weight $\psi_{\lambda}$ has asymptotics $\lambda^{-b_{\alpha}}$. In the estimate of $\mathbf{u}$, we gain the factor $\lambda$ in $\lambda^{-a_{\alpha}+1}$ since $\mathbf{u}$ is 1 order more regular than $\nabla \mathbf{u}$.

### 7.2. Hölder estimates of $\nabla \mathbf{u}, \mathbf{u}$. Denote

$$
\begin{align*}
& x_{c 2}=2 x_{c}, \mu=1 / 8, \quad \Omega_{1} \triangleq\left[x_{c 2},(1+\mu) x_{c 2}\right] \times\left[0, x_{c 2}\right] \cup\left[0,(1+\mu) x_{c 2}\right] \times\left\{x_{c 2}\right\} \\
& \Omega_{2} \triangleq\left\{x_{c 2}\right\} \times\left[0,(1+\mu) x_{c 2}\right] \cup\left[0, x_{c 2}\right] \times\left[x_{c 2},(1+\mu) x_{c 2}\right] . \tag{7.17}
\end{align*}
$$

Firstly, we choose $\lambda$ and rescale $x=\lambda \hat{x}$ with $|\hat{x}|_{\infty}=x_{c 2}$. In the $C_{x}^{1 / 2}$ estimate, for $z=\lambda \hat{z}$ with $x_{1} \leq z_{1}, z_{2}=x_{2}$, if $|\hat{x}-\hat{z}| \leq \mu x_{c 2}$, we have $\hat{z} \in \Omega_{1}$. In the $C_{y}^{1 / 2}$ estimate, for $z=\lambda \hat{z}$ with $x_{2} \leq z_{2}, x_{1}=z_{1}$, we have $z \in \Omega_{2}$. For $|\hat{x}-\hat{z}|>\mu x_{c 2}$, we apply the triangle inequality to estimate $\nabla \mathbf{u}(x)-\nabla \mathbf{u}(z)$ directly. In Section 4.5 in Part II [2], we discuss the estimate of the derivatives of the regular part of the integrand in (7.1)

$$
\partial_{\hat{x}_{i}} J(\hat{x}, \hat{y}), \quad J=K(\hat{x}, \hat{y}) p_{\lambda}(\hat{x}), \text { or } J=K^{C}(\hat{x}, \hat{y})\left(p_{\lambda}(\hat{x})-p_{\lambda}(\hat{y})\right)+K^{N C}(\hat{x}, \hat{y}) p_{\lambda}(\hat{x})
$$

with weight $p(x)=\psi(x)$, where $K^{C}, K^{N C}$ are parts of the symmetrized kernel $K^{\text {sym (7.13) }}$ determined by the distance between the $y$ and the singularity $x$. See Section 4.1.5 in Part II [2]. Using such estimates for $\partial_{\hat{x}_{i}} J$, we can estimate the integral of $\partial_{\hat{x}_{i}} J$ in terms of $\|\omega \varphi\|_{\infty}$ using the method in Section 7.1.1 and the method in Section 4.3 in Part II [2] with weight $\varphi_{\infty, \alpha}$ (7.8). It particular, these parts are bounded by

$$
C(\hat{x}) \lambda^{b_{\alpha}-a_{\alpha}}\|\omega \varphi\|_{\infty}
$$

We have the power $\lambda^{b_{\alpha}}$ from the weight $p_{\lambda}=\psi_{\lambda}$ (7.7). Here, $\alpha=1$ means that we estimate $x=\lambda \hat{x}$ for very small $x$ and $\lambda$, and $\alpha=n$ means that we estimate $x=\lambda \hat{x}$ for very large $x$ and $\lambda$. We use the same notatons below.

Remark 7.2. We estimate $\partial_{\hat{x}_{i}} J$ rather than $\partial_{x_{i}} J$. We have the scaling relation (17.2) between these two quantities.

The remaining part is the integral in the singular region (see $I_{5}$ in Section 4.3.4 in Part II [2])

$$
I_{5}\left(\hat{x}, k_{2}\right)=\int_{R\left(k_{2}\right)} K(\hat{x}-\hat{y})\left(\psi_{\lambda}(\hat{x})-\psi_{\lambda}(\hat{y})\right) W_{\lambda}(\hat{y}) d \hat{y}
$$

In Part II [2], we discuss the case of finite $\lambda$ and choose $\lambda=1$ for simplicity. The case of general $\lambda$ is given above using the rescaling relation (7.1). For the Hölder estimate, in Sections 4.3
in Part II [2], we decompose $I_{5}$ into several parts and estimate the piecewise $L^{\infty}$ norm or the piecewise derivatives bounds for each part. For each part, we can bound it using the weights

$$
\psi_{\lambda}(\hat{x}) \partial_{\hat{x}_{1}}^{i} \partial_{\hat{x}_{2}}^{j} \psi_{\lambda}^{-1}(\hat{x}), \quad \psi_{\lambda}(\hat{x}) / \varphi_{\lambda}(\hat{x})
$$

the norm $\left\|W_{\lambda} \varphi_{\lambda}\right\|_{L^{\infty}}$ and the semi-norm $\left[\omega_{\lambda} \psi_{\lambda}\right]_{C_{x_{i}}^{1 / 2}}$. The above weights are the same as those in (7.15), and we establish the uniform estimates in Appendix A.2, A. 3 in Part II [2]. Using the scaling relation (7.2), we can bound each part by

$$
C_{1}(\hat{x}) \lambda^{b_{\alpha}-a_{\alpha}}\|\omega \varphi\|_{L^{\infty}}+C_{2}(\hat{x}) \lambda^{1 / 2}[\omega \psi]_{C_{x_{l}}^{1 / 2}}
$$

for $l=1$ or 2 . In the $C_{x_{l}}^{1 / 2}$ estimate, we only need the semi-norm $[\omega \psi]_{C_{x_{l}}^{1 / 2}}$. The above estimates allow us to estimate different parts in the Hölder estimate in Section 4.3 in Part II [2]. Similar to the $L^{\infty}$ estimate in Section 7.1] we can apply essentially the same estimate in [3] for the case of finite $\lambda$ to the case of $\lambda \rightarrow 0$ or $\lambda \rightarrow \infty$ except that we need to use the asymptotic properties of the weights (7.7), (7.8), (7.15) instead of using $\varphi_{\lambda}^{-1}, \partial_{x}^{i} \partial_{y}^{j} \psi_{\lambda}^{-1}$ etc for a fixed $\lambda$, and track the power.

Similarly, in the Hölder estimate of $\psi_{u, \lambda}(x) \mathbf{u}(\lambda x)$ (see Section 4.3.8 in Part II [2]), we can bound the piecewise $L^{\infty}$ norm or derivatives of each part by

$$
\begin{equation*}
C(\hat{x}) \lambda^{1+c_{\alpha}-a_{\alpha}}\|\omega \varphi\|_{\infty} \tag{7.18}
\end{equation*}
$$

where $c_{\alpha}$ is the leading order power of $\psi_{u, \lambda}$ (7.7). We refer to Remark 7.1 for tracking the power in the upper bounds.

Once we obtain the above bounds for each parts, we assemble the Hölder estimate following Section 4.6 in Part II [2] and the scaling relation (7.2). See Section 7.4.
7.3. Assemble $L^{\infty}$ estimate of $\mathbf{u}, \nabla \mathbf{u}$ in $\mathbb{R}_{2}^{++}$. We use the method in Section 4.2 in Part II [2] to estimate the piecewise bounds of $\mathbf{u}(x), \nabla \mathbf{u}(x)$ for $\hat{x}$ in each $h_{x} \times h_{x} \operatorname{grid} B_{i_{1} j_{1}}\left(h_{x}\right) \subset$ $\left[0,2 x_{c}\right]^{2} \backslash\left[0, x_{c}\right]^{2}$

$$
x=\lambda \hat{x}, \quad \hat{x} \in\left[0,2 x_{c}\right]^{2} \backslash\left[0, x_{c}\right]^{2}=\cup_{i_{1}, j_{1}} B_{i_{1} j_{1}}\left(h_{x}\right),
$$

with finitely many $\lambda=\lambda_{i}, i=1,2, . ., n_{1}$. This allows us to bound $\mathbf{u}, \nabla \mathbf{u}$ for $x \in \lambda_{n_{1}}\left[0,2 x_{c}\right]^{2} \backslash \lambda_{1}\left[0, x_{c}\right]^{2}$. To assemble the estimate in $\mathbb{R}_{2}^{++}$, we first pass the estimates from the dyadic mesh to the mesh (7.3) for computing the approximate steady state. For each mesh $Q_{i j}$ (7.3) with $\lambda_{1} x_{c} \leq$ $\max \left(y_{i}, y_{j}\right) \leq \max \left(y_{i+1}, y_{j+1}\right) \leq 2 \lambda_{n_{1}} x_{c}$, we can cover $Q_{i j}$ by finitely many dyadic mesh $M_{l i_{1} j_{1}}$ (7.6). Then we use

$$
\|f\|_{L^{\infty}\left(Q_{i j}\right)} \leq \max _{Q_{i j} \cap M_{l i_{1} j_{1}} \neq \emptyset}\|f\|_{L^{\infty}\left(M_{l, i_{1}, j_{1}}\right)}
$$

to obtain the piecewise bound on $Q_{i j}$. In our implementation, we loop over $l, i_{1}, j_{1}$ and update the bound in $Q_{i j}$ if $Q_{i j} \cap M_{l i_{1} j_{1}} \neq \emptyset$. The same method applies to piecewise bounds of weighted velocity $\rho_{10} \mathbf{u}, \psi_{1} \nabla \mathbf{u}$ for some weights since we have piecewise bounds for the weights. See Appendix A.2, A. 3 in Part II [2].
7.3.1. Near-field and far-field. We need to further bound $\mathbf{u}, \nabla \mathbf{u}$ for $x \in \lambda_{1}\left[0, x_{c}\right]^{2}$ and $|x|_{\infty} \geq$ $2 \lambda_{n} x_{c}$. We focus on the weighted estimate $(p \nabla \mathbf{u})(x)$. The estimate for $\mathbf{u}$ is similar. To estimate $\mathbf{u}, \nabla \mathbf{u}$ in the far field, we use the estimate in Section 7.1 with $\lambda \geq \lambda_{*}, \lambda_{*} x_{c} \leq 2 \lambda_{n} x_{c}$.

We focus on the estimate of the singular part (7.16) in the far-field. The estimates of other parts are easier. In the first estimate, we assume that $x$ is in the computational domain $x=$ $\lambda \hat{x} \in\left[y_{i}, y_{i+1}\right] \times\left[y_{j}, y_{j+1}\right]=Q_{i j}$ with $\lambda=\frac{|x|_{\infty}}{x_{c}}$ (7.4) and $|\hat{x}|_{\infty}=x_{c}$. We can cover the range of $\hat{x}$ by finitely many intervals $\Gamma_{1, i_{1}}, \Gamma_{2, j_{1}}$ in (7.6). For $\hat{x}$ in each interval, from (7.16), we have

$$
\begin{equation*}
\left|(\nabla \mathbf{u})_{S}(x)\right| \leq E_{1}\left(\lambda^{-a_{n}}\left(C_{0}+C_{1} \log \frac{b}{a}+C_{2}|a|\right)+C_{3}|a|^{1 / 2} \lambda^{-b_{n}+1 / 2}\right) \triangleq g(\lambda, a) \cdot E_{1} \tag{7.19}
\end{equation*}
$$

for some constants $C_{i}$ depending on the interval, where $a_{n}, b_{n}$ are the slowest decay power in $\varphi, \psi$ (7.7). Suppose that

$$
\begin{equation*}
p(x) \leq c|x|^{\alpha} \tag{7.20}
\end{equation*}
$$

We can obtain the upper and lower bound of $\lambda /|x|=O(1), \lambda,|x|$ for $x \in Q_{i j}$. We first bound $\lambda^{\alpha}$ by $\max \left(\lambda_{l}^{\alpha}, \lambda_{u}^{\alpha}\right)$ and then optimize the estimate (7.19) over finite many $a=a_{i}$, established in (7.16). In the second estimate, we minimize the upper bound in (7.19) by choosing

$$
\begin{equation*}
a=\min \left(h, \min \left(f\left(\lambda_{l}\right), f\left(\lambda_{u}\right)\right)\right), \quad f(\lambda)=\left(\frac{2 C_{1}}{C_{3}}\right)^{2} \lambda^{2\left(b_{n}-a_{n}\right)-1} \tag{7.21}
\end{equation*}
$$

With the above choice of $a$, the upper bound in (7.19) is of order $\lambda^{-a_{n}} \log \lambda$. We further use the piecewise bounds for $\lambda /|x|,|x|$, and (7.20) to obtain piecewise bounds of $p \nabla \mathbf{u}(x)$ for $x \in Q_{i j}$ and $\hat{x}$ in one of the intervals $\Gamma_{1, i_{1}}, \Gamma_{2, j_{1}}$. Taking the maximum of different cases yields the piecewise estimates in $Q_{i j}$.

For $x$ sufficiently large $\max x_{i} \geq R, x$ is outside the domain spanned by $Q_{i j}$. We take $\lambda_{*}=\frac{R}{x_{c}}$. In this case, we estimate $p(x) \nabla \mathbf{u}(x)$ uniformly for $x$ with $|x|_{\infty} \geq R$ (7.4). We assume (7.20) for all $x$ with $\max x_{i} \geq R$ and require $\alpha<\min \left(a_{n}, 0\right)$. We optimize (7.19) by choosing $a=f(\lambda)$ (7.21).

Recall the power $\left(a_{\alpha}, b_{\alpha}\right)$ of the weights (7.7). Since $2\left(b_{n}-a_{n}\right)-1 \leq-\frac{1}{2}$ and $\lambda$ is sufficiently large, we have $a=f(\lambda) \leq h$ in our application. With the above choice of $a$, we get

$$
C_{3} a^{1 / 2} \lambda^{-b_{n}+1 / 2}=C_{3} \frac{2 C_{1}}{C_{3}} \lambda^{-b_{n}+1 / 2+b_{n}-a_{n}-1 / 2}=2 C_{1} \lambda^{-a_{n}}
$$

and $g(\lambda, a)$ (7.19) reduces to

$$
g(\lambda, a)=\lambda^{-a_{n}}\left(C_{0}+C_{1} \log \frac{b}{a}+C_{2}\left(\frac{2 C_{1}}{C_{3}}\right)^{2} \lambda^{2\left(b_{n}-a_{n}\right)-1}+2 C_{1}\right)
$$

Estimate of the logarithm term. We need to further control the term $\lambda^{-a_{n}} \log \frac{b}{a}$. Denote

$$
F(\lambda, b, C, d, \beta)=\log (b / A(\lambda)) \lambda^{C}, \quad A(\lambda)=d \lambda^{-\beta}
$$

for some $\beta>0>C$. We maximize it over $\lambda \geq \lambda_{*}$. Clearly, we have

$$
\frac{d}{d \lambda} F(\lambda)=\lambda^{C-1} C \log \frac{b}{A(\lambda)}-\lambda^{C} \frac{A^{\prime}}{A}=\lambda^{C-1} C \log \frac{b}{A(\lambda)}+\beta \lambda^{C-1}=\lambda^{C-1}\left(C \log \frac{b}{A(\lambda)}+\beta\right)
$$

The critical point $\lambda_{c}$ is given by

$$
A\left(\lambda_{c}\right)=b e^{\beta / C}, \quad \lambda_{c}=\left(\frac{d}{A\left(\lambda_{c}\right)}\right)^{1 / \beta}=\left(\frac{d}{b}\right)^{1 / \beta} e^{-1 / C}
$$

For $\lambda \leq \lambda_{c}, F(\lambda)$ is increasing. For $\lambda \geq \lambda_{c}, F(\lambda)$ is decreasing. Thus for $\lambda \geq \lambda_{*}$, we get

$$
F(\lambda, b, C, d, e) \leq F\left(\max \left(\lambda_{c}, \lambda_{*}\right), b, C, d, e\right)
$$

Finally, using $|x| \geq \max _{i}\left(x_{i}\right)=\lambda x_{c}, \alpha \leq \min \left(a_{n}, 0\right), \lambda \geq \lambda_{*}$, and $p \leq c|x|^{\alpha}$ (7.20), we obtain

$$
\begin{aligned}
|p(\nabla \mathbf{u})| & \leq c x_{c}^{\alpha} g(\lambda, b) \lambda^{\alpha} E_{1} \leq c x_{c}^{\alpha} E_{1}\left(\left(C_{0}+2 C_{1}\right) \lambda_{*}^{\alpha-a_{n}}\right. \\
& \left.+C_{2}\left(\frac{2 C_{1}}{C_{3}}\right)^{2} \lambda_{*}^{2\left(b_{n}-a_{n}\right)-1+\alpha-a_{n}}+C_{1} F\left(\lambda, b, \alpha-a_{n},\left(\frac{2 C_{1}}{C_{3}}\right)^{2}, 2\left(-b_{n}+a_{n}\right)+1\right)\right)
\end{aligned}
$$

where $F(\lambda)$ is further bounded using the above estimate.
Estimate of the regular part. From Section7.1.1, the regular part of $\nabla \mathbf{u}$ with approximation terms can be bounded by

$$
\left|(\nabla \mathbf{u})_{R}\right| \leq C_{1}(\hat{x}) \lambda^{-a_{n}}+C_{2}(\hat{x})
$$

where $C_{i}(\hat{x})$ are some piecewise constants and $C_{2}(\hat{x})$ can be negative. For a weight satisfying (7.20) and $x \in Q_{i j}=\left[y_{i}, y_{i+1}\right] \times\left[y_{j}, j_{j+1}\right]$, it is easy to obtain the upper and lower bounds for $r=|x|$ and $\lambda=\frac{\max x_{i}}{x_{c}}$. Then using

$$
\begin{equation*}
\min \left(C t_{l}^{\alpha}, C t_{u}^{\alpha}\right) \leq C t^{\alpha} \leq \max \left(C t_{l}^{\alpha}, C t_{u}^{\alpha}\right), \quad(f g)_{u}=\max \left(f_{l} g_{l}, f_{l} g_{u}, f_{u} g_{l}, f_{u} g_{u}\right) \tag{7.22}
\end{equation*}
$$

for any $C$, $\alpha$, we obtain the upper bound in $Q_{i j}$. For $x$ sufficiently large max $x_{i} \geq R$, we assume that the weight $p$ satisfies (7.20) with $\alpha<a_{n}$ so that $\left|p(\nabla \mathbf{u})_{R}\right|$ decays for large $x$. In this case, we have $r_{l}=R, r_{u}=\infty, \lambda_{l}=\frac{R}{x_{c}}, \lambda_{u}=\infty$, and the estimates follow from (7.22).

The weighted estimate of $\nabla \mathbf{u}$ in the near field and the estimate for $\mathbf{u}$ are similar.

Estimate of $u_{A} /|x|^{1 / 2}$. In the energy estimate, we need to control $u_{A} /|x|^{1 / 2}$ in the far-field, where $u_{A}=-\partial_{y}\left(\phi+L_{12}(0) x y\right)$ (7.11). We want to control $\frac{u_{A}}{x^{1 / 2}} r^{\gamma}$ for some $\gamma<-1$ and $|x|_{\infty} \geq R$ (7.4). Firstly, using the previous methods and $u_{x, A}=u_{A, x}$ (7.12), we obtain

$$
\left|u_{A, x}(x)\right| r^{\gamma+1} \leq C_{2}, \quad\left|u_{A}(x)\right| r^{\gamma} \leq C_{1}, \quad|x|_{\infty} \geq R
$$

Denote $r=|x|, \beta=\arctan \frac{x_{2}}{x_{1}}$. For $x_{1} \geq x_{2}$, we get $\beta \leq \pi / 4, x_{1}=|x|_{\infty} \geq R, x_{1} \geq r / \sqrt{2}$. Thus, we get

$$
\begin{equation*}
\frac{\left|u_{A}\right|}{\left|x_{1}\right|^{1 / 2}} r^{\gamma} \leq \frac{C_{1}}{(r \cos \beta)^{1 / 2}}, \beta \in\left[0, \frac{\pi}{2}\right], \quad \frac{\left|u_{A}\right|}{\left|x_{1}\right|^{1 / 2}} r^{\gamma} \leq \min \left(\frac{C_{1}}{R^{1 / 2}}, \frac{C_{1} 2^{1 / 4}}{r^{1 / 2}}\right), \quad \beta \leq \frac{\pi}{4} \tag{7.23}
\end{equation*}
$$

Using the estimate for $u_{A, x}$ and $\gamma+1<0$, we yield

$$
\left|u_{A}\right|=\int_{0}^{x_{1}}\left|u_{A, x}\left(z, x_{2}\right)\right| d x_{1} \leq \int_{0}^{x_{1}} C_{2}\left|\left(z, x_{2}\right)\right|^{-\gamma-1} d x_{1} \leq C_{2} x_{1} r^{-\gamma-1}, \quad \frac{\left|u_{A}\right| r^{\gamma}}{\left|x_{1}\right|^{1 / 2}} \leq C_{2} \frac{\left|x_{1}\right|^{1 / 2}}{r} \leq \frac{C_{2}(\cos \beta)^{1 / 2}}{r^{1 / 2}}
$$

Combining the above two estimates, for $x_{2}>x_{1}$, we establish

$$
\frac{\left|u_{A}\right| r^{\gamma}}{\left|x_{1}\right|^{1 / 2}} \leq r^{-1 / 2} \min \left(\frac{C_{1}}{(\cos \beta)^{1 / 2}}, C_{2}(\cos \beta)^{1 / 2}\right) \leq R^{-1 / 2} \min \left(\sqrt{C_{1} C_{2}}, \frac{C_{1}}{\left(\cos \beta^{u}\right)^{1 / 2}}, C_{2}\left(\cos \beta^{l}\right)^{1 / 2}\right)
$$

Combining (7.23) and the above estimate, we establish the estimate for $u_{A} /\left|x_{1}\right|^{1 / 2}$.
7.4. Assemble the Hölder estimate in $\mathbb{R}_{2}^{+}$. In the remaining part of this section, $\mathbf{u}, \nabla \mathbf{u}$ denote the velocity with approximations, i.e. $\mathbf{u}_{A},(\nabla \mathbf{u})_{A}$. We do not write down the approximation terms to simplify the notations.

In Section 4.6 in Part II [2], we discuss how to assemble the estimates of different parts of (7.1) to obtain the $C_{x_{i}}^{1 / 2}$ estimate $\delta_{i}(f, x, z)$ (7.24)

$$
\begin{equation*}
\delta_{i}(f, x, z) \triangleq \frac{|f(x)-f(z)|}{|x-z|^{1 / 2}}, \quad z_{i}>x_{i}, \quad z_{3-i}=x_{3-i} \tag{7.24}
\end{equation*}
$$

for $f$ being $\mathbf{u}, \nabla \mathbf{u}$ with the approximation terms, $x=\lambda \hat{x}, z=\lambda \hat{z}$ with $\hat{x} \in\left[0,2 x_{c}\right]^{2} \backslash\left[0, x_{c}\right]^{2}, \mid \hat{z}-$ $\hat{x} \mid \leq 2 \nu x_{c}$, and $\lambda=\lambda_{1}, \lambda_{2}, . . \lambda_{n_{1}}$. For $x, z$ with $2 \nu x_{c} \leq|\hat{x}-\hat{z}| \leq 2 \nu_{2} x_{c}$, we get $|x-z| \geq 2 \nu \lambda x_{c} \geq$ $\nu|x|_{\infty}$ and apply triangle inequality to estimate $\delta_{i}(f, x, z)$ for $\hat{x}, \hat{z}$ in a larger domain. Then, in the $C_{x}^{1 / 2}$ estimate, we obtain the piecewise estimate of $\delta_{i}(f, x, z)$ on the dyadic mesh (7.6)

$$
\begin{equation*}
x \in M_{l p q}, z \in M_{l r q}, p \leq r \leq p+m_{1}, \hat{x} \in B_{p q}\left(h_{x}\right) \subset\left[0,2 x_{c}\right]^{2} \backslash\left[0, x_{c}\right]^{2} \tag{7.25}
\end{equation*}
$$

In the above meshes, $l$ denotes the scaling factor $\lambda_{l}$. We have the same subindex $q$ for $x, z$ since $x_{2}=z_{2}$. Next, we pass the estimate from the dyadic mesh to the mesh $Q_{i j}$ (7.3).
7.4.1. The Hölder estimate in the bulk. Recall $M_{l, i_{1}, j_{1}}$ from (7.6). We focus on the $C_{x}^{1 / 2}$ estimate. For $x \in Q_{i_{1} j_{1}} \subset \lambda_{n_{2}}\left[0,2 x_{c}\right]^{2} \backslash \lambda_{1}\left[0, x_{c}\right]^{2}$, and $z \in Q_{i_{2}, j_{1}}$ close to $x i_{1} \leq i_{2} \leq i_{1}+m$, by covering $Q_{i_{1} j_{1}}, Q_{i_{2} j_{1}}$, we obtain

$$
\begin{equation*}
\max _{x \in Q_{i_{1} j_{1}}, z \in Q_{i_{2}, j_{1}}}\left|\delta_{1}(f, x, z)\right| \leq \max _{l}\left(\max _{x \in M_{l p q} \cap Q_{i_{1} j_{1}} \neq \emptyset, z \in M_{l s q} \cap Q_{i_{2} j_{1}} \neq \emptyset} \delta_{1}(f, x, z)\right) . \tag{7.26}
\end{equation*}
$$

Suppose that $Q_{i_{1} j_{1}} \subset \cup M_{l p q}$. For each $x \in M_{l p q}$, we have $\hat{x} \in B_{p q}, \hat{z}_{2}=\hat{x}_{2} \in I_{q}$. Denote

$$
I_{q}=\left[q h_{x},(q+1) h_{x}\right], \quad S_{z} \triangleq\left\{a \in \mathbb{R}^{2}: a_{2} \in \lambda_{l} I_{q}\right\} \cap Q_{i_{2} j_{1}}, \quad C_{z} \triangleq \cup_{p \leq p_{2} \leq p+m_{1}} M_{l p_{2} q}
$$

The set $S_{z}$ is the location of $z \in Q_{i_{2} j_{1}}$ since $z_{2}=x_{2}$, and $C_{z}$ is the location of $z$ we have computed $\delta_{1}(f, x, z)$ in the dyadic mesh (7.25). For each $Q_{i_{2} j_{1}} i_{2} \leq i_{1}+m$, we have two cases

$$
\text { (a) } y_{i_{2}+1} \leq \lambda_{l}\left(p+m_{1}+1\right) h_{x}, \quad \text { (b) } y_{i_{2}+1}>\lambda_{l}\left(p+m_{1}+1\right) h_{x}
$$

where $y$ is the mesh (7.3). In case (a), $z$ is close to $x$ or $i_{2}$ close to $i_{1}$, e.g. $i_{2}=i_{1}$. We estimate $\delta_{1}(f, x, z)$ using the estimate of $\delta_{1}(f, x, z)$ on the dyadic mesh (7.25) $C_{z}$ we have computed.

In case (b), the mesh $C_{z}$ does not cover $S_{z}$, the range of $z$ with $z_{1} \in\left[y_{i_{2}}, y_{i_{2}+1}\right], z_{2} \in I_{q}$. For $z_{1} \in\left[y_{i_{2}}, \lambda_{l}\left(p+m_{1}+1\right) h_{x}\right], C_{z}$ covers parts of $S_{z}$ (or none), and we use the estimate on the
dyadic mesh (7.25) to estimate $\delta_{1}(f, x, z)$. For $z_{1} \in\left[\max \left(y_{i_{2}}, \lambda_{l}\left(p+m_{1}+1\right) h_{x}\right), y_{i_{2}+1}\right]$, since $x \in Q_{i_{1} j_{1}} \cap M_{l p q}$ and we choose $\lambda_{l}=2^{k_{l}}$ for $k_{l} \in Z$, we have

$$
\begin{equation*}
x_{1} \leq \lambda_{l}(p+1) h_{x}, \max \left(y_{i 1}, y_{j 1}\right) \leq|x|_{\infty}=\lambda_{l}|\hat{x}|_{\infty} \leq \lambda_{l} 2 x_{c}, \quad \lambda_{l} \geq 2^{\left\lceil\log _{2}\left(\frac{\max \left(y_{i_{1}}, y_{j_{1}}\right)}{2 x_{c}}\right\rceil\right.} \triangleq \lambda_{i_{1}, j_{1}} \tag{7.27}
\end{equation*}
$$

where $\lceil a\rceil$ is the smallest integer no less than $a$. It follows

$$
z_{1}-x_{1} \geq \lambda_{l} m_{1} h_{x} \geq \frac{m_{1} h_{x}}{2 x_{c}}|x|_{\infty} \geq \frac{m_{1} h_{x}}{2 x_{c}} \max \left(y_{i_{1}}, y_{j_{1}}\right), \quad z_{1}-x_{1} \geq \max \left(y_{i_{2}}-y_{i_{1}+1}, 0\right)
$$

We apply the triangle inequality, the piecewise bound of $f=\nabla \mathbf{u}$ in $Q_{i_{2} j_{1}}, Q_{i_{1} j_{1}}$ established in Section 7.3, and the above lower bound on $|x-z|$ to estimate $\delta_{1}(f, x, z)$. Since $x_{1} \in \lambda_{l} h_{x}[p, p+1]$ $\left(x \in M_{l p q}\right)$, conditions $y_{i_{2}+1}>\lambda_{l}\left(p+m_{1}+1\right) h_{x}, x \in Q_{i_{1} j_{1}}$, imply

$$
y_{i_{2}+1}-y_{i_{1}} \geq y_{i_{2}+1}-x_{1} \geq y_{i_{2}+1}-\lambda_{l}(p+1) h_{x} \geq \lambda_{l} m_{1} h_{x} \geq m_{1} h_{x} 2^{\left\lceil\log _{2}\left(\frac{\max \left(y_{i_{1}}, y_{j_{1}}\right)}{2 x_{c}}\right\rceil\right.} .
$$

Recall $\lambda_{i_{1}, j_{1}}$ from (7.27). For $i_{1} \leq i_{2}$, we introduce

$$
D\left(i_{1}, i_{2}, j_{1}\right)=\max \left(\lambda_{i_{1}, j_{1}} h_{x}, \max \left(y_{i_{2}}-y_{i_{1}+1}, 0\right)\right), \quad S \triangleq\left\{\left(i_{1}, i_{2}, j_{1}\right): y_{i_{2}+1}-y_{i_{1}} \geq \lambda_{i_{1}, j_{1}} h_{x} m_{1}\right\}
$$

From the above discussion, we can modify the estimate (7.26) as follows

$$
\begin{gathered}
\max _{x \in Q_{i_{1} j_{1}}, z \in Q_{i_{1}, j_{2}}}\left|\delta_{1}(f, x, z)\right| \leq \max \left\{\max _{l}\left(\max _{x \in M_{l p q} \cap Q_{i_{1} j_{1}} \neq \emptyset, z \in M_{l p_{2} q} \cap Q_{i_{2} j_{1}} \neq \emptyset, p \leq p_{2} \leq p+m_{1}} \delta_{1}(f, x, z)\right),\right. \\
\left.\mathbf{1}_{S}\left(i_{1}, i_{2}, j_{1}\right) \frac{\|f\|_{\infty\left(Q_{i_{1} j_{1}}\right)}+\|f\|_{\infty}\left(Q_{i_{2} j_{1}}\right)}{D_{i_{1}, i_{2}, j_{1}}^{1 / 2}}\right\} .
\end{gathered}
$$

Note that $D\left(i_{1}, i_{2}, j_{1}\right), \lambda_{i_{1}, j_{1}}, S$, and the second bound in the maximum are independent of the dyadic estimates, and only depend on the piecewise estimates on mesh $Q_{i j}$. The above estimate allows us to obtain a piecewise $C_{x}^{1 / 2}$ estimates for $x \in Q_{i_{1}, j_{1}}, z \in Q_{i_{2}, j_{1}}$ with $\left|i_{1}-i_{2}\right| \leq m$ and $x$ in the bulk of the domain. Similarly, we obtain the piecewise bounds for the $C_{y}^{1 / 2}$ estimate.
7.4.2. The Hölder estimate in the far-field and near-field for small distance. We focus on the estimate for $\nabla \mathbf{u}$. The estimate of $\mathbf{u}$ is similar and easier. Firstly, in the $C_{x_{i}}^{1 / 2}$ estimate using (6.6) and the estimates in Section 7.2, we can decompose (7.1) into several parts and bound the piecewise $L^{\infty}$ norm or derivatives in $\hat{x}$ for each part in a small domain $B_{i_{1} j_{1}}\left(h_{x}\right) \subset \Omega_{i}$ (7.4), (7.17) by

$$
C_{1}(\hat{x}) \lambda^{b_{\alpha}-a_{\alpha}}\|\omega \varphi\|_{L^{\infty}}+C_{2}(\hat{x}) \lambda^{1 / 2}[\omega \psi]_{C_{x_{i}}^{1 / 2}} \leq \lambda^{1 / 2}\left(C_{1}(\hat{x}) \lambda^{b_{\alpha}-a_{\alpha}-1 / 2}+\gamma_{i} C_{2}(\hat{x})\right) E_{1}
$$

Recall the power $a_{\alpha}, b_{\alpha}$ of the weights (7.7). We have

$$
b_{1}-a_{1}-\frac{1}{2} \geq-2+\frac{5}{2}-\frac{1}{2} \geq 0, \quad b_{n}-a_{n}-\frac{1}{2} \leq-\frac{1}{6}+\frac{1}{6}-\frac{1}{2} \leq-\frac{1}{2} \leq 0
$$

In the near field $\lambda \leq \lambda_{*} \ll 1$ or the far field $\lambda \geq \lambda_{*}$, using $\lambda^{b_{\alpha}-a_{\alpha}-1 / 2} \leq \lambda_{*}^{b_{\alpha}-a_{\alpha}-1 / 2}$, we can bound the estimate of each part uniformly in $\lambda$ by

$$
\lambda^{1 / 2}\left(C_{1}(\hat{x}) \lambda_{*}^{b_{\alpha}-a_{\alpha}-1 / 2}+\gamma_{i} C_{2}(\hat{x})\right) E_{1} .
$$

The power $\lambda^{1 / 2}$ is factorized out in the estimates. Thus we can treat the Hölder estimate $\delta_{i}(f, \lambda \hat{x}, \lambda \hat{z})$ (7.24) for $\lambda \geq \lambda_{*}$ or $\lambda \leq \lambda_{*}$ in the same way as that for finite $\lambda$ case using the $\lambda$-independent bound $\left(C_{1}(\hat{x}) \lambda_{*}^{b_{\alpha}-a_{\alpha}-1 / 2}+\gamma_{i} C_{2}(\hat{x})\right) E_{1}$ and tracking the power $\lambda^{1 / 2}$. Using the scaling relation (7.2) and the estimates in Section 4.6 in Part II [2], we obtain the weighted Hölder estimate $\delta_{i}(f, \lambda \hat{x}, \lambda \hat{z})(7.24)$ for $f=\psi \nabla \mathbf{u}$ and $|\hat{x}|_{\infty}=x_{c 2},|\hat{z}-\hat{x}| \leq \mu x_{c 2}$ uniformly for $\lambda \leq \lambda_{*}$ or $\lambda \geq \lambda_{*}$. Note that the power $\lambda^{1 / 2}$ in the above estimate and that in (7.2) are exactly canceled.

Large distance. The above argument applies to obtain Hölder estimate for $x, z$ close relative to $|x|$. To estimate $\frac{|f(\lambda \hat{x})-f(\lambda \hat{z})|}{|\lambda(\hat{x}-\hat{z})|^{1 / 2}}$ with large $|\hat{x}-\hat{z}|,|\hat{x}|=x_{c 2}$, we need to bound $f(\lambda \hat{z})$ for large $\hat{z}$. We focus on $C_{x}^{1 / 2}$ estimate of $\nabla \mathbf{u}$. Other estimates are similar.
$L^{\infty}$ estimate. From the above discussions, we seek a bound

$$
\begin{equation*}
|f(\lambda \hat{z})| \leq \lambda^{\beta} C(\hat{z}) \tag{7.28}
\end{equation*}
$$

for $f=\psi \nabla \mathbf{u}$ with approximation terms uniformly in $\lambda \geq \lambda_{*}, \beta \leq \frac{1}{2}$ or $\lambda \leq \lambda_{*}, \beta \geq \frac{1}{2}$, and $\hat{z}$ in a large domain, e.g. $\left[0, x_{c 3}\right]^{2} \backslash\left[0, x_{c 2}\right]^{2}$ (7.6).

Recall the estimate of the regular parts, e.g. (7.14), and singular parts (7.16) in Section 7.1. We first optimize the bound in (7.16) with a fixed $\lambda=\lambda_{*}$ over $a=a_{1}, a_{2}, . ., a_{N}$. Denote by $a_{*}$ be the minimizer among $a_{i}$. Choosing $a=a_{*}$ in (7.16), we can bound $(\nabla \mathbf{u})_{S}$ by

$$
E_{1}\left(\lambda^{-a_{\alpha}} C_{1}(\hat{x})+\lambda^{-b_{\alpha}+1 / 2} C_{2}(\hat{x})\right)
$$

for another piecewise constants $C_{i}(\hat{x})$. For the nonsingular parts in Section 7.1.1, for $\lambda \in\left[\lambda_{l}, \lambda_{u}\right]$, we can bound it by

$$
E_{1}\left(C_{3}(\hat{x}) \lambda^{-a_{\alpha}}-C_{4}(\hat{x})\right) \leq E_{1} \lambda^{-a_{\alpha}}\left(C_{3}(\hat{x})-C_{4}(\hat{x}) \min \left(\lambda_{l}^{a_{\alpha}}, \lambda_{u}^{a_{\alpha}}\right)\right)=C_{5}(\hat{x}) \lambda_{l}^{a_{\alpha}} E_{1}
$$

If $\lambda_{u}=\infty$, we simply drop the term $-C_{4}(\hat{x})$. Using this method, we can bound $\nabla \mathbf{u}$ by

$$
\begin{equation*}
E_{1} \cdot \sum_{i \leq N} C_{i}(\hat{x}) \lambda^{\beta_{i}}, \quad N=2, \beta=\left(-a_{\alpha},-b_{\alpha}+1 / 2\right), \quad|\hat{x}|=x_{c}, \quad x=\lambda \hat{x} \tag{7.29}
\end{equation*}
$$

for some piecewise constant $C_{1}(\hat{x}), C_{2}(\hat{x})$. We do not further optimize the estimate (7.16) over all small $a$ since the above estimate is good enough for our purpose and is simpler. For $\mathbf{u}$, we can obtain similar estimates and we only have the term $C_{1}(\hat{x}) \lambda^{-a_{\alpha}+1}$. Next, for the weight

$$
\begin{equation*}
\psi_{\lambda}(x) \leq \psi_{\infty, u}(x) \lambda^{b_{\alpha}} \tag{7.30}
\end{equation*}
$$

we estimate the piecewise bound

$$
(\psi \nabla \mathbf{u})_{\tau}(\hat{z}) \leq C_{1, i j} \tau^{b_{\alpha}-a_{\alpha}}+C_{2, i j} \tau^{1 / 2}
$$

for $\hat{z}$ in each grid $B_{i j}\left(h_{x}\right) \subset\left[0, x_{c 3}\right]^{2} \backslash\left[0, x_{c 2}\right]^{2}, \tau \leq \lambda_{*}, \alpha=1$ or $\tau \geq \lambda_{*}, \alpha=n$, which gives the desired estimate (7.28). To apply (7.29), we need to rescale $\tau \hat{z}=\lambda \tilde{z}$ with $|\tilde{z}|_{\infty}=x_{c}$. Without loss of generality, we assume $\hat{z}_{1} \geq \hat{z}_{2}$. For $\hat{z} \in B_{i j}\left(h_{x}\right)$, we have

$$
\lambda=\frac{|\tau \hat{z}|_{\infty}}{x_{c}}=\frac{\tau \hat{z}_{1}}{x_{c}} \in \tau\left[i h_{x} / x_{c},(i+1) h_{x} / x_{c}\right]
$$

Applying (7.29) and $\psi_{\tau}(\hat{z}) \leq \tau^{b_{\alpha}} \psi_{\infty, u}(\hat{z})$ (7.30), we yield

$$
\begin{equation*}
|(\psi \nabla \mathbf{u})(\tau \hat{z})| \leq E_{1} \sum_{l \leq N} \tau^{\beta_{l}+b_{\alpha}} C_{l}(\tilde{z}) \max \left(\left(i h_{x} / x_{c}\right)^{\beta_{l}},\left((i+1) h_{x} / x_{c}\right)^{\beta_{l}}\right) \psi_{\infty, u}(\hat{z}), \tilde{z}=\frac{\hat{z} \cdot x_{c}}{|\hat{z}|_{\infty}} \tag{7.31}
\end{equation*}
$$

We cover the range of $\tilde{z}$ by the intervals $\Gamma_{p, q}\left(x_{c}\right)$ (7.6) and apply the piecewise bound of $C_{i}(\hat{x})$ in $\Gamma_{p, q}\left(x_{c}\right)$ to bound $C_{i}(\tilde{z})$. Similarly, we derive the piecewise bound of $(\psi \nabla \mathbf{u})_{\tau}(\hat{z})$ for $\hat{z} \in \Gamma\left(x_{c 2}\right)$ (7.6). Since $\hat{z} / 2 \in \Gamma\left(x_{c 2} / 2\right)=\Gamma\left(x_{c}\right)$ and $\tau \hat{z}=(2 \tau) \cdot \hat{z} / 2$, applying the above estimates, we yield

$$
|(\psi \nabla \mathbf{u})(\tau \hat{z})| \leq E_{1} \sum_{l \leq N} \tau^{\beta_{l}+b_{\alpha}} C_{l}(\tilde{z}) 2^{\beta_{l}} \psi_{\infty, u}(\hat{z})
$$

Using the above estimate, we obtain the uniform bound for $z=\lambda \hat{z}$ with $|\hat{z}|=x_{c 2}$ and $\lambda \geq \lambda_{*}$

$$
\begin{equation*}
\max _{\hat{z} \in \Gamma_{k, i}\left(x_{c 2}\right), i} \frac{|(\psi \nabla \mathbf{u})(\lambda \hat{z})|}{|\lambda \hat{z}|_{\infty}^{1 / 2}} \leq E_{1} \max _{\hat{z} \in \Gamma_{k, i}\left(x_{c 2}\right), i} \sum_{l \leq N} \lambda^{\beta_{l}+b_{\alpha}-1 / 2} \frac{C_{l}(\hat{z})}{\left|x_{c 2}\right|^{1 / 2}} \psi_{\infty, u}(\hat{z}) 2^{\beta_{l}} \triangleq E_{1} \cdot F_{h a, k}(\lambda) \tag{7.32}
\end{equation*}
$$

From (7.29), it is easy to check that the power $\lambda^{\beta_{l}+b_{\alpha}-1 / 2}$ and $F_{i}(\lambda)$ are increasing in $\lambda$ for $\lambda \leq \lambda_{*}, \alpha=1$ and decreasing in $\lambda$ for $\lambda \geq \lambda_{*}, \alpha=n$. The functions $F_{h a, k}(\lambda)$ allow us to control $\psi \nabla \mathbf{u}(z)$ for $z_{2} \geq z_{1}$ and $z_{1} \leq z_{2}$ uniformly in $\lambda$.
7.4.3. The Hölder estimate with large distance. Now, we are in a position to perform $C_{x}^{1 / 2}$ estimate $\delta_{1}(f, x, z)$ with large $|x-z|$ relative to $|x|$. Denote $f=\psi \nabla \mathbf{u}$. Firstly, using (7.31), we can obtain piecewise bound of $f(\lambda \hat{z})$ for $\hat{z} \in\left[0, x_{c 3}\right]^{2} \backslash\left[0, x_{c 2}\right]^{2}$. Using the triangle inequality, we can bound

$$
\delta_{1}(f, \lambda \hat{x}, \lambda \hat{z}), \quad|\hat{x}|_{\infty}=x_{c 2}, \quad \hat{z}_{2}=\hat{x}_{2}, \quad \hat{z}_{1}-\hat{x}_{1} \leq m_{3} h_{x}
$$

uniformly in $\lambda$, where $m_{3}$ a parameter given in (7.6). Note that we have estimated $\delta_{1}(f, \lambda \hat{x}, \lambda \hat{z})$ based on Hölder regularity of $\omega$ for $|\hat{z}-\hat{x}| \leq \mu\left|x_{c 2}\right|$ at the beginning of Section 7.4.2 In such a case, we have two estimates and we will optimize them. These allow us to estimate $\delta_{1}(f, \hat{x}, \hat{z})$ with $|\hat{x}-\hat{z}| \leq m_{3} h_{x}=x_{c 3}-x_{c 2}$.

Finally, we consider the case

$$
\begin{equation*}
|\hat{z}-\hat{x}| \geq m_{3} h_{x} \geq x_{c 2}=|\hat{x}|_{\infty} \tag{7.33}
\end{equation*}
$$

We consider the far-field estimate $\lambda \geq \lambda_{*}$. The estimate in the near-field is similar. We have two estimates. In the first estimate, (1) if $\hat{x}_{1} \geq \hat{x}_{2}$, we get $\hat{z}_{1} \geq \hat{x}_{1}+m_{3} h_{x}>\hat{x}_{2}=\hat{z}_{2}$; (2) if $\hat{x}_{1} \leq \hat{x}_{2}$, we get $\hat{z}_{1} \geq \hat{x}_{1}+m_{3} h_{x}$. In case (l), we use (7.32) with $F_{h a, l}$ to bound $f_{\lambda}(\hat{x})$. In both cases, we use $\max _{s=1,2}\left(F_{h a, s}(\lambda \hat{z})\right.$ to bound $f(\lambda \hat{z})$. Using the triangle inequality and $|\hat{z}-\hat{x}| \geq\left|m_{3} h_{x}\right|^{1 / 2}$, we can bound $\delta_{1}(f, x, z)$.

We have an improved estimate for $x \in Q_{i_{1} j_{1}}, z \in Q_{i_{2} j_{1}}$ (7.3). Suppose that $|x|_{\infty} \geq|x|_{\infty, l},|z| \geq$ $|z|_{\infty, l}$. We consider two scenarios (1) $x_{1} \geq x_{2}$, (2) $x_{2} \leq x_{1}$. In case (1), we get $z_{1} \geq x_{1} \geq x_{2}=z_{2}$. Suppose that

$$
\begin{equation*}
|\hat{z}-\hat{x}|=\hat{z}_{1}-\hat{x}_{1} \geq \tau \hat{x}_{1} \tag{7.34}
\end{equation*}
$$

From (7.33), we have $\tau \gtrsim 1$. For example, if $x_{1} \in\left[y_{i}, y_{i+1}\right], z_{1} \in\left[y_{j}, y_{j+1}\right], j \geq i+1$ for the mesh (7.3), we get $\frac{\hat{z}_{1}-\hat{x}_{1}}{\hat{x}_{1}} \geq \frac{y_{j}}{y_{i+1}}-1$. Using (7.32) and the fact that $F_{h a, .}(\lambda)$ is decreasing, we yield

$$
\begin{aligned}
\delta_{1}(f, \lambda \hat{x}, \lambda \hat{z}) & \leq \frac{F_{h a, l}\left(\frac{|x|_{\infty}, l}{x_{c 2}}\right)|x|_{\infty}^{1 / 2}+F_{h a, l}\left(\frac{|z|_{\infty}, l}{x_{c 2}}\right)|x|_{\infty}^{1 / 2}}{|x-z|^{1 / 2}} \cdot E_{1} \\
& \leq\left(F_{h a, 1}\left(\frac{|x|_{\infty, l}}{x_{c 2}}\right)\left(\frac{\hat{x}_{1}}{|\hat{z}-\hat{x}|}\right)^{1 / 2}+F_{h a, 1}\left(\frac{|z|_{\infty, l}}{x_{c 2}}\right)\left(\frac{\hat{z}_{1}}{|\hat{z}-\hat{x}|}\right)^{1 / 2}\right) E_{1} .
\end{aligned}
$$

From (7.34), we derive

$$
\frac{\hat{x}_{1}}{\hat{z}_{1}-\hat{x}_{1}} \leq \tau^{-1}, \quad \hat{z}_{1} \geq(1+\tau) \hat{x}_{1}, \quad \hat{z}_{1}-\hat{x}_{1} \geq\left(1-\frac{1}{\tau+1}\right) \hat{z}_{1}=\frac{\tau}{\tau+1} \hat{z}_{1}, \quad \frac{\hat{z}_{1}}{\hat{z}_{1}-\hat{x}_{1}} \leq \frac{\tau+1}{\tau}
$$

and the upper bounds of $\frac{\hat{x}_{1}}{\hat{z}_{1}-\hat{x}_{1}}, \frac{\hat{z}_{1}}{z_{1}-\hat{x}_{1}}$ are decreasing in $\tau$. Combining the above two estimates, we yield the estimate of $\delta_{1}(f, x, z)$.

In case (2) $x_{1} \leq x_{2}$, since $z_{1} \geq x_{1}+x_{c 3}-x_{c 2} \geq 2 x_{c 2}>x_{2}=z_{2}$, we get $|z|_{\infty}=z_{1}$ and

$$
\begin{equation*}
\delta_{1}(f, \lambda \hat{x}, \lambda \hat{z}) \leq E_{1} \max _{s=1,2} F_{h a, s}\left(\frac{|x|_{\infty, l}}{x_{c 2}}\right)\left(\frac{\hat{x}_{2}}{\left|\hat{z}_{1}-\hat{x}_{1}\right|}\right)^{\frac{1}{2}}+E_{1} \max _{s=1,2} F_{h a, s}\left(\frac{|z|_{\infty, l}}{x_{c 2}}\right)\left(\frac{\hat{z}_{1}}{\left|\hat{z}_{1}-\hat{x}_{1}\right|}\right)^{\frac{1}{2}} \tag{7.35}
\end{equation*}
$$

We further bound the ratio. Suppose that $\hat{x}_{2} / \hat{x}_{1} \in\left[y_{l} / x_{u}, y_{u} / x_{l}\right]$. Since $\hat{x}_{2}=|\hat{x}|=x_{c 2}$ and $m_{3} h_{x} \geq \hat{x}_{2}$ (7.33), we get
$\frac{\hat{z}_{1}}{\hat{x}_{2}} \geq \frac{\hat{x}_{1}+m_{3} h_{x}}{\hat{x}_{2}} \geq \frac{x_{l}}{y_{u}}+\frac{m_{3} h_{x}}{x_{c 2}}, \frac{\hat{z}_{1}}{\hat{x}_{1}} \geq 1+\frac{m_{3} h_{x}}{\hat{x}_{1}} \geq 1+\max \left(\frac{m_{3} h_{x}}{x_{c 2}}, \frac{\hat{x}_{2}}{\hat{x}_{1}}\right) \geq 1+\max \left(\frac{m_{3} h_{x}}{x_{c 2}}, \frac{y_{l}}{x_{u}}\right)$.
Combining the above estimates and using

$$
\frac{\hat{x}_{2}}{\left|\hat{z}_{1}-\hat{x}_{1}\right|}=\frac{\hat{x}_{2} / \hat{z}_{1}}{1-\hat{x}_{1} / \hat{z}_{1}}, \quad \frac{\hat{z}_{1}}{\left|\hat{z}_{1}-\hat{x}_{1}\right|}=\frac{1}{1-\hat{x}_{1} / \hat{z}_{1}}
$$

we obtain the bound for $\delta_{1}(f, \lambda \hat{x}, \lambda \hat{z})$. From the estimates of the above two cases, if $\left|\hat{z}_{1}-\hat{x}_{1}\right| \gg$ $|\hat{x}|_{\infty}$, we have $\left|\delta_{1}(f, x, z)\right| \leq C \max _{s} F_{h a, s}\left(\frac{|z|_{\infty, l}}{x_{c 2}}\right)$ with $C \approx 1$.

Covering the large domain. Using the above Hölder estimate $\delta_{1}(f, \lambda \hat{x}, \lambda \hat{z})$ for $\hat{x}, \hat{z}$ uniformly in $\lambda$ and the covering argument in Section 7.4.1 we can obtain piecewise estimate

$$
\delta_{1}(f, x, z), \quad x \in Q_{i_{1} j_{1}}, z \in Q_{i_{2} j_{1}}, \quad i_{1} \leq i_{2} \leq i_{1}+m
$$

on the mesh (7.3) for large $\max \left(i_{1}, j_{1}\right)$. For $x \in Q_{i_{1} j_{1}}, z \in Q_{i_{2} j_{1}}$, we can derive the upper and lower bounds for $x_{i}, z_{i}$ and can estimate the ratio $\hat{z}_{1} / \hat{x}_{1}$ and $\hat{x}_{2} / \hat{x}_{1}$ used in the above estimates (7.34), (7.35) for large $\left|\hat{z}_{1}-\hat{x}_{1}\right|$.

Infinite region. To estimate $\delta_{1}(f, x, z)$ with $|x|_{\infty} \geq R$ with sufficiently large $R$, we simply apply the estimate in Section 7.4 .2 for small distance $|\hat{x}-\hat{z}|$, the estimate above (7.33) for $|\hat{x}-\hat{z}| \leq m_{3} h_{x}$, and the first estimate below (7.33) for $|\hat{x}-\hat{z}| \geq m_{3} h_{x}$. Since we do not have upper bound for $x, z$, we do not apply the improved estimate in Section 7.4.3,

The $C_{y}^{1 / 2}$ estimates for $\nabla \mathbf{u}$ are completely similar.
Hölder estimate of $\mathbf{u}$. The Hölder estimates of $\mathbf{u}$ are completely similar. From (7.18), (7.2), and Section (7.2), we can obtain Hölder estimate of $\psi_{u, \lambda} \tilde{\mathbf{u}}_{\lambda}$ as follows

$$
\frac{\left|\psi_{u, \lambda} \mathbf{u}_{\lambda}(\hat{x})-\psi_{u, \lambda} \mathbf{u}_{\lambda}(\hat{z})\right|}{|\lambda \hat{x}-\lambda \hat{z}|^{1 / 2}} \leq \lambda^{c_{\alpha}-a_{\alpha}+1 / 2} C(\hat{x}, \hat{z})
$$

uniformly for $\lambda \leq \lambda_{*}, \alpha=1$ or $\lambda \geq \lambda_{*}, \alpha=n$ for $|\hat{x}|_{\infty}=x_{c 2},|\hat{x}-\hat{z}| \leq \mu x_{c 2}$, see (7.17). We lose a power $\lambda^{1 / 2}$ due to (7.2). The power $c_{1}, c_{n}$ are given in (7.7). In particular, we have

$$
\begin{equation*}
c_{1}-a_{1}+\frac{1}{2} \geq-\frac{5}{2}+\frac{5}{2}+\frac{1}{2}>0, \quad c_{n}-a_{n}+\frac{1}{2} \leq-\frac{7}{6}+\frac{1}{2}+\frac{1}{2}<0 \tag{7.36}
\end{equation*}
$$

In the near-field $\lambda \leq \lambda_{*}$ or in the far-field $\lambda \geq \lambda_{*}$, we have a uniform estimate $\lambda^{c_{\alpha}-a_{\alpha}+1 / 2} \leq$ $\lambda_{*}^{c_{\alpha}-a_{\alpha}+1 / 2}$. For a large distance, we apply $L^{\infty}$ estimate similar to that in Section 7.4.3, We can obtain a piecewise $L^{\infty}$ estimate similar to the estimate above (7.32) as follows

$$
\left|\left(\psi_{u} \mathbf{u}\right)(\lambda \hat{z})\right| \leq E_{1} \lambda^{c_{\alpha-a_{\alpha}+1}} C(\hat{z}), \quad \hat{z} \in\left[0, x_{c 3}\right]^{2} \backslash\left[0, x_{c 2}\right]^{2}
$$

and another estimate similar to (7.32). For $\mathbf{u}$, since the kernel is locally integrable, we only have one term related to $\|\omega \varphi\|_{\infty}$ in the above bound. From the power law (7.36) and the scaling relation (7.2), using the argument for the Hölder estimate of $\nabla \mathbf{u}$, we obtain Hölder estimate $\delta_{i}(f, \lambda \hat{x}, \lambda \hat{z})$ for large $|\hat{x}-\hat{z}|$ uniformly in $\lambda$.
7.5. Estimate using other norms. Similar to Section 6.3, we estimate $\mathbf{u}_{A},(\nabla \mathbf{u})_{A}$ using another combiniation of norms, e.g. $\left.\| \omega \varphi_{1, g}\right] \|_{\infty},\left[\omega \psi_{1}\right]_{C_{x_{i}}^{1 / 2}}$. We use the estimates in previous sections of Section 7 to obtain the estimates in the near-field and the far-field and bound different norms using the energy (6.14) or the direct estimate for the error (6.15). In addition to the $L^{\infty}$ estimate mentioned in Section 6.3, we use $\left\|\varepsilon \varphi_{\text {elli }}\right\|_{\infty},\left[\varepsilon \psi_{1}\right]_{C_{x_{i}}^{1 / 2}}$ for the Hölder estimate of $\mathbf{u}_{A}(\varepsilon),(\nabla \mathbf{u})_{A}(\varepsilon)$, where $\varepsilon=\bar{\omega}-\bar{\phi}^{N}, \varepsilon=\hat{\omega}-\hat{\phi}^{N}$ is the error of solving the Poisson equations. We use these estimates to control the nonlocal error.

## 8. Estimating the piecewise bounds of functions

In this appendix, we estimate $\|f\|_{L^{\infty}(D)}$ in the domain $D=[a, b] \times[c, d]$ given the grid point values of $f$ in $D$ and the derivatives bound $\left\|\partial_{x}^{i} \partial_{y}^{j} f\right\|_{L^{\infty}}$. We want to obtain an error term as small as possible without evaluating $f$ on too many grid points and its high order derivatives, which are expensive for some complicated function $f$, e.g. the residual error $f=\left(\partial_{t}-\mathcal{L}\right) \widehat{W}$ in Section 3 in Part II [2]. Based on these $L^{\infty}$ estimates, we further develop the Hölder estimate in Appendix E in Part II [2]. We use these estimates to verify the smallness of the residual error, e.g. $f=\left(\partial_{t}-\mathcal{L}\right) \widehat{W}$, in suitable energy norm.

We will develop three estimates based on the Newton polynomial, the Lagrangian interpolating polynomial, and the Hermite interpolation. Each method has its own advantages. For the Newton and the Lagrangian method, to obtain 4-th order error estimates, we only need to evaluate $4 \times 4$ grid point values of $f$.
(a) For the Newton method, we have a sharp error bound with a much smaller constant than that of the Lagrangian method.
(b) For the Lagrangian method, it is easier and more efficient to estimate the Lagrangian interpolating polynomials for grid points $\left(x_{i}, y_{j}\right)$ in a general position.

In some situation, we need to estimate both $f$ and $\nabla f$. We use grid point values $f\left(x_{i}, y_{j}\right)$ and $\nabla f\left(x_{i}, y_{j}\right)$ to build 4 -th and 5 -th order interpolating polynomials based on the Hermite interpolation. The 4 -th order error estimate is as sharp as the Newton method. Moreover, in the 4 -th order Newton or Lagrangian interpolation, we need $f \in C^{4}\left[x_{0}, x_{3}\right]$. When $f$ is only piecewise smooth in $\left[x_{i}, x_{i+1}\right], i \leq 3$, we cannot use these two methods. Instead, we evaluate $f\left(x_{i}\right), f^{\prime}\left(x_{i}\right)$ and construct the Hermite interpolation in each interval $\left[x_{i}, x_{i+1}\right]$. One disadvantage is that the estimate of the interpolating polynomials is more complicated and takes longer time in practice.

We do not pursue higher order error estimates since most of these estimates are applied to estimate the residual errors, e.g. $\left(\partial_{t}-\mathcal{L}\right) \widehat{W}$ in Section $3[2$, which is only piecewise smooth, and we do not use very small $h$ in the whole computational domain.
8.1. Estimates based on the Newton polynomials. Given $x_{0}, x_{1}, x_{2}, . ., x_{k}$, we first define the divided differences recursively
$f[x]=f(x), \quad f[x, y]=\frac{f(y)-f(x)}{y-x}, \quad f\left[x_{i}, x_{1}, . ., x_{j+1}\right]=\frac{f\left[x_{i+1}, x_{i+2}, . ., x_{j+1}\right]-f\left(x_{i}, x_{i+1}, . ., x_{j}\right)}{x_{j+1}-x_{i}}$.
8.1.1. The Newton polynomials in $1 D$. We first discuss how to bound $f(x)$ in 1 D . We consider the domain $[a, b]$ and denote

$$
z_{0}=a, \quad h=(b-a) / 3, \quad z_{i}=z_{0}+i h
$$

Denote

$$
\begin{align*}
& D_{1, i} f=f\left(z_{i+1}\right)-f\left(z_{i}\right), i=0,1,2, \quad D_{2, i} f=D_{1, i+1} f-D_{1, i} f, i=0,1 \\
& D_{3} f=D_{2,1} f-D_{2,0} f=f\left(z_{3}\right)-3 f\left(z_{2}\right)+3 f\left(z_{1}\right)-f\left(z_{0}\right) \tag{8.1}
\end{align*}
$$

Let $\left\{x_{i}\right\}_{i=0}^{3}$ be a permutation of $\left\{z_{i}\right\}_{i=0}^{3}$. We construct the Newton polynomial

$$
\begin{align*}
P(x)= & \left(f\left[x_{0}\right]+f\left[x_{0}, x_{1}\right]\left(x-x_{0}\right)\right)+f\left[x_{0}, x_{1}, x_{2}\right]\left(x-x_{0}\right)\left(x-x_{1}\right) \\
& +f\left[x_{0}, x_{1}, x_{2}, x_{3}\right]\left(x-x_{0}\right)\left(x-x_{1}\right)\left(x-x_{2}\right) \triangleq l(x)+q(x)+c(x), \tag{8.2}
\end{align*}
$$

where $l(x), q(x), c(x)$ denote the linear, quadratic, and the cubic parts, respectively. We remark that the above Newton polynomial agrees with the Lagrangian polynomial interpolating $\left(z_{i}, f\left(z_{i}\right)\right)$.

By standard error analysis of the Newton interpolation, the error part $R(x)$ can be bounded as follows

$$
\begin{equation*}
|f(x)-P(x)| \leq \frac{1}{24}\left\|\partial_{x}^{4} f\right\|_{L^{\infty}[a, b]} \max _{x \in[a, b]}\left|\Pi_{0 \leq i \leq 3}\left(x-x_{i}\right)\right|=\frac{1}{24}\left\|\partial_{x}^{4} f\right\|_{L^{\infty}[a, b]} \frac{(b-a)^{4}}{81} \tag{8.3}
\end{equation*}
$$

To obtain the last equality, using the definition of $x_{i}, z_{i}$, we write $z=a+t h, t \in[0,3]$ and get

$$
\begin{equation*}
\max _{x \in[a, b]}\left|\Pi_{0 \leq i \leq 3}\left(x-x_{i}\right)\right|=\max _{z \in[a, b]}\left|\Pi_{0 \leq i \leq 3}\left(z-z_{i}\right)\right|=\max _{t \in[0,3]} h^{4}\left|\Pi_{0 \leq i \leq 3}(t-i)\right| \leq h^{4} \tag{8.4}
\end{equation*}
$$

where we have used (8.71) in Lemma 8.1 in the last inequality.
To bound $f(x)$, given the derivative bound of $f$ and the above estimate, we only need to control $P(x)$. We choose different permutation $\left\{x_{i}\right\}_{i=0}^{3}$ of $\left\{z_{i}\right\}_{i=0}^{3}$ for $z$ in different part of $[a, b]$ :

$$
\begin{aligned}
& x_{i}=z_{i}, z \in\left[z_{0}, z_{1}\right], \quad\left(x_{0}, x_{1}, x_{2}, x_{3}\right)=\left(z_{2}, z_{1}, z_{0}, z_{3}\right), z \in\left[z_{1}, z_{2}\right], \\
& \left(x_{0}, x_{1}, x_{2}, x_{3}\right)=\left(z_{3}, z_{2}, z_{1}, z_{0}\right), z \in\left[z_{2}, z_{3}\right] .
\end{aligned}
$$

Let $I_{z}$ be the interval with endpoints $x_{0}, x_{1}$. We have $z \in I_{z}$. Since $l(x)$ in (8.2) is linear with $l\left(x_{i}\right)=f\left(x_{i}\right)$ and $\left|\left(x-x_{0}\right)\left(x-x_{1}\right)\right| \leq \frac{\left(x_{1}-x_{0}\right)^{2}}{4}$, we get

$$
|l(z)| \leq \max \left(\left|f\left(x_{0}\right)\right|,\left|f\left(x_{1}\right)\right|\right), \quad \max _{z \in I_{z}}|q(z)| \leq\left|f\left[x_{0}, x_{1}, x_{2}\right]\right| \frac{\left(x_{1}-x_{0}\right)^{2}}{4}=\left|f\left[x_{0}, x_{1}, x_{2}\right]\right| \frac{h^{2}}{4}
$$

Since $x_{0}, x_{1}, x_{2}$ are three consecutive points with distance $h$ and $I \triangleq\left[\min \left(x_{0}, x_{1}, x_{2}\right), \max \left(x_{0}, x_{1}, x_{2}\right)\right]$ covers $z$, we have

$$
\begin{equation*}
\max _{z \in I}\left|\left(z-x_{0}\right)\left(z-x_{1}\right)\left(z-x_{2}\right)\right|=\max _{z \in[0,2 h]}|z(z-h)(z-2 h)| \leq \frac{2}{3 \sqrt{3}} h^{3} \tag{8.5}
\end{equation*}
$$

where we have used (8.69) in Lemma 8.1 in the last inequality.
Next, we use (8.1) to simplify $f\left[x_{i}, x_{i+1}, . ., x_{j}\right]$. For each case, a direct calculation yields

$$
\begin{aligned}
& z \in\left[z_{0}, z_{1}\right]:\left|f\left[z_{0}, z_{1}, z_{2}\right]\right|=\left|\frac{f\left[z_{2}, z_{1}\right]-f\left[z_{1}, z_{0}\right]}{z_{2}-z_{0}}\right|=\left|\frac{1}{2 h^{2}}\left(D_{1,1} f-D_{1,0} f\right)\right|=\frac{1}{2 h^{2}}\left|D_{2,0} f\right|, \\
& z \in\left[z_{1}, z_{2}\right]:\left|f\left(z_{2}, z_{1}, z_{0}\right)\right|=\left|\frac{f\left[z_{0}, z_{1}\right]-f\left[z_{1}, z_{2}\right]}{z_{0}-z_{2}}\right|=\left|\frac{1}{2 h^{2}}\left(D_{1,1} f-D_{1,0} f\right)\right|=\frac{1}{2 h^{2}}\left|D_{2,0} f\right|
\end{aligned}
$$

Similarly, for $z \in\left[z_{2}, z_{3}\right]$, we get

$$
\left|f\left(z_{3}, z_{2}, z_{1}\right)\right|=\frac{1}{2 h^{2}}\left|D_{2,1} f\right|
$$

A direct calculation yields $\left|f\left[x_{0}, x_{1}, x_{2}, x_{3}\right]\right|=\frac{1}{6 h^{3}}\left|D_{3} f\right|$ for any permutation $\left\{x_{i}\right\}$ of $\left\{z_{i}\right\}$. Thus, for $c(x)$ (8.2), using (8.5) and the above estimate, we get

$$
\max _{z \in[a, b]}|c(x)| \leq \frac{2}{3 \sqrt{3}} h^{3}\left|f\left[x_{0}, x_{1}, x_{2}, x_{3}\right]\right|=\frac{2}{3 \sqrt{3}} h^{3} \cdot \frac{1}{6 h^{3}}\left|D_{3} f\right|=\frac{1}{9 \sqrt{3}}\left|D_{3} f\right|
$$

Combining the above estimates, we obtain
$|P(x)| \leq \max \left(\max _{i=0,1,2}\left|f\left(z_{i}\right)\right|+c_{1}\left|D_{2,0} f\right|, \max _{i=1,2,3}\left|f\left(z_{i}\right)\right|+c_{1}\left|D_{2,1} f\right|\right)+c_{2}\left|D_{3} f\right|, c_{1}=\frac{1}{8}, c_{2}=\frac{1}{9 \sqrt{3}}$.
8.1.2. A quadratic interpolation. We also need a cubic interpolation in the Hermite interpolation in Section 8.4. Given $x_{0}<x_{1}<x_{2}$ with $x_{2}-x_{1}=x_{1}-x_{0}=h$, we define

$$
\begin{equation*}
N_{2}\left(f, x_{0}, x_{1}, x_{2}\right)(x) \triangleq f\left(x_{0}\right)+f\left[x_{0}, x_{1}\right]\left(x-x_{0}\right)+f\left[x_{0}, x_{1}, x_{2}\right]\left(x-x_{0}\right)\left(x-x_{1}\right) \tag{8.7}
\end{equation*}
$$ and construct $P(x)=N_{2}\left(f, x_{0}, x_{1}, x_{2}\right)(x)$.

We have an error estimate similar to (8.3) for $x \in\left[x_{0}, x_{2}\right], h=x_{1}-x_{0}$

$$
\begin{equation*}
|P(x)-f(x)| \leq \frac{\left\|\partial_{x}^{3} f\right\|_{L^{\infty}\left[x_{0}, x_{2}\right]}}{6} \max _{x \in\left[x_{0}, x_{2}\right]}\left|\Pi_{i=0,1,2}\left(x-x_{i}\right)\right| \leq \frac{\left\|\partial_{x}^{3} f\right\|_{L^{\infty}\left[x_{0}, x_{2}\right]}}{6} \frac{2 h^{3}}{3 \sqrt{3}}=\frac{\left\|\partial_{x}^{3} f\right\|_{L^{\infty}\left[x_{0}, x_{2}\right]} h^{3}}{9 \sqrt{3}} \tag{8.8}
\end{equation*}
$$

where we have used (8.5) in the last inequality.
Using the same estimates in Section 8.1.1 for the linear part and quadratic part, we obtain

$$
\begin{align*}
& x \in\left[x_{0}, x_{1}\right]:|P(x)| \leq \max \left(\left|f\left(x_{0}\right)\right|,\left|f\left(x_{1}\right)\right|\right)+\frac{1}{8}\left|f\left(x_{0}\right)-2 f\left(x_{1}\right)+f\left(x_{2}\right)\right|,  \tag{8.9}\\
& x \in\left[x_{0}, x_{1}\right]:|P(x)| \leq \max \left(\left|f\left(x_{0}\right)\right|,\left|f\left(x_{1}\right)\right|\right)+\frac{1}{8}\left|f\left(x_{0}\right)-2 f\left(x_{1}\right)+f\left(x_{2}\right)\right|
\end{align*}
$$

Note that using the notation (8.1), we have $\left|D_{2,0} f\right|=\left|f\left(x_{0}\right)-2 f\left(x_{1}\right)+f\left(x_{2}\right)\right|$.
8.1.3. Generalization to 2D. Denote

$$
D=[a, b] \times[c, d], \quad x_{i}=a+i h_{1}, \quad y_{j}=c+j h_{2}, \quad h_{1}=(b-a) / 3, \quad h_{2}=(d-c) / 3
$$

Suppose that $f\left(x_{i}, y_{j}\right)$ and $\left\|\partial_{x}^{k} \partial^{l} f\right\|_{L^{\infty}}, k+l \leq 4$ are given. Firstly, we treat $y$ as a parameter and interpolate $f(x, y)$ in $x$. Denote

$$
\begin{aligned}
& D_{i, 1} f(y)=f\left(x_{i+1}, y\right)-f\left(x_{i}, y\right), 0 \leq i \leq 2, \quad D_{i, 2} f(y)=D_{i+1,1} f(y)-D_{i, 1} f(y), 0 \leq i \leq 1 \\
& D_{3} f(y)=D_{2,1} f(y)-D_{2,0} f(y)
\end{aligned}
$$

Applying (8.3), (8.6), we get

$$
\begin{aligned}
\max _{x \in[a, b]}|f(x, y)| \leq & \max \left(\max _{i=0,1,2}\left|f\left(x_{i}, y\right)\right|+c_{1}\left|D_{2,0} f(y)\right|, \max _{i=1,2,3}\left|f\left(x_{i}, y\right)\right|+c_{1}\left|D_{2,1} f(y)\right|\right) \\
& +c_{2}\left|D_{3} f(y)\right|+\frac{1}{24} h_{1}^{4}| | \partial_{x}^{4} f \|_{L^{\infty}(D)} .
\end{aligned}
$$

Note that $f\left(x_{i}, y\right), D_{i, j} f(y)$ are 1D functions in $y$ and their grid point values on $y_{j}$ can be obtained from $f\left(x_{i}, y_{j}\right)$. We further apply (8.3), (8.6) to estimating $\|g\|_{L^{\infty}[c, d]}$ with $g=$ $f\left(x_{i}, y\right), D_{i, j} f(y)$. Maximizing the above estimate over $y$ yields the bound for $f$.
8.2. Estimates based on the Lagrangian interpolation. If the grid points $x_{i}$ are not equispaced, the estimates of the Newton polynomials can be more complicated. We develop another estimate based on the Lagrangian interpolating polynomials. Although these two interpolating polynomials on ( $x_{i}, f\left(x_{i}\right)$ ) are equivalent, the Lagrangian formulation is easier to estimate.

Firstly, let $p_{i}(x), q_{j}(y)$ be the Lagrange interpolating polynomials associated to the points $x_{1}<. .<x_{k} \in[a, b], y_{1}<y_{2}<\ldots<y_{k} \in[c, d]$

$$
\begin{equation*}
p_{i}(x)=\Pi_{j \neq i} \frac{x-x_{j}}{x_{i}-x_{j}}, \quad q_{i}(y)=\Pi_{j \neq i} \frac{y-y_{j}}{y_{i}-y_{j}} \tag{8.10}
\end{equation*}
$$

For any $(x, y) \in D$, we consider the following decomposition by first interpolating $f$ in $x$ and then in $y$

$$
\begin{align*}
f(x, y) & =\sum_{i=1}^{k} f\left(x_{i}, y\right) p_{i}(x)+R_{1}(x, y)=\sum_{i=1}^{k}\left(\sum_{j=1}^{k} f\left(x_{i}, y_{j}\right) q_{j}(y)+R_{2}\left(x_{i}, y\right)\right) p_{i}(x)+R_{1}(x, y)  \tag{8.11}\\
& =\sum_{i, j=1}^{k} p_{i}(x) q_{j}(y) f\left(x_{i}, y_{j}\right)+\left(\sum_{i} R_{2}\left(x_{i}, y\right) p_{i}(x)+R_{1}(x, y)\right) \triangleq I+I I .
\end{align*}
$$

By standard error analysis of the Lagrange interpolation, the error part $R_{1}(x, y), R_{2}(x, y)$ can be bounded as follows

$$
\begin{gather*}
\left|R_{1}(x, y)\right| \leq \frac{1}{k!}\left\|\partial_{x}^{k} f(x, y)\right\|_{L^{\infty}(D)} \max _{x \in[a, b]}\left|\Pi_{i=1}^{k}\left(x-x_{i}\right)\right| \\
\left|R_{2}\left(x_{i}, y\right)\right| \leq \frac{1}{k!}\left\|\partial_{y}^{k} f(x, y)\right\|_{L^{\infty}(D)} \max _{y \in[c, d]}\left|\Pi_{i=1}^{k}\left(y-y_{i}\right)\right| \tag{8.12}
\end{gather*}
$$

Denote

$$
\begin{equation*}
C_{1}=\max _{x \in[a, b]} \sum_{i=1}^{k}\left|p_{i}(x)\right|, \quad a_{i j}=f\left(x_{i}, y_{j}\right) \tag{8.13}
\end{equation*}
$$

Note that the value $C_{1}$ only depends on the ratio $\frac{x_{i+1}-x_{i}}{b-a}, i=1, . ., k$, since from (8.10), we have

$$
p_{i}(x)=\Pi_{j \neq i} \frac{x-x_{j}}{x_{i}-x_{j}}=\Pi_{j \neq i} \frac{t-t_{j}}{t_{i}-t_{j}}, \quad t=\frac{x-a}{b-a}, \quad t_{j}=\frac{x_{j}-a}{b-a}
$$

We will choose the grid points with $\frac{x_{i+1}-x_{i}}{b-a}=\frac{y_{i+1}-y_{i}}{d-c}$ so that we also have $C_{1}=\max _{y \in[c, d]} \sum_{i=1}^{k}\left|q_{j}(y)\right|$. See (8.17). We have the following trivial estimate for any $c_{j}$

$$
\begin{equation*}
\left|\sum_{i} p_{i}(x) c_{i}\right| \leq \sum_{i}\left|p_{i}(x)\right| \max \left|c_{i}\right| \leq C_{1} \max \left|c_{i}\right|, \quad\left|\sum_{j} q_{j}(y) c_{j}\right| \leq C_{1} \max \left|c_{i}\right| \tag{8.14}
\end{equation*}
$$

Next, we estimate $I$. Since $\sum_{i} p_{i}(x)=\sum_{j} q_{j}(y)=1$, we expect that $I \approx f\left(x_{i}, y_{j}\right)+$ $O\left(\max \left(h_{1}, h_{2}\right)\right)$, where $h_{1}=b-a, h_{2}=d-c$. Thus, for some $m$ to be chosen, we further decompose it into the mean and the variation and apply (8.14) to obtain

$$
\begin{aligned}
|I| & =\left|m+\sum_{i, j=1}^{k} p_{i}(x) q_{j}(y)\left(a_{i j}-m\right)\right| \leq|m|+\max _{i, j}\left|a_{i j}-m\right| \sum_{i}\left|p_{i}(x)\right| \sum_{j}\left|q_{j}(y)\right| \\
& =|m|+C_{1}^{2} \max _{i, j}\left|a_{i j}-m\right|
\end{aligned}
$$

We use the following trivial equality for $b_{1}, b_{2}, . ., b_{n}$

$$
\begin{equation*}
\max _{i}\left|b_{i}-b\right|=\frac{1}{2}\left(\max b_{i}-\min b_{i}\right), \quad b=\frac{1}{2}\left(\max _{i} b_{i}+\min _{i} b_{i}\right) \tag{8.15}
\end{equation*}
$$

which can be proved by ordering $b_{i}$. Thus, we optimize the estimate of $I$ by choosing $m=$ $\frac{1}{2}\left(\max _{i, j} a_{i j}+\min _{i, j} a_{i j}\right)$.

We can obtain a sharper estimate as follows

$$
\begin{align*}
|I| & =\left|\sum_{j} q_{j}(y)\left(\bar{a}_{j}+\sum_{i} p_{i}(x)\left(a_{i j}-\bar{a}_{j}\right)\right)\right|=\left|\sum_{j} q_{j}(y)\left(\bar{a}_{j}+S_{j}(x)\right)\right|  \tag{8.16}\\
& \leq|\bar{a}|+\left|\sum_{j} q_{j}(y)\left(\bar{a}_{j}-\bar{a}\right)\right|+\left|\sum_{j} q_{j}(y) S_{j}(x)\right|, \quad S_{j}(x)=\sum_{i} p_{i}(x)\left(a_{i j}-\bar{a}_{j}\right) .
\end{align*}
$$

For a fixed $j$, we choose

$$
\bar{a}_{j}=\frac{\max _{i} a_{i j}+\min _{i} a_{i j}}{2}, \quad \bar{a}=\frac{\max _{j} \bar{a}_{j}+\min _{j} \bar{a}_{j}}{2} .
$$

Applying (8.14), (8.15), and the definition of $S_{j}(x)$ in (8.16), we yield

$$
\left|\sum_{j} q_{j}(y) S_{j}(x)\right| \leq C_{1} \cdot \max _{j}\left|S_{j}(x)\right|, \quad\left|S_{j}(x)\right| \leq C_{1} \max _{i}\left|a_{i j}-\bar{a}_{j}\right|=\frac{C_{1}}{2}\left|\max _{i} a_{i j}-\min _{i} a_{i j}\right|
$$

Similarly, we have

$$
\left|\sum_{j} q_{j}(x)\left(\bar{a}_{j}-\bar{a}\right)\right| \leq C_{1} \max _{j}\left|\bar{a}_{j}-\bar{a}\right|=\frac{C_{1}}{2}\left(\max _{j} \bar{a}_{j}-\min _{j} \bar{a}_{j}\right)
$$

Combining two parts, we yield an improved estimate for $|I|$

$$
|I| \leq \frac{1}{2}\left|\max _{j} \bar{a}_{j}+\min _{j} \bar{a}_{j}\right|+\frac{C_{1}}{2}\left(\max _{j} \bar{a}_{j}-\min _{j} \bar{a}_{j}\right)+\frac{C_{1}^{2}}{2} \max _{j}\left|\max _{i} a_{i j}-\min _{i} a_{i j}\right| .
$$

The above estimate is better if $a_{i j}=f\left(x_{i}, y_{j}\right)$ is smooth in $x$. Similarly, we can first sum over $p_{i}(x)$ and then sum over $q_{j}(y)$ in (8.16) to obtain another improved estimate.

We apply the above method to the fourth and third order estimate of $f$ on $[a, b] \times[c, d]$. We choose

$$
\left(x_{1}, x_{2}, x_{3}, x_{4}\right)=\left(a, a+\frac{1}{3} h_{1}, a+\frac{2}{3} h_{1}, b\right), \quad\left(y_{1}, y_{2}, y_{3}, y_{4}\right)=\left(c, c+\frac{1}{3} h_{2}, c+\frac{2}{3} h_{2}, d\right)
$$

for the fourth order estimate, and

$$
\begin{equation*}
\left(x_{1}, x_{2}, x_{3}\right)=\left(a+\frac{h_{1}}{32}, a+\frac{h_{1}}{2}, a+\frac{31}{32} h_{1}\right), \quad\left(y_{1}, y_{2}, y_{3}\right)=\left(c+\frac{h_{2}}{32}, c+\frac{h_{2}}{2}, c+\frac{31}{32} h_{2}\right) \tag{8.17}
\end{equation*}
$$

for the third order estimate, where $h_{1}=b-a, h_{2}=d-c$. To estimate the constant $C_{1}$ (8.13) in each case, since the interpolation polynomial $\left.p_{i}(x) 8.10\right)$ has degree at most 3 and $C_{1}$ only depends on the ratio $\frac{x_{i+1}-x_{i}}{b-a}$, we can first assume $I=[a, b]=[0,3]$ or $[0,1]$ and partition $I$ into $N$ small sub-intervals $I_{j}$ with $\left|I_{j}\right|=|I| / N$. Then we estimate the piecewise bound of $p_{i}(x)$ in $I_{j}$ using (8.6) and then $\sum\left|p_{j}\right|$ using the triangle inequality. We get

$$
C_{1}^{(4)} \leq 1.635, \quad C_{2}^{(3)} \leq 1.276
$$

for the fourth order and third order case, respectively. To estimate the 3 rd order error term in (8.12), we use (8.6) again to estimate the third order polynomial $p(t)$ below and yield

$$
\begin{aligned}
& \max _{x \in[a, b]} \Pi_{i=1}^{3}\left|x-x_{i}\right|=h_{1}^{3} \max _{t \in[0,1]}|p(t)|=C_{3} h_{1}^{3}, \max _{x \in[a, b]} \Pi_{i=1}^{3}\left|y-y_{i}\right|=C_{3} h_{2}^{3}, p(t)=\left(t-\frac{1}{32}\right)\left(t-\frac{1}{2}\right)\left(t-\frac{31}{32}\right), \\
& C_{3}^{(3)} \leq 0.0397
\end{aligned}
$$

For the fourth order error term, since $\left\{x_{i}\right\}$ are equi-spacing, following (8.4), we get

$$
\max _{x \in[a, b]} \prod_{i=1}^{4}\left|x-x_{i}\right|=\left(\frac{h_{1}}{3}\right)^{4} \max _{t \in[0,3]}|t(t-1)(t-2)(t-3)| \leq\left(\frac{h_{1}}{3}\right)^{4}
$$

Using (8.12), (8.14), and the above estimate, for the 3rd order estimate, we get

$$
|I I| \leq C_{2}^{(3)} \max _{i}\left\|R_{2}\left(x_{i}, y\right)\right\|_{L^{\infty}}+\left\|R_{1}\right\|_{L^{\infty}} \leq \frac{1}{6} \cdot C_{3}^{(3)}\left(C_{2}^{(3)}\left\|\partial_{y}^{3} f\right\|_{L^{\infty}(D)} h_{2}^{3}+\left\|\partial_{x}^{3} f\right\|_{L^{\infty}(D)} h_{1}^{3}\right)
$$

We can also first interpolate $f$ in $y$ and then in $x$ (8.11) to obtain another form of $I I$. Similar estimates yield

$$
|I I| \leq \frac{1}{6} \cdot C_{3}^{(3)}\left(C_{2}^{(3)}\left\|\partial_{x}^{3} f\right\|_{L^{\infty}(D)} h_{1}^{3}+\left\|\partial_{y}^{3} f\right\|_{L^{\infty}(D)} h_{2}^{3}\right)
$$

We minimize the above two estimates to bound II. Similar arguments apply to the fourth order estimate.
8.3. Hermite interpolation in 1D. We first discuss the Hermite interpolation in 1D, and then generalize it to 2D. Consider $x_{0}<x_{1}<x_{2}$ with $x_{2}-x_{1}=x_{1}-x_{0}$. Denote by $p_{i}, q_{i}$ the cubic polynomials such that

$$
\begin{array}{ccc}
p_{i}\left(x_{i}\right) 1, & p_{i}\left(x_{1-i}\right)=0, i=0,1, & p_{i}^{\prime}\left(x_{j}\right)=0, j=0,1, \\
q_{i}^{\prime}\left(x_{i}\right)=1, & q_{i}^{\prime}\left(x_{1-i}\right)=0, i=0,1, & q_{i}\left(x_{j}\right)=0, j=0,1,
\end{array}
$$

and

$$
\begin{equation*}
l_{i}=f\left(x_{i}\right), \quad m_{i}=h f^{\prime}\left(x_{i}\right), \quad h=x_{1}-x_{0} . \tag{8.18}
\end{equation*}
$$

We consider the $4-t h$ and $5-$ th order Hermite interpolations for $f$

$$
\begin{align*}
H_{4}\left(f, x_{0}, x_{1}\right)(x) & =\sum_{i=0,1}\left(f\left(x_{i}\right) p_{i}(x)+f^{\prime}\left(x_{i}\right) q_{i}(x)\right), \\
H_{5}\left(f, x_{0}, x_{1}, x_{2}\right)(x) & =H_{4}(x)+\left(f\left(x_{2}\right)-H_{4}\left(x_{2}\right)\right) \frac{\left(x-x_{0}\right)^{2}\left(x-x_{1}\right)^{2}}{\left(x_{2}-x_{0}\right)^{2}\left(x_{2}-x_{1}\right)^{2}} . \tag{8.19}
\end{align*}
$$

For simplicity, we drop the dependence of $f, x_{i}$. Note that the coefficients of the polynomials $p_{i}, q_{i}$ depend on $x_{0}, x_{1}, x_{2}$ only. It is easy to see that

$$
H_{4}\left(x_{i}\right)=H_{5}\left(x_{i}\right)=f\left(x_{i}\right), \quad H_{4}^{\prime}\left(x_{i}\right)=H_{5}^{\prime}\left(x_{i}\right)=f^{\prime}\left(x_{i}\right), i=0,1, \quad H_{5}\left(x_{2}\right)=f\left(x_{2}\right) .
$$

8.3.1. Estimates of the interpolation error. For $\left[x_{0}, x_{2}\right]$, we have the standard error estimate

$$
\begin{equation*}
\left|f(x)-H_{4}(x)\right| \leq \frac{1}{24}\left\|\partial_{x}^{4} f\right\|_{L^{\infty}\left(\left[x_{0}, x_{2}\right]\right)}\left(x-x_{0}\right)^{2}\left(x-x_{1}\right)^{2}, \quad x \in\left[x_{0}, x_{2}\right] . \tag{8.20}
\end{equation*}
$$

For $x \in\left[x_{0}, x_{1}\right]$, we have the following error estimates with further simplifications

$$
\begin{align*}
& \left|f(x)-H_{4}(x)\right| \leq \frac{1}{24}\left\|\partial_{x}^{4} f\right\|_{L^{\infty}\left(\left[x_{0}, x_{1}\right]\right)}\left(x-x_{0}\right)^{2}\left(x-x_{1}\right)^{2} \leq \frac{h^{4}}{384}\left\|\partial_{x}^{4} f\right\|_{L^{\infty}\left(\left[x_{0}, x_{1}\right]\right)}  \tag{8.21}\\
& \left|f(x)-H_{5}(x)\right| \leq \frac{\left\|\partial_{x}^{5} f\right\|_{L^{\infty}\left(\left[x_{0}, x_{2}\right]\right)}}{120}\left|\left(x-x_{0}\right)^{2}\left(x-x_{1}\right)^{2}\left(x-x_{2}\right)\right| \leq \frac{h^{5}}{1200}\left\|\partial_{x}^{5} f\right\|_{L^{\infty}\left(\left[x_{0}, x_{2}\right]\right)},
\end{align*}
$$

where we have used (8.67), (8.72) in Lemma 8.1 in the last inequality. The proof of the first inequality is standard and easier. We consider the second estimate. For any $t \in\left[x_{0}, x_{2}\right]$, denote

$$
R_{t}(x)=f(x)-H_{5}(x)-\left(f(t)-H_{5}(t)\right) \frac{\left(x-x_{0}\right)^{2}\left(x-x_{1}\right)^{2}\left(x-x_{2}\right)}{\left(t-x_{0}\right)^{2}\left(t-x_{1}\right)^{2}\left(t-x_{2}\right)} .
$$

Clearly, we have $R_{t}\left(x_{i}\right)=0, i=0,1,2, R_{t}^{\prime}\left(x_{i}\right)=0, i=0,1, R_{t}(t)=0$. Thus, $R_{t}$ has 6 zeros. For $f \in C^{4,1}$, applying the Rolle's theorem repeatedly up to $\partial_{x}^{4} f$, we yield $\xi_{1} \neq \xi_{2} \in\left(x_{0}, x_{2}\right)$ with

$$
0=\partial_{x}^{4} f\left(\xi_{i}\right)-C_{1}-\frac{\left(f(t)-H_{5}(t)\right)\left(120 \xi_{i}-C_{2}\right)}{\left(t-x_{0}\right)^{2}\left(t-x_{1}\right)^{2}\left(t-x_{2}\right)}
$$

where we have used $\partial_{x}^{4} H_{5}(x)=C_{1}, \partial_{x}^{4}\left(x-x_{0}\right)^{2}\left(x-x_{1}\right)^{2}\left(x-x_{2}\right)=120 x-C_{2}$. Rewritting the above identities and computing the difference, we obtain
$\left|f(t)-H_{5}(t)\right|=\left|\frac{\partial_{x}^{4} f\left(\xi_{2}\right)-\partial_{x}^{4} f\left(\xi_{1}\right)}{120\left(\xi_{2}-\xi_{1}\right)}\left(t-x_{0}\right)^{2}\left(t-x_{1}\right)^{2}\left(t-x_{2}\right)\right| \leq \frac{\left\|\partial_{x}^{5} f\right\|_{L^{\infty}\left[x_{0}, x_{2}\right]}}{120}\left|\left(t-x_{0}\right)^{2}\left(t-x_{1}\right)^{2}\left(t-x_{2}\right)\right|$.
Since $t$ is arbitrary, this proves the second estimate in (8.21). The first estimate can be proved similarly. Next, we estimate the interpolating polynomials $H_{4}, H_{5}$.

We should compare the first estimate (8.21) with (8.3). In (8.3), $x_{1}-x_{0}=\frac{b-a}{3}=h$ and the upper bound is $\frac{1}{24}\left\|\partial_{x}^{4} f\right\|_{\infty} h^{4}$. In (8.21), for $x \in\left[x_{0}, x_{1}\right]$, using $\left|\left(x-x_{0}\right)\left(x-x_{1}\right)\right| \leq \frac{\left(x_{1}-x_{0}\right)^{2}}{4}$, we obtain an extra small factor $\frac{1}{16}$. This is one of the main advantages of the Hermite interpolation.
8.3.2. Estimate $H_{4}, H_{5}$. We consider $x \in\left[x_{0}, x_{1}\right]$. Recall $l_{i}, m_{i}$ from (8.18). We have

$$
\begin{align*}
G_{4}(t) & \triangleq H_{4}\left(x_{0}+t h\right)=H_{4}(x), \quad t=\frac{x-x_{0}}{h} \\
G_{4}(t) & =\left(l_{0}(1-t)+l_{1} t\right)+\left(t^{2}(t-1)\left(m_{1}-\left(l_{1}-l_{0}\right)\right)+(t-1)^{2} t\left(m_{0}-\left(l_{1}-l_{0}\right)\right)\right)  \tag{8.22}\\
& \triangleq I(t)+I I(t)
\end{align*}
$$

To show that $G_{4}$ defined via the first identity has the second expression, we only need to verify that the expression satisfies $G_{4}(i)=l_{i}=f\left(x_{i}\right), \partial_{t} G_{4}(i)=m_{i}=h f^{\prime}\left(x_{i}\right), i=0$, 1 , which is obvious. Then both expressions are Hermite polynomials interpolating $\left(x_{i}, f\left(x_{i}\right)\right),\left(x_{i}, f^{\prime}\left(x_{i}\right)\right)$, and thus they must be the same. To estimate $H_{4}(x)$ on $\left[x_{0}, x_{1}\right]$, we only need to estimate $G_{4}(t)$ on $[0,1]$. The estimate of the linear part is trivial

$$
|I(t)|=\left|l_{0}(1-t)+l_{1} t\right| \leq \max \left(\left|l_{0}\right|,\left|l_{1}\right|\right)
$$

The second part $I I$ is treated as error and we want to obtain a sharp constant. Denote

$$
\begin{equation*}
M_{1}=\max \left(\left|m_{1}-m_{0}\right|,\left|m_{1}-\left(l_{1}-l_{0}\right)\right|,\left|m_{0}-\left(l_{1}-l_{0}\right)\right|\right) \tag{8.23}
\end{equation*}
$$

For $0 \leq t \leq \frac{1}{2}$, using $t(t-1)^{2} \leq \frac{4}{27}$ (8.70), we have

$$
\begin{aligned}
I I(t) & =t(t-1)\left(t\left(m_{1}-\left(l_{1}-l_{0}\right)\right)+(t-1)\left(m_{0}-\left(l_{1}-l_{0}\right)\right)\right. \\
& =t(t-1)\left(t\left(m_{1}-m_{0}\right)+(2 t-1)\left(m_{0}-\left(l_{1}-l_{0}\right)\right)\right) \\
\mid I I(t) & \leq M_{1}|t(t-1)|(t+|1-2 t|)=M_{1} t(1-t)^{2} \leq \frac{4}{27} M_{1}
\end{aligned}
$$

Similarly, for $t \in[1 / 2,1]$, writing $m_{0}-\left(l_{1}-l_{0}\right)=m_{0}-m_{1}+\left(m_{1}-\left(l_{1}-l_{0}\right)\right)$, we get

$$
\begin{aligned}
I I(t) & =t(t-1)\left((t-1)\left(m_{0}-m_{1}\right)+(2 t-1)\left(m_{1}-\left(l_{1}-l_{0}\right)\right)\right) \\
|I I(t)| & \leq|t(t-1)|(|t-1|+|2 t-1|) M_{1}=t(1-t)(1-t+2 t-1) M_{1}=t^{2}(1-t) M_{1} \leq \frac{4}{27} M_{1}
\end{aligned}
$$

To obtain the last inequality, we apply (8.70) with $s=1-t$. Therefore, we prove

$$
\begin{equation*}
\left|H_{4}(x)\right|=\left|G_{4}(t)\right| \leq \max \left(\left|l_{0}\right|,\left|l_{1}\right|\right)+\frac{4}{27} \max \left(\left|m_{1}-m_{0}\right|,\left|m_{1}-\left(l_{1}-l_{0}\right)\right|,\left|m_{0}-\left(l_{1}-l_{0}\right)\right|\right) \tag{8.24}
\end{equation*}
$$

For $H_{5}$, we estimate the extra term in (8.19). Since $x_{2}-x_{1}=x_{1}-x_{0}=h$, we have

$$
H_{4}\left(x_{2}\right)=G_{4}(2)=-l_{0}+2 l_{1}+4\left(m_{1}-\left(l_{1}-l_{0}\right)\right)+2\left(m_{0}-\left(l_{1}-l_{0}\right)\right)=l_{0}+2 m_{0}+4\left(m_{1}-\left(l_{1}-l_{0}\right)\right)
$$

Since $x=x_{0}+t h \in\left[x_{0}, x_{1}\right]$, we have $t \in[0,1],|t(1-t)| \leq \frac{1}{4}$,

$$
\begin{equation*}
\frac{\left(x-x_{0}\right)^{2}\left(x-x_{1}\right)^{2}}{\left(x_{2}-x_{0}\right)^{2}\left(x_{1}-x_{0}\right)^{2}}=\frac{t^{2}(t-1)^{2}}{4} \leq \frac{1}{64} \tag{8.25}
\end{equation*}
$$

We obtain

$$
\begin{equation*}
\max _{x \in\left[x_{0}, x_{1}\right]}\left|H_{5}\right| \leq \max _{x \in\left[x_{0}, x_{1}\right]}\left|H_{4}(x)\right|+\frac{1}{64}\left|f\left(x_{2}\right)-H_{4}\left(x_{2}\right)\right| . \tag{8.26}
\end{equation*}
$$

8.3.3. Estimate derivatives in $1 D$. In this subsection, we discuss how to estimate $\partial f(x)$ with fourth order error term using the Hermite interpolation $\partial H_{5}(x)$. We consider $\partial_{x}$ without loss of generality. Firstly, since $f(x)-H_{5}(x)$ has five zeros: two zeros at $x_{0}$, two zeros at $x_{1}$, and one zero at $x_{2}$, we know that $\partial_{x}\left(f(x)-H_{5}(x)\right)$ has four zeros: $x_{0}<\xi<x_{1}<\eta$. Using the Rolle's theorem and an argument similar to that in Section 8.3.1, we get

$$
\left|\partial_{x}\left(f(x)-H_{5}(x)\right)\right| \leq \frac{1}{24}\left\|\partial_{x}^{5} f\right\|_{L^{\infty}\left[x_{0}, x_{2}\right]}\left|\left(x-x_{0}\right)\left(x-x_{1}\right)(x-\xi)(x-\eta)\right|
$$

Next, for $x \in\left[x_{0}, x_{1}\right]$, we simplify the upper bound. Clearly, we have $|x-\xi| \leq \max \left(\left|x-x_{0}\right|, \mid x_{1}-\right.$ $x \mid),|x-\eta|=\eta-x \leq x_{2}-x$. We yield

$$
\begin{aligned}
p(x) & \triangleq\left|\left(x-x_{0}\right)\left(x-x_{1}\right)(x-\xi)(x-\eta) \leq\left|\left(x-x_{0}\right)\left(x-x_{1}\right)\left(x-x_{2}\right)\right| \max \left(\left|x-x_{0}\right|,\left|x-x_{1}\right|\right)\right. \\
& =h^{4}|t(1-t)(2-t)| \max (|1-t|, t) \triangleq h^{4} q(t), \quad t=\left(x-x_{0}\right) h^{-1}
\end{aligned}
$$

If $t \leq \frac{1}{2}$, we denote $s=(1-t)^{2} \in[0,1]$ and get

$$
q(t)=t(1-t)^{2}(2-t)=\left(2 t-t^{2}\right) s=(1-s) s \leq \frac{1}{4}
$$

If $t \in[1 / 2,1]$, we get

$$
q(t)=t(1-t) \cdot t(2-t) \leq \frac{1}{4} \cdot 1=\frac{1}{4}
$$

Thus, for $x \in\left[x_{0}, x_{1}\right]$, we prove the error estimate

$$
\begin{equation*}
\left|\partial_{x}\left(f(x)-H_{5}(x)\right)\right| \leq \frac{h^{4}}{96}\left\|\partial_{x}^{5} f\right\|_{L^{\infty}\left[x_{0}, x_{2}\right]} \tag{8.27}
\end{equation*}
$$

Next, we estimate $\partial_{x} H_{5}$. Recall $t=\frac{x-x_{0}}{h}$. Using (8.19) and the chain rule, we get

$$
\begin{align*}
& H_{5}(x)=H_{4}(x)+\left(f\left(x_{2}\right)-H_{4}\left(x_{2}\right)\right) \frac{t^{2}(t-1)^{2}}{4}=G_{4}(t)+\left(f\left(x_{2}\right)-H_{4}\left(x_{2}\right)\right) \frac{t^{2}(t-1)^{2}}{4}  \tag{8.28}\\
& \partial_{x} H_{5}(x)=\frac{1}{h}\left(\partial_{t} G_{4}(t)+\left(f\left(x_{2}\right)-H_{4}\left(x_{2}\right)\right) \partial_{t} \frac{t^{2}(t-1)^{2}}{4}\right) \triangleq \frac{1}{h}(I(t)+I I(t))
\end{align*}
$$

We estimate two parts separately. Using (8.22), we get

$$
\begin{equation*}
I(t) \triangleq \partial_{t} G_{4}=\left(l_{1}-l_{0}\right)+\left(3 t^{2}-2 t\right)\left(m_{1}-\left(l_{1}-l_{0}\right)\right)+\left(3 t^{2}-4 t+1\right)\left(m_{0}-\left(l_{1}-l_{0}\right)\right) \tag{8.29}
\end{equation*}
$$

Note that $l_{1}-l_{0}, m_{1}, m_{0}$ are approximations of $h f^{\prime}(x)$ and have cancellations. We discuss different $t \in[0,1]$ to exploit the cancellations.

Denote $a \vee b=\max (a, b)$. If $t \leq \frac{1}{3}$, we get

$$
I(t)=\left(4 t-3 t^{2}\right)\left(l_{1}-l_{0}\right)+\left(3 t^{2}-4 t+1\right) m_{0}+\left(2 t-3 t^{2}\right)\left(l_{1}-l_{0}-m_{1}\right)
$$

The first two terms are the main terms, and the last term is the error term. Since $t \leq \frac{1}{3}$, we get

$$
4 t-3 t^{2}>0, \quad 3 t^{2}-4 t+1=(1-3 t)(1-t) \geq 0, \quad 0 \leq 2 t-3 t^{2}=3 t\left(\frac{2}{3}-t\right) \leq \frac{1}{3}
$$

where the last inequality is equivalent to $3\left(t-\frac{1}{3}\right)^{2} \geq 0$. It follows

$$
\begin{equation*}
|I(t)| \leq\left(\left|l_{1}-l_{0}\right| \vee m_{0}\right) \cdot\left(4 t-3 t^{2}+3 t^{2}-4 t+1\right)+\frac{1}{3}\left|l_{1}-l_{0}-m_{1}\right|=\left|l_{1}-l_{0}\right| \vee m_{0}+\frac{1}{3}\left|l_{1}-l_{0}-m_{1}\right| \tag{8.30}
\end{equation*}
$$

The estimate of $t \in[2 / 3,1]$ is similar by swapping $t$ and $1-t, m_{0}$ and $m_{1}, p_{0}$ and $p_{1}$. We get

$$
\begin{equation*}
|I(t)| \leq\left|l_{1}-l_{0}\right| \vee m_{1}+\frac{1}{3}\left|l_{1}-l_{0}-m_{0}\right| \tag{8.31}
\end{equation*}
$$

For $t \in[1 / 3,2 / 3]$, we rewrite $I(t)(8.29)$ as follows

$$
I(t)=l_{1}-l_{0}+\left(4 t-3 t^{2}-1\right)\left(l_{1}-l_{0}-m_{0}\right)+\left(2 t-3 t^{2}\right)\left(l_{1}-l_{0}-m_{1}\right)
$$

Since $t \in[1 / 3,2 / 3]$, we get

$$
\begin{aligned}
& 4 t-3 t^{2}-1=(3 t-1)(1-t) \geq 0, \quad 2 t-3 t^{2}=t(2-3 t) \geq 0 \\
& 0 \leq 4 t-3 t^{2}-1+2 t-3 t^{2}=6 t-6 t^{2}-1=6 t(1-t)-1 \leq 6 \cdot \frac{1}{4}-1=\frac{3}{2}-1 \leq \frac{1}{2}
\end{aligned}
$$

Thus, we obtain

$$
\begin{equation*}
I(t) \leq\left|l_{1}-l_{0}\right|+\frac{1}{2}\left(\left|l_{1}-l_{0}-m_{0}\right| \vee\left|l_{1}-l_{0}-m_{1}\right|\right) \tag{8.32}
\end{equation*}
$$

Combining three cases (8.30)-(8.32), we obtain the bound for $\partial_{t} G_{4}$
$\frac{1}{h}\left|\partial_{t} G_{4}\right| \leq \max \left(\max _{i=0,1}\left(\left|l_{1}-l_{0}\right| \vee m_{i}+\frac{1}{3}\left|l_{1}-l_{0}-m_{1-i}\right|,\left|l_{1}-l_{0}\right|+\frac{1}{2}\left(\left|l_{1}-l_{0}-m_{0}\right| \vee\left|l_{1}-l_{0}-m_{1}\right|\right)\right)\right.$.
For the second term in (8.28), using (8.69) with $2 t \in[0,2]$ (we consider $x \in\left[x_{0}, x_{1}\right]$ ), we get

$$
\begin{equation*}
\partial_{t} \frac{t^{2}(t-1)^{2}}{4}=t(t-1)\left(t-\frac{1}{2}\right),\left|t(t-1)\left(t-\frac{1}{2}\right)\right|=\frac{1}{8}|2 t(1-2 t)(2-2 t)| \leq \frac{1}{8} \cdot \frac{2}{3 \sqrt{3}} \leq \frac{1}{12 \sqrt{3}} \tag{8.34}
\end{equation*}
$$

which implies

$$
\begin{equation*}
\left|\frac{1}{h} I I(t)\right| \leq \frac{1}{12 \sqrt{3} h}\left|f\left(x_{2}\right)-H_{4}\left(x_{2}\right)\right| \tag{8.35}
\end{equation*}
$$

Combining the estimates (8.30)-(8.35), we obtain the estimate for $\partial_{x} H_{5}$.
8.3.4. A 4-th order interpolation in $1 D$. We will also use a 4-th order interpolation $H_{4}(x)$ to approximate $f$ and $\partial_{x} f$. Below, we restrict the derivation to $x \in\left[x_{0}, x_{1}\right]$. We have estimated $f-H_{4}$ in (8.21) and $H_{4}$ in (8.24). For the derivatives, we have

$$
\partial_{x} f=\partial_{x}\left(f-H_{4}\right)+\partial_{x} H_{4}=\partial_{x}\left(f-H_{4}\right)+\frac{1}{h} \partial_{t} G_{4} \triangleq I I_{1}+I I_{2}
$$

We have estimated $I I_{2}=\partial_{t} G_{4}$ in (8.29) and (8.33). For $I I_{1}$, following the argument at the beginning of Section 8.3.3 and 8.35) and using Rolle's theorem, we get

$$
\left|\partial_{x}\left(f(x)-H_{4}(x)\right)\right| \leq \frac{1}{6}\left\|\partial_{x}^{4} f\right\|_{L^{\infty}\left[x_{0}, x_{1}\right]}\left|\left(x-x_{0}\right)\left(x-x_{1}\right)(x-\xi(x))\right| \triangleq \frac{1}{6}\left\|\partial_{x}^{4} f\right\|_{L^{\infty}\left[x_{0}, x_{1}\right]} Q(x)
$$

for some $\xi(x) \in\left[x_{0}, x_{1}\right]$. Without loss of generality, we assume that $x \leq \frac{x_{0}+x_{1}}{2}$. Using $h=$ $x_{1}-x_{0},|x-\xi(x)| \leq \max \left(x_{1}-x, x-x_{0}\right)=x_{1}-x, x=x_{0}+t \cdot h$, and (8.70), we get

$$
\begin{align*}
& Q(x) \leq\left(x-x_{0}\right)\left(x_{1}-x\right)^{2}=h^{3} t(1-t)^{2} \leq \frac{4}{27} h^{3}  \tag{8.36}\\
& \left|\partial_{x}\left(f(x)-H_{4}(x)\right)\right| \leq \frac{4}{27} \cdot \frac{1}{6} h^{3}\left\|\partial_{x}^{4} f\right\|_{L^{\infty}\left[x_{0}, x_{1}\right]}=\frac{2 h^{3}}{81}\left\|\partial_{x}^{4} f\right\|_{L^{\infty}\left[x_{0}, x_{1}\right]}
\end{align*}
$$

8.4. Hermite interpolation in 2D. The estimate in 2D is more involved. Consider

$$
x_{0}<x_{1}<x_{2}, y_{0}<y_{1}<y_{2}, x_{2}-x_{1}=x_{1}-x_{0}, y_{2}-y_{1}=y_{1}-y_{0}
$$

The domain $D$ can be decomposed into 4 blocks with size $h_{1} \times h_{2}$. For the 5 -th order interpolation, we interpolate $f$ in $(x, y) \in\left[x_{0}, x_{1}\right] \times\left[y_{0}, y_{1}\right]$ without loss of generality. For the 4 -th order interpolation, we only need the value of $f$ in 1 block $\left[x_{0}, x_{1}\right] \times\left[y_{0}, y_{1}\right]$. Denote

$$
\begin{align*}
& D=\left[x_{0}, x_{2}\right] \times\left[y_{0}, y_{2}\right], h_{1}=x_{1}-x_{0}, h_{2}=y_{1}-y_{0} \\
& t=\left(x-x_{0}\right) h_{1}^{-1}, s=\left(y-y_{0}\right) h_{2}^{-1} \tag{8.37}
\end{align*}
$$

8.4.1. Estimate the $L^{\infty}$ norm. We first consider the estimate of $\|f\|_{L^{\infty}}$. We treat $y$ as a parameter and use (8.19) to construct the 4-th and 5-th order interpolations in $x: H_{4}(x, y), H_{5}(x, y)$. Using the formula in (8.22), we have

$$
\begin{align*}
G_{5}(t, s) \triangleq & H_{5}(x, y)=H_{4}(x, y)+\left(f\left(x_{2}, y\right)-H_{4}\left(x_{2}, y\right)\right) \frac{t^{2}(t-1)^{2}}{4} \\
H_{4}(x, y)= & f\left(x_{0}, y\right)(1-t)+f\left(x_{1}, y\right) t+t^{2}(t-1)\left(h f^{\prime}\left(x_{1}, y\right)-\left(f\left(x_{1}, y\right)-f\left(x_{0}, y\right)\right)\right)  \tag{8.38}\\
& +(t-1)^{2} t\left(h f^{\prime}\left(x_{0}, y\right)-\left(f\left(x_{1}, y\right)-f\left(x_{0}, y\right)\right)\right)
\end{align*}
$$

Using (8.21), we have the error bound in $x$

$$
\begin{equation*}
\left|H_{5}(x, y)-f(x, y)\right| \leq \frac{1}{1200}\left\|\partial_{x}^{5} f\right\|_{L^{\infty}(D)} h_{1}^{5} \tag{8.39}
\end{equation*}
$$

We need to further interpolate $G_{5}(t, s), H_{5}(x, y)$ in the $y$ direction. To achieve the overall 5 -th order error, we do not need to apply a high order interpolation to each coefficient. For the linear term, we apply the 5 -th order Hermite interpolation to $f\left(x_{i}, y\right)$ in $y$ using $f\left(x_{i}, y_{j}\right), j=0,1,2$ and $\partial_{y} f\left(x_{i}, y_{j}\right), j=0,1$ for $i=0,1$ and denote it by $A_{i}^{(5)}(y)$, i.e.

$$
\begin{equation*}
A_{i}^{(5)}(y)=H_{5}\left(f\left(x_{i}, \cdot\right), y_{0}, y_{1}, y_{2}\right)(y) \tag{8.40}
\end{equation*}
$$

using the notation in (8.19). Applying (8.21), we have the error bound in $y$

$$
\begin{equation*}
\left|f\left(x_{i}, y\right)-A_{i}^{(5)}(y)\right| \leq \frac{1}{1200}\left\|\partial_{y}^{5} f\right\|_{L^{\infty}(D)} h_{2}^{5} \tag{8.41}
\end{equation*}
$$

Denote

$$
\begin{equation*}
M_{i}(y) \triangleq h f^{\prime}\left(x_{i}, y\right)-\left(f\left(x_{1}, y\right)-f\left(x_{0}, y\right)\right) \tag{8.42}
\end{equation*}
$$

For $M_{i}(y), i=1,2$, it is of order $h_{1}^{2}$. Thus, we apply the cubic interpolation in Section 8.1.2 to these functions in the $y$ direction on grids $y_{0}, y_{1}, y_{2}$ and denote it by $Q_{i}(y)$. Using the notation (8.7), we have

$$
\begin{equation*}
Q_{i}(y)=N_{2}\left(M_{i}, y_{0}, y_{1}, y_{2}\right)(y) \tag{8.43}
\end{equation*}
$$

Applying (8.8), we have the error bound

$$
\begin{equation*}
\left.\left|M_{i}(y)-Q_{i}(y)\right| \leq \frac{h_{2}^{3}}{9 \sqrt{3}} \| \partial_{y}^{3} M_{i}(y)\right)\left\|_{L^{\infty}\left(\left[y_{0}, y_{2}\right]\right)} \leq \frac{h_{2}^{3} h_{1}^{2}}{18 \sqrt{3}}\right\| \partial_{x}^{2} \partial_{y}^{3} f \|_{L^{\infty}(D)} \tag{8.44}
\end{equation*}
$$

where we have used the following estimate with $c=a, b, g=\partial_{y}^{3} f(\cdot, y)$ in the last inequality

$$
\begin{equation*}
\left|(b-a) g^{\prime}(c)-(g(b)-g(a))\right|=\left|\int_{a}^{b} g^{\prime}(c)-g^{\prime}(s) d s\right| \leq\left\|g^{\prime \prime}\right\|_{L^{\infty}[a, b]} \int_{a}^{b}|c-s| d s=\frac{(b-a)^{2}}{2}\left\|g^{\prime \prime}\right\|_{L^{\infty}[a, b]} \tag{8.45}
\end{equation*}
$$

The last term $f\left(x_{2}, y\right)-H_{4}\left(x_{2}, y\right)$ is already very small and of order $h_{1}^{4}$. From (8.38), since $\partial_{y}$ commutes with $t=\left(x-x_{0}\right) h^{-1}$ and $\partial_{x}$, for a fixed $y, \partial_{y} H_{4}(x, y)$ is the Hermite polynomials for $\partial_{y} f(x, y)$ in $x$. We use the first order estimate in $y$ and then (8.20) with $\partial_{y} f$ and $x=x_{2}$ to obtain

$$
\begin{align*}
\left|f\left(x_{2}, y\right)-H_{4}\left(x_{2}, y\right)\right| & \leq \max _{j=0,1}\left|f\left(x_{2}, y_{j}\right)-H_{4}\left(x_{2}, y_{j}\right)\right|+\frac{h_{2}}{2}\left|\partial_{y}\left(f\left(x_{2}, y\right)-H_{4}\left(x_{2}, y\right)\right)\right| \\
& \leq \max _{j=0,1}\left|f\left(x_{2}, y_{j}\right)-H_{4}\left(x_{2}, y_{j}\right)\right|+\frac{h_{2} h_{1}^{4}}{2 \cdot 6}\left\|\partial_{y} \partial_{x}^{4} f\right\|_{L^{\infty}(D)} \tag{8.46}
\end{align*}
$$

where we have used $x_{2}-x_{0}=2 h_{1}, x_{1}-x_{0}=h_{1}, \frac{\left(x_{2}-x_{0}\right)^{2}\left(x_{1}-x_{0}\right)^{2}}{24}=\frac{4 h_{1}^{2} \cdot h_{1}^{2}}{24}=\frac{h_{1}^{4}}{6}$ in (8.20).
Estimate the 2D interpolating polynomials for $f$. We obtain the 2D interpolating polynomials in $(x, y) \in\left[x_{0}, x_{1}\right] \times\left[y_{0}, y_{1}\right]$ for $H_{4}(x, y)$ as follows
(8.47)
$P_{5}(x, y)=\left(A_{0}^{(5)}(y)(1-t)+A_{1}^{(5)}(y) t\right)+\left(t^{2}(t-1) Q_{1}(y)+t(t-1)^{2} Q_{0}(y)\right)=I(t, y)+I I(t, y)$.
We restrict $t \in[0,1], x \in\left[x_{0}, x_{1}\right], y \in\left[y_{0}, y_{1}\right]$. Applying (8.39) for $H_{5}-f$, (8.46) for $H_{4}-H_{5}$ in (8.38), (8.41)-(8.44) for $H_{4}-P_{5}, t+(1-t)=1$, we have the following error bound

> (8.48)
$\left|P_{5}(x, y)-f(x, y)\right| \leq\left|H_{4}(x, y)-H_{5}(x, y)\right|+\left|H_{5}(x, y)-f(x, y)\right|+\left|H_{4}(x, y)-P_{5}(x, y)\right|$,
$\left|H_{4}-H_{5}\right| \leq \frac{1}{64}\left|f\left(x_{2}, y\right)-H_{4}\left(x_{2}, y\right)\right| \leq \frac{1}{64} \max _{j=0,1}\left|f\left(x_{2}, y_{j}\right)-H_{4}\left(x_{2}, y_{j}\right)\right|+\frac{h_{2} h_{1}^{4}}{768}\left\|\partial_{y} \partial_{x}^{4} f\right\|_{L^{\infty}(D)}$,
$\left|H_{5}-f\right|+\left|H_{4}-P_{5}\right| \leq \frac{1}{72 \sqrt{3}} h_{1}^{2} h_{2}^{3}\left\|\partial_{x}^{2} \partial_{y}^{3} f\right\|_{L^{\infty}(D)}+\frac{1}{1200}\left(h_{1}^{5}\left\|\partial_{x}^{5} f\right\|_{L^{\infty}(D)}+h_{2}^{5}\left\|\partial_{y}^{5} f\right\|_{L^{\infty}(D)}\right)$,
where the constant $\frac{1}{64}$ in $H_{4}-H_{5}$ is from (8.25), $\frac{1}{768}$ comes from (8.46) and $\frac{1}{768}=\frac{1}{64} \cdot \frac{1}{16}, \frac{1}{72 \sqrt{3}}$ from $\frac{1}{18 \sqrt{3}}\left(t^{2}|t-1|+(t-1)^{2} t\right)=\frac{1}{18 \sqrt{3}}(1-t) t \leq \frac{1}{72 \sqrt{3}}$ (8.44). To estimate $P_{5}(x, y)$ (8.47), we use estimates similar to (8.24). The estimate of the linear part is trivial

$$
\begin{equation*}
|I(t, y)| \leq \max _{i=0,1}\left\|A_{i}^{(5)}(y)\right\|_{L^{\infty}\left[y_{0}, y_{1}\right]}, \quad y \in\left[y_{0}, y_{1}\right] \tag{8.49}
\end{equation*}
$$

Since $A_{i}(y)(8.40)$ is the Hermite polynomial in $y$, we can use the method in Section 8.3 to estimate it. For $I I$, following the derivations between (8.23) to (8.24) and by considering two cases: $t \leq \frac{1}{2}$ and $t>\frac{1}{2}$, we obtain

$$
\begin{equation*}
|I I(t, y)| \leq \frac{4}{27} \max \left(\left|Q_{1}(y)-Q_{0}(y)\right|,\left|Q_{1}(y)\right|,\left|Q_{0}(y)\right|\right) \tag{8.50}
\end{equation*}
$$

Since $Q_{i}, Q_{1}-Q_{0}$ are quadratic interpolating polynomials of $M_{i}, M_{1}-M_{0}$ (8.42) on $y_{0}, y_{1}, y_{2}$, we can use (8.9) to estimate the $L^{\infty}\left[y_{0}, y_{1}\right]$ norm.
8.4.2. Estimates of the $\partial f(x, y)$ in $2 D$. Now, we consider how to estimate $\partial f(x, y)$ using the Hermite interpolation. We consider $\partial_{x}$ without loss of generality. Recall the notation (8.37). We first fix $y$ as a parameter and interpolate $f(x, y)$ in $x$ using the same method as (8.38). Using the error estimate (8.27), we yield

$$
\begin{equation*}
\left|\partial_{x}\left(f(x, y)-H_{5}(x, y)\right)\right| \leq \frac{h_{1}^{4}}{96}\left\|\partial_{x}^{5} f\right\|_{L^{\infty}(D)} \tag{8.51}
\end{equation*}
$$

Using the computation (8.28), (8.29), (8.34) in Section 8.3 .3 and the notation (8.18) for $m_{i}, l_{i}$, we have

$$
\begin{align*}
h_{1} \partial_{x} H_{5}(x, y)= & h_{1} \partial_{x} H_{4}(x, y)+\left(f\left(x_{2}, y\right)-H_{4}\left(x_{2}, y\right)\right) t(t-1)\left(t-\frac{1}{2}\right) \\
h_{1} \partial_{x} H_{4}(x, y)= & f\left(x_{1}, y\right)-f\left(x_{0}, y\right)+\left(3 t^{2}-2 t\right)\left(h f^{\prime}\left(x_{1}, y\right)-\left(f\left(x_{1}, y\right)-f\left(x_{0}, y\right)\right)\right)  \tag{8.52}\\
& +\left(3 t^{2}-4 t+1\right)\left(h f^{\prime}\left(x_{0}, y\right)-\left(f\left(x_{1}, y\right)-f\left(x_{0}, y\right)\right)\right) \\
= & f\left(x_{0}, y\right)-f\left(x_{1}, y\right)+\left(3 t^{2}-2 t\right) M_{1}(y)+\left(3 t^{2}-4 t+1\right)\left(M_{0}(y)\right.
\end{align*}
$$

where $t=\frac{x-x_{0}}{h_{1}}$ (8.37), and we have used $M_{i}$ (8.42) to simplify the presentation.
Next, we interpolate the above functions in $y$. We want to achieve an overall 4 -th order approximation for $\partial_{x} H_{4}(x, y)$. For $f\left(x_{1}, y\right)-f\left(x_{0}, y\right)$, we use the 4 -th order Hermite interpolation in $y$ based on the grid point values $f\left(x_{1}, y_{j}\right)-f\left(x_{0}, y_{j}\right), \partial_{y}\left(f\left(x_{1}, y_{j}\right)-f\left(x_{0}, y_{j}\right)\right), j=0,1$ and denote it as $B(y)$, i.e.

$$
\begin{equation*}
B(y) \triangleq H_{4}\left(f\left(x_{1}, \cdot\right)-f\left(x_{0}, \cdot\right), y_{0}, y_{1}\right) \tag{8.53}
\end{equation*}
$$

using the notation (8.19). By (8.21), we have the error estimate

$$
\begin{equation*}
\left|\frac{B(y)}{h_{1}}-\frac{f\left(x_{1}, y\right)-f\left(x_{0}, y\right)}{h_{1}}\right| \leq \frac{h_{2}^{4}}{384}\left\|\partial_{y}^{4} \frac{f\left(x_{1}, y\right)-f\left(x_{0}, y\right)}{h_{1}}\right\|_{L^{\infty}\left(\left[y_{0}, y_{1}\right]\right)} \leq \frac{h_{2}^{4}}{384}\left\|\partial_{y}^{4} \partial_{x} f\right\|_{L^{\infty}(D)} \tag{8.54}
\end{equation*}
$$

For $M_{i}$, we apply the same quadratic interpolation $Q_{i}$ in $y$ (8.43). Using the error bound (8.44) and (8.68) in Lemma 8.1 we obtain

$$
\begin{aligned}
& h_{1}^{-1}\left|\left(3 t^{2}-2 t\right) M_{1}(y)+\left(3 t^{2}-4 t+1\right) M_{0}(y)-\left(3 t^{2}-2 t\right) Q_{1}(y)-\left(3 t^{2}-4 t+1\right) Q_{0}(y)\right| \\
\leq & h_{1}^{-1}\left(\left|3 t^{2}-2 t\right|+\left|3 t^{2}-4 t+1\right|\right) \max _{i=1,2}\left|Q_{i}-M_{i}\right| \leq \frac{h_{2}^{3} h_{1}}{18 \sqrt{3}}\left\|\partial_{x}^{2} \partial_{y}^{3} f\right\|_{L^{\infty}(D)} .
\end{aligned}
$$

For $f\left(x_{2}, y\right)-H_{4}\left(x_{2}, y\right)$, we use the same estimate (8.46), which along with (8.34) implies

$$
\begin{align*}
\frac{1}{h_{1}}\left|\left(f\left(x_{2}, y\right)-H_{4}\left(x_{2}, y\right)\right) t(t-1)\left(t-\frac{1}{2}\right)\right| \leq & \frac{1}{12 \sqrt{3} h_{1}} \max _{j=0,1}\left|f\left(x_{2}, y_{j}\right)-H_{4}\left(x_{2}, y_{j}\right)\right| \\
& +\frac{h_{2} h_{1}^{3}}{12 \cdot 12 \sqrt{3}}\left\|\partial_{y} \partial_{x}^{4} f\right\|_{L^{\infty}(D)} \tag{8.55}
\end{align*}
$$

Estimate the 2D interpolating polynomials for $\partial_{x} f$. Now, we use (8.53), (8.43) to construct the interpolating polynomials for $h_{1} \partial_{x} H_{4}$

$$
\begin{equation*}
S_{4}(x, y)=B(y)+\left(3 t^{2}-2 t\right) Q_{1}(y)+\left(3 t^{2}-4 t+1\right) Q_{0}(y) \tag{8.56}
\end{equation*}
$$

Combining the estimate 8.54 and using the triangle inequality, we can estimate the error $\frac{1}{h_{1}} S_{4}(x, y)-f(x, y)$.

It remains to estimate $S_{4}(x, y)$. We further decompose the above approximation as the linear part and the nonlinear part in $y$. The linear part in $y$ is the main term, and we want to obtain a sharper estimate. Since $B$ is the $4-t h$ order Hermite interpolation in $y$ (8.53), we can apply the decomposition (8.22) into the linear and the nonlinear parts to $B$. Since $Q_{i}$ is the quadratic interpolation of $M_{i}$ (8.42), (8.43), we can apply the decomposition (8.7) into the linear and the quadratic terms to $Q_{i}$

$$
\begin{equation*}
S_{4}=S_{l i n}+S_{n l i n}, \quad S_{\sigma}(t, y) \triangleq B_{\sigma}(y)+\left(3 t^{2}-2 t\right) Q_{1, \sigma}(y)+\left(3 t^{2}-4 t+1\right) Q_{0, \sigma}(y) \tag{8.57}
\end{equation*}
$$

where $\sigma \in\{\operatorname{lin}, n l i n\}, f_{l i n}, f_{n l i n}$ denote the linear part and nonlinear part in $y$, respectively.

Since $S_{\text {lin }}$ is linear in $y$ and $B\left(y_{j}\right)=f\left(x_{1}, y_{j}\right)-f\left(x_{0}, y_{j}\right), Q_{i}\left(y_{j}\right)=M_{i}\left(y_{j}\right)$ for $j=0,1$ (the interpolating polynomials agree with the functions on the grid points), we get

$$
\begin{align*}
\left|S_{\text {lin }}(t, y)\right| & \leq \max _{i=0,1}\left|B_{\text {lin }}\left(y_{i}\right)+\left(3 t^{2}-2 t\right) Q_{1, l i n}\left(y_{i}\right)+\left(3 t^{2}-4 t+1\right) Q_{0, l i n}\left(y_{i}\right)\right|  \tag{8.58}\\
& =\max _{i=0,1}\left|f\left(x_{1}, y_{i}\right)-f\left(x_{0}, y_{i}\right)+\left(3 t^{2}-2 t\right) M_{1}\left(y_{i}\right)+\left(3 t^{2}-4 t+1\right) M_{0}\left(y_{i}\right)\right|
\end{align*}
$$

Recall the definition of $M_{i}$ in (8.42). The polynomials inside $|\cdot|$ is the 1D polynomial in $t$ with the same form as $I(t)$ (8.29). We estimate it using (8.33), 8.35) and following Section 8.3.3.

For the nonlinear part, we have

$$
\begin{equation*}
S_{n l i n}=B_{n l i n}(y)+\left(3 t^{2}-2 t\right) Q_{1, n l i n}(y)+\left(3 t^{2}-4 t+1\right) Q_{0, n l i n}(y) \tag{8.59}
\end{equation*}
$$

we recall the definition of $B(8.53)$ and $Q$ (8.43). The estimate of $B_{n l i n}$ follows the method of estimating $I I(t)$ in (8.22), (8.24) with $l_{i}, m_{i}$ replaced by

$$
\tilde{l}_{i}=f\left(x_{1}, y_{i}\right)-f\left(x_{0}, y_{i}\right), \quad \tilde{m}_{i}=h_{2} \partial_{y}\left(f\left(x_{1}, y_{i}\right)-f\left(x_{0}, y_{i}\right)\right)
$$

which gives

$$
\begin{equation*}
\left|B_{n l i n}(y)\right| \leq \frac{4}{27} \max \left(\left|\tilde{m}_{0}, \tilde{m}_{1}\right|,\left|\tilde{m}_{0}-\left(\tilde{l}_{1}-\tilde{l}_{0}\right)\right|,\left|\tilde{m}_{1}-\left(\tilde{l}_{1}-\tilde{l}_{0}\right)\right|\right) \tag{8.60}
\end{equation*}
$$

The estimate of the $Q_{\text {nlin }}$ follows that in (8.9) and Section 8.1.2 which gives

$$
\left|Q_{i, n l i n}\right| \leq \frac{1}{8}\left|M_{i}\left(y_{0}\right)-2 M_{i}\left(y_{1}\right)+M_{i}\left(y_{2}\right)\right|
$$

Using (8.68) in Lemma 8.1] for $t \in[0,1]$, we prove

$$
\begin{aligned}
& \left|\left(3 t^{2}-2 t\right) Q_{1, n l i n}(y)+\left(3 t^{2}-4 t+1\right) Q_{0, n l i n}(y)\right| \\
\leq & \left(\left|3 t^{2}-2 t\right|+\left|3 t^{2}-4 t+1\right|\right) \max _{i=0,1}\left|Q_{i, n l i n}(y)\right| \leq \frac{1}{8} \max _{i=0,1}\left|M_{i}\left(y_{0}\right)-2 M_{i}\left(y_{1}\right)+M_{i}\left(y_{2}\right)\right|
\end{aligned}
$$

Thus, we yield the estimate of $S_{\text {nlin }}$. Combining the above estimates of $S_{l i n}$ and $S_{n l i n}$ and using (8.57), we obtain the estimate of $S_{4}$. We remark that one needs to further divide the above bounds by $\frac{1}{h_{1}}$ to get the bound for $\frac{S_{4}}{h_{1}}$.
8.4.3. The fourth order estimate. Denote $D=\left[x_{0}, x_{1}\right] \times\left[y_{0}, y_{1}\right]$. We only need the value of $f$ in $D$. The above method interpolates $f(x, y)$ with a fifth order error estimate. We also use a simpler fourth order method to estimate $f(x, y)$. Recall the interpolation polynomials $H_{4}$ and $M_{i}$ in $x$ defined in (8.38), (8.42). We consider the following interpolation for $f(x, y)$

$$
\begin{equation*}
P_{4}(x, y)=\left(A_{0}^{(4)}(y)(1-t)+A_{1}^{(4)}(y) t\right)+\left(t^{2}(t-1) l_{1}(y)+t(t-1)^{2} l_{0}(y)\right)=I(t, y)+I I(t, y) \tag{8.61}
\end{equation*}
$$ similar to 8.47), where $l_{i}$ is a linear interpolation of $M_{i}$ in $y, A^{(4)}$ is the 4 -th order Hermite interpolation (8.19) of $f\left(x_{i}, y\right)$ in $y \operatorname{using} f\left(x_{i}, y_{j}\right), j=0,1$ and $\partial_{y} f\left(x_{i}, y_{j}\right), j=0,1$ for $i=0,1$

$$
\begin{equation*}
A_{i}^{(4)}(y)=H_{4}\left(f\left(x_{i}, \cdot\right), y_{0}, y_{1}\right)(y) \tag{8.62}
\end{equation*}
$$

For a fixed $t$, applying the linear interpolation error bound for $M_{i}-l_{i}$ and (8.45), we yield

$$
\begin{equation*}
\left|l_{i}-M_{i}\right| \leq \frac{h_{2}^{2}}{8}\left\|\partial_{y y} M_{i}\right\|_{L^{\infty}(D)} \leq \frac{h_{2}^{2} h_{1}^{2}}{16}\left\|\partial_{x}^{2} \partial_{y}^{2} f\right\|_{L^{\infty}(D)}, \quad i=1,2 \tag{8.63}
\end{equation*}
$$

which along with (8.21) for interpolation error $A_{i}^{(4)}-f\left(x_{i}, y\right)$ in $y, t(1-t) \leq \frac{1}{4}$ implies
$\left|H_{4}-P_{4}\right| \leq \frac{h_{2}^{4}}{384}\left\|\partial_{y}^{4} f\right\|_{L^{\infty}(D)}+\left(\left|t^{2}(t-1)\right|+\left|t(1-t)^{2}\right|\right) \max _{i}\left(\left|l_{i}-M_{i}\right|\right) \leq \frac{h_{2}^{4}}{384}\left\|\partial_{y}^{4} f\right\|_{L^{\infty}(D)}+\frac{h_{1}^{2} h_{2}^{2}}{64}\left\|\partial_{x}^{2} \partial_{y}^{2} f\right\|_{L^{\infty}(D)}$.
Applying (8.21) for the interpolation error $H_{4}-f$ in $x$, we obtain
$\left|P_{4}-f\right| \leq\left|H_{4}-f\right|+\left|H_{4}(x, y)-P_{4}(x, y)\right| \leq \frac{1}{384}\left(h_{1}^{4}\left\|\partial_{x}^{4} f\right\|_{L^{\infty}(D)}+h_{2}^{4}\left\|\partial_{y}^{4} f\right\|_{L^{\infty}(D)}\right)+\frac{h_{1}^{2} h_{2}^{2}}{64}\left\|\partial_{x}^{2} \partial_{y}^{2} f\right\|_{L^{\infty}(D)}$.
To estimate $P_{4}$, we follow the estimates (8.49) and (8.50)

$$
|I(y)| \leq \max _{i=0,1}\left|A_{i}^{(4)}(y)\right|_{L^{\infty}\left[y_{0}, y_{1}\right]}, \quad|I I(y)| \leq \frac{4}{27} \max \left(\left|l_{1}(y)-l_{0}(y)\right|,\left|l_{0}(y)\right|,\left|l_{1}(y)\right|\right)
$$

The estimate of the 4 -th order interpolation polynomial $A_{i}^{(4)}(y)$ in $y$ follows (8.24). Since $l_{i}(y), l_{1}(y)-l_{0}(y)$ is linear in $y$, the maximum of each term is achieved at the endpoint $y=$ $y_{0}, y=y_{1}$, and we can bound $l_{i}, l_{1}(y)-l_{0}(y)$ and $I I(t)$.
Estimate $\partial f$. We use $\partial H_{4}(x, y)$ to approximate $\partial f$, which has the formula (8.52). We further approximate $h_{1} \partial_{x} H_{4}$ by constructing interpolation similar to (8.56)

$$
\tilde{S}_{4}=B(y)+\left(3 t^{2}-2 t\right) l_{1}(y)+\left(3 t^{2}-4 t+1\right) l_{0}(y)
$$

with $Q_{i}$ in (8.56) replaced by the linear interpolation of $M_{i}$ (8.52), $l_{i}, B(y)$ is the same as (8.53). Using (8.54), (8.63), and (8.68), we yield

$$
\left.\begin{array}{l}
\left|\partial_{x} f-\frac{\tilde{S}}{h_{1}}\right| \leq\left|\frac{\tilde{S}_{4}}{h_{1}}-\partial_{x} H_{4}\right|+\left|\partial_{x} f-\partial_{x} H_{4}\right|=I+I I \\
|I|
\end{array}\right) \leq\left|\frac{B(y)}{h_{1}}-\frac{f\left(x_{1}, y\right)-f\left(x_{0}, y\right)}{h_{1}}\right|+\frac{1}{h_{1}}\left(\left|3 t^{2}-2 t\right|+\left|3 t^{2}-4 t+1\right|\right) \max _{i}\left|M_{i}-l_{i}\right|
$$

where $D=\left[x_{0}, x_{1}\right] \times\left[y_{0}, y_{1}\right]$. For $I I$, we apply (8.36). It remains to estimate $\tilde{S}$. Using the notation in (8.57), $Q_{i, l i n}=l_{i}, B=B_{l i n}+B_{\text {nlin }}$, we obtain

$$
\tilde{S}_{4}=B+\left(3 t^{2}-2 t\right) Q_{1, l i n}+\left(3 t^{2}-4 t+1\right) Q_{0, l i n}=S_{l i n}+B_{n l i n}
$$

and then estimate $\tilde{S}_{4}$ using (8.58) and (8.60). The estimate for $\partial_{y} f$ is similar.
8.5. Weighted estimates of a function using derivatives. In our estimate of the residual error or some norms, we need to estimate $F(x) \rho(x)$ near $x=0$ with a singular weight $\rho$. In this section, we discuss how to use the estimate of the derivatives of $F$ to estimate the weighted norm of $F$. Note that $\partial_{x}^{i} \partial_{y}^{j} F$ can be estimated by the methods in Sections 8.1 8.4,

Typical behavior of $\rho$ near $x=0$ is $|x|^{-k}, k=2, \frac{5}{2}, 3$ or $|x|^{-k} x_{1}^{-1 / 2}$. See (6.4), (6.3). By decomposing $\rho=|x|^{-i} x_{1}^{-j / 2} \rho_{m}$, where $\rho_{m}$ is regular near $x=0$, we only need to estimate $F|x|^{-i} x^{-j / 2}$. Denote

$$
E_{x}(F, i)(x, y) \triangleq \frac{1}{x^{i+1}} \int_{0}^{x} F(z, y)(x-z)^{i} d z, \quad E_{y}(F, i)(x, y) \triangleq \frac{1}{y^{i+1}} \int_{0}^{y} F(x, z)(y-z)^{i} d z
$$

Using Fubini's Theorem, we get

$$
E_{x}\left(E_{y}(F, j), i\right)=E_{y}\left(E_{x}(F, i), j\right)=\frac{1}{x^{i+1} y^{j+1}} \int_{0}^{x} \int_{0}^{y} F(x-t)^{i}(y-s)^{j} d t d s \triangleq E_{i j}(F)(x, y)
$$

Using the piecewise bound of $F$, we can bound these functions easily. See 8.66).
For $F(x, y)$ odd in $x$ and $\nabla^{k} F(0)=0$ for $k \leq 2$, we have the following identities

$$
\begin{aligned}
F(x, y) & =\int_{0}^{y} F_{y}(x, z) d z+F(x, 0) \\
& =\int_{0}^{x} \int_{0}^{y} F_{x y y}(t, s)(y-s) d t d s+y \int_{0}^{x} F_{x x y}(t, 0)(x-t) d t+\frac{1}{2} \int_{0}^{x} F_{x x x}(t, 0)(x-t)^{2} d t
\end{aligned}
$$

Using the average operators, we yield

$$
|F(x, y)| \leq E_{x y}\left(\left|F_{x y y}\right|, 0,1\right) x y^{2}+x^{2} y E_{x}\left(\left|F_{x x y}(\cdot, 0)\right|, 1\right)+\frac{x^{3}}{2} E_{x}\left(\left|F_{x x x}(\cdot, 0)\right|, 2\right)(x, 0)
$$

Denote $\beta=\arctan \left(\frac{y}{x}\right), r=\left(x^{2}+y^{2}\right)^{1 / 2}$. Using these estimates, for $a+b=3, b \leq 1$, we get

$$
\begin{align*}
\frac{|F(x, y)|}{r^{a} x^{b}} \leq & \frac{1}{2} E_{x}\left(\left|F_{x x x}\right|, 2\right)(x, 0) \cos ^{3-b}(\beta)+E_{x y}\left(\left|F_{x y y}\right|, 0,1\right) \cos \beta^{1-b} \sin ^{2} \beta  \tag{8.64}\\
& +E_{x}\left(\left|F_{x x y}(\cdot, 0)\right|, 1\right) \cos ^{2-b} \sin \beta
\end{align*}
$$

Since we have the bounds for these coefficients, e.g., $E_{x y}\left(\left|F_{x x y}\right|, 0,1\right)$, by maximizing $\beta \in[0, \pi / 2]$, we obtain the bounds for $\frac{F}{r^{3}}$ and $\frac{F}{r^{5 / 2} x^{1 / 2}}$.

Similarly, we can bound $\frac{\partial_{x}^{i} \partial_{y}^{j} F}{r^{k}}, i+j+k \leq 3$. For odd $F$, we have (8.65)

$$
\begin{aligned}
|F| & =\left|\int_{0}^{x} F_{x}(z, y) d z\right| \leq E_{x}\left(\left|F_{x}\right|, 0\right) x \\
|F| & =\left|\int_{0}^{y} F_{y}(x, z) d z+F(x, 0)\right|=\left|\int_{0}^{x} \int_{0}^{y} F_{x y}(t, s) d x d y+\int_{0}^{x} F_{x x}(t, 0)(x-t) d t\right| \\
& \leq E_{x y}\left(\left|F_{x y}\right|, 0,0\right) x y+E_{x}\left(\left|F_{x x}(\cdot, 0)\right|, 1\right)(x, 0) x^{2}, \quad \nabla^{k} F(0)=0, k=0,1 \\
\left|F_{x}\right| & =\left|\int_{0}^{y} F_{x y y}(x, z)(y-z) d z+y \int_{0}^{x} F_{x x y}(z, 0) d z+\int_{0}^{x} F_{x x x}(z, 0)(x-z) d z\right| \\
& \leq E_{y}\left(\left|F_{x y y}\right|, 1\right) y^{2}+x y E_{x}\left(\left|F_{x x y}\right|(\cdot, 0) \mid, 0\right)+x^{2} E_{x}\left(\left|F_{x x x}(\cdot, 0)\right|, 1\right), \quad \nabla^{k} F(0)=0, k \leq 2
\end{aligned}
$$

Using estimate similar to (8.64), we can bound $\frac{F}{|x|^{2}}, \frac{F}{|x|^{3 / 2}\left|x_{1}\right|^{1 / 2}}, \frac{F}{|x|}$.
Weighted derivatives. Similarly, we estimate $\frac{\partial_{x_{i}} F}{|x|^{2}}, \frac{\partial_{x_{i}} F x_{j}^{1 / 2}}{|x|^{5 / 2}}$. For odd $F$ with $\nabla^{k} F=0, k \leq 2$, using (8.65) with $F$ replaced by $F_{y}$, we get

$$
\left|F_{y}\right| \leq E_{x y}\left(\left|F_{x y y}\right|, 0,0\right) x y+E_{x}\left(\mid F_{x x y}(\cdot, 0), 1\right)(x, 0) x^{2}
$$

Then, we can use the method in (8.64) to estimate

$$
\frac{\left|x_{j}\right|^{\alpha} \partial_{x_{i}} F}{|x|^{2+\alpha}}=(g(\beta))^{\alpha} \frac{\partial_{x_{i}} F}{|x|^{2}}, \quad g(\beta)=\cos \beta, j=1, \text { or } \sin \beta, j=2
$$

We also need to estimate $\frac{F_{x}}{|x|}$ and $\frac{F_{y}}{|x|}$. Using (8.65) with $F$ replaced by $F_{y}$, we get

$$
\left|F_{y}\right| \leq E_{x}\left(\left|F_{x y}\right|, 0\right) x
$$

For $\frac{F_{x}}{|x|}$, we have two cases. If $F(x, 0) \equiv 0$, we yield

$$
\left|F_{x}\right| \leq E_{y}\left(\left|F_{x y}, 0\right|\right) y
$$

Without the vanishing conditions, we require $\nabla F(0,0)=0$ and yield

$$
\begin{aligned}
\left|F_{x}(x, y)\right| & =\left|\int_{0}^{y} F_{x y}(x, z) d z+F_{x}(x, 0)\right|=\left|\int_{0}^{y} F_{x y}(x, z) d z+\int_{0}^{x} F_{x x}(z, 0) d z\right| \\
& \leq E_{y}\left(\left|F_{x y}\right|, 0\right) y+E_{x}\left(\left|F_{x x}\right|(\cdot, 0), 0\right) x
\end{aligned}
$$

Then we apply the method in (8.64) to estimate $\frac{\partial_{x_{i}} F}{|x|}$.
Piecewise bound for powers. Let $x_{i}=i h, I_{i}=\left[x_{i-1}, x_{i}\right]$. Suppose that $0 \leq f$ with $f(x) \leq$ $f_{i}, x \in I_{i}, i \geq 1$. We can obtain a piecewise bound for

$$
\begin{equation*}
I_{k}(f)=\frac{1}{x^{k+1}} \int_{0}^{x} f(z)(x-z)^{k} d z=\frac{1}{x^{k+1}} \int_{0}^{x} f(x-z) z^{k} d z=\int_{0}^{1} f(x(1-t)) t^{k} d t \tag{8.66}
\end{equation*}
$$

from above for $x \in I_{m}=[(m-1) h, m h]$, where we have used a change of variable $z=t x$. We fix $x \in I_{m}=[(m-1) h, m h]$. We partition $[0,1]$ into $J_{i}=[(i-1) / m, i / m], 1 \leq i \leq m$. Since for $t \in[(i-1) / m, i / m]$, we have $(m-1)(m-i)=m^{2}-m(i+1)+i \geq m(m-i-1)$ and $x(1-t) \geq(m-1) h\left(1-\frac{i}{m}\right)=\frac{(m-1)(m-i) h}{m} \geq(m-i-1) h, \quad x(1-t) \leq m h\left(1-\frac{i-1}{m}\right)=(m-i+1) h$.

Thus for $t \in J_{i}$, we get $x(1-t) \in I_{m-i} \cup I_{m-i+1}, I_{0}=\emptyset, f(x(1-t)) \leq \max \left(f_{m-i}, f_{m-i+1}\right)$. It follows
$I_{k}(f)=\sum_{1 \leq j \leq m} \int_{J_{i}} f(x(1-t)) t^{k} d t \leq \frac{1}{k+1} \sum_{1 \leq i \leq m} \frac{1}{m^{k+1}}\left(i^{k+1}-(i-1)^{k+1}\right) \max \left(f_{m-i}, f_{m-i+1}\right), f_{0}=0$.
8.6. Estimate of some explicit polynomials. We use the following bounds for some polynomials in the error estimate of the interpolation.
Lemma 8.1. We have the following estimates

$$
\begin{align*}
& |(t-a)(t-b)| \leq \frac{(b-a)^{2}}{4}, \quad t \in[a, b]  \tag{8.67}\\
& \left|3 t^{2}-2 t\right|+\left|3 t^{2}-4 t+1\right| \leq 1, \quad t \in[0,1]  \tag{8.68}\\
& |t(t-1)(t-2)| \leq \frac{2}{3 \sqrt{3}}, \quad t \in[0,2]  \tag{8.69}\\
& t(t-1)^{2} \leq \frac{4}{27}, \quad t \in[0,1]  \tag{8.70}\\
& |t(t-1)(t-2)(t-3)| \leq 1, \quad t \in[0,3]  \tag{8.71}\\
& \left|t^{2}(t-1)^{2}(t-2)\right| \leq \frac{1}{10}, \quad t \in[0,1] \tag{8.72}
\end{align*}
$$

Proof. The proof of (8.67) follows from the inequality of arithmetic and geometric means (AMGM) or a direct calculation.

For (8.68), firstly, we note that $|a|+|b|=|a+b|$ or $|a-b|$. It suffices to prove $|a+b| \leq 1$ and $|a-b| \leq 1$ for $a=3 t^{2}-2 t, b=3 t^{2}-4 t+1$. Since $t^{2}-t \in[-1 / 4,0], 2 t-1 \in[-1,1]$, we have

$$
a+b=6 t^{2}-6 t+1 \in[-1 / 2,1], \quad a-b=2 t-1 \in[-1,1]
$$

which implies $|a+b|,|a-b| \leq 1$. We prove the desired result.
Denote $s=t-1 \in[-1,1]$. Then using the AM-GM inequality, we have

$$
t^{2}(t-1)^{2}(t-2)^{2}=(t-1)^{2}\left(t^{2}-2 t\right)^{2}=\frac{1}{2} 2 s^{2}\left(1-s^{2}\right)^{2} \leq \frac{1}{2}\left(\frac{2 s^{2}+2\left(1-s^{2}\right)}{3}\right)^{3}=\frac{4}{27}
$$

Taking the squart root on both sides proves (8.69).
To prove (8.70), applying the AM-GM inequality, we get

$$
t(1-t)^{2}=\frac{1}{2} 2 t(1-t)^{2} \leq \frac{1}{2}\left(\frac{2 t+2(1-t)}{3}\right)^{3}=\frac{4}{27}
$$

Denote $s=t(3-t) \in\left[0, \frac{9}{4}\right]$. Then we have $|s-1|^{2} \in[0,2]$ and

$$
|(t-1)(t-2) t(t-3)|=\left|s\left(t^{2}-3 t+2\right)\right|=|(2-s) s|=\left|1-(s-1)^{2}\right| \leq 1
$$

which implies (8.71).
To prove (8.72), we use (8.67) with $a=0, b=1$ and (8.69) to obtain

$$
\left|t^{2}(t-1)^{2}(t-2)\right| \leq \frac{1}{4} \frac{2}{3 \sqrt{3}}=\frac{1}{6 \sqrt{3}}<\frac{1}{10}
$$

where the last inequality follows from $(6 \sqrt{3})^{2}=108>100=10^{2}$.

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