# STABLE NEARLY SELF-SIMILAR BLOWUP OF THE 2D BOUSSINESQ AND 3D EULER EQUATIONS WITH SMOOTH DATA I: ANALYSIS 

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#### Abstract

Inspired by the numerical evidence of a potential 3D Euler singularity 65 66, we prove finite time blowup of the 2D Boussinesq and 3D axisymmetric Euler equations with smooth initial data of finite energy and boundary. There are several essential difficulties in proving finite time blowup of the 3D Euler equations with smooth initial data. One of the essential difficulties is to control a number of nonlocal terms that do not seem to offer any damping effect. Another essential difficulty is that the strong advection normal to the boundary introduces a large growth factor for the perturbation if we use weighted $L^{2}$ or $H^{k}$ estimates. We overcome this difficulty by using a combination of a weighted $L^{\infty}$ norm and a weighted $C^{1 / 2}$ norm, and develop sharp functional inequalities using the symmetry properties of the kernels and some techniques from optimal transport. Moreover we decompose the linearized operator into a leading order operator plus a finite rank operator. The leading order operator is designed in such a way that we can obtain sharp stability estimates. The contribution from the finite rank operator to linear stability can be estimated by constructing approximate solutions in space-time. This enables us to establish nonlinear stability of the approximate self-similar profile and prove stable nearly self-similar blowup of the 2D Boussinesq and 3D Euler equations with smooth initial data and boundary.


## 1. Introduction

The question whether the 3 D incompressible Euler equations can develop a finite time singularity from smooth initial data of finite energy is one of the most outstanding open questions in the theory of nonlinear partial differential equations and fluid dynamics. The main difficulty is due to the presence of the vortex stretching term in the vorticity equation:

$$
\begin{equation*}
\boldsymbol{\omega}_{t}+\mathbf{u} \cdot \nabla \boldsymbol{\omega}=\boldsymbol{\omega} \cdot \nabla \mathbf{u} \tag{1.1}
\end{equation*}
$$

where $\boldsymbol{\omega}=\nabla \times \mathbf{u}$ is the vorticity vector of the fluid, and $\mathbf{u}$ is related to $\boldsymbol{\omega}$ via the BiotSavart law. The velocity gradient $\nabla \mathbf{u}$ formally has the same scaling as vorticity $\boldsymbol{\omega}$. Thus the vortex stretching term has a nonlocal quadratic nonlinearity in terms of vorticity. However, the nonlocal nature of the vortex stretching term can lead to dynamic depletion of the nonlinear vortex stretching, which could prevent a finite time blowup, see e.g. [25, 34, 57]. The interested readers may consult the excellent surveys [24, 43, 54, 60, 67] and the references therein.

The blowup analysis presented in this paper is inspired by the computation of Luo-Hou [65,66] in which they presented some convincing numerical evidence that the 3D axisymmetric Euler equations with smooth initial data and boundary develop a potential finite time singularity. Inspired by the recent breakthrough of Elgindi 35 (see also 36) on the blowup of the axisymmetric Euler equations without swirl for $C^{1, \alpha}$ initial velocity, we have proved asymptotically self-similar blowup of the 2D Boussinesq equations and the nearly self-similar blowup of the 3D axisymmetric Euler equations with $C^{1, \alpha}$ velocity and boundary in [17. The blowup analysis presented in [17] takes advantage of the $C^{1, \alpha}$ velocity in an essential way and does not generalize to prove the Hou-Luo blowup scenario with smooth initial data. The results presented in this paper provide the first rigorous proof of stable nearly self-similar blowup of the 2D Boussinesq and 3D Euler equations with smooth data and boundary.

The main results of this paper are stated by the two informal theorems below. The more precise and stronger statement of Theorem 1 will be given by Theorem 3 in Section 2 and the precise statement of Theorem 2 will be given Theorem 4 in Section 6.

Theorem 1. Let $\theta$, $\mathbf{u}$ and $\omega$ be the density, velocity and vorticity in the 2D Boussinesq equations (2.3) -(2.5), respectively. There is a family of smooth initial data $\left(\theta_{0}, \omega_{0}\right)$ with $\theta_{0}$ being even and $\omega_{0}$ being odd, such that the solution of the Boussinesq equations develops a singularity in finite time $T<+\infty$. The initial velocity field $\mathbf{u}_{0}$ has finite energy. The blowup solution $(\theta(t), \omega(t))$ is nearly self-similar in the sense that $(\theta(t), \omega(t))$ with suitable dynamic rescaling is close to an approximate blowup profile $(\bar{\theta}, \bar{\omega})$ up to the blowup time. Moreover, the blowup is stable for initial data $\left(\theta_{0}, \omega_{0}\right)$ close to $(\bar{\theta}, \bar{\omega})$ in some weighted $L^{\infty}$ and $C^{1 / 2}$ norm.

Theorem 2. Consider the $3 D$ axisymmetric Euler equations in the cylinder $r, z \in[0,1] \times \mathbb{T}$. Let $u^{\theta}$ and $\omega^{\theta}$ be the angular velocity and angular vorticity, respectively. The solution of the 3D Euler equations (2.1)-(2.2) develops a nearly self-similar blowup (in the sense described in Theorem 11) in finite time for some smooth initial data $\omega_{0}^{\theta}$, $u_{0}^{\theta}$ supported away from the symmetry axis $r=0$. The initial velocity field has finite energy, $u_{0}^{\theta}$ and $\omega_{0}^{\theta}$ are odd and periodic in $z$. The blowup is stable for initial data $\left(u_{0}^{\theta}, \omega_{0}^{\theta}\right)$ that are close to the approximate blowup profile $\left(\bar{u}^{\theta}, \bar{\omega}^{\theta}\right)$ after proper rescaling subject to some constraint on the initial support size.
1.1. A novel framework of analysis with computer assistance. One of our main contributions is to introduce a novel framework of analysis that enables us to obtain sharp stability estimates. In our analysis, we combine sharp functional inequalities, energy estimates, and approximate space-time solutions constructed numerically with rigorous error control. We follow the framework in [17, 19, 20 to establish finite time blowup of the 2D Boussinesq and 3D Euler equations by proving the nonlinear stability of an approximate steady state of the dynamic rescaling formulation. A very important first step is to construct an approximate steady state with sufficiently small residual errors. We achieve this by decomposing the solution into a semianalytic part capturing the far field behavior of the solution and a numerically computed part with compact support. The approximate steady state gives an approximate self-similar profile. See more discussions in Section 7. We remark that there has been some recent exciting development of using a physics-informed neural network (PINN) to construct an approximate steady state of the 2D Boussinesq equations, see [89].

Establishing linear stability of the approximate steady state is the most crucial step in our blowup analysis. One essential difficulty is that the advection normal to the boundary for smooth initial data introduces a large growth factor if we use weighted $L^{2}$ or $H^{k}$ energy estimates similar to [17, 19, 20, 35, see more discussions in Section 2 . To overcome it, we choose a weighted $L^{\infty}$ norm to extract the maximal amount of damping from the local terms without suffering from the destabilizing effect due to advection normal to the boundary [55-57]. In order to close the energy estimates, we use a combination of the weighted $L^{\infty}$ norm and the weighted $C^{1 / 2}$ norm.

To estimate the nonlocal terms, we derive sharp $C^{1 / 2}$ estimates for $\nabla \mathbf{u}$ using the symmetry properties of the kernels and some techniques from optimal transport [87, 88]. We decompose the Biot-Savart law into two parts. The main part captures the most singular part of the BiotSavart law, and we apply the sharp functional inequalities for its $C^{1 / 2}$ estimate. The terms from the second part are more regular. We can approximate them by a finite rank operator and obtain sharp estimates by constructing space-time solutions with rigorous error control.

We use the 2D Boussinesq equations to give a high level description of the linear stability analysis using this new framework of analysis. More discussions and motivation will be provided in Section 2, Let $\bar{\omega}, \bar{\theta}$ be an approximate steady state. We denote $W=\left(\omega, \theta_{x}, \theta_{y}\right)$ and decompose $W=\bar{W}+\widetilde{W}$ with $\bar{W}=\left(\bar{\omega}, \bar{\theta}_{x}, \bar{\theta}_{y}\right)$. We further denote by $\mathcal{L}$ the linearized operator around $\bar{W}$ that governs the perturbation $\widetilde{W}$ in the dynamic rescaling formulation (see Section 2.5),

$$
\begin{equation*}
\widetilde{W}_{t}=\mathcal{L}(\widetilde{W}) \tag{1.2}
\end{equation*}
$$

where the coefficients of $\mathcal{L}$ depend on the approximate steady state $\bar{W}$. We further decompose the linearized operator $\mathcal{L}$ into a leading order operator $\mathcal{L}_{0}$ plus a finite rank perturbation $\mathcal{K}$, i.e $\mathcal{L}=\mathcal{L}_{0}+\mathcal{K}$. The leading order operator $\mathcal{L}_{0}$ is constructed in such way that we can obtain sharp stability estimates using weighted estimates and sharp functional inequalities.

In Part I of our paper, we perform the weighted energy estimates. In our analysis, we decompose $\widetilde{W}=\widetilde{W}_{1}+\widetilde{W}_{2}$. The first term $\widetilde{W}_{1}$ captures the main part of the perturbation,
which is essentially governed by the leading order operator $\mathcal{L}_{0}$ with a weak coupling to $\widetilde{W}_{2}$ through nonlinear interaction. The second term $\widetilde{W}_{2}$ captures the contribution from the finite rank operator. Our stability analysis is performed mainly for $\widetilde{W}_{1}$ since $\widetilde{W}_{2}$ is driven by $\widetilde{W}_{1}$ (see (1.3) below). We establish nonlinear stability using the stability lemma (see Lemma A.2 and Section (2.3), which depends on various constants in the estimates. For this purpose, we need to obtain relatively sharp energy estimates for the leading order operator $\mathcal{L}_{0}$ by subtracting $\mathcal{L}$ from a finite rank operator $\mathcal{K}$. Without subtracting $\mathcal{K}$, we would not be able to obtain linear and nonlinear stability of the approximate self-similar profile.

The constants in the weighted energy estimates depend on the approximate steady state that we constructed numerically in Section 7 and the singular weights that we use. The approximate steady state is represented based on piecewise polynomials. We can obtain rigorous bounds for its high order derivatives. Such bounds in turn provide rigorous bounds for lower order derivatives, the pointwise values and various integrals involving the approximate steady state by using standard numerical analysis. See more discussions for the main ideas below (2.15). In Part II of our paper [15], we will provide sharp and rigorous upper bounds for these constants and the residual error of the approximate steady state. In Section 4 of Part II [15], we also estimate the velocity in the regular case by bounding various integrals with computer assistance. These sharp estimates of the constants enable us to prove that the inequalities in our stability lemma hold for our approximate self-similar profile. Thus we can complete the stability analysis of the approximate self-similar profile and prove the nearly self-similar blowup of the 2D Boussinesq and 3D Euler equations. See Section 2.3 for more discussions of our blowup analysis.

We use the following toy model to illustrate the main ideas by considering $\mathcal{K}$ as a rank-one operator $\mathcal{K}(\widetilde{W})=a(x) P(\widetilde{W})$ for some operator $P$ satisfying (i) $P(\widetilde{W})$ is constant in space; (ii) $\|P(\widetilde{W})\| \leq c\|\widetilde{W}\|$. Given initial data $\widetilde{W}_{0}$, we decompose (1.2) as follows

$$
\begin{align*}
& \partial_{t} \widetilde{W}_{1}(t)=\mathcal{L}_{0} \widetilde{W}_{1}, \quad \widetilde{W}_{1}(0)=\widetilde{W}_{0} \\
& \partial_{t} \widetilde{W}_{2}(t)=\mathcal{L} \widetilde{W}_{2}+a(x) P\left(\widetilde{W}_{1}(t)\right), \quad \widetilde{W}_{2}(0)=0 \tag{1.3}
\end{align*}
$$

It is easy to see that $\widetilde{W}=\widetilde{W}_{1}+\widetilde{W}_{2}$ solves (1.2) with initial data $\widetilde{W}_{0}$. The second part $\widetilde{W}_{2}$ is driven by the rank-one forcing term $a(x) P\left(\widetilde{W}_{1}(t)\right)$. Using Duhamel's principle, the fact that $P\left(\widetilde{W}_{1}(t)\right)$ is constant in space, we yield

$$
\begin{equation*}
\widetilde{W}_{2}(t)=\int_{0}^{t} P\left(\widetilde{W}_{1}(s)\right) e^{\mathcal{L}(t-s)} a(x) d s \tag{1.4}
\end{equation*}
$$

Since the leading operator $\mathcal{L}_{0}$ has the desired stability property by construction, $\widetilde{W}_{1}(t)=$ $e^{\mathcal{L}_{0}(t)} \widetilde{W}_{0}$ decays in $L^{\infty}(\varphi)$ ( $\varphi$ is a singular weight) and we can control $P\left(\widetilde{W}_{1}(s)\right)$. By checking the decay of $e^{\mathcal{L}(t)} a(x)$ in the energy space for large $t$, we can obtain the stability estimate of $\widetilde{W}_{2}$. A crucial idea in the estimate of $\widetilde{W}_{2}$ is to bridge the energy estimates and numerical PDEs via an approximate solution in space and time. Note that $e^{\mathcal{L}(t)} a(x)$ is equivalent to solving the linear evolution equation $v_{t}=\mathcal{L}(v)$ with initial data $v_{0}=a(x)$. Due to the rapid decay of the linearized equation, we solve this initial value problem using a numerical scheme up to a modest time. The stability property of $\widetilde{W}_{1}$ allows us to control the numerical error in computing $e^{\mathcal{L}(t)} a(x)$ and obtain sharp stability estimates for $\widetilde{W}_{2}$.

We remark that we have used the approximate steady state in an essential way in establishing the linear stability of the approximate self-similar blowup profile. Moreover, the stability factor (or the damping factor $\lambda$ ) that we obtain in Lemma A. 1 is quite small. Without obtaining relatively sharp upper bounds for the constants in the energy estimates that depend on the approximate self-similar profile, we would not have been able to apply the stability Lemma A. 2 to prove nonlinear stability. If we attempt to prove finite time blowup of the 3D Euler equations around a generic blowup profile $\bar{U}$ without specific information about this blowup profile, the stability conditions for $\bar{U}$ in Lemma A.2 may not be satisfied. Thus it seems quite difficult to prove stable blowup without using any computer assistance.

We note that in obtaining sharp bound on the blow-up rate for the critical nonlinear Schrödinger equation (see e.g. [72]), the property of the ground state solution has been used in an essential way. Since we do not have an explicit ground state for the 3D Euler equation, the role of an approximate steady state with a small residual error that we constructed numerically plays a role similar to the ground state in the study of blowup of other nonlinear PDEs, including the nonlinear Schrödinger equation [72] and the Keller-Segel system [23.

To pass from the 2D Boussinesq equations to the 3D axisymmetric Euler equations, we follow the same ideas presented in our previous work [17] by controlling the support of the solution to be in a small region close to the boundary and does not intersect the symmetry axis. The asymptotic scaling properties of the Biot-Savart kernels are exactly the same as those of the Biot-Savart kernels for the 2D Boussinesq equations up to some asymptotically small terms after making appropriate changes of variables. We will provide some additional estimates to control these asymptotically small terms and prove the blowup of the 3D Euler equations.
1.2. Comparison between our method of analysis and the topological argument. Our method of analysis shares some similarity with the recently developed blowup analysis using a topological argument, see e.g. 69, 73 75. In the topological argument, one also constructs a compact perturbation operator $\mathcal{K}$ to the linearized operator $\mathcal{L}$. After subtracting the compact perturbation operator from the linearized operator, one can establish linear stability of the leading operator $\mathcal{L}_{0}$ in some Hilbert space. The compact perturbation operator can be approximated by a finite rank operator. This method has been successfully used to prove blowup of several nonlinear PDEs with potentially finitely many unstable directions.

The main difference between our method of analysis and the topological argument is in the way we estimate the finite rank operator $\mathcal{K}$. First of all, in our framework, we do not require the energy space to be a Hilbert space. The main innovation of our approach is that we develop a constructive method of analysis to establish stability of the finite rank operator by solving a finite number of decoupled linear PDEs in space-time with rigorous error control. In comparison, a typical topological argument may only allow one to establish stability of the leading order operator $\mathcal{L}_{0}$ at the expenses of creating potentially finitely many unstable directions induced by the finite rank operator. Moreover, if one attempts to establish stability of the leading order operator $\mathcal{L}_{0}$ using a high order Sobolev norm $H^{k}$, it would be extremely difficult to construct an approximate self-similar profile with a small residual error in $H^{k}$ with a large $k$, e.g. $k \geq 14$. See Section 2.7.1 for more discussion.
1.3. Review of literature. There has been a lot of effort in studying 3D Euler singularities using various simplified models. Several 1D models, including the Constantin-Lax-Majda (CLM) model [26, the De Gregorio (DG) model 32 33, the gCLM model [79] and the Hou-Li model [56], have been introduced to study the effect of advection and vortex stretching in the 3D Euler equations. Singularity formation has been established for the CLM model in [26], for the DG model with smooth data in [19] and with $C^{1-}$ data in [11], and for the gCLM model with various parameters in [7, 10, 12, 19, 37, 39, 83]. In [21, the authors proved the blowup of the Hou-Luo model proposed in [66]. In [20], Chen-Hou-Huang proved the asymptotically self-similar blowup of the Hou-Luo model by extending the method of analysis established for the finite time blowup of the De Gregorio model by the same authors in [19]. Inspired by their work on the vortex sheet singularity [5], Caflisch and Siegel have studied complex singularity for 3D Euler equation, see [4,85] and also [80] for the complex singularities for 2D Euler equation.

In [22] 47, 49], the authors proposed several simplified models to study the Hou-Luo blowup scenario 65, 66 and established finite time blowup of these models. In these works, the velocity is determined by a simplified Biot-Savart law in a form similar to the key lemma in the seminal work of Kiselev-Sverak [59]. In [38, 40, Elgindi and Jeong proved finite time blowup for the 2D Boussinesq and 3D axisymmetric Euler equations in a domain with a corner using $\dot{C}^{0, \alpha}$ data. There has been some recent progress in searching for potential Euler and Navier-Stokes singularity in the interior domain, see 5053 .

There has been some interesting recent results on the potential instability of the Euler blowup solutions, see [62, 86]. In a recent paper [14], we showed that the blowup solutions of the 2D

Boussinesq and 3D Euler equations with $C^{1 \alpha}$ velocity considered in [17, 35] are also unstable using the notion of stability introduced in [62,86. The blowup analysis in [17,35] is based on the stability of a self-similar blowup profile using the dynamic rescaling formulation. In comparison, the linear stability in [17,35] is performed by directly linearizing the 3D Euler equations around a particular blowup solution with a fixed blowup time $T$ in the original physical variables.

The rest of the paper is organized as follows. Sections 25 will be devoted to the blowup analysis for the 2D Boussinesq equations and Section 6 will be devoted to the blowup analysis for the 3D Euler equations. In Section2, we provide detailed discussions and some key ingredients in establishing linear stability of an approximate profile using various simplified models. In Section 3. we develop sharp Hölder estimates using optimal transport. In Section 4 we introduce the $L^{\infty}$-based finite rank perturbation method. Section 5 is devoted to energy estimates and Section 7 is devoted to the construction of an approximate self-similar profile using the dynamic rescaling formulation. Some technical estimates and derivations are deferred to the Appendix.

## 2. Linear stability analysis and the main ideas

In this section, we will outline the main ingredients in our stability analysis. We will mainly focus on the 2D Boussinesq equations. As in [17, 19, 20, we will use the dynamic rescaling formulation for the 2D Boussinesq equations in an essential way. The most essential part of our analysis lies in the linear stability. We need to use a number of techniques to extract the damping effect from the linearized operator around the approximate steady state of the dynamic rescaling equations and obtain sharp estimates of various nonlocal terms. Since the damping coefficients we obtain are relatively small, we need to construct an approximate steady state with a very small residual error. This is extremely challenging since the solution is supported on the upper half plane with a slowly decaying tail in the far field.

Passing from linear stability to nonlinear stability is relatively easier by treating the nonlinear terms and residual error as small perturbations to the linear damping terms. See Section 5.9 We generalize the analysis of the 2D Boussinesq equations to the 3D Euler by controlling their differences, which are asymptotically small, see Section 6 .

Denote by $\omega^{\theta}, u^{\theta}$ and $\phi^{\theta}$ the angular vorticity, angular velocity, and angular stream function, respectively. The 3D axisymmetric Euler equations are given below:

$$
\begin{equation*}
\partial_{t}\left(r u^{\theta}\right)+u^{r}\left(r u^{\theta}\right)_{r}+u^{z}\left(r u^{\theta}\right)_{z}=0, \quad \partial_{t}\left(\frac{\omega^{\theta}}{r}\right)+u^{r}\left(\frac{\omega^{\theta}}{r}\right)_{r}+u^{z}\left(\frac{\omega^{\theta}}{r}\right)_{z}=\frac{1}{r^{4}} \partial_{z}\left(\left(r u^{\theta}\right)^{2}\right) \tag{2.1}
\end{equation*}
$$

where the radial velocity $u^{r}$ and the axial velocity $u^{\theta}$ are given by the Biot-Savart law:

$$
\begin{equation*}
-\left(\partial_{r r}+\frac{1}{r} \partial_{r}+\partial_{z z}\right) \phi^{\theta}+\frac{1}{r^{2}} \phi^{\theta}=\omega^{\theta}, \quad u^{r}=-\phi_{z}^{\theta}, \quad u^{z}=\phi_{r}^{\theta}+\frac{1}{r} \phi^{\theta}, \tag{2.2}
\end{equation*}
$$

with the no-flow boundary condition $\phi^{\theta}(1, z)=0$ on the solid boundary $r=1$ and a periodic boundary condition in $z$. For 3D Euler blowup that occurs at the boundary $r=1$, we know that the axisymmetric Euler equations have scaling properties asymptotically the same as those of the 2D Boussinesq equations 67]. Thus, we also study the 2D Boussinesq equations on the upper half space:

$$
\begin{align*}
\omega_{t}+\mathbf{u} \cdot \nabla \omega & =\theta_{x}  \tag{2.3}\\
\theta_{t}+\mathbf{u} \cdot \nabla \theta & =0 \tag{2.4}
\end{align*}
$$

where the velocity field $\mathbf{u}=(u, v)^{T}: \mathbb{R}_{+}^{2} \times[0, T) \rightarrow \mathbb{R}_{+}^{2}$ is determined via the Biot-Savart law

$$
\begin{equation*}
-\Delta \phi=\omega, \quad u=-\phi_{y}, \quad v=\phi_{x} \tag{2.5}
\end{equation*}
$$

where $\phi$ is the stream function with the no-flow boundary condition $\phi(x, 0)=0$ at $y=0$. By making the change of variables $\tilde{\theta} \triangleq\left(r u^{\theta}\right)^{2}, \tilde{\omega}=\omega^{\theta} / r$, we can see that $\tilde{\theta}$ and $\tilde{\omega}$ satisfy the 2 D Boussinesq equations up to the leading order for $r \geq r_{0}>0$.
2.1. Dynamic rescaling formulation. Following [17, 19, 20, we consider the dynamic rescaling formulation of the 2D Boussinesq equations. Let $\omega(x, t), \theta(x, t), \mathbf{u}(x, t)$ be the solutions of (2.3)-(2.5). Then it is easy to show that

$$
\begin{align*}
& \tilde{\omega}(x, \tau)=C_{\omega}(\tau) \omega\left(C_{l}(\tau) x, t(\tau)\right), \quad \tilde{\theta}(x, \tau)=C_{\theta}(\tau) \theta\left(C_{l}(\tau) x, t(\tau)\right) \\
& \tilde{\mathbf{u}}(x, \tau)=C_{\omega}(\tau) C_{l}(\tau)^{-1} \mathbf{u}\left(C_{l}(\tau) x, t(\tau)\right) \tag{2.6}
\end{align*}
$$

are the solutions to the dynamic rescaling equations

$$
\begin{equation*}
\tilde{\omega}_{\tau}(x, \tau)+\left(c_{l}(\tau) \mathbf{x}+\tilde{\mathbf{u}}\right) \cdot \nabla \tilde{\omega}=c_{\omega}(\tau) \tilde{\omega}+\tilde{\theta}_{x}, \quad \tilde{\theta}_{\tau}(x, \tau)+\left(c_{l}(\tau) \mathbf{x}+\tilde{\mathbf{u}}\right) \cdot \nabla \tilde{\theta}=c_{\theta} \tilde{\theta} \tag{2.7}
\end{equation*}
$$

where $\tilde{\mathbf{u}}=(\tilde{u}, \tilde{v})^{T}=\nabla^{\perp}(-\Delta)^{-1} \tilde{\omega}, \mathbf{x}=(x, y)^{T}$,

$$
\begin{equation*}
C_{\omega}(\tau)=\exp \left(\int_{0}^{\tau} c_{\omega}(s) d \tau\right), C_{l}(\tau)=\exp \left(\int_{0}^{\tau}-c_{l}(s) d s\right), C_{\theta}=\exp \left(\int_{0}^{\tau} c_{\theta}(s) d \tau\right) \tag{2.8}
\end{equation*}
$$

$t(\tau)=\int_{0}^{\tau} C_{\omega}(\tau) d \tau$ and the rescaling parameters $c_{l}(\tau), c_{\theta}(\tau), c_{\omega}(\tau)$ satisfy [17]

$$
\begin{equation*}
c_{\theta}(\tau)=c_{l}(\tau)+2 c_{\omega}(\tau) \tag{2.9}
\end{equation*}
$$

We have the freedom to choose the time-dependent scaling parameters $c_{l}(\tau)$ and $c_{\omega}(\tau)$ according to some normalization conditions. These two free scaling parameters are related to the fact that Boussinesq equations have scaling-invariant property with two parameters. The 3D Euler equations enjoy the same property. See [17]. After we determine the normalization conditions for $c_{l}(\tau)$ and $c_{\omega}(\tau)$, the dynamic rescaling equation is completely determined and the solution of the dynamic rescaling equation is equivalent to that of the original equation using the scaling relationship described in (2.6)-(2.8), as long as $c_{l}(\tau)$ and $c_{\omega}(\tau)$ remain finite.

We remark that the dynamic rescaling formulation was introduced in 63, 71] to study the self-similar blowup of the nonlinear Schrödinger equations. This formulation is also called the modulation technique in the literature and has been developed by Merle, Raphael, Martel, Zaag and others. It has been a very effective tool to analyze the formation of singularities for many problems like the nonlinear Schrödinger equation [58, 72, compressible Euler equations [2],3, the nonlinear wave equation [77], the nonlinear heat equation [76], the generalized KdV equation 68], and other dispersive problems. Recently, this method has been applied to study singularity formation in incompressible fluids [17, 35] and related models [10-12, 19].

To simplify our presentation, we still use $t$ to denote the rescaled time in (2.7) and simplify $\tilde{\omega}, \tilde{\theta}$ as $\omega, \theta$

$$
\begin{equation*}
\omega_{t}+\left(c_{l} x+\mathbf{u}\right) \cdot \nabla \omega=\theta_{x}+c_{\omega} \omega, \quad \theta_{t}+\left(c_{l} x+\mathbf{u}\right) \cdot \nabla \theta=c_{\theta} \theta \tag{2.10}
\end{equation*}
$$

Following [20], we impose the following normalization conditions on $c_{\omega}, c_{l}$

$$
\begin{equation*}
c_{l}=2 \frac{\theta_{x x}(0)}{\omega_{x}(0)}, \quad c_{\omega}=\frac{1}{2} c_{l}+u_{x}(0), \quad c_{\theta}=c_{l}+2 c_{\omega} \tag{2.11}
\end{equation*}
$$

For smooth data, these two normalization conditions play the role of enforcing

$$
\begin{equation*}
\theta_{x x}(t, 0)=\theta_{x x}(0,0), \quad \omega_{x}(t, 0)=\omega_{x}(0,0) \tag{2.12}
\end{equation*}
$$

for all time. In fact, we can derive the ODEs of $\theta_{x x}(t, 0)$ and $\omega_{x}(t, 0)$

$$
\frac{d}{d t} \omega_{x}(t, 0)=\left(c_{\omega}-c_{l}-u_{x}(0)\right) \omega_{x}(t, 0)+\theta_{x x}(t, 0), \quad \frac{d}{d t} \theta_{x x}(t, 0)=\left(c_{\theta}-2\left(c_{l}+u_{x}(0)\right)\right) \theta_{x x}(t, 0)
$$

where we use $\left.v\right|_{y=0}=0, v_{x}(t, 0)=0$. Under the conditions (2.11), the right hand sides vanish.
2.2. Main Result. In this section, we state our main result for the 2D Boussinesq equations. We first introduce some notations and define our energy. Let $\psi_{i}, \varphi_{i}, \psi_{i, g}, g_{i}$ be the singular weights defined in (C.1), (C.3), (C.4), $[\cdot]_{C_{g_{i}}^{1 / 2}}$ the Hölder seminorm (2.21) in $\mathbb{R}_{2}^{++}$, and $\mu_{i j}, \tau_{1}, \tau_{2}$
be the parameters chosen in (C.5). We define the energy $E$ on three variables $f_{1}, f_{2}, f_{3}$ as follows (2.13)
$P_{1}=\max _{1 \leq i \leq 3}\left\|f_{i} \varphi_{i}\right\|_{\infty}, \quad P_{2}=\tau_{1}^{-1} \max \left(\left\|f_{1} \psi_{1}\right\|_{C_{g_{1}}^{1 / 2}}, \mu_{1}\left\|f_{2} \psi_{2}\right\|_{C_{g_{2}}^{1 / 2}}, \mu_{2}\left\|f_{3} \psi_{3}\right\|_{C_{g_{3}}^{1 / 2}}, \sqrt{2}\left\|f_{1} \psi_{1}\left|x_{1}\right|^{-\frac{1}{2}}\right\|_{\infty}\right)$
$P_{3}=\tau_{2} \max \left(\mu_{4}\left\|f_{1} \varphi_{g 1}\right\|_{\infty},\left\|f_{2} \varphi_{g, 2}\right\|_{\infty},\left\|f_{3} \varphi_{g, 3}\right\|_{\infty}\right)$,
$P_{4} \triangleq \max \left(\mu_{51}^{-1}\left|u_{x}\left(f_{1} \chi_{\text {ode }}\right)(0)\right|, \mu_{52}^{-1}\left|u_{x}\left(f_{2} \chi_{o d e}\right)(0)\right|, \mu_{6}^{-1}\left|u_{x}\left(f_{1}\right)(0)\right|, \mu_{62}^{-1}\left|u_{x}\left(f_{2}\right)(0)\right|, \mu_{7}^{-1}\left|f_{2, x y}(0)\right|, \mu_{8}^{-1}\left|f_{1, x y}(0)\right|\right)$, $E=\max \left(P_{1}, P_{2}, P_{3}, P_{4}\right)$,
where $u_{x}(f)(0)=-\frac{4}{\pi} \int_{\mathbb{R}_{2}^{++}} \frac{y_{1} y_{2}}{|y|^{4}} f(y) d y, \chi_{\text {ode }}$ is defined in (5.64).
Theorem 3. Let $\left(\bar{\theta}, \bar{\omega}, \overline{\mathbf{u}}, \bar{c}_{l}, \bar{c}_{\omega}\right)$ be the approximate self-similar profile constructed in Section 7 and $E_{*}=5 \cdot 10^{-6}$. Assume that even initial data $\theta_{0}$ and odd $\omega_{0}$ of (2.10) satisfy $E\left(\omega_{0}-\bar{\omega}, \theta_{0, x}\right.$ $\left.\bar{\theta}_{x}, \theta_{0, y}-\bar{\theta}_{y}\right)<E_{*}$. We have

$$
\|\omega-\bar{\omega}\|_{L^{\infty}},\left\|\theta_{x}-\bar{\theta}_{x}\right\|_{L^{\infty}},\left\|\theta_{y}-\bar{\theta}_{y}\right\|_{\infty}<200 E_{*}, \quad\left|u_{x}(t, 0)-\bar{u}_{x}(0)\right|, \quad\left|\bar{c}_{\omega}-c_{\omega}\right|<100 E_{*}
$$

for all time. In particular, we can choose smooth initial data $\omega_{0}, \theta_{0} \in C_{c}^{\infty}$ in this class with finite energy $\left\|\mathbf{u}_{0}\right\|_{L^{2}}<+\infty$ such that the solution to the physical equations (2.3)-(2.5) with these initial data blows up in finite time $T$.
2.3. The main steps in the proof of Theorem 3. We will follow the framework in $[17,19,20]$ to establish finite time blowup by proving the nonlinear stability of an approximate steady state to (2.10). We divide the proof of Theorem 3 into proving the following lemmas. The requirement of smallness of the residual error is incorporated in the conditions A.11), e.g. the term $a_{i j, 3}$, in Lemma 2.4. We define the $C^{1 / 2}$ semi-norm in (2.20), the approximate solution $\hat{W}_{2}$ (4.19), residual operator $\overline{\mathcal{R}}_{i}$ in (2.19), and the energy norm $E_{4}(5.70)$ in Section 5 for energy estimates.

Lemma 2.1. There exists a nontrivial approximate steady state ( $\bar{\omega}, \bar{\theta}, \bar{c}_{l}, \bar{c}_{\omega}$ ) to (2.10), (2.11) with $\bar{\omega}, \bar{\theta} \in C^{4,1}$ and residual errors $\overline{\mathcal{F}}_{i}, i=1,2,3$ sufficiently small in some energy norm.

The construction of an approximate self-similar profile in Lemma 2.1 is provided in Section 7. the estimate of residual error is given in Appendix C. 4 in Part II [15, and the properties of $\left(\bar{\omega}, \bar{\theta}, \bar{c}_{l}, \bar{c}_{\omega}\right)$ are discussed in Section 2.4.

Lemma 2.2. Let $\omega$ be odd in $x_{1}$. Denote $\delta(f, x, z)=f(x)-f(z)$. There exists finite rank approximations $\hat{\mathbf{u}}, \widehat{\nabla \mathbf{u}}$ for $\mathbf{u}(\omega), \nabla \mathbf{u}(\omega)$ with rank less than 50 such that we have the following weighted $L^{\infty}$ and directional Hölder estimate for $f=u, v, \partial_{l} u, \partial_{l} v, x, z \in \mathbb{R}_{2}^{++}, i=1,2, \gamma_{i}>0$

$$
\begin{align*}
\left|\rho_{f}(f-\hat{f})(x)\right| & \leq C_{f, \infty}\left(x, \varphi, \psi_{1}, \gamma\right) \max \left(\|\omega \varphi\|_{\infty}, s_{f} \max _{j=1,2} \gamma_{j}\left[\omega \psi_{1}\right]_{C_{x_{j}}^{1 / 2}\left(\mathbb{R}_{2}^{+}\right)}\right) \\
\frac{\left|\delta\left(\psi_{f}(f-\hat{f}), x, z\right)\right|}{|x-z|^{1 / 2}} & \leq C_{f, i}\left(x, z, \varphi, \psi_{1}, \gamma\right) \max \left(\|\omega \varphi\|_{\infty}, s_{f} \max _{j=1,2} \gamma_{j}\left[\omega \psi_{1}\right]_{C_{x_{j}}^{1 / 2}\left(\mathbb{R}_{2}^{+}\right)}\right) \tag{2.14}
\end{align*}
$$

with $x_{3-i}=z_{3-i}$, where $s_{f}=0$ for $f=u, v, s_{f}=1$ for $f=\partial_{l} u, \partial_{l} v$, the functions $C(x), C(x, z)$ depend on $\gamma$, the weights, and the approximations, the singular weights $\varphi=\varphi_{1}, \varphi_{g, 1}, \varphi_{\text {elli }}, \psi_{\partial u}=$ $\psi_{1}, \psi_{u}$ are defined in (C.3), (C.4), (C.2), the weight $\rho_{10}$ for $\mathbf{u}$ and the weight for $\rho_{i j}$ for $\nabla \mathbf{u}$ with $i+j=2$ are given in (C.2). In the estimate of $f=u, v$, we do not need the Hölder semi-norm and $s_{f}=0$. Moreover, $C(x), C(x, z)$ are bounded in any compact domain of $\mathbb{R}_{2}^{++}$. We have an additional estimate for $\rho_{4}(u-\hat{u})$ similar to the above with $\rho_{4}(\mathbb{C} .2)$ singular along $x_{1}=0$.

Since the weights $\rho_{10} \sim|x|^{-3}, \psi_{1} \sim|x|^{-2}, \psi_{u}$ are singular near $x=0$, without subtracting the approximation $\hat{f}$ from $f, \rho_{f} f$ is not bounded near $x=0$. Based on these finite rank approximations, we can decompose the perturbations.

We also apply similar estimates for the nonlocal error, e.g. $\mathbf{u}(\bar{\varepsilon})$ and $\bar{\varepsilon}$ is the error of solving the Poisson equations. Since we can estimate piecewise bounds of $\bar{\varepsilon}$ following Section 3.6 in Part II [15], instead of using a global norm, we improve the estimate using localized norms, which are much smaller than the global norm. See Lemma 2.3 and Section 4.7 in Part II [15].

Lemma 2.3. There exists $m<50$ approximate solutions $\hat{F}_{i}$ to the linearized equations $\partial_{t} W=$ $\mathcal{L} W$ of (2.10) around $\left(\bar{\omega}, \bar{\theta}, \bar{c}_{l}, \bar{c}_{\omega}\right)$ in Lemma 2.1 from given initial data $\bar{F}_{i}$ with residual error $\hat{F}_{i}(0)-\bar{F}_{i},\left(\partial_{t}-\mathcal{L}\right) \hat{F}_{i}(t)$ small in the energy norm. Furthermore we can decompose the perturbation $W=W_{1}+\widehat{W}_{2}$ with the following properties: (a) $\widehat{W}_{2}$ is constructed based on $\widehat{F}_{i}$, and (4.19) ; (b) $W_{1}$ satisfies equations with the leading order linearized operator $(\mathcal{L}-\mathcal{K}) W_{1}$ up to the small residual operator $\mathcal{R}$ (4.20), (4.21) for some finite rank operator $\mathcal{K}$, and $W_{1}$ depends on $\widehat{W}_{2}$ weakly at the linear level via $\mathcal{R}$. The functionals $a_{i}\left(W_{1}\right), a_{n l, i}(W)$ in the construction of $\widehat{W}_{2}$ and $\mathcal{K}$ (4.19) are related to the finite rank approximations in Lemma 2.2.

Moreover, there exists an energy $E_{4}(t)$ for $W_{1}, W$ (see (5.70)) that controls the weighted $L^{\infty}$ and $C^{1 / 2}$ seminorm of $W_{1}$ such that under the bootstrap assumption $E_{4}(t)<E_{* 0}$ with $E_{* 0}>0$, using the estimates in Lemma 2.2, we can establish nonlinear energy estimates for $E_{4}(t)$.

If the bounds in Lemma 2.2 are tight, and the residual error in the constructions of $(\bar{\omega}, \bar{\theta}), \widehat{F}_{i}$ are small enough, we can use Lemma A. 2 to obtain nonlinear stability.

Lemma 2.4. For $E_{*}=5 \cdot 10^{-6}$, the coefficients in the nonlinear energy estimates of $E_{4}(t)$ satisfy the conditions (A.11), and the statements in Theorem 3 hold true.

We verfiy the inequalities for the stability conditions stated in Lemma 2.4 in Part II [15].
Estimates of nonlocal terms. To establish the nonlinear stability conditions (A.11) in Lemma 2.4. we need to obtain sharp constants in the estimates in Lemma 2.2. The form of the upper bound is related to the energy (2.13), (5.70). Although the upper bounds in Lemma 2.2 are equivalent for different $\gamma$, we choose $\gamma$ according to the energy (2.13), (5.70).

The proof of Lemma 2.2 consists of several steps. Given $\omega \in C^{1 / 2}$, we have $\mathbf{u} \in C^{3 / 2}, \nabla \mathbf{u} \in$ $C^{1 / 2}$. Firstly, in Section 3 and Appendix B we use some methods from optimal transport to establish sharp $C^{1 / 2}$ estimate of $\nabla \mathbf{u}$ and $\nabla \mathbf{u}$ with the localized kernels, e.g. $u_{x}(x, a, b)$ defined in (3.4), which captures the most singular part in the estimates in Lemma 2.2. We remark that $\psi_{f}=\psi_{1}$ in (2.14) for $f=\nabla \mathbf{u}$. We can derive the upper bounds for these sharp $C^{1 / 2}$ estimates of $\nabla \mathbf{u}$ in terms of some explicit $L^{1}$ integrals independent of the weights $\varphi, \psi_{i}$. In Section 5 of the supplementary material II in Part II [16, we will estimate these explicit integrals using integral formulas, numerical quadrature, and obtain the constants for these bounds rigorously.

We can derive the damping terms from the local terms in $\mathcal{L}-\mathcal{K}$ and $\mathcal{L}$ in Sections 5.3.5.5 without using Lemma 2.2 and $\widehat{F}_{i}$ in Lemma 2.3. With the sharp $C^{1 / 2}$ estimates, we can establish the stability conditions (A.11) in the Hölder energy estimate for a fixed $x$ with $|x-z| \rightarrow 0$. We can accomplish this without using the estimates of the more regular part of $\mathbf{u}, \nabla \mathbf{u}$ in Lemma 2.2 discussed below and $\widehat{F}_{i}$ from Part II [15], which are more regular and vanish in the $C^{1 / 2}$ estimate as $|x-z| \rightarrow 0$. See more discussions in Section 2.8.2. This estimate captures the stability of the leading order terms in terms of regularity and is the cornerstone for the entire nonlinear stability analysis. We further develop several methods to control the more regular terms.

Other parts of the estimates in Lemma [2.2, e.g. $I=\psi u_{x}(\omega)-u_{x, a, b}(\omega \psi)-\psi \hat{u}_{x}(\omega)$, involve the velocity with desingularized kernels, which are more regular. In the second step and in Section 4.3, we construct the finite rank approximation $\hat{f}$ for $f, f=\mathbf{u}, \nabla \mathbf{u}$ so that we have better estimates of $I$ than the case without approximation.

In the third step, we perform $C^{1 / 2}$ estimate of the regular part $I$. The term $I$ is only LogLipschitz and is similar to $J(\omega)(s, 0)$ below with $a=0$
$J(\omega)(s, a)=\int_{a \leq \max _{i}\left|s_{i}-y_{i}\right|} K(s, y) \omega(y) d y, \quad|K(s, y)| \leq C_{1}|s-y|^{-1}, \quad|\partial K(s, y)| \leq C_{2}|s-y|^{-2}$.
In the $C^{1 / 2}$ estimate of $I$, for $x, z \in \mathbb{R}_{2}^{++}$, we decompose $I(s)=I_{R}(s, a)+I_{S}(s, a)$. The first part $I_{R}(s, a)$ corresponds to the regular part $I$ with a distance $|a|$ away from the singularity, e.g. $J(f)(s, a)$. The second part $I_{S}(s, a)$ is a singular part $I_{S}(s, a)$ similar to $J(f)(s, 0)-J(f)(s, a)$. For $I_{R}(a, s)$, using the norm $\|\omega \varphi\|_{\infty}$, we reduce estimating its piecewise Lipschitz norm to estimating certain explicit integrals depending on the weight $\varphi$. See Section 5.2 for more details. For $I_{S}(s)$, we estimate its piecewise $L^{\infty}$ norm by estimating integrals depending on the weights.

This allows us to obtain

$$
\begin{aligned}
& \left|\partial I_{R}(s, a)\right| \leq C_{1}(s) \log a^{-1}\|\omega \varphi\|_{\infty}, \quad\left|I_{S}(s, a)\right| \leq C_{2}(s)|a|\|\omega \varphi\|_{L^{\infty}} \\
& \frac{|I(x)-I(z)|}{|x-z|^{1 / 2}} \leq \frac{\left|I_{R}(x)-I_{R}(z)\right|+\left|I_{S}(x)-I_{S}(z)\right|}{|x-z|^{1 / 2}}\|\omega \varphi\|_{L^{\infty}} \leq\left(C_{3}(x, z) \log a^{-1}|x-z|^{\frac{1}{2}}+\frac{C_{4}(x, z)|a|}{|x-z|^{1 / 2}}\right)\|\omega \varphi\|_{L^{\infty}} .
\end{aligned}
$$

To obtain a sharp estimate, we need to choose $a \sim|x-z|$. We perform a sequence of decompositions by choosing different size $a$ and obtain an estimate similar to the above. We minimize different estimates by selecting $a \sim|x-z|$, and obtain the desired $C^{1 / 2}$ estimate. Since different (semi)norms contribute to the upper bounds in Lemma 2.2 with different weights, we also perform improved estimates of $I$ using a small portion of the Hölder norm and optimize different estimates. The $L^{\infty}$ estimate is simpler. We refer to Section 5.2 for more details.

Since we reduce the estimate of the regular part $I$ to bounding explicit $L^{1}$ integrals depending on the weights, we perform the decompositions and estimate the integrals in Section 4 in Part II [15] using the scaling symmetries of the kernels, the symmetrization of the integrands, and numerical analysis, e.g. the Trapezoidal rule with rigorous error control. Note that the $C^{1 / 2}$ seminorm in the $C^{1 / 2}$ estimate in Lemma 2.2 is mostly used to control the most singular part in step one, and we estimate them in Section 3. For the regular part in such an estimate, we can mainly use the norm in (2.14) with $s_{f}=0$.

We remark that controlling the zero-order singular integral operator $\nabla \mathbf{u}=\nabla^{\perp}(-\Delta)^{-1} \omega$ is a challenging problem. Singularity formation of a model problem from smooth data

$$
\omega_{t}=\mathcal{K}(\omega) \omega, \quad x \in \mathbb{R}^{2}
$$

that captures this difficulty is listed as an open problem in [46] and discussed in [27], where $\mathcal{K}$ is some zero order Calderon-Zygmund operator in 2D. The 2D Boussinesq and 3D Euler equations contain several more nonlocal terms and are much more complicated.
$L^{\infty}$-based finite rank perturbation and energy estimates. Given an initial datum $\bar{F}_{i}(0)$, in Section 3 in Part II [15], we first construct a numerical solution $\hat{F}_{i}\left(t_{k}\right)$ at discrete time $t_{k}$ up to a finite time $T_{i}$ and then extend it to infinite time by setting the solution to zero beyond $T_{i}$. Then we interpolate the solution $\hat{F}_{i}\left(t_{k}\right)$ using a cubic polynomial in time so that we have a solution $\hat{F}_{i}(t)$ for any $t>0$, which is piecewise smooth in $t$. Due to numerical error, the solution $\hat{F}_{i}(t)$ does not preserve the vanishing order $O\left(|x|^{2}\right)$ and the residual error $\hat{F}_{i}(0)-\bar{F}_{i},\left(\partial_{t}-\mathcal{L}\right) F_{i}(t)$ does not vanish $O\left(|x|^{3}\right)$ near 0 . As a result, they are not in the energy space.

To overcome this difficulty, we perform two analytic rank-one corrections to $\hat{F}_{i}(t)$ near 0 to enforce $\hat{F}_{i}(t)=O\left(|x|^{2}\right)$ and make sure that the residual error is in the energy space. We further decompose the residual error into the local part $\mathcal{R}_{\text {loc,i }}$ and nonlocal error. For the local part error $\mathcal{R}_{\text {loc, } i}$ and $\hat{F}_{i}$, we estimate them in Section 3 in Part II [15] using method from numerical analysis. Based on these constructions, we construct the approximate solution $\widehat{W}_{2}$ and the residual operator (4.20) and estimate them under the bootstrap assumption, see Section 5.7. We combine the estimate of the nonlocal error and the nonlinear energy estimate in Section 5.8 .

Similarly, for the residual error in Lemma 2.1, we decompose it into the local part $\overline{\mathcal{F}}_{l o c, i}$ and the nonlocal part. The estimate of $\overline{\mathcal{F}}_{l o c, i}$ is established in Appendix C. 4 in Part II [15]. The estimate of the error $\mathcal{R}_{l o c, i}$ follows a similar argument. We combine the estimate of nonlocal error with the energy estimate in Section 5.8 .

The method behind Lemma 2.3 is a $L^{\infty}$-based finite rank perturbation, which we will develop in Section 4 This method allows us to decompose the perturbation and perform energy estimate on $W_{1}$ with a linearized operator $\mathcal{L}-\mathcal{K}$, which is a finite rank perturbation of the original linearized operator $\mathcal{L}$. By designing $\mathcal{K}$ to approximate the nonlocal terms, we can obtain much better linear stability estimates for $\mathcal{L}-\mathcal{K}$.

The variable $\widehat{W}_{2}$ (4.19) plays an auxiliary role only, and we do not perform energy estimate on it directly. In Section 5, we perform the energy estimates and design the whole energy $E_{4}$ (5.70). We further bound the upper bounds in Lemma 2.2 using the energy $E_{4}$. With these estimates, we can derive the coefficients in the stability conditions A.11) and Lemma 2.4. The full inequalities contain the energy estimates of several norms, and are given in Appendix D.

Using the estimates of the constants in Lemma 2.2, the estimate of $\widehat{F}_{i}$ in Lemma 2.3, the estimate of the local part of the residual error and the residual operator, which are established in Part II [15], we obtain the concrete values of the inequalities in Appendix D which only depend on the weights and the approximate steady state, and further verify that they hold true with computer assistance in Part II [15]. The codes can be found in [13]. See more discussions Appendix D.

After we show that the stability conditions (A.11) are satisfied, we obtain nonlinear stability estimates $E_{4}(t)<E_{*}$ for all $t>0$ using Lemma A.2 which implies the bounds stated in Theorem 3. The remaining steps of obtaining finite time blowup from smooth initial data and finite energy follows [19, 20 and a rescaling argument.

Note that all the nonlocal terms in the linearized equations are not small. Without obtaining sharp $C^{1 / 2}$ estimates, even if we use the energy $E_{4}$ (5.70), the stability conditions in (A.11) and Lemma 2.4 fail in the weighted Hölder estimate (see Section (5.4) at some $x$ with $|x-z| \rightarrow 0$. Without the finite rank approximations for the nonlocal terms in Lemma 2.2, 2.3, the stability conditions for the weighted $L^{\infty}$ estimates also fail.

Rigorous piecewise bounds. In our energy estimates, we need to derive rigorous and tight piecewise bounds of various quantities involving the approximate steady state, singular weights, and several explicit functions. One of the main ideas is to use the second order error estimate

$$
\begin{equation*}
\max _{x \in\left[x_{l}, x_{u}\right]}|f(x)| \leq \max \left(\left|f\left(x_{l}\right)\right|,\left|f\left(x_{u}\right)\right|\right)+\frac{h^{2}}{8}\left\|f_{x x}\right\|_{L^{\infty}\left(\left[x_{l}, x_{u}\right]\right)}, \quad h=x_{u}-x_{l} \tag{2.15}
\end{equation*}
$$

to obtain a piecewise sharp bound of $f$ on $\left[x_{l}, x_{u}\right]$, see e.g. Appendix C. 2 in [15]. If we can obtain a rough bound for $f_{x x}$ on $[a, b]$, by partitioning $[a, b]$ into small intervals $\left[x_{i l}, x_{i u}\right]$, evaluating $f$ on finitely many grid points $x_{i l}, x_{i u}$, and using the above estimates, we can obtain tight bound of $f$ on $[a, b]$. Similarly, by estimating $\partial_{x}^{k} f$ and applying the above estimate recursively, we can obtain a tight bound for $\partial_{x}^{i} f$ with $i \leq k-1$. Note that for a polynomial with degree less than $d$, we have $\partial_{x}^{k} f \equiv 0, k \geq d+1$. Our approximate steady state is represented by piecewise polynomials (See Section 7), and we apply these estimates. For several explicit functions, we estimate the higher order derivatives using induction. Using the Leibniz rule and the triangle inequalities, we can estimate higher order derivatives for more complicated functions. We further develop various higher order error estimates (error terms $\left.C_{k} h^{k}, k=3,4,5\right)$ using numerical analysis. See more details in Appendix in Paper II [15]. To track the round off error in the computation, we use interval arithmetic [78, 81].

Computer-assisted proof has played an important role in the analysis of several PDE problems, especially in computing explicit tight bounds of complicated (singular) integrals [9, 29, 45] or bounding the norms of linear operators [6, 41]. We refer to [44] for an excellent survey on computer-assisted proofs in establishing rigorous analysis for PDEs and refer to [44, 78, 81, for related works using the interval arithmetic and computer assistance in analysis of PDEs.

Note that our approach to establish stability analysis with computer assistance is different from existing computer-assisted approaches, e.g. [8, where the stability is established by quantifying the spectral gap of a given operator numerically. We do not use direct computation to quantify the spectral gap of the linearized operator since our linearized operator is not compact.

In the remaining of this section, we will use a number of simplified models to illustrate and motivate the main ideas behind our stability analysis.

Notations and operators. The upper bar notation is reserved for the approximate steady state, e.g. $\bar{\omega}, \bar{\theta}$. We introduce the bilinear operator $B_{o p, i}((\mathbf{u}, M), G)$ for $(\mathbf{u}, M), G=\left(G_{1}, G_{2}, G_{3}\right)$

$$
\begin{align*}
& \mathcal{B}_{o p, 1}=-\mathbf{u} \cdot \nabla G_{1}+M_{11}(0) G_{1}, \quad \mathcal{B}_{o p, 2}=-\mathbf{u} \cdot \nabla G_{2}+2 M_{11}(0) G_{2}-M_{11} G_{2}-M_{21} G_{3}  \tag{2.16}\\
& \mathcal{B}_{o p, 3}=-\mathbf{u} \cdot \nabla G_{3}+2 M_{11}(0) G_{3}-M_{12} G_{2}-M_{22} G_{3}
\end{align*}
$$

If $M=\nabla \mathbf{u}, M_{11}=u_{x}, M_{12}=u_{y}, M_{21}=v_{x}, M_{22}=v_{y}$, then we drop $M$ to simplify the notation (2.17)

$$
\begin{aligned}
& \mathcal{B}_{o p, 1}(\mathbf{u}, G)=-\mathbf{u} \cdot \nabla G_{1}+u_{x}(0) G_{1}, \mathcal{B}_{o p, 2}(\mathbf{u}, G)=-\mathbf{u} \cdot \nabla G_{2}+2 u_{x}(0) G_{2}-u_{x} G_{2}-v_{x} G_{3} \\
& \mathcal{B}_{o p, 3}(\mathbf{u}, G)=-\mathbf{u} \cdot \nabla G_{3}+2 u_{x}(0) G_{3}-u_{y} G_{2}-v_{y} G_{3}
\end{aligned}
$$

We introduce the notations for the nonlinear terms

$$
\begin{align*}
& \mathcal{N}_{i}=\mathcal{B}_{o p, i}(\mathbf{u},(\omega, \eta, \xi)), \quad \mathcal{N}_{1}=-\mathbf{u} \cdot \nabla \omega+u_{x}(0) \omega \\
& \mathcal{N}_{2}=-\mathbf{u} \cdot \nabla \eta-u_{x} \eta-v_{x} \xi+2 u_{x}(0) \eta, \quad \mathcal{N}_{3}=-\mathbf{u} \cdot \nabla \xi-u_{y} \eta-v_{y} \xi+2 u_{x}(0) \xi \tag{2.18}
\end{align*}
$$

Without specification, $\mathcal{N}_{i}$ depends on $(\omega, \eta, \xi)$. Given the approximate steady state $\bar{\omega}, \bar{\theta}, \bar{c}_{l}, \bar{c}_{\omega}$, we denote by $\overline{\mathcal{F}}_{i}$ and $\bar{F}_{\omega}, \bar{F}_{\theta}$ the residual error

$$
\begin{align*}
& \bar{F}_{\omega}=-\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \bar{\omega}+\bar{\theta}_{x}+\bar{c}_{\omega} \bar{\omega}, \quad \bar{F}_{\theta}=-\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \bar{\theta}+\bar{c}_{\theta} \bar{\theta} \\
& \overline{\mathcal{F}}_{1} \triangleq \bar{F}_{\omega}, \quad \overline{\mathcal{F}}_{2} \triangleq \partial_{x} \bar{F}_{\theta}, \quad \overline{\mathcal{F}}_{3} \triangleq \partial_{y} \bar{F}_{\theta} \tag{2.19}
\end{align*}
$$

Denote by $C_{x}^{\alpha}, C_{y}^{\alpha}$ the partial Hölder seminorms

$$
\begin{equation*}
[\omega]_{C_{x}^{\alpha}(D)} \triangleq \sup _{x, z \in D, x_{2}=z_{2}} \frac{|\omega(x)-\omega(z)|}{\left|x_{1}-z_{1}\right|^{\alpha}}, \quad[\omega]_{C_{y}^{\alpha}(D)} \triangleq \sup _{x, z \in D, x_{1}=z_{1}} \frac{|\omega(x)-\omega(z)|}{\left|x_{2}-z_{2}\right|^{\alpha}} . \tag{2.20}
\end{equation*}
$$

Given a weight $g(h): \mathbb{R}^{2} \rightarrow \mathbb{R}_{+}$that is $-\alpha$-homogeneous, i.e. $g\left(\lambda h_{1}, \lambda h_{2}\right)=\lambda^{-\alpha} g(h)$, e.g., $g(h)=|h|^{-\alpha}$, we define the weighted Hölder seminorm

$$
\begin{equation*}
[\omega]_{C_{g}^{\alpha}(D)}=\sup _{x, z \in D}|(\omega(x)-\omega(z)) g(x-z)| \tag{2.21}
\end{equation*}
$$

We will mostly use $D=\mathbb{R}_{++}^{2}$. In this case, we drop $D$ to simplify the notations.
We define $\langle f, g\rangle$ the inner product in $\mathbb{R}_{++}^{2}$

$$
\begin{equation*}
\langle f, g\rangle=\int_{\mathbb{R}_{++}^{2}} f(x) g(x) d x \tag{2.22}
\end{equation*}
$$

2.4. Basic properties of the approximate steady state. Following the ideas in [19, 20, we construct the approximate steady state $\left(\bar{\omega}, \bar{\theta}, \bar{c}_{\omega}, \bar{c}_{l}\right)$ of the dynamic rescaling equations (2.10), (2.11) by solving them numerically for a long enough time. In Figure 1, we plot the approximate steady state $\bar{\omega}, \bar{\theta}_{x}$ on the grid points. We plot the variable $\bar{\theta}_{x}$ rather than $\bar{\theta}$ since $\bar{\theta}$ grows in the far-field. Given the approximate steady state, we construct the numerical stream function $\bar{\phi}^{N}$ by solving the Poisson equations. Then we can derive the residual (2.19) up to the error in solving the Poisson equations. In Figure 2, we plot the piecewise rigorous bound of the weighted $L^{\infty}\left(\varphi_{i}\right)$ norm of $\overline{\mathcal{F}}_{i}, i=1,2$. Since $\varphi_{1}, \varphi_{2}$ are very singular near $x=0$ with leading order $|x|^{-2.4}\left|x_{1}\right|^{-\frac{1}{2}}, c|x|^{-\frac{5}{2}}\left|x_{1}\right|^{-\frac{1}{2}}$, the weighted $L^{\infty}\left(\varphi_{i}\right)$ norm of $\overline{\mathcal{F}}_{i}$ is relatively large near the origin. The $L^{\infty}\left(\varphi_{i}\right)$ norms are used in the energy estimate (2.13). We remark that the unweighted errors of $\overline{\mathcal{F}}_{1}, \overline{\mathcal{F}}_{2}$ are very small near the origin, less than $2 \cdot 10^{-12}$ since we use a uniform fine grid near the origin. We defer the details of numerical computation to Section 7. Here, we list some important properties of the approximate steady state.


Figure 1. Approximate steady state in the near-field. Left figure: profile $\bar{\omega}$; right figure: $\bar{\theta}_{x}$.


Figure 2. Weighted residual errors of the approximate steady state. Left figure: piecewise rigorous $L^{\infty}\left(\varphi_{1}\right)$ bound of $\mathcal{F}_{1}$ in the $\omega$ equation. Right figure: piecewise rigorous $L^{\infty}\left(\varphi_{2}\right)$ bound of $\mathcal{F}_{2}$ in the $\theta_{x}$ equation

Exponents. The exponents and the velocity near the origin satisfy

$$
\begin{equation*}
\bar{c}_{l} \approx 3.00649898, \quad \bar{c}_{\omega} \approx-1.02942516, \quad \bar{u}_{x}(0) \approx-2.532674, \quad \bar{v}_{x}(0)=0 \tag{2.23}
\end{equation*}
$$

We remark that the ratio $\bar{c}_{l} / \bar{c}_{\omega} \approx-2.9205600$ is very close to the one reported by Hou-Luo [65, 66].
Regularity and representation. The variables $\bar{\omega}, \bar{\phi}$ are odd in $x$ and $\bar{\theta}$ is even in $x$. Denote by $\phi$ the stream function. One should not confuse the stream function $\phi$ with singular weights $\varphi_{i}, \psi_{i}$, etc. The approximate steady state $(\bar{\omega}, \bar{\theta}, \bar{\phi})$ is represented by piecewise fifth order polynomials $\bar{\omega}_{2}, \bar{\theta}_{2}, \bar{\phi}_{2}$ supported in $\left[0, D_{1}\right]^{2}$ with $D_{1} \approx 10^{15}$, semi-analytic parts $\bar{\omega}_{1}, \bar{\theta}_{1}, \bar{\phi}_{1}$ that capture the far-field behavior of the solutions, an analytic part $\bar{\phi}_{3}$ that captures $\bar{\phi}$ near $x=0$ to reduce the round-off error, and a small rank-one correction $\bar{\phi}_{\text {cor }}$ such that $\bar{\omega}-(-\Delta) \bar{\phi}=O\left(|x|^{2}\right)$ near 0

$$
\bar{\omega}=\bar{\omega}_{1}+\bar{\omega}_{2}, \quad \bar{\theta}=\bar{\theta}_{1}+\bar{\theta}_{2}, \quad \bar{\phi}=\bar{\phi}_{1}+\bar{\phi}_{2}+\bar{\phi}_{3}+\bar{\phi}_{c o r} .
$$

See (7.2). In particular, we have $\bar{\omega}, \bar{\theta}, \bar{\phi} \in C^{4,1}$. The solution enjoys the decay rate

$$
\bar{\omega} \sim r^{\alpha}, \quad \bar{\theta} \sim r^{1+2 \alpha}, \quad \alpha \approx \bar{c}_{\omega} / \bar{c}_{l} .
$$

Anisotropic. The solutions $\bar{\theta}$ and $\bar{\omega}$ are anisotropic in the sense that the $y$-derivative of the profile is much smaller than the $x$-derivative, especially in the near field $(x, y) \in[0,1]^{2}$ :

$$
\begin{equation*}
\left|\bar{\theta}_{y}\right|<c_{3}\left|\bar{\theta}_{x}\right|, \quad c_{3} \approx 0.16, \quad\left|\bar{\omega}_{y}\right|<c_{4}\left|\bar{\omega}_{x}\right|, \quad c_{4} \approx 0.23 \tag{2.24}
\end{equation*}
$$

These anisotropic properties are similar to those for the $C^{1, \alpha}$ singular solution 17 .
The advection. The advection in (2.10) satisfies the following important inequalities

$$
\bar{c}_{l} x+\bar{u}(x, y) \geq c_{1} x, \quad c_{1} \approx 0.47, \quad \bar{c}_{l} y+\bar{v}(x, y) \geq c_{2} y, \quad c_{2} \approx 3
$$

for all $x, y \in \mathbb{R}_{++}^{2}$. For $x, y \in \mathbb{R}_{++}^{2}$ near the origin, we have

$$
\bar{c}_{l} x+\bar{u}(x, y) \approx 0.47 x, \quad c_{l} y+\bar{v}(x, y) \approx 5.54 y
$$

2.5. Linearized equations. Linearizing (2.10) around ( $\bar{\omega}, \bar{\theta}, \overline{\mathbf{u}}, \bar{c}_{l}, \bar{c}_{\omega}$ ), we yield

$$
\begin{align*}
\omega_{t} & =-\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \omega+\theta_{x}+\bar{c}_{\omega} \omega-\mathbf{u} \cdot \nabla \bar{\omega}+c_{\omega} \bar{\omega}+\bar{F}_{\omega}+N(\omega) \\
\theta_{t} & =-\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \theta+\bar{c}_{\theta} \theta+c_{\theta} \bar{\theta}-\mathbf{u} \cdot \nabla \bar{\theta}+\bar{F}_{\theta}+N(\theta), \quad \mathbf{u}=\nabla^{\perp}(-\Delta)^{-1} \omega \tag{2.25}
\end{align*}
$$

where $\bar{F}_{\omega}, \bar{F}_{\theta}$ are the residual errors (2.19), and $N(\omega), N(\theta)$ are the nonlinear terms

$$
N(\omega)=-\mathbf{u} \cdot \nabla \omega+c_{\omega} \omega=\mathcal{N}_{1}, \quad N(\theta)=-\mathbf{u} \cdot \nabla \theta+c_{\theta} \theta
$$

where we have used the notation $\mathcal{N}_{1}(2.18)$ and the following normalization conditions for the perturbing $c_{l}, c_{\omega}$ from (2.11)

$$
\begin{equation*}
c_{\omega}=u_{x}(0), \quad c_{l} \equiv 0, \quad c_{\theta}=c_{l}+2 c_{\omega} \tag{2.26}
\end{equation*}
$$

Since $\omega, \nabla \theta$ have similar regularity, we study the system of $\left(\omega, \theta_{x}, \theta_{y}\right)$ and denote

$$
\begin{equation*}
\eta=\theta_{x}, \quad \xi=\theta_{y} \tag{2.27}
\end{equation*}
$$

Taking derivatives on the $\theta$ equation in (2.25) and using the notations (2.18), (2.19), we obtain

$$
\begin{align*}
\partial_{t} \eta & =-\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \eta+\left(2 \bar{c}_{\omega}-\bar{u}_{x}\right) \eta-\bar{v}_{x} \xi-\mathbf{u}_{x} \cdot \nabla \bar{\theta}-\mathbf{u} \cdot \nabla \bar{\theta}_{x}+2 c_{\omega} \bar{\theta}_{x}+\mathcal{N}_{2}+\overline{\mathcal{F}}_{2} \\
\partial_{t} \xi & =-\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \xi+\left(2 \bar{c}_{\omega}+\bar{u}_{x}\right) \xi-\bar{u}_{y} \eta-\mathbf{u}_{y} \cdot \nabla \bar{\theta}-\mathbf{u} \cdot \nabla \bar{\theta}_{y}+2 c_{\omega} \bar{\theta}_{y}+\mathcal{N}_{3}+\overline{\mathcal{F}}_{3} \tag{2.28}
\end{align*}
$$

where we have used $c_{\theta}=c_{l}+2 c_{\omega}$. Due to the normalization conditions (2.12) and the odd symmetries of $\theta_{x}, \omega$ we have the following vanishing conditions near the origin

$$
\begin{equation*}
\omega=O\left(|x|^{2}\right), \quad \theta_{x}=O\left(|x|^{2}\right), \quad \theta_{y}=O\left(|x|^{2}\right) \tag{2.29}
\end{equation*}
$$

Analyzing the linear stability of the above system is extremely challenging since it contains several nonlocal terms, which are not small. Note that numerical evidence of linear stability of the above system has been reported by Liu 64] (see Section 3.4), who showed that the eigenvalues of the discretized linearized operator has negative real parts bounded away from 0 .
2.6. Main terms of the system. Firstly, we identify the main terms in the system (2.25), (2.28).
2.6.1. Anisotropy in the $x, y$ directions. Since the solutions are anisotropic (2.24) in the near field, the $y$-derivatives of the solution, e.g., $\bar{\omega}_{y}, \bar{\theta}_{y}, \bar{\theta}_{x y}$, are relatively small.

In (2.28), $\xi=\theta_{y}$ enjoys much better stability estimates than those of $\eta=\theta_{x}$ due to the flow structure: compression along the $x$-direction and outward flow along the $y$-direction. Indeed, since $\bar{u}_{x}(0) \approx-2.5$ near the origin and $\bar{c}_{\omega} \approx-1$ (2.23), we have

$$
\left(2 \bar{c}_{\omega}-\bar{u}_{x}\right) \eta \approx 0.5 \eta, \quad\left(2 \bar{c}_{\omega}+\bar{u}_{x}\right) \xi \approx-5.5 \xi
$$

These terms contribute to a growing term in the equation of $\eta$ and a large damping term in the $\xi$ equation. These anisotropic properties are similar to those for the $C^{1, \alpha}$ singular solution [17.
2.6.2. Weak coupling. Note that $\bar{v}_{x} \approx 0$ near $0(2.23)$ and $\bar{v}(x, 0)=0$ due to the boundary condition, $\bar{v}_{x}$ is quite small in the near field and near the boundary. Therefore, $\xi$ is weakly coupled to the equation of $\eta$ in such a region, which is the most difficult region of the analysis. See the discussion in Section 2.7.2. This coupling structure between $\eta$ and $\xi$ is consistent with that of the $C^{1, \alpha}$ singular solution in [17, where $\bar{v}_{x} \xi$ is a lower order term in the $\eta$ equation.

Using the above analysis and dropping the smaller terms and the $\xi$ equation, we identify the main terms in the linear part of the system (2.25)

$$
\begin{align*}
\omega_{t} & =-\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \omega+\eta+\bar{c}_{\omega} \omega-u \bar{\omega}_{x}+c_{\omega} \bar{\omega}+\mathcal{R}_{\omega} \\
\partial_{t} \eta & =-\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \eta+\left(2 \bar{c}_{\omega}-\bar{u}_{x}\right) \eta-u_{x} \bar{\theta}_{x}-u \bar{\theta}_{x x}+2 c_{\omega} \bar{\theta}_{x}+\mathcal{R}_{\eta} \tag{2.30}
\end{align*}
$$

where $\mathcal{R}_{\omega}, \mathcal{R}_{\eta}$ denote the remaining terms in the equations. The above system is very similar to that in the Hou-Luo model [20] with similar coefficients $\bar{\omega}, \bar{\theta}$ near the boundary.
2.7. The local parts and functional spaces. Following 17, 19, 20, we will perform weighted energy estimate in some suitable space $X$ and derive the damping terms in the weighted energy estimate from the local terms, especially the advection term $\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla f$ in (2.30). See Section 2 in [19] for an example. The principle of choosing the appropriate energy space $X$ is the following. Firstly, the local part of the linearized equations should be stable in space $X$. Secondly, we can estimate the nonlocal terms in $X$ effectively.
2.7.1. A toy model for the local term. To understand the linear stability, we first focus on the local terms in the main system (2.30). We drop the nonlocal terms involving $u, u_{x}$ and the remaining terms in (2.30) and approximate (2.30) near the origin by the following model in $\mathbb{R}_{2}^{++}$

$$
\begin{align*}
\omega_{t}+\left(a_{1} x \partial_{x}+a_{2} y \partial_{y}\right) \omega & =-\omega+\eta \\
\eta_{t}+\left(a_{1} x \partial_{x}+a_{2} y \partial_{y}\right) \eta & =a_{3} \eta, \quad a_{1}=0.5, a_{2}=5.5, a_{3}=0.5 \tag{2.31}
\end{align*}
$$

with $\omega, \eta$ being odd in $x$, where we have used (2.23) to obtain approximations

$$
\bar{c}_{l} x+\bar{u} \approx\left(\bar{c}_{l}+\bar{u}_{x}(0)\right) x \approx 0.5 x, \quad \bar{c}_{l} y+\bar{v} \approx 5.5 y, \quad 2 \bar{c}_{\omega}-\bar{u}_{x}(0) \approx 0.5
$$

Weighted $L^{\infty}$ space. Since $a_{3}>0, a_{3} \eta$ in (2.31) contributes to a growing term. We consider weighted $L^{\infty}$ estimates to take advantage of the transport structure. Suppose that $\varphi=r^{-\gamma}, r=$ $\left(x^{2}+y^{2}\right)^{1 / 2}$. Multiplying the $\eta$ equation with $\varphi$ and a direct calculation yield

$$
\begin{equation*}
\partial_{t}(\eta \varphi)+\left(a_{1} x \partial_{x}+a_{2} y \partial_{y}\right)(\eta \varphi)=\left(a_{3} \varphi+a_{1} x \partial_{x} \varphi+a_{2} y \partial_{y} \varphi\right) \eta \triangleq a(\gamma) \eta \varphi \tag{2.32}
\end{equation*}
$$

Since $x \partial_{x} \varphi=-\gamma x^{2} r^{-\gamma-2}, y \partial_{y} \varphi=-\gamma y^{2} r^{-\gamma-2}$ and $a_{2} \geq a_{1}$, we get

$$
\begin{equation*}
a(\gamma)=a_{3}+\frac{a_{1} x \partial_{x} \varphi+a_{2} y \partial_{y}}{\varphi}=a_{3}-\gamma \frac{a_{1} x^{2}+a_{2} y^{2}}{x^{2}+y^{2}} \leq a_{3}-a_{1} \gamma \tag{2.33}
\end{equation*}
$$

Since $a_{1}=0.5, a_{2}=5.5, a_{3}=0.5$, to obtain a damping factor $a(\gamma) \leq 0$, we can choose $\gamma \geq 1$. Notice that for the system (2.25), (2.28), $\omega, \eta$ vanish at least quadratically near 0 (2.29). Therefore, we can choose $\gamma \geq 2$ to derive the damping terms in the $\eta$ equation.

For the system in (2.31), performing $L^{\infty}$ estimate with weight $\varphi=r^{-\gamma}$ and $\gamma>1$ on both equations, we get

$$
\begin{equation*}
\frac{d}{d t}\|\omega \varphi\|_{\infty} \leq\left(-1-a_{1} \gamma\right)\|\omega \varphi\|_{\infty}+\|\eta \varphi\|_{\infty}, \quad \frac{d}{d t}\|\eta \varphi\|_{\infty} \leq\left(a_{3}-a_{1} \gamma\right)\|\eta \varphi\|_{\infty} \tag{2.34}
\end{equation*}
$$

It is easy to further obtain that $\max \left(\|\omega \varphi\|_{\infty},\|\eta \varphi\|_{\infty}\right)$ decays exponentially fast.
From (2.33), since $a_{2}$ is much larger than $a_{1}$, as the ratio $\lambda=y / x$ increases, we get a much larger damping factor

$$
a(\gamma, \lambda)=a_{3}-\gamma \frac{a_{1}+a_{2} \lambda^{2}}{1+\lambda^{2}}, \quad a(\gamma, 0)=a_{3}-a_{1} \gamma=\frac{1}{2}-\frac{1}{2} \gamma, a(\gamma, \infty)=a_{3}-\gamma a_{2}=\frac{1}{2}-\frac{11}{2} \gamma
$$

In [17, 19, 20, 35, the stability analysis is based on some weighted $L^{2}$ spaces. However, if one performs weighted $L^{2}$ estimate of (2.32) with singular weight $\varphi=x^{-\alpha} y^{-\beta}$, using integration by parts, the $y$-advection contributes

$$
I=-\int \eta a_{2} y \partial_{y} \eta \varphi=\frac{a_{2}(1-\beta)}{2} \int \eta^{2} \varphi
$$

to the energy estimate of $\int \eta^{2} \varphi$. Since $\omega(x, 0) \neq 0, \eta(x, 0) \neq 0$, we need to choose $\beta<1$ so that the energy is well-defined. If $\beta$ is close to 1 , in our later estimates of nonlocal terms, e.g. $u_{x}$, since $u_{x}(x, 0) \neq 0$, we expect an estimate with a large constant: $\left\|u_{x} \varphi^{1 / 2}\right\|_{2} \leq C(1-\beta)^{-1 / 2}\left\|\omega \varphi^{1 / 2}\right\|_{2}$. If $1-\beta$ is not small, $I$ is a large growing factor in the energy estimate. This forces one to choose a very singular weight to extract a damping term for $\eta$, e.g. $\alpha \geq 14$ if $\beta=0$, a new difficulty which is absent in [17, 19, 20, 35]. We overcome it by using $L^{\infty}$ type estimates and develop a set of estimates for the nonlocal terms in some appropriate functional spaces.

A potential $L^{2}$ based approach is to perform sufficiently high order $H^{k}$ estimate. Taking a partial derivative $\partial_{x}^{i} \partial_{y}^{j}$ plays a role similar to a singular weight $x^{-i} y^{-j}$. However, to derive a damping term for $\partial_{x}^{k} \eta$ in this model, one needs to take $k \geq 14$. Due to the mixed derivative terms $\partial_{x}^{i} \partial_{y}^{j} \eta$, it is not clear if $H^{k}$ estimates can be closed for (2.32) with $k$ not very large. This approach can lead to many more terms in the system (2.25), (2.28), e.g. the mixed derivative terms and $\partial_{x}^{i}\left(u_{x} \bar{\theta}_{x}\right)$, which can be difficult to control. Moreover, due to the boundary, $\partial_{y}$ does not commute with the nonlocal operator $\nabla^{\perp}(-\Delta)^{-1}$ for the velocity. Thus, in the $H^{k}$ estimates, it is not clear if we can obtain stability for the leading order operator. Furthermore, constructing an approximate steady state with a small residual error in $H^{k}$ with a large $k$, e.g. $k \geq 14$, is extremely challenging.

Weighted Hölder estimate. Since $\nabla \mathbf{u}=\nabla \nabla^{\perp}(-\Delta)^{-1} \omega$ in (2.28) is not bounded from $L^{\infty} \rightarrow$ $L^{\infty}$, to close the estimates, we perform weighted $C^{\alpha}$ estimates. We have a simple identity.
Lemma 2.5. Suppose that $f$ satisfies

$$
\begin{equation*}
\partial_{t} f+b(x) \cdot \nabla f=c(x) f(x)+\mathcal{R}, \quad x \in \mathbb{R}_{+}^{2} \tag{2.35}
\end{equation*}
$$

Given some weights $g\left(x_{1}, x_{2}\right)$ even in $x_{1}, x_{2}$ and $\varphi$, we denote the operator $\delta$ and function $F$
$\delta(p)(x, z)=p(x)-p(z), \quad F(x, z, t)=\delta(f \varphi)(x, z) g(x-z), \quad d(x)=c(x)+\frac{b \cdot \nabla \varphi}{\varphi}, x, z \in \mathbb{R}_{+}^{2}$.
Then we have

$$
\begin{align*}
\partial_{t} F+\left(b(x) \cdot \nabla_{x}+b(z) \cdot \nabla_{z}\right) & F=\left(d(x)+\frac{(b(x)-b(z)) \cdot(\nabla g)(x-z)}{g(x-z)}\right) F  \tag{2.36}\\
+ & (d(x)-d(z)) g(x-z)(f \varphi)(z)+\delta(\mathcal{R} \varphi)(x, z) g(x-z)
\end{align*}
$$

The proof follows a direct calculation and is deferred to Appendix A.1 We treat the first term on the right hand side of (2.36) as a damping term. The term $\frac{b \cdot \nabla \varphi}{\varphi}$ in $d(x)$ is the damping term from the singular weight $\varphi(x)$. We apply Lemma 2.5 to the $\eta$ equation in (2.31). Denote $\varphi_{2}=|x|^{-\gamma_{2}}, g(h)=|h|^{-\alpha}, h \in \mathbb{R}^{2}, \quad b(x)=\left(a_{1} x_{1}, a_{2} x_{2}\right), \quad F=\left(\left(\eta \varphi_{2}\right)(x)-\left(\eta \varphi_{2}\right)(z)\right) g(x-z)$ for $x=\left(x_{1}, x_{2}\right), z=\left(z_{1}, z_{2}\right) \in \mathbb{R}_{+}^{2}$. Using the identity (2.33) and definitions of $g, b$, we get

$$
\begin{align*}
& d(x)=a_{3}+\frac{b(x) \cdot \nabla \varphi_{2}}{\varphi_{2}}=a\left(\gamma_{2}\right), \quad h_{i} \partial_{i} g=-\alpha \frac{h_{i}^{2}}{|h|^{2+\alpha}}=-\alpha \frac{h_{i}^{2}}{|h|^{2}} g \\
& b(x)-b(z)=\left(a_{1}\left(x_{1}-z_{1}\right), a_{2}\left(x_{2}-z_{2}\right)\right),  \tag{2.37}\\
& \frac{(b(x)-b(z)) \cdot(\nabla g)(x-z)}{g(x-z)}=-\alpha \frac{a_{1}\left(x_{1}-z_{1}\right)^{2}+a_{2}\left(x_{2}-z_{2}\right)^{2}}{|x-z|^{2}} \triangleq e(\alpha, x, z)
\end{align*}
$$

Thus, $e(\alpha, x, z) F$ in (2.36) is also a damping term, which comes from the Hölder function $g$. Using Lemma 2.5 with $\mathcal{R}=0$, we yield
$\partial_{t} F+\left(b(x) \nabla_{x}+b(z) \nabla_{z}\right) F=\left(a\left(\gamma_{2}\right)(x)+e(\alpha, x, z)\right) F+\left(a\left(\gamma_{2}\right)(x)-a\left(\gamma_{2}\right)(z)\right) g(x-z)\left(\eta \varphi_{2}\right)(z)$.
From definition (2.33), $d(x)=a\left(\gamma_{2}\right)(x)$ is not in $C^{\alpha}$. Instead, we estimate $I_{4}=\left(a\left(\gamma_{2}\right)(x)-\right.$ $\left.a\left(\gamma_{2}\right)(z)\right) g(x-z)|z|^{\alpha}$. Since $a\left(\gamma_{2}\right)(x)|x|^{\alpha},|x|^{\alpha} \in C^{\alpha}$, we yield

$$
\left|I_{4}\right|=\left|\left(a\left(\gamma_{2}\right)(x)|x|^{\alpha}-a\left(\gamma_{2}\right)(z)|z|^{\alpha}\right) g(|x-z|)+a\left(\gamma_{2}\right)(x)\left(|z|^{\alpha}-|x|^{\alpha}\right) g(|x-z|)\right| \leq C_{\alpha}
$$

Combining the $L^{\infty}\left(|x|^{-\gamma}\right)$ estimate of $\eta$ with $\gamma=\gamma_{2}+\alpha$ and $\varphi_{2}|x|^{-\alpha}=|x|^{-\gamma_{2}-\alpha}=|x|^{-\gamma}$, and using $a\left(\gamma_{2}\right) \leq a_{3}-a_{1} \gamma_{2}, e(\alpha, x, z) \leq-a_{1} \alpha$ (2.33), (2.37), we obtain

$$
\frac{d}{d t}\|F\|_{L^{\infty}(x, z)} \leq\left(a_{3}-a_{1}\left(\gamma_{2}+\alpha\right)\right)\|F\|_{L^{\infty}(x, z)}+C_{\alpha}\left\|\eta|z|^{-\gamma}\right\|_{\infty}
$$

and yield the weighted $C^{\alpha}$ estimate for $\eta$. Similarly, we obtain weighted $C^{\alpha}$ for $\omega$. Since $a_{1}=a_{3}=\frac{1}{2}$, choosing $\gamma_{2}+\alpha=\gamma>1$ and combining the above estimate and (2.34), we establish stability estimate for the model (2.32) in a combination of weighted $L^{\infty}$ and $C^{\alpha}$ spaces.
2.7.2. Anisotropy of the flow and the most difficult scenario. Motivated by the above analysis, we will design the functional spaces $X$ as a combination of weighted $L^{\infty}$ and $C_{g}^{1 / 2}$ spaces (2.21). The system (2.25), (2.28) is much more complicated than the model problem (2.31) since it involves variables coefficients and several nonlocal terms. Similar to [17, 19, 20, we will design the weights $\varphi_{i}, \psi_{i}$ as linear combination of different powers $|x|^{-\alpha_{i}}$ to take into account the behavior in the near field and the far field.

Denote $f=(\omega, \eta, \xi)$. We will perform $L^{\infty}$ estimate for $f_{i} \varphi_{i}$ and $C^{1 / 2}$ estimate for $f_{i} \psi_{i}$. Moreover, we choose the $L^{\infty}$ weight $\varphi_{i}$ at least $|x|^{-\frac{1}{2}}$ more singular near $x=0$ than the Hölder weight $\psi_{i}$ since the damping coefficients similar to $a\left(\gamma_{2}\right)$ in (2.37) is not $C^{1 / 2}$. See (C.3), (C.1).

From the above analysis of the model problem, (2.33), and (2.37), we see that the estimate is anisotropic in $x$ and $y$. In the near field, from (2.33), if $y / x$ is not small, we get a much larger damping term. Moreover, the solution $(\bar{\omega}, \nabla \bar{\theta})$ decays in the far-field. Thus the most difficult region for the analysis is a sector $\Sigma_{S}$ near the boundary, e.g. $(x, y):|(x, y)| \leq 2, y / x \leq 0.1$.

From (2.37), we also get a much larger damping factor if $\left|x_{2}-z_{2}\right| /\left|x_{1}-z_{1}\right|$ is large. This implies that the Hölder estimate in $y$ direction enjoys much better estimates than those in the $x$ direction. Therefore, the most difficult part of the Hölder estimate is in the horizontal direction.
2.7.3. Vanishing order of the perturbation. From (2.29), the perturbation $\omega, \eta, \xi$ vanishes quadratically near $x=0$. To obtain larger damping factors, from the model problem (2.31) and (2.33), we can choose a larger $\gamma$. We will decompose the perturbation $f_{i}$ into two parts

$$
f_{i}=f_{i, 1}+f_{i, 2}
$$

where $f_{i, 1}$ captures the main part of $f_{1}$ and vanishes to the order $O\left(|x|^{3}\right)$ near $x=0$, and $f_{i, 2}$ accounts for the contribution from some finite rank operators. For example, if we choose $\omega_{2}=\omega_{x y}(0) x y \chi(x, y)$ for some cutoff function $\chi \in C_{c}^{\infty}$ with $\chi=1$ near $x=0$, then $\omega_{1}=$ $\omega-\omega_{2}=O\left(|x|^{3}\right)$ near 0 . In this problem, cubic vanishing order is good enough for our stability analysis. See more discussions in Section 4.2.4 and (4.21).

We will perform energy estimates on $f_{i, 1}$ and use space-time estimates for $f_{i, 2}$.
2.8. Estimate the nonlocal terms $\nabla \mathbf{u}$. In the stability analysis, we need to estimate the nonlocal terms $\mathbf{u}, \nabla \mathbf{u}$ in (2.25), (2.28). Although we have standard $C^{\alpha}$ estimates for the Riesz transform $\nabla \nabla^{\perp}(-\Delta)^{-1} \omega$, the constants usually are not given explicitly, and they are not sharp enough for our purposes.

To obtain sharp $C^{\alpha}$ estimates for $\nabla \mathbf{u}$, e have a crucial observation that we can use techniques from optimal transport to obtain sharp estimates. This is another reason why we choose a weighted $C^{1 / 2}$ space in the energy estimates. Note that optimal transport has been applied to establish many sharp functional inequalities and study functional inequalities in details, e.g., the reverse Brascamp-Lieb inequality [1], the Sobolev and Gagliardo-Nirenberg inequalities [28], the isoperimetric inequalities 42. See also the excellent books [87, 88 , for more details.

We focus on $u_{x}$ from $u_{x} \bar{\theta}_{x}$ in the main system (2.30). This term is the most difficult nonlocal term to estimate since other nonlocal terms $u \bar{\omega}_{x}, u \bar{\theta}_{x x}$ in (2.30) are more regular. Note that in the leading order system for the $C^{1, \alpha}$ singular solution [17, $u_{x} \bar{\theta}_{x}$ is also the main nonlocal term. Other nonlocal terms involving $\mathbf{u}, \nabla \mathbf{u}$ contain a small factor $\alpha$.

Denote by $u_{x}(x, a, b)$ the localized version of $u_{x}$

$$
\begin{equation*}
u_{x}(x, a, b) \triangleq-\frac{1}{\pi} P . V . \int_{\left|x_{1}-y_{1}\right| \leq a,\left|x_{2}-y_{2}\right| \leq b} K_{1}(x-y) W(y) d y, \quad K_{1}(s)=\frac{s_{1} s_{2}}{|s|^{4}} \tag{2.38}
\end{equation*}
$$

where $W$ is an odd extension of $\omega$ in $y$ from $\mathbb{R}_{2}^{+}$to $\mathbb{R}^{2}$ (3.3).
We decompose $u_{x}=u_{x, S}+u_{x, R}$ into the more regular part $u_{x, R}$ and $u_{x, S}(x)=u_{x}(x, a, b)$ that captures the most singular part of $u_{x}$ in the Biot-Savart law. Using the odd symmetry property of $K_{1}(s)$ in $s_{1}, s_{2}$ and some techniques from optimal transport, we establish sharp estimates for the singular term $u_{x, S}$ in Lemma 3.1 uniformly in $a, b$. Similarly, we have established a sharp estimate of $u_{x}$ in the $C_{y}^{1 / 2}$ seminorm and the estimates of $u_{y}, v_{x}$ in the Hölder seminorms. We estimate the regular part $u_{x, R}$ following the discussion in Section 2.3
2.8.1. Weighted estimates. In the stability analysis, we need to estimate the weighted $C^{\alpha}$ norm of the nonlocal terms. We focus on estimating $u_{x} \psi$. We observe that the commutator

$$
\begin{align*}
{\left[u_{x}(\cdot, a, b), \psi\right](\omega) } & \triangleq u_{x}(\omega)(x, a, b) \psi(x)-u_{x}(\omega \psi)(x, a, b) \\
& =-\frac{1}{\pi} \int_{\left|x_{1}-y_{1}\right| \leq a,\left|x_{2}-y_{2}\right| \leq b} K_{1}(x-y) W(y)(\psi(x)-\psi(y)) d y \tag{2.39}
\end{align*}
$$

is more regular. Therefore, we have the decomposition

$$
\begin{equation*}
u_{x}(\omega)(x, a, b) \psi=u_{x}(\omega \psi)(x, a, b)+\left[u_{x}(\cdot, a, b), \psi\right](\omega) \tag{2.40}
\end{equation*}
$$

For the first term on the right hand side, we can apply the sharp Hölder estimate in Section 3 , Given that $\omega$ is in some weighted $L^{\infty}$ space, since $K_{1}(x-y)(\psi(x)-\psi(y)$ has a singularity of order $\frac{1}{|x-y|}$, the second term is log-Lipschitz and is more regular than the first term. Therefore, we can estimate its $C^{1 / 2}$ seminorm by the weighted $L^{\infty}$ norm of $\omega$. In particular, if $a$ and $b$ are small, we obtain a small factor of order $(\max (a, b))^{1 / 2}$ in this estimate.
2.8.2. The singular scenario. To understand if we can obtain linear stability by treating the nonlocal terms as a small perturbation using the sharp functional inequalities, we consider the following model by dropping the nonlinear and error terms in (2.25), (2.28)

$$
\begin{align*}
\omega_{t} & =-\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \omega+\bar{c}_{\omega} \omega+\eta \\
\eta_{t} & =-\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \eta+\left(2 \bar{c}_{\omega}-\bar{u}_{x}\right) \eta-\left(u_{x}-\hat{u}_{x}\right) \bar{\theta}_{x}-\left(v_{x}-\hat{v}_{x}\right) \bar{\theta}_{y} \tag{2.41}
\end{align*}
$$

We also remove the term $\bar{v}_{x} \xi$ in the $\eta$ equation due to the weak coupling discussed in Section 2.6.2 and the more regular nonlocal terms, e.g. u, which are small in the following estimates. See the discussion around (2.47). Here $\hat{u}_{x}, \hat{v}_{x}$ approximate $u_{x}, v_{x}$ near 0:

$$
\begin{equation*}
\hat{u}_{x}=\left(u_{x}(0)+C_{u_{x}}(x) \mathcal{K}_{00}\right) \chi(x), \quad \hat{v}_{x}=C_{v_{x}}(x) \mathcal{K}_{00} \chi \tag{2.42}
\end{equation*}
$$

The above approximation can be obtained by Taylor expansions and satisfies

$$
\begin{equation*}
u_{x}-\hat{u}_{x}=O\left(|x|^{5 / 2}\right), \quad v_{x}-\hat{v}_{x}=O\left(|x|^{5 / 2}\right) \tag{2.43}
\end{equation*}
$$

for $\omega=O\left(|x|^{5 / 2}\right)$ near $x=0$. These vanishing orders can be justified for $\omega$ in our energy class. The functions $C_{u_{x}}(x), C_{v_{x}}(x)$ and the rank-one operator $\mathcal{K}_{00}$ are defined in (4.26), (4.25), and $\chi$ is some compactly supported cutoff function with $\chi=1$ near $x=0$. The above approximation is a simplification of the finite rank approximation of the velocity in Section 4.3 ,

For initial perturbation $\omega_{0}, \eta_{0}$ with vanishing order $O\left(|x|^{3}\right)$ near $x=0$, using (2.43), we can show that these vanishing conditions can be preserved. See Sections 2.7 .3 and 4.2 for more discussions regarding the vanishing order.
Goal of the estimates and heuristic. In the following weighted Hölder estimates, we consider the difficult scenario discussed in Section 2.7 .2 for $x, z$ with $x_{2}=z_{2}$, and the most singular scenario where $x$ and $z$ are sufficiently close. In this case, using the sharp estimates in Lemmas 3.1 and 3.4 we can establish the linear stability condition (A.3) for (2.41).

In this scenario, we can interpret the following estimates as taking a half derivative $D$ on (2.45). If $D$ applies to a regular term, which is Lipschitz, we almost get 0 . If $D$ acts on the nonlocal terms, e.g. $u_{x}-\hat{u}_{x}$, since $\hat{u}_{x}$ is more regular and $\left[u_{x}\right]_{C_{x}^{1 / 2}}$ can be bounded using Lemma 3.1, we treat it as $2.55\left[\omega \psi_{1}\right]_{C_{x}^{1 / 2}}$. If $D$ applies to the local term, we use the energy to bound it.

To reduce the technicality from the singular weights $\psi_{i}(2.44)$ near 0 and simplify the discussion, we consider $x$ not too close to 0 . For $x$ close to 0 , we actually have a large damping coefficient, see Figure 9, Due to (2.43), $\psi_{1}\left(u_{x}-\hat{u}_{x}\right) \in C^{1 / 2}$, and $\nabla \bar{\theta}=O(|x|)$ near 0 , we have $I=\psi_{2}\left(u_{x}-\hat{u}_{x}\right) \bar{\theta}_{x}=O(|x|)$ and gain a small factor $|x|^{1 / 2}$ for the $C^{1 / 2}$ estimate of $I$ with small $|x-z|$ near 0. Similar estimates apply to $\left(v_{x}-\hat{v}_{x}\right) \bar{\theta}_{y}$ in (2.41). One can also treat this setting by first fixing $x$ and then considering $z$ with $|x-z| \ll|x|$.

Following the ideas in Section [2.7.2, we design the weights (C.1) for the $C^{1 / 2}$ estimate with

$$
\begin{equation*}
\psi_{1} \sim|x|^{-2}, \quad \psi_{2} \sim p_{1}|x|^{-5 / 2} \tag{2.44}
\end{equation*}
$$

near $x=0$ for some parameter $p_{1}$. Next, we perform the weighted $C^{1 / 2}$ estimate. Denote

$$
\bar{b}(x)=\left(\bar{c}_{l} x+\bar{u}, \bar{c}_{l} y+\bar{v}\right)
$$

Derivations for the local terms. For a pair $(x, z)$, applying Lemma 2.5, we get

$$
\begin{align*}
& \partial_{t} \delta\left(\omega \psi_{1}\right) g_{1}+\left(\bar{b}(x) \cdot \nabla_{x}+\bar{b}(z) \cdot \nabla_{z}\right)\left(\delta\left(\omega \psi_{1}\right) g_{1}\right)=\bar{e}_{1}(x, z) \delta\left(\omega \psi_{1}\right) g_{1}+B_{1}(x, z) \\
& \partial_{t} \delta\left(\eta \psi_{2}\right) g_{2}+\left(\bar{b}(x) \cdot \nabla_{x}+\bar{b}(z) \cdot \nabla_{z}\right)\left(\delta\left(\eta \psi_{2}\right) g_{2}\right)=\bar{e}_{2}(x, z) \delta\left(\eta \psi_{2}\right) g_{2}+B_{2}(x, z) \\
& B_{1}(x, z) \triangleq \delta\left(\eta \psi_{1}\right) g_{1}(x-z)+\delta\left(\bar{d}_{1}\right) g_{1}(x-z)\left(\omega \psi_{1}\right)(z)  \tag{2.45}\\
& B_{2}(x, z) \triangleq-\delta\left(\left(u_{x}-\hat{u}_{x}\right) \bar{\theta}_{x} \psi_{2}+\left(v_{x}-\hat{v}_{x}\right) \bar{\theta}_{y} \psi_{2}\right) g_{2}+\delta\left(\bar{d}_{2}\right) g_{2}(x-z)\left(\eta \psi_{2}\right)(z)
\end{align*}
$$

where $B_{i}$ denotes the bad terms, $g_{i}=g_{i}(x-z), \delta f=f(x)-f(z)$, and

$$
\begin{equation*}
\bar{d}_{1}=\bar{c}_{\omega}+\frac{\bar{b} \cdot \nabla \psi_{1}}{\psi_{1}}, d_{2}=2 \bar{c}_{\omega}-\bar{u}_{x}+\frac{\bar{b} \cdot \nabla \psi_{2}}{\psi_{2}}, \bar{e}_{i}(x, z)=\bar{d}_{i}(x)+\frac{(\bar{b}(x)-\bar{b}(z))\left(\nabla g_{i}\right)(x-z)}{g_{i}(x-z)} \tag{2.46}
\end{equation*}
$$

Estimate the nonlocal terms. Suppose that $\omega \varphi_{1}, \eta \varphi_{2} \in L^{\infty}$ and $\delta\left(\omega \psi_{1}\right) g_{1}, \delta\left(\eta \psi_{2}\right) g_{2} \in L^{\infty}$. To estimate $\delta_{2}\left(\left(u_{x}-\hat{u}_{x}\right) \bar{\theta}_{x} \psi_{2}\right) g_{2}$, we introduce $A(x)=\bar{\theta}_{x} \frac{\psi_{2}}{\psi_{1}}$ and rewrite the difference as follows

$$
\begin{aligned}
\delta\left(\left(u_{x}-\hat{u}_{x}\right) \bar{\theta}_{x} \psi_{2}\right) g_{2} & =\delta\left(\left(u_{x}-\hat{u}_{x}\right) \psi_{1} A(x)\right) g_{2} \\
& =\left(\delta\left(\left(u_{x}-\hat{u}_{x}\right) \psi_{1}\right) A(x)+\left(u_{x}-\hat{u}_{x}\right)(z) \psi_{1}(z) \delta(A)\right) g_{2} \triangleq I+I I
\end{aligned}
$$

The second term $I I$ is a regular term since the half derivative acts on the coefficient. Indeed, since we consider $x$ not too close to 0 , the coefficient $\bar{\theta}_{x} \frac{\psi_{2}}{\psi_{1}}$ is Lipschitz. Since $|x-z|$ is sufficiently small and $\left(u_{x}-\hat{u}_{x}\right) \psi_{1}$ is bounded (2.43), (2.44), in this case, we get $|I I| \approx 0$.

For $x$ away from 0 , the weight $\psi$ is nonsingular and $|A(x)| \lesssim 1$. Recall that the commutator $\left[u_{x}, \psi_{1}\right](\omega)(2.39)$ is more regular and $\hat{u}_{x}(2.42)$ is regular. Applying the above argument to estimate the regular term, Lemma 3.1 for $u_{x}\left(\omega \psi_{1}\right)$, and $g_{2}\left(x_{1}-z_{1}, 0\right)=\left|x_{1}-z_{1}\right|^{-1 / 2}$, we get

$$
I \approx \delta\left(u_{x}(\omega) \psi_{1}\right) g_{2} A(x) \approx \delta\left(u_{x}\left(\omega \psi_{1}\right)\right) g_{2} A(x), \quad\left|\delta\left(u_{x}\left(\omega \psi_{1}\right)\right) g_{2} A(x)\right| \leq 2.55\left[\omega \psi_{1}\right]_{C_{x}^{1 / 2}}\left|\bar{\theta}_{x} \frac{\psi_{2}}{\psi_{1}}(x)\right|
$$

Similarly, applying Lemma 3.4 with $\tau=0.582$, we have

$$
\delta\left(v_{x} \bar{\theta}_{y} \psi_{2}\right) g_{2} \approx \delta\left(v_{x}\left(\omega \psi_{1}\right)\right)\left|x_{1}-z_{1}\right|^{-1 / 2} \bar{\theta}_{y} \frac{\psi_{2}}{\psi_{1}} \triangleq m, \quad m \leq 2.53\left\|\delta\left(\omega \psi_{1}\right) g_{1}\right\|_{L^{\infty}}\left|\bar{\theta}_{y} \frac{\psi_{2}}{\psi_{1}}(x)\right|
$$

Estimate the regular and remaining terms. For $|x-z|$ sufficiently small, the more regular term vanishes in this estimate. For example, for $u \bar{\omega}_{x}$ in (2.25) (not included in (2.45)), we will approximate $\mathbf{u}$ using some finite rank operators $\hat{\mathbf{u}}$ similar to (2.42) and estimate $(u-\hat{u}) \bar{\omega}_{x}$, which vanishes $|x|^{7 / 2}$ near $x=0$. See Sections 4.2 and 4.3. We can control the log-Lipschitz norm or $C^{4 / 5}$ norm of $u$ in some weighted space using $\left\|\omega \varphi_{1}\right\|_{L^{\infty}}$. Thus, for $|x|$ not too close to 0 and $|x-z|$ sufficiently small, we get

$$
\begin{equation*}
\left|\delta\left((u-\hat{u}) \bar{\omega}_{x} \psi_{1}\right) g_{1}\right| \approx 0 \tag{2.47}
\end{equation*}
$$

The same idea applies to other regular terms in (2.45), e.g. $\delta\left(\bar{d}_{1}\right) g_{1}(x-z)\left(\omega \psi_{1}\right), \delta\left(\bar{d}_{2}\right) g_{2}(x-$ $z)\left(\eta \psi_{2}\right)$. For $\delta\left(\eta \psi_{1}\right) g_{1}$ in the $\omega$ equation (2.45), we get

$$
\delta\left(\psi_{1} \eta\right) g_{1}=\delta\left(\frac{\psi_{1}}{\psi_{2}} \eta \psi_{2}\right) g_{1} \approx \frac{\psi_{1}}{\psi_{2}}(x) \delta\left(\eta \psi_{2}\right)\left|x_{1}-z_{1}\right|^{-1 / 2} \triangleq m_{1}, \quad\left|m_{1}\right| \leq\left|\frac{\psi_{1}}{\psi_{2}}(x)\right|\left[\eta \psi_{2}\right]_{C_{x}^{1 / 2}}
$$

We remark that, if $z_{1}=x_{1}$, all the above approximations become equality since the difference $\delta(f)|x-z|^{-1 / 2}$ becomes 0 for $f$ being $C^{\beta}, \beta>1 / 2$ around $x$.

Summarize the estimates. For $x_{2}=z_{2}$ with $\left|x_{1}-z_{1}\right|$ sufficiently small, the damping terms $\bar{e}_{1}, \bar{e}_{2}$ in (2.45) can be simplified as

$$
\bar{e}_{i}(x)=\bar{d}_{i}(x)-\frac{1}{2} \frac{\bar{b}_{1}(x)-\bar{b}_{1}(z)}{x_{1}-z_{1}} \approx \bar{d}_{i}(x)-\frac{1}{2} \partial_{1} \bar{b}_{1}(x)=\bar{d}_{i}(x)-\frac{1}{2}\left(\bar{c}_{l}+\bar{u}_{x}(x)\right)
$$

We apply the stability Lemma A. 1 and choose weight $\mu_{1}$

$$
\begin{equation*}
\mu_{1}=0.668 \tag{2.48}
\end{equation*}
$$

to change the weight between $\delta\left(\omega \psi_{1}\right) g_{1}, \delta\left(\eta \psi_{2}\right) g_{2}$ in the energy estimate (2.45) so that the damping term dominates (A.3). When $x_{2}=z_{2}$ with $\left|x_{1}-z_{1}\right| \rightarrow 0$, the above estimate implies

$$
\begin{align*}
& \left|B_{1}(x, x)\right| \leq S_{1}(x) \cdot \mu_{1}\left[\theta_{x} \psi_{2}\right]_{C_{x}^{1 / 2}}, \quad \mu_{1}\left|B_{2}(x, x)\right| \leq S_{2}(x)\left\|\omega \psi_{1}\right\|_{C_{g}^{1 / 2}} \\
& S_{1}=\left|\frac{\psi_{1}}{\mu_{1} \psi_{2}}(x)\right|, \quad S_{2}=\mu_{1}\left(2.55\left|\bar{\theta}_{x}\right|+2.53\left|\bar{\theta}_{y}\right|\right) \frac{\psi_{2}}{\psi_{1}}(x) \tag{2.49}
\end{align*}
$$

From (A.3) and Lemma A.1, if the coefficient of the damping term is larger than that of the bad term, we can obtain stability. Indeed, for some $c_{1}, c_{2}>0$, we have

$$
\begin{equation*}
\bar{e}_{i}+S_{i} \leq-c_{i} \tag{2.50}
\end{equation*}
$$

In Figure 3, we plot the grid point values of $S_{1}, S_{2}$ (2.49), and $-\bar{e}_{1},-\bar{e}_{2}$ on the boundary $y=0$ for $x<5$. The estimates away from the boundary and for large $x$ are much better due to the larger damping from $\frac{\bar{b} \cdot \nabla \psi_{i}}{\psi_{i}}$ and the decay of the profile. See Section 2.7.2


Figure 3. Coefficients $\bar{e}_{1}, \bar{e}_{2}$ of the damping terms and the estimates of the bad terms $B_{1}, B_{2}$

For $|x-z|$ sufficiently small with other ratio $\left|x_{2}-z_{2}\right| /\left|x_{1}-z_{1}\right|$, from Section 2.7.2 we obtain a larger damping term from $\frac{\delta \bar{b} \cdot \nabla g_{i}}{g_{i}}$ (2.46) and better estimates. Using the sharp functional inequalities in Section 3, we can also verify that the damping terms dominate. The case of $|x-z|$ not too small is estimated in Section 55 For larger $|x-z|$, we have better constants in the Hölder estimates for $\nabla \mathbf{u}$ from Lemma 3.1]3.3. See more discussions below (5.24) for this case.

Remark 2.6. To obtain the stability condition (2.50), one can choose weights similar to $\psi_{1}, \psi_{2}$ with other parameters, or other weights. From the derivations in Section 2.7.1 the gap $-c_{i}-$ $\left(\bar{e}_{i}+S_{i}\right)$ can be larger for weight $|x|^{\gamma}$ with smaller $\gamma$. In $\psi_{1}, \psi_{2}$ (C.1), the power with the largest exponent is $|x|^{1 / 6}$, which leads to a smaller gap for $x$ near 0.6. We choose this growing weight so that we have a stronger control of the perturbation in the far-field, which leads to smaller constants in the nonlinear estimates and makes it easier to control the nonlinear estimates.
2.8.3. A model problem for localized velocity and energy estimate. We consider the following model problem to illustrate the ideas of our overall energy estimate and motivate the localization of velocity (2.38)

$$
\begin{equation*}
\omega_{t}=-d(x) \omega+a(x) u_{x}(\omega, \varepsilon)(x) \tag{2.51}
\end{equation*}
$$

Here $u_{x}(\omega, \varepsilon)$ denotes the localized velocity $u_{x}(\omega, a, b)(2.38)$ with $b=a=\varepsilon$. We assume
$(2.52)-d(x)+m|a(x)| \leq-c_{1}<0, \quad-d(x) \leq-c_{2}<0, \quad d(x),|a(x)|,\|d\|_{C^{1 / 2}},\|a\|_{C^{1 / 2}} \leq c_{3}$,
for some constant $c_{1}, c_{2}, c_{3}>0$, where $m=\max \left(C_{1}(\infty), \frac{1}{2} C_{1}(\infty)+C_{2}(\infty)\right) \leq 2.64$. The constants $C_{1}(\cdot), C_{2}(\cdot)$ are defined in Lemmas 3.1, 3.3. From Lemmas 3.1, 3.3, we have

$$
\begin{equation*}
\left[u_{x}(x, a, b)\right]_{C^{1 / 2}} \leq m[\omega]_{C^{1 / 2}} \tag{2.53}
\end{equation*}
$$

uniformly for any $a, b$. The first condition in (2.52) corresponds to the stability condition (2.50) in the $C^{1 / 2}$ estimate with small $|x-y|$. In (2.51), we remove the transport terms for simplicity since in the weighted energy estimate, it contributes to the damping terms (see Sections 2.7.1, 2.8.2) similar to $-d(x) \omega$. The nonlocal term $u_{x}(\omega, \varepsilon)$ models other nonlocal terms $\nabla \mathbf{u}$ in (2.25), (2.28). We remove the more regular nonlocal terms in (2.25), (2.28), e.g. u, $c_{\omega}$, and the nonsingular part $u_{x, R}=u_{x}-u_{x}(\omega, \varepsilon)$, which will be estimated using the methods in Sections 4.14.2,

We argue that if $\varepsilon$ is small enough, we can establish linear stability. We use $C$ to denote absolute constants independent of $c_{i}, \varepsilon$. Using the formula of $u_{x}(\omega, \varepsilon)$ (2.38), we get

$$
\begin{equation*}
\left|u_{x}(\omega, \varepsilon)(x)\right| \leq C \varepsilon^{1 / 2}[\omega]_{C^{1 / 2}},\left|a(x) u_{x}\right| \leq C c_{3} \varepsilon^{1 / 2}[\omega]_{C^{1 / 2}},-d(x) \leq-c_{2} \tag{2.54}
\end{equation*}
$$

The $L^{\infty}$ estimate of $\omega$ is almost closed due to the small parameter $\varepsilon$. Denote $\delta(f)(x, z)=$ $f(x)-f(z)$. In the Hölder estimate, using (2.52), and a direct calculation, we yield

$$
-\frac{\delta(d \omega)(x, z)}{|x-z|^{1 / 2}}=-d(x) \frac{\delta(\omega)}{|x-z|^{1 / 2}}-\frac{d(x)-d(z)}{|x-z|^{1 / 2}} \omega(z), \quad\left|\frac{d(x)-d(z)}{|x-z|^{1 / 2}} \omega(z)\right| \leq c_{3}\|\omega\|_{L^{\infty}}
$$

For $a(x) u_{x}(x, \varepsilon)$, using (2.52), (2.53), and the above $L^{\infty}$ estimate on $u_{x}$, we obtain
$\left|\frac{\delta\left(a u_{x}(\omega, \varepsilon)\right)}{|x-z|^{1 / 2}}\right| \leq a(x)\left|\frac{\delta\left(u_{x}(\omega, \varepsilon)\right)}{|x-z|^{1 / 2}}\right|+\left|\frac{a(x)-a(z)}{|x-z|^{1 / 2}}\right|\left|u_{x}(\omega, \varepsilon)(z)\right| \leq|a(x)| m[\omega]_{C^{1 / 2}}+C c_{3} \varepsilon^{1 / 2}[\omega]_{C^{1 / 2}}$.
It follows
$\partial_{t} \frac{\delta \omega}{|x-z|^{1 / 2}}=-d(x) \frac{\delta \omega}{|x-z|^{1 / 2}}+B(x, z), \quad|B(x, z)| \leq\left(|a(x) m|+C c_{3} \varepsilon^{1 / 2}\right)[\omega]_{C^{1 / 2}}+c_{3}| | \omega \mid \|_{L^{\infty}}$.
The first term is a damping term. To apply Lemma A. 1 for linear stability, we construct energy $E=\max \left(\|\omega\|_{L^{\infty}}, \tau\|\omega\|_{C^{1 / 2}}\right)$ with $\tau$ to be determined. Using the estimates (2.54) and (2.55), we reduce conditions (A.3) to

$$
\begin{equation*}
c_{2}-C c_{3} \varepsilon^{1 / 2} \tau^{-1} \geq \lambda, \quad d(x)-|a(x)| m-C c_{3} \varepsilon^{1 / 2}-\tau c_{3} \geq \lambda \tag{2.56}
\end{equation*}
$$

for some $\lambda>0$. Since $d(x)-|a(x)| m \geq c_{1}$, for $\varepsilon$ small enough such that

$$
C c_{3} \varepsilon^{1 / 2}<\frac{c_{2}}{2}, \quad C c_{3} \varepsilon^{1 / 2} \cdot c_{3}<\frac{c_{2} c_{1}}{2}
$$

we get $d(x)-|a(x)| m-C c_{3} \varepsilon^{1 / 2} \geq \frac{c_{1}}{2}$. Then we can choose

$$
\begin{equation*}
\tau=\sqrt{\frac{C \varepsilon^{1 / 2} c_{1}}{2 c_{2}}} \tag{2.57}
\end{equation*}
$$

to achieve the stability condition (2.56) for some $\lambda>0$.
Interpretation of the estimates. Since the $L^{\infty}$ estimate (2.54) is almost closed, we can formally treat $\|\omega\|_{L^{\infty}}$ as an a-priori estimate. Then we choose the weight $\tau \sim \varepsilon^{1 / 4}$ for the energy $[\omega]_{C^{1 / 2}}$ so that $c_{3} \tau\|\omega\|_{L^{\infty}}(2.55)$ is small and close the Hölder estimate. In our energy estimate of (2.25), (2.28), we follow similar ideas and will approximate the regular terms so that we can establish the $L^{\infty}$ estimate with a small cost of the Hölder norm of $\omega$ similar to (2.54), and then put a small weight to the Hölder norm in the energy for the Hölder estimate.

We will track the constants and choosing the weight, e.g. $\tau$ in (2.57), much more carefully so that we do not need to choose $\varepsilon$ to be too small, or approximate the regular terms using finite rank operators with a very high rank to get a small approximation error in a suitable norm. This will reduce our computation cost significantly. See Sections 4.1 and 4.2.

## 3. Sharp Hölder estimate via optimal transport

In this section, we derive the sharp Hölder $C^{1 / 2}$ estimate for $\nabla \mathbf{u}$ using the symmetry properties of the kernels and some techniques from optimal transport. We note that novel functional inequalities on similar Biot-Savart laws have played a crucial role in the important works [35, 59 . Those estimates enable the authors to control the velocity effectively. The sharp Hölder estimates play a similar role in our work.

The natural approach to obtain the Hölder estimate of $\nabla \mathbf{u}$ in $\mathbb{R}_{2}^{++}$is to estimate $\nabla \mathbf{u}(x)-$ $\nabla \mathbf{u}(z)$ for all pairs $x, z \in \mathbb{R}_{++}^{2}$, which has a dimension of 4 . Yet, it is very difficult to obtain a sharp estimate since the kernel in $\nabla \mathbf{u}(x)-\nabla \mathbf{u}(z)$ for arbitrary $x, z$ has a complicated sign structure and destroys some symmetry properties of the kernels in $\nabla^{\perp}(-\Delta)^{-1}$. Instead, we will estimate the $C_{x}^{1 / 2}$ and $C_{y}^{1 / 2}$ seminorms (2.20) due to the following important observations. Firstly, the linearized operators (2.25), (2.28) are anisotropic in $x$ and $y$. See Section 2.7.2 We have much larger damping factors along the $y$ direction. Therefore, a sharp Hölder estimate of $\nabla \mathbf{u}$ in the $x$ direction, i.e. $[\nabla \mathbf{u}]_{C_{x}^{1 / 2}}$ is much more important. Secondly, if we estimate the $C_{x}^{1 / 2}$ or $C_{y}^{1 / 2}$ seminorm (2.20), where we assume $x_{1}=z_{1}$ or $x_{2}=z_{2}$, we reduce the dimension of $(x, z)$ from 4 to 3 . Moreover, the kernel in $\nabla \mathbf{u}(x)-\nabla \mathbf{u}(z)$ enjoys better symmetry properties and the sign properties are much simpler. These properties allow us to reduce estimating the 2D integral into estimating many 1D integrals. After we estimate $[\nabla \mathbf{u}]_{C_{x}^{1 / 2}},[\nabla \mathbf{u}]_{C_{y}^{1 / 2}}$, using the triangle inequality, we can obtain the estimate of $[\nabla \mathbf{u}]_{C^{1 / 2}}$.

The kernels associated with $\nabla u$ are given by

$$
\begin{equation*}
K_{1}(y) \triangleq \frac{y_{1} y_{2}}{|y|^{4}}, \quad K_{2}(y) \triangleq \frac{1}{2} \frac{y_{1}^{2}-y_{2}^{2}}{|y|^{4}}, \quad G(y)=-\frac{1}{2} \log |y| \tag{3.1}
\end{equation*}
$$

where $\frac{1}{\pi} G$ is the Green function of $-\Delta$ in $\mathbb{R}^{2}$. Note that $\partial_{1} \partial_{2} G=K_{1}, \quad \partial_{1}^{2} G=K_{2}$.
Denote by $K_{i, s}$ the symmetrized kernel

$$
\begin{equation*}
K_{i, s}(x, y) \triangleq K_{i}(x-y)-K_{i}\left(x_{1}+y_{1}, x_{2}-y_{2}\right)-K_{i}\left(x_{1}-y_{1}, x_{2}+y_{2}\right)+K_{i}(x+y) \tag{3.2}
\end{equation*}
$$

Consider the odd extension of $\omega$ in $y$ from $\mathbb{R}_{2}^{+}$to $\mathbb{R}_{2}$

$$
\begin{equation*}
W(y)=\omega(y) \text { for } y_{2} \geq 0, \quad W(y)=-\omega\left(y_{1},-y_{2}\right) \text { for } y_{2}<0 \tag{3.3}
\end{equation*}
$$

$W$ is odd in both $y_{1}$ and $y_{2}$ variables. Clearly, $u_{x}$ can be written as

$$
u_{x}(x)=-\frac{1}{\pi} \int_{\mathbb{R}^{2}} K_{1}(x-y) W(y) d y=-\frac{1}{\pi} \int_{\mathbb{R}_{2}^{++}} \omega(y) K_{1, s}(x, y) d y
$$

For any $a, b_{1}, b_{2}>0$, we consider the localized velocity

$$
\begin{align*}
Q_{a, b_{1}, b_{2}}(x) & \triangleq\left[x_{1}-a, x_{1}+a\right] \times\left[x_{2}-b_{1}, x_{2}+b_{2}\right], \\
u_{x}\left(x, a, b_{1}, b_{2}\right) & \triangleq-\frac{1}{2 \pi} P . V \cdot \int_{y \in Q_{a, b_{1}, b_{2}(x)}} \frac{2\left(x_{1}-y_{1}\right)\left(x_{2}-y_{2}\right)}{|x-y|^{2}} \omega(y) d y, \\
u_{y}(x, a) & \triangleq \frac{1}{2 \pi} P . V \cdot \int_{y \in Q_{a, a}(x)} \frac{\left(x_{1}-y_{1}\right)^{2}-\left(x_{2}-y_{2}\right)^{2}}{|x-y|^{4}} \omega(y) d y+\frac{\omega(x)}{2},  \tag{3.4}\\
v_{x}(x, a) & \triangleq \frac{1}{2 \pi} P . V \cdot \int_{y \in Q_{a, a}(x)} \frac{\left(x_{1}-y_{1}\right)^{2}-\left(x_{2}-y_{2}\right)^{2}}{|x-y|^{4}} \omega(y) d y-\frac{\omega(x)}{2} .
\end{align*}
$$

If $b_{1}=b_{2}=b$, we simplify $u_{x}\left(x, a, b_{1}, b_{2}\right)$ as $u_{x}(x, a, b)$; if $b_{1}=b_{2}=a$, we further simplify $u_{x}\left(x, a, b_{1}, b_{2}\right)$ as $u_{x}(x, a)$.
3.1. Hölder estimates of the velocity. We have the following estimates for $\nabla \mathbf{u}$. We will discuss the ideas in Section 3.2 and the proof in Sections 3.3, 3.4 and Appendix B We localize the velocity in (3.4) to obtain improvement of the constant $C_{1}\left(\frac{b}{|x-z|}\right)$ when $|x-z|$ is large.

Lemma 3.1 (Estimate of $\left.\left[u_{x}\right]_{C_{x}^{1 / 2}}\right)$. For any $b_{1}, b_{2}>0, a \geq \frac{1}{2}\left|x_{1}-z_{1}\right|, x=\left(x_{1}, x_{2}\right), z=$ $\left(z_{1}, x_{2}\right) \in \mathbb{R}_{2}^{+}$, and $D$ covering $\left(Q_{a, b_{1}, b_{2}}(x) \cup Q_{a, b_{1}, b_{2}}(z)\right) \cap \mathbb{R}_{2}^{+}$(3.4), we have

$$
\frac{\left|u_{x}\left(x, a, b_{1}, b_{2}\right)-u_{x}\left(z, a, b_{1}, b_{2}\right)\right|}{|x-z|^{1 / 2}} \leq C_{1}\left(\frac{b}{|x-z|}\right)[\omega]_{C_{x}^{1 / 2}(D)}
$$

where $b=\max \left(b_{1}, b_{2}\right)$ and $C_{1}(a)$ is an increasing function given by

$$
\begin{aligned}
C_{1}(b) & \left.=\frac{4}{\pi}\left|\int_{0}^{b} d s_{2} \int_{f\left(s_{2}\right)}^{\infty}\right| T\left(s_{1}, s_{2}\right)-\left.s_{1}\right|^{1 / 2} \Delta\left(s_{1}, s_{2}\right) d s_{1} \right\rvert\, \\
\Delta\left(s_{1}, s_{2}\right) & =\frac{\left(s_{1}+1 / 2\right) s_{2}}{\left(\left(s_{1}+1 / 2\right)^{2}+s_{2}^{2}\right)^{2}}-\frac{\left(s_{1}-1 / 2\right) s_{2}}{\left(\left(s_{1}-1 / 2\right)^{2}+s_{2}^{2}\right)^{2}} .
\end{aligned}
$$

Here, $f\left(s_{2}\right)$ is the unique solution in $[0, \infty)$ satisfying $\Delta\left(f\left(s_{2}\right), s_{2}\right)=0$ and $T\left(s_{1}, s_{2}\right)$ is the unique solution in $\left[0, f\left(s_{2}\right)\right)$ that solves

$$
\int_{T\left(s_{1}, s_{2}\right)}^{s_{1}} \Delta\left(s_{1}, s_{2}\right) d s_{1}=0
$$

for $s_{1}>f\left(s_{2}\right)$. In particular, $T\left(s_{1}, s_{2}\right)$ can be obtained explicitly by solving a cubic equation and $C_{1}(b) \leq 2.55$ for any $b>0$.

If $a<\frac{1}{2}\left|x_{1}-z_{1}\right|$, the singular region is small. We can simply apply the triangle inequality to estimate each term. We localize the seminorm to region $D$ since we only use the seminorm to control $w(x)-w(z)$ for $x, z \in D$. The same reasoning applies to the following lemmas.

Remark 3.2. The above Lemma can be further generalized to the localized velocity
$\omega * K_{1}(s) 1_{-a_{1} \leq s \leq a_{2},-b_{1} \leq s_{2} \leq b_{2}}$, i.e., we do not need $a_{1}=a_{2}$ in (3.4). The proof follows from the same argument. Yet, we will only use the special case $a_{1}=a_{2}$ in our later estimates.

The upper bounds in the following Lemmas involve $[\omega]_{C_{x}^{1 / 2}}$ and $[\omega]_{C_{y}^{1 / 2}}$. We will further bound it using the energy norm.

Lemma 3.3 (Estimate of $\left.\left[u_{x}\right]_{C_{y}^{1 / 2}}\right)$. For any $a, b \geq|x-z|, x=\left(x_{1}, x_{2}\right), z=\left(x_{1}, z_{2}\right) \in \mathbb{R}_{2}^{+}$, and $D$ covering $\left(Q_{a, b}(x) \cup Q_{a, b}(z)\right) \cap \mathbb{R}_{2}^{+}$(3.4), we have

$$
\frac{\left|u_{x}(x, a, b)-u_{x}(z, a, b)\right|}{|x-z|^{1 / 2}} \leq \frac{1}{2} C_{1}\left(\frac{a}{|x-z|}\right)[\omega]_{C_{y}^{1 / 2}(D)}+C_{2}\left(\frac{a}{|x-z|}\right)[\omega]_{C_{x}^{1 / 2}(D)}
$$

where $C_{1}(a)$ is defined in the previous Lemma and $C_{2}(a)$ is given by

$$
C_{2}(a)=\frac{\sqrt{2}}{\pi} \int_{0}^{a} \int_{0}^{\infty} y_{1}^{1 / 2}\left|\frac{y_{1}\left(1 / 2-y_{2}\right)}{\left(y_{1}^{2}+\left(1 / 2-y_{2}\right)^{2}\right)^{2}}+\frac{y_{1}\left(1 / 2+y_{2}\right)}{\left(y_{1}^{2}+\left(1 / 2+y_{2}\right)^{2}\right)^{2}}\right| d y
$$

In particular, $C_{2}(a) \leq \frac{4.26}{\pi}$.
Next we estimate the other kernel. We remark that for $u_{y}(x, a)$ and $v_{x}(y, a)$, the estimates are different due to the local term related to $\omega$ (3.4).

Lemma $3.4\left(C_{x}^{1 / 2}\right.$ estimate of $\left.u_{y}, v_{x}\right)$. For any $a \geq 2|x-z|, x=\left(x_{1}, x_{2}\right), z=\left(z_{1}, x_{2}\right) \in \mathbb{R}_{2}^{+}, D$ covering $\left(Q_{a}(x) \cup Q_{a}(z)\right) \cap \mathbb{R}_{2}^{+}$(3.4), and $\tau>0$, we have

$$
\begin{aligned}
\frac{\left|v_{x}(x, a)-v_{x}(z, a)\right|}{|x-z|^{1 / 2}} & \leq \frac{1}{\pi} C_{1}(\tau) \max \left([\omega]_{C_{x}^{1 / 2}(D)}, \tau^{-1}[\omega]_{C_{y}^{1 / 2}(D)}\right) \\
\frac{\left|u_{y}(x, a)-u_{y}(z, a)\right|}{|x-z|^{1 / 2}} & \leq \frac{1}{\pi} C_{2}(\tau) \max \left([\omega]_{C_{x}^{1 / 2}(D)}, \tau^{-1}[\omega]_{C_{y}^{1 / 2}(D)}\right)
\end{aligned}
$$

for some constant $C_{1}(\tau), C_{2}(\tau)>0$ with $\frac{1}{\pi} C_{1}(0.582) \leq 2.53$ and $\frac{1}{\pi} C_{2}(0.582) \leq 1.55$.
In the proof of the above Lemma, we provide the upper bounds for $C_{1}(\tau), C_{2}(\tau)$, which can be computed. Although the estimates are equivalent for different $\tau$, we choose $\tau$ according to the weight $g_{1}$ in the Hölder seminorm $\left[\omega \psi_{1}\right]_{C_{g_{1}}^{1 / 2}}$. In practice, we choose $\tau=g_{1}(0,1) / g_{1}(1,0)$ which is close to 0.582 .

In general, the localized $u_{y}$ is not in $C_{y}^{1 / 2}$ due to the presence of the boundary and the discontinuity of $W$ cross $y=0$. Thus, we consider the estimate without localizing the kernel.
Lemma 3.5 ( $C_{y}^{1 / 2}$ estimate of $\left.u_{y}, v_{x}\right)$. For $x=\left(x_{1}, x_{2}\right), z=\left(x_{1}, z_{2}\right) \in \mathbb{R}_{2}^{+}, D$ covering $\left(Q_{a}(x) \cup Q_{a}(z)\right) \cap \mathbb{R}_{2}^{+}$(3.4), and any $\tau>0$, we have

$$
\begin{aligned}
& \frac{\left|v_{x}(x, \infty)-v_{x}(z, \infty)\right|}{|x-z|^{1 / 2}} \leq \frac{C_{3}(\tau)}{\pi} \max \left([\omega]_{C_{x}^{1 / 2}(D)}, \tau^{-1}[\omega]_{C_{y}^{1 / 2}(D)}\right) \\
& \frac{\left|u_{y}(x, \infty)-u_{y}(z, \infty)\right|}{|x-z|^{1 / 2}} \leq \frac{C_{4}(\tau)}{\pi} \max \left([\omega]_{C_{x}^{1 / 2}(D)}, \tau^{-1}[\omega]_{C_{y}^{1 / 2}(D)}\right)
\end{aligned}
$$

for some constant $C_{3}(\tau), C_{4}(\tau)>0$. We have $\frac{1}{\pi} C_{3}(0.582) \leq 2.60, \frac{1}{\pi} C_{4}(0.582) \leq 2.61$.
3.2. Connection to optimal transport and ideas of the proof. A key observation is that the Hölder estimate is related to an optimal transport problem. We illustrate the ideas by proving a sharp Hölder estimate of the Hilbert transform. The Hilbert transform can be seen as an approximation of $u_{x}(\omega)$, which is exact if $\omega(x, y)$ is constant in $y$ [21,65.

We estimate $\frac{1}{|x-z|^{1 / 2}}|H f(x)-H f(z)|$ by $[f]_{C^{1 / 2}}$. Due to translation and scaling symmetry, we can assume $x=1, z=-1$ without loss of generality. Then we need to estimate

$$
\begin{equation*}
S=H f(1)-H f(-1)=\frac{1}{\pi} \int_{\mathbb{R}}\left(\frac{1}{1-y}+\frac{1}{1+y}\right) f(y) d y=\frac{2}{\pi} \int_{\mathbb{R}} \frac{1}{1-y^{2}} f(y) d y \tag{3.5}
\end{equation*}
$$

The kernel $k(y)=\frac{2}{1-y^{2}}$ is positive on $(-1,1)$ and negative for $|y|>1$, and satisfies $\int k(y) d y=0$.

Denote $k^{ \pm}(y)=\max ( \pm k(y), 0)$. An estimate of $S$ using $[f]_{C^{1 / 2}}$ is equivalent to estimating the transportation cost of moving the positive region of $k(y)$ with measure $k^{+}(y) d y$ to its negative region with measure $k^{-}(y) d y$ with distant function $c(x, y)=|x-y|^{1 / 2}$.

For example, if $k(y)=\delta_{1}(y)+\delta_{2}(y)-\delta_{3}(y)-\delta_{4}(y)$, where $\delta_{a}(x)$ is the Dirac function centered at $a$, then we get

$$
\left|\int k(y) f(y) d y\right|=|f(1)+f(2)-f(3)-f(4)| \leq|f(2)-f(3)|+|f(1)-f(4)| \leq(\sqrt{1}+\sqrt{3})\|f\|_{C^{1 / 2}}
$$

The above estimate can be interpreted as moving the mass from 2 to 3 and 1 to 4 with cost function $|x-y|^{1 / 2}\|f\|_{C^{1 / 2}}$. Using the language of optimal transport, to obtain sharp estimate of $S$ (3.5), we are seeking a measurable map $T$ such that $T \# k^{+} d y=k^{-} d y$, where $(T \# \mu)(A)=\mu(T(A))$ for a measurable set $A$, and the following cost

$$
C(T)=\int_{k(y) \geq 0}|T(y)-y|^{1 / 2} k(y) d y\|f\|_{C^{1 / 2}}
$$

is as small as possible. Based on the above discussion, we have the following transportation lemma, which will be used repeatedly in the Hölder estimate.

Lemma 3.6 (Transportation Lemma). Suppose that there exists $c \in(a, b)$ such that $f<0$ on $(a, c), f>0$ on $(c, b), f|x-c|^{\alpha} \in L_{l o c}^{1}$ with $\int_{a}^{b} f(x) d x=0$. For $\alpha \in(0,1), g \in C^{\alpha}(a, b)$, we have

$$
\left|\int_{a}^{b} f(x) g(x) d x\right| \leq \int_{c}^{b}|f(x)||x-T(x)|^{\alpha} d x[g]_{C_{x}^{\alpha}}=\int_{a}^{c}|f(x) \| x-T(x)|^{\alpha} d x[g]_{C^{\alpha}}
$$

where $T(x)$ solves $\int_{x}^{T(x)} f(s) d s=0$.
We use the fact that $|h|^{\alpha}$ is concave for $\alpha \in(0,1)$ to design the map $T$. In our later estimates of $\nabla \mathbf{u}$, we will use the above Lemma with $\alpha=\frac{1}{2}$

Proof. Firstly, we want to understand how to construct the map $T$. Note that for $x_{1}<x_{2}<$ $x_{3}<x_{4}$ and $\alpha \in(0,1)$, we have

$$
\begin{aligned}
& \left|\int\left(\delta_{x_{1}}+\delta_{x_{2}}-\delta_{x_{3}}-\delta_{x_{4}}\right) g(x) d x\right|=\left|g\left(x_{1}\right)+g\left(x_{2}\right)-g\left(x_{3}\right)-g\left(x_{4}\right)\right| \\
\leq & \min \left(\left|x_{1}-x_{3}\right|^{\alpha}+\left|x_{2}-x_{4}\right|^{\alpha},\left|x_{1}-x_{4}\right|^{\alpha}+\left|x_{2}-x_{3}\right|^{\alpha}\right)[g]_{C_{x}^{\alpha}} \\
= & \left(\left|x_{1}-x_{4}\right|^{\alpha}+\left|x_{2}-x_{3}\right|^{\alpha}\right)[g]_{C^{\alpha}}
\end{aligned}
$$

The above estimate indicates that to find an optimal map $T$ moving $\left(\delta_{x_{1}}+\delta_{x_{2}}\right) d x$ to $\left(\delta_{x_{3}}+\delta_{x_{4}}\right) d x$ with cost $|x-y|^{\alpha}$, we should choose $T\left(x_{1}\right)=x_{4}, T\left(x_{2}\right)=x_{3}$, which implies that $T(x)$ is decreasing in $x$. Due to conservation of mass and the sign properties of $f$, a natural construction of $T:(a, c) \rightarrow(c, b)$ is given by

$$
\begin{equation*}
\int_{x}^{T(x)} f(x) d x=0 \tag{3.6}
\end{equation*}
$$

for $x<c$, which implies $T^{\prime}(x) f(T(x))=f(x)$ for smooth $f$. The idea of the above map is to move the mass in the positive region to its closest negative region that has not been occupied due to the monotonicity of $T$. Using a change of variable $y=T(x):(a, c) \rightarrow(c, b]$, we get

$$
\int_{c}^{b} f(x) g(x) d x=-\int_{a}^{c} f(T(x)) g(T(x)) T^{\prime}(x) d x=-\int_{a}^{c} f(x) g(T(x)) d x
$$

It follows

$$
\left|\int_{a}^{b} f g\right|=\left|\int_{a}^{c} f(x)(g(T(x))-g(x)) d x\right| \leq \int_{a}^{c}|f(x) \| T(x)-x|^{\alpha} d x[g]_{C_{x}^{\alpha}}
$$

Similarly, we can define $T:(c, b) \rightarrow(a, c)$ by solving $\int_{T(x)}^{x} f(s) d s=0$, which is also equivalent to (3.6). The first inequality in Lemma 3.6 follows from the same argument.
3.2.1. $C^{1 / 2}$ estimate of the Hilbert transform. We use the Hilbert transform as an example to illustrate Lemma 3.6. We apply Lemma 3.6 with $f=k(y), g=f(y)$ to estimate (3.5). For any $y>0$, we construct the transportation map $T(y)$ by solving

$$
0=\int_{x}^{T(x)} k(y) d y=\int_{x}^{T(x)} \frac{1}{1-y^{2}} d y=0
$$

which implies

$$
\frac{x+1}{1-x}=\frac{T+1}{T-1}, \quad T(x)=\frac{1}{x}
$$

where we have used $T(x)>1$ if $x<1$ and $T(x)<1$ if $x>1$ due to the sign of $\frac{1}{1-y^{2}}$. This map also applies to $y<0$. Applying this map to (3.5), we yield

$$
\begin{align*}
|S| & =\left|\frac{2}{\pi} \int_{y>1} k(y)(f(y)-f(T(y))) d y+\frac{2}{\pi} \int_{y<-1} k(y)(f(y)-f(T(y))) d y\right| \\
& \leq \frac{4}{\pi} \int_{y>1}|k(y)||y-T(y)|^{1 / 2} d y[f]_{C^{1 / 2}}=\frac{4}{\pi} \int_{y>1} \frac{1}{\left|y^{2}-1\right|} \frac{\left|y^{2}-1\right|^{1 / 2}}{|y|^{1 / 2}} d y[f]_{C^{1 / 2}}  \tag{3.7}\\
& =[f]_{C^{1 / 2}} \frac{4}{\pi} \int_{y>1} \frac{1}{\left|y^{2}-1\right|^{1 / 2} y^{1 / 2}} d y=C[f]_{C^{1 / 2}}, \quad \frac{C}{\sqrt{2}} \approx 2.37
\end{align*}
$$

Since $x, z$ are arbitrary, we yield $[H f]_{C^{1 / 2}} \leq \frac{C}{\sqrt{2}}[f]_{C^{1 / 2}}$. The equality achieves if $\mid f(y)-$ $f\left(\frac{1}{y}\right)\left|=\left|y-\frac{1}{y}\right|^{1 / 2}[f]_{C^{1 / 2}}\right.$ for all $y>0$ and $y<0$. Since the Hilbert transform satisfies $H(H f)=$ $-f$, the sharp constant in $[H f]_{C^{1 / 2}} \leq C_{*}[f]_{C^{1 / 2}}$ satisfies $C_{*} \geq 1$.

In the following subsections, we prove Lemmas 3.1 , 3.3 for $u_{x}$, which is the main nonlocal term in (2.30). The proofs of Lemmas 3.4, 3.5 are similar but technical due to the presence of boundary, which are deferred to Appendix B

To apply Lemma 3.6 to the Hölder estimate of $\nabla \mathbf{u}$, we need two steps. Firstly, we identify the sign of the kernel $K$ in the integral of $\nabla \mathbf{u}(x)-\nabla \mathbf{u}(z)$. Next, we fix a variable in the 2D integral in one direction, e.g. fix $x=a$, and then apply Lemma 3.6 to estimate the 1D integral in the other direction, e.g., on the line $\{(a, y): y \in \mathbb{R}\}$. One may generalize Lemma 3.6 to 2D and construct the 2D optimal transport map directly. Yet, the domain where the kernel $K$ is positive or negative is complicated. To avoid this difficulty, we build the 2 D transport map using the 1D Lemma 3.6 repeatedly. The odd symmetry of the kernel $K_{1}(s)$ in $s_{1}$ enables us to apply this approach to obtain sharp estimate of $u_{x}$ effectively. See Remark 3.7
3.3. Estimate of $\left[u_{x}\right]_{C_{x}^{1 / 2}}$. In the $C_{x}^{1 / 2}$ estimate of $u_{x}$, we have $x_{2}=z_{2}$. In the case without localization of the kernel, using the scaling symmetry and translation invariance, we only need to estimate the following

$$
\begin{equation*}
u_{x}\left(\frac{1}{2}, x_{2}\right)-u_{x}\left(-\frac{1}{2}, x_{2}\right)=-\frac{1}{\pi} P . V \cdot \int_{\mathbb{R}^{2}} K(s) W\left(s_{1}, x_{2}-s_{2}\right) d s \tag{3.8}
\end{equation*}
$$

for any $x_{2}$, where $W$ is an odd extension of $\omega$ from $\mathbb{R}_{2}^{+}$to $\mathbb{R}^{2}$, and $K(s)$ is given by

$$
\begin{equation*}
K(s)=K_{1}\left(s_{1}+\frac{1}{2}, s_{2}\right)-K_{1}\left(s_{1}-\frac{1}{2}, s_{2}\right)=\frac{\left(s_{1}+\frac{1}{2}\right) s_{2}}{\left(\left(s_{1}+\frac{1}{2}\right)^{2}+s_{2}^{2}\right)^{2}}-\frac{\left(s_{1}-\frac{1}{2}\right) s_{2}}{\left(\left(s_{1}-\frac{1}{2}\right)^{2}+s_{2}^{2}\right)^{2}} \tag{3.9}
\end{equation*}
$$

Since $K(s)$ is odd in $s_{2}$, we consider $s_{2} \geq 0$ without loss of generality. We will only use the Hölder seminorm of $W,[W]_{C_{x}^{1 / 2}}$, to estimate the above quantity. Note that $[W]_{C_{x}^{1 / 2}\left(\mathbb{R}^{2}\right)}=$ $[\omega]_{C_{x}^{1 / 2}\left(\mathbb{R}_{+}^{2}\right)}$. Without loss of generality, we can assume that $x_{2}=0$.

A direct calculation yields

$$
K(s)=\frac{s_{2} \Delta_{1}\left(s_{1}, s_{2}\right)}{\left(\left(s_{1}+\frac{1}{2}\right)^{2}+s_{2}^{2}\right)^{2}\left(\left(s_{1}-\frac{1}{2}\right)^{2}+s_{2}^{2}\right)^{2}}, \quad \Delta_{1}=s_{2}^{4}-2 s_{1}^{2} s_{2}^{2}-3 s_{1}^{4}+\frac{1}{2} s_{1}^{2}+\frac{1}{2} s_{2}^{2}+\frac{1}{16}
$$

For a fixed $s_{2}, \Delta_{1}\left(s_{1}, s_{2}\right)=0$ implies

$$
\begin{equation*}
s_{1}=f\left(s_{2}\right)=\left(\frac{\frac{1}{2}-2 s_{2}^{2}+\sqrt{16 s_{2}^{4}+4 s_{2}^{2}+1}}{6}\right)^{1 / 2} \tag{3.10}
\end{equation*}
$$

Moreover, for $s_{1}, s_{2} \geq 0$, it is easy to see that $\Delta_{1}\left(s_{1}, s_{2}\right) \geq 0$ if and only if

$$
\begin{equation*}
K\left(s_{1}, s_{2}\right) \geq 0 \text { for } s_{1} \in\left[0, f\left(s_{2}\right)\right], \quad K\left(s_{1}, s_{2}\right) \leq 0 \text { for } s_{1} \geq f\left(s_{2}\right) \tag{3.11}
\end{equation*}
$$

See Figure 4 for an illustration of $\operatorname{sgn}(K(s))$.


Figure 4. The black curve illustrates $s_{1}=f\left(s_{2}\right)$ (3.10) (but does not agree with the exact function), and $\pm$ indicates $\operatorname{sgn}(K(s))$ in different regions. The blue arrows indicate the direction of 1D transportation plan.

Note that the sign changes if $s_{2} \leq 0$ since $K$ is odd in $s_{2}$. Since $K_{1}$ is odd in $s_{1}$, we get

$$
\begin{aligned}
\int_{0}^{\infty} K\left(s_{1}, s_{2}\right) d s_{1} & =\int_{0}^{\infty} K_{1}\left(s_{1}+\frac{1}{2}, s_{2}\right)-K_{1}\left(s_{1}-\frac{1}{2}, s_{2}\right) d s_{1} \\
& =\int_{1 / 2}^{\infty} K_{1}\left(s_{1}, s_{2}\right) d s_{1}-\int_{-1 / 2}^{\infty} K\left(s_{1}, s_{2}\right) d s_{1}=-\int_{-1 / 2}^{1 / 2} K_{1}\left(s_{1}, s_{2}\right) d s_{1}=0
\end{aligned}
$$

To estimate the integral in (3.8), we first fix $s_{2}$ and then apply Lemma 3.6 to estimate
$I\left(s_{2}\right)=\int_{\mathbb{R}} K\left(s_{1}, s_{2}\right) W\left(s_{1}, x_{2}-s_{2}\right) d s_{1}=\left(\int_{\mathbb{R}_{-}}+\int_{R_{+}}\right) K\left(s_{1}, s_{2}\right) W\left(s_{1}, x_{2}-s_{2}\right) d s_{1} \triangleq I_{-}\left(s_{2}\right)+I_{+}\left(s_{2}\right)$.
We do so for the following reason. Near the singularity, from Taylor expansion of (3.10): $s_{1}=\frac{1}{2}+O\left(s_{2}^{4}\right)$, the curve $\Gamma=\left\{s: s_{1}=f\left(s_{2}\right)\right\}$ is close to a straight line in the vertical direction. See Figure 4 for an illustration. Similar to the idea below (3.6), an effective plan in 2D is to move the mass in the positive region to its closest possible negative region that has not been occupied. Thus, we expect that an effective 2D transport plan $(x, y) \rightarrow T(x, y)$ is orthogonal to the curve $\Gamma$ and thus almost parallel to the $x$ direction.
Remark 3.7. The fact that near the singularity $s=\left( \pm \frac{1}{2}, 0\right)$, the curve $\Gamma$ is almost vertical is due to the odd symmetry of $K_{1}\left(s_{1}, s_{2}\right)$ in $s_{1}$. In fact, from (3.9), for $s$ close to $\left(\frac{1}{2}, 0\right)$, we have $K(s) \approx-K_{1}\left(s_{1}-\frac{1}{2}, s_{2}\right)$, whose sign is determined by $\operatorname{sgn}\left(s_{1}-\frac{1}{2}\right)$.

Since $K\left(s_{1}, s_{2}\right)$ is even in $s_{1}$, we can estimate $I_{+}\left(s_{2}\right), I_{-}\left(s_{2}\right)$ in the same way. To apply Lemma 3.6, we first construct $T\left(\cdot, s_{2}\right)$ on $[0, \infty)$ by solving

$$
\begin{equation*}
\int_{T\left(s_{1}, s_{2}\right)}^{s_{1}} K\left(t, s_{2}\right) d t=0 \tag{3.13}
\end{equation*}
$$

We will show later that this equation has a unique solution of $T$ on $[0, \infty]$ for $s_{1}>0$. Then applying Lemma 3.6 to $I_{+}\left(s_{2}\right)$ and using $[W]_{C_{x}^{1 / 2}}=[\omega]_{C_{x}^{1 / 2}}$, we get

$$
\left|I_{+}\left(s_{2}\right)\right| \leq[\omega]_{C_{x}^{1 / 2}}\left|\int_{f\left(s_{2}\right)}^{\infty} K\left(s_{1}, s_{2}\right)\right| T\left(s_{1}, s_{2}\right)-\left.s_{1}\right|^{1 / 2} d s_{1} \mid \triangleq M\left(s_{2}\right)
$$

See the blue arrows in Figure 4 for an illustration of this transportation plan. Since $K\left(s_{1}, s_{2}\right)$ is even in $s_{1}$, the estimate of $I_{-}\left(s_{2}\right)$ in (3.12) is the same: $I_{-}\left(s_{2}\right) \leq M\left(s_{2}\right)$. Since $K(s)$ is odd in $s_{2}$, from (3.13), we get $T\left(s_{1},-s_{2}\right)=T\left(s_{1}, s_{2}\right)$. Therefore, the estimate of $I\left(s_{1}, s_{2}\right)$ is the same as $I\left(s_{1},-s_{2}\right)$ : $\left|I\left(s_{1}, s_{2}\right)\right| \leq 2 M\left(\left|s_{2}\right|\right)$. Integrating the estimate of $I(s)$ over $s_{2}$, we yield $\left.\left|-\frac{1}{\pi} P \cdot V \cdot \int_{\mathbb{R}^{2}} K(s) W\left(s_{1}, x_{2}-s_{2}\right) d s\right| \leq \frac{4}{\pi}[\omega]_{C_{x}^{1 / 2}}\left|\int_{0}^{\infty} d s_{2} \int_{f\left(s_{2}\right)}^{\infty} K\left(s_{1}, s_{2}\right)\right| T\left(s_{1}, s_{2}\right)-\left.s_{1}\right|^{1 / 2} d s_{1} \right\rvert\,$, which along with (3.8) prove Lemma 3.1 in the case of $a=b_{1}=b_{2}=\infty$.
3.3.1. Formula of $T$. From (3.9) and $\frac{x y}{\left(x^{2}+y^{2}\right)^{2}}=-\frac{1}{2} \partial_{x} \frac{y}{x^{2}+y^{2}}$, equation (3.13) is equivalent to

$$
s_{2}\left(\frac{1}{\left(T+\frac{1}{2}\right)^{2}+s_{2}^{2}}-\frac{1}{\left(T-\frac{1}{2}\right)^{2}+s_{2}^{2}}\right)=s_{2}\left(\frac{1}{\left(s_{1}+\frac{1}{2}\right)^{2}+s_{2}^{2}}-\frac{1}{\left(s_{1}-\frac{1}{2}\right)^{2}+s_{2}^{2}}\right)
$$

where we have simplified $T\left(s_{1}, s_{2}\right)$ as $T$. For $s_{2} \neq 0$, expanding the identity yields

$$
\begin{equation*}
0=-1-8 T s_{1}+16 T s_{1}\left(T^{2}+T s_{1}+s_{1}^{2}\right)-8 s_{2}^{2}\left(1-4 s_{1} T+2 s_{2}^{2}\right) \tag{3.14}
\end{equation*}
$$

The above equation is cubic in $T$, and thus can be solved explicitly. In Appendix B.3, we show that it has a unique real root and derive its solution formula.

Remark 3.8. In the special case where $\omega(x, y)$ is constant in $y$, we have $u_{x}(\omega)(x, 0)=H \omega(x)$, which has been observed in [21, 65]. Thus, the optimal constant in Lemma 3.1 must be larger than that of the Hilbert transform (3.7). Here, we can obtain upper bound $C_{1}(b) \leq 2.55$, which is very close to that of the Hilbert transform $C / \sqrt{2} \approx 2.37$ (3.7), which reflects the effectiveness of applying 1 D transport maps to construct the 2 D transport map in this setting.
3.3.2. Localized estimate of $u_{x}$. Next, we estimate $u_{x}\left(x, a, b_{1}, b_{2}\right)-u_{x}\left(z, a, b_{1}, b_{2}\right)$ with $x_{2}=z_{2}$ using $[W]_{C_{x}^{1 / 2}}$. The estimate consists of following steps. Firstly, we identify the sign of the kernel similar to those between (3.8) and (3.10). Secondly, we construct the transportation map along the $x$ direction and derive the transportation cost. Thirdly, we compare the transportation cost in the case with kernel localization and the case without kernel localization using the properties of the transportation maps, and show that the cost with kernel localization is smaller.

Without loss of generality, we assume $x_{1}=\frac{1}{2}, z_{1}=-\frac{1}{2}, x_{2}=0$. Denote

$$
I_{a} \triangleq[-a, a], \quad I_{b}=\left[-b_{1}, b_{2}\right], \quad Q \triangleq I_{a} \times I_{b}, \quad b=\max \left(b_{1}, b_{2}\right)
$$

Since we assume $a \geq \frac{1}{2}\left|x_{1}-z_{1}\right|$ in Lemma 3.1 we have

$$
\begin{equation*}
a \geq 1 / 2 \tag{3.15}
\end{equation*}
$$

The kernel associated with $u_{x}\left(\frac{1}{2}, x_{2}\right)-u_{x}\left(-\frac{1}{2}, x_{2}\right)$ (3.4) becomes

$$
\begin{align*}
K_{a, b}\left(s_{1}, s_{2}\right)= & \mathbf{1}_{s_{2} \in I_{b}}\left(\frac{\left(s_{1}+\frac{1}{2}\right) s_{2}}{\left(\left(s_{1}+\frac{1}{2}\right)^{2}+s_{2}^{2}\right)^{2}} \mathbf{1}_{s_{1}+\frac{1}{2} \in I_{a}}-\frac{\left(s_{1}-\frac{1}{2}\right) s_{2}}{\left(\left(s_{1}-\frac{1}{2}\right)^{2}+s_{2}^{2}\right)^{2}} \mathbf{1}_{s_{1}-\frac{1}{2} \in I_{a}}\right) \\
= & \left(\left(K_{1}\left(s_{1}+1 / 2, s_{2}\right)-K_{1}\left(s_{1}-1 / 2, s_{2}\right)\right) \mathbf{1}_{s+1 / 2 \in Q}\right.  \tag{3.16}\\
& \left.-K_{1}\left(s_{1}-1 / 2, s_{2}\right)\left(\mathbf{1}_{s-1 / 2 \in Q}-\mathbf{1}_{s+1 / 2 \in Q}\right)\right) .
\end{align*}
$$

Since $a \geq \frac{1}{2}$ and $K_{1}(s)$ is odd in $s_{1}$, for fixed $s_{2}$, we have
$\int_{0}^{\infty} K_{a, b}\left(s_{1}, s_{2}\right) d s_{1}=\left(\int_{1 / 2}^{a}-\int_{-1 / 2}^{a}\right) K_{1}(s) d s_{1}=-\int_{-1 / 2}^{1 / 2} K_{1}(s) d s_{1}=0, \quad \int_{-\infty}^{0} K_{a, b}\left(s_{1}, s_{2}\right) d s_{1}=0$.
Similar to the case without localization, for each $s_{2}$, we consider the transportation from the positive part of $K_{a, b}$ to its negative part. Firstly, we identify the sign of $K_{a, b}$. We restrict to $s_{2} \in\left[-b_{1}, b_{2}\right]$ and $s_{2} \neq 0$ since otherwise $K_{a, b}=0$. We focus on $s_{1}, s_{2} \geq 0$ and the estimate for $s_{1}<0$ or $s_{2}<0$ is the same. Since $a>\frac{1}{2}$, we always have

$$
\begin{equation*}
s_{1} \pm 1 / 2>-a, \quad \text { for } \quad s_{1} \geq 0 \tag{3.17}
\end{equation*}
$$

Thus, for $s_{1} \geq 0$, we can neglect the constraint $s_{1} \pm \frac{1}{2} \geq-a$ in the localization in (3.16).
Case 1: $a \in(1 / 2,1]$. Clearly, $K_{a, b}\left(s_{1}, s_{2}\right)>0$ for $s_{1}<\frac{1}{2}$ since both kernels in (3.16) are non-negative. For $s_{1} \geq \frac{1}{2}$, since $a \leq 1$, we get

$$
K_{a, b}\left(s_{1}, s_{2}\right)=-K_{1}\left(s_{1}-\frac{1}{2}, s_{2}\right) \mathbf{1}_{s-1 / 2 \in Q} \leq 0
$$

In this case, we denote $s_{c}\left(s_{2}\right)=\frac{1}{2}$.

Case 2: $a \in\left(1, f\left(s_{2}\right)+\frac{1}{2}\right)$. Recall $f\left(s_{2}\right)$ from (3.10). For $s_{1}>a-\frac{1}{2}>\frac{1}{2}$, we get

$$
K_{a, b}=-K_{1}\left(s_{1}-\frac{1}{2}, s_{2}\right) \mathbf{1}_{s-1 / 2 \in Q} \leq 0
$$

For $s_{1} \leq a-\frac{1}{2}<f\left(s_{2}\right)$, using (3.11) and $s_{1}+\frac{1}{2} \leq a$, we obtain

$$
K_{a, b}=K_{1}\left(s_{1}+\frac{1}{2}, s_{2}\right)-K_{1}\left(s_{1}-\frac{1}{2}, s_{2}\right) \geq 0
$$

We denote $s_{c}\left(s_{2}\right)=a-\frac{1}{2}$.
Case 3: $a \geq f\left(s_{2}\right)+\frac{1}{2}$. For $s_{1}<f\left(s_{2}\right)$, using (3.11) and $s_{1} \pm \frac{1}{2}<a$, we get

$$
K_{a, b}=K_{1}\left(s_{1}+\frac{1}{2}, s_{2}\right)-K_{1}\left(s_{1}-\frac{1}{2}, s_{2}\right) \geq 0
$$

For $s_{1} \geq f\left(s_{2}\right)>\frac{1}{2}$, since $K_{1}\left(s_{1}+\frac{1}{2}, s_{2}\right)-K_{1}\left(s_{1}-\frac{1}{2}, s_{2}\right) \leq 0$ (3.11) and

$$
\mathbf{1}_{s-1 / 2 \in Q}-\mathbf{1}_{s+1 / 2 \in Q}=\mathbf{1}_{s_{1}-1 / 2 \leq a}-\mathbf{1}_{s+1 / 2 \leq a} \geq 0
$$

we get

$$
\left.K_{a, b} \leq-K_{1}\left(s_{1}-1 / 2, s_{2}\right)\left(\mathbf{1}_{s-1 / 2 \in Q}-\mathbf{1}_{s+1 / 2 \in Q}\right)\right) \leq 0
$$

We denote $s_{c}\left(s_{2}\right)=f\left(s_{2}\right)$. In summary, for fixed $s_{2}$, we define

$$
\begin{align*}
& s_{c}\left(s_{2}\right)=\frac{1}{2}, \text { if } a \in\left(\frac{1}{2}, 1\right], \quad s_{c}\left(s_{2}\right)=a-\frac{1}{2}, \quad \text { if } a \in\left(1, f\left(s_{2}\right)+\frac{1}{2}\right),  \tag{3.18}\\
& s_{c}\left(s_{2}\right)=f\left(s_{2}\right), \text { if } a \geq f\left(s_{2}\right)+1 / 2
\end{align*}
$$

which satisfies

$$
\begin{equation*}
K_{a, b}\left(s_{1}, s_{2}\right) \geq 0, s_{1} \in\left[0, s_{c}\right], \quad K_{a, b}\left(s_{1}, s_{2}\right) \leq 0, s_{1} \in\left[s_{c}, \infty\right], \quad s_{c}\left(s_{2}\right) \leq f\left(s_{2}\right) \tag{3.19}
\end{equation*}
$$

where the last inequality follows from the definition of $s_{c}$ and $f\left(s_{2}\right) \geq \frac{1}{2}$ (3.10).
In each case $i=1,2,3$, we construct the transport map by solving

$$
\begin{equation*}
\int_{T_{i}\left(s_{1}, s_{2}\right)}^{s_{1}} K_{a, b}\left(x, s_{2}\right) d x=0, \quad T_{i} \leq a+\frac{1}{2} \tag{3.20}
\end{equation*}
$$

We add the restriction $T_{i} \leq a+\frac{1}{2}$ since $K_{a, b}(s)=0$ for $s_{1}>a+\frac{1}{2}$ by definition (3.16). Applying Lemma 3.6 in the $s_{1}$ direction and using $K_{a, b}(s)=0$ for $\left|s_{2}\right| \geq b$, we yield

$$
I_{i} \triangleq\left|\int_{s_{1} \geq 0, s_{2} \geq 0} K_{a, b}\left(s_{1}, s_{2}\right) \omega\left(s_{1},-s_{2}\right) d s\right| \leq \int_{0}^{b} \int_{0}^{s_{c}\left(s_{2}\right)}\left|K_{a, b}(s)\right|\left|T_{i}(s)-s_{1}\right|^{1 / 2} d s \cdot[\omega]_{C_{x}^{1 / 2}}
$$

3.3.3. Comparison of the cost. Next, we show that the cost can be bounded uniformly by the cost of the case without localization

$$
\begin{equation*}
I_{i} \leq \int_{0}^{b} \int_{f\left(s_{2}\right)}^{\infty}\left|K(s) \| T(s)-s_{1}\right|^{1 / 2} d s \cdot[\omega]_{C_{x}^{1 / 2}} \tag{3.21}
\end{equation*}
$$

where $T$ is defined in (3.13). It suffices to prove

$$
\begin{equation*}
J_{i} \triangleq \int_{0}^{s_{c}\left(s_{2}\right)}\left|K_{a, b}(s)\right|\left|T_{i}(s)-s_{1}\right|^{1 / 2} d s \leq \int_{0}^{f\left(s_{2}\right)}\left|K(s) \| T(s)-s_{1}\right|^{1 / 2} d s \tag{3.22}
\end{equation*}
$$

for any $s_{2}$. We focus on $\left|s_{2}\right| \leq a$ and $s_{2} \neq 0$. The intuition behind the above inequality is that if the mass is localized, we should get "cheaper" transportation cost than the case without localization since the transportation distance is shorter. To justify these heuristics, we compare the kernels and will prove

$$
\begin{equation*}
\left|K_{a, b}(s)\right| \leq|K(s)|, \quad s_{1} \in\left[0, s_{c}\left(s_{2}\right)\right] \tag{3.23}
\end{equation*}
$$

and use (3.13) and (3.20) to compare $T_{i}$ and $T$

$$
\begin{equation*}
s_{1} \leq s_{c}\left(s_{2}\right) \leq T_{i}(s) \leq T(s), \quad s_{1} \in\left[0, s_{c}\left(s_{2}\right)\right] \tag{3.24}
\end{equation*}
$$

and thus $T_{i}(s)-s_{1} \leq T(s)-s_{1}$. Clearly, inequality (3.22) follows from (3.23) and (3.24).

Compare the kernels. From (3.19) and (3.11), since $s_{c}\left(s_{2}\right) \leq f\left(s_{2}\right)$, we get $K_{a, b}(s), K(s) \geq 0$ for $s_{1} \in\left[0, s_{c}\left(s_{2}\right)\right]$. Hence, for fixed $s_{2} \in\left[-b_{1}, b_{2}\right]$, (3.23) is equivalent to

$$
0 \leq K(s)-K_{a, b}(s)=K_{1}\left(s_{1}+\frac{1}{2}, s_{2}\right)\left(1-\mathbf{1}_{s_{1}+1 / 2 \in I_{a}}\right)-K_{1}\left(s_{1}-\frac{1}{2}, s_{2}\right)\left(1-\mathbf{1}_{s_{1}-1 / 2 \in I_{a}}\right) \triangleq I
$$

From the definition of (3.18) and (3.17), for $s_{1} \in\left[0, s_{c}\left(s_{2}\right)\right]$, we have

$$
s_{1} \pm 1 / 2 \geq-a, \quad s_{1}-1 / 2 \leq a, \quad 1-\mathbf{1}_{s_{1}-1 / 2 \in I_{a}}=0
$$

which along with $K_{1}\left(s_{1}+\frac{1}{2}, s_{2}\right) \geq 0$ (3.16) implies (3.23)

$$
I=K_{1}\left(s_{1}+\frac{1}{2}, s_{2}\right)\left(1-\mathbf{1}_{s_{1}+1 / 2 \in I_{a}}\right) \geq 0
$$

Remark 3.9. In the above derivations, we consider $s_{2} \geq 0$. If $s_{2} \leq 0$, one needs to track the sign to prove inequality (3.23).

Compare the maps. To prove (3.24), our idea is to use the equations (3.13), (3.20) and the sign of the kernels $K_{a, b}, K$ to compare $T_{i}$ and $T$.

We fix $s_{2}>0$ in the following derivations. To simplify the notation, we simplify $T\left(s_{1}, s_{2}\right)$ as $T\left(s_{1}\right)$ in some places. Since $T_{i}, T$ (3.13), (3.20) are decreasing and $s_{c}\left(s_{2}\right)$ is a fixed point for $T_{i}\left(\cdot, s_{2}\right)$, for $s_{1} \leq s_{c}\left(s_{2}\right)$, we get

$$
\begin{equation*}
T_{i}\left(s_{1}, s_{2}\right) \geq T_{i}\left(s_{c}\left(s_{2}\right), s_{2}\right)=s_{c}\left(s_{2}\right), \quad T\left(s_{1}, s_{2}\right) \geq T\left(f\left(s_{2}\right), s_{2}\right)=f\left(s_{2}\right) \geq s_{c}\left(s_{2}\right) \tag{3.25}
\end{equation*}
$$

Moreover, from (3.13), (3.20), we have

$$
\begin{equation*}
T_{i}\left(T_{i}\left(s_{1}\right)\right)=s_{1}, \quad T\left(T\left(s_{1}\right)\right)=s_{1} \tag{3.26}
\end{equation*}
$$

Denote

$$
\begin{equation*}
K^{ \pm}=K\left(s_{1} \pm \frac{1}{2}, s_{2}\right), \quad K_{a, b}^{+}=K_{1}\left(s_{1}+\frac{1}{2}, s_{2}\right) \mathbf{1}_{s_{1}+\frac{1}{2} \leq a}, \quad K_{a, b}^{-}=K_{1}\left(s_{1}-\frac{1}{2}, s_{2}\right) \mathbf{1}_{s_{1}-\frac{1}{2} \leq a} \tag{3.27}
\end{equation*}
$$

We remark that $K^{-}$is not non-negative but $K^{+}$is positive. By definition, we have

$$
\begin{align*}
& K_{a, b}=K_{a, b}^{+}-K_{a, b}^{-}, \quad K=K^{+}-K^{-} \\
& K^{+}(s) \geq 0, s_{1}, s_{2} \geq 0, \quad K^{-}(s) \geq 0, s_{1} \geq 1 / 2, s_{2} \geq 0 \tag{3.28}
\end{align*}
$$

Next, we study each case in the order of $3,2,1$ to prove (3.24).
Case 3: $a \geq f\left(s_{2}\right)+\frac{1}{2}$. In this case, recall $s_{c}\left(s_{2}\right)=f\left(s_{2}\right)$ from (3.18).
For $s_{1} \leq a-\frac{1}{2}$, we get $K_{a, b}=K$ (3.16). Hence, equations (3.13) and (3.20) are the same for $s_{1} \leq a-1 / 2$, and we get

$$
\begin{equation*}
T_{3}\left(s_{1}, s_{2}\right)=T\left(s_{1}, s_{2}\right), \quad s_{1} \in\left[T\left(a-\frac{1}{2}\right), a-\frac{1}{2}\right] \tag{3.29}
\end{equation*}
$$

It follows (3.24) for $s_{1} \in\left[T(a-1 / 2), f\left(s_{2}\right)\right]$. We recall that from (3.25), $a-1 / 2 \geq f\left(s_{2}\right)$ and $T(a-1 / 2)=T_{3}(a-1 / 2)$, we have

$$
\begin{equation*}
T(a-1 / 2) \leq T\left(f\left(s_{2}\right)\right)=f\left(s_{2}\right) \leq a-1 / 2, \quad T\left(s_{1}\right), T_{3}\left(s_{1}\right) \geq a-1 / 2, \quad s_{1} \leq T(a-1 / 2) \tag{3.30}
\end{equation*}
$$

Next, we compare $T\left(s_{1}\right), T_{3}\left(s_{1}\right)$ for $s_{1}<T(a-1 / 2) \leq f\left(s_{2}\right)$. From (3.13), (3.20), and $T(a-$ $1 / 2)=T_{3}(a-1 / 2) \leq a-1 / 2$, we have

$$
\int_{T(a-1 / 2)}^{a-1 / 2} K(s) d s_{1}=\int_{T_{3}(a-1 / 2)}^{a-1 / 2} K_{a, b}(s) d s_{1}=\int_{T(a-1 / 2)}^{a-1 / 2} K_{a, b}(s) d s_{1}=0
$$

Moreover, from (3.16) and (3.20), we have

$$
\begin{aligned}
& K_{a, b}\left(t, s_{2}\right)=-K_{a, b}^{-}\left(t, s_{2}\right)=-K^{-}\left(t, s_{2}\right), \quad t \in\left[a-1 / 2, T_{3}\left(s_{1}\right)\right] \subset[1-1 / 2, a+1 / 2] \\
& K_{a, b}\left(t, s_{2}\right)=K\left(t, s_{2}\right), \quad t \leq T(a-1 / 2) \leq a-1 / 2
\end{aligned}
$$

Plugging the above identities in (3.13), (3.20) for $s_{1} \leq T(a-1 / 2)$, we yield

$$
\begin{aligned}
& 0=\int_{s_{1}}^{T\left(s_{1}\right)} K\left(t, s_{2}\right) d t=\int_{s_{1}}^{T(a-1 / 2)} K\left(t, s_{2}\right)+\int_{a-1 / 2}^{T} K\left(t, s_{2}\right) d t \\
& 0=\int_{s_{1}}^{T_{3}\left(s_{1}\right)} K_{a, b}\left(t, s_{2}\right) d t=\int_{s_{1}}^{T(a-1 / 2)} K\left(t, s_{2}\right)-\int_{a-1 / 2}^{T_{3}\left(s_{1}\right)} K^{-}\left(t, s_{2}\right) d t
\end{aligned}
$$

Note that $K=K^{+}-K^{-}$. Calculating the difference between the two identities yields

$$
0=\int_{a-1 / 2}^{T\left(s_{1}\right)}\left(K^{+}-K^{-}\right) d s+\int_{a-1 / 2}^{T_{3}\left(s_{1}\right)} K^{-}=\int_{a-1 / 2}^{T\left(s_{1}\right)} K^{+} d s_{1}+\int_{T\left(s_{1}\right)}^{T_{3}\left(s_{1}\right)} K^{-} d s_{1}
$$

Recall $s_{2} \neq 0$ and from (3.30), we obtain $T_{3}\left(s_{1}\right), T\left(s_{1}\right) \geq a-1 / 2 \geq 1 / 2$. From (3.28), we yield $K^{+}>0$ and $K_{-}>0$ for $s_{1} \geq \min \left(a-1 / 2, T_{3}, T\right)$. Since $T$ is decreasing and

$$
T\left(s_{1}\right) \geq T(T(a-1 / 2))=a-1 / 2, \quad s_{1} \leq T(a-1 / 2)
$$

the first integral is non-negative. We prove $T\left(s_{1}\right) \geq T_{3}\left(s_{1}\right)$ for $s_{1} \leq T(a-1 / 2)$, which along with (3.29) implies (3.24).

The proof in the case 1,2 is completely similar.
Case 2: $a \in\left(1, f\left(s_{2}\right)+\frac{1}{2}\right)$. Recall $s_{c}\left(s_{2}\right)=a-\frac{1}{2} \leq f\left(s_{2}\right)$ from (3.18) and (3.25). For any $s_{1} \leq a-\frac{1}{2} \leq f\left(s_{2}\right)$, using (3.13), (3.20), and an argument similar to that in case 3, we yield

$$
\begin{aligned}
& 0=\int_{s_{1}}^{T_{2}\left(s_{1}\right)} K_{a, b}\left(t, s_{2}\right) d t=\int_{s_{1}}^{a-1 / 2} K\left(t, s_{2}\right) d t-\int_{a-1 / 2}^{T_{2}\left(s_{1}\right)} K^{-}\left(t, s_{2}\right) d t \\
& 0=\int_{s_{1}}^{T\left(s_{1}\right)} K\left(t, s_{2}\right) d t=\int_{s_{1}}^{a-1 / 2} K\left(t, s_{2}\right) d t+\int_{a-1 / 2}^{T\left(s_{1}\right)}\left(K^{+}-K^{-}\right)\left(t, s_{2}\right) d t
\end{aligned}
$$

where we have used $K_{a, b}\left(t, s_{2}\right)=-K^{-}\left(t, s_{2}\right)$ for $t \geq a-1 / 2$ (3.16), (3.27) in the second equality.
Comparing the difference between two identities yields
$0=\int_{a-1 / 2}^{T\left(s_{1}\right)} K^{+}-K^{-}\left(t, s_{2}\right) d t+\int_{a-1 / 2}^{T_{2}\left(s_{1}\right)} K^{-}\left(t, s_{2}\right) d t=\int_{a-1 / 2}^{T\left(s_{1}\right)} K^{+}\left(t, s_{2}\right) d t+\int_{T\left(s_{1}\right)}^{T_{2}\left(s_{1}\right)} K^{-}\left(t, s_{2}\right) d t$.
Recall from (3.25) that $T\left(s_{1}\right), T_{2}\left(s_{1}\right) \geq s_{c}\left(s_{2}\right)=a-1 / 2$ for $s_{1} \leq a-1 / 2$. For $s_{2} \neq 0$ and $s_{1} \geq \min \left(T, T_{2}, a-1 / 2\right)=a-1 / 2>1 / 2$, we have $K^{-}>0, K^{+}>0$ (3.28). We obtain $T\left(s_{1}\right) \geq T_{2}\left(s_{1}\right)$, which implies (3.24).

Case 1: $a \in(1 / 2,1]$. In this case, $s_{c}\left(s_{2}\right)=\frac{1}{2}<f\left(s_{2}\right)$. From (3.16), we yield

$$
\begin{aligned}
0 & \leq K_{a, b}=\mathbf{1}_{s_{1}+1 / 2 \leq a} K_{1}\left(s_{1}+1 / 2, s_{2}\right)-K_{1}\left(s_{1}-1 / 2, s_{2}\right) \\
& \leq K_{1}\left(s_{1}+1 / 2, s_{2}\right)-K_{1}\left(s_{1}-1 / 2, s_{2}\right)=K, \quad s_{1} \in[0,1 / 2] \\
K_{a, b} & =-K_{1}\left(s_{1}-1 / 2, s_{2}\right)=-K^{-}\left(s_{1}, s_{2}\right), \quad K^{-}\left(s_{1}, s_{2}\right) \geq 0, \quad s_{1} \in[1 / 2, a+1 / 2]
\end{aligned}
$$

For any $s_{1}<\frac{1}{2}$ and $s_{2} \neq 0$, from (3.25), we get $T_{1}\left(s_{1}\right) \geq 1 / 2, T\left(s_{1}\right) \geq f\left(s_{2}\right)>1 / 2$. Using (3.13), (3.20) and the above estimates for $K_{a, b}$, we yield

$$
0=\int_{s_{1}}^{1 / 2} K_{a, b}\left(t, s_{2}\right) d t-\int_{1 / 2}^{T_{1}\left(s_{1}\right)} K^{-}\left(t, s_{2}\right) d t=\int_{s_{1}}^{1 / 2} K\left(t, s_{2}\right) d t+\int_{1 / 2}^{T}\left(K^{+}-K^{-}\right)\left(t, s_{2}\right) d t
$$

It follows

$$
0=\int_{s_{1}}^{1 / 2}\left(K-K_{a, b}\right)\left(t, s_{2}\right) d t+\int_{1 / 2}^{T} K^{+}+\int_{T}^{T_{1}} K^{-}\left(t, s_{2}\right) d t \triangleq I I_{1}+I I_{2}+I I_{3}
$$

From (3.23) and $K_{a, b}, K>0$ on $t \in\left[0, s_{c}\left(s_{2}\right)\right]=[0,1 / 2]$, we get $I I_{1} \geq 0$. Note that $K^{-}, K^{+}>0$ for $s_{1}>1 / 2$ (3.11), (3.19). Since $T_{1}, T>1 / 2$, we must obtain $T\left(s_{1}\right) \geq T_{1}\left(s_{1}\right)$, which implies (3.24).

We have proved (3.24) in all three cases, which implies $\left|T(s)-s_{1}\right| \geq\left|T_{i}(s)-s_{1}\right|$. Combining this estimate and (3.23), we prove (3.21) and conclude the proof of Lemma 3.1.
3.4. Estimate of $\left[u_{x}\right]_{C_{y}^{1 / 2}}$. Recall from Lemma 3.3 that $b_{1}=b_{2}=b$ and $x_{1}=z_{1}$ in this case. Without loss of generality, we assume $z_{2}=m+1 / 2, x_{2}=m-1 / 2$ and $x_{1}=y_{1}=0$ for some $m \geq 1 / 2$. We have

$$
u_{x}(z)-u_{x}(x)=\frac{1}{\pi} \int_{\mathbb{R}^{2}} W(y)\left(K_{a, b}\left(y_{1}, y_{2}-(m-1 / 2)\right)-K_{a, b}\left(y_{1}, y_{2}-(m+1 / 2)\right)\right) d y
$$

where $W$ is the odd extension of $\omega$ in $\mathbb{R}^{2}$ (3.3). Note that $W$ is not Hölder in the $y$-direction near $y_{2}=0$, we cannot use the same method as that in the estimate of $\left[u_{x}\right]_{C_{x}}^{1 / 2}$. On the other hand, since $W \in C_{y}^{1 / 2}(\mathbb{R} \times[m, \infty))$, we can apply the previous method to obtain

$$
\left|\frac{1}{\pi} \int_{y_{2} \geq m} W(y)\left(K_{a, b}\left(y_{1}, y_{2}-(m-1 / 2)\right)-K_{a, b}\left(y_{1}, y_{2}-(m+1 / 2)\right) d y\right)\right| \leq \frac{1}{2} C_{1}(a)\|\omega\|_{C_{y}^{1 / 2}}
$$

Rotating the coordinate by 90 degree, we obtain the case studied in Section 3.3 ,
It remains to estimate

$$
\begin{aligned}
I(b) & =\frac{1}{\pi} \int_{y_{2} \leq m} W(y)\left(K_{a, b}\left(y_{1}, y_{2}-(m-1 / 2)\right)-K_{a, b}\left(y_{1}, y_{2}-(m+1 / 2)\right) d y\right) \\
& =\frac{1}{\pi} \int_{y_{2} \leq 0} W\left(y_{1}, y_{2}+m\right)\left(K_{a, b}\left(y_{1}, y_{2}+1 / 2\right)-K_{a, b}\left(y_{1}, y_{2}-1 / 2\right) d y\right) .
\end{aligned}
$$

Since $W$ is not Hölder continuous across $y=0$, we use $[W]_{C_{x}^{1 / 2}}$ to control $I$. Our idea is to compare the integral $I(b)$ with the case $b=\infty, I(\infty)$. To do so, we need a monotonicity Lemma.
Lemma 3.10. Suppose $f, f g \in L^{1}$ and $g \geq 0$ is monotone increasing on $[0, \infty]$. For any $0 \leq k \leq b \leq c$, we have

$$
\int_{b-k}^{b+k}|f(x-k)| g(x) d x \leq \int_{b-k}^{c-k}|f(x-k)-f(x+k)| g(x) d x+\int_{c-k}^{c+k}|f(x-k)| g(x) d x .
$$

Proof. Denote by $R, L$ the right and the left hand side of the above inequality, respectively. We have

$$
\begin{aligned}
R-L & \geq \int_{b-k}^{c-k}(|f(x-k)|-|f(x+k)|) g(x) d x+\int_{c-k}^{c+k}|f(x-k)| g(x) d x-\int_{b-k}^{b+k}|f(x-k)| g(x) d x \\
& =\int_{b+k}^{c+k}|f(x-k)| g(x) d x-\int_{b-k}^{c-k}|f(x+k)| g(x) d x=\int_{b}^{c}|f(x)|(g(x+k)-g(x-k)) d x .
\end{aligned}
$$

Since $g$ is increasing on $[0, \infty)$, we prove $R \geq L$.
Now, we are in a position to estimate $I$. Since $K_{a, b}\left(y_{1}, y_{2}\right)$ is odd in $y_{1}$, we yield

$$
|I| \leq \frac{1}{\pi} \int_{y_{2} \leq 0, y_{1} \geq 0} \sqrt{2 y_{1}}\left|K_{a, b}\left(y_{1}, y_{2}+1 / 2\right)-K_{a, b}\left(y_{1}, y_{2}-1 / 2\right)\right| d y \cdot[\omega]_{C_{x}^{1 / 2}}
$$

For a fixed $y_{1}$ with $\left|y_{1}\right| \leq a$ and $b \geq 1 / 2$, using the definition of $K_{a, b}$ (3.4), the odd symmetry $K_{a, b}\left(y_{1}, y_{2}+1 / 2\right)-K_{a, b}\left(y_{1}, y_{2}-1 / 2\right.$ in $y_{2}$, and Lemma 3.10 with $k=1 / 2$ and $c=\infty$, we get

$$
\begin{aligned}
& \int_{y_{2} \leq 0}\left|K_{a, b}\left(y_{1}, y_{2}+1 / 2\right)-K_{a, b}\left(y_{1}, y_{2}-1 / 2\right)\right| d y_{2} \\
= & \int_{y_{2} \geq 0}\left|K_{a, b}\left(y_{1}, y_{2}+1 / 2\right)-K_{a, b}\left(y_{1}, y_{2}-1 / 2\right)\right| d y_{2} \\
= & \int_{0}^{b-1 / 2}\left|K_{1}\left(y_{1}, y_{2}+1 / 2\right)-K_{1}\left(y_{1}, y_{2}-1 / 2\right)\right| d y_{2}+\int_{b-1 / 2}^{b+1 / 2}\left|K_{1}\left(y_{1}, y_{2}-1 / 2\right)\right| d y_{2} \\
\leq & \int_{0}^{\infty}\left|K_{1}\left(y_{1}, y_{2}+1 / 2\right)-K_{1}\left(y_{1}, y_{2}-1 / 2\right)\right| d y_{2} .
\end{aligned}
$$

Since $K_{a, b}(y)=0$ for $\left|y_{1}\right| \geq a$, integrating the above inequality in $y_{1}$ from 0 to $a$, we prove

$$
|I| \leq \frac{1}{\pi} \int_{0}^{a} \int_{0}^{\infty} \sqrt{2 y_{1}}\left|K_{1}\left(y_{1}, y_{2}+1 / 2\right)-K_{1}\left(y_{1}, y_{2}-1 / 2\right)\right| d y \cdot[\omega]_{C_{x}^{1 / 2}}
$$

## 4. $L^{\infty}$-BASED FINITE RANK PERTURBATION

In this Section, we provide further details how we decompose the linearized operator into a leading order operator $\mathcal{L}_{0}$ plus the finite rank perturbation operator $\mathcal{K}$. We then discuss how to perform the $L^{\infty}$-based finite rank perturbation to the linearized equations introduced in (2.25), and then apply it to estimate the more regular nonlocal terms in (2.25), (2.28).
4.1. A toy model with a nonlocal term. We use a model problem to illustrate the ideas of stability analysis of a linearized equation perturbed from a simpler linearized equation. Consider

$$
\begin{equation*}
f_{t}=\mathcal{L}_{0} f+a(x) P(f) \triangleq \mathcal{L} f, \quad P(f)=\int_{\mathbb{R}_{2}^{++}} f g d x \tag{4.1}
\end{equation*}
$$

in $\mathbb{R}_{2}^{++}$, where $a, g$ are some given time-independent functions. Operator $\mathcal{L}_{0}$ models the local terms in (2.25), (2.28), and the rank-one operator $a(x) P(f)$ models the nonlocal terms. We assume that $\mathcal{L}_{0}$ is linearly stable in $L^{\infty}(\varphi)$ with some singular weight $\varphi$, which can be studied following Section 2.7, and $a(x) \in L^{\infty}(\varphi)$. We want to understand the long time behavior and the stability of the above model using the information of $\mathcal{L}_{0}$.

A natural attempt is to use Duhamel's principle and the semi-group $e^{\mathcal{L}_{0} t}$ to represent the solution to 4.1). However, $a(x)$ is not small and $a(x) P(f)$ cannot be treated as a small perturbation. Another attempt is to project $f$ onto some space $Y$ orthogonal to $g(x)$ or $a(x)$ so that the nonlocal term is 0 in $Y$. However, the projection is not compatible with our $L^{\infty}$-based estimates.
4.1.1. Rank-one perturbation. Following the ideas in Section 1.1 we decompose (4.1) as follows

$$
\begin{align*}
& \partial_{t} f_{1}(t)=\mathcal{L}_{0} f_{1}, \quad f_{1}(0)=f_{0} \\
& \partial_{t} f_{2}(t)=\mathcal{L} f_{2}+a(x) P\left(f_{1}(t)\right), \quad f_{2}(0)=0 \tag{4.2}
\end{align*}
$$

for initial data $f_{0}$, and then represent $f_{2}$ using Duhamel's principle

$$
\begin{equation*}
f_{2}(t)=\int_{0}^{t} P\left(f_{1}(s)\right) e^{\mathcal{L}(t-s)} a(x) d s \tag{4.3}
\end{equation*}
$$

If $e^{\mathcal{L} t} a(x)$ decays in $L^{\infty}(\varphi)$ for large $t$, we can establish $L^{\infty}(\varphi)$ stability estimate of $\mathcal{L}$.
Note that by choosing zero initial data for $f_{2}$ and using the fact that $P\left(f_{1}(s)\right)$ is independent of space, we can solve $f_{2}$ for an arbitrary forcing coefficient $P\left(f_{1}(s)\right)$.

Similar idea appears in the $T(1)$ [30, $T(b)$ 31, 70 theorems in harmonic analysis. Roughly, it states that for a linear operator $T$ associated with a standard kernel $K$, proving the $L^{2}$ boundedness of $T$ reduces to proving $T(1)$ or $T(b) \in B M O$. Here, using energy estimate to establish the stability of $\mathcal{L}_{0}$ (4.1) is similar to extracting certain properties of $T$ from a standard kernel. Testing the decay of $e^{\mathcal{L} t} a$ from some initial data $a$ to obtain its stability is similar to testing $T$ on 1 or $b$ to obtain the $L^{2}$ boundedness of $T$. Our idea also relates to the Sherman-Morrison formula [84] which connects the invertibility of $A \in R^{n \times n}$ and its rank-one perturbation.
4.1.2. Decay of $e^{\mathcal{L} t}$ a and constructing approximate solution to $f_{2}$. Though the operator $\mathcal{L}$ and $a(x)$ (4.1) are given, it is difficult to prove decay of $e^{\mathcal{L} t} a$ in the weighted norm analytically since $\mathcal{L}$ is nonlocal. The operator $\mathcal{L}$ for the Boussinesq system (2.25) is even more complicated.

An alternative approach is to solve (4.1) numerically from initial data $a(x)$ to obtain an approximate solution $\hat{g}(t, x)$. Then by showing the error $e^{\mathcal{L} t} a-\hat{g}(t, x)$ is small and verifying the decay of $\hat{g}(t)$, we obtain the decay estimates of $e^{\mathcal{L} t} a$. The difficulty lies in estimating the error in the weighted norm rigorously. Standard a-priori error estimate provides a bound

$$
\left|e^{\mathcal{L} t} a(x)-\hat{g}(t, x)\right| \leq C_{1}\left(h^{m}+k^{n}\right) e^{C_{2} t}
$$

for some constants $C_{1}, C_{2}$ depending on $a(x)$ and $\mathcal{L}$, where $h$ is the mesh size, $k$ is the time step in the computation, and $m, n$ relate to the order of the numerical scheme. However, $C_{1}, C_{2}$ are not easy to estimate and can be quite large, and $t$ is not small since we want to obtain decay estimates of $e^{\mathcal{L} t} a$ for suitably large $t$, e.g., $t \geq 10$. Thus, the factor $e^{C_{2} t}$ can be very large, and the above estimate is not practical.

Instead, we seek a-posteriori error estimate. Firstly, we solve (4.1) numerically and obtain a numerical solution $\hat{g}\left(t_{k}, x\right)$ at time $t_{k}$, which is represented by piecewise polynomials and thus defined globally in $x$. Then we interpolate the solution $\hat{g}\left(t_{k}, x\right)$ in time $t$ using piecewise cubic polynomials to obtain solution $\hat{g}(t, x)$ defined on $[0, T] \times \mathbb{R}_{2}^{+}$. We introduce the residual error and a residual operator $\mathcal{R}$ related to the nonlocal term $P\left(f_{1}(t)\right)$ in (4.2)

$$
\begin{align*}
e(t, x) & \triangleq\left(\partial_{t}-\mathcal{L}\right) \hat{g}, \quad e_{0}(x)=\hat{g}(0)-g_{0}, \quad g_{0}=a(x) \\
\mathcal{R}\left(f_{1}, t\right) & =P\left(f_{1}(t)\right) e_{0}(x)+\int_{0}^{t} P\left(f_{1}(s)\right) e(t-s, x) d s \tag{4.4}
\end{align*}
$$

Since $\hat{g}$ is defined everywhere in space and time, we can estimate $e(t, x)$ and $e_{0}(x)$. Using the approximate solution $\hat{g}(t, x)$ for $e^{\mathcal{L} t} g_{0}$, we construct the approximate solution to $f_{2}$ (4.3)

$$
\begin{equation*}
\hat{f}_{2}(t)=\int_{0}^{t} P\left(f_{1}(s)\right) \hat{g}(t-s) d s \tag{4.5}
\end{equation*}
$$

By definition and (4.4), we have

$$
\begin{aligned}
\left(\partial_{t}-\mathcal{L}\right) \hat{f}_{2} & =P\left(f_{1}(t)\right) \hat{g}(0)+\int_{0}^{t} P\left(f_{1}(s)\right)\left(\partial_{t}-\mathcal{L}\right) \hat{g}(t-s) d s \\
& =P\left(f_{1}(t)\right)\left(a(x)+e_{0}\right)+\int_{0}^{t} P\left(f_{1}(s)\right) e(t-s, x) d s=P\left(f_{1}(t)\right) a(x)+\mathcal{R}\left(f_{1}, t\right)
\end{aligned}
$$

If the error $e(t, x)$ and $e_{0}$ are small, we can show that the norm of the residual operator is small in some suitable functional space

$$
\begin{equation*}
\left\|\mathcal{R}\left(f_{1}, t\right)\right\|_{X} \leq \varepsilon\left\|f_{1}\right\|_{X}, \quad \varepsilon \ll 1 \tag{4.6}
\end{equation*}
$$

Now, we modify the decomposition (4.2) as follows $\left(f=f_{1}+\hat{f}_{2}\right)$

$$
\begin{align*}
& \partial_{t} f_{1}=\mathcal{L}_{0} f_{1}-\mathcal{R}\left(f_{1}, t\right), \quad f_{1}(0)=f_{0} \\
& \partial_{t} \hat{f}_{2}=\mathcal{L} \hat{f}_{2}+a(x) P\left(f_{1}(t)\right)+\mathcal{R}\left(f_{1}, t\right) \tag{4.7}
\end{align*}
$$

We remark that the solution (4.5) constructed by the numerical solution $\hat{g}$ solves the second equation exactly. Now, due to the smallness (4.6), we can apply the stability estimate of $\mathcal{L}_{0}$ and treat $\mathcal{R}\left(f_{1}, t\right)$ as perturbation to obtain stability estimate of $f_{1}$.

Remark 4.1. Using the decomposition (4.2), (4.7), constructing an approximating solution $\hat{g}$ to $e^{\mathcal{L} t} a(x)$, and testing its decay, we replace a difficult nonlocal term in the original problem (4.1) by a small error term $\mathcal{R}\left(f_{1}, t\right)$ in (4.7) that can be treated as a small perturbation. Moreover, we do not need to assume any specific form about the rank-one operator $a(x) P\left(f_{1}(t)\right)$.
4.2. Finite rank perturbations to the linearized operators. We generalize the idea in the previous subsection to the Boussinesq equations. We modify the operator $\mathcal{L}$ in (2.25), (2.28) by a finite rank operator $\mathcal{K}$ with rank $N$ by testing $\mathcal{L}$ on $N$ suitable functions. Then we perform linear stability analysis on $\mathcal{L}-\mathcal{K}$, which serves as the role of stability estimate of $\mathcal{L}_{0}$ in the model problem in Section 4.1. These finite rank operators approximate the contributions from the more regular terms in (2.25), (2.28), e.g., u $\cdot \nabla \bar{\omega}, \mathbf{u} \cdot \nabla \theta_{x}$, which we neglect in Section 2.8 and can be seen as compact operators of $\omega$ in some suitable weighted spaces.

Since we will perform weighted estimates with singular weights near $x=0$, we rewrite (2.25), (2.28) such that each term has the right vanishing order. We introduce the following notations [17,20]

$$
\begin{equation*}
\tilde{u}=u-u_{x}(0) x, \quad \tilde{v} \triangleq v-v_{y}(0) y=v+u_{x}(0) y \tag{4.8}
\end{equation*}
$$

Since $\omega_{x}(0)=0$ (2.29), $\omega_{x}(0)=-\Delta \phi_{x}(0), \phi(x, 0)=0$, and $\phi$ is odd in $x$, we yield

$$
\phi=O\left(|x|^{4}\right), \quad \tilde{u}=O\left(|x|^{3}\right), \quad \tilde{v}=O\left(|x|^{3}\right), \quad \nabla \tilde{\mathbf{u}}=O\left(|x|^{2}\right)
$$

for perturbations regular enough. Recall $c_{\omega}=u_{x}(0)$ (2.26). Using $\tilde{u}_{x}=u_{x}-u_{x}(0), \tilde{u}_{y}=$ $u_{y}, \tilde{v}_{x}=v_{x}, \tilde{v}_{y}=v_{y}+u_{x}(0)$,

$$
\begin{aligned}
-\mathbf{u} \cdot \nabla \bar{\omega}+c_{\omega} \bar{\omega} & =-\tilde{\mathbf{u}} \cdot \nabla \bar{\omega}+c_{\omega}\left(\bar{\omega}-x \bar{\omega}_{x}+y \bar{\omega}_{y}\right) \\
-\mathbf{u} \cdot \nabla \bar{\theta}_{x}-\mathbf{u}_{x} \cdot \bar{\theta}+2 c_{\omega} \bar{\theta}_{x} & =-\tilde{\mathbf{u}} \cdot \nabla \bar{\theta}_{x}-\tilde{\mathbf{u}}_{x} \cdot \nabla \bar{\theta}+c_{\omega}\left(\bar{\theta}_{x}-x \bar{\theta}_{x x}+y \bar{\theta}_{x y}\right), \\
-\mathbf{u} \cdot \nabla \bar{\theta}_{y}-\mathbf{u}_{y} \cdot \nabla \bar{\theta}+2 c_{\omega} \bar{\theta}_{y} & =-\tilde{\mathbf{u}} \cdot \nabla \bar{\theta}_{y}-\tilde{\mathbf{u}}_{y} \cdot \nabla \bar{\theta}+c_{\omega}\left(\bar{\theta}_{y}-x \bar{\theta}_{x y}+y \bar{\theta}_{y y}\right)
\end{aligned}
$$

and denoting

$$
\begin{equation*}
\bar{f}_{c_{\omega}, 1}=\bar{\omega}-x \bar{\omega}_{x}+y \bar{\omega}_{y}, \quad \bar{f}_{c_{\omega}, 2}=\bar{\theta}_{x}-x \bar{\theta}_{x x}+y \bar{\theta}_{x y}, \quad \bar{f}_{c_{\omega}, 3}=3 \bar{\theta}_{y}-x \bar{\theta}_{x y}+y \bar{\theta}_{y y} \tag{4.9}
\end{equation*}
$$

we can rewrite (2.25), (2.28) as follows

$$
\begin{align*}
\partial_{t} \omega= & -\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \omega+\bar{c}_{\omega} \omega+\eta-\tilde{\mathbf{u}} \cdot \nabla \bar{\omega}+c_{\omega} \bar{f}_{c_{\omega}, 1}+\mathcal{N}_{1}+\overline{\mathcal{F}}_{1} \triangleq \mathcal{L}_{1}+\mathcal{N}_{1}+\overline{\mathcal{F}}_{1} \\
\partial_{t} \eta= & -\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \eta+\left(2 \bar{c}_{\omega}-\bar{u}_{x}\right) \eta-\bar{v}_{x} \xi-\tilde{\mathbf{u}}_{x} \cdot \nabla \bar{\theta}-\tilde{\mathbf{u}} \cdot \nabla \bar{\theta}_{x}+c_{\omega} \bar{f}_{c_{\omega}, 2} \\
& +\partial_{x}\left(N_{\theta}+\bar{F}_{\theta}\right) \triangleq \mathcal{L}_{2}+\mathcal{N}_{2}+\overline{\mathcal{F}}_{2}  \tag{4.10}\\
\partial_{t} \xi= & -\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \xi+\left(2 \bar{c}_{\omega}+\bar{u}_{x}\right) \xi-\bar{u}_{y} \eta-\tilde{\mathbf{u}}_{y} \cdot \nabla \bar{\theta}-\tilde{\mathbf{u}} \cdot \nabla \bar{\theta}_{y}+c_{\omega} \bar{f}_{c_{\omega}, 3} \\
& +\partial_{y}\left(N_{\theta}+\bar{F}_{\theta}\right) \triangleq \mathcal{L}_{3}+\mathcal{N}_{3}+\overline{\mathcal{F}}_{3}
\end{align*}
$$

The nonlocal terms $c_{\omega} \bar{f}_{c_{\omega}, i},-\tilde{\mathbf{u}} \cdot \nabla f$ for $f=\bar{\omega}, \bar{\theta}_{x}, \bar{\theta}_{y}$, and $\nabla \tilde{\mathbf{u}}_{R}$, the nonsingular part of the integral, are more regular than $\omega$. We will choose finite rank operators to approximate them.
4.2.1. Correction near the origin. We discuss in Section 2.7.3 that to obtain better stability factors, we choose more singular weights for the stability analysis. We consider the following corrections

$$
\begin{align*}
& \mathcal{K}_{1 i}(\omega) \triangleq c_{\omega}(\omega) \bar{f}_{c_{\omega}, i}, \\
& N F_{1}(\omega, \eta, \xi)=\left(c_{\omega} \omega_{x y}(0)+\partial_{x y} \overline{\mathcal{F}}_{1}(0)\right) f_{\chi, 1}, \quad f_{\chi, 1} \triangleq \Delta\left(x y^{3} \chi_{N F} / 6\right), \\
& N F_{2}(\omega, \eta, \xi)=\left(c_{\omega} \eta_{x y}(0)+\partial_{x y} \overline{\mathcal{F}}_{2}(0)\right) f_{\chi, 2}, \quad f_{\chi, 2} \triangleq x y \chi_{N F},  \tag{4.11}\\
& N F_{3}(\omega, \eta, \xi)=\left(c_{\omega} \xi_{x x}(0)+\partial_{x x} \overline{\mathcal{F}}_{3}(0)\right) f_{\chi, 3}, \quad f_{\chi, 3} \triangleq \frac{x^{2}}{2} \chi_{N F},
\end{align*}
$$

where $\chi_{N F}$ is some cutoff function with $\chi_{N F}=1+O\left(|x|^{4}\right)$ near $x=0$ constructed in (C.9). The form of $f_{\chi, 1}$ allows us to get $\mathbf{u}\left(f_{\chi, 1}\right)=\nabla^{\perp}(-\Delta)^{-1}\left(f_{\chi, 1}\right)$ analytically, and we have $f_{\chi, 1}=$ $x y+$ h.o.t. The operator $\mathcal{K}_{1 i}$ is a correction to the linear part, and $N F_{i}$ is a correction to the nonlinear term and the residual in (4.10), respectively. After subtracting $\mathcal{K}_{1 i}$ and $N F_{i}$ from (4.10), the resulting equations preserve the vanishing conditions $\omega, \eta, \xi=O\left(|x|^{3}\right)$.

We can derive the ODE for $\omega_{x y}(0), \theta_{x x y}(0)$ using (2.10)

$$
\begin{align*}
\frac{d}{d t} \omega_{x y}(0) & =\left(-2 c_{l}+c_{\omega}\right) \omega_{x y}(0)-\omega_{x}^{2}(0)+\theta_{x x y}(0) \\
\frac{d}{d t} \theta_{x x y}(0) & =\left(-2 c_{l}+2 c_{\omega}-u_{x}(0)\right) \theta_{x x y}(0)-2 \omega_{x}(0) \theta_{x x}(0) \tag{4.12}
\end{align*}
$$

Since $\omega_{x}(0), \theta_{x x}(0)$ are preserved (2.12), to estimate $\omega_{x y}(0), \theta_{x x y}(0)$, using (4.12), we only need to control $c_{l}, c_{\omega}, u_{x}(0)$ rather than some higher order norm of $\omega, \theta$, e.g. $\|\omega\|_{C^{2}},\|\theta\|_{C^{3}}$.
4.2.2. Approximation of the velocity. For $f=u, v, u_{x}, u_{y}, v_{x}, v_{y}$, we will construct in (4.38), (4.29), (4.37) in Section 4.3 the finite rank approximations $\hat{\hat{f}}$ for $\tilde{f}$ so that we get smaller constants $C$ in the weighted estimate of $\tilde{f}-\hat{\tilde{f}}$ using the energy $\|\omega \varphi\|_{L^{\infty},},\left\|\omega \psi_{1}\right\|_{C_{x}^{1 / 2}},\left\|\omega \psi_{1}\right\|_{C_{y}^{1 / 2}}$.

We remark that for these operators, we do not have

$$
\partial_{x}^{i} \partial_{y}^{j} \hat{u}=\widehat{\partial_{x}^{i} \partial_{y}^{j} u}, \quad \partial_{x}^{i} \partial_{y}^{j} \hat{v}=\widehat{\partial_{x}^{i} \partial_{y}^{j} v}
$$

for $i+j=1$. These approximations contribute to the following finite rank operators

$$
\begin{equation*}
\mathcal{K}_{21}=-\hat{\tilde{\mathbf{u}}} \cdot \nabla \bar{\omega}, \quad \mathcal{K}_{22}=-\hat{\tilde{\mathbf{u}}}_{x} \cdot \nabla \bar{\theta}-\hat{\tilde{\mathbf{u}}} \cdot \bar{\nabla} \theta_{x}, \quad \mathcal{K}_{23}=-\hat{\tilde{\mathbf{u}}}_{y} \cdot \nabla \bar{\theta}-\hat{\tilde{\mathbf{u}}} \cdot \nabla \bar{\theta}_{y}, \tag{4.13}
\end{equation*}
$$

which are designed to capture the contributions from the regular nonlocal terms.
4.2.3. Decomposition of the system. Denote $W_{1}=\left(\omega_{1}, \eta_{1}, \xi_{1}\right), W_{2}=\left(\omega_{2}, \eta_{2}, \xi_{2}\right)$. Recall the notations (2.18) and (2.19). Following Section 4.1 and (4.2), we decompose (4.10) as follows

$$
\begin{align*}
\partial_{t} W_{1, i} & =\left(\mathcal{L}_{i}-\mathcal{K}_{1 i}-\mathcal{K}_{2 i}\right) W_{1}+\mathcal{N}_{i}\left(W_{1}+W_{2}\right)+\overline{\mathcal{F}}_{i}-N F\left(W_{1}+W_{2}\right) \\
\partial_{t} W_{2, i} & =\mathcal{L}_{i} W_{2}+\mathcal{K}_{1 i}\left(W_{1}\right)+\mathcal{K}_{2 i}\left(W_{1}\right)+N F_{i}\left(W_{1}+W_{2}\right)  \tag{4.14}\\
\left.W_{1}\right|_{t=0} & =\left(\omega_{0}, \eta_{0}, \xi_{0}\right),\left.\quad W_{2}\right|_{t=0}=(0,0,0)
\end{align*}
$$

with $\omega_{0}, \eta_{0}, \xi_{0}$ being the initial perturbation with vanishing order $O\left(|x|^{3}\right)$. We have

$$
\partial_{t}\left(W_{1}+W_{2}\right)=\mathcal{L}_{i}\left(W_{1}+W_{2}\right)+\mathcal{N}_{i}\left(W_{1}+W_{2}\right)+\overline{\mathcal{F}}_{i}
$$

which are the same equations as (4.10). Since $W_{1}+W_{2}$ has initial data ( $\omega_{0}, \eta_{0}, \xi_{0}$ ), $W_{1}+W_{2}$ solves (4.10) with the given initial data. Using the definitions (4.11) and a Taylor expansion near $x=0$, we obtain that the vanishing conditions $\omega_{1}, \eta_{1}, \xi_{1}=O\left(|x|^{3}\right)$ are preserved.

Remark 4.2. Although $W_{1,2}+W_{2,2}=\theta_{x}, W_{1,3}+W_{2,3}=\theta_{y}$, since the finite rank operators $\mathcal{K}_{i j}$ we choose do not satisfy similar partial derivative relations, the solution to (4.14) does not satisfy $\partial_{y} W_{i, 2}=\partial_{x} W_{i, 3}$ for $i=1$ or $i=2$.

Let us motivate the decomposition (4.14). At the linear level, we choose finite rank operators $\mathcal{K}_{1 i}, \mathcal{K}_{2 i}$ to approximate $\mathcal{L}_{i}$. Then $\mathcal{L}_{i}-\mathcal{K}_{1 i}-\mathcal{K}_{2 i}, \mathcal{L}_{i}$ serve as the $\mathcal{L}_{0}, \mathcal{L}$ operators in the model problem (4.1), respectively. The decomposition of the solutions $W_{1}, W_{2}$ is similar to (4.2). Since we want to perform energy estimate on $W_{1}$ using more singular weights, we correct the nonlinear terms and the forcing terms in the first equation in (4.14). Although $N F_{i}\left(W_{1}+W_{2}\right)$ involves nonlinear factors, e.g. $c_{\omega}\left(\omega_{1}+\omega_{2}\right) \partial_{x y}\left(\omega_{1}+\omega_{2}\right)(0)$, since these factors are constant in space, we can still apply Duhamel's formula in (4.3) to $N F_{i}\left(W_{1}+W_{2}\right)$, i.e.,

$$
\int_{0}^{t} e^{\mathcal{L}(t-s)}\left(c_{\omega} \partial_{x y}\left(\omega_{1}(s)+\omega_{2}(s)\right)(0) \bar{f}\right) d s=\int_{0}^{t} c_{\omega} \partial_{x y}\left(\omega_{1}(s)+\omega_{2}(s)\right)(0) e^{\mathcal{L}(t-s)} \bar{f} d s
$$

and obtain the formula of $W_{2}$ in (4.14).
Avoiding the loss of derivatives. Note that in the equation of $W_{1}$ in (4.14), it contains the nonlinear terms $\mathbf{u}\left(W_{1}+W_{2}\right) \cdot \nabla\left(W_{1}+W_{2}\right)$. In general, the term $\mathbf{u} \cdot \nabla W_{2}$ can lead to loss of derivatives. Note that $W_{2}$ in (4.14) is driven by the forcing terms of the following forms

$$
\begin{equation*}
\sum_{1 \leq i \leq N} a_{i}\left(W_{1}, W_{2}\right)(t) f_{i} \tag{4.15}
\end{equation*}
$$

for some $N$, time-dependent scalars (independent of $x) a_{i}\left(W_{1}, W_{2}\right)(t)$, and time-independent functions $f_{i}$, e.g. $c_{\omega}\left(W_{1}\right) \bar{f}_{c_{\omega}, i}$ in (4.11). By choosing smoother $f_{i}$ in the approximation, we can obtain solution $W_{2}$ smooth enough for our energy estimates and overcome the above difficulty.
4.2.4. Constructing the approximate solution of $W_{2}$ and modifying the decomposition. Following the ideas in Section 4.1.2, instead of solving the $W_{2}$ equations in (4.14) exactly, we solve them with an error term. Assume that we have the following representations for the operators

$$
\begin{equation*}
\overrightarrow{\mathcal{K}}_{1 j}\left(W_{1}\right)+\overrightarrow{\mathcal{K}}_{2 j}\left(W_{1}\right)=\sum_{1 \leq i \leq n_{1}} a_{i}\left(W_{1}\right)(t) \bar{F}_{i j} \tag{4.16}
\end{equation*}
$$

where $\overrightarrow{\mathcal{K}}_{i}=\left(\mathcal{K}_{i 1}, \mathcal{K}_{i 2}, \mathcal{K}_{i 3}\right), \bar{F}_{i}(x)=\left(\bar{f}_{i, 1}(x), \bar{f}_{i, 2}(x), \bar{f}_{i, 3}(x)\right): \mathbb{R}_{+}^{2} \rightarrow \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, and $a_{i}\left(W_{1}\right)(t)$ is some linear functional on $W_{1}$. For example, the formula (4.11) can be written as

$$
a\left(W_{1}\right)(t)\left(\bar{f}_{c_{\omega}, 1}, \bar{f}_{c_{\omega}, 2}, \bar{f}_{c_{\omega}, 3}\right), \quad a\left(W_{1}\right)(t)=c_{\omega}\left(W_{1}\right)=u_{x}\left(W_{1}(t)\right)(0)
$$

where we have used (2.26) for $c_{\omega}$. Recall the operators $N F_{i}$ and functions $f_{\chi, 1}$ from (4.11). Writing (4.11) as vectors, we have

$$
\begin{align*}
N F(\omega)= & \left(c_{\omega} \partial_{x y} \omega(0)+\partial_{x y} \overline{\mathcal{F}}_{1}(0)\right)\left(f_{\chi, 1}, 0,0\right)+\left(c_{\omega} \partial_{x y} \eta(0)+\partial_{x y} \overline{\mathcal{F}}_{2}(0)\right)\left(0, f_{\chi, 2}, 0\right) \\
& +\left(c_{\omega} \partial_{x x} \xi(0)+\partial_{x x} \overline{\mathcal{F}}_{3}(0)\right)\left(0,0, f_{\chi, 3}\right) \triangleq \sum_{1 \leq i \leq 3} a_{n l, i}(W(t)) F_{\chi, i}, \tag{4.17}
\end{align*}
$$

where
(4.18)

$$
a_{n l, \cdot}=\left(c_{\omega} \partial_{x y} \omega(0)+\partial_{x y} \overline{\mathcal{F}}_{1}(0), c_{\omega} \partial_{x y} \eta(0)+\partial_{x y} \overline{\mathcal{F}}_{2}(0), c_{\omega} \partial_{x x} \xi(0)+\partial_{x x} \overline{\mathcal{F}}_{3}(0)\right), \quad F_{\chi, i}=f_{\chi, i} e_{i}
$$

and $e_{1}, e_{2}, e_{3}$ are the standard basis for $\mathbb{R}^{3}$. Denote by $\hat{F}_{i}(t, x)$ and $\hat{F}_{\chi, i}(t, x)$ the approximation of $e^{\mathcal{L} t} \bar{F}_{i}$ and $e^{\mathcal{L} t} \bar{F}_{\chi, i}$. Following (4.4) and (4.5), and using the idea in (4.15), we construct the approximate solution to $W_{2}$ in (4.14) as follows

$$
\begin{equation*}
\hat{W}_{2}(t)=\sum_{i \leq n_{1}} \int_{0}^{t} a_{i}\left(W_{1}(s)\right) \hat{F}_{i}(t-s) d s+\sum_{i \leq 3} \int_{0}^{t} a_{n l, i}\left(W_{1}(s)+\hat{W}_{2}(s)\right) \hat{F}_{\chi, i}(t-s) d s \tag{4.19}
\end{equation*}
$$

We introduce the residual operator

$$
\begin{align*}
& \mathcal{R}_{l}\left(W_{1}\right) \triangleq \sum_{i=1}^{n_{1}}\left(a_{i}\left(W_{1}(t)\right)\left(\hat{F}_{i}(0)-\bar{F}_{i}\right)+\int_{0}^{t} a_{i}\left(W_{1}(s)\right)\left(\left(\partial_{t}-\mathcal{L}\right) \hat{F}_{i}\right)(t-s) d s\right) \\
& \mathcal{R}_{n l}(W) \triangleq \sum_{i=1}^{3}\left(a_{n l, i}(W(t))\left(\hat{F}_{\chi, i}(0)-\bar{F}_{\chi, i}\right)+\int_{0}^{t} a_{n l, i}(W(s))\left(\left(\partial_{t}-\mathcal{L}\right) \hat{F}_{\chi, i}\right)(t-s) d s\right)  \tag{4.20}\\
& \mathcal{R}\left(W_{1}, \hat{W}_{2}\right) \triangleq \mathcal{R}_{l}\left(W_{1}\right)+\mathcal{R}_{n l}\left(W_{1}+\hat{W}_{2}\right)
\end{align*}
$$

where $\mathcal{R}_{l}, \mathcal{R}_{n l}$ denote the linear and the nonlinear parts, respectively. Note that $\mathcal{R}\left(W_{1}, \hat{W}_{2}\right)(x)$ is a vector in $\mathbb{R}^{3}$.

Remark 4.3. Given $W_{1}, \hat{F}_{i}, \hat{F}_{\chi, i}$, the solution $\hat{W}_{2}$ (4.19) is not completely determined since the second part depends on $\hat{W}_{2}$. At the linear level, $\hat{W}_{2}(4.19)$ is determined. Since the second part depends on $\hat{W}_{2}$ nonlinearly, we will show that it is much smaller than the linear part and control $a_{n l, i}(W(s))$ using a bootstrap condition (5.72). Then we can still use (4.19) to estimate $\hat{W}_{2}$.

Similar to (4.7), using the above operators, we modify the decomposition (4.14) as follows

$$
\begin{align*}
\partial_{t} W_{1, i} & =\left(\mathcal{L}_{i}-\mathcal{K}_{1 i}-\mathcal{K}_{2 i}\right)\left(W_{1}\right)+\mathcal{N}_{i}\left(W_{1}+\hat{W}_{2}\right)+\overline{\mathcal{F}}_{i}-N F_{i}\left(W_{1}+\hat{W}_{2}\right)-\mathcal{R}_{i}\left(W_{1}, \hat{W}_{2}\right) \\
\partial_{t} \hat{W}_{2, i} & =\mathcal{L}_{i} \hat{W}_{2}+\mathcal{K}_{1 i}\left(W_{1}\right)+\mathcal{K}_{2 i}\left(W_{1}\right)+N F_{i}\left(W_{1}+\hat{W}_{2}\right)+\mathcal{R}_{i}\left(W_{1}, \hat{W}_{2}\right)  \tag{4.21}\\
\left.W_{1}\right|_{t=0} & =\left(\omega_{0}, \eta_{0}, \xi_{0}\right),\left.\quad \hat{W}_{2}\right|_{t=0}=(0,0,0)
\end{align*}
$$

where $\mathcal{R}=\left(\mathcal{R}_{1}, \mathcal{R}_{2}, \mathcal{R}_{3}\right)$. The above decomposition is a nonlinear generalization of (4.7). We solve the $\hat{W}_{2}$ equation using the formula (4.19) exactly. It is easy to see that $W_{1}+\hat{W}_{2}$ solves (4.10) from initial data $\left(\omega_{0}, \eta_{0}, \xi_{0}\right)$. If the error in (4.20), e.g. $\hat{F}_{i}(0)-\hat{F}_{i},\left(\partial_{t}-\mathcal{L}\right) \hat{F}_{i}$, is small, we expect that the following estimates for $\mathcal{R}$ :

$$
\left\|\mathcal{R}\left(W_{1}, \hat{W}_{2}\right)\right\|_{X} \leq \varepsilon\left(\left\|W_{1}\right\|_{X}+\left\|W_{1}+\hat{W}_{2}\right\|_{X}^{2}+\bar{\varepsilon}\right)
$$

in some suitable weighted space $X$ with very small $\varepsilon, \bar{\varepsilon}$, where the second and the third terms come from the estimate of $a_{n l, i}\left(W_{1}+\hat{W}_{2}\right)$ defined in (4.17). Since $\overline{\mathcal{F}}_{i}$ is the residual error of the profile, for $i+j=2, \partial_{x}^{i} \partial_{y}^{j} \overline{\mathcal{F}}(0)$ is small and contributes to the small factor $\bar{\varepsilon}$. Then, the residual operator $\mathcal{R}$ can be treated as a small perturbation in (4.21). In particular, at the linear level, $\hat{W}_{2}$ is almost decoupled from the $W_{1}$ equation.

We construct approximate solution $\hat{F}_{i}$ and $\hat{F}_{\chi, i}$ with errors vanishing cubically near $x=0$ :

$$
\begin{equation*}
\hat{F}_{i}(0)-\bar{F}_{i}, \quad\left(\partial_{t}-\mathcal{L}\right) \hat{F}_{i}=O\left(|x|^{3}\right), \quad \hat{F}_{\chi, i}(0)-\bar{F}_{\chi, i}, \quad\left(\partial_{t}-\mathcal{L}\right) \hat{F}_{\chi i}=O\left(|x|^{3}\right) \tag{4.22}
\end{equation*}
$$

and estimate the local part of the residual error in weighted functional spaces for the energy estimate rigorously in Section 3 in Part II [15]. We combine the estimate of nonlocal error with energy estimate in Section 5.8 .

For initial perturbation $\omega_{0}, \eta_{0}, \xi_{0}=O\left(|x|^{3}\right)$, from the definitions 4.11) and the above vanishing order of the error, we obtain that the vanishing conditions $\omega_{1}, \eta_{1}, \xi_{1}=O\left(|x|^{3}\right)$ are preserved. Thus, we can perform energy estimates on $W_{1}$ using singular weights of order $|x|^{-3}$ near $x=0$. See Section 2.7.3 for more discussions on the vanishing order. We will perform the energy estimates in Section 5

Remark 4.4. Since $\hat{F}_{i}$ is the numerical solution to $\partial_{t} F_{i}=\mathcal{L} F_{i}$, the initial data and coefficients of $\mathcal{L}$ are smooth enough, in principle, by choosing a high order numerical scheme with sufficiently small mesh size and timestep, one can make the error (4.22) to be arbitrarily small. Then the residual operators in (4.20), (4.21) are sufficiently small compared to the perturbation $W_{1}, W$.

We present the formula of different initial data $\bar{F}_{i}, \bar{F}_{\chi, i}$ for the finite rank perturbation (4.19), (4.20) in Appendix C.2.1. In Figure 5, we present $\log \|\hat{f}(t) \rho\|_{\infty}, \rho=|x|^{-2}+1$ with discrete $L^{\infty}$ norm computed over the gird points to illustrate the decay of $\left\|e^{\mathcal{L} t} f \cdot \rho\right\|_{\infty}$. See Section 4.1.1. Over a time period $T=12$, the solution in the weighted norm decreases by a factor about $e^{-10} \approx 4.5 \cdot 10^{-5}$. In Part II [15], we use the method in Section 4.1.2 to estimate the decay rigorously. The exponential decay of $\|\hat{f}(t) \rho\|_{\infty}$ in time is consistent with the numerical evidence of linear stability reported by Liu [64] (see Section 3.4). By constructing approximate space-time solution to $e^{\mathcal{L} t} f$, we establish this spectral property rigorously.


Figure 5. Plot of $\log \left\|\hat{f}\left(t_{i}\right) \rho\right\|_{L^{\infty}}, \hat{f}=\hat{\omega}, \hat{\eta}, \hat{\xi}, \rho(x)=|x|^{-2}+1$ on grid points over time $t_{i}=0.48 i$ up to $T=12$. Red, blue, and black curves represent $\hat{\omega}, \hat{\eta}, \hat{\xi}$, respectively. Left: the first mode related to $\bar{f}_{c_{\omega}, i}$ (4.11). Right: all modes.
4.3. Approximating the regular part of the velocity. We want to construct a finite rank approximation $\mathcal{K}(\omega)$ of $\int_{\mathbb{R}^{2}} K_{f}(x-y) \omega(y) d y$ so that we can estimate

$$
\begin{equation*}
\left|\int_{\mathbb{R}^{2}} K_{f}(x-y) \omega(y) d y-\mathcal{K}(\omega)\right| \leq C_{1}(x, \gamma) \max \left(\left\|\omega \varphi_{1}\right\|_{L^{\infty},}, s_{f} \max _{i=1,2} \gamma_{i}\left[\omega \psi_{1}\right]_{C_{x_{i}}^{1 / 2}}\right) \tag{4.23}
\end{equation*}
$$

with $s_{f}=0$ for $f=\mathbf{u}$ and $s_{f}=1$ for $f=\nabla \mathbf{u}$ and small constant $C_{1}(x, \gamma)$ for some given weights and $\gamma$, where $K_{f}$ is the kernel for $\partial_{x}^{i} \partial_{y}^{j}(-\Delta)^{-1} \omega, i+j \leq 2$ and the Hölder seminorms $C_{x}^{1 / 2}, C_{y}^{1 / 2}$ are defined in (2.20).

Since $K_{f}(z)$ is smooth away from $z=0$, a natural approach is to approximate the nonsingular part of $K(x-y)$ by interpolating $K(x-y)$ on finite many points $x_{i}$ :

$$
K(x-y) \mathbf{1}_{|x-y| \geq \varepsilon} \approx \sum_{1 \leq i \leq n} \chi_{i}(x) K\left(x_{i}-y\right) \mathbf{1}_{\left|x_{i}-y\right| \geq \varepsilon}
$$

where $\chi_{i}$ is some cutoff function localized to $x_{i}$. The above right hand sides lead to the finite rank operator

$$
\begin{equation*}
\mathcal{K}(\omega)=\sum_{i=1}^{n} \chi_{i}(x) \int K\left(x_{i}-y\right) \mathbf{1}_{\left|x_{i}-y\right| \geq{ }_{\varepsilon}} \omega(y) d y \tag{4.24}
\end{equation*}
$$

We will construct the bulk part of the approximation in Section4.3.2 based on (4.24). Due to the decay of the coefficients of $\mathbf{u}, \nabla \mathbf{u}$ in (4.10), e.g. $\nabla \bar{\omega}$ in $\mathbf{u} \cdot \nabla \bar{\omega}, \nabla \bar{\theta}$ in $\nabla \mathbf{u} \cdot \nabla \bar{\theta}$, these nonlocal terms are small for large $|x|$. Thus, we only need to approximate $\mathbf{u}, \nabla \mathbf{u}$ for $x$ in the near field, especially for $x$ close to the boundary due to the anisotropy of the flow. See Section 2.7.2,

Regularity of the velocity. For $\mathbf{u}=\nabla^{\perp}(-\Delta)^{-1} \omega$, given $\omega$ in some weighted $L^{\infty}(\varphi)$ space, $\mathbf{u}$ is log-Lipschitz. Thus we can approximate $\mathbf{u}(x)$ in $C^{\beta} \cap L^{\infty}$ for any $\beta<1$ by interpolating discrete points $\mathbf{u}\left(x_{i}\right), i=1,2, \ldots, n$ with $n$ sufficiently large. The $C^{\beta} \cap L^{\infty}$ norm of the approximation error can be bounded by $c\|\omega \varphi\|_{\infty}$ with a small constant $c$. Similarly, for $\nabla \mathbf{u}=\nabla \nabla^{\perp}(-\Delta)^{-1} \omega$, given $\omega$ in some weighted $L^{\infty}$ space, the nonsingular part of $\nabla \mathbf{u}, K(z) \mathbf{1}_{|z| \geq \varepsilon} * \omega$, is Lipschitz. Thus we can approximate it in $C^{1 / 2} \cap L^{\infty}$. Since $\nabla \mathbf{u}=\nabla \nabla^{\perp}(-\Delta)^{-1} \omega$ is not bounded from $L^{\infty}$ to $L^{\infty}$, for the singular part of $\nabla \mathbf{u}, K(z) \mathbf{1}_{|z| \leq \varepsilon} * \omega$, we need to use the Hölder regularity of $\omega$ to control it. These motivate (4.23).
4.3.1. Approximation near 0 . Since we will perform weighted energy estimates with singular weights and the velocity $\mathbf{u}, \nabla \mathbf{u}$ do not vanish near $x=0$ with high order, we first approximate $\mathbf{u}, \nabla \mathbf{u}$ by its leading order behavior at $x=0$.

In our energy estimate, we consider perturbation $\omega$ with vanishing order $O\left(|x|^{2+\alpha}\right)$ for some $\alpha>0$ near $x=0$. Recall $-\Delta \psi=\omega$ and $\mathbf{u}=\nabla^{\perp} \psi$. Using Taylor expansion and

$$
0=\omega_{x}(0)=-\psi_{x x x}(0)-\psi_{x y y}(0), \quad 0=\omega_{x y}(0)=-\psi_{x x x y}(0)-\psi_{x y y y}(0)
$$

we get
$\psi(x, y)=\psi_{x y}(0) x y+\frac{1}{6}\left(\psi_{x x x y}(0) x^{3} y+\psi_{x y y y}(0) x y^{3}\right)+$ h.o.t. $=\psi_{x y}(0) x y+\frac{1}{6} \psi_{x x x y}(0)\left(x^{3} y-x y^{3}\right)+$ h.o.t.
We can represent $\psi_{x x x y}(0)$ as an integral of $\omega$
$\psi_{x x x y}(0)=\frac{2}{\pi} \int_{\mathbb{R}_{++}^{2}} \omega(y) K_{00}(y) d y, \quad K_{00}(y) \triangleq \frac{24 y_{1} y_{2}\left(y_{1}^{2}-y_{2}^{2}\right)}{|y|^{8}}, \quad \mathcal{K}_{00}(\omega) \triangleq \frac{1}{\pi} \int_{\mathbb{R}_{++}^{2}} K_{00}(y) \omega(y) d y$.
For $\omega=O\left(|x|^{2+\alpha}\right)$ with a suitable decay, the above integral is well-defined. By definition, we have $\psi_{x x x y}(0)=2 \mathcal{K}_{00}$. We use $\mathcal{K}_{00}$ as a short hand notation for $\mathcal{K}_{00}(\omega)$. Note that $u_{x}(0)=$ $-\psi_{x y}(0)$. Using the above formulas, near 0 , the leading order term for $\nabla \mathbf{u}$ and $\mathbf{u}$ are given by
$u=-\psi_{y}=u_{x}(0) x-\left(\frac{1}{3} x^{3}-x y^{2}\right) \mathcal{K}_{00}+$ h.o.t., $\quad v=\psi_{x}=-u_{x}(0) y+\left(x^{2} y-\frac{1}{3} y^{3}\right) \mathcal{K}_{00}+$ h.o.t., $u_{x}=u_{x}(0)-\left(x^{2}-y^{2}\right) \mathcal{K}_{00}+$ h.o.t., $\quad v_{x}=2 x y \mathcal{K}_{00}+$ h.o.t., $\quad u_{y}=-\psi_{y y}=2 x y \mathcal{K}_{00}+$ h.o.t.

By introducing

$$
\begin{align*}
C_{u 0} & =x, \quad C_{v 0}=-y, \quad C_{u_{x} 0}=1, \quad C_{u_{y} 0}=C_{v_{x} 0}=0 \\
C_{u} & =-\left(\frac{1}{3} x^{3}-x y^{2}\right), \quad C_{v}=x^{2} y-\frac{1}{3} y^{3}  \tag{4.26}\\
C_{u_{x}} & =-\left(x^{2}-y^{2}\right), \quad C_{v_{x}}=2 x y, \quad C_{u_{y}}=2 x y
\end{align*}
$$

we can rewrite the above leading order formulas as

$$
\begin{equation*}
f(x, y)=u_{x}(0) C_{f 0}(x, y)+\mathcal{K}_{00} C_{f}(x, y)+\text { h.o.t., } \quad f=u, v, u_{x}, v_{x}, u_{y} \tag{4.27}
\end{equation*}
$$

For $v_{y}$, we will use $v_{y}=-u_{x}$ to estimate it. We will localize the above leading order terms to construct the approximation term near 0 in the next subsection.
4.3.2. Approximation along the boundary. Let $\chi$ be the cutoff function constructed in (C.6) and $\tilde{\chi}=1-\chi$. They satisfy

$$
\chi(x)=0, \quad \tilde{\chi}(x)=1, \quad x \leq 0, \quad \chi(x)=1, \quad \tilde{\chi}(x)=0, \quad x \geq 1
$$

Given $0<x_{0}<x_{1}<\ldots<x_{n}<x_{n+1}$ and $y_{0}>0$, we construct the cutoff functions
$\chi_{0}=\tilde{\chi}\left(\frac{x-x_{0}}{x_{1}-x_{0}}\right) \tilde{\chi}\left(\frac{y-y_{0}}{y_{0}}\right), \quad \chi_{i}=\left(\chi\left(\frac{x-x_{i-1}}{x_{i}-x_{i-1}}\right) \mathbf{1}_{x \leq x_{i}}+\tilde{\chi}\left(\frac{x-x_{i}}{x_{i+1}-x_{i}}\right) \mathbf{1}_{x \geq x_{i}}\right) \tilde{\chi}\left(\frac{y-y_{0}}{y_{0}}\right), 1 \leq i \leq n$.
We impose the cutoff function $\tilde{\chi}\left(\frac{y-y_{0}}{y_{0}}\right)$ so that $\chi_{i}$ is supported near $y=0$. By definition, for $x \leq x_{n}, y \leq y_{0}$, we have

$$
\sum_{i \leq n} \chi_{i}(x, y)=1
$$

We want to approximate $\mathbf{u}, \nabla \mathbf{u}$ such that the remainders $\mathbf{u}-\mathbf{u}_{\text {app }}$ vanish near $x=0$ with high order. See Section 4.3.1. To preserve these vanishing orders in the approximations and obtain smoother approximations, we consider the following approximation, which modifies (4.24)
$\hat{f}_{1}(x, y) \triangleq C_{f 0}(x, y) u_{x}(0)+\hat{f}_{10}(x, y), \quad \hat{f}_{10} \triangleq C_{f}(x, y)\left(\mathcal{K}_{00} \chi_{0}(x, y)+\sum_{1 \leq i \leq n} \frac{\hat{f}_{N S}\left(x_{i}, 0\right)}{C_{f}\left(x_{i}, 0\right)} \chi_{i}(x, y)\right)$,
where

$$
\begin{equation*}
\hat{f}_{N S}\left(x_{i}, 0\right)=\int_{y \in \mathbb{R}^{2}, \max \left(\left|y_{1}-x_{i}, y_{2}\right|\right) \geq t_{i}} K_{f}\left(\left(x_{i}, 0\right)-y\right) \omega(y) d y-C_{f 0}\left(x_{i}, 0\right) u_{x}(0) \tag{4.30}
\end{equation*}
$$

$K_{f}$ is the kernel for $f=u, v, u_{x}, v_{x}, u_{y}, N S$ is short for nonsingular, and $C_{f 0}$ and $C_{f}$ are the coefficients of the leading order approximations of $\partial_{x}^{i} \partial_{y}^{j} \psi$ near $x=0$. See (4.27) and (4.26). We add the functions $C_{f}(x, y)$ in (4.29) so that $\hat{f}_{1}$ has the same vanishing order as that of $f$.

We construct the above approximations along the boundary for $u, u_{x}$. For $v, v_{x}, u_{y}$, since the associated coefficients are relatively small, e.g. $\bar{\omega}_{y}$ in $v \bar{\omega}_{y}$ and $\bar{\theta}_{y}$ in $v_{x} \bar{\theta}_{y}$ (4.10), we only construct the approximation term $C_{f}(x, y) \mathcal{K}_{00} \chi_{0}(x, y)$ near 0 . Now, by definition, we have

$$
\begin{align*}
f\left(x_{i}, 0\right)-\hat{f}_{1}\left(x_{i}, 0\right) & =f\left(x_{i}, 0\right)-C_{f 0}\left(x_{i}, 0\right) u_{x}(0)-\hat{f}_{N S}\left(x_{i}, 0\right), \quad 0 \leq i \leq n \\
f(x, y)-\hat{f}_{1}(x, y) & =f(x, y)-C_{f 0}(x, y) u_{x}(0)-C_{f}(x, y) \mathcal{K}_{00}, \quad \text { for } x \leq x_{0}, y \leq y_{0} \tag{4.31}
\end{align*}
$$

The first identity shows that $\hat{f}_{1}$ is an interpolation of the non-singular part of $K_{f} * \omega$, which is similar to (4.24). Here, we consider a weighted version of (4.24) with weight $C_{f}(x)$ so that the approximation has the right vanishing order near $x=0$. The second identity shows that near $x=0$, the approximation $\hat{f}_{1}$ captures the leading order behavior of $f$ near $x=0$ (4.27). Thus, $\hat{f}_{1}$ can approximate $f$ near the points $(0,0),\left(x_{i}, 0\right), 0 \leq i \leq n$.
4.3.3. Approximation in the far-field. To improve the far-field estimate, instead of using $u_{x}(0)$ to approximate $u, v, \partial u$ (4.30), we use the truncated version of $u_{x}(0)$

$$
\begin{equation*}
I_{n} \triangleq-\frac{4}{\pi} \int_{\max \left(y_{1}, y_{2}\right) \geq R_{n}} \frac{y_{1} y_{2}}{|y|^{4}} \omega(y) d y \tag{4.32}
\end{equation*}
$$

The above approximation is similar to the leading order term of the velocity derived in [59]. For $f=u_{x}$, the leading order part of the kernel $K(s)=-\frac{1}{\pi} \frac{s_{1} s_{2}}{|s|^{4}}$ with symmetrization is given by

$$
\begin{align*}
K^{\text {sym }}(x, y) & =K(x-y)+K(x+y)-K\left(x_{1}-y_{1}, x_{2}+y_{2}\right)-K\left(x_{1}+y_{1}, x_{2}-y_{2}\right) \\
& =-\frac{4}{\pi} \frac{y_{1} y_{2}}{|y|^{4}}+\text { l.o.t. } \tag{4.33}
\end{align*}
$$

for max $\left|y_{i}\right| \geq C \max \left|x_{i}\right|$ with large $C$. For $f=u, v, v_{y}$, using a similar argument, we obtain the leading order part of the associated kernel $K_{f}$

$$
\begin{equation*}
K_{f}^{s y m}=-C_{f 0} \frac{4}{\pi} \frac{y_{1} y_{2}}{|y|^{4}}+\text { l.o.t. } \tag{4.34}
\end{equation*}
$$

for $\max \left|y_{i}\right| \geq C \max \left|x_{i}\right|$ with large $C$, where $C_{f 0}$ is defined in (4.26). When $|y| /|x|$ is small, the function $-\frac{4}{\pi} \frac{y_{1} y_{2}}{|y|^{4}}$ does not approximate $K^{s y m}(x, y)$ well, and thus we truncate it

$$
\begin{equation*}
\int_{\max \left|y_{i}\right| \geq C \max \left|x_{i}\right|}-\frac{y_{1} y_{2}}{|y|^{4}} \omega(y) d y \tag{4.35}
\end{equation*}
$$

The above operator does not have a finite rank due to the hard cutoff function $\mathbf{1}_{|y| \geq C|x|}$. To approximate it by a finite rank operators, we approximate $\mathbf{1}_{|y| \geq C|x|}$ by a smooth cutoff function

$$
g(x, y)=\sum_{i} \mathbf{1}_{\left|y_{1}\right| \vee\left|y_{2}\right| \geq R_{i}} \chi_{i}(x)
$$

such that $\chi_{i}(x)$ is localized to the domain with $|x|$ comparable to $R_{i}$. Then for $x$ close to $R_{i}$, we obtain $g(x, y) \approx \mathbf{1}_{\left|y_{1}\right| \vee\left|y_{2}\right| \geq R_{i}} \approx \mathbf{1}_{\left|y_{1}\right| \vee\left|y_{2}\right| \geq|x|}$.

More specifically, given $R_{0}<R_{2}<.<R_{m}$, we construct cutoff functions as follows

$$
\begin{align*}
\chi_{i}^{R}(x, y) & =\tilde{\chi}\left(\frac{x-R_{i}}{R_{i+1}-R_{i}}\right) \tilde{\chi}\left(\frac{y-R_{i}}{R_{i+1}-R_{i}}\right)-\tilde{\chi}\left(\frac{x-R_{i-1}}{R_{i}-R_{i-1}}\right) \tilde{\chi}\left(\frac{y-R_{i-1}}{R_{i}-R_{i-1}}\right), 1 \leq i \leq m-1  \tag{4.36}\\
\chi_{m}^{R}(x, y) & =1-\tilde{\chi}\left(\frac{x-R_{m-1}}{R_{m}-R_{m-1}}\right) \tilde{\chi}\left(\frac{y-R_{m-1}}{R_{m}-R_{m-1}}\right), \quad S_{i}^{R}=1-\tilde{\chi}\left(\frac{x-R_{i-1}}{R_{i}-R_{i-1}}\right) \tilde{\chi}\left(\frac{y-R_{i-1}}{R_{i}-R_{i-1}}\right)
\end{align*}
$$

By definition, $\chi_{i}^{R}$ is supported in the annulus $\left[0, R_{i+1}\right]^{2} \backslash\left[0, R_{i-1}\right]^{2}, 1 \leq i \leq m-1, \chi_{m}^{R}=1$ for $\max (x, y) \geq R_{m}, S_{i}^{R}$ is supported in $\mathbb{R}_{++}^{2} \backslash\left[0, R_{i-1}\right]^{2}$ with $S_{i}^{R}=1$ for $\max (x, y) \geq R_{i}$. Moreover, we have $\sum_{i \leq m} \chi_{i}^{R}(x, y)=1$ for $\max (x, y) \geq R_{1}$. Now, we construct the second approximation

$$
\begin{align*}
\hat{f}_{2}(x, y) & =C_{f 0}(x, y)\left(1-\chi_{t o t}(x, y)\right)\left(\sum_{1 \leq i \leq m} \chi_{i}^{R}(x, y)\left(I_{i}-u_{x}(0)\right)\right)  \tag{4.37}\\
\chi_{t o t}(x, y) & =\sum_{0 \leq i \leq n} \chi_{i}(x, y)
\end{align*}
$$

where $\chi_{t o t}(x, y)$ is the sum of the cutoff functions for the first approximation in Section4.3.2, For $\chi_{t o t}(x, y)=0$ and $x=R_{i}, y \in\left[0, R_{i}\right]$, from (4.30), we get $\hat{f}_{1}=C_{f 0}(x) u_{x}(0), \hat{f}_{2}=C_{f 0}\left(I_{i}-u_{x}(0)\right)$,

$$
\begin{aligned}
f-\hat{f}_{1}-\hat{f}_{2} & =f-C_{f 0} u_{x}(0)-C_{f 0}\left(I_{i}-u_{x}(0)\right)=\int\left(K_{f}(x-y) \omega(y) d y-C_{f 0} I_{i}\right. \\
& =\int_{\mathbb{R}_{++}^{2}}\left(K_{f}^{s y m}(x, y)+\frac{4}{\pi} C_{f 0}(x) \frac{y_{1} y_{2}}{|y|^{4}} \mathbf{1}_{\max \left(\left|y_{1}\right|,\left|y_{2}\right|\right) \geq R_{i}}\right) \omega(y) d y
\end{aligned}
$$

where $K_{f}^{\text {sym }}$ is the symmetrized kernel (4.33). Therefore, for large $|x|$, the approximation $\hat{f}_{1}+\hat{f}_{2}$ can be seen as a smooth interpolation of the main term related to (4.34), (4.35) with $C=1$.

Remark 4.5. We remark that the lower order term in (4.33), (4.34) is about $|x| /|y|$ of the main term, which is not small if $|y|$ and $|x|$ are comparable. As a result, the above finite rank approximation (4.37) can only approximate part of the integral in $f$. Nevertheless, it allows us to obtain a better estimate of $f-\hat{f}_{1}-\hat{f}_{2}$ than $f$, which is sufficient for our purpose.

Combining (4.29), (4.37), we construct the following approximation for $f=\mathbf{u}, \nabla \mathbf{u}$ and for $\tilde{f}=f-C_{f 0} u_{x}(0)$

$$
\begin{equation*}
\hat{f}(\omega) \triangleq \hat{f}_{1}(\omega)+\hat{f}_{2}(\omega), \quad \tilde{\hat{f}} \triangleq \hat{f}(\omega)-C_{f 0} u_{x}(0)=\hat{f}_{10}(\omega)+\hat{f}_{2}(\omega) \tag{4.38}
\end{equation*}
$$

where the notations $\tilde{u}, \tilde{v}$ are introduced in (4.8). Clearly, $\hat{f}, \hat{\tilde{f}}$ is a finite rank operator of $\omega$.
4.3.4. Reformulation of the approximations. Recall $\chi_{i}$ defined in (4.28). We introduce

$$
S_{j}=\sum_{i \leq j} \chi_{i}, \quad \chi_{j}=S_{j}-S_{j-1}
$$

By definition, we have $S_{j}(x, y)=1$ for $x \leq x_{j}, y \leq y_{0}$ and $S_{j}(x, y)=0$ for $x \geq x_{j+1}$. By rewriting the summation, we can rewrite $\hat{f}_{10}$ in (4.29) as follows

$$
\begin{align*}
& \hat{f}_{10}=C_{f}\left(\mathcal{K}_{00} S_{0}+\sum_{1 \leq i \leq n} \frac{\hat{f}\left(x_{i}, 0\right)}{C_{f}\left(x_{i}, 0\right)}\left(S_{i}-S_{i-1}\right)\right)  \tag{4.39}\\
& =C_{f}(x, y)\left(S_{0}\left(\mathcal{K}_{00}-\frac{\hat{f}\left(x_{1}, 0\right)}{C_{f}\left(x_{1}, 0\right)}\right)+S_{n} \frac{\hat{f}\left(x_{n}, 0\right)}{C_{f}\left(x_{n}, 0\right)}+\sum_{1 \leq i \leq n-1} S_{i}\left(\frac{\hat{f}\left(x_{i}, 0\right)}{C_{f}\left(x_{i}, 0\right)}-\frac{\hat{f}\left(x_{i+1}, 0\right)}{C_{f}\left(x_{i+1}, 0\right)}\right)\right)
\end{align*}
$$

We have dropped the dependence of $C_{f}, S_{i}$ on $x, y$ to simplify the notations. An advantage of the above formula is that we exploit the cancellation among $\mathcal{K}_{00}, \frac{\hat{f}\left(x_{i}, 0\right)}{C_{f}\left(x_{i}, 0\right)}$.

Similarly, using $\chi_{i}^{R}=S_{i}^{R}-S_{i+1}^{R}, i \leq m-1, S_{m}^{R}=\chi_{m}^{R}$ (4.36), we can rewrite the second approximation (4.37) as follows

$$
\begin{align*}
\hat{f}_{2} & =C_{f 0}\left(1-\chi_{t o t}\left(\sum_{1 \leq i \leq m-1}\left(S_{i}^{R}-S_{i+1}^{R}\right)\left(I_{i}-u_{x}(0)\right)+S_{m}^{R}\left(I_{m}-u_{x}(0)\right)\right)\right.  \tag{4.40}\\
& =C_{f 0}\left(1-\chi_{t o t}\right)\left(S_{1}^{R}\left(I_{1}-u_{x}(0)\right)+\sum_{2 \leq i \leq m} S_{i}^{R}\left(I_{i}-I_{i-1}\right)\right),
\end{align*}
$$

where we have dropped the dependence of $\hat{f}_{2}, \chi_{t o t}, S_{i}^{R}$ on $x, y$. An advantage of the above formula is that we have better estimate of $I_{i}-I_{i-1}$ than $I_{i}-u_{x}(0)$.

In our estimates of the approximated velocity $f-\hat{f}_{1}-\hat{f}_{2}$ in Section 4 of Part II [15] we use (4.29), (4.37) for $\hat{f}_{i}$ since there are at most two nontrivial summands in each formula for a fixed $x$. On the other hand, in the construction of $\widehat{W}_{2}$ (4.19), we use the second formula (4.39), (4.40) and choose the coefficient $C_{f}(x, y) S_{i}, C_{f 0}\left(1-\chi_{t o t}\right) S_{i}^{R}$ as the initial data since the coefficients of each rank-one term given by $C_{f}(x, y) S_{i}, C_{f 0}\left(1-\chi_{t o t}\right) S_{i}^{R}$ are numerically more regular than $C_{f}(x, y) \chi_{i}$ and $C_{f 0}\left(1-\chi_{t o t}\right) \chi_{i}^{R}\left(S_{i}(x, y)\right.$ has one slope in $x$ and $\chi_{i}(x, y)$ has two $)$. Moreover, we exploit the cancellation among different integrals from different approximation terms. It allows us to obtain smaller errors in solving the linearized equations.

In our energy estimates for $\hat{f}_{1}$ in the nonlinear stability analysis, we perform two estimates based on these two formulas and optimize the estimates. In the estimate of $\hat{f}_{2}$, we use (4.40).

We list the parameters $x_{i}, t_{i}, R_{i}$ in the above approximations in Appendix C.2 To obtain sharp estimate of the constants in (4.23) and $\mathbf{u}-\hat{\mathbf{u}}, \nabla \mathbf{u}-\widehat{\nabla \mathbf{u}}$ in the energy estimates, which are important for us to reduce the number of approximation terms and obtain sharp energy estimates, we will estimate the integrals with computer assistance. We will discuss them in details in Section 4 of Part II [15]. We remark that we do not need to construct too many approximation terms. The total number of approximation terms we choose is less than 50.

## 5. Energy estimates

Recall the decomposition (4.21) in Section 4.2, In this section, we perform energy estimates of $W_{1}$ following the ideas and some derivations in Sections 2.7.1, 2.8.2, 2.8.3 The goal of the energy estimates is to control several weighted $L^{\infty}$ norms of $\omega, \eta, \xi$ and their weighted Hölder norms and establish the estimates (A.3) for the coefficients in the estimates. The condition (A.3) means that the damping term is stronger than the bad terms. Then we can further establish stability using the stability Lemma A. 1.
5.1. The main equation. After choosing a suitable approximation for the velocity $\mathbf{u}$ and using the approach described in Section 4.2, the main equations (4.21) for $\omega_{1}, \eta_{1}, \xi_{1}$ read

$$
\begin{align*}
\partial_{t} \omega_{1}+\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \omega_{1} & =\bar{c}_{\omega} \omega_{1}+\eta_{1}-\mathbf{u}_{A}\left(\omega_{1}\right) \cdot \nabla \bar{\omega} \\
\partial_{t} \eta_{1}+\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \eta_{1} & =\left(2 \bar{c}_{\omega}-\bar{u}_{x}\right) \eta_{1}-\bar{v}_{x} \xi_{1}-\mathbf{u}_{A} \cdot \nabla \bar{\theta}_{x}-\mathbf{u}_{x, A} \cdot \nabla \bar{\theta}  \tag{5.1}\\
\partial_{t} \xi_{1}+\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \xi_{1} & =\left(2 \bar{c}_{\omega}-\bar{v}_{x}\right) \xi_{1}-\bar{u}_{y} \eta_{1}-\mathbf{u}_{A} \cdot \nabla \bar{\theta}_{y}-\mathbf{u}_{y, A} \cdot \nabla \bar{\theta}
\end{align*}
$$

where $\overline{\mathbf{u}}=(\bar{u}, \bar{v})$, and $\mathbf{u}_{A}\left(\omega_{1}\right)$ is the velocity after subtracting the approximation term $\tilde{\hat{\mathbf{u}}}$ defined in (4.29), (4.37), (4.38), (4.13)

$$
\begin{equation*}
\mathbf{u}_{A}(f) \triangleq \tilde{\mathbf{u}}(f)-\tilde{\hat{\mathbf{u}}}(f), \quad \mathbf{u}_{x, A} \triangleq \tilde{\mathbf{u}}_{x}(f)-\tilde{\hat{\mathbf{u}}}_{x}(f), \quad \mathbf{u}_{y, A} \triangleq \tilde{\mathbf{u}}_{y}(f)-\tilde{\hat{\mathbf{u}}}_{y}(f) \tag{5.2}
\end{equation*}
$$

Note that we do not have $\partial_{x} \mathbf{u}_{A}=\mathbf{u}_{x, A}$ since we choose the approximations for $\mathbf{u}, \mathbf{u}_{x}, \mathbf{u}_{y}$ separately. Similarly, we do not have $\partial_{y} \eta_{1}=\partial_{x} \xi_{1}$. We make these finite rank perturbations to the velocity and the linearized equations by subtracting $\mathcal{K}_{2 i}$ (4.13) from $\mathcal{L}$ in (4.21). We also remove the $c_{\omega} \bar{f}_{\chi, i}$ terms (4.11) in (4.10) by subtracting $\mathcal{K}_{1 i}$ from $\mathcal{L}$ in (4.21). At this stage, we have dropped the remaining term $\mathcal{R}_{i}, N F\left(W_{1}+W_{2}\right)$, part of the nonlinear terms $\mathcal{N}_{i}$, the error term $\overline{\mathcal{F}}_{i}$ in (4.21) to simplify the presentation.

We adopt the notation from (4.21) and introduce $\bar{W}$

$$
\begin{equation*}
W_{1,1}=\omega_{1}, \quad W_{1,2}=\eta_{1}, \quad W_{1,3}=\xi_{1}, \quad \bar{W}=\left(\bar{\omega}, \bar{\theta}_{x}, \bar{\theta}_{y}\right) \tag{5.3}
\end{equation*}
$$

For initial perturbations that satisfy $\omega_{1,0}, \eta_{1,0}, \xi_{1,0}=O\left(|x|^{3}\right)$, the system (4.21) preserve these vanishing orders. See more discussions in Section 4.2.4

We introduce $\mathcal{T}_{d}(\rho), d_{i}(\varphi)$ to denote the coefficients of the damping terms and $b(x)$ to denote the coefficient of the advection

$$
\begin{align*}
& b(x)=\bar{c}_{l} x+\overline{\mathbf{u}}, \quad \mathcal{T}_{d}(\rho)=\rho^{-1}\left(\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \rho\right)=\rho^{-1}(b \cdot \nabla \rho),  \tag{5.4}\\
& d_{1, L}(\rho)=\mathcal{T}_{d}(\rho)+\bar{c}_{\omega}, \quad d_{2, L}(\rho)=\mathcal{T}_{d}(\rho)+2 \bar{c}_{\omega}-\bar{u}_{x}, \quad d_{3, L}(\rho)=\mathcal{T}_{d}(\rho)+2 \bar{c}_{\omega}+\bar{u}_{x} .
\end{align*}
$$

The terms $d_{i, L}(\rho)$ appear naturally in the weighted $L^{\infty}(\rho)$ estimates of $W_{1, i}$. See below (5.6). The subscript $L$ is short for linear.

In the equation of $W_{1, i}$, we treat the terms other than the local terms of $W_{1, i}$ in (5.1) as bad terms

$$
\begin{align*}
& B_{1}\left(W_{1}\right) \triangleq \eta_{1}-\mathbf{u}_{A}\left(\omega_{1}\right) \cdot \nabla \bar{\omega}, \quad B_{2}\left(W_{1}\right) \triangleq-\bar{v}_{x} \xi_{1}-\mathbf{u}_{A} \cdot \nabla \bar{\theta}_{x}-\mathbf{u}_{x, A} \cdot \nabla \bar{\theta}, \\
& B_{3}\left(W_{1}\right) \triangleq-\bar{u}_{y} \eta_{1}-\mathbf{u}_{A} \cdot \nabla \bar{\theta}_{y}-\mathbf{u}_{y, A} \cdot \nabla \bar{\theta} . \tag{5.5}
\end{align*}
$$

With the above notations, we can simplify (5.1) as follows

$$
\partial_{t} W_{1, i}+b \cdot \nabla W_{1, i}=d_{i, L}(1) W_{1, i}+B_{i},
$$

where $d_{i}(1)$ acts on constant function 1 and $\mathcal{T}_{d}(1)=0$. The weighted quantity satisfies

$$
\begin{equation*}
\partial_{t}\left(W_{1, i} \rho\right)+b \cdot \nabla\left(W_{1, i} \rho\right)=d_{i, L}(\rho) W_{1, i} \rho+B_{i} \rho . \tag{5.6}
\end{equation*}
$$

We choose the following weights for the weighted $C^{1 / 2}$ estimate

$$
\begin{align*}
\psi_{1} & =p_{11}|x|^{-2}+p_{12}|x|^{-1 / 2}+p_{13}|x|^{-1 / 6} \\
\psi_{i} & =p_{i 1}|x|^{-5 / 2}+p_{i 2}|x|^{-1}+p_{i 3}|x|^{-1 / 2}+p_{i 4}|x|^{1 / 6}, i=2,3 \tag{5.7}
\end{align*}
$$

where $p_{i j}$ are given in (C.1). The above weights can be determined by the analysis of the singular scenario in Section 2.8.2 where we consider the Hölder estimate for any pair $x, z \in \mathbb{R}_{2}^{++}$with $x_{2}=z_{2}$ and $|x-z|$ being sufficiently small.
Remark 5.1. The reader should not confuse the weights $\psi_{i}$ with the notation for the stream function $\phi=(-\Delta)^{-1} \omega$. In this paper, we rarely use the stream function.

From Section 2.8.2 we know that in the scenario when $x_{2}=z_{2}$ and $|x-z|$ is sufficiently small, we have enough damping to obtain the stability estimate. See (2.50) and (A.3) in Lemma A.1 To estimate the more regular case when $|x-z|$ is not small, we need to control the weighted $L^{\infty}$ norm of $\omega_{1}, \eta_{1}, \xi_{1}$. We will follow the ideas in Sections 2.8.3 4.2 to first show that the weighted $L^{\infty}$ estimates with suitable weights are almost close. We then combine the $L^{\infty}$ and $C^{1 / 2}$ estimates to close the stability estimate. We will show that in the more regular case when $|x-z|$ is not small, the damping factor in the Hölder estimate, i.e., $\lambda$ in A.3), is similar or even larger than $c_{1}, c_{2}$ in (2.50). Therefore, from $c_{1}, c_{2}$ in (2.50), we can get a good estimate of the stability factor $\lambda_{*} \approx c_{1}, c_{2}$ in our overall energy estimates based on A.3) and Lemma A. 1

$$
\begin{equation*}
\lambda_{*} \in[0.035,0.08] . \tag{5.8}
\end{equation*}
$$

5.1.1. Guidelines of choosing the Hölder weights. To choose the parameters in the above weights (5.7), we first choose different powers so that we can control the solution in the near-field and the far-field. Then we choose the coefficients $p_{i j}$ such that we can obtain the damping terms from the local parts following the derivations in the weighted Hölder estimates in Section 2.7.1 Next, we use the estimates in Section 2.8.2 and treat the nonlocal terms as bad terms (5.5). We further optimize the coefficients so that we can obtain (2.50) with $c_{1}, c_{2}$ as large as possible. These ideas are similar to those presented in [20], and we refer to [20] for more discussions.

Next, we determine the weights $g_{i}$ in the Hölder seminorm $C_{g_{i}}^{1 / 2}$ (2.21). For $g$ to be determined, we use Lemma 3.17Lemma 3.5 and the triangle inequality

$$
g(x-z)|f(x)-f(z)| \leq g(x-z)\left(\left|f\left(x_{1}, x_{2}\right)-f\left(z_{1}, x_{2}\right)\right|+\left|f\left(z_{1}, x_{2}\right)-f\left(z_{1}, z_{2}\right)\right|\right)
$$

for Hölder estimate of $\nabla \mathbf{u}$. In general, applying triangle inequality in the Hölder estimate leads to a larger constant. For example, if $[f]_{C_{x}^{1 / 2}} \leq A,[f]_{C_{y}^{1 / 2}} \leq A$, a direct estimate yields $[f]_{C^{1 / 2}} \leq \sqrt{2} A$, which has an extra factor $\sqrt{2}$. One way to avoid this overestimate is to choose

$$
\begin{equation*}
g\left(h_{1}, h_{2}\right)=\left(\left|h_{1}\right|^{1 / 2}+c\left|h_{2}\right|^{1 / 2}\right)^{-1} \tag{5.9}
\end{equation*}
$$

for some constant $c$ in the weighted Hölder estimate. In the above example, one can choose $c=1$ and obtain $g(x-z)|f(x)-f(z)| \leq A$ for any $x, z$. However, in the weighted Hölder estimate, the damping factor from the weight $g$

$$
d_{g}=g(x-z)^{-1}(b(x)-b(z)) \cdot(\nabla g)(x-z)
$$

can be unbounded since $\left(\partial_{1} g\right)\left(0, h_{2}\right)=\infty, \partial_{2}\left(h_{1}, 0\right)=\infty$. Alternatively, we modify (5.9) as follows

$$
\begin{equation*}
g_{i}(h)=q_{i 0}(h) g_{i 0}(1,0)^{-1}, \quad g_{i 0}(h)=\left(\sqrt{\left|h_{1}\right|+q_{i 1}\left|h_{2}\right|}+q_{i 3} \sqrt{\left|h_{2}\right|+q_{i 2}\left|h_{1}\right|}\right)^{-1} \tag{5.10}
\end{equation*}
$$

for some small $q_{i 1}, q_{i 2}$. We divide $g_{i 0}(1,0)$ to normalize $g_{i}(1,0)=1$. To exploit the anisotropy of the flow (see Section 2.7.2), we choose $q_{i 3}<1$. The parameters $q_{i j}$ are given in (C.1).

We remark that we still have a larger constant when we estimate $g(x-z)(\nabla \mathbf{u}(x)-\nabla \mathbf{u}(z))$ for general $(x, z)$ than the case $x_{1}=z_{1}$ or $x_{2}=z_{2}$. Yet, since we also gain more damping from the above $d_{g}$ when $\left|x_{2}-z_{2}\right| /\left|x_{1}-z_{1}\right|$ is not too small, we can still show that the damping term dominates other nonlocal terms.

For $\eta$ and $\xi$, we choose $g_{3}(h)=g_{2}(h)$. To determine the parameters $g_{i j}$, we first find $x \in \mathbb{R}_{2}^{++}$ where we have the least damping in the case when $|x-z|$ is sufficiently small with $x_{2}=z_{2}$. That is, we find $x_{*}$ such that the left hand side of (2.50) achieves the maximum at $x_{*}$. Then near $x_{*}$, we perform the Hölder estimates with other ratio $\left|z_{2}-x_{2}\right| /\left|z_{1}-x_{1}\right|$ and keeping $|x-z|$ small. In this case, similar to the analysis in Section 2.8.2, the more regular terms vanish. We choose $q_{i j}$ so that the damping factor is larger than or close to the one in the case of $x_{2}=z_{2}$.
5.2. Ideas of estimating the nonlocal terms. In the energy estimates, we need to perform weighted $L^{\infty}$ and Hölder estimates on the velocity $\mathbf{u}_{A}, \mathbf{u}_{A, x}, \mathbf{u}_{A, y}$ (5.2) given that $\omega_{1} \varphi_{1} \in$ $L^{\infty}, \omega_{1} \psi_{1} \in C^{1 / 2}$ for some weights $\varphi_{1}, \psi_{1}$. For $f=\mathbf{u}_{A},(\nabla \mathbf{u})_{A}$, it can be written as

$$
I(f)(x)=\int_{\mathbb{R}^{2}} K_{f}(x, y) \Omega_{1}(y) d y
$$

for some kernel $K_{f}$, where $\Omega_{1}$ is the odd extension of $\omega_{1}$ in $y$ from $\mathbb{R}_{+}^{2}$ to $\mathbb{R}^{2}$ (3.3). In the case without the approximation terms, the formulas of $\nabla \mathbf{u}$ are given in (3.4). For $f=\mathbf{u}_{A}$, the kernel involves $\nabla^{\perp} \log |y|$ and has a singularity of order $|x|^{-1}$, which is locally integrable. To obtain a sharp weighted estimate of $\mathbf{u}_{A}$ with some singular weight $\rho$, since $\Omega_{1}$ is odd in $y_{1}, y_{2}$, we symmetrize the kernel and then apply the $L^{\infty}$ estimate

$$
\begin{align*}
K_{f}^{s y m}(x, y) & =K_{f}(x, y)-K_{f}\left(x,-y_{1}, y_{2}\right)-K_{f}\left(x, y_{1},-y_{2}\right)+K_{f}(x, y) \\
|\rho(x) I(\mathbf{u})(x)| & =\rho(x)\left|\int_{\mathbb{R}_{++}^{2}} K_{\mathbf{u}}^{s y m} \omega_{1}(y) d y\right| \leq \rho(x) \int_{\mathbb{R}_{++}^{2}}\left|K_{\mathbf{u}}^{s y m}\right| \varphi_{1}^{-1}(y) d y \cdot\left\|\omega_{1} \varphi_{1}\right\|_{L^{\infty}}  \tag{5.11}\\
& \triangleq \rho(x) C(\mathbf{u}, x)\left\|\omega_{1} \varphi_{1}\right\|_{L^{\infty}}
\end{align*}
$$

where $C(\mathbf{u}, x)$ denotes the last integral on the second line. The above estimate is sharp in the sense that for a fixed $x$, the equality can be achieved if $\omega \varphi_{1}(y)=C \operatorname{sgn}\left(K^{s y m}(x, y)\right)$ for some constant $C$. For a given weight $\varphi_{1}$, the constant $C(\mathbf{u}, x)$ is independent of $\omega_{1}$ and is an integral of some explicit function. We can estimate it effectively for all $x$ using the scaling symmetry of the kernel and numerical computation with rigorous error estimates.

For $f=(\nabla \mathbf{u})_{A}$, the kernel has a singularity of order $|x|^{-2}$, which is not integrable near the singularity. We decompose the integral $I(f)$ into the nonsingular part (NS) and the singular part (S) with singular region $R$ centered around $x$ with radius $r(x)$

$$
\begin{align*}
I(f) & =I_{N S}(f)+I_{S}(f), \quad R(x)=\left\{y: \max _{i=1,2}\left|x_{i}-y_{i}\right| \leq r(x)\right\} \\
I_{N S}(f) & =\int_{R^{c}} K_{f}(x, y) \Omega_{1}(y) d y, \quad I_{S}(f)=\int_{R} K_{f}(x, y) \Omega_{1}(y) d y \tag{5.12}
\end{align*}
$$

In the weighted $L^{\infty}$ estimate of $(\nabla \mathbf{u})_{A}$, we use the above idea and $\left\|\omega \varphi_{1}\right\|_{L^{\infty}}$ to estimate $I_{N S}$. For the singular part, we further decompose it using the identity related to the commutator, e.g., (2.40). We apply the above $L^{\infty}$ estimate (5.11) to the regular term. The singular term related to $\nabla \mathbf{u}\left(\omega \psi_{1}\right)$ is estimated using $\left\|\omega \psi_{1}\right\|_{C^{1 / 2}}$. For example, we have the following estimate

$$
\begin{aligned}
\left|\int_{\left|s_{1}\right|,\left|s_{2}\right| \leq \tau} \frac{s_{1} s_{2}}{|s|^{4}}\left(\omega \psi_{1}\right)(x-s) d s\right| & =\left\lvert\, \int_{0 \leq s_{1},\left|s_{2}\right| \leq \tau} \frac{s_{1} s_{2}}{|s|^{4}}\left(\left(\omega \psi_{1}\right)(x-s)-\left(\omega \psi_{1}\right)\left(x_{1}+s_{1}, x_{2}-s_{2}\right) d s \mid\right.\right. \\
& \leq C \tau^{1 / 2}\left[\omega \psi_{1}\right]_{C_{x}^{1 / 2}}
\end{aligned}
$$

where $C$ is some constant related to the kernel and is independent of $\tau$. In short, we can estimate $\rho I(f)$ with some singular weight as follows

$$
\begin{equation*}
|\rho(x) I(f)(x)| \leq C_{1}(x)| | \omega \varphi_{1} \|_{L^{\infty}}+C_{2}(x)\left[\omega \psi_{1}\right]_{C_{x}^{1 / 2}}+C_{3}(x)\left[\omega \psi_{1}\right]_{C_{y}^{1 / 2}} \tag{5.13}
\end{equation*}
$$

for some constant $C_{i}(x)$. For any $\gamma_{j}>0$, we can bound the right hand side using the norm in 2.14 In particular, we can bound it using the energy $E_{1}$ (5.21).

The weighted Hölder estimate is more involved. For $(\nabla \mathbf{u})_{A}$, we again decompose it into the regular part and the singular part. For the singular part, we will use the sharp Hölder estimates in Lemma 3.1.Lemma 3.5. The nonsingular part is locally Lipschitz. We can estimate its Lipschitz norm by computing suitable integrals and using ideas similar to the above. The estimate for $\mathbf{u}_{A}$ is easier since it is more regular. We refer the details to Section 4 of Part II [15].
5.2.1. Scaling symmetry and rescaled integral. In the above computation of the integrals, e.g., (5.11), there are two singularities. Firstly, the weight $\rho(x)$ is singular near 0 , which can amplify the error in the computation of the integral $\int_{\mathbb{R}_{2}^{+}}\left|K_{\mathbf{u}}^{s y m}(x, y) \rho(y)^{-1}\right| d y$ significantly. Secondly, the kernel $K_{\mathbf{u}}(x, y), K_{\nabla \mathbf{u}}$ are singular near $y=x$. If there are only a few $x$, one can design a mesh that is adapted to the singularity $y=x$ and then apply the standard quadrature rule. However, it is very difficult to apply this method to compute the integrals for all $x$. A crucial observation is that the kernel $K(x, y)$ enjoys scaling symmetry, which enables us to restrict the singularity $x$ in a finite domain away from 0 by choosing suitable rescaling.

Denote $f_{\lambda}(x) \triangleq f(\lambda x)$. We consider the kernels about $\nabla u$, which are singular of order 2 and satisfy $K(\lambda x, \lambda y)=\lambda^{-2} K(x, y)$. For $\lambda$ to be chosen, applying a change of variables $y=\lambda \hat{y}, x=\lambda \hat{x}$, we get
$\rho(x) \int_{\mathbb{R}_{2}^{++}} K^{\text {sym }}(x, y) \omega(y) d y=\rho(\lambda \hat{x}) \int_{\mathbb{R}_{2}^{++}} K^{\text {sym }}(\lambda \hat{x}, \lambda \hat{y}) \omega(\lambda \hat{y}) \lambda^{2} d \hat{y}=\rho_{\lambda}(\hat{x}) \int_{\mathbb{R}_{2}^{++}} K^{\text {sym }}(\hat{x}, \hat{y}) \omega_{\lambda}(\hat{y}) d \hat{y}$.
Now, applying the $L^{\infty}$ estimates, we obtain

$$
\left|\rho(x) \int_{\mathbb{R}_{2}^{++}} K^{\text {sym }}(x, y) \omega(y) d y\right| \leq\left\|\omega_{\lambda} \varphi_{1, \lambda}\right\|_{L^{\infty} \rho_{\lambda}(\hat{x})} \int_{\mathbb{R}_{2}^{++}}\left|K^{\text {sym }}(\hat{x}, \hat{y})\right| \varphi_{1, \lambda}(\hat{y})^{-1} d \hat{y}
$$

Note that $\left\|\omega_{\lambda} \varphi_{\lambda}\right\|_{L^{\infty}}=\|\omega \varphi\|_{L^{\infty}}$. Hence, to establish the estimate, it suffices to compute the rescaled integral. The advantage of the above integral compared to the one without rescaling is that the integral is singular at the rescaled point $\hat{x}$, which can be restricted to some finite domain by choosing suitable rescaling parameter $\lambda$. As a result, we can design an adaptive mesh which is dense in the $O(1)$ region to compute the integrals and we do not need to re-mesh in the computation of integrals with different $\hat{x}$. In addition, $\hat{x}$ can be chosen to be away from 0 , e.g. $|\hat{x}| \asymp 1$, so that $\rho_{\lambda}(\hat{x})$ is not singular in $\hat{x}$. For example, we can write $|x|^{-2}=\lambda^{-2}|\hat{x}|^{-2}$ by choosing $\lambda=|x| /|\hat{x}|$ with $|\hat{x}| \asymp 1$. The above rescaling argument enables us to overcome the difficulties caused by the singularities in our computation. We refer more details to Section 4 of Part II 15].
5.3. Weighted $L^{\infty}$ estimate with decaying weights. We first perform weighted $L^{\infty}$ estimate with decaying weights $\varphi_{i}$ below to obtain more damping in the energy estimates. See the weighted $L^{\infty}$ estimate in the model problem in Section 2.7.1 for more motivations. We choose
the following weights

$$
\begin{align*}
\varphi_{1} & =\left(p_{41}|x|^{-2.4}+p_{42}|x|^{-1 / 2}\right)\left|x_{1}\right|^{-1 / 2}+p_{43}|x|^{-1 / 6}, \quad \varphi_{4}=\psi_{1}\left|x_{1}\right|^{-1 / 2} \\
\varphi_{i-3} & =\left(p_{i 1}|x|^{-5 / 2}+p_{i 2}|x|^{-3 / 2}+p_{i 3}|x|^{-1 / 6}\right)\left|x_{1}\right|^{-1 / 2}+p_{i 4}|x|^{-1 / 4}+p_{i 5}|x|^{1 / 7}, i=5,6 \tag{5.14}
\end{align*}
$$

with parameters $p_{i j}$ given in (C.3). We apply $\psi_{1}, \varphi_{1}$ for $\omega, \psi_{2}, \varphi_{2}$ for $\eta$, and $\psi_{3}, \varphi_{3}$ for $\xi$. We will use $\varphi_{4}$ in Section 5.3.4 for an additional weighted $L^{\infty}$ estimate of $\omega_{1}$. We will discuss the ideas of choosing $\varphi_{i}$ in Section 5.3.3.

Using the weights $\varphi_{1}, \psi_{1}$, we can estimate the constants in the weighted estimate of $\mathbf{u}_{A},(\nabla \mathbf{u})_{A}$ $C_{i j, k}$ in (5.11), (5.13) following the ideas in Section 5.2

$$
\begin{equation*}
\left|\rho_{i j} f_{i j}(x)\right| \leq C_{i j, 1}(x)\left\|\omega \varphi_{1}\right\|_{\infty}+C_{i j, 2}(x)\left[\omega \psi_{1}\right]_{C_{x}^{1 / 2}}+C_{i j, 3}(x)\left[\omega \psi_{1}\right]_{C_{y}^{1 / 2}} \tag{5.15}
\end{equation*}
$$

where $f_{01}=u_{A}, f_{10}=v_{A}, f_{11}=u_{x, A}$, etc. We use these indexes since $u=-\partial_{y} \phi, v=\partial_{x} \phi, u_{x}=$ $-\partial_{x} \partial_{y} \phi$ etc, where $\phi$ is the stream function. We add the weight $\rho_{i j}$ to capture the vanishing order near 0 and decays of $\mathbf{u}_{A}, \nabla \mathbf{u}_{A}$

$$
\begin{equation*}
\rho_{10}=\rho_{01}=|x|^{-3}+|x|^{-7 / 6}, \quad \rho_{i j}=\psi_{1}, i+j=2 \tag{5.16}
\end{equation*}
$$

For $(\nabla \mathbf{u})_{A}$, we choose $\rho_{i j}=\psi_{1}, i+j=2$, since we need to estimate $(\nabla \mathbf{u})_{A} \psi_{1}$ using the Hölder norm of $\omega_{1} \psi_{1}$ and $\nabla \mathbf{u}$ and $\omega_{1}$ are of the same order. To control $\mathbf{u}_{A}$, we do not need to use the Hölder seminorm and have

$$
\begin{equation*}
C_{i j, 2}(x)=C_{i j, 3}(x)=0, \quad i+j=1 \tag{5.17}
\end{equation*}
$$

Note that $\varphi_{i}$ contains the singular term $\left|x_{1}\right|^{-1 / 2}$ (5.14) and $\bar{\omega}_{x}, \bar{\theta}_{x x}$ do not vanish on $x_{1}=0$. To bound $u_{A} \bar{\omega}_{x} \varphi_{1}, u_{A} \bar{\theta}_{x x} \varphi_{2}$ in the energy estimate of (5.1), we use the odd symmetry of $u_{A}$ in $x_{1}$ and $u_{A}\left(0, x_{2}\right)=0$ to absorb the singularity. Since $u_{A}$ is 1 order more regular than $\omega$, we can develop estimate $\left|u_{A}\right| \lesssim\left|x_{1}\right| \cdot|\log | x\left|\left\|\left|\mid \omega \varphi_{1} \|_{\infty}\right.\right.\right.$. In particular, we estimate $\rho_{4} u_{A}$ using $\left\|\omega \varphi_{1}\right\|_{\infty}$ with $\rho_{4}(\mathbb{C} .2)$ capturing $\left|x_{1}\right|^{-1 / 2}$. This estimate is covered in Lemma [2.2, and we perform the estimate in Appendix B. 4 in Part II [15. We optimize this estimate and (5.15) for $u_{A}$. As a result, the constant $C_{01}(x)$ bounding $u_{A} \rho_{10}=u_{A} \rho_{4} \cdot \frac{\rho_{10}}{\rho_{4}}$ vanishes along $x_{1}=0$.
5.3.1. Piecewise upper bounds. We discretize a very large domain $[0, D]^{2}$ in $\mathbb{R}_{2}^{++}$using the same mesh $y$ in Section 7 for computing the profiles. Using the method in Section 4 in Part II [15], we can obtain piecewise bounds of $\mathbf{u}_{A},(\nabla \mathbf{u})_{A}$ and $\rho_{10} \mathbf{u}_{A}, \rho_{20}(\nabla \mathbf{u})_{A}$ in each grid $\left[y_{i}, y_{i+1}\right] \times\left[y_{j}, y_{j+1}\right]$. In particular, $C_{i j, 1}, C_{i j, 2}, C_{i j, 3}$ in the upper bound (5.14) are piecewise constants. We track these bounds using $n \times n$ matrices. The estimate in the far-field $x \notin[0, D]^{2}$ is much easier since the coefficients of the nonlocal terms in (5.1) have fast decay and are very small. The same ideas apply to all other estimates.
Operators and functions. To simplify the notations, we introduce some operators and functions. We define

$$
\begin{align*}
& \mathcal{T}_{u}(f)(x)=C_{01,1}\left|f_{x}\right|+C_{10,1}\left|f_{y}\right|, \quad C\left(\mathbf{u}_{x}, i\right) \triangleq C_{11, i}\left|\bar{\theta}_{x}\right|+C_{20, i}\left|\bar{\theta}_{y}\right| \\
& C\left(\mathbf{u}_{y}, i\right) \triangleq C_{02, i}\left|\bar{\theta}_{x}\right|+C_{11, i}\left|\bar{\theta}_{y}\right|, \quad C(f, a) \triangleq C(f, 1) a_{1}+C(f, 2) a_{2}+C(f, 3) a_{3} \tag{5.18}
\end{align*}
$$

for $\mu \in \mathbb{R}^{3}$ and $f=\mathbf{u}_{x}$ or $f=\mathbf{u}_{y}$. We will use $\mathcal{T}_{u}$ for the estimate of $\mathbf{u}_{A} \cdot \nabla f, C\left(\mathbf{u}_{x}, i\right)$ for $\mathbf{u}_{A, x} \cdot \nabla \bar{\theta}$, and $C\left(\mathbf{u}_{y}, i\right)$ for $\mathbf{u}_{A, y} \cdot \nabla \bar{\theta}$. Note that $C(f, \mu)$ is linear in $\mu$.

Following (5.6), we derive the weighted $L^{\infty}$ estimates for $W_{1, i}=\omega_{1}, \eta_{1}, \xi_{1}$ (5.3) in (5.1)

$$
\begin{equation*}
\partial_{t}\left(W_{1, i} \varphi_{i}\right)+\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla\left(W_{1, i} \varphi_{i}\right)=d_{i, L}\left(\varphi_{i}\right) W_{1, i} \varphi_{i}+B_{i}(x) \varphi_{i} \tag{5.19}
\end{equation*}
$$

where we have used the operators (5.4) $d_{i}(\cdot)$ to denote the coefficient of the damping terms, and $B_{i}(x)$ are the bad terms defined in (5.5). We estimate $B_{i}(x)$ directly using the above pointwise
estimates for the nonlocal terms

$$
\begin{align*}
\left|B_{1}(x)\right| \leq & \frac{\varphi_{1}}{\varphi_{2}}\left\|\eta_{1} \varphi_{2}\right\|_{\infty}+\frac{\varphi_{1}}{\rho_{10}} \mathcal{T}_{u}(\bar{\omega})\left\|\omega_{1} \varphi_{1}\right\|_{\infty} \\
\left|B_{2}(x)\right| \leq & \frac{\varphi_{2}}{\varphi_{3}}\left|\bar{v}_{x}\left\|\mid \xi_{1} \varphi_{3}\right\|_{\infty}+\frac{\varphi_{2}}{\rho_{10}} \mathcal{T}_{u}\left(\bar{\theta}_{x}\right)\left\|\omega_{1} \varphi_{1}\right\|_{\infty}\right. \\
& +\frac{\varphi_{2}}{\psi_{1}}\left(C\left(\mathbf{u}_{x}, 1\right)\left\|\omega_{1} \varphi_{1}\right\|_{\infty}+C\left(\mathbf{u}_{x}, 2\right)\left[\omega_{1} \psi_{1}\right]_{C_{x}^{1 / 2}}+C\left(\mathbf{u}_{x}, 3\right)\left[\omega_{1} \psi_{1}\right]_{C_{y}^{1 / 2}}\right)  \tag{5.20}\\
\left|B_{3}(x)\right| \leq & \frac{\varphi_{3}}{\varphi_{2}} \left\lvert\, \bar{u}_{y}\| \| \eta_{1} \varphi_{2}\left\|_{\infty}+\frac{\varphi_{3}}{\rho_{10}} \mathcal{T}_{u}\left(\bar{\theta}_{y}\right)\right\| \omega_{1} \varphi_{1}\right. \|_{\infty} \\
& +\frac{\varphi_{3}}{\psi_{1}}\left(C\left(\mathbf{u}_{y}, 1\right)\left\|\omega_{1} \varphi_{1}\right\|_{\infty}+C\left(\mathbf{u}_{y}, 2\right)\left[\omega_{1} \psi_{1}\right]_{C_{x}^{1 / 2}}+C\left(\mathbf{u}_{y}, 3\right)\left[\omega_{1} \psi_{1}\right]_{C_{y}^{1 / 2}}\right)
\end{align*}
$$

5.3.2. Weights between the $L^{\infty}$ norm and the Hölder norm. We cannot close the $L^{\infty}$ estimate since the estimate of $\nabla \mathbf{u}_{A}$ involves $\left[\omega_{1} \psi_{1}\right]_{C_{x_{i}}^{1 / 2}}$. To close our weighted $L^{\infty}$ and Hölder estimate using Lemma A.1 we need to choose weights $\mu_{i}$ among different norms such that (A.3) holds.

Recall the weighted Hölder seminorm from (2.21). We introduce the first energy

$$
\begin{equation*}
E_{1}(t)=\max \left(\max _{i}\left\|W_{1, i} \varphi_{i}\right\|_{\infty}, \tau_{1}^{-1} \max \left(\left[\omega_{1} \psi_{1}\right]_{C_{g_{1}}^{1 / 2}}, \sqrt{2}\left\|\omega \varphi_{4}\right\|_{\infty}\right)\right), \varphi_{4}=\psi_{1}\left|x_{1}\right|^{-1 / 2} \tag{5.21}
\end{equation*}
$$

Note that the Hölder seminorm $[\cdot]_{C_{g_{1}^{1 / 2}}}$ is only defined in $\mathbb{R}_{2}^{++}$. We add the extra $L^{\infty}$ norm $\left\|\omega \psi_{1}\left|x_{1}\right|^{-1 / 2}\right\|_{\infty}$ to control $[\omega]_{C_{x}^{1 / 2}\left(\mathbb{R}_{2}^{+}\right)}$. See more discussions in Section 5.3.3. Since $g_{1}\left(h_{1}, 0\right)=$ $\left|h_{1}\right|^{-\frac{1}{2}}, g_{1}\left(0, h_{2}\right)=g_{1}(0,1)\left|h_{2}\right|^{-\frac{1}{2}}$ (5.10), using (5.25) and the estimate in Section5.3.3, we obtain $E_{1}(t) \geq \tau_{1}^{-1} \max \left(\left[\omega_{1} \psi_{1}\right]_{C_{g_{1}}^{1 / 2}}, \sqrt{2}\left\|\omega \psi_{1}\left|x_{1}\right|^{-\frac{1}{2}}\right\|_{L^{\infty}}\right) \geq \tau_{1}^{-1}\left[\omega_{1} \psi_{1}\right]_{C_{x}^{1 / 2}\left(\mathbb{R}_{2}^{+}\right)}, \tau_{1}^{-1} g_{1}(0,1)\left[\omega_{1} \psi_{1}\right]_{C_{y}^{1 / 2}}$.

Using $E_{1}(t)$ and the notation (5.18), we can simplify the estimate (5.15) for $\nabla \mathbf{u}_{A}$ as follows (5.22)

$$
\left|\rho_{i j} f_{i j}\right| \leq\left(C\left(f_{i j}, 1\right)+C\left(f_{i j}, 2\right) \tau_{1}+C\left(f_{i j}, 3\right) \tau_{1} g_{1}(0,1)^{-1}\right) E_{1}=E_{1} \cdot C\left(f_{i j},\left(1, \tau_{1}, \tau_{1} g_{1}(0,1)^{-1}\right)\right)
$$

The above estimate relates to (2.14) in Lemma 2.2. Similarly, the bound in $B_{i}$ can be simplified as follows

$$
\left.C(f, 2)\left[\omega_{1} \psi_{1}\right]_{C_{x}^{1 / 2}}+C(f, 3)\left[\omega_{1} \psi_{1}\right]_{C_{y}^{1 / 2}}\right) \leq \tau_{1} C\left(f,\left(0,1, g_{1}(0,1)^{-1}\right)\right) E_{1}(t), f=\mathbf{u}_{x}, \mathbf{u}_{y}
$$

where $C(f, \mu)$ is defined in (5.18). The constraint (A.3) for the $\eta$ equation becomes

$$
\begin{equation*}
\left(-d_{2, L}\left(\varphi_{2}\right)-\frac{\varphi_{2}}{\varphi_{3}}\left|\bar{v}_{x}\right|-\frac{\varphi_{2}}{\rho_{10}} \mathcal{T}_{u}\left(\bar{\theta}_{x}\right)-\frac{\varphi_{2}}{\psi_{1}} C\left(\mathbf{u}_{x}, 1\right)\right)-\tau_{1} \frac{\varphi_{2}}{\psi_{1}} C\left(\mathbf{u}_{x},\left(0,1, g_{1}(0,1)^{-1}\right)\right) \geq \lambda \tag{5.23}
\end{equation*}
$$

Similarly, we have another constraint for $\tau_{1}$ from the estimate of $\xi_{1}$. We want to obtain an overall stability factor $\lambda_{*}(5.8)$ and thus choose $\lambda \approx \bar{\lambda}_{*}$. We choose the largest $\tau_{1}$ such that the inequality (5.23) and a similar inequality for $\xi_{1}$ hold. The idea to choose large $\tau_{1}$ (or small $\tau_{1}^{-1}$ ) is similar to that in (2.57) for the model problem, where the weight $\tau$ for the Hölder norm is small. We choose the largest $\tau_{1}$ so that in the Hölder estimate for $\tau_{1}^{-1} \omega_{1} \psi_{1}$, we have the small factor $\tau_{1}^{-1}$ associated with the weighted $L^{\infty} \operatorname{norm}_{\max _{i}\left\|W_{1, i} \varphi_{i}\right\|_{\infty} \text { in A.3). In our estimate, }}$ ( we can choose

$$
\begin{equation*}
\tau_{1}=5 \tag{5.24}
\end{equation*}
$$

Although $\tau_{1}$ is not very large, it is enough for us to show that the estimate of the more regular case in the Hölder estimate, i.e. $|x-z|$ is not very small, is similar to or even better than that in the singular case when $|x-z|$ is small. There are three reasons. Firstly, we get the above small factor $\tau_{1}^{-1}$ when we estimate the more regular terms using the weighted $L^{\infty}$ norm. Secondly, as $|x-z|$ increases, due to our localized estimates in Lemmas 3.1] 3.4, the constants in the estimates of the nonlocal terms decrease. Thirdly, for $|x-z|$ not too small, the Hölder estimate of nonlocal terms $(\nabla \mathbf{u})_{A}$ using (5.22) and triangle inequality can provide estimates better than Lemmas 3.1 3.4. Note that from (5.23), choosing a larger $\tau_{1}$ requires better estimates on the nonlocal terms, e.g., smaller $C\left(u_{x}, i\right)$. For this reason, we need to approximate the nonlocal terms $\mathbf{u}$ with finite rank operators with a higher rank, which increases the computation cost. Due to this consideration, we choose a moderate $\tau_{1}$.

In Figure 6 we plot the rigorous piecewise lower bounds of the damping terms, e.g. $-d_{2}\left(\varphi_{2}\right)$ (5.23), the estimates of the bad terms, i.e. the sum of the terms with negative sign (5.23), and the rigorous piecewise lower bound of the remaining damping factors (the left hand side of (5.23)) along the boundary. The $\xi_{1}$ variable enjoys a much better estimates near the boundary, so we do not plot it.

We choose the approximation terms $\hat{\mathbf{u}}, \widehat{\nabla \mathbf{u}}$ for $\mathbf{u}, \nabla \mathbf{u}$ along the boundary in Section 4.3 such that the weighted estimates of $\mathbf{u}-\hat{\mathbf{u}}, \nabla \mathbf{u}-\widehat{\nabla \mathbf{u}}$ are small. Near the center of the approximation terms, $x_{i}$ in (4.29), we have better estimates of the bad terms. In Figure 6, the points $x_{i}$ are near the local minimum of the blue dashed curve. Since the coefficients of the nonlocal terms (5.1), e.g., $\bar{\omega}_{x}, \bar{\theta}_{x}$, decay for large $x$, in addition to the approximations near boundary (4.29), we only construct 7 approximation terms (4.37), (4.28) in a much larger domain $[0,200]^{2}$, and we do not need to construct approximations for large $x$.

We choose $\varphi_{1}$ slightly weaker than $\varphi_{2}$ near the origin (5.14) such that $\varphi_{1} / \varphi_{2}\left\|\eta_{1} \varphi_{2}\right\|_{\infty}$ is small, and we can obtain large stability factors for both $\omega$ and $\eta$, which are larger than 0.7 . This allows us to control a larger weighted residual error near the origin.

In Figure 7, we plot rigorous piecewise lower bounds of the stability factors, e.g., the left hand side of (5.23), in $L^{\infty}\left(\varphi_{i}\right)$ estimates of $\omega_{1}, \eta_{1}$ in the near-field. Due to the anisotropy of the flow, the damping terms and the stability factors are larger if the angle $y / x$ is large. See Section 2.7.2. These plots help us visualize the estimates.

To justify the inequalities (5.23), we follow the methods in 1920 and derive piecewise bounds of different functions based on the estimates of the approximate steady state and the weights in Appendix C and Appendix A. 1 of Part II [15].


Figure 6. Weighted $L^{\infty}$ estimates with slowly decay weights. Left figure: estimates near $0, x \in[0,1.8]$; Right figure: estimates in a larger domain, $x \in$ $[0,35]$. The red curves shows the coefficient of the damping term $-d_{1}\left(\varphi_{1}\right)$, and the estimate of $B_{1}$; the blue curves are for $-d_{2}\left(\varphi_{2}\right), B_{2}$ in the estimates of $\eta_{1}$. The green and the black curves are the stability factors in the estimate of $\omega_{1}$ and $\eta_{1}$.
5.3.3. Order of choosing the parameters. We have discussed how to choose the Hölder estimate in Section 5.1.1. For $\varphi_{i}$, we first choose the weight $\varphi_{1}$ for $\omega_{1}$ consisting of different powers to take into account the vanishing order of $\omega_{1}$ near 0 and its decay in the far field. We add the power $\left|x_{1}\right|^{-1 / 2}$ in $\varphi_{1}, \varphi_{2}, \varphi_{3}$ (5.14) since we need to control $\left\|\omega_{1}\left|x_{1}\right|^{-1 / 2} \psi_{1}\right\|_{\infty}$ for the Hölder estimate. See Section 5.3.4. In the $L^{\infty}$ estimate of $\omega_{1}\left|x_{1}\right|^{-1 / 2} \psi_{1}$, we need to control $\eta_{1}\left|x_{1}\right|^{-1 / 2} \psi_{1}$ and other weighted quantities with weights singular at $x_{1}=0$. Thus, we add the weight $\left|x_{1}\right|^{-1 / 2}$ in $\varphi_{i}$. We adjust the parameters in $\varphi_{1}$ so that we have a good damping factor $d_{1}(x)$ from the local term for $\omega_{1}$. Then we can estimate the nonlocal terms and the constants (5.15). Once we obtain the estimates for $\nabla \mathbf{u}_{A}, \mathbf{u}_{A}$, we choose the exponents of different powers in $\varphi_{2}$ and adjust the parameters so that we have better stability factors in the weighted $L^{\infty}$ estimate and choose a larger $\tau_{1}$ (5.23). Since the equations of $\xi_{1}$ and $\eta_{1}$ are similar, we choose the same combination


Figure 7. Weighted $L^{\infty}$ estimates in the near-field. Left: stability factors in the estimate of $\omega_{1}$. Right: stability factor in the estimate of $\eta_{1}$.
of powers in $\varphi_{2}$ and $\varphi_{3}$ (5.14). Moreover, since $\xi_{1}$ is weakly coupled with $\omega_{1}$ and $\eta_{1}$ (see Section 2.6.2) and enjoys much better stability estimate $\left(2 \bar{c}_{\omega}+\bar{u}_{x} \approx-5.5\right.$ near $x=0$ in (5.1)), we determine the parameters in $\varphi_{3}$ after we obtain $\varphi_{1}, \varphi_{2}$.
5.3.4. Weighted $L^{\infty}$ estimate related to the Hölder norm. To simplify our energy estimate, using the symmetry of $W_{1, i}$ in the $x$, we will only perform Hölder estimates in $\mathbb{R}_{++}^{2}$, which control

$$
I_{x}=\frac{\left(W_{1, i} \psi_{i}\right)\left(x_{1}, x_{2}\right)-\left(W_{1, i} \psi_{i}\right)\left(z_{1}, x_{2}\right)}{\left|x_{1}-z_{1}\right|^{1 / 2}}, \quad I_{y}=\frac{\left(W_{1, i} \psi_{i}\right)\left(x_{1}, x_{2}\right)-\left(W_{1, i} \psi_{i}\right)\left(x_{1}, z_{2}\right)}{\left|x_{2}-z_{2}\right|^{1 / 2}}
$$

for $\left(x_{1}, x_{2}\right),\left(z_{1}, x_{2}\right),\left(x_{1}, z_{2}\right) \in \mathbb{R}_{++}^{2}$. Due to the symmetry in the $x$ direction, we have $\left|I_{y}\left(x_{1}, x_{2}, z_{2}\right)\right|=$ $\left|I_{y}\left(-x_{1}, x_{2}, z_{2}\right)\right|$ and can control $[\omega]_{C_{y}^{1 / 2}\left(\mathbb{R}_{+}^{2}\right)}$. To control the weighted Hölder norm of $\nabla \mathbf{u}\left(\omega_{1}\right)$ in $\mathbb{R}_{++}^{2}$, we need to control $\left[\omega_{1} \psi_{1}\right]_{C_{x_{i}}^{1 / 2}\left(\mathbb{R}_{+}^{2}\right)}$ since $\nabla \mathbf{u}$ is nonlocal. Yet, the above estimate does not directly control $I_{x}\left(x_{1}, x_{2}, z_{1}\right)$ with $x_{1} z_{1}<0$ or $\left[\omega_{1} \psi_{1}\right]_{C_{x}^{1 / 2}\left(\mathbb{R}_{+}^{2}\right)}$. In fact, it is easy to obtain that

$$
[f]_{C_{x}^{1 / 2}\left(\mathbb{R}_{+}^{2}\right)} \leq \sqrt{2}[f]_{C_{x}^{1 / 2}\left(R_{++}^{2}\right)}
$$

for an odd function $f$, which leads to an extra factor $\sqrt{2}$. Instead, to further control $I_{x}(\omega)$ with $x_{1}<0<z_{1}$, since $F \triangleq \omega_{1} \psi_{1}$ is odd and $\left|x_{1}-z_{1}\right|=\left|x_{1}\right|+\left|z_{1}\right|$, we have

$$
\begin{align*}
\frac{\left|F\left(-x_{1}, x_{2}\right)+F\left(z_{1}, x_{2}\right)\right|}{\left|x_{1}-z_{1}\right|^{1 / 2}} & \leq \max \left(\frac{2\left|F\left(-x_{1}, x_{2}\right)\right|}{\left|2 x_{1}\right|^{1 / 2}}, \frac{2\left|F\left(z_{1}, x_{2}\right)\right|}{\left|2 z_{1}\right|^{1 / 2}}\right) \frac{\left|x_{1}\right|^{1 / 2}+\left|z_{1}\right|^{1 / 2}}{\left(2\left|x_{1}\right|+2\left|z_{1}\right|\right)^{1 / 2}}  \tag{5.25}\\
& \leq \max \left(\frac{2\left|F\left(-x_{1}, x_{2}\right)\right|}{\left|2 x_{1}\right|^{1 / 2}}, \frac{2\left|F\left(z_{1}, x_{2}\right)\right|}{\left|2 z_{1}\right|^{1 / 2}}\right)
\end{align*}
$$

where we have used the Cauchy-Schwarz inequality in the last inequality. Therefore, it suffices to control $\left\|\omega \psi_{1}\left|x_{1}\right|^{-1 / 2}\right\|_{L^{\infty}}$.

In view of (5.25) and $[\omega]_{C_{g_{1}}^{1 / 2}} \geq[\omega]_{C_{x}^{1 / 2}\left(\mathbb{R}_{++}^{2}\right)}$ (5.21), we include the norm $\tau_{1}^{-1} \sqrt{2}\left\|\omega \varphi_{4}\right\|_{\infty}$ with a specific weight $\tau_{1}^{-1} \sqrt{2}$ in $E_{1}(5.21)$ so that $E_{1}(t) \geq \tau_{1}^{-1}[\omega]_{C_{x}^{1 / 2}\left(\mathbb{R}_{2}^{+}\right)}$. We perform $L^{\infty}\left(\varphi_{4}\right)$ estimate of $\omega_{1}$ using the estimates of nonlocal terms and derivations in (5.20) and Section 5.3

$$
\begin{aligned}
& \partial_{t}\left(\omega_{1} \varphi_{4}\right)+\left(\bar{c}_{l} x+\overline{\mathbf{u}}+\mathbf{u}\right) \cdot \nabla\left(\omega_{1} \varphi_{4}\right)=-d_{4, L}(x)\left(\omega_{1} \varphi_{4}\right)+B_{1}(x) \varphi_{4} \\
& d_{4, L}(x) \triangleq d_{1, L}\left(\varphi_{4}\right)=\mathcal{T}_{d}\left(\varphi_{4}\right)+\bar{c}_{\omega}, \quad\left|B_{1}(x) \varphi_{4}\right| \leq \frac{\varphi_{4}}{\varphi_{2}}\left\|\eta_{1} \varphi_{2}\right\|_{\infty}+\frac{\varphi_{4}}{\rho_{10}} \mathcal{T}_{u}(\bar{\omega})\left\|\omega_{1} \varphi_{1}\right\|_{\infty}
\end{aligned}
$$

where the operator $\mathcal{T}_{d}, d_{1, L}(\cdot)$ is defined in (5.4), and $\mathcal{T}_{\mathbf{u}}(\bar{\omega})$ is defined in (5.18). The condition (A.3) for $\left\|\omega \varphi_{4}\right\|_{\infty}$ with weight $\tau_{1}^{-1} \sqrt{2}$ becomes

$$
\begin{equation*}
-d_{4, L}-\frac{\sqrt{2}}{\tau_{1}}\left(\frac{\varphi_{4}}{\varphi_{2}}+\frac{\varphi_{4}}{\rho_{10}} \mathcal{T}_{u}(\bar{\omega})\right) \geq \lambda \tag{5.26}
\end{equation*}
$$

for some $\lambda>0$. From (C.1), (C.3), we have $\varphi_{4} / \varphi_{2}=\left|x_{1}\right|^{-1 / 2} \psi_{1} / \varphi_{2} \lesssim 1$. Here, we have a much larger damping factor compared to that in the weighted $L^{\infty}\left(\varphi_{i}\right)$ estimate, e.g. (5.23), since we have a smaller parameter $\frac{\sqrt{2}}{\tau_{1}}$ for the bad term.
5.4. Weighted Hölder estimates. Recall the weights $\psi_{i}$ for $\omega_{1}, \eta_{1}, \xi_{1}$ (C.1) and $g_{i}(h)$ (5.10) in the weighted $C^{1 / 2}$ estimates, the notation $W_{1, i}$ (5.3), and the simplified equation (5.6). The goal of the weighted Hölder estimate is to control $\left\|W_{1, i} \psi_{i}\right\|_{C_{g_{i}}^{1 / 2}}$, where $[\cdot]_{C_{g_{i}}^{1 / 2}}$ is defined in (2.21), which along with the weighted $L^{\infty}$ estimate, we can control the second energy

$$
\begin{align*}
E_{2}(t) & \triangleq \max \left(E_{1}(t), \tau_{1}^{-1} \max \left(\left[W_{1,1} \psi_{1}\right]_{C_{g_{1}}^{1 / 2}}, \mu_{1}\left[W_{1,2} \psi_{2}\right]_{C_{g_{2}}^{1 / 2}}, \mu_{2}\left[W_{1,3} \psi_{3}\right]_{C_{g_{3}}^{1 / 2}}\right)\right.  \tag{5.27}\\
\mu_{h} & =\tau_{1}^{-1}\left(1, \mu_{1}, \mu_{2}\right)
\end{align*}
$$

for the weights $\mu_{1}, \mu_{2}$ determined by analyzing the most singular scenario in Section 2.8.2 (2.48). They are given in (C.5). The energy $E_{1}$ is defined in (5.21). In fact, these two factors can be absorbed in the definition of $\psi_{2}, \psi_{3}$. We have normalized the coefficient of the most singular power in $\psi_{1}$ to be 1 .

Following the derivations in the weighted Hölder estimates in Section 2.7 .1 and using (5.6) and Lemma 2.5, we derive the following for $W_{1, i} \psi_{i}$ and any $x, z \in \mathbb{R}_{++}^{2}$

$$
\begin{array}{r}
\partial_{t} H_{i}+\left(b(x) \cdot \nabla_{x}+b(z) \cdot \nabla_{z}\right) H_{i}=\left(\left(d_{i}\left(\psi_{i}\right) W_{1, i} \psi_{i}\right)(x)-\left(d_{i, L}\left(\psi_{i}\right) W_{1, i} \psi_{i}\right)(z)\right) g_{i}(x-z) \\
+d_{g, i} H_{i}+\left(\left(B_{i} \psi_{i}\right)(x)-\left(B_{i} \psi_{i}\right)(z)\right) g_{i}(x-z) \triangleq I_{1}+I_{2}+I_{3} \triangleq R_{i} \tag{5.28}
\end{array}
$$

where $b(x)$ is the coefficient of the advection (5.4), $d_{g_{i}}$ is the damping factor from $g_{i}$ in the Hölder estimate, and $J_{i}, H_{i}$ are given below

$$
\begin{equation*}
J_{i} \triangleq W_{1, i} \psi_{i}, \quad H_{i}(x, z)=\left(J_{i}(x)-J_{i}(z)\right) g_{i}(x-z), d_{g, i} \triangleq \frac{(b(x)-b(z)) \cdot\left(\nabla g_{i}\right)(x-z)}{g_{i}(x-z)} \tag{5.29}
\end{equation*}
$$

The factor $B_{i}$ is the bad term defined in (5.5), and $d_{i, L}$ is defined in (5.4)

$$
\begin{equation*}
d_{1, L}\left(\psi_{1}\right)=\mathcal{T}_{d}\left(\psi_{1}\right)+\bar{c}_{\omega}, d_{2, L}\left(\psi_{2}\right)=\mathcal{T}_{d}\left(\psi_{2}\right)+2 \bar{c}_{\omega}-\bar{u}_{x}, d_{3, L}\left(\psi_{3}\right)=\mathcal{T}_{d}\left(\psi_{3}\right)+2 \bar{c}_{\omega}+\bar{u}_{x} \tag{5.30}
\end{equation*}
$$

We note that the second term $I_{2}$ in (5.28) is already a damping term. See Section 2.7.1] and discussion below Lemma 2.5. To further simplify the notation, we introduce

$$
\begin{equation*}
a_{i}(x)=d_{i, L}\left(\psi_{i}\right)(x) \tag{5.31}
\end{equation*}
$$

5.4.1. Basic Hölder estimates. For the Hölder estimate of a variable $|f(x)-f(z)| g(x-z)$, we will mostly use its $C_{x}^{1 / 2}, C_{y}^{1 / 2}$ estimates and then apply the triangle inequality. We discuss some basic estimates. We use the following notations

$$
\begin{equation*}
\delta_{i}(f, x, z) \triangleq \frac{|f(x)-f(z)|}{|x-z|^{1 / 2}}, \quad z_{i}>x_{i}, z_{3-i}=x_{3-i}, \quad \delta(f)(x, z) \triangleq f(x)-f(z) \tag{5.32}
\end{equation*}
$$

By abusing the notations of $\delta_{i}$, we denote by $\delta_{i}(f, g, x, z)$ a basic $C^{1 / 2}$ estimate for product

$$
\begin{equation*}
\delta_{i}(f, g, x, z)=\min _{(a, b)=(x, z),(z, x)} \delta_{i}(f, x, z)|g(a)|+\delta_{i}(g, x, z)|f(b)|, \quad x_{3-i}=z_{3-i} \tag{5.33}
\end{equation*}
$$

If $g=1$, we get $\delta_{i}(f, g, x, z)=\delta_{i}(f, x, z)$. Using the triangle inequality, we get

$$
\begin{align*}
& \delta(f g)(x, z)=\delta(f)(x, z) g(a)+\delta(g)(x, z) f(b), \quad(a, b)=(x, z), \quad(z, x) \\
& \delta_{i}(f g, x, z) \leq \delta_{i}(f, g, x, z), \quad x_{3-i}=z_{3-i} \tag{5.34}
\end{align*}
$$

Given the piecewise $C_{x}^{1 / 2}, C_{y}^{1 / 2}$ estimates of $f$, we use the following method for the piecewise Hölder estimate of $f$ with Hölder weight $g$ and two points $(x, z), h=z-x$

$$
\begin{align*}
& |(f(z)-f(x)) \rho(h)|=|(f(z)-f(w)+f(w)-f(x)) \rho(h)| \\
\leq & \left(\delta_{1}(f, w, x)\left|h_{1}\right|^{1 / 2}+\delta_{2}(f, z, w)\left|h_{2}\right|^{1 / 2}\right) \rho(h), w=\left(z_{1}, x_{2}\right), \rho(h)=g(h), \text { or } \rho \equiv 1 \tag{5.35}
\end{align*}
$$

The function $\left|h_{i}\right|^{1 / 2} g(h)$ is 0-homogeneous and we apply the method in Section D. 2 to estimate it. For $\tilde{w}=\left(x_{1}, z_{2}\right)$, we derive another estimate. We optimize two estimates for $|\delta(f)(x, z)| g(x-$


Figure 8. Left, right figures correspond to the locations of $(x, z)$ in cases (1) $s \geq 0$, (2) $s<0, s=\left(z_{1}-x_{1}\right)\left(z_{2}-x_{2}\right)$. The red line and blue line represent two choices of the $C_{x}^{k}, C_{y}^{k}, k=\frac{1}{2}$ or 1 estimates used to estimate $f(x)-f(z)$.
$z)$. In Figure 8, we illustrate the locations of $x, z$ and the $C_{x_{i}}^{1 / 2}\left(\left(\delta_{i}(f, p, q)\right)\right)$ estimates and the triangle inequality used to estimate $\delta(f, x, z) g(h)$. We introduce $\delta_{\square}$ to denote this estimate and similar estimate for product

$$
\begin{align*}
\delta_{\square}(f, g, x, z, s) \triangleq & \min \left(\delta_{1}\left(f, g, x,\left(z_{1}, x_{2}\right)\right)\left|s_{1}\right|^{1 / 2}+\delta_{2}\left(f, g,\left(z_{1}, x_{2}\right), z\right)\left|s_{2}\right|^{1 / 2},\right. \\
& \left.\delta_{2}\left(f, g, x,\left(x_{1}, z_{2}\right)\right)\left|s_{2}\right|^{1 / 2}+\delta_{1}\left(f, g,\left(x_{1}, z_{2}\right), z\right)\left|s_{1}\right|^{1 / 2}\right),  \tag{5.36}\\
\delta_{\square}(f, x, z, s) \triangleq & \delta_{\square}(f, 1, x, z, s),
\end{align*}
$$

where $\delta_{i}(f, g, x, z)$ is defined in (5.33), and we simplify the notation if $g=1$. We use the notation $\square$ since it mimics Figure 8 and indicates that we use $C_{x}^{1 / 2}, C_{y}^{1 / 2}$ estimates to obtain the $C^{1 / 2}$ estimate. By definition, we get

$$
\begin{equation*}
|\delta(f)(x, z)| \leq \delta_{\square}(f, x, z, h), \quad \delta(f g)(x, z) \leq \delta_{\square}(f, g, x, z, h) \tag{5.37}
\end{equation*}
$$

Note that $\delta_{i}(f, x, z), \delta_{i}(f, g, x, z), \delta_{\square}(f, x, z, s), \delta_{\square}(f, g, x, z, s)$ are symmetric in $x, z$. We introduce an extra variable $s$ to reduce bounding $\delta_{\square}(f, x, z, x-z) g(x-z)$ to estimating $\delta_{i}(f, x, z)$ and $\left|x_{j}-z_{j}\right|^{1 / 2} g_{i}(x-z)$ separately, and $\delta_{\square}(f, x, z, s) g_{i}(s)$ is 0 -homogeneous in $s$. See Section 5.4.5. We drop the dependence of $\delta, \delta_{\square}, \delta_{i}$ on $x, z$ when there is no confusion. Using the above basic estimates, we only need to estimate its $C_{x_{i}}^{1 / 2}$ seminorm for most terms.
5.4.2. Estimate the explicit coefficients. In the Hölder estimates, we need to estimate $(\bar{p} q)(x)-$ $(\bar{p} q)(z)) g(x-z)$ for some coefficient $\bar{p}$, perturbation $q$, e.g. $q=\omega_{1}, \eta_{1}$, and some weight $g$, e.g. $g=g_{i}$. The coefficient $\bar{p}$ depends on the weights $\psi_{i}, \varphi_{i}$ and the approximate steady state only. In particular, $\bar{p}$ is quite smooth in a local region. Note that the approximate steady state, the singular weights and their derivatives can be estimated effectively using the method in Appendix C and Appendix A. 1 of Part II [15]. We estimate the piecewise $C_{x}^{1 / 2}$ and $C_{y}^{1 / 2}$ seminorms of $\bar{p}(x)$ using the method in Appendix E.6, E. 7 of Part II 15, and then use (5.36), (5.37) to estimate $g(x-z) \delta(\bar{p})$. For example, given $x, z$, we have

$$
|\bar{p}(x)-\bar{p}(z)| \leq \delta_{i}(\bar{p}, x, z)\left|x_{i}-z_{i}\right|^{1 / 2}=A_{i}(x, z)\left|x_{i}-z_{i}\right|^{1 / 2}, \quad x_{3-i}=z_{3-i}
$$

for some constants $A_{i}$ depending on the weights and the approximate steady state. We discretize the domain $\mathbb{R}_{++}^{2}$ using the same mesh $y_{0}<y_{1}<\ldots, y_{n}$ in our computation for the profile in Section 7 and estimate these constants for $x \in Q_{1}, z \in Q_{2}$ for different grids uniformly. Therefore, we can track the piecewise bounds $A_{1}\left(x_{1}, x_{2}, z_{1}, x_{2}\right)\left(z_{2}=x_{2}\right)$ for $x_{1}, x_{2}, z_{1}$ in each cube $I_{i} \times I_{j} \times I_{i+k}, I_{i}=\left[y_{i}, y_{i+1}\right], 0 \leq k \leq m-1$ using $n \times n \times m$ matrices. We have another estimate in (5.36), (5.37) by choosing another path from $x$ to $z$ and we optimize two estimates. We restrict $z_{1}$ within $m$ grids from $x_{1}$ since for $z_{1}$ far-apart from $x_{1},\left|z_{1}-x_{1}\right|$ is not small, we can apply the triangle inequality to obtain the piecewise Hölder estimate.

In general, such an estimate has some overestimates. Yet, since the problem is anisotropic in the $x$ and $y$ directions, in the worst case scenario where $\left|x_{2}-z_{2}\right|$ is much smaller than $\left|x_{1}-z_{1}\right|$, this simple estimate is effective. See also Section 5.1.1

Although the weights $\psi_{i}, \varphi_{i}$ are singular near $x=0$, from the estimates in the most singular scenario in Section 2.8.2 (see Figure 3), we have better estimates near $x=0$. Thus, the more
challenging part of our estimates comes from the region where $x$ is away from 0 , e.g. $x$ around 0.5. In such a case, we can simply treat the weights $\psi_{i}, \varphi_{i}$ as smooth functions.

Now, using (5.32), we obtain

$$
\begin{equation*}
P \triangleq \delta(\bar{p} q) g(x-z)=(\bar{p}(x) \delta(q)+\delta(\bar{p}) q(z)) g(x-z) \triangleq P_{1}+P_{2} \tag{5.38}
\end{equation*}
$$

The second term is more regular. We can use the weighted $L^{\infty}$ norm of $q$ to control it. For the first term, we bound it using the weighted Hölder seminorm. Below, we discuss different cases. In all cases, the estimate of $P_{2}$ is much smaller than that of $P_{1}$ when $|x-z|$ is small. Moreover, we have another decomposition in (5.34). We optimize these two estimates using (5.36), (5.37).
5.4.3. Estimate of $I_{1}$. Recall $I_{1}$ from (5.28). Note that $H_{i}=\delta\left(J_{i}\right) g_{i}(x-z)$ is the energy we want to control. We have

$$
\begin{align*}
I_{1} & =\delta\left(a_{i} J_{i}\right) g_{i}(x-z)=\left(a_{i}(x) \delta\left(J_{i}\right)+\delta\left(a_{i}\right) J_{i}(z)\right) g_{i}(x-z) \\
& =a_{i}(x) H_{i}+\delta\left(a_{i}\right) g_{i}(x-z) J_{i}(z) \triangleq I_{11}+I_{12} \tag{5.39}
\end{align*}
$$

The first term is a damping term. We can control $J_{i}(x)$ using the weighted $L^{\infty}$ norm in the energy $E_{1}$ (5.21)

$$
\left|J_{i}(z)\right|=\left|\left(W_{1, i} \psi_{i}\right)(z)\right| \leq\left\|W_{1, i} \varphi_{i}\right\|_{\infty} \frac{\psi_{i}(z)}{\varphi_{i}(z)} \leq E_{1} \frac{\psi_{i}(z)}{\varphi_{i}(z)}
$$

Since $a_{i}$ is a given function with an explicit expression, we follow Section 5.4 .2 and estimate $\delta\left(a_{i}\right) g_{i}(x-z)$ using the method in Appendix E of Part II [15] and (5.36), (5.37). In particular, when $|x-z|$ is small, $\delta\left(a_{i}\right) g_{i}(x-z)$ is very small. It follows

$$
\begin{equation*}
\left|I_{12}\right| \leq\left|\delta\left(a_{i}\right) g_{i}(x-z)\right| \frac{\psi_{i}(z)}{\varphi_{i}(z)} E_{1} \tag{5.40}
\end{equation*}
$$

Similarly, we can also define $\tilde{I}_{11}=a_{i}(z) H_{i}$ and $\tilde{I}_{12}=\delta\left(a_{i}\right) g_{i}(x-z) J_{i}(x)$ and obtain

$$
\begin{equation*}
I_{1}=\tilde{I}_{11}+\tilde{I}_{12}, \quad \tilde{I}_{11}=a_{i}(z) H_{i}, \quad\left|\tilde{I}_{12}\right| \leq\left|\delta\left(a_{i}\right) g_{i}(x-z)\right| \frac{\psi_{i}(x)}{\varphi_{i}(x)} E_{1} \tag{5.41}
\end{equation*}
$$

We choose one of the above estimates according to the relative size of the following terms

$$
\begin{equation*}
m_{1}=a_{i}(x)+\mu_{h, i}^{-1}\left|\delta\left(a_{i}\right) g_{i}(x-z)\right| \frac{\psi_{i}(z)}{\varphi_{i}(z)}, \quad m_{2}=a_{i}(z)+\mu_{h, i}^{-1}\left|\delta\left(a_{i}\right) g_{i}(x-z)\right| \frac{\psi_{i}(x)}{\varphi_{i}(x)} \tag{5.42}
\end{equation*}
$$

where $\mu_{h}$ is the weight of the Hölder seminorm in (5.27). We use the decomposition (5.39) and its estimates if $m_{1}$ is smaller. We choose (5.41) if $m_{2}$ is smaller. We use this optimization to maximize the left hand side of (A.3) (the sign is different) and obtain a better stability factor, since the estimate of $I_{1}$ contributes exactly $-\min \left(m_{1}, m_{2}\right)$ to the left hand side of (A.3). We will use similar optimizations several times to get better stability factors, see, e.g. (5.33), (5.36). Following the discussions and ideas in Section 5.4.2, we can track the piecewise bounds of the above functions and estimates, e.g. $a_{i}, \tau_{1}^{-1}\left|\delta\left(a_{i}\right) g_{i}(x-z)\right| \frac{\psi_{i}(z)}{\varphi_{i}(z)}$.

Remark 5.2. Since $\psi_{i}$ (C.1) is singular, $a_{i}(x)=d_{i, L}$ (5.30), (5.4) is not $C^{1 / 2}$ near $x=0$. Yet, since we choose $\psi_{i}, \varphi_{i}$ with $\psi_{i} / \varphi_{i} \lesssim|x|^{1 / 2}$ (C.1), (C.3), the extra power $|x|^{1 / 2}$ compensates the low regularity of $a_{i}$ and we still have $\delta(a, x, z)|x-z|^{-1 / 2}|x|^{1 / 2} \in L^{\infty}$. In Section 8.4 of the supplementary material I [18] (contained in this paper), we perform an improved estimates near 0 and bound the explicit functions $\min \left(m_{1}, m_{2}\right)$ (5.41) from above. Since the bad terms (5.5) are very small near $x=0$ due to the vanishing coefficients, e.g. $\bar{\theta}_{x}$, this technical difficulty only has a tiny effect on the stability estimate. See Figure 9, We optimize the improved estimate with the previous one.
5.4.4. Estimate of $I_{3}$. Recall $I_{3}$ from (5.28) and $B_{i}$ from (5.5). The term $B_{i}$ involves both the local term and nonlocal terms. We treat them as bad terms and estimate them separately.

Estimate of the local part. We focus on $\eta_{1}$ in $B_{1}$. Other terms $-\bar{v}_{x} \xi_{1}$ in $B_{2},-\bar{u}_{y} \eta_{1}$ in $B_{3}$ (5.5) can be estimated similarly. Note that the weights are different for $\omega_{1}, \eta_{1}$. We rewrite the difference as follows
$\delta\left(\eta_{1} \psi_{1}\right) g_{1}(x-z)=\delta\left(\eta_{1} \psi_{2} \frac{\psi_{1}}{\psi_{2}}\right) g_{1}(x-z)=\left(\delta\left(\eta_{1} \psi_{2}\right) \frac{\psi_{1}}{\psi_{2}}(x)+\left(\eta_{1} \psi_{2}\right)(z) \delta\left(\frac{\psi_{1}}{\psi_{2}}\right)\right) g_{1}(x-z) \triangleq P_{1}+P_{2}$.
The term $P_{2}$ is more regular. We follow Section 5.4.2 and use (5.34)-(5.37) to estimate $\delta\left(\frac{\psi_{1}}{\psi_{2}}\right) g_{1}(x-z)$. Using the weighted $L^{\infty}$ norm of $\eta_{1}$ and the energy $E_{2}$ (5.27), we obtain

$$
\begin{equation*}
\left|P_{2}\right| \leq\left\|\eta_{1} \varphi_{2}\right\|_{\infty} \frac{\psi_{2}(z)}{\varphi_{2}(z)}\left|\delta_{\square}\left(\frac{\psi_{1}}{\psi_{2}}, h\right) g_{1}(x-z)\right| \leq E_{2} \frac{\psi_{2}(z)}{\varphi_{2}(z)}\left|\delta_{\square}\left(\frac{\psi_{1}}{\psi_{2}}, h\right) g_{1}(h)\right|, h=x-z \tag{5.43}
\end{equation*}
$$

Following Section 5.4.2 we can track the piecewise bound of the coefficient in the above upper bound. For $P_{1}$, we have

$$
\begin{align*}
\left|P_{1}\right| & \leq \frac{\psi_{1}}{\psi_{2}}(x) \frac{g_{1}(x-z)}{g_{2}(x-z)}\left|\delta\left(\eta_{1} \psi_{2}\right) g_{2}(x-z)\right| \leq \frac{\psi_{1}}{\psi_{2}}(x)\left|\frac{g_{1}(x-z)}{g_{2}(x-z)}\right|| | \eta_{1} \psi_{2} \|_{C_{g_{2}}^{1 / 2}} \\
& \leq \frac{\psi_{1}}{\psi_{2}}(x)\left|\frac{g_{1}(x-z)}{g_{2}(x-z)}\right| E_{2} \tau_{1} \mu_{2}^{-1} \tag{5.44}
\end{align*}
$$

where we have used the energy $E_{2}(5.27)$ in the last inequality. We note that in the estimate of $\tau^{-1}\left[\omega_{1} \psi_{1}\right]_{C_{g_{2}}^{1 / 2}}$, we have the term $\tau_{1}^{-1} P_{1}$. The weight $\tau_{1}^{-1}$ cancels $\tau_{1}$ in the above upper bound. Using another decomposition in (5.34), we get another estimate and we optimize them.

Note that $g_{1}$ and $g_{2}$ are equivalent to $|h|^{-1 / 2}$ and homogeneous of order $-1 / 2$. The quantity $\left|\frac{g_{1}(x-z)}{g_{2}(x-z)}\right|$ only depends on the ratio between $x_{1}-z_{1}, x_{2}-z_{2}$. We also track this ratio.

For large $|x-z|$, we have a trivial estimate

$$
\begin{equation*}
\left|\delta\left(\eta_{1} \psi_{1}\right) g_{1}(x-z)\right| \leq\left\|\eta_{1} \varphi_{2}\right\|_{\infty}\left(\frac{\psi_{1}}{\varphi_{2}}(x)+\frac{\psi_{1}}{\varphi_{2}}(z)\right) g_{1}(x-z) \leq E_{2}\left(\frac{\psi_{1}}{\varphi_{2}}(x)+\frac{\psi_{1}}{\varphi_{2}}(z)\right) g_{1}(x-z) \tag{5.45}
\end{equation*}
$$

Estimate of other local terms. Recall $W_{1}=\left(\omega_{1}, \eta_{1}, \xi_{1}\right)$ from (5.3) and the weights $\mu_{h}, \psi_{i}$ (C.1) in the energy $E_{2}$ (5.27). For $x_{1} \leq z_{1}$ and $f W_{1, i} \psi_{i}$ with $f \in C^{1 / 2}$, using the energy $\left\|W_{1, i} \varphi_{i}\right\|_{\infty}, \mu_{h, i}\left[W_{1, i} \psi_{i}\right]_{C_{g_{i}}^{1 / 2}} \leq E_{2}$ (5.70), (5.34), (5.37), we perform its $C^{1 / 2}$ estimate as follows

$$
\begin{align*}
& \left|\mu_{h, j} g_{j}(h) \delta\left(f W_{1, i} \psi_{i}, x, z\right)\right| \leq \mu_{h, j} g_{j}(h)\left(\left|\delta(f) W_{1, i} \psi_{i}(z)\right|+\left|f(x) \delta\left(W_{1, i} \psi_{i}\right)\right|\right) \\
& \quad \leq \frac{\mu_{h, j} g_{j}(h)}{\mu_{h, i} g_{i}(h)}|f(x)| E_{2}+\mu_{h, j} g_{j}(h) \delta_{\square}(f, x, z, h) \frac{\psi_{i}}{\varphi_{i}}(z) E_{2}, \quad h=x-z \tag{5.46}
\end{align*}
$$

If $i=j$, the first term reduces to $|f(x)| E_{2}$. We only pick one decomposition in (5.34) with the coefficient of $\delta\left(W_{1, i} \psi_{i}\right)$ evaluating at $x$, i.e. $f(x)$, to simplify the estimates. Note that $x_{1} \leq z_{1}$. We apply (5.46) to $\bar{v}_{x} \xi_{1} \psi_{2}, \bar{u}_{y} \eta \psi_{3}$ in (5.5), (5.28) with $i=2,3$ and in the nonlinear estimates in Section 5.9.
Estimate of the nonlocal part. To control the nonlocal terms in $B_{i}$, we use the sharp $C_{x}^{1 / 2}$ and $C_{y}^{1 / 2}$ estimates in Section 3 for the most singular part and the estimates in Section 4 of Part II [15] for the more regular part. We focus on the estimate of $-u_{x, A} \bar{\theta}_{x}$ in $B_{2}$ (5.5), which contributes to the largest part in the estimate. Using (5.33), (5.36), (5.37), we get

$$
\begin{equation*}
\left|\delta\left(u_{x, A} \bar{\theta}_{x} \psi_{2}\right)\right| \leq \delta_{\square}\left(u_{x, A} \psi_{1}, \frac{\psi_{2}}{\psi_{1}} \bar{\theta}_{x}, h\right), \quad P_{4}=\frac{\psi_{2}}{\psi_{1}} \bar{\theta}_{x} \tag{5.47}
\end{equation*}
$$

and it suffices to estimate $\delta_{i}\left(u_{x, A} \psi_{1}\right), \delta_{i}\left(\frac{\psi_{2}}{\psi_{1}} \bar{\theta}_{x}\right)$ and the $L^{\infty}$ bounds of $u_{x, A} \psi_{1}, P_{4}$. For $u_{x, A}(p), p=$ $x, z$, we use the estimate in Section 5.3. The term $P_{4}$ is more regular. It has vanishing order $|x|^{1 / 2}$ near $x=0$ and is in $C^{1 / 2}$. We follow Section 5.4 .2 to estimate it. In particular, we have

$$
\begin{equation*}
\left|\left(u_{x, A} \psi_{1}\right)(p) \cdot \delta_{i}\left(P_{4}\right)\right| \leq C_{i}(x, z, p) E_{1}, \quad p=x, z, \quad i=1,2, x_{3-i}=z_{3-i} \tag{5.48}
\end{equation*}
$$

for some functions $C_{1}(x, z, p), C_{2}(x, z, p), p=x, z$ depending on the weights and the approximate profile. See Section 5.3.1. Again, we can obtain piecewise upper bound of these functions.

For $u_{x, A} \psi_{1}$, applying the $C_{x}^{1 / 2}$ and $C_{y}^{1 / 2}$ estimates in Section 3, Section 4 of Part II [15], we obtain

$$
\begin{equation*}
\delta_{i}\left(u_{x, A} \psi_{1}\right) \leq C_{2+i}(x, z) \max \left(\tau_{1}\left\|\omega_{1} \varphi_{1}\right\|_{\infty},\left[\omega_{1} \psi_{1}\right]_{C_{g_{1}}^{1 / 2}}\right) \leq C_{2+i}(x, z) \tau_{1} E_{2}, x_{3-i}=z_{3-i} \tag{5.49}
\end{equation*}
$$

for some constants $C_{3}, C_{4}$ depending on the weights. We remark that the constants $C_{3}, C_{4}$ are very close to the constants provided by the sharp Hölder estimates in Section 3 when $|x-z|$ is small. In the estimate of $\tau_{1}^{-1} \mu_{1}\left[\eta_{1} \psi_{1}\right]_{C_{g_{2}}^{1 / 2}}$ in the energy (5.27), the weight $\tau_{1}^{-1}$ cancels $\tau_{1}$ in the above upper bound. See A.3). Again, we can obtain these piecewise upper bounds and track them carefully. See Section 5.3.1. Plugging the above estimates and the piecewise $L^{\infty}$ estimate of $P_{4}$ in (5.47) and using (5.37), (5.34), we yield the estimate for $u_{x, A} \bar{\theta}_{x} \psi_{2}$.

When $|x-z|$ is not small, we can apply the triangle inequality and the $L^{\infty}$ estimate of $u_{x, A}$ in Section 5.3 to obtain another bound. In practice, we only need to apply the above Hölder estimate when $|x-z|$ is small, e.g. $x, z$ are within 40 mesh grids designed in Section[7] Beyond such a range, the $L^{\infty}$ estimate already provides a better estimate.
5.4.5. Summarize the estimates. Similarly, we can obtain the linear estimates for other terms in (5.28) and present them in (D.11) with the modification in Section 5.8 to track the nonlocal error. At this moment, the reader can treat $\overline{\mathbf{u}}^{N}=\overline{\mathbf{u}}$ in (D.11). In particular, for the right hand sides in (5.28), when $x, z$ are close, we obtain the following estimates

$$
R_{i}=\left(d_{g, i}(x, z)+a_{i}\left(p_{x, z}\right)\right) H_{i}+\tilde{B}_{i}, \quad \tilde{B}_{i}=\hat{I}_{12}+\delta\left(B_{i} \psi_{i}\right) g_{i}(x-z)
$$

where $\left(p, \hat{I}_{12}\right)=\left(x, I_{12}\right)$ or $\left(z, \tilde{I}_{12}\right)$ depending on the size of $m_{1}, m_{2}$ in (5.42), and $\tilde{B}_{i}$ combines the term $\hat{I}_{12}$ (5.39) or (5.41) and $I_{3}$. We can estimate it as follows

$$
\begin{equation*}
\left|\tilde{B}_{i}\right| \leq C_{i}(x, z, h) E_{2}, \quad h=x-z \tag{5.50}
\end{equation*}
$$

where the coefficients $C_{i}(x, z, h)$ depend on the weights and the approximate steady state and are 0-homogeneous in $h$. For example, it involve $g_{i}(h)\left|h_{j}\right|^{1 / 2}$ via $g_{j}(h) \delta_{\square}$ (5.37) in (5.47), $\frac{g_{1}(h)}{g_{2}(h)}$ in (5.44). We can obtain piecewise upper bounds of the coefficients $C_{i}(x, z, h)$ in the above estimates and (D.11) following the discussions in Sections 5.3.1, 5.4.2 and 5.4.3, and track their dependence on $x, z$ using matrices, and $h$ using 0 -homogeneous functions. In this case, the linear stability condition (A.3) becomes

$$
\begin{equation*}
-d_{g, i}(x, z)-a_{i}\left(p_{x, z}\right)-\mu_{h, i} C_{i}(x, z, h)>\lambda, \quad \mu_{h}=\tau_{1}^{-1}\left(1, \mu_{1}, \mu_{2}\right) \tag{5.51}
\end{equation*}
$$

uniformly in $x, z$ for some $\lambda>0$. where $\mu_{h}$ is the weight of the Hölder seminorm in (5.27).
Checking the stability conditions. According to Lemma A. 1 to obtain linear stability, we need to check the conditions (A.3) or (5.51). We use the following method to check such a condition. We discretize a large domain $[0, D]^{2}$ into small grid cells $Q_{i j}=I_{i} \times I_{j}$ using the same mesh $y_{0}<y_{1}<. .<y_{n}$ as that in Section 7

Firstly, we fix the locations of $x, z$ to some grid cells: $x \in Q_{i j}, z \in Q_{k l}$, and can derive the piecewise bounds $C_{i}(x, z, h)$ in $x, z$. The bound (5.45) using the triangle inequality (and (D.14) similarly) involves $g_{1}(x-z)$ and is not homogeneous in $x-z$. We bound it using monotonicity of $g$. See Section D.2 We still need to control functions in $C_{i}(x, z, h$ ) (5.50), (D.11) involving $h=x-z$. Since these functions are 0 -homogeneous in $h$, we only need to further consider the ratio between $r_{1}=x_{1}-z_{1}, r_{2}=x_{2}-z_{2}$. Similar considerations apply to the damping factors $d_{g, i}$ (5.28),

$$
d_{g, i} \leq C_{1}(x, z) F_{1}(x-z)+C_{2}(x, z) F_{2}(x-z)=C_{1}(x, z) f_{1}(x-z)+C_{2}(x, z) f_{2}(x-z)
$$

for some 0 -homogeneous functions $f_{1}, f_{2}$ and piecewise constant bounds $C_{1}, C_{2}$, see Section 8.4.2 in the Supplementary Material I [18] (contained in this paper). We consider four different cases depending on the sign of $r_{1} / r_{2}$ and the size between $\left|r_{1}\right|,\left|r_{2}\right|$. We focus on the $r_{1} / r_{2}>0$ and $r_{2} \leq r_{1}$ to illustrate the ideas. In such a case, we can normalize $r_{1}=1$ and $0 \leq r_{2} \leq 1$. Now the problem reduces to checking the inequality in 1D. Since these functions have monotone properties, e.g. $g_{i}\left(1, r_{2}\right)$ is decreasing in $r_{2}$, these inequalities can be checked by partitioning $r_{2} \in[0,1]$ into smaller intervals $r_{2} \in\left[b_{i}, b_{i+1}\right], 0=b_{0}<b_{1}<\ldots<b_{N}=1$. For $x \in Q_{1}, z \in Q_{2}$,
we can bound the ratio $\left|z_{2}-x_{2}\right| /\left|x_{1}-z_{1}\right|$ using the piecewise upper and lower bounds for $x_{i}, z_{j}$. Thus, for $r_{2} / r_{1}$ out of such a range, we do not need to check (5.51) for such a case, or we just mark it as correct.

Note that when $|x-z|$ is far away, we will have a much better estimate due to the improvement from the sharp Hölder estimates in Lemmas 3.1]3.4 In practice, for $x \in Q_{i_{1}, i_{2}} z \in Q_{j_{1}, j_{2}}$ with $\max \left(\left|i_{1}-j_{1}\right|,\left|i_{2}-j_{2}\right|\right) \geq 15$, we already have much better stability factors.

In Figure 9, we plot the piecewise rigorous estimates along the boundary with $r_{2} / r_{1}=0$. Here, we consider $x \in I_{i} \times I_{0}, z \in I_{j} \times I_{0}, j-i=2$, where $I_{i}=\left[y_{i}, y_{i+1}\right]$ is a small interval. This corresponds to the case where we have the smallest damping. Other cases with small $j-i$ are similar and the estimate is better. The estimate of the bad terms in the $\eta_{1}$ equation is very close to the one in the most singular scenario based on the sharp inequalities. In some cases, we have better estimates since $|x-z|$ is far away and the improvement of constants for the localized velocity from Lemmas 3.1/3.3. For larger ratio $\left|r_{2} / r_{1}\right|$, we have larger stability factors than the case of $\left|r_{2} / r_{1}\right|$ being very small due to the anisotropy of the flow. See Section 2.7.2,


Figure 9. Weighted Hölder estimates. Left figure: estimates near $0, x \in$ $[0,1.8]$; Right figure: estimates in a larger domain, $x \in[0,30]$. The red curves show the coefficient of the damping term for $\delta\left(W_{1,1} \psi_{1}\right) g_{1}(x-z)$ and the estimate of the bad terms; the blue curves are for the Hölder estimate of $W_{1,2} \psi_{2}$. The green curve is the same as the bound that we estimated in the most singular scenario based on the sharp inequalities. The magenta and the black curves are the stability factors in the Hölder estimate for $\omega_{1}, \eta_{1}$. The stability factors are larger than 0.08.

In Figure 10, we consider $x \in I_{i} \times I_{0}, z \in I_{j} \times I_{0}$ with $j-i=10$. The stability factor for $\eta_{1}$ shown by the black curve becomes much larger and is larger than 0.3 .


Figure 10. Weighted Hölder estimates with larger $|x-z|$
5.5. Weighted $L^{\infty}$ estimates with growing weights. To close the nonlinear estimate in (4.10), (4.21), we need to control $\|\omega\|_{\infty},\|\nabla \theta\|_{\infty},\left\|W_{1, i}\right\|_{\infty},\|\nabla \mathbf{u}\|_{L^{\infty}}$. Since $\varphi_{1}$ (5.21) decays for large $|x|$ (see (5.14)), the energy $E_{2}$ (5.27) does not control $\left\|W_{1, i}\right\|_{\infty}$. Thus, we further perform weighted $L^{\infty}\left(\varphi_{g, i}\right)$ estimates with the following weights $1 \lesssim \varphi_{g i}$ stronger than $\varphi_{i}$ in the far-field

$$
\begin{align*}
& \varphi_{g 1}=\varphi_{1}+p_{71}|x|^{1 / 16}, \quad \varphi_{g 2}=\varphi_{2}+p_{81}|x|^{1 / 4}+p_{82}|x|^{1 / 3+\alpha_{g, n}} \\
& \varphi_{g 3}=\varphi_{3}+p_{91}|x|^{1 / 4}+p_{92}|x|^{\alpha_{g, n}}, \quad \alpha_{g, n}=1 / 3+10^{-3} \tag{5.52}
\end{align*}
$$

Since we have established weighted $L^{\infty}$ and $C^{1 / 2}$ stability estimates at the linear level, which can be treated as a-priori bounds, the following estimate is relatively simple. The subscript " $g$ " is short for "grow". For $\varphi_{g 1}, \varphi_{g 2}, \varphi_{g 3}$, the main growing terms are $|x|^{1 / 16},|x|^{1 / 4}$, and are used to close the nonlinear weighted $L^{\infty}$ estimate. The last terms in $\varphi_{g 2}, \varphi_{g 3}$ have a larger growth rate but with much smaller coefficients, $p_{92} \ll p_{91}, p_{82} \ll p_{81}$, and are used to close the nonlinear $C^{1 / 2}$ estimate. See (5.97).

To choose $p_{i j}$, we first check that the damping coefficients in the weighted $L^{\infty}\left(\varphi_{g i}\right)$ estimate (5.53)
$d_{g i}(x)=d_{i, L}\left(\varphi_{g i}\right), d_{g 1}(x)=\mathcal{T}_{d}\left(\varphi_{g 1}\right)+\bar{c}_{\omega}, d_{g 2}(x)=\mathcal{T}_{d}\left(\varphi_{g 2}\right)+2 \bar{c}_{\omega}-\bar{u}_{x}, d_{g 3}(x)=\mathcal{T}_{d}\left(\varphi_{g 3}\right)+2 \bar{c}_{\omega}+\bar{u}_{x}$,
are negative and bounded by some $d_{i}$ with $d_{i}<0$, where $\mathcal{T}_{d}$ is defined in (5.4). For $|x|$ large enough, since $\overline{\mathbf{u}}$ has sublinear growth and $\mathbf{u}$ is small, the leading order terms of $d_{g i}$ are given by

$$
d_{g 1}=\bar{c}_{l} \alpha_{1}+\bar{c}_{\omega}+\text { l.o.t., } d_{g 2}=\bar{c}_{l} \alpha_{2}+2 \bar{c}_{\omega}+\text { l.o.t., } d_{g 3}=\bar{c}_{l} \alpha_{3}+2 \bar{c}_{\omega}+\text { l.o.t }
$$

with $\bar{c}_{l} \approx 3, \bar{c}_{\omega} \approx 1$, where $\alpha_{i}$ is the exponent of the last power in $\varphi_{g i}$ (5.52). In particular, the main terms are negative. We can choose $p_{71}=1$ and first determine the power $p_{81}|x|^{1 / 4}, p_{91}|x|^{1 / 4}$ in $\varphi_{g 2}, \varphi_{g 3}$ by setting $p_{82}=p_{92}=0$ and to obtain a damping factor $d_{g i}<0$ not too close to 0 . Then we choose the last power $p_{82}|x|^{\alpha_{g, n}}, p_{92}|x|^{\alpha_{g, n}}$ with much smaller parameters $p_{82}, p_{92}$, and we still have $d_{g i}<0, i=2,3$ not too close to 0 . The parameters are given in (C.4).

For some weight parameters $\tau_{2}, \mu_{4}$ to be determined, we consider a new energy

$$
\begin{equation*}
E_{3}(t)=\max \left(E_{2}(t), \tau_{2} \max \left(\mu_{2}\left\|\omega_{1} \varphi_{g 1}\right\|_{\infty},\left\|\eta_{1} \varphi_{g, 2}\right\|_{\infty},\left\|\xi_{1} \varphi_{g, 3}\right\|_{\infty}\right)\right) \tag{5.54}
\end{equation*}
$$

where $E_{2}(t)$ is defined in (5.27). To control $\left\|\nabla \mathbf{u}_{A}\right\|_{L^{\infty}}$, we further use $\left\|\omega_{1} \varphi_{g 1}\right\|_{\infty}$ and the Hölder norm of $\omega_{1} \psi_{1}$ to derive another estimate of $\mathbf{u}_{A}, \nabla \mathbf{u}_{A}$

$$
\begin{equation*}
\left|\rho_{i j} f_{i j}(x)\right| \leq C_{g i j, 1}(x)\left\|\omega \varphi_{g 1}\right\|_{\infty}+C_{g i j, 2}(x)\left[\omega \psi_{1}\right]_{C_{x}^{1 / 2}}+C_{g i j, 3}(x)\left[\omega \psi_{1}\right]_{C_{y}^{1 / 2}} \tag{5.55}
\end{equation*}
$$

where $g$ is short for "grow", $f_{01}=u_{A}, f_{10}=v_{A}$, etc similar to those in (5.15). Similar to (5.17), we do not need the Hölder norm to control $\mathbf{u}_{A}: C_{g i j, 2}(x)=C_{g i j, 3}(x)=0, i+j=1$. Since $\varphi_{g, i}$ is growing, for large $|x|$, the above estimate is better than (5.22), and we can obtain $\left\|f_{i j}\right\|_{L^{\infty}} \lesssim \max \left(\left\|\omega \varphi_{g 1}\right\|_{\infty},\left[\omega \psi_{1}\right]_{C^{1 / 2}}\right)$ from (5.55) with constant depending on the weights.

We optimize the estimates (5.55), (5.22) and use the energy $E_{3} \geq E_{1}$ to obtain

$$
\begin{align*}
\left|\rho_{i j} f_{i j}(x)\right| \leq & C_{g i j}\left(\tau_{2} \mu_{4}\right)(x) E_{3}(t) \\
C_{g i j}(\kappa)(x)= & \min \left(C_{g i j, 1}(x) \kappa^{-1}+C_{g i j, 2}(x) \tau_{1}+C_{g i j, 3}(x) \tau_{1} g_{1}(0,1)^{-1}\right.  \tag{5.56}\\
& \left.C\left(f_{i j},\left[1, \tau_{1}, \tau_{1} g_{1}(0,1)^{-1}\right]\right)\right)
\end{align*}
$$

Since $\tau_{1}$ has been chosen, $C_{g i j}$ depends on $\kappa$ only. Performing weighted $L^{\infty}\left(\varphi_{g i}\right)$ estimate yields

$$
\partial_{t}\left(\mu_{g, i} W_{1, i} \varphi_{g i}\right)+\left(\bar{c}_{l} x+\overline{\mathbf{u}}+\mathbf{u}\right) \cdot \nabla\left(\mu_{g, i} W_{1, i} \varphi_{g i}\right)=-d_{g i}(x)\left(\mu_{g, i} W_{1, i} \varphi_{g i}\right)+\mu_{g, i} B_{g i}(x)
$$

with $\nu_{g}=\left(\tau_{2} \mu_{4}, \tau_{2}, \tau_{2}\right)$ and damping terms $d_{i}$ (5.53).
Using the energy $E_{3}$ we can obtain pointwise bounds for $\omega_{1}, \eta_{1}, \xi_{1}$, e.g.

$$
\tau_{2} \mu_{4}\left|\varphi_{g 1} \eta\right| \leq \tau_{2} \mu_{4} \varphi_{g 1}\left(\varphi_{2}^{-1}| | \eta_{1} \varphi_{2}\left\|_{\infty} \wedge \tau_{2}^{-1} \varphi_{g 2}^{-1}\right\| \tau_{2} \eta \varphi_{g 2} \|_{\infty}\right) \leq \mu_{4} \varphi_{g 1}\left(\tau_{2} \varphi_{2}^{-1} \wedge \varphi_{g 2}^{-1}\right) E_{3}(t)
$$

where $a \wedge b=\min (a, b)$. To simplify the notation, we introduce $\mathcal{T}_{u, g}$ similar to $\mathcal{T}_{u}$ in (5.18)

$$
\mathcal{T}_{u, g}(f, \tau)=C_{g 01}(\tau)\left|f_{x}\right|+C_{g 01}(\tau)\left|f_{y}\right|
$$

to control $\mathbf{u}_{A} \cdot \nabla f$. Applying the above pointwise bounds and (5.56) for $\mathbf{u}_{A}, \nabla \mathbf{u}_{A}$, we yield (5.57)
$\kappa_{i}\left|B_{g i}(x)\right| \leq \kappa_{i} A_{g i}(x) E_{3}(t), \quad i=1,2,3$,
$\tau_{2} \mu_{4} A_{g 1}=\mu_{4} \varphi_{g 1}\left(\tau_{2} \varphi_{2}^{-1} \wedge \varphi_{g 2}^{-1}\right)+\tau_{2} \mu_{4} \frac{\varphi_{g 1}}{\rho_{01}} \mathcal{T}_{u, g}\left(\bar{\omega}, \tau_{2} \mu_{4}\right)$,
$\tau_{2} A_{g 2}=\varphi_{g 2}\left|\bar{v}_{x}\right|\left(\tau_{2} \varphi_{3}^{-1} \wedge \varphi_{g 3}^{-1}\right)+\tau_{2} \frac{\varphi_{g 2}}{\rho_{10}} \mathcal{T}_{u, g}\left(\bar{\theta}_{x}, \tau_{2} \mu_{4}\right)+\tau_{2} \frac{\varphi_{g 2}}{\psi_{1}}\left(C_{g 11}\left(\tau_{2} \mu_{4}\right)\left|\bar{\theta}_{x}\right|+C_{g 20}\left(\tau_{2} \mu_{4}\right)\left|\bar{\theta}_{y}\right|\right)$,
$\tau_{2} A_{g 3}=\varphi_{g 3}\left|\bar{u}_{y}\right|\left(\tau_{2} \varphi_{2}^{-1} \wedge \varphi_{g 2}^{-1}\right)+\tau_{2} \frac{\varphi_{g 3}}{\rho_{10}} \mathcal{T}_{u, g}\left(\bar{\theta}_{y}, \tau_{2} \mu_{4}\right)+\tau_{2} \frac{\varphi_{g 3}}{\psi_{1}}\left(C_{g 02}\left(\tau_{2} \mu_{4}\right)\left|\bar{\theta}_{x}\right|+C_{g 11}\left(\tau_{2} \mu_{4}\right)\left|\bar{\theta}_{y}\right|\right)$.
Now, the inequality (A.3) for $\mid W_{1, i} \varphi_{g i} \|_{\infty}$ with weights $\left(\tau_{2} \mu_{4}, \tau_{2}, \tau_{2}\right)$ reads

$$
\begin{equation*}
-d_{g 1}(x)-\mu_{4} \tau_{2} A_{g 1} \geq \lambda,-d_{g 2}(x)-\tau_{2} A_{g 2} \geq \lambda,-d_{g 3}(x)-\tau_{2} A_{g 3} \geq \lambda, d_{g i}(x)=d_{i, L}\left(\varphi_{g i}\right) \tag{5.58}
\end{equation*}
$$

for some $\lambda>0$. We have chosen $p_{i j}$ and the weights are fixed. Since the coefficients $\nabla \bar{\omega}, \nabla \bar{\theta}, \nabla^{2} \bar{\theta}, \nabla \overline{\mathbf{u}}$ decay and the second bound in (5.56) is independent of $\tau_{2}, \mu_{4}$, using the asymptotics of the weights, one can obtain that the above estimates go to 0 as $\tau_{2} \rightarrow 0$ uniformly for $\mu_{4} \leq 1$, e.g.

$$
\varphi_{g 2}\left|\bar{v}_{x}\right|\left(\tau_{2} \varphi_{3}^{-1} \wedge \varphi_{g 3}^{-1}\right) \rightarrow 0, \quad \tau_{2} \frac{\varphi_{g 2}}{\rho_{10}} \mathcal{T}_{u, g}\left(\bar{\theta}_{x}, \tau_{2} \mu_{4}\right) \rightarrow 0
$$

Thus, we can choose a small $\tau_{2}$ to first achieve the second and third stability condition in (5.58) with $\lambda$ similar to that in (5.23). Similarly, for a fixed $\tau_{2}$, as $\mu_{4} \rightarrow 0$, we get $\tau_{2} \mu_{4} A_{g 1} \rightarrow 0$. We can choose a small $\mu_{4}$ to achieve the first condition in (5.58). Note that we do not simply set $\mu_{4}=1$ since it will force us to choose a smaller $\tau_{2}$ to satisfy all three conditions, which lead to a weaker energy $E_{3}$ (5.54) and larger constants in later nonlinear estimates. We adjust $\tau_{2}, \mu_{4}$ under constraint (5.58) to obtain $\tau_{2}, \tau_{2} \mu_{4}$ not too small. The parameters $\tau_{2}, \mu_{4}$ are given in (C.5). We remark that the choices of weights $\varphi_{g i}$ and $\tau_{2}, \mu_{4}$ mainly affect the contants in the nonlinear estimates in the far-field, e.g. $|x| \geq 10^{4}$, since the weight $\varphi_{i}$ (5.14) in the energy (5.21), (5.54) is stronger than $\mu_{g, i} \varphi_{g i}$ for $|x|$ not very large. We can afford larger constants due to much larger damping coefficients in the far-field.
5.6. Estimate of some linear functionals. In the previous sections, we have performed the weighted $L^{\infty}$ and $C^{1 / 2}$ estimates on $W_{1, i}$ for the main equations (5.1) and established the stability estimates provided that (5.23), (5.58), (5.26), (5.51) hold. To close the energy estimates of (4.21), we need to further estimate the residual operators $\mathcal{R}$ (4.20). The error part related to the approximate solution constructed numerically, e.g. $\hat{F}_{i}(0)-\bar{F}_{i}$, will be estimated in Section 3 of Part II [15]. To control $\mathcal{R}$, we need to control the functional $a_{i}(W)$ and $a_{n l, i}(W)$.

For the linear functional $a_{i}(W)$, we have two types. The first type is $c_{\omega}\left(\omega_{1}\right)$ from $\mathcal{K}_{1 i}\left(\omega_{1}\right)$ (4.11). The second type is from $\mathcal{K}_{i 2}\left(\omega_{1}\right)$ (4.13) for the approximation of $\tilde{\mathbf{u}}, \tilde{\nabla} \mathbf{u}$ (4.37), (4.29), (4.38). For $a_{n l, i}$ defined in (4.18), we need to control $c_{\omega}\left(W_{1}+\hat{W}_{2}\right)$ and $\partial^{2}\left(W_{1}+\hat{W}_{2}\right)(0)$. For the second type of term, it is given by the integral

$$
\int_{\mathbb{R}_{++}^{2}} \omega_{1}(y) p(y) d y
$$

for some function $p(y)$ that has a fast decay, e.g. it has a decay rate $|y|^{-4}$. We have two equivalent formulas (4.29), (4.39) (4.37), (4.40) for approximating $\tilde{\mathbf{u}}, \tilde{\nabla} \mathbf{u}$. Then we estimate it directly using the norms $\left\|\omega \varphi_{1}\right\|_{\infty},\left\|\omega \varphi_{g 1}\right\|_{\infty}$ in the energy and pointwise estimate (5.59).

Next, we estimate $c_{\omega}\left(\omega_{1}\right), c_{\omega}\left(\omega_{1}+\hat{\omega}_{2}\right), \partial^{2}\left(W_{1}+\hat{W}_{2}\right)(0)$. With the estimates of these terms, since the coefficients in (4.29), (4.39) (4.37), 4.40), e.g. $C_{f}(x) \chi_{i}, C_{f} S_{i}$, are given smooth functions with the appropriate vanishing order near $x=0$, we can estimate their derivatives and weighted norms following Appendix E of Part II [15] and then obtain the estimate of $\hat{\mathbf{u}}, \widehat{\nabla \mathbf{u}}$ (4.38).

From the definition of the energy (5.21), (5.54), we have the pointwise control

$$
\begin{align*}
& \left|W_{1, i}(x)\right| \leq \varphi_{M i}^{-1}(x) E_{3}(t), \quad W_{1,1}=\omega_{1}, W_{i, 2}=\eta_{1}, W_{i, 3}=\xi_{1} \\
& \varphi_{M 1}(x) \triangleq \max \left(\varphi_{1}(x), \tau_{2} \mu_{4} \varphi_{g 1}(x)\right), \quad \varphi_{M i}(x) \triangleq \max \left(\varphi_{i}(x), \tau_{2} \varphi_{g i}(x)\right), i=2,3 \tag{5.59}
\end{align*}
$$

Recall the inner product (2.22). Controlling the normalization factor (2.26)

$$
\begin{equation*}
c_{\omega}(\omega)=u_{x}(0)=-\frac{4}{\pi} \int_{\mathbb{R}_{2}^{++}} \frac{y_{1} y_{2}}{|y|^{4}} \omega(y) d y=-\frac{4}{\pi}\left\langle\omega, f_{*}\right\rangle, \quad f_{*}(y)=\frac{y_{1} y_{2}}{|y|^{4}} . \tag{5.60}
\end{equation*}
$$

effectively is nontrivial since the integrand $\frac{y_{1} y_{2}}{|y|^{4}}$ decays slowly (it is not in $L^{1}$ ) and our weight for $\omega_{1}$ is very weak in the far-field. See (5.52) and (5.54). If we use the pointwise estimate (5.59) directly to bound the integral, we get $\left|c_{\omega}\left(\omega_{1}\right)\right| \leq C_{1} E_{3}$ with $C_{1}$ about $170-300$, which contributes directly to the main nonlinear terms. See the discussion around (5.94). Although this estimate only enters the energy estimates via the residual operators and nonlinear terms, a larger constant forces us to obtain a smaller residual error in the computation to close the estimates. To ease the computation burden, we seek a more effective estimate based on the ODE of $c_{\omega}\left(\omega_{1}\right)$. For the same reason, we also derive a sharper estimate of $c_{\omega}(\omega)$ in Section 5.6.2.
5.6.1. Controlling of $c_{\omega}\left(\omega_{1}\right)$. Following [17,20, we perform the estimates based on the ODE of $c_{\omega}$. Using the main equations (5.1) and (4.21) we can derive the evolution of $\left\langle\omega_{1}, f_{*}\right\rangle,\left\langle\eta_{1}, f_{*}\right\rangle$

$$
\begin{align*}
& \frac{d}{d t}\left\langle\omega_{1}, f_{*}\right\rangle=\bar{c}_{\omega}\left\langle\omega_{1}, f_{*}\right\rangle+\left\langle\eta_{1}, f_{*}\right\rangle+\left\langle\Gamma_{1}, f_{*}\right\rangle, \quad \frac{d}{d t}\left\langle\eta_{1}, f_{*}\right\rangle=2 \bar{c}_{\omega}\left\langle\eta_{1}, f_{*}\right\rangle+\left\langle\Gamma_{2}, f_{*}\right\rangle \\
& \left\langle\Gamma_{1}, f_{*}\right\rangle \triangleq\left\langle\Gamma_{1, M}+\mathcal{N}_{1}+\mathcal{F}_{1}-N F_{1}-\mathcal{R}_{1}, f_{*}\right\rangle, \quad \Gamma_{1, M}=-\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \omega_{1}-\mathbf{u}_{A} \cdot \nabla \bar{\omega}  \tag{5.61}\\
& \left\langle\Gamma_{2}, f_{*}\right\rangle \triangleq\left\langle\Gamma_{2, M}+\mathcal{N}_{2}+\overline{\mathcal{F}}_{2}-N F_{2}-\mathcal{R}_{2}, f_{*}\right\rangle, \quad \Gamma_{2, M}=-\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \eta_{1}+B_{2}\left(W_{1}\right)
\end{align*}
$$

where $\langle\cdot, \cdot\rangle(2.22)$ is the standard inner product on $\mathbb{R}_{++}^{2}, B_{2}\left(W_{1}\right)$ denotes the bad term (5.5), and $\mathcal{N}_{i}, \overline{\mathcal{F}}_{i}, N F_{i}, \mathcal{R}_{i}$ are the nonlinear terms (2.18), residual error (2.19), rank-one correction (4.11), and residual operator (4.20). The transport term $\mathbf{u} \cdot \nabla W_{1, i}$ in (5.1) is contained in $\mathcal{N}_{i}$ (2.18). We derive the ODE of $\left\langle\eta_{1}, \varphi_{*}\right\rangle$ to control it in the first equation. The main terms for $\Gamma_{i}$ are given by $\Gamma_{i, M}$ from the main linearized equations (5.1). Using integration by parts, we get

$$
\begin{equation*}
-\int\left(\bar{c}_{l} y+\overline{\mathbf{u}}\right) \cdot \nabla g(y) f_{*}(y) d y=\int g(y) \nabla \cdot\left(\left(\bar{c}_{l} y+\overline{\mathbf{u}}\right) f_{*}(y)\right) d y=\int g(y) \overline{\mathbf{u}} \cdot \nabla f_{*}(y) d y \tag{5.62}
\end{equation*}
$$

where we have used $\nabla \cdot\left(y f_{*}(y)\right)=0$, which is an algebraic property of $f_{*}$ (5.60), and $\nabla \cdot\left(\overline{\mathbf{u}} f_{*}\right)=$ $\overline{\mathbf{u}} \cdot \nabla f_{*}$.

The first terms on the right hand side are damping terms since $\bar{c}_{\omega} \approx-1$. The advantage of the above ODE system is that the integrands in the linear part, e.g. $g(y) \overline{\mathbf{u}} \cdot \nabla f_{*}$, have faster decay than $g(y) \varphi_{*}$ since $\overline{\mathbf{u}}$ grows sublinearly $O\left(|x|^{\gamma}\right)$ with $\gamma \approx \frac{2}{3}$. For the nonlocal terms in (5.61) and $B_{2}$ (5.5) involving $\mathbf{u}_{A}, \nabla \mathbf{u}_{A}$, we apply the estimates (5.56). For the local terms in (5.61) $B_{2}$ (5.5) other than $\left\langle\omega_{1}, f_{*}\right\rangle,\left\langle\eta_{1}, f_{*}\right\rangle$, we use (5.59) to estimate them.

For the nonlinear and error terms (5.61), we treat them as perturbation and estimate them using integration by parts and pointwise estimate similar to those for $\left\langle\mathbf{u}_{A} \cdot \nabla \bar{\omega}, f_{*}\right\rangle$ and (5.62).

Improvement. We can further improve the above estimate by decomposing

$$
\begin{equation*}
\left\langle\omega_{1}, f_{*}\right\rangle=\left\langle\omega_{1}, \chi_{o d e} f_{*}\right\rangle+\left\langle\omega_{1},\left(1-\chi_{\text {ode }}\right) f_{*}\right\rangle, \quad\left\langle\eta_{1}, f_{*}\right\rangle=\left\langle\eta_{1}, \chi_{\text {ode }} f_{*}\right\rangle+\left\langle\eta_{1},\left(1-\chi_{\text {ode }}\right) f_{*}\right\rangle \tag{5.63}
\end{equation*}
$$

where $\chi_{o d e}$ is a smooth cutoff function supported away from the origin. We derive the ODEs for $\left\langle\omega_{1}, \chi_{\text {ode }} f_{*}\right\rangle,\left\langle\eta_{1}, \chi_{\text {ode }} f_{*}\right\rangle$ similar to (5.61), and perform energy estimates on these terms. The main difference is the advection term. Instead of having (5.62), we yield

$$
-\int\left(\bar{c}_{l} y+\overline{\mathbf{u}}\right) \cdot \nabla g(y)\left(f_{*} \chi_{o d e}\right)(y) d y=\int g(y) \nabla \cdot\left(\left(\bar{c}_{l} y+\overline{\mathbf{u}}\right) f_{*} \chi_{o d e}\right) d y
$$

Out of the support of $1-\chi_{\text {ode }}$, we get $\chi_{\text {ode }}=1$ and yield the same integrand as (5.62). For $\left\langle\omega_{1},\left(1-\chi_{\text {ode }}\right) f_{*}\right\rangle$ with integrand supported near 0 , we estimate it directly using the pointwise estimate (5.59). We perform the above decomposition since the estimate via the ODE system is only more effective than the pointwise estimate (5.59) to the control the far-field part of the integral $\int \omega_{1} f_{*}$ since the integrand in the ODE system has faster decay. We choose

$$
\begin{equation*}
\chi_{\text {ode }}(x, y)=1-\chi_{e}\left(\left(x-\nu_{31}\right) / \nu_{31}\right) \chi_{e}\left(\left(y-\nu_{32}\right) / \nu_{32}\right), \quad \nu_{31}=80, \nu_{32}=1200 \tag{5.64}
\end{equation*}
$$

where $\chi_{e}$ is the cutoff function defined in C.6). Following the estimates discussed above and using (5.54), we can control the main part as follows

$$
\begin{equation*}
\left|\left\langle\Gamma_{i, M}, f_{*} \chi_{o d e}\right\rangle\right| \leq \mu_{5 i, 1} E_{3}(t), \quad i=1,2 \tag{5.65}
\end{equation*}
$$

for some constant $\mu_{51, i}$. At this step, if we neglect the remaining parts $\left\langle\Gamma_{i}-\Gamma_{i, M}, f_{*}\right\rangle$ from the residual error and nonlinear terms which are much smaller, according to (A.3), we can choose small $\mu_{52}$ and then $\mu_{51}$

$$
\begin{equation*}
\mu_{52} \mu_{52,1}<\left|2 \bar{c}_{\omega}\right|, \quad \mu_{51}\left(\mu_{51,1}+\mu_{52}^{-1}\right)<\left|\bar{c}_{\omega}\right|, \quad E_{\text {new }}=\max \left(E_{3}, \mu_{51} m_{1}, \mu_{52} m_{2}\right) \tag{5.66}
\end{equation*}
$$

and obtain linear stability for $E_{\text {new }}$. The factor $\mu_{51,1}+\mu_{52}^{-1}$ comes from $\left\langle\eta_{1}+\Gamma_{1, M}, f_{*}\right\rangle \leq$ $\left(\mu_{51,1}+\mu_{52}^{-1}\right) E_{n e w}$. To close the nonlinear estimates (A.11), due to the remaining terms, we will choose a slightly smaller weights. The weights are mostly determined by the above estimates.
5.6.2. Controlling $c_{\omega}(\omega)$. Recall that $W_{1}+\widehat{W}_{2}=(\omega, \eta, \xi)$ is the solution to (4.10) and $c_{\omega}(\omega)=$ $c_{\omega}\left(W_{1}\right)+c_{\omega}\left(\widehat{W}_{2}\right)$. We use similar ideas to estimate $c_{\omega}(\omega)$ by deriving the ODEs of $c_{\omega}\left(\omega_{1}\right)$ and $c_{\omega}\left(\widehat{W}_{2}\right)$ separately. We have derived the ODE of $c_{\omega}\left(\omega_{1}\right)$ in (5.61). For $c_{\omega}\left(\widehat{W}_{2}\right)$, since $b_{i}(s) \triangleq a_{i}\left(W_{1}(s), \hat{W}_{2}(s)\right)$ depending on $\left(W_{1}, \widehat{W}_{2}\right)$ in (4.19) is spatial-independent, we use the formula (4.19) and linearity to get

$$
\begin{equation*}
c_{\omega}\left(\widehat{W}_{2}\right)=u_{x}\left(\widehat{W}_{2}\right)(0)=\sum_{i \leq n_{1}+3} \int_{0}^{t \wedge T_{i}} b_{i}(t-s) c_{\omega}\left(\widehat{F}_{i}\right)(s) d s \triangleq \sum_{i \leq n_{1}+3} I_{i}, b_{i}=a_{i}\left(W_{1}, \hat{W}_{2}\right) \tag{5.67}
\end{equation*}
$$

We add the constraint $s \leq T_{i}$ in the integral since the term $c_{\omega}\left(\widehat{F}_{i}\right)(t)$ we constructed is supported in $\left[0, T_{i}\right]$. See Section 3.5 of Part II [15]. We label the first approximation term as $a_{1}(s) \widehat{F}_{1}=$ $c_{\omega}\left(W_{1}(s)\right) \bar{f}_{c_{\omega}, i}$ chosen in (4.11). To simplify the notation, we denote $b_{n_{1}+i} \widehat{F}_{n_{1}+i}=a_{n l, i} \widehat{F}_{\chi, i}(t-$ $s), i=1,2,3$. For each term $I_{i}$, taking derivatives and using $\partial_{t} b_{i}(t-s)=-\partial_{s} b_{i}(t-s)$ and integration by parts, we yield

$$
\begin{aligned}
\frac{d}{d t} I_{i}= & \mathbf{1}_{t<T_{i}} b_{i}\left(t-t \wedge T_{i}\right) c_{\omega}\left(\widehat{F}_{i}\left(t \wedge T_{i}\right)\right)+\int_{0}^{t \wedge T_{i}}-\partial_{s} b_{i}(t-s) c_{\omega}\left(\widehat{F}_{i}(s)\right) d s \\
= & \mathbf{1}_{t<T_{i}} b_{i}\left(t-t \wedge T_{i}\right) c_{\omega}\left(\widehat{F}_{i}\left(t \wedge T_{i}\right)\right)+\int_{0}^{t \wedge T_{i}} b_{i}(t-s) \partial_{s} c_{\omega}\left(\hat{F}_{i}(s)\right) d s \\
& -b_{i}\left(t-t \wedge T_{i}\right) c_{\omega}\left(\widehat{F}_{i}\left(t \wedge T_{i}\right)\right)+b_{i}(t) c_{\omega}\left(\widehat{F}_{i}(0)\right) \\
= & b_{i}(t) c_{\omega}\left(\hat{F}_{i}(0)\right)+\mathbf{1}_{t \geq T_{i}} b_{i}\left(t-T_{i}\right) c_{\omega}\left(\widehat{F}_{i}\left(T_{i}\right)\right)+\int_{0}^{t \wedge T_{i}} b_{i}(t-s) \partial_{s} c_{\omega}\left(\widehat{F}_{i}(s)\right) d s \\
\triangleq & b_{i}(t) c_{\omega}\left(\widehat{F}_{i}(0)\right)+I_{b, i}
\end{aligned}
$$

The term $I_{b, i}$ is treated as a bad term. For $i=1$, the term $b_{1}(t) c_{\omega}\left(\hat{F}_{1}(0)\right)$ with $c_{\omega}\left(\hat{F}_{1}(0)\right) \approx-2.5$ provides an additional damping term for $b_{1}(t)=c_{\omega}\left(W_{1}(t)\right)$, which is the main reason why we combine the estimate of $c_{\omega}\left(W_{1}\right)$ and $c_{\omega}\left(\widehat{W}_{2}\right)$. Denote

$$
\bar{\lambda}_{c_{\omega}}=c_{\omega}\left(\widehat{F}_{1}(0)\right)+\bar{c}_{\omega}, \quad B_{c_{\omega}, 1}=\frac{4}{\pi}\left\langle\Gamma_{1}, f_{*}\right\rangle
$$

Since $c_{\omega}(g)=-\frac{4}{\pi}\left\langle g, f_{*}\right\rangle$ (5.60), multiplying (5.61) by $-\frac{4}{\pi}$ and then combining it with the above derivations for $c_{\omega}\left(\hat{W}_{2}\right)$, we get
$\frac{d}{d t}\left(c_{\omega}\left(W_{1}\right)+c_{\omega}\left(\hat{W}_{2}\right)\right)=\bar{c}_{\omega} c_{\omega}\left(W_{1}\right)+B_{c_{\omega}, 1}-\frac{4}{\pi}\left\langle\eta_{1}, f_{*}\right\rangle+c_{\omega}\left(\hat{F}_{1}(0)\right) b_{1}(t)+\sum_{i \geq 2} c_{\omega}\left(\hat{F}_{i}(0)\right) b_{i}(t)+\sum_{i \geq 1} I_{b, i}$.

Since $b_{1}(t)=c_{\omega}\left(W_{1}(t)\right)$, we can combine the two terms of $c_{\omega}\left(W_{1}\right)$. Adding $\bar{\lambda}_{c_{\omega}} c_{\omega}\left(\widehat{W}_{2}\right)$ and subtracting it on the RHS using (5.67), and then using $\omega=W_{1}+\widehat{W}_{2}$, we yield

$$
\begin{align*}
& \frac{d}{d t} c_{\omega}(\omega)=\bar{\lambda}_{c_{\omega}} c_{\omega}(\omega)+B_{c_{\omega}, 1}-\frac{4}{\pi}\left\langle\eta_{1}, f_{*}\right\rangle+\sum_{i \geq 2} c_{\omega}\left(\hat{F}_{i}(0)\right) b_{i}(t) \\
& +\sum_{i \geq 1} \mathbf{1}_{t \geq T_{i}} b_{i}\left(t-T_{i}\right) c_{\omega}\left(\hat{F}_{i}\left(T_{i}\right)\right)+\int_{0}^{t \wedge T_{i}} b_{i}(t-s)\left(\partial_{s} c_{\omega}\left(\hat{F}_{i}(s)\right)-\bar{\lambda}_{c_{\omega}} c_{\omega}\left(\hat{F}_{i}(s)\right)\right) d s . \tag{5.68}
\end{align*}
$$

We estimate $B_{c_{\omega}, 1}$ using the method in Section 5.6.1, and $b_{i}(s)$ using the bootstrap bounds (5.78). For $\frac{4}{\pi}\left\langle\eta_{1}, f_{*}\right\rangle$, we use the ODE (5.61) and the method in Section 5.6.1] The approximate terms $c_{\omega}\left(\hat{F}_{i}\left(T_{i}\right)\right), \partial_{s} c_{\omega}\left(\hat{F}_{i}(s)\right)-\bar{\lambda}_{c_{\omega}} c_{\omega}\left(\hat{F}_{i}(s)\right)$ are piecewise cubic polynomials constructed numerically, which we can estimate using the method in Section 5.7. We have estimated $c_{\omega}\left(W_{1}\right)$ in Section 5.6.1 at the linear level. Using the above estimates and following the discussion around (5.66), at the linear level, we can determine the weights for $c_{\omega}(\omega)$ and $\left\langle\eta_{1}, f_{*}\right\rangle$ in the energy.

If we estimate $c_{\omega}\left(W_{1}\right), c_{\omega}\left(\widehat{W}_{2}\right)$ separately, we need to add a much smaller weight for $c_{\omega}(\omega)$ in the energy, which leads to a constant about three times larger for the nonlinear estimates.
5.6.3. Controlling $\omega_{x y}(0), \eta_{x y}(0), \xi_{x x}(0)$. To control $\omega_{x y}(0), \eta_{x y}(0), \xi_{x x}(0)$, we first note that $\eta_{x y}(0)=\xi_{x x}(0)=\theta_{x x y}(0)$ since the solution to (4.10) satisfies $\eta=\theta_{x}, \xi=\theta_{y}$ (2.27).

Recall the ODEs for the full solution $\omega_{x y}(0), \theta_{x x y}(0)$ in (4.12). Linearizing it around the approximate steady state and using the normalization conditions (2.26), (2.29), we yield the equations for the perturbations

$$
\begin{align*}
& \frac{d}{d t} \omega_{x y}(0)=\left(-2 \bar{c}_{l}+\bar{c}_{\omega}\right) \omega_{x y}(0)+\theta_{x x y}(0)+c_{\omega} \bar{\omega}_{x y}(0)+c_{\omega} \omega_{x y}(0)+\bar{\partial}_{x y} \mathcal{F}_{1}(0)  \tag{5.69}\\
& \frac{d}{d t} \theta_{x x y}(0)=\left(-2 \bar{c}_{l}+2 \bar{c}_{\omega}-\bar{u}_{x}(0)\right) \theta_{x x y}(0)+c_{\omega} \bar{\theta}_{x x y}(0)+c_{\omega} \theta_{x x y}(0)+\bar{\partial}_{x y} \mathcal{F}_{2}(0)
\end{align*}
$$

where $\overline{\mathcal{F}}_{i}$ is defined in (2.19). Note that the matrix involving $\omega_{x y}(0), \theta_{x x y}(0)$ has negative eigenvalues. We can first estimate $\theta_{x x y}(0)$ and then $\omega_{x y}(0)$. Using the above ODEs, at the linear level, we can determine the weights for $\theta_{x x y}(0), \omega_{x y}(0)$ in the energy.

To handle the nonlinear and error terms in (55.61), the ODE of $c_{\omega}$, and (55.69) later, we choose the weights of the functionals in Sections 5.6.1, 5.6.2 in the energy slightly smaller than those determined by the linear estimates, and define the final energy $E_{4}(t)$

$$
\begin{align*}
& E_{4}(t) \triangleq \max \left(E_{3}(t), \mu_{51}^{-1}\left|c_{\omega}\left(\omega_{1} \chi_{o d e}\right)\right|, \mu_{52}^{-1}\left|c_{\omega}\left(\eta_{1} \chi_{o d e}\right)\right|, \mu_{6}^{-1}\left|c_{\omega}(\omega)\right|, \mu_{62}^{-1}\left|c_{\omega}\left(\eta_{1}\right)\right|,\right.  \tag{5.70}\\
& \left.\mu_{7}^{-1}\left|\theta_{x x y}(0)\right|, \mu_{8}^{-1}\left|\omega_{x y}(0)\right|\right), \quad \mu_{6}=61, \mu_{7}=9.5, \mu_{8}=4.5
\end{align*}
$$

where the energy $E_{3}$ is defined in (5.54), $\chi_{o d e}$ is defined in (5.64), $\mu_{i j}$ are given in (C.5), and we have used the notation (5.60) to simplify the functionals estimated in Section 5.6.1 See also (5.21), (5.27). We remark that the variables $c_{\omega}\left(\omega_{1} \chi\right), c_{\omega}\left(\eta_{1} \chi\right), c_{\omega}\left(\eta_{1}\right)$ and parameters $\mu_{51}, \mu_{52}, \mu_{62}$ are intermediate parameters and are used only in the ODEs in Sections 5.6.1 5.6.2 along with (5.63) to control

$$
\begin{equation*}
\left|c_{\omega}\left(\omega_{1}\right)\right|<\mu_{5} E_{4}, \quad\left|c_{\omega}(\omega)\right|<\mu_{6} E_{4}, \quad \mu_{5}=76 . \tag{5.71}
\end{equation*}
$$

To estimate the nonlinear mode $a_{n l, i}\left(W_{1}+\hat{W}_{2}\right)$ for (4.11), we impose the bootstrap bound

$$
\begin{equation*}
\left|c_{\omega} \omega_{x y}(0)+\partial_{x y} \overline{\mathcal{F}}_{1}(0)\right|<5 \mu_{6} \cdot E_{*}, \quad\left|c_{\omega} \theta_{x x y}(0)+\partial_{x y} \overline{\mathcal{F}}_{2}(0)\right|<10 \mu_{6} \cdot E_{*} \tag{5.72}
\end{equation*}
$$

where $E_{*}$ will be chosen in (5.101). Under the bootstrap assumptions $E_{4}(t)<E_{*}$, we will verify the stronger estimate

$$
\begin{align*}
& \left|c_{\omega} \omega_{x y}(0)+\partial_{x y} \overline{\mathcal{F}}_{1}(0)\right|<\mu_{8} \mu_{6} E_{*}+\left|\partial_{x y} \overline{\mathcal{F}}_{1}(0)\right|<5 \mu_{6} E_{*},  \tag{5.73}\\
& \left|c_{\omega} \theta_{x x y}(0)+\partial_{x y} \overline{\mathcal{F}}_{2}(0)\right|<\mu_{7} \mu_{6} E_{*}+\left|\partial_{x y} \overline{\mathcal{F}}_{2}(0)\right|<10 \mu_{6} E_{*} .
\end{align*}
$$

5.7. Estimate $\widehat{W}_{2}$ and the residual operator. Using the method in Section 3 in part II, for intial data $\bar{G}_{i}(0)=\bar{F}_{i}(0), \bar{F}_{\chi, i}$ given in Appendix C.2.1 and the spatial independent factors $b_{i}(s)=a_{i}\left(W_{1}(s)\right), a_{n l, i}(W(s))$ in (4.20), (4.19), we construct an approximate space-time solution $\hat{F}_{i}, \hat{W}_{2}$ and its associate approximate stream function $\hat{\phi}^{N}$ ) and error $\hat{\varepsilon}$
(5.74)
$\hat{W}_{2}=\sum_{i \leq n} \int b_{i}(t-s) \hat{G}_{i}(s) d s, \quad \hat{\phi}^{N}=\sum_{i \leq n} \int b_{i}(t-s) \hat{\phi}_{i}^{N}(s) d s, \quad \hat{\varepsilon}=\sum_{i \leq n} \int b_{i}(t-s)\left(\hat{G}_{i}+\Delta \hat{\phi}_{i}^{N}\right)(s) d s$,
with residual error in the $j$-th equation given by

$$
\begin{align*}
& \mathcal{R}_{j}(t)=\mathcal{R}_{l o c, 0, j}(t)+\mathcal{R}_{j}^{\bar{\varepsilon}}-D_{j}^{2} \mathcal{R}_{j}^{\bar{\varepsilon}}(0) f_{j, 2}+\mathcal{R}_{j}^{\hat{\varepsilon}}-D_{j}^{2} \mathcal{R}_{j}^{\hat{\varepsilon}}(0) f_{j 2}, \quad D^{2}=\left(\partial_{x y}, \partial_{x y}, \partial_{x x}\right)  \tag{5.75}\\
& \mathcal{R}_{l o c, 0, j}=\sum_{i \leq n} \int_{0}^{t} c_{i}(t-s) \mathcal{R}_{n u m, i, j}(s) d s, \quad \mathcal{R}_{j}^{\bar{\varepsilon}}=\mathcal{B}_{o p, j}\left(\mathbf{u}(\bar{\varepsilon}), \hat{W}_{2}\right), \quad \mathcal{R}_{j}^{\hat{\varepsilon}}=\mathcal{B}_{o p, j}(\mathbf{u}(\hat{\varepsilon}), \bar{W})
\end{align*}
$$

where $\bar{W}, \chi_{i 2}$ are given in (5.3), (C.9), and $\bar{\varepsilon}=\bar{\omega}-(-\Delta) \bar{\phi}^{N}$ is given in (5.80), $\mathcal{R}_{n u m, i}=$ $O\left(|x|^{3}\right)$ depends on the numerical solution $\hat{\phi}_{i}^{N}, \hat{G}_{i}$ locally, and we have absorbed the initial error $\hat{G}_{i}-\bar{G}_{i}(0)($ see (4.20) $)$ in $\mathcal{R}_{\text {num }}$. Moreover, in Part II, we have estimated

$$
\begin{equation*}
\int_{0}^{\infty}\left|\partial_{x}^{i} \partial_{y}^{j} f_{l}(s)\right| d s, \quad f_{l}=\hat{F}_{k, l}, \hat{\phi}_{i}^{N}, \hat{G}_{i}, \hat{G}_{i}+\Delta \phi_{i}^{N}, \mathcal{R}_{n u m, l} \tag{5.76}
\end{equation*}
$$

For later estimates, we add and subtract $f_{\chi, j}$ in $\mathcal{R}_{j}^{\bar{\varepsilon}}$ in (5.75)

$$
\begin{equation*}
\mathcal{R}_{j}(t)=\mathcal{R}_{l o c, 0, j}(t)+D_{j}^{2} \mathcal{R}_{j}^{\bar{\varepsilon}}(0)\left(f_{\chi, 2}-f_{j, 2}\right)+\left(\mathcal{R}_{j}^{\bar{\varepsilon}}-D_{j}^{2} \mathcal{R}_{j}^{\bar{\varepsilon}}(0) f_{\chi, 2}\right)+\left(\mathcal{R}_{j}^{\hat{\varepsilon}}-D_{j}^{2} \mathcal{R}_{j}^{\hat{\varepsilon}}(0) f_{j 2}\right) \tag{5.77}
\end{equation*}
$$

We can control the spatial-independent factor $b_{i}(s)$ using the energy estimate $\left|b_{i}(s)\right|<c_{i} E_{4}(s)$ discussed in Section 5.6. Since we will use a bootstrap argument to show that $E_{4}(s)<E_{*}$ for all time $s$, under such an assumption we have

$$
\begin{equation*}
\left|b_{i}(s)\right| \leq c_{i} E_{4}(t) \tag{5.78}
\end{equation*}
$$

for some threshold $E_{*}$ to be determined. Then we can control the local terms, e.g.

$$
\begin{equation*}
\left|\partial_{x}^{i} \partial_{y}^{j} \mathcal{R}_{l o c, j}(t)\right| \leq \sum_{i \leq n} \sup _{s<t}\left|b_{i}(s)\right| \int_{0}^{t}\left|\partial_{x}^{i} \partial_{y}^{j} \mathcal{R}_{n u m, l}(s)\right| d s \leq E_{4}(t) \cdot \sum_{i \leq n}\left|c_{i}\right| \int_{0}^{\infty}\left|\partial_{x}^{i} \partial_{y}^{j} \mathcal{R}_{n u m, l}(s)\right| d s \tag{5.79}
\end{equation*}
$$

uniformly in $t$ using monotonicity. The error and similar quantities are integrable in time since the approximate solution $\hat{G}_{i}(s)$ and residual error $\hat{\mathcal{R}}_{n u m, i}$ can be decomposed into $A_{i}(t)+B_{i}(t)$ with $A_{i}(t)$ compactly supported in time, and $B_{i}(t)$ decays exponentially fast in $t$. See Section 3.5 of Part II [15] for more discussions. Moreover, we completely decouple the numerical solution $\hat{\mathcal{R}}_{i}$ and the time-depend factor $b_{i}(t)$.
Remark 5.3. Using linearity and the triangle inequality, we can assemble the estimates for $\mathcal{R}$ from the estimates of each mode $\int_{0}^{t} b_{i}(s) \hat{\mathcal{R}}_{i}(t-s) d s$. In practice, this means that we can implement the above estimate for each individual mode completely in parallel.

Although we estimate $\partial_{1}^{\alpha} \partial_{2}^{\beta} G, \partial_{1}^{\alpha} \partial_{2}^{\beta} \mathcal{R}$ (5.74), (5.75) by applying triangle inequality and combining the estimates of different modes (5.79), such an estimate does not lead to a constant of $O(n)$ since different solutions are large in different regions. In fact, when we construct approximations for the velocity in Section 4.3, we apply some partition of unity. The coefficients of different approximations are large in different regions. These coefficients are the initial conditions for the approximate solution (4.19) (see Appendix C.2.1). We can exploit these properties in the above estimates for $\mathcal{R}, G(5.79)$ and do not obtain a large constant.
5.8. Estimate of the nonlocal error and modified decomposition. To construct the approximate steady state for the velocity, since $\overline{\mathbf{u}}=\nabla^{\perp} \bar{\phi}, \bar{\phi}=(-\Delta)^{-1} \bar{\omega}$ depend on $\bar{\omega}$ nonlocally, we solve $-\Delta \bar{\phi}=\bar{\omega}$ numerically to obtain the numerical stream function $\bar{\phi}^{N}$ (see Section 77), which has an error. In the residual operator, we have a similar error (5.74). To estimate these
errors effectively, we combine the estimates of nonlocal error and $\mathbf{u}(\omega)$ in the energy estimate. Other errors depend on the numerical bases locally, e.g. piecewise polynomials and semi-analytic functions, which we can estimate using standard numerical analysis and the methods in Section 3.6 in Part II [15]. Denote

$$
\begin{equation*}
\bar{\varepsilon}=\bar{\omega}-(-\Delta) \bar{\phi}^{N}, \quad \overline{\mathbf{u}}^{N}=\mathbf{u}\left(-\Delta \bar{\phi}^{N}\right)=\nabla^{\perp} \bar{\phi}^{N}, \quad \bar{c}_{\omega}^{N}=\bar{u}_{x}^{N}(0)+\frac{\bar{c}_{l}}{2} \tag{5.80}
\end{equation*}
$$

We introduce $\bar{c}_{\omega}^{N}$ since $\bar{c}_{\omega}$ chosen in (2.11) depends on $u_{x}(\bar{\omega})(0)$. The solution $\bar{\phi}^{N}$ we constructed only satisfies $\bar{\varepsilon}=O\left(|x|^{2}\right)$. Recall the finite rank approximation $\hat{\mathbf{u}}, \widehat{\nabla \mathbf{u}}$ (4.38). To apply the functional inequalities to $\mathbf{u}(\bar{\varepsilon})$, we correct $\bar{\varepsilon}$ near 0 . Similar consideration applies to $\hat{\varepsilon}$ (5.74). For an error $\varepsilon=\bar{\varepsilon}$ or $\hat{\varepsilon}$, and some cutoff function $\chi_{\varepsilon}=1+O\left(|x|^{4}\right)$ near 0 , we decompose

$$
\begin{align*}
& \varepsilon=\varepsilon_{1}+\varepsilon_{2}, \quad \varepsilon_{2}=\varepsilon_{x y}(0) \Delta\left(\frac{x^{3} y}{2} \chi_{\varepsilon}\right), \quad \mathbf{u}\left(\varepsilon_{2}\right)=\nabla^{\perp}(-\Delta)^{-1} \varepsilon_{2}=\frac{1}{2} \varepsilon_{x y}(0) \nabla^{\perp}\left(x^{3} y \chi_{\varepsilon}\right)  \tag{5.81}\\
& \mathbf{u}(\varepsilon)=\mathbf{u}\left(\varepsilon_{1}\right)+\mathbf{u}\left(\varepsilon_{2}\right)=\mathbf{u}_{A}\left(\varepsilon_{1}\right)+\hat{\mathbf{u}}\left(\varepsilon_{1}\right)+\mathbf{u}\left(\varepsilon_{2}\right), \quad u_{x}(\varepsilon)(0)=u_{x}\left(\varepsilon_{1}\right)(0)
\end{align*}
$$

where $u_{x}\left(\varepsilon_{2}\right)(0)=-\varepsilon_{x y}(0) /\left.2 \cdot \partial_{y}\left(x^{3} y \chi_{\varepsilon}\right)\right|_{x, y=0}=0$. We perform a similar decomposition for $\nabla \mathbf{u}(\varepsilon)$. We choose $\chi_{\varepsilon}$ in the above form such that $\varepsilon_{2}=\chi_{x y}(0) x y+O\left(|x|^{4}\right)$ and we can obtain $\mathbf{u}\left(\chi_{\varepsilon}\right)$ explicitly. We choose $\chi_{\bar{\varepsilon}}$ for $\bar{\varepsilon}$ and $\chi_{\hat{\varepsilon}}$ for $\hat{\varepsilon}$ in (C.9) and they have different parameters. We obtain weighted $L^{\infty}$ estimate of $\mathbf{u}_{A}\left(\varepsilon_{1}\right)$ and $C^{1 / 2}$ estimate of $\psi_{1}(\nabla \mathbf{u})_{A}\left(\varepsilon_{1}\right), \psi_{u} \mathbf{u}_{A}\left(\varepsilon_{1}\right)$ with $\varepsilon_{1}=\bar{\varepsilon}_{1}, \hat{\varepsilon}_{1}$ using the functional inequalities in Lemma 2.2 with norms $\left\|\omega \varphi_{\text {elli }}\right\|_{\infty},\left[\omega \psi_{1}\right]_{C_{x_{i}}^{1 / 2}}$ (C.2), (C.1).

Let $\omega$ be the perturbation without decomposition. Recall $\mathbf{u}(f)=\mathbf{u}_{A}(f)+\hat{\mathbf{u}}(f)=\mathbf{u}_{A}(f)+$ $u_{x}(f)(0)(x,-y)+\tilde{\hat{\mathbf{u}}}(f)$ from (4.8), (4.38), (5.2). We combine these errors and perturbations and perform the following decompositions

$$
\begin{align*}
& \mathbf{U}=(U, V)=\mathbf{u}(\omega+\bar{\varepsilon}), \quad \mathbf{U}_{A}=\mathbf{u}_{A}\left(\omega_{1}+\bar{\varepsilon}_{1}+\hat{\varepsilon}_{1}\right), \quad \mathbf{U}=\mathbf{U}_{A}+U_{x}(0)(x,-y)+\mathbf{U}_{a p p} \\
& \mathbf{U}_{a p p}=\tilde{\hat{\mathbf{u}}}\left(\omega_{1}+\bar{\varepsilon}_{1}+\hat{\varepsilon}_{1}\right)+\mathbf{u}\left(\bar{\varepsilon}_{2}+\hat{\varepsilon}_{2}\right)+\tilde{\mathbf{u}}\left(-\Delta \hat{\phi}^{N}\right) \tag{5.82}
\end{align*}
$$

We do not put $\mathbf{u}\left((-\Delta) \hat{\phi}^{N}\right)$ in $\mathbf{U}_{A}$, and use $\mathbf{U}_{A}$ to denote the variable we estimate using Lemma 2.2. Similarly, we decompose $\nabla \mathbf{U}$ and define $(\nabla \mathbf{U})_{A},(\nabla \mathbf{U})_{\text {app }}$. The term $\mathbf{U}_{\text {app }}$ is more regular, and will be used later in Section 5.9 for nonlinear estimate. The terms $\hat{\mathbf{u}}\left(\varepsilon_{1}\right), \tilde{\hat{\mathbf{u}}}\left(\varepsilon_{1}\right), \mathbf{u}\left(\varepsilon_{2}\right), \varepsilon=\bar{\varepsilon}, \hat{\varepsilon}$ only depend on $\varepsilon$ via $\varepsilon_{x y}(0)$ and finite many integrals $(<50)$ (4.38) with smooth coefficients. We estimate the piecewise bounds of $\varepsilon$ following Section 3.6 in Part II 15], then estimate these integrals and piecewise $C^{3}$ bounds of these terms. We estimate $\tilde{\hat{\mathbf{u}}}\left(\omega_{1}\right)$ following Section 5.6 and $\mathbf{u}\left(\hat{\phi}^{N}\right)=\nabla^{\perp} \hat{\phi}^{N}$ using (5.76), (5.79). We factor out $U_{x}(0)(x,-y)$ in (5.82) since our estimate for such a term is larger than others. See Section 5.9.1. Since we have piecewise bounds of $\hat{\varepsilon}$ by $C_{\varepsilon}(x) E_{4}(5.76)$, (5.79), (5.78) and of $\bar{\varepsilon}$ by $C_{\bar{\varepsilon}}(x)$ with very small $C_{\bar{\varepsilon}}(x)$, when we combine the estimates of $\omega, \varepsilon, \hat{\varepsilon}$, the upper bounds of $\mathbf{U}$ are given by, e.g.

$$
\begin{equation*}
\left|U_{A}(x)\right| \leq C(x) E_{1}+C_{\varepsilon}(x) E_{4}+C_{\bar{\varepsilon}}(x) \tag{5.83}
\end{equation*}
$$

To verify nonlinear stability A.11, we further bound $E_{1}, E_{4}$ by the bootstrap threshold $E_{*}$ (5.101). Then the above upper bound become a concrete value. In practice, we track this value to combine the estimate of $\mathbf{u}(\omega), \mathbf{u}(\bar{\varepsilon}), \mathbf{u}(\hat{\varepsilon})$.

From (2.11), (2.12) and the above decomposition, we have

$$
\begin{equation*}
\overline{\mathbf{u}} \triangleq \mathbf{u}(\bar{\omega})=\overline{\mathbf{u}}^{N}+\mathbf{u}(\bar{\varepsilon}), \quad \mathbf{u}(\omega+\bar{\omega})=\overline{\mathbf{u}}^{N}+\mathbf{U}, \quad \bar{c}_{\omega}+c_{\omega}(\omega)=\bar{c}_{\omega}^{N}+U_{x}(0) \tag{5.84}
\end{equation*}
$$

Next, we use the above decomposition and (5.75), (5.77) to rewrite the decomposition (4.21).
Modified nonlinear terms. Firstly, we combine the nonlocal error from $I-D_{i}^{2} I(0) f_{\chi, i}$ for $I=\mathcal{B}_{o p, i}\left(\mathbf{u}(\bar{\varepsilon}), \widehat{W}_{2}\right)$ from the residual operator (5.75), (5.77), (4.21), the terms involving $\mathbf{u}(\bar{\varepsilon})$ in the linearized equations of (5.1), (5.5)

$$
\begin{align*}
& -\mathbf{u}(\bar{\varepsilon}) \cdot \nabla \omega_{1}+c_{\omega}(\bar{\omega}) \omega_{1}=\mathcal{B}_{o p, 1}\left(\mathbf{u}(\bar{\varepsilon}), W_{1}\right) \\
& -\mathbf{u}(\bar{\varepsilon}) \cdot \nabla \eta_{1}+2 c_{\omega}(\bar{\omega}) \eta_{1}-u_{x}(\bar{\varepsilon}) \eta_{1}-v_{x}(\bar{\varepsilon}) \xi_{1}=\mathcal{B}_{o p, 2}\left(\mathbf{u}(\bar{\varepsilon}), W_{1}\right)  \tag{5.85}\\
& -\mathbf{u}(\bar{\varepsilon}) \cdot \nabla \xi_{1}+2 c_{\omega}(\bar{\omega}) \xi_{1}-u_{y}(\bar{\varepsilon}) \eta_{1}-v_{y}(\bar{\varepsilon}) \xi_{1}=\mathcal{B}_{o p, 3}\left(\mathbf{u}(\bar{\varepsilon}), W_{1}\right)
\end{align*}
$$

which has vanishing order $O\left(|x|^{3}\right)$ and thus $D_{i}^{2} B_{o p, i}\left(\mathbf{u}(\bar{\varepsilon}), W_{1}\right)(0)=0$, and the nonlinear term $\mathcal{N}_{i}$ (2.18) with the nonlinear rank-one correction (4.11) in (4.21), we yield

$$
\begin{align*}
I_{i} & =\mathcal{B}_{o p, i}\left(\mathbf{u}(\bar{\varepsilon}), \hat{W}_{2}\right)+\mathcal{B}_{o p, i}\left(\mathbf{u}(\bar{\varepsilon}), W_{1}\right)+\mathcal{B}_{o p, i}(\mathbf{u}(\omega),(\omega, \eta, \xi)) \\
& =\mathcal{B}_{o p, i}(\mathbf{u}(\bar{\varepsilon})+\mathbf{u}(\omega),(\omega, \eta, \xi))=\mathcal{B}_{o p, i}(\mathbf{U},(\omega, \eta, \xi))  \tag{5.86}\\
\tilde{\mathcal{N}}_{i} & =I_{i}-D_{i}^{2} I_{i}(0) f_{\chi, i}=\mathcal{B}_{o p, i}\left(\mathbf{U}, W_{1}\right)+\mathcal{B}_{o p, i}\left(\mathbf{U}, \hat{W}_{2}\right)-U_{x}(0) D_{i}^{2}\left(W_{1, i}+\widehat{W}_{2, i}\right)(0) f_{\chi, i}
\end{align*}
$$

where we have used $W_{1}+\hat{W}_{2}=(\omega, \eta, \xi), D^{2}=\left(\partial_{x y}, \partial_{x y}, \partial_{x}^{2}\right)$, the fact that $B_{o p i}$ (2.17) is bilinear, and $\mathbf{U}=U_{x}(0)(x,-y)+O\left(|x|^{2}\right), \omega, \eta, \xi=O\left(|x|^{2}\right)$ to obtain

$$
D_{i}^{2} \mathcal{B}_{o p, i}(\mathbf{U},(\omega, \eta, \xi))(0)=U_{x}(0) V_{i}, \quad V=\left(\partial_{x y} \omega(0), \partial_{x y} \eta(0), \partial_{x x} \xi(0)\right)
$$

We remark that the full solution $\eta=\theta_{x}, \xi=\theta_{y}$ satisfies $\eta_{y}=\xi_{x}$.
Modified residual error. We decompose residual error (2.19) and the remaining part in (5.77) (see also (5.75)) into the essentially local part and nonlocl part. Recall the general bilinear operator (2.16) $\mathcal{B}_{o p}\left(\mathbf{u}_{A},(\nabla \mathbf{u})_{A}, G\right)$. We decomposed $\overline{\mathbf{u}}=\mathbf{u}(\bar{\omega})=\left(\overline{\mathbf{u}}-\mathbf{u}_{A}\left(\bar{\varepsilon}_{1}\right)\right)+\mathbf{u}_{A}\left(\bar{\varepsilon}_{1}\right)$ (5.81) and modify (2.19), (4.21) by replacing $\overline{\mathbf{u}}$ by $\overline{\mathbf{u}}^{N}-\mathbf{u}_{A}\left(\bar{\varepsilon}_{1}\right)$

$$
\begin{align*}
\overline{\mathcal{F}}_{l o c, i} & =I I_{i}-D_{i}^{2} I I_{i}(0) f_{\chi, i}=I I_{i}-D_{i}^{2} \overline{\mathcal{F}}_{i}(0) f_{\chi, i} \\
I I_{i} & =\overline{\mathcal{F}}_{i}-\mathcal{B}_{o p, i}\left(\left(\mathbf{u}_{A}\left(\bar{\varepsilon}_{1}\right),(\nabla \mathbf{u}){ }_{A}\left(\bar{\varepsilon}_{1}\right)\right), \bar{W}\right) \\
I I_{1} & =-\left(\bar{c}_{l} x+\overline{\mathbf{u}}-\mathbf{u}_{A}\left(\bar{\varepsilon}_{1}\right)\right) \cdot \nabla \bar{\omega}+\bar{\theta}_{x}+\bar{c}_{\omega} \bar{\omega}  \tag{5.87}\\
I I_{2} & =-\left(\bar{c}_{l} x+\overline{\mathbf{u}}-\mathbf{u}_{A}\left(\bar{\varepsilon}_{1}\right)\right) \cdot \nabla \bar{\theta}_{x}+2 \bar{c}_{\omega} \bar{\theta}_{x}-\left(\bar{u}_{x}-u_{x, A}\left(\bar{\varepsilon}_{1}\right)\right) \bar{\theta}_{x}-\left(\bar{v}_{x}-v_{x, A}\left(\bar{\varepsilon}_{1}\right)\right) \bar{\theta}_{y} \\
I I_{3} & =-\left(\bar{c}_{l} x+\overline{\mathbf{u}}-\mathbf{u}_{A}\left(\bar{\varepsilon}_{1}\right)\right) \cdot \nabla \bar{\theta}_{y}+2 \bar{c}_{\omega} \bar{\theta}_{y}-\left(\bar{u}_{y}-u_{y, A}\left(\bar{\varepsilon}_{1}\right)\right) \bar{\theta}_{x}-\left(\bar{v}_{y}-v_{y, A}\left(\bar{\varepsilon}_{1}\right)\right) \bar{\theta}_{y}
\end{align*}
$$

From our construction in Section 4.3, we have $\mathbf{u}_{A}=O\left(|x|^{3}\right),(\nabla \mathbf{u})_{A}=0$. Thus we have $D_{i}^{2} I I_{i}(0)=D_{i}^{2} \overline{\mathcal{F}}_{i}(0)$. From (5.81), (5.84) and the discussion below, the above error $\overline{\mathcal{F}}_{\text {loc }, i}$ essentially depends on the numerical construction locally. Similarly, we decompose the remaining part of the residual error in (5.75) as follows

$$
\begin{align*}
\mathcal{R}_{l o c, i}= & \mathcal{R}_{l o c, 0, i}+D_{i}^{2} \mathcal{B}_{o p, i}\left(\mathbf{u}(\bar{\varepsilon}), \hat{W}_{2}\right)(0)\left(\chi_{i 2}-f_{\chi, i}\right) \\
& +\mathcal{B}_{o p, i}(\mathbf{u}(\hat{\varepsilon}), \bar{W})-D_{i}^{2} \mathcal{B}_{o p, i}(\mathbf{u}(\hat{\varepsilon}), \bar{W})(0) f_{i 2}-\mathcal{B}_{o p, i}\left(\mathbf{u}_{A}\left(\hat{\varepsilon}_{1}\right),(\nabla \mathbf{u})_{A}\left(\hat{\varepsilon}_{1}\right), \bar{W}\right) \tag{5.88}
\end{align*}
$$

The remaining part $\mathcal{B}_{o p, i}(\mathbf{u}(\hat{\varepsilon}), \bar{W})-\mathcal{B}_{o p, i}\left(\mathbf{u}_{A}\left(\hat{\varepsilon}_{1}\right),(\nabla \mathbf{u})_{A}\left(\hat{\varepsilon}_{1}\right), \bar{W}\right)$. is essentially local.
Modified bad terms. Recall $\mathbf{U}_{A}$ from (5.82). We combine the terms $\mathbf{u}_{A}\left(\hat{\varepsilon}_{1}\right)$ (5.88), $\mathbf{u}_{A}\left(\bar{\varepsilon}_{1}\right)$ (5.87) and the bad terms (5.5) with $\overline{\mathbf{u}}$ replaced by $\overline{\mathbf{u}}^{N}$

$$
\begin{align*}
B_{\text {modi }, i}(x)= & B_{i}(x)-\mathcal{B}_{o p, i}\left(\mathbf{u}(\bar{\omega}), W_{1}\right)+\mathcal{B}_{o p, i}\left(\mathbf{u}_{A}(\bar{\varepsilon}),(\nabla \mathbf{u})_{A}(\bar{\varepsilon}), \bar{W}\right) \\
& +\mathcal{B}_{o p, i}\left(\mathbf{u}_{A}(\hat{\varepsilon}),(\nabla \mathbf{u})_{A}(\hat{\varepsilon}), \bar{W}\right) \\
B_{\text {modi }, 1}(x) \triangleq & \eta_{1}-\mathbf{U}_{A} \cdot \nabla \bar{\omega}, \quad B_{\text {modi }, 2}(x) \triangleq-\bar{v}_{x}^{N} \xi_{1}-\mathbf{U}_{A} \cdot \nabla \bar{\theta}_{x}-\mathbf{U}_{x, A} \cdot \nabla \bar{\theta}  \tag{5.89}\\
B_{\text {modi }, 3}(x) \triangleq & -\bar{u}_{y}^{N} \eta_{1}-\mathbf{U}_{A} \cdot \nabla \bar{\theta}_{y}-\mathbf{U}_{y, A} \cdot \nabla \bar{\theta}
\end{align*}
$$

We replace $\overline{\mathbf{u}}$ by $\overline{\mathbf{u}}^{N}$ since we put the difference $\mathbf{u}(\bar{\varepsilon})$ to (5.86).
Using the above decompositions, we modify the linearized equations of (4.21), (5.1) as follows

$$
\begin{align*}
\partial_{t} \omega_{1}+\left(\bar{c}_{l} x+\overline{\mathbf{u}}^{N}\right) \cdot \nabla \omega_{1} & =\bar{c}_{\omega}^{N} \omega_{1}+B_{m o d i, 1}+\tilde{N}_{1}+\overline{\mathcal{F}}_{l o c, 1}-\mathcal{R}_{l o c, 1} \\
\partial_{t} \eta_{1}+\left(\bar{c}_{l} x+\overline{\mathbf{u}}^{N}\right) \cdot \nabla \eta_{1} & =\left(2 \bar{c}_{\omega}^{N}-\bar{u}_{x}^{N}\right) \eta_{1}+B_{m o d i, 2}+\tilde{N}_{2}+\overline{\mathcal{F}}_{l o c, 2}-\mathcal{R}_{l o c, 2}  \tag{5.90}\\
\partial_{t} \xi_{1}+\left(\bar{c}_{l} x+\overline{\mathbf{u}}^{N}\right) \cdot \nabla \xi_{1} & =\left(2 \bar{c}_{\omega}^{N}-\bar{v}_{x}^{N}\right) \xi_{1}+B_{m o d i, 3}+\tilde{N}_{3}+\overline{\mathcal{F}}_{l o c, 3}-\mathcal{R}_{l o c, 3}
\end{align*}
$$

The linear energy estimates in Sections 5.3/5.6] can be rederived directly for (5.90) in terms of $\overline{\mathbf{u}}^{N}, \bar{c}_{\omega}^{N}$, and we obtain (D.4), (D.8), (D.11). Note that we also modify the damping coefficients $d_{i, L}$ (5.4) to $d_{i, L}^{n u m}$ (D.3).

Remark 5.4. The errors $\hat{\varepsilon}_{1}, \bar{\varepsilon}_{1}$ (5.81) are much smaller than $\omega_{1}$ at the bootstrap threshold $E_{*}$ (5.101) in the region where we have small damping factor. We combine the estimates of error terms (5.81) and the perturbation to simplify the nonlocal error estimate significantly. For
readability, the reader can simply treat $\hat{\varepsilon}_{1}, \bar{\varepsilon}_{1}$ as 0 and $\overline{\mathbf{u}}^{N}=\overline{\mathbf{u}}=\mathbf{u}(\bar{\omega}), \mathbf{U}=\mathbf{u}(\omega), \mathbf{U}_{A}=\mathbf{u}_{A}\left(\omega_{1}\right)$. We do not apply standard estimates for the operator $\nabla^{\perp}(-\Delta)^{-1} \varepsilon, \nabla^{\perp}(-\Delta)^{-1}(\varepsilon)$ to obtain the error bounds of $\mathbf{u}(\varepsilon), \nabla \mathbf{u}(\varepsilon)$ from those of $\varepsilon$ since such error estimates are not small enough to close the estimate, and we need weighted estimates for the error.

Using (5.79) and the methods in Section 3.6 and following the estimate in Appendix C. 4 of Part II [15], we can control the local part of the residual operator $\mathcal{R}_{\text {loc, } i}$ (5.88). In Figure 11] we plot the rigorous piecewise $L^{\infty}\left(\varphi_{i}\right)$ bound $C_{k l}$ for $\mathcal{R}_{l o c, i} \leq C_{k l} E_{4}$ in $\left.\left[y_{k}, y_{k+1}\right] \times\left[y_{l}, y_{l+1}\right]\right]$ with adaptive mesh $y_{k}$ in Section 7 .

In the near-field $|x| \leq 10^{4}<y_{550}$, we have $\mathcal{R}_{l o c, 1} \varphi_{1} \leq 0.008 E_{4}, \mathcal{R}_{l o c, 2} \varphi_{2} \leq 0.009 E_{4}$. We have $\left|\mathcal{R}_{l o c, 3}\right| \varphi_{3} \leq 0.002 E_{4}$ for $x$ in the mesh. The near-field region with a large weighted error is about $\left[0, y_{100}\right]^{2}$ with $y_{100} \approx 0.4863$. In such a region, the error is much smaller than the remaining damping part in the weighted $L^{\infty}\left(\varphi_{i}\right)$ estimate. See Figure 6, In the far-field region with a large weighted error $\left(\mathcal{R}_{1} \phi_{1} \leq 0.016 E_{4}, \mathcal{R}_{2} \phi_{2} \leq 0.009 E_{4}\right)$, we have $|x| \geq 10^{4}$ and have a large damping coefficient. We can further reduce the error in the far-field by performing error estimates with a finer mesh and use a larger computational domain. The estimate from the nonlinear modes $R_{n l}$ (4.20) is very small compared to the above bounds, and we have bounded it under the bootstrap assumption $E_{4}<E_{*}=5 \cdot 10^{-6}$, which will be discussed in Section 5.9,


Figure 11. Weighted estimate of the local residual operator under adaptive mesh. Left: piecewise rigorous bound for $\mathcal{R}_{l o c, 1} \varphi_{1}$ in the $\omega$ equation. Right: piecewise rigorous bound for $\mathcal{R}_{l o c, 2} \varphi_{2}$ in the $\eta$ equation.

Using the piecewise weighted $L^{\infty}$ and $C^{1}$ bound of $\mathcal{R}_{l o c, i}$ and the method in Appendix E of Part II [15], we derive the piecewise $C_{x_{i}}^{1 / 2}$ estimate of $\mathcal{R}_{l o c, i} \psi_{i}$. We combine the Hölder estimate of $\mathcal{R}_{\text {loc, } i} \psi_{i}$ with the energy estimate in Section 5.4. Such an estimate is very small compared to the least damping coefficients (near $x=(0.5,0)$ see Figure 9) since the estimates of $\mathcal{R}_{\text {loc }, i}$ are much smaller near $x=(0.5,0)$, and we have a small factor $\tau_{1}^{-1}$ for $\mathcal{R}_{l o c, i} \psi_{i}$ from the weight $\tau_{1}$ in $E_{2}$ (5.27) in the weighted Hölder energy estimate.
5.9. Nonlinear estimates. Using the energy $E_{4}$ (5.70), we can control the $L^{\infty}$ norm of $W_{1, i}, \nabla \mathbf{U}, \hat{W}_{2}$ following Sections 5.6, 5.7 and close the nonlinear estimates. To establish nonlinear stability, we need to check the condition (A.11). The nonlinear estimates to be established are similar to the following

$$
\frac{d}{d t} E_{4} \leq-\lambda E_{4}+C E_{4}^{2}+\varepsilon
$$

Here, $-\lambda E_{4}^{2}$ with $\lambda>0$ comes from linear stability, $C E_{4}^{2}$ with some constant $C(\bar{\omega}, \bar{\theta}, \psi, \varphi)>0$ controls the nonlinear terms, and $\varepsilon$ is the weighted norm of the residual error of the approximate steady state. To close the bootstrap argument $E_{4}(t)<E^{*}$ with some threshold $E^{*}>0$, a sufficient condition is that $\varepsilon<\varepsilon^{*}=\lambda^{2} /(4 C)$, which provides an upper bound on the required accuracy of the approximate steady state. Condition (A.11) provides similar constraints on
the error $\bar{\varepsilon}(x)$ for different $x$. A significant difference between this step and the previous linear stability estimate is that we have a small parameter $\varepsilon$. As long as $\varepsilon$ is sufficiently small, thanks to the linear damping term $-\lambda E_{4}$, we can afford a large constant $C(\bar{\omega}, \bar{\theta}, \psi, \varphi)$ in the estimate of the nonlinear terms and close the nonlinear estimates. We discuss the construction of approximate steady state with small error $\varepsilon$ in Section 7 . We refer more discussion of this philosophy to [19|20]. Thus, the nonlinear stability estimate of $W_{1}=\left(\omega_{1}, \eta_{1}, \xi_{1}\right)$ in (4.21) is much simpler.

We perform energy estimate on (5.90) modifying the decomposition in (4.21). In the estimate of the weighted quantity $W_{1, i} \rho$, we have

$$
\begin{equation*}
\rho\left(\mathbf{U} \cdot \nabla W_{1, i}\right)=\mathbf{U} \cdot \nabla\left(W_{1, i} \rho\right)+\rho^{-1}(\mathbf{U} \cdot \nabla \rho)\left(\omega_{1} \rho\right), \quad \mathcal{T}_{d, N}(\rho) \triangleq \rho^{-1}(\mathbf{U} \cdot \nabla \rho) \tag{5.91}
\end{equation*}
$$

From Lemma A.2, we do not need to estimate the first advection term. See (5.6) and (5.4). Using the above computation, the decomposition (5.82) $\mathbf{U}=\tilde{\mathbf{U}}+U_{x}(0)(x,-y)$ and rewriting $2 U_{x}(0) \eta_{1}-U_{x} \eta_{1}-V_{x} \xi_{1}=U_{x}(0) \eta_{1}-\tilde{U}_{x} \eta_{1}-\tilde{V}_{x} \xi_{1}, \quad 2 U_{x}(0) \xi_{1}-U_{y} \eta_{1}-V_{y} \xi_{1}=3 U_{x}(0) \xi_{1}-\tilde{U}_{y} \eta_{1}-\tilde{V}_{y} \xi_{1}$ in $\mathcal{B}_{o p, i}\left(\mathbf{U}, W_{1}\right)$ (5.86), (2.17), we need to estimate the following nonlinear terms

$$
\begin{align*}
& \tilde{\mathcal{N}}_{1}\left(\rho_{1}\right)=\left(\mathbf{U} \cdot \nabla \rho_{1}\right) \cdot \omega_{1}+U_{x}(0) \omega_{1} \rho_{1}+\mathcal{B}_{o p, 3}\left(\mathbf{U}, \hat{W}_{2}\right) \rho_{1}-U_{x}(0) \omega_{x y}(0) f_{\chi, 1} \rho_{1}  \tag{5.92}\\
& \tilde{\mathcal{N}}_{2}\left(\rho_{2}\right)=\left(\mathbf{U} \cdot \nabla \rho_{2}\right) \cdot \eta_{1}+\left(U_{x}(0) \eta_{1}-\tilde{U}_{x} \eta_{1}-\tilde{V}_{x} \xi_{1}\right) \rho_{2}+\mathcal{B}_{o p, 2}\left(\mathbf{U}, \widehat{W}_{2}\right) \rho_{2}-U_{x}(0) \eta_{x y}(0) f_{\chi, 2} \rho_{2} \\
& \tilde{\mathcal{N}}_{3}\left(\rho_{3}\right)=\left(\mathbf{U} \cdot \nabla \rho_{3}\right) \cdot \xi_{1}+\left(3 U_{x}(0) \xi_{1}-\tilde{U}_{y} \eta_{1}-\tilde{V}_{y} \xi_{1}\right) \rho_{3}+\mathcal{B}_{o p, 3}\left(\mathbf{U}, \widehat{W}_{2}\right) \rho_{3}-U_{x}(0) \xi_{x x}(0) f_{\chi, 3} \rho_{3}
\end{align*}
$$

Recall from the discussion in Section 4.2 .4 that $\omega=\omega_{1}+\hat{\omega}_{2}$ and $\omega_{1}=O\left(|x|^{3}\right)$ near 0 . We have $\omega_{x y}(0)=\partial_{x y} \hat{\omega}_{20}$. Using (2.17), (5.82), we further decompose the nonlinear terms of $\mathbf{U}, \hat{W}_{2}$

$$
\begin{align*}
& \mathcal{B}_{o p, i}\left(\mathbf{U}, \hat{W}_{2}\right)-U_{x}(0) D_{i}^{2} \hat{W}_{2, i}(0) f_{\chi, i}=\mathcal{B}_{o p, i}\left(\tilde{\mathbf{U}}, \hat{W}_{2}\right)+I I_{i} \\
& I I_{i}=\mathcal{B}_{o p, i}\left(U_{x}(0)(x,-y), \hat{W}_{2}\right)-U_{x}(0) D_{i}^{2} \hat{W}_{2, i}(0) f_{\chi, i}=U_{x}(0) \hat{W}_{2, i, M}  \tag{5.93}\\
& \hat{W}_{2, \cdot, M}=\left(\hat{\omega}_{2}, \hat{\eta}_{2}, \hat{\xi}_{2}\right), \quad \hat{\omega}_{2, M} \triangleq \hat{\omega}_{2}-x \partial_{x} \hat{\omega}_{2}+y \partial_{y} \hat{\omega}_{2}-\hat{\omega}_{2, x y}(0) f_{\chi, 1} \\
& \hat{\eta}_{2, M} \triangleq \hat{\eta}_{2}-x \partial_{x} \hat{\eta}_{2}+y \partial_{y} \hat{\eta}_{2}-\hat{\eta}_{2, x y}(0) f_{\chi, 2}, \quad \hat{\xi}_{2, M} \triangleq 3 \hat{\xi}_{2}-x \partial_{x} \hat{\xi}_{2}+y \partial_{y} \hat{\xi}_{2}-\hat{\xi}_{2, x x}(0) f_{\chi, 3}
\end{align*}
$$

Note that in $\mathcal{B}_{o p, i}\left(\tilde{\mathbf{U}}, \hat{W}_{2}\right)$ in (2.17), the term $\tilde{U}_{x}(0)=0$ vanishes.
Using the above derivations and (5.90), for $C .(x)=\left(\bar{c}_{\omega}^{N}, 2 \bar{c}_{\omega}^{N}-\bar{u}_{x}^{N}, 2 \bar{c}_{\omega}^{N}+\bar{u}_{x}^{N}\right)$, we get
$\partial_{t}\left(W_{1, i} \rho_{i}\right)+\left(\bar{c}_{l} x+\overline{\mathbf{u}}^{N}+\mathbf{U}\right) \cdot \nabla\left(W_{1, i} \rho_{i}\right)=C_{i}(x) W_{1, i} \rho_{i}+B_{m o d i, i} \rho_{i}+\tilde{N}_{i}\left(\rho_{i}\right)+\left(\overline{\mathcal{F}}_{l o c, i}+\mathcal{R}_{l o c, i}\right) \rho_{i}$.
5.9.1. The main nonlinear term. Recall that we have large constants in the estimate of $c_{\omega}(\omega), c_{\omega}\left(W_{1}\right)$

$$
\begin{equation*}
\left|c_{\omega}(\omega)\right|=\left|u_{x}(\omega)(0)\right| \leq \mu_{6} E_{4}, \quad\left|c_{\omega}\left(\omega_{1}\right)\right|=\left|u_{x}\left(\omega_{1}\right)(0)\right| \leq \mu_{5} E_{4} \tag{5.94}
\end{equation*}
$$

using the energy $E_{4}$ (5.70). Compared to $c_{\omega}(\omega)$, at the bootstrap threshold $E_{4}=E_{*}$ (5.101), the error $\left|u_{x}(\bar{\varepsilon})\right|<10^{-2} E_{*}$ is much smaller and we have $U_{x}(0) \approx c_{\omega}$. From (5.78) and Section 5.7. we need to pay a large constant $c_{1}=\mu_{5}$ in our estimate of $\hat{\omega}_{2}$. Then for $U_{x}(0) \widehat{W}_{2, i, M}$ in (5.93), we have a large constant $\mu_{5} \mu_{6}$, with $\mu_{5} \mu_{6} \approx 4700$. In comparison, for $\omega_{1}$, using $E_{4}$ (5.70), we have $\omega_{1} \varphi_{1} \leq E_{4}$ with constant 1 . Similarly, the velocity with approximation $\mathbf{U}_{A} \approx \mathbf{u}_{A}\left(\omega_{1}\right)$ (5.2), (5.82) also has size of order 1.

Note that we also have a large constant $\mu_{6}^{2}$ in the estimate of nonlinear terms for $\mathcal{B}_{o p, i}\left(\tilde{\mathbf{U}}, \widehat{W}_{2}\right)$ from (5.78) since $\tilde{\mathbf{U}}$ contains $\mathbf{u}\left(\hat{\omega}_{2}\right)$. Since $\tilde{\mathbf{U}}=O\left(|x|^{3}\right), \widehat{W}_{2}=O\left(|x|^{2}\right)$, these nonlinear terms have a higher vanishing order $O\left(|x|^{4}\right)$ near $x=0$. Since $\hat{\omega}_{2}$ decays and the weights are singular near 0 , our estimates of these nonlinear term are smaller than $U_{x}(0) \widehat{W}_{2, i, M}$ (5.93), and thus the latter is the main nonlinear term in (5.92), (5.93).
5.9.2. $L^{\infty}$ estimates. Using (5.82), (4.8), we decompose $\tilde{\mathbf{U}}$

$$
\begin{align*}
& \tilde{\mathbf{U}}=\mathbf{U}_{A}+\mathbf{U}_{a p p} \\
& \mathcal{B}_{o p, i}\left(\tilde{\mathbf{U}}, \widehat{W}_{2}\right)=\mathcal{B}_{o p, i}\left(\mathbf{U}_{A},(\nabla \mathbf{U})_{A}, \widehat{W}_{2}\right)+\mathcal{B}_{o p, i}\left(\mathbf{U}_{a p p},(\nabla \mathbf{U})_{a p p}, \widehat{W}_{2}\right) \triangleq I_{1}+I_{2} \tag{5.95}
\end{align*}
$$

We estimate $C^{3}$ bounds of the $\widehat{W}_{2}$ terms in (5.93), (5.92) using (5.79), (5.76) and following Section 3.7 of Part II [15]. Then we can estimate $\hat{W}_{2, M, i}$ (5.93) and apply the same estimate for
the nonlocal terms $\left(\mathbf{u}_{A}, \bar{W}\right)$ (5.1), (4.10) in Section 5.3 to $I_{1}$. From the discussion below (5.82), we can estimate piecewise $C^{3}$ bounds for $\mathbf{U}_{\text {app }}$. Then we obtain the estimate for $I_{2}$. Since for $W_{1, i}=O\left(|x|^{\beta}\right)$ with $W_{1}=\left(\omega_{1}, \eta_{1}, \xi_{1}\right)$ near $x=0$ with $\beta \in[5 / 2,3]$, each term involving $W_{1, i}$ in (5.92) vanishes $O\left(|x|^{\beta}\right)$ near $x=0$, and we can estimate their weighted $L^{\infty}\left(\varphi_{1}\right)$ norm using the energy. Moreover, we have piecewise $L^{\infty}$ bound for $\nabla \mathbf{U}, \tilde{\nabla} \mathbf{U}$. See the discussion around (5.82).

We estimate the nonlinear terms from the transport term (5.91) in $L^{\infty}(\rho)$ estimate of $W_{1, i}$ for $\left(c, W_{1, i}, \rho\right)=\left(1, W_{1, i}, \varphi_{i}\right),\left(\mu_{g, i}, W_{1, i}, \varphi_{g, i}\right),\left(\sqrt{2} \tau_{1}^{-1}, W_{1,1}, \varphi_{4}\right)$ (5.21) as follows

$$
\begin{equation*}
c\left|\frac{(\mathbf{U} \cdot \nabla \rho)}{\rho} W_{1, i} \rho\right| \leq\left(\left|\frac{U}{x} \frac{x \partial_{x} \rho}{\rho}\right|+\left\lvert\, \frac{V}{y} \frac{y \partial_{y} \rho}{\rho}\right.\right)\left\|c W_{1, i} \rho\right\|_{\infty} \leq\left(\left|\frac{U}{x} \frac{x \partial_{x} \rho_{i}}{\rho_{i}}\right|+\left\lvert\, \frac{V}{y} \frac{y \partial_{y} \rho_{i}}{\rho_{i}}\right.\right) E_{4} \tag{5.96}
\end{equation*}
$$

and estimate piecewise $L^{\infty}$ bounds for $U / x, V / y, \partial_{x_{j}} \rho_{i} / \rho_{i}$.
Remark 5.5. In the nonlinear estimates, we optimize two estimates of $\mathbf{u}_{A}\left(\omega_{1}\right)$ using the functional inequalities based on $\left\|\omega_{1} \varphi_{1}\right\|_{\infty}$ in Section 5.3 and $\left\|\omega_{1} \varphi_{g, 1}\right\|_{\infty}$ in Section 5.5 such that $\nabla \mathbf{u}_{A} \in$ $L^{\infty}$. Similarly, we apply two functional inequalities to $\mathbf{u}_{A}\left(\varepsilon_{1}\right), \varepsilon_{1}=\bar{\varepsilon}_{1}, \hat{\varepsilon}_{1}$ in Section5.8 using $\left\|\varepsilon_{1} \varphi_{\text {elli }}\right\|_{\infty},\left\|\varepsilon_{1} \varphi_{g, 1}\right\|_{\infty}$ so that $\nabla \mathbf{U} \in L^{\infty}$.

### 5.9.3. Hölder estimate of typical terms.

Nonlinear terms involving $\mathbf{U}, W_{1, i}$. We focus on a typical term $\tilde{U}_{x} \eta_{1}$ in (5.92). Using the $C^{1 / 2}$ estimate of $\psi_{1} \mathbf{U}_{A}$ by the energy, $C^{1 / 2}$ esttimate of $\psi_{1}^{-1}, \mathbf{U}_{\text {app }}$ (see below (5.82)), and (5.36), we have $C^{1 / 2}$ estimate $\tilde{\mathbf{U}}$. Then we estimate $\delta\left(\tilde{U}_{x} \cdot \eta_{1} \psi_{2}\right)$ using (5.46). For $x, z$ in the far-field, we need another decomposition and estimate since $\psi_{1}^{-1}$ (C.1) in this estimate is not bounded. We can still estimate $\delta\left(\tilde{U}_{a p p, x} \cdot \eta_{1} \psi_{2}\right)$ using (5.46). For $U_{x, A} \eta_{1} \psi_{2}$, we use (5.32), (5.34) to get

$$
\delta\left(U_{x, A} \eta_{1} \psi_{2}\right)=\delta\left(U_{x, A} \psi_{1} \cdot \eta_{1} \frac{\psi_{2}}{\psi_{1}}\right)=\delta\left(U_{x, A} \psi_{1}\right) \frac{\eta_{1} \psi_{2}}{\psi_{1}}+\left(U_{x, A} \psi_{1}\right)(z) \delta\left(\eta_{1} \frac{\psi_{2}}{\psi_{1}}\right)=I_{1}+I_{2}
$$

To bound $I_{1}$, using the energy $E_{4}$, we can bound $C^{1 / 2}$ of $U_{x, A} \psi_{1}$ and $\frac{\eta_{1} \psi_{2}}{\psi_{1}}$ in $L^{\infty}$

$$
\begin{equation*}
\left|\eta_{1} \psi_{2} / \psi_{1}\right| \leq\left\|\eta_{1} \varphi_{g, 2}\right\|_{\infty} \psi_{2} /\left(\psi_{1} \varphi_{g, 2}\right) \tag{5.97}
\end{equation*}
$$

To ensure $\psi_{2} / \psi_{1} / \varphi_{g 2} \in L^{\infty}$, by comparing the far-field behavior, $\psi_{2}(x) \sim|x|^{1 / 6}, \psi_{1} \sim|x|^{-1 / 6}$, we need $\varphi_{g 2} \gtrsim|x|^{1 / 3}$ and thus we choose $\alpha_{g, n}=\frac{1}{3}+10^{-8}$ in (5.52). For $I_{2}$ we use

$$
\left|I_{2}\right| \leq\left(U_{x, A} \psi_{1}\right)(z)\left(\psi^{-1}(z)\left|\delta\left(\eta_{1} \psi_{2}\right)\right|+\delta\left(\psi^{-1}\right)\left|\eta_{1} \psi_{2}\right|(x)\right)
$$

Since $U_{x, A} \in L^{\infty}$, we can bound both terms using the energy $E_{4}$.
The estimate of nonlinear term $\mathcal{T}_{d, N}$ (5.91) from the transport term is similar. To estimate $\mathcal{T}_{d, N}\left(\psi_{i}\right)\left(W_{1, i} \psi_{i}\right)$, we use the energy $E_{4}$ and apply (5.46). We only need to control $\mathcal{T}_{d, N}\left(\psi_{i}\right)$. Using (5.82), we perform the decomposition

$$
\begin{equation*}
\mathcal{T}_{d, N}\left(\psi_{i}\right)=\frac{\mathbf{U} \cdot \nabla \psi_{i}}{\psi_{i}}=U_{x}(0) \frac{\left(x \partial_{x} \psi_{i}-y \partial_{y} \psi_{i}\right)}{\psi_{i}}+\frac{\mathbf{U}_{A} \cdot \nabla \psi_{i}}{\psi_{i}}+\frac{\mathbf{U}_{a p p} \cdot \nabla \psi_{i}}{\psi_{i}} \triangleq \mathcal{T}_{c_{\omega}}+\mathcal{T}_{u A}+\mathcal{T}_{u R} \tag{5.98}
\end{equation*}
$$

For $\mathcal{T}_{u A}$, since we have piecewise $C^{1 / 2}$ estimates of $\mathbf{U}_{A} \psi_{u}, C^{3}$ estimate of $\mathbf{U}_{a p p}$ with $\mathbf{U}_{a p p}=$ $O\left(|x|^{3}\right)$, we decompose it as follows

$$
\begin{equation*}
\mathcal{T}_{u A}=\left(U_{A} \psi_{u}\right) \cdot \frac{\partial_{x} \psi_{i}}{\psi_{i} \psi_{u}}+\left(V_{A} \psi_{u}\right) \cdot \frac{\partial_{y} \psi_{i}}{\psi_{i} \psi_{u}}, \quad \mathcal{T}_{u R}=\frac{\mathbf{U}_{a p p}}{|x|^{2}} \cdot|x|^{2} \frac{\nabla \psi_{i}}{\psi_{i}} \tag{5.99}
\end{equation*}
$$

and then estimate each product $f_{1} \cdot f_{2}$ using (5.36), (5.37). The explicit function $\mathcal{T}_{c_{\omega}}$ is not $C^{1 / 2}$ near 0 , but we can bound $\delta\left(\mathcal{T}_{c_{\omega}}, x, z\right) g(x-z)|p|^{1 / 2}$ for $p=x, z$. See Remark 5.2 and Section 8.5.2 in the supplementary material I [18].

Nonlinear terms involving $\tilde{\mathbf{U}}, \hat{W}_{2}$. We estimate a typical nonlinear terms $P_{6} \mathbf{U}_{A} \cdot \nabla \widehat{W}_{2, i} \psi_{i}$ (2.17), (5.92) in $I_{1}$ (5.95). Using (5.36)-(5.37), we get

$$
\left|\delta\left(P_{6}\right)\right| \leq \delta_{\square}\left(U_{A} \psi_{u}, \partial_{x} \hat{W}_{2, i} \frac{\psi_{i}}{\psi_{u}}, h\right)+\delta_{\square}\left(V_{A} \psi_{u}, \partial_{y} \hat{W}_{2, i} \frac{\psi_{i}}{\psi_{u}}, h\right) .
$$

Near $0, \nabla \widehat{W}_{2, i} \frac{\psi_{i}}{\psi_{u}}$ has a vanishing order $O(|x|)$, and we can estimate its $C^{1}$ bound. We bound $\delta_{i}\left(U_{A} \psi_{u}\right)$ using the energy. For $I_{2}$, since $\widehat{W}_{2, i}=O\left(|x|^{2}\right), \mathbf{U}_{\text {app }}=O\left(|x|^{3}\right)$ and we have their $C^{3}$ bounds. From (2.16), we get

$$
\mathcal{B}_{o p, i}\left(\mathbf{U}_{A},(\nabla \mathbf{U})_{A}, \hat{W}_{2}\right)=O\left(|x|^{4}\right)
$$

and we can estimate the $C^{1}$ bound of $\mathcal{B}_{o p, i} \psi_{i}$ and then its $C^{1 / 2}$ bound using (5.36)-(5.37). The main terms (5.93) have a vanishing order $O\left(|x|^{3}\right)$, and we can estimate their $C^{3}$ bounds. Since $\psi_{i} \lesssim|x|^{-5 / 2}$, we can estimate $C^{1 / 2}$ bound of $\psi_{i} \hat{W}_{2, i, M}$. In Section 8.9 of the supplementary material I [18], we discuss the piecewise $C^{1 / 2}$ estimates of $f(x) /|x|^{5 / 2}$ for $f \in C^{3}$ with $f=$ $O\left(|x|^{3}\right)$.

The estimate of other nonlinear terms are similar and relatively straightforward based on (5.34)-(5.36). We refer more details to Section 8.5 in the supplementary material I [18]. Note that the estimate of the main term (5.93) is simple, and we can afford a much larger constant in the estimates of terms other than the main terms (5.93).

Other nonlinear estimates. The term $U_{x}(0) \omega_{1}, U_{x}(0) \eta_{1}$ in (5.92) in the ODEs of $c_{\omega}(f q)$ (5.61) with cutoff $q=\chi_{\text {ode ( (5.64) }}$ and $q=1$ (5.68) contribute to $U_{x}(0)\left\langle\omega_{1}, f_{*} q\right\rangle, f=\omega_{1}, \eta_{1}$, which can be bounded by the energy $E_{4}$ (5.71), (5.70) directly

$$
\begin{align*}
& \frac{4}{\pi}\left|U_{x}(0)\right|\left|\left\langle\omega_{1}, f_{*} q\right\rangle\right|=\left|U_{x}(0) c_{\omega}\left(\omega_{1} q\right)\right| \leq \gamma_{1}\left|U_{x}(0)\right| E_{4} \\
& \left|U_{x}(0)\right| \cdot \frac{4}{\pi}\left|\left\langle\eta_{1}, f_{*} q\right\rangle\right|=\left|U_{x}(0) c_{\omega}\left(\eta_{1} q\right)\right| \leq \gamma_{2}\left|U_{x}(0)\right| E_{4} \tag{5.100}
\end{align*}
$$

where $\left(q, \gamma_{1}, \gamma_{2}\right)=\left(\chi_{\text {ode }}, \mu_{5,1}, \mu_{5,2}\right),\left(1, \mu_{5}, \mu_{62}\right)$. The estimates of other nonlinear terms in these ODEs follow Section 5.9 .2 and the argument in Section 5.6, e.g. integration by parts.

For the energy estimates beyond our computational domain $[0, D]^{2}, D \geq 10^{15}$, we estimate the asymptotics of the profile (7.2) in Appendix C. 3 and the nonlocal terms in Section 4.5 in Part II [15]. Since the coefficients of the nonlocal terms decay, e.g. $\nabla \bar{\omega}, \nabla \bar{\theta}$, the equations (5.1), (4.21) are essentially local in the far-field. We have much larger damping factors and can afford much larger constants in the estimate of nonlocal terms. We refer the far-field estimates to Sections 8.6, 8.7 in the supplementary material I 18 .

In Figure 12, we plot the rigorous piecewise bounds $C_{k l}$ for the full nonlinear terms $|J| \leq$ $C_{k l} E_{4}^{2}$ in mesh $\left[y_{k}, y_{k+1}\right] \times\left[y_{l}, y_{l+1}\right]$ covering regions $\left[0,10^{15}\right]^{2}$, in the $\omega$ equation, and similar terms for $\eta, \xi$ equations. The largest terms for these three equations are bounded by $8300,8300,5000$, respectively. For $x$ very small and $y$ very large (adaptive mesh $y_{2, n}>10^{6}$ for $n \geq 600$ ), we have a jump in the estimate, especially in the $\xi$-equation. It is due to the piecewise estimate of $\varphi_{3} / \varphi_{2}($ see (C.3) $)$ in the estimate of nonlinear term $\left|u_{y} \eta_{1} \varphi_{3}\right| \leq\left|u_{y} \frac{\varphi_{3}}{\varphi_{2}}\left\|\mid \eta_{1} \varphi_{2}\right\|_{\infty}\right.$. Both weights involve $\left|x_{1}\right|^{-1 / 2}$ singular along $x_{1}=0$. We can refine the estimate to get a smoother bound. Yet, since we have a large damping factor $\geq 1$ in that region (very far-field), we can afford a constant that is even 20 times larger $\left(-E_{*}+20 \cdot 5000 E_{*}^{2} \leq-0.5 E_{*}\right)$ and do not need to refine the estimate.


Figure 12. Weighted $L^{\infty}\left(\varphi_{i}\right)$ estimate of the nonlinear terms.

Estimate of the residual error of the profile. For the residual error $\overline{\mathcal{F}}_{l o c, i}$ in (5.87) modified from (4.21), (2.19), it is essentially local and its estimate follows standard numerical analysis. We estimate them following Sections 3.6 of Part II [15] with some details in Appendix C. 4 of Part II [15]. We have plotted rigorous piecewise bounds for the local part $\overline{\mathcal{F}}_{l o c, i} \varphi_{i}, i=1,2$ in Figure 2, Note that the weighted residual error away from the first few grids and in the bulk region is very small $\left(\leq 5 \cdot 10^{-8}\right)$ relative to the bootstrap threshold $E_{*}$ (5.101).
Estimate in the region with small stability factor. In the linear weighted $L^{\infty}\left(\varphi_{i}\right)$ estimate, we have a minimum stability factor about 0.04 . We have a small stability factor below 0.08 only in the bulk region $D_{B}=[0,1000]^{2} \backslash[0,1]^{2}$. Since it is away from 0 , the singular weight $\varphi_{i}$ becomes much smaller and both the estimate of nonlinear terms and the residual error becomes much smaller in $D_{B}$. See Figures 2, 12 Similar discussion applies to the Hölder estimate.
Remark 5.6. An advantage of the stability condition (A.11) is that it depends on the estimate locally. Thus, we do not need to compare the minimum damping coefficients with the $L^{\infty}$ bound of the nonlinear terms and error terms or

$$
\min _{x, z} a_{i i}(x, z, t) E_{*}-\max _{x, z} \sum_{j \neq i}\left(\left|a_{i j}\right| E_{*}+\left|a_{i j, 2}\right| E_{*}^{2}+\left|a_{i j, 3}(x, z, t)\right|\right)>\varepsilon_{0}
$$

for some $\varepsilon_{0}>0$, which is a much tighter constraint for stability.
We remark that we have large damping factors in the far-field since the coefficients for the nonlocal terms in the linearized equations (4.10), (5.1) decay. Thus, it is much easier to obtain the stability condition (A.11) for large $x$.
5.9.4. Nonlinear stability and finite time blowup. To close the nonlinear estimates, for the bootstrap argument in Lemma A.2 we choose the threshold

$$
\begin{equation*}
E_{*}=5 \cdot 10^{-6} \tag{5.101}
\end{equation*}
$$

We choose the bootstrap threshold guided by the quadratic inequality on $E_{*}$ (A.2) for $x$ (or $x, z$ ) in the region with small damping coefficients. Under this bootstrap threshold, the largest part of the nonlinear terms in the weighted $L^{\infty}\left(\varphi_{1}\right)$ estimates are bounded by $(0.0415,0.0415,0.025) E_{4}$ (see Figure 12), and we can close the nonlinear estimates. See Figures 14, 15 for the stability conditions of $L^{\infty}\left(\varphi_{i}\right), L^{\infty}\left(\varphi_{g i}\right)$ estimates.

Using Lemma A.2, we can obtain that if the initial perturbation satisfies

$$
E_{4}\left(\omega_{1}(0), \eta_{1}(0), \xi_{1}(0)\right)<E_{*}
$$

then we have

$$
E_{4}\left(\omega_{1}(t), \eta_{1}(t), \xi_{1}(t)\right)<E_{*}
$$

for all time $t>0$. With the estimates of $W_{1}$, we can control $\widehat{W}_{2}$ using the estimates in Sections 5.6. 5.7 In particular, we can obtain

$$
\begin{equation*}
\left\|W_{1, i}+\widehat{W}_{2, i}\right\|_{\infty}<200 E_{*}, \quad\left|c_{\omega}(\omega)\right|<100 E_{*} \tag{5.102}
\end{equation*}
$$

The bounds for $W_{1, i}, c_{\omega}(\omega)$ follows from the definition of the energy (5.21), (5.54), (5.70). From the definitions of the weights $\varphi_{i}, \varphi_{g, i}, \mu_{g, i}$ (C.3), (C.4), (C.5), it is easy to see that $\left|W_{1, i}\right| \leq$ $\mu_{g, i}^{-1} \varphi_{g, i}^{-1} E_{4}<100 E_{*}$. We verify $\left|\widehat{W}_{2, i}\right|<100 E_{4}$ and collect this inequality in (D.17). Recall the normalization condition (2.26). We also have $\left|u_{x}(0)\right|=\left|c_{\omega}\right|<100 E_{*}$.

Moreover, since we choose 0 initial condition for $\hat{W}_{2}$, we have $W_{1}=(\omega, \eta, \xi)=\left(\omega, \theta_{x}, \theta_{y}\right)$ at the initial time. Therefore, we prove the estimates in Theorem 3. Passing from the stability analysis to finite time blowup follows the standard rescaling argument [17, 19, 20 .

## 6. Finite time blowup of 3D axisymmetric Euler equations with solid boundary

In this section, we prove the finite blowup of the axisymmetric Euler equations with smooth initial data and boundary. We will follow the same proof strategy as in our previous work [17. We first review the setup of the problem. In Section 6.1, we reformulate the 3D Euler equations using the dynamic rescaling formulation and discuss the connection between the 3D Euler and 2D

Boussinesq; see e.g. 67]. In Section 6.2, we establish the localized elliptic estimates. In Section 6.4. we will construct initial data and control the support of the solution under some bootstrap assumptions. With these estimates, the rest of the proof follows essentially the nonlinear stability analysis of the 2D Boussinesq equations and we will sketch the part of the analysis that is different from the 2D Boussinesq equations.

Notations. In this section, we use $x_{1}, x_{2}, x_{3}$ to denote the Cartesian coordinates in $\mathbb{R}^{3}$, and

$$
\begin{equation*}
r=\sqrt{x_{1}^{2}+x_{2}^{2}}, \quad z=x_{3}, \quad \vartheta=\arctan \left(x_{2} / x_{1}\right) \tag{6.1}
\end{equation*}
$$

to denote the cylindrical coordinates. Let $\mathbf{u}$ be the axi-symmetric velocity and $\boldsymbol{\omega}=\nabla \times \mathbf{u}$ be the vorticity vector. In the cylindrical coordinates, we have the following representation

$$
\mathbf{u}(r, z)=u^{r}(r, z) \mathbf{e}_{r}+u^{\theta}(r, z) \mathbf{e}_{\theta}+u^{z}(r, z) \mathbf{e}_{z}, \quad \boldsymbol{\omega}=\omega^{r}(r, z) \mathbf{e}_{r}+\omega^{\theta}(r, z) \mathbf{e}_{\theta}+\omega^{z}(r, z) \mathbf{e}_{z}
$$

where $\mathbf{e}_{r}, \mathbf{e}_{\theta}$ and $\mathbf{e}_{z}$ are the standard orthonormal vectors defining the cylindrical coordinates,

$$
\mathbf{e}_{r}=\left(\frac{x_{1}}{r}, \frac{x_{2}}{r}, 0\right)^{T}, \quad \mathbf{e}_{\theta}=\left(\frac{x_{2}}{r},-\frac{x_{1}}{r}, 0\right)^{T}, \quad \mathbf{e}_{z}=(0,0,1)^{T}
$$

We study the 3 D axisymmetric Euler equations in a cylinder $D=\{(r, z): r \in[0,1], z \in$ $\mathbb{T}\}, \mathbb{T}=\mathbb{R} /(2 \mathbb{Z})$ that is periodic in $z$. The 3D axisymmetric Euler equations are given below:

$$
\begin{equation*}
\partial_{t}\left(r u^{\theta}\right)+u^{r}\left(r u^{\theta}\right)_{r}+u^{z}\left(r u^{\theta}\right)_{z}=0, \quad \partial_{t}\left(\frac{\omega^{\theta}}{r}\right)+u^{r}\left(\frac{\omega^{\theta}}{r}\right)_{r}+u^{z}\left(\frac{\omega^{\theta}}{r}\right)_{z}=\frac{1}{r^{4}} \partial_{z}\left(\left(r u^{\theta}\right)^{2}\right) \tag{6.2}
\end{equation*}
$$

The radial and axial components of the velocity can be recovered from the Biot-Savart law

$$
\begin{equation*}
-\left(\partial_{r r}+\frac{1}{r} \partial_{r}+\partial_{z z}\right) \tilde{\phi}+\frac{1}{r^{2}} \tilde{\phi}=\omega^{\theta}, \quad u^{r}=-\tilde{\phi}_{z}, \quad u^{z}=\tilde{\phi}_{r}+\frac{1}{r} \tilde{\phi} \tag{6.3}
\end{equation*}
$$

with a no-flow boundary condition on the solid boundary $r=1$

$$
\begin{equation*}
\tilde{\phi}(1, z)=0 \tag{6.4}
\end{equation*}
$$

and a periodic boundary condition in $z$.
We consider solution $\omega^{\theta}$ with odd symmetry in $z_{2}$, which is preserved by the equations dynamically. Then $\tilde{\phi}$ is also odd in $z$. Moreover, since $\tilde{\phi}$ is 2 -periodic in $z$, we obtain

$$
\begin{equation*}
\tilde{\phi}(r, 2 k-1)=0 . \quad \text { for all } k \in \mathbb{Z} \tag{6.5}
\end{equation*}
$$

This setup of the problem is essentially the same as that in 65, 66].
Due to the periodicity in $z$ direction, it suffices to consider the equations in the first period $D_{1}=\{(r, z): r \in[0,1],|z| \leq 1\}$. We have the following pointwise estimate on $\tilde{\phi}$ from [17], which will be used to estimate $\tilde{\phi}$ away from the $\operatorname{supp}\left(\omega^{\theta}\right)$ in Section 6.2,

Lemma 6.1. Let $\tilde{\phi}$ be a solution of (6.3)-(6.4), and $\omega^{\theta} \in C^{\alpha}\left(D_{1}\right)$ for some $\alpha>0$ be odd in $z$ with $\operatorname{supp}\left(\omega^{\theta}\right) \cap D_{1} \subset\left\{(r, z):(r-1)^{2}+z^{2}<1 / 4\right\}$. For $\frac{1}{4}<r \leq 1,|z| \leq 1$, we have

$$
|\tilde{\phi}(r, z)| \lesssim \int_{D_{1}}\left|\omega^{\theta}\left(r_{1}, z_{1}\right)\right|\left(1+\left|\log \left(\left(r-r_{1}\right)^{2}+\left(z-z_{1}\right)^{2}\right)\right|\right) r_{1} d r_{1} d z_{1}
$$

If the initial data $u^{\theta}$ of (6.2)-(6.4) is non-negative, $u^{\theta}$ remains non-negative before the blowup, if it exists. Then, $u^{\theta}$ can be uniquely determined by $\left(u^{\theta}\right)^{2}$. We introduce the following variables

$$
\begin{equation*}
\tilde{\theta} \triangleq\left(r u^{\theta}\right)^{2}, \quad \tilde{\omega}=\omega^{\theta} / r \tag{6.6}
\end{equation*}
$$

We reformulate (6.2)-(6.4) as

$$
\begin{align*}
& \partial_{t} \tilde{\theta}+u^{r} \tilde{\theta}_{r}+u^{z} \tilde{\theta}_{z}=0, \quad \partial_{t} \tilde{\omega}+u^{r} \tilde{\omega}_{r}+u^{z} \tilde{\omega}_{z}=\frac{1}{r^{4}} \tilde{\theta}_{z} \\
&-\left(\partial_{r}^{2}+\frac{1}{r} \partial_{r}+\partial_{z}^{2}-\frac{1}{r^{2}}\right) \tilde{\phi}=r \tilde{\omega}, \quad \tilde{\phi}(1, z)=0, \quad u^{r}=-\tilde{\phi}_{z}, \quad u^{z}=\frac{1}{r} \tilde{\phi}+\tilde{\phi}_{r} \tag{6.7}
\end{align*}
$$

6.1. Dynamic rescaling formulation. We introduce new coordinates $(x, y)$ centered at $r=$ $1, z=0$ and its related polar coordinates

$$
\begin{equation*}
x=C_{l}(\tau)^{-1} z, \quad y=(1-r) C_{l}(\tau)^{-1} \tag{6.8}
\end{equation*}
$$

where $C_{l}(\tau)$ is defined below (6.11). By definition, we have

$$
\begin{equation*}
z=C_{l}(\tau) x, \quad r=1-C_{l}(\tau) y \tag{6.9}
\end{equation*}
$$

We consider the following dynamic rescaling formulation centered at $r=1, z=0$

$$
\begin{align*}
\theta(x, y, \tau) & =C_{\theta}(\tau) \tilde{\theta}\left(1-C_{l}(\tau) y, C_{l}(\tau) x, t(\tau)\right) \\
\omega(x, y, \tau) & =C_{\omega}(\tau) \tilde{\omega}\left(1-C_{l}(\tau) y, C_{l}(\tau) x, t(\tau)\right)  \tag{6.10}\\
\phi(x, y, \tau) & =C_{\omega}(\tau) C_{l}(\tau)^{-2} \tilde{\phi}\left(1-C_{l}(\tau) y, C_{l}(\tau) x, t(\tau)\right)
\end{align*}
$$

where $C_{l}(\tau), C_{\theta}(\tau), C_{\omega}(\tau), t(\tau)$ are given by $C_{\theta}=C_{l}^{-1}(0) C_{\omega}^{2}(0) \exp \left(\int_{0}^{\tau} c_{\theta}(s) d \tau\right)$,

$$
\begin{equation*}
C_{\omega}(\tau)=C_{\omega}(0) \exp \left(\int_{0}^{\tau} c_{\omega}(s) d \tau\right), C_{l}(\tau)=C_{l}(0) \exp \left(\int_{0}^{\tau}-c_{l}(s) d s\right), t(\tau)=\int_{0}^{\tau} C_{\omega}(\tau) d \tau \tag{6.11}
\end{equation*}
$$

and the rescaling parameters $c_{l}(\tau), c_{\theta}(\tau), c_{\omega}(\tau)$ satisfy $c_{\theta}(\tau)=c_{l}(\tau)+2 c_{\omega}(\tau)$. We remark that $C_{\theta}(\tau)$ is determined by $C_{l}, C_{\omega}$ via $C_{\theta}=C_{\omega}^{2} C_{l}^{-1}$. We have this relation due to the same reason as that of (2.9). We choose $(r, z)=(1,0)$ as the center of the above transform since the singular solution is concentrated near this point. Since we rescale the cylinder $D_{1}=\{(r, z): r \leq 1,|z| \leq$ $1\}$, from (6.8), the domain for $(x, y)$ is

$$
\begin{equation*}
\tilde{D}_{1} \triangleq\left\{(x, y):|x| \leq C_{l}^{-1}, y \in\left[0, C_{l}^{-1}\right]\right\} \tag{6.12}
\end{equation*}
$$

We have a minus sign for $\partial_{y}$

$$
\partial_{y} \theta=-C_{\theta} C_{l}(\tau) \tilde{\theta}_{r}, \quad \partial_{y} \omega=-C_{\omega} C_{l}(\tau) \tilde{\omega}_{r}, \quad \partial_{y} \phi=-C_{\omega} C_{l}(\tau)^{-1} \tilde{\phi}_{r}
$$

Let $(\tilde{\theta}, \tilde{\omega})$ be a solution of (6.7). It is easy to show that $\omega, \theta$ satisfy

$$
\theta_{t}+c_{l} \mathbf{x} \cdot \nabla \theta+\left(-u^{r}\right) \theta_{y}+u^{z} \theta_{x}=c_{\theta} \theta, \quad \omega_{t}+c_{l} \mathbf{x} \cdot \nabla \omega+\left(-u^{r}\right) \omega_{y}+u^{z} \omega_{x}=c_{\omega} \omega+\frac{1}{r^{4}} \theta_{x}
$$

The Biot-Savart law in (6.7) depends on the rescaling parameter $C_{l}, \tau$

$$
-\left(\partial_{x x}+\partial_{y y}\right) \phi+\frac{1}{r} C_{l} \partial_{y} \phi+\frac{1}{r^{2}} C_{l}^{2} \phi=r \omega, \quad u^{r}(r, x)=-\phi_{x}, \quad u^{z}(r, x)=\frac{1}{r} C_{l}(\tau) \phi-\phi_{y}
$$

where $r=1-C_{l}(\tau) y(6.9)$. We introduce $u=u^{z}, v=-u^{r}$. Then, we can further simplify

$$
\begin{align*}
& \theta_{t}+\left(c_{l} \mathbf{x}+\mathbf{u} \cdot \nabla\right) \theta=c_{\theta} \theta, \quad \omega_{t}+\left(c_{l} \mathbf{x}+\mathbf{u} \cdot \nabla\right) \omega=\theta_{x}+\frac{1-r^{4}}{r^{4}} \theta_{x} \\
& -\left(\partial_{x x}+\partial_{y y}\right) \phi+\frac{1}{r} C_{l} \partial_{y} \phi+\frac{1}{r^{2}} C_{l}^{2} \phi=r \omega, \quad u(x, y)=-\phi_{y}+\frac{1}{r} C_{l} \phi, \quad v=\phi_{x} \tag{6.13}
\end{align*}
$$

with boundary condition $\phi(x, 0) \equiv 0$. If $C_{l}$ is extremely small, we expect that the above equations are essentially the same as the dynamic rescaling formulation (2.10) of the Boussinesq equations. We look for solutions of (6.13) with the following symmetry

$$
\omega(x, y)=-\omega(-x, y), \quad \theta(x, y)=\theta(-x, y)
$$

Obviously, the equations preserve these symmetry properties and thus it suffices to solve (6.13) on $x, y \geq 0$ with boundary condition $\phi(x, 0)=\phi(y, 0)=0$ for the elliptic equation.

We now state a more precise version of Theorem 2 below.
Theorem 4. Let $\left(\bar{\theta}_{0}, \bar{\omega}_{0}, \overline{\mathbf{u}}, \bar{c}_{l}, \bar{c}_{\omega}\right)$ be the approximate self-similar profile constructed in Section 6.4 .2 and $E_{*}=5 \cdot 10^{-6}$. Assume that even initial data $\theta_{0}$ and odd $\omega_{0}$ of (6.13) compactly supported with size $S(0)$ to be defined in Definition 6.2 satisfy

$$
E\left(\omega_{0}-\bar{\omega}_{0}, \theta_{0, x}-\bar{\theta}_{0, x}, \theta_{0, y}-\bar{\theta}_{0, y}\right)<E_{*}
$$

where $E$ is defined in (2.13). For $E_{*}=5 \cdot 10^{-6}$, there exists a constant $C(S(0))$ depending on $S(0)$ such that if the initial rescaling factor $C_{l}(0)$ (6.11) satisfies $C_{l}(0)<C(S(0))$, we have

$$
\left\|\omega-\bar{\omega}_{0}\right\|_{L^{\infty}},\left\|\theta_{x}-\bar{\theta}_{0, x}\right\|_{L^{\infty}},\left\|\theta_{y}-\bar{\theta}_{0, y}\right\|_{\infty}<200 E_{*},\left|u_{x}(t, 0)-\bar{u}_{x}(0)\right|,\left|\bar{c}_{\omega}-c_{\omega}\right|<100 E_{*}
$$

for all time. In particular, we can choose smooth initial data $\omega_{0}, \theta_{0} \in C_{c}^{\infty}$ in this class with finite energy $\left\|\mathbf{u}_{0}\right\|_{L^{2}}<+\infty$ such that the solution to the physical equations (2.3)-(2.5) with these initial data blows up in finite time $T$.

We need to choose a small rescaling factor $C_{l}(0)$ so that the solution in the physical space is confined in the cylinder, which is not scaling invariant.
6.2. The elliptic estimates. In this subsection, we follow the ideas in [17] to estimate the elliptic equation with time-dependent coefficients in (6.13). We first estimate $\phi$ away from $\operatorname{supp}(\omega)$. Then we localize the elliptic equation and perform weighted $L^{\infty}$ and Hölder estimate. We will show that within the support of $\omega, \theta$, the estimates for the velocity are the same as those in the 2 D Boussinesq equations up to a lower order term, which can be made arbitrary small. Throughout this Section, we assume that $\omega(x, y)$ is odd in $x$.

Definition 6.2. We define the size of support of $(\theta, \omega)$ of (6.13)

$$
S(\tau)=\max \left(\operatorname{essinf}\left\{\rho: \theta(x, y, \tau)=0, \omega(x, y, \tau)=0 \text { for } x^{2}+y^{2} \geq \rho^{2}\right\}, 1\right)
$$

We take the maximum in the definition so that $S(\tau) \geq 1$, which simplifies some later estimates. After rescaling the spatial variable, the support of $(\tilde{\theta}, \tilde{\omega})$ of (6.7) satisfies

$$
\operatorname{supp} \tilde{\theta}(t(\tau)), \operatorname{supp} \tilde{\omega}(t(\tau)) \subset\left\{(r, z):\left((r-1)^{2}+z^{2}\right)^{1 / 2} \leq C_{l}(\tau) S(\tau)\right\}
$$

We will construct initial data of (6.13) with compact support $S(0)<+\infty$ and follow [17 to prove that $C_{l}(\tau) S(\tau)$ remains sufficiently small for all $\tau>0$.

Remark 6.3. There are several small parameters $C_{l}(\tau), C_{l}(\tau) S(\tau)$ in the following estimates. We will choose $C_{l}(0)$ to be very small at the final step of the proof. This allows us to prove that $C_{l}(\tau), C_{l}(\tau) S(\tau)$ are very small. One can essentially regard $C_{l}(\tau) \approx 0$. Recall the relation (6.9) about $r$. In the support of the solution, we have $r=1-C_{l} \rho \sin (\beta) \approx 1$. We treat the error terms in these approximations as small perturbations.

The elliptic equation in (6.13) contains the first order term $\frac{1}{r} C_{l} \partial_{y} \psi$, which leads to a few technical difficulties in the elliptic estimate. To overcome it, we multiply the equation with an integrating factor $r^{1 / 2}$. Using $\partial_{y} r^{1 / 2}=-C_{l} r^{-1 / 2} / 2, \partial_{y y} r^{1 / 2}=-\frac{1}{4} C_{l}^{2} r^{-3 / 2}$,

$$
\partial_{y y}\left(\phi r^{1 / 2}\right)=r^{1 / 2} \partial_{y y} \phi+2 \partial_{y} \phi \partial_{y} r^{1 / 2}+\phi \partial_{y y} r^{1 / 2}=r^{1 / 2} \partial_{y y} \phi-\frac{C_{l}}{r^{1 / 2}} \partial_{y} \phi-\frac{1}{4} C_{l}^{2} r^{-3 / 2} \phi
$$

we can rewrite (6.13) as follows

$$
-\Delta\left(\phi r^{1 / 2}\right)+\frac{a C_{l}^{2}}{r^{2}} \phi r^{1 / 2}=\omega r^{3 / 2}, \quad a=\frac{3}{4}
$$

Note that within the support of $\omega, \theta, r, r^{-1}$ are smooth. Once we obtain the estimate of $\phi r^{1 / 2}$, we can recover the estimate of $\phi$. We rewrite the above equation as follows

$$
\begin{equation*}
-\Delta \phi_{1}=\Omega_{1}-\frac{a C_{l}^{2}}{r^{2}} \phi_{1}, \quad \Omega_{1}=\omega r^{3 / 2}, \quad \phi_{1}=\phi r^{1 / 2} \tag{6.14}
\end{equation*}
$$

Our goal is to show that $\phi_{1}$ and $\phi$ enjoy estimates similar to those for $\left(-\Delta_{2 D}\right)^{-1} \omega$, then we can generalize the analysis for 2D Boussinesq to 3D Euler equations.
6.2.1. Estimate of $\phi$ away from the support. To localize the elliptic equations, we first estimate $\phi$ away from the support of the solution. Based on Lemma 6.1. we have the following estimate.

Lemma 6.4. Suppose that the assumptions in Lemma 6.1 hold true. Let $S(\tau)$ be the support size of $\omega(\tau), \theta(\tau)$. Assume $C_{l}(\tau) S(\tau)<\frac{1}{4}$. For any $|x|>2 S$ and $\beta \in[0,1)$, the solution to (6.13) satisfies

$$
|\phi(x)| \lesssim\left\|\omega\left(1+|x|^{\beta}\right)\right\|_{L^{\infty}}\left(1+\left|\log \left(C_{l}|x|\right)\right|\right) S^{2-\beta} .
$$

Since $x$ is away from the support, the proof follows from the rescaling relation (6.10) and the estimate in Lemma 6.1 by putting $\omega(y)\left(1+|y|^{\beta}\right)$ in $L^{\infty}$. We defer the proof to Appendix C.3.
6.2.2. Localize the elliptic equation. We will take advantage of the fact that $C_{l}(\tau) S(\tau)$ can be extremely small and localize the elliptic equation. Firstly, we assume that $C_{l}(\tau) S(\tau)<\frac{1}{4}$. Recall the relation (6.9) about $r$. Within the support, we have $r=1-C_{l} y \geq \frac{3}{4}, r^{-1} \lesssim 1$.

Let $\chi(\cdot): \mathbb{R}_{2}^{+} \rightarrow[0,1]$ be a smooth cutoff function even in $x_{1}$, such that $\chi(x)=1$ for $|x| \leq 1$, $\chi(x)=0$ for $|x| \geq 2$. It is easy to verify that

$$
\begin{equation*}
\left|\nabla^{k} \chi(x / R)\right| \lesssim R^{-k} \mathbf{1}_{R \leq|x| \leq 2 R} \tag{6.15}
\end{equation*}
$$

for $1 \leq k \leq 5$. Next, we choose several radii and define the related cutoff function

$$
\begin{equation*}
R_{i}=4^{-i} C_{l}^{-1}, \quad \chi_{i}(x) \triangleq \chi\left(x / R_{i}\right), \quad i \leq 5 \tag{6.16}
\end{equation*}
$$

By definition, we have $\chi_{i}=1$ in the support of $\chi_{i+1}$. Multiplying (6.14) with $\chi_{i}$, we obtain the equation of $\phi_{1} \chi_{i}$

$$
\begin{equation*}
-\Delta\left(\phi_{1} \chi_{i}\right)=\Omega_{1} \chi_{i}-Z_{1, i}-Z_{2, i}, \quad Z_{1, i} \triangleq 2 \nabla \phi_{1} \cdot \nabla \chi_{i}+\phi_{1} \Delta \chi_{i}, \quad Z_{2, i} \triangleq \frac{a C_{l}^{2}}{r^{2}} \phi_{1} \chi_{i} \tag{6.17}
\end{equation*}
$$

with boundary condition

$$
\left(\phi_{1} \chi_{i}\right)(0, y)=0, \quad\left(\phi_{1} \chi_{i}\right)(x, 0)=0
$$

After we localize the elliptic equation, (6.17) can be seen as an elliptic equation in $\mathbb{R}_{2}^{+}$with compactly supported source term. Since the solution $\phi_{1} \chi_{i}$ decays for large $|x|$, it agrees with the solution defined by the Green function $\log (|x-y|)$ in the upper half space:
$\left((-\Delta)^{-1} f\right)(x)=-\frac{1}{2 \pi} \int_{\mathbb{R}_{2}^{+}}\left(\log |x-y|-\log \left|\left(x_{1}-y_{1}, x_{2}+y_{2}\right)\right|\right) f(y) d y=-\frac{1}{2 \pi} \int_{\mathbb{R}_{2}} \log |x-y| F(y) d y$. where $F$ is the odd extension of $f$ from $\mathbb{R}_{2}^{+}$to $\mathbb{R}_{2}$. Similar formula also holds for $\nabla(-\Delta)^{-1} f$

$$
\begin{equation*}
\partial_{i}(-\Delta)^{-1} f=-\frac{1}{2 \pi} \int_{\mathbb{R}_{2}} \frac{x_{i}-y_{i}}{|x-y|^{2}} F(y) d y \tag{6.19}
\end{equation*}
$$

Ideas of the estimates. We will assume that $\Omega_{1}$ is in a suitable weighted $L^{\infty}$ and Hölder space. Our goal is to show that the terms on the right hand side of (6.17) except for $\Omega \chi_{i}$ are very small in such a space. Then we can obtain the estimate for $\phi_{1} \chi_{i}$ by inverting $-\Delta$, which is similar to that of $(-\Delta)^{-1}\left(\Omega_{1} \chi_{i}\right)$. We will also require that the support satisfies

$$
\begin{equation*}
S(\tau)<R_{i}, \quad \text { or } C_{l} S<4^{-i}, \quad i \leq 5 \tag{6.20}
\end{equation*}
$$

so that $\Omega_{1} \chi_{i}=\Omega_{1}$. We will choose $C_{l} S$ to be sufficiently small.
We need to estimate the $L^{\infty}$ norm of $\nabla \phi_{1}$ and its Hölder norm. We will first estimate $\nabla \phi_{1}$ for $|x| \leq R_{2}$, and then $\nabla^{2} \phi_{1}$ for $|x| \leq R_{3}$. Once we obtain the estimates of $\nabla \phi_{1}, \nabla^{2} \phi_{1}$, due to the small parameters on the right hand side of (6.17) and the decay of the solution, we establish the desired estimate. We need several weighted estimates of the Laplacian in $\mathbb{R}_{2}^{+}$.
Lemma 6.5. Suppose that $-\Delta \phi=\omega$ in $\mathbb{R}_{2}^{+}, \omega$ is odd, and $\phi$ satisfies the Dirichlet boundary condition. For $\alpha>0$ and $\beta \in(0,1)$, we have

$$
|\nabla \phi| \lesssim \alpha, \beta|x| \wedge|x|^{1-\beta}| | \omega\left(|x|^{-\alpha}+|x|^{\beta}\right) \|_{L^{\infty}}
$$

For $\alpha \in(0,2)$, we have

$$
\begin{array}{r}
\left|\nabla\left(\phi-\phi_{x y}(0) x_{1} x_{2}\right)\right| \lesssim \alpha, \beta|x|^{1+\alpha}| | \omega\left(|x|^{-\alpha}+|x|^{\beta}\right) \|_{L^{\infty}} \\
\left|\phi-\phi_{x y}(0) x_{1} x_{2}\right| \lesssim \alpha, \beta|x|^{2+\alpha}| | \omega\left(|x|^{-\alpha}+|x|^{\beta}\right) \|_{L^{\infty}}
\end{array}
$$

For $\alpha \in(2,3]$, we have

$$
\begin{array}{r}
\left|\nabla\left(\phi-\phi_{x y}(0) x_{1} x_{2}\right)\right| \lesssim_{\alpha, \beta}|x|^{3}| | \omega\left(|x|^{-\alpha}+|x|^{\beta}\right) \|_{L^{\infty}} \\
\left|\phi-\phi_{x y}(0) x_{1} x_{2}\right| \lesssim_{\alpha, \beta}|x|^{4}| | \omega\left(|x|^{-\alpha}+|x|^{\beta}\right) \|_{L^{\infty}} .
\end{array}
$$

We will mostly use $\alpha=1,2,2.9, \beta=\frac{1}{16}$ relates to the weight $\varphi_{1}, \varphi_{g, 1}$ for $\omega_{1}$ in the 2D Boussinesq equations (C.3), (C.4). We prove the first estimate below and defer the proof of the second and third to Appendix C.3, which are similar.

Proof. In the following proof, the implicit constant in $\lesssim$ can depend on $\alpha, \beta$. We drop it to simplify the notations. Denote by $M=\left\|\omega\left(|x|^{-\alpha}+|x|^{\beta}\right)\right\|_{\infty}$. Clearly, we have

$$
\begin{align*}
|\omega| & \leq \min \left(|x|^{\alpha},|x|^{-\beta}\right) M \\
\partial_{i} \phi & =C_{i} \int K_{i}(x-y) W(y) d y, \quad K_{i}(z)=\frac{z_{i}}{|z|^{2}} \tag{6.21}
\end{align*}
$$

where $W$ is the odd extension of $\omega$ from $\mathbb{R}_{2}^{+}$to $\mathbb{R}_{2}$. Then $W$ is odd in both $x$ and $y$. Without loss of generality, we consider $i=2$. For a fixed $x$, we partition the integral into three regions:

$$
Q_{1}=\{y:|y| \geq 2|x|\}, \quad Q_{2}=\{y:|y-x| \leq|x| / 2\}, \quad Q_{3}=\left(Q_{1} \cup Q_{2}\right)^{c}
$$

In $Q_{1}$, symmetrizing the kernel, we need to estimate

$$
I_{1} \triangleq \int_{Q_{1}} K_{2}(x-y) W(y) d y=\int_{Q_{1}, y_{1} \geq 0}\left(K_{2}\left(x_{1}-y_{1}, x_{2}-y_{2}\right)-K_{2}\left(x_{1}+y_{1}, x_{2}-y_{2}\right)\right) W(y) d y
$$

Since $K_{2}(z)$ is odd in $z_{2}$ and even in $z_{1},\left|\nabla K_{2}(z)\right| \lesssim|z|^{-2}$, for $|y| \geq 2|x|$, we get

$$
\left|K_{2}\left(x_{1}-y_{1}, x_{2}-y_{2}\right)-K_{2}\left(x_{1}+y_{1}, x_{2}-y_{2}\right)\right|=\left|K_{2}\left(y_{1}-x_{1}, x_{2}-y_{2}\right)-K_{2}\left(x_{1}+y_{1}, x_{2}-y_{2}\right)\right| \lesssim \frac{\left|x_{1}\right|}{|y|^{2}}
$$

Using (6.21), we get

$$
\left|I_{1}\right| \lesssim M\left|x_{1}\right| \int_{|y| \geq 2|x|}|y|^{-2} \min \left(|y|^{\alpha},|y|^{-\beta}\right) d y \lesssim M\left|x_{1}\right| \min \left(1,|x|^{-\beta}\right) \lesssim M \min \left(|x|,|x|^{1-\beta}\right)
$$

In $Q_{2}$, since $|x-y| \leq|x| / 2$, we have $|y| \asymp|x|$, and $\min \left(|x|^{\alpha},|x|^{-\beta}\right) \asymp \min \left(|y|^{\alpha},|y|^{-\beta}\right)$. It follows

$$
\left|\int_{Q_{2}}\right| K_{2}(x-y) W(y)\left|\leq M \int_{|x-y| \leq|x| / 2}\right| x-\left.y\right|^{-1} \min \left(|y|^{\alpha},|y|^{-\beta}\right) d y \lesssim M \min \left(|x|^{\alpha},|x|^{-\beta}\right)|x|
$$

In $Q_{3}$, we have $|x| / 2 \leq|x-y| \leq 3|x|,|y| \leq 2|x|$. Using this estimate and (6.21), we obtain

$$
\begin{aligned}
\left|\int_{Q_{3}} K(x-y) W(y) d y\right| & \lesssim M \int_{Q_{3}}|x-y|^{-1} \min \left(|y|^{\alpha},|y|^{-\beta}\right) d y \lesssim M|x|^{-1} \int_{|y| \leq 2|x|} \min \left(|y|^{\alpha},|y|^{-\beta}\right) d y \\
& \lesssim M|x|^{-1} \min \left(|x|^{2+\alpha},|x|^{2-\beta}\right) \lesssim M \min \left(|x|^{1+\alpha},|x|^{1-\beta}\right)
\end{aligned}
$$

Combining the above estimates, we prove the first estimate in Lemma 6.5.
6.2.3. Estimate of $\nabla \phi_{1}$. We have the following estimate of $\nabla \phi_{1}$ in $|x| \leq R_{2}$.

Proposition 6.6. Let $\phi_{1}$ be the solution in (6.14) and $\alpha>0, \beta \in(0,1)$. There exists some absolute constant $\nu_{1}(\alpha, \beta)<\frac{1}{32}$ such that if $C_{l}(\tau)(1+S(\tau))<\nu_{1}$, we have

$$
\max _{|x| \leq 2 R_{2}}\left|\nabla \phi_{1}\right|\left(|x|^{-1}+|x|^{-1+\beta}\right) \lesssim\left\|\Omega_{1}\left(|x|^{-\alpha}+|x|^{\beta}\right)\right\|_{\infty} .
$$

For $8 S \leq|x| \leq R_{1} / 2$ away from the support of $\omega$, we have an improved estimate

$$
\begin{equation*}
\left|\nabla \phi_{1}\right| \lesssim\left\|\Omega_{1}\left(|x|^{-\alpha}+|x|^{\beta}\right)\right\|_{\infty}\left(\frac{S^{3-\beta}}{|x|^{2}}+|x|^{1-\beta}\left(C_{l} S\right)^{2-\beta}\right) \tag{6.22}
\end{equation*}
$$

In the following estimate, since $r$ is sufficiently close to 1 within the support of $\omega$, we can simply treat $\Omega_{1}=\omega r^{3 / 2}$ and $\omega$ as the same.

Proof. We choose $i=1$ in (6.17). Denote

$$
B_{1} \triangleq \max _{|x| \leq 2 R_{2}}\left|\nabla \phi_{1}\left(|x|^{-1}+|x|^{-1+\beta}\right)\right|, \quad M_{1} \triangleq\left\|\Omega_{1}\left(|x|^{-\alpha}+|x|^{\beta}\right)\right\|_{\infty}
$$

Inverting $-\Delta$ and then apply $\nabla$, we obtain

$$
\begin{align*}
& \nabla\left(\phi_{1} \chi_{1}\right)=\nabla(-\Delta)^{-1}\left(\Omega_{1} \chi_{1}\right)-\nabla(-\Delta)^{-1} Z_{1,1}-\nabla(-\Delta)^{-1} Z_{2,1} \triangleq I_{1}+I_{2}+I_{3} \\
& Z_{1,1}=2 \nabla \phi_{1} \cdot \nabla \chi_{1}+\phi_{1} \Delta \chi_{i}, \quad Z_{2,1}=\frac{a C_{l}^{2}}{r^{2}} \phi_{1} \chi_{i} \tag{6.23}
\end{align*}
$$

Our goal is to prove the following estimate

$$
\begin{equation*}
B_{1} \leq C_{\alpha, \beta}\left(M_{1}+\left(C_{l} S\right)^{2} \cdot B_{1}\right) \tag{6.24}
\end{equation*}
$$

Then as long as $C_{l} S$ is small, we can obtain the bound for $B_{1}$.
For $I_{1}, I_{3}$, applying Lemma 6.5, we get
$\left|I_{1}\right| \leq \min \left(|x|,|x|^{1-\beta}\right) M_{1}, \quad\left|I_{3}\right| \leq \min \left(|x|,|x|^{1-\beta}\right)\left\|Z_{2,1}\left(|x|^{-1}+|x|^{\beta}\right)\right\|_{\infty}, \quad Z_{2,1} \quad=\frac{a C_{l}^{2}}{r^{2}} \phi_{1} \chi_{1}$.
It suffices to bound the norm of $Z_{2,1}$. For $|x| \leq 2 S<R_{2}$ (6.20), using the definition of $B_{1}$, $\phi(0)=0$, and integration, we get

$$
\left|\phi_{1}\right| \lesssim B_{1} \min \left(|x|^{2},|x|^{2-\beta}\right)
$$

which along with $r^{-1} \lesssim 1$ within the support of $\chi_{1}$ yields

$$
\left|Z_{2,1}\right|\left(|x|^{-1}+|x|^{\beta}\right) \lesssim B_{1} C_{l}^{2} \min \left(|x|^{2},|x|^{2-\beta}\right)\left(|x|^{-1}+|x|^{\beta}\right) \lesssim B_{1} C_{l}^{2}\left(|x|^{2}+|x|\right) \lesssim B_{1} C_{l}^{2} S^{2}
$$

For $|x| \geq 2 S>2$, using Lemma 6.4, we yield

$$
\left|Z_{2,1}\left(|x|^{-1}+|x|^{\beta}\right)\right| \lesssim C_{l}^{2}\left(1+\left|\log \left(C_{l}|x|\right)\right|\right) S^{2-\beta}|x|^{\beta} \mathbf{1}_{|x| \leq 2 R_{1}}\left\|\omega\left(1+|x|^{-\beta}\right)\right\|_{\infty}
$$

By definition, we have $C_{l} R_{1} \in[0,1 / 4]$ (6.16). Within the support of $\omega,\left|\omega r^{3 / 2}\right|=\left|\Omega_{1}\right|$ and $\omega$ are equivalent. Hence, we obtain $\left(1+\left|\log C_{l} x\right|\right)\left|C_{l} x\right|^{\beta} \lesssim 1$ and

$$
\begin{equation*}
\left|Z_{2,1}\left(|x|^{-1}+|x|^{\beta}\right)\right| \lesssim\left(C_{l} S\right)^{2-\beta}| | \Omega_{1}\left(1+|x|^{\beta}\right) \|_{\infty} \tag{6.25}
\end{equation*}
$$

Next, we estimate $I_{2}$. Since $R_{2} \leq \frac{R_{1}}{4}$, for $|x| \leq 2 R_{2}$ and $y \in \operatorname{supp}\left(Z_{1,1}\right)$, we have $2|x| \leq|y|$. We estimate a typical term in $Z_{1,1}$. To use the formula (6.18), (6.19), we extend $\phi_{1}, \chi_{1}$ naturally from $\mathbb{R}_{2}^{+}$to $\mathbb{R}^{2}$ as an odd, and even function, respectively. For $i, j \in\{1,2\}$, using integration by parts, we get

$$
J \triangleq \partial_{i}(-\Delta)^{-1}\left(\partial_{j} \phi_{1} \partial_{j} \chi_{1}\right)=C \int_{\mathbb{R}_{2}} \frac{x_{i}-y_{i}}{|x-y|^{2}}\left(\partial_{j} \phi_{1} \partial_{j} \chi_{1}\right)=-C \int_{\mathbb{R}_{2}} \partial_{j}\left(\frac{x_{i}-y_{i}}{|x-y|^{2}} \partial_{j} \chi_{1}\right) \phi_{1} \triangleq J_{1}+J_{2}
$$

$J_{1} \triangleq-C \int_{\mathbb{R}_{2}} \partial_{j} \frac{x_{i}-y_{i}}{|x-y|^{2}} \partial_{j} \chi_{1} \phi_{1}, \quad J_{2} \triangleq-C \int_{\mathbb{R}_{2}} \frac{x_{i}-y_{i}}{|x-y|^{2}} \partial_{j}^{2} \chi_{1} \phi_{1}$.
Since the singularity $x$ is away from the support of the integrand, the singular integral kernel is smooth. We estimate the first term with $i=2, j=1$. Estimates of other cases and the second term are similar. Denote $K(z)=\frac{z_{1} z_{2}}{|z|^{4}}$. Using the fact that $\partial_{1} \chi_{1} \phi_{1}$ is even in $y_{1}$ and symmetrizing the kernel in $y_{1}$, we get
$J_{1}=C \int K(x-y) \partial_{1} \chi_{1} \phi_{1}(y) d y=C \int_{y_{1} \geq 0}\left(K\left(x_{1}-y_{1}, x_{2}-y_{2}\right)+K\left(x_{1}+y_{1}, x_{2}-y_{2}\right)\right) \partial_{1} \chi_{1} \phi_{1}(y) d y$, where we have used that $\partial_{1} \chi_{1} \phi_{1}$ are even in $y_{1}$. Since $K(z)$ is odd in $z_{1}$ and $|y| \geq 2|x|$ for $y$ in the support of the integrand, we get
$\left|K\left(x_{1}-y_{1}, x_{2}-y_{2}\right)+K\left(x_{1}+y_{1}, x_{2}-y_{2}\right)\right|=\left|K\left(y_{1}+x_{1}, x_{2}-y_{2}\right)-K\left(y_{1}-x_{1}, x_{2}-y_{2}\right)\right| \lesssim \frac{2 x_{1}}{|y|^{3}}$.
Using Lemma 6.4 (6.15), and (6.16), we get

$$
\left|\partial_{1} \chi_{1} \phi_{1}\right| \lesssim \frac{1}{R_{1}} S^{2-\beta} \mathbf{1}_{R_{1} \leq|y| \leq 2 R_{1}} M_{1}
$$

It follows

$$
\left|J_{1}\right| \lesssim \frac{|x| S^{2-\beta}}{R_{1}} M_{1} \int \mathbf{1}_{R_{1} \leq|y| \leq 2 R_{1}}|y|^{-3} d y \lesssim \frac{|x| S^{2-\beta}}{R_{1}^{2}} M_{1}
$$

Using a similar symmetrization argument, the fact that $x$ is away from the singularity of the kernel when $|x| \leq 2 R_{2}$, and $|x| \leq R_{1}, R_{1} \asymp C_{l}^{-1}$ (6.16), we obtain

$$
\begin{equation*}
\left|\nabla(-\Delta)^{-1} Z_{1,1}\right| \lesssim \frac{|x| S^{2-\beta}}{R_{1}^{2}} M_{1} \lesssim|x| C_{l}^{2} S^{2-\beta} M_{1} \lesssim \min \left(|x|,|x|^{1-\beta}\right)\left(C_{l} S\right)^{2-\beta} M_{1} \tag{6.26}
\end{equation*}
$$

Combining the above estimate and using $C_{l} S \leq 1$, we obtain

$$
\left|\nabla \phi_{1}\right| \lesssim \alpha, \beta \min \left(|x|,|x|^{1-\beta}\right)\left(M_{1}+\left(C_{l} S\right)^{2-\beta} B_{1}\right)
$$

Using (6.23), and taking the maximum of $x$ over $|x| \leq 2 R_{2}$, we prove (6.24), which further implies the desired result.

Improved estimate. For $8 S \leq|x| \leq R_{1} / 2$, we refine the estimate of $I_{1}$ and $I_{3}$. In $I_{1}$, for $y$ in the support of $\Omega_{1}$, we have $|x| \geq 2|y|$. For $K(z)=\frac{z_{i}}{|z|^{2}}$, using the same symmetrization argument, we get

$$
\begin{aligned}
\left|\int_{\mathbb{R}_{2}} K(x-y) \Omega_{1} d y\right| & =\left|\int_{y_{1} \geq 0}\left(K\left(x_{1}-y_{1}, x_{2}-y_{2}\right)-K\left(x_{1}+y_{1}, x_{2}-y_{2}\right)\right) \Omega_{1} d y\right| \\
& \lesssim \int_{y_{1} \geq 0} \frac{2 y_{1}}{|x|^{2}}\left|\Omega_{1}(y)\right| d y \lesssim|x|^{-2} M_{1} \int_{|y| \leq S}|y| \min \left(|y|^{\alpha},|y|^{-\beta}\right) d y \lesssim M_{1}|x|^{-2} S^{3-\beta}
\end{aligned}
$$

The term $\nabla(-\Delta)^{-1} Z_{1,1}$ is already estimated in (6.26). For $Z_{2,1}$ and $I_{3}$ (6.23), we improve the estimate of $\left\|Z_{2,1}\left(|x|^{-\alpha}+|x|^{\beta}\right)\right\|_{\infty}$. For $|x| \geq 2 S$, we have the estimate (6.25). For $|x| \leq 2 S$, using the first estimate in Proposition 6.6 we just proved and using $\nabla \phi_{1}$ to bound $\phi_{1}$, we obtain $\left|Z_{2,1}\right|\left(|x|^{-\alpha}+|x|^{\beta}\right) \lesssim C_{l}^{2} \min \left(|x|^{2},|x|^{2-\beta}\right)\left(|x|^{-\alpha}+|x|^{\beta}\right) M_{1} \lesssim C_{l}^{2}\left(|x|^{2-\alpha}+|x|^{2}\right) M_{1} \lesssim\left(C_{l} S\right)^{2-\beta} M_{1}$.
Combining the estimates in two cases, we get

$$
\left\|Z_{2,1} \mid\left(|x|^{-\alpha}+|x|^{\beta}\right)\right\|_{\infty} \lesssim\left(C_{l} S\right)^{2-\beta} M_{1}
$$

Applying Lemma 6.5 again, we obtain

$$
\left|\nabla^{\perp}(-\Delta)^{-1} Z_{2,1}\right| \lesssim|x|^{1-\beta}\left(C_{l} S\right)^{2-\beta} M_{1}
$$

Combining the above estimates and (6.26), we prove (6.22).
6.2.4. Estimate of $\nabla^{2} \phi_{1}$. Based on the estimate in Proposition 6.6 for $\nabla \phi_{1}$, we further estimate $\nabla^{2} \phi_{1}$ for $|x| \leq R_{3}$.
Proposition 6.7. Let $\phi_{1}$ be the solution in (6.14), $\beta \in(0,1)$, and $\alpha \in(0,1]$. There exists some absolute constant $\nu_{2}(\alpha, \beta)<\frac{1}{32}$ such that if $C_{l}(\tau)(1+S(\tau))<\nu_{2}$, we have

$$
\left|\nabla^{2}\left(\phi_{1} \chi_{3}\right)-\nabla^{2}(-\Delta)^{-1} \Omega_{1}\right| \lesssim \alpha, \beta C_{l}^{\beta}| | \Omega_{1}\left(|x|^{-\alpha}+|x|^{\beta}\right) \|_{\infty}
$$

In particular, we have

$$
\begin{equation*}
\partial_{x y}\left(\phi_{1}\right)=\partial_{x y} \phi(0), \quad\left|\partial_{x y} \phi_{1}(0)-\partial_{x y}(-\Delta)^{-1} \Omega_{1}(0)\right| \lesssim \alpha, \beta C_{l}^{\beta}\left\|\Omega_{1}\left(|x|^{-\alpha}+|x|^{\beta}\right)\right\|_{\infty} \tag{6.27}
\end{equation*}
$$

For $\Omega_{1}$ being the perturbation, we will further bound $\nabla^{2}(-\Delta)^{-1} \Omega_{1}$ using the energy defined in the Boussinesq equation (5.70).
Proof. We consider (6.17) with $i=2$. Denote

$$
M_{1}=\left\|\Omega_{1}\left(|x|^{-\alpha}+|x|^{\beta}\right)\right\|_{\infty}
$$

Using (6.17), we have

$$
\nabla^{2}\left(\phi_{1} \chi_{2}\right)=\nabla^{2}(-\Delta)^{-1} \Omega_{1}-\nabla^{2}(-\Delta)^{-1} Z_{1,2}-\nabla^{2}(-\Delta)^{-1} Z_{2,2} \triangleq I_{1}+I_{2}+I_{3}
$$

where we have used $\Omega_{1} \chi_{i}=\Omega_{1}$ by requiring $C_{l} S$ small. We only need to estimate $I_{2}, I_{3}$. The estimate of $I_{2}$ is similar to that in the proof of Proposition 6.6. We consider the typical term

$$
J=\partial_{12}(-\Delta)^{-1}\left(\partial_{1} \phi_{1} \partial_{1} \chi_{2}\right)
$$

For $|x| \leq R_{3}$, it is away from the support of $\partial_{1} \phi_{1} \partial_{1} \chi_{2}$. Denote $K(z)=\frac{z_{1} z_{2}}{|z|^{4}}$. We have

$$
J=C \int K(x-y)\left(\partial_{1} \phi_{1} \partial_{1} \chi_{2}\right)(y) d y
$$

Using Proposition 6.6 and (6.15), we get

$$
\begin{equation*}
\left|\partial_{1} \phi_{1} \partial_{1} \chi_{2}\right| \lesssim R_{2}^{-1} \mathbf{1}_{R_{2} \leq|y| \leq 2 R_{2}}\left|\partial_{1} \phi_{1}\right| \lesssim M_{1} R_{2}^{-1} R_{2}^{1-\beta} \mathbf{1}_{R_{2} \leq|y| \leq 2 R_{2}} \lesssim M_{1} R_{2}^{-\beta} \mathbf{1}_{R_{2} \leq|y| \leq 2 R_{2}} \tag{6.28}
\end{equation*}
$$

Since $|x| \leq R_{3} \leq R_{2} / 4 \leq|y| / 4$ and $|K(x-y)| \lesssim|y-x|^{-2} \lesssim|y|^{-2}$, we get

$$
|J| \lesssim M_{1} R_{2}^{-\beta} \int_{R_{2} \leq|y| \leq 2 R_{2}}|K(x-y)| d y \lesssim M_{1} R_{2}^{-\beta} \int_{R_{2} \leq|y| \leq 2 R_{2}}|y|^{-2} d y \lesssim M_{1} R_{2}^{-\beta}
$$

Similarly, we can obtain

$$
\left|I_{2}\right| \lesssim M_{1} R_{2}^{-\beta} \lesssim M_{1} C_{l}^{\beta}
$$

where we have used (6.16) to obtain the last inequality.
For $I_{3}$, we estimate $\partial_{12}(-\Delta)^{-1} Z_{2,2}$. Recall $Z_{2,2}=\frac{a C_{l}^{2}}{r} \phi_{1} \chi_{2}$ from (6.17). Other derivatives are similar. By definition, we have

$$
\partial_{12}(-\Delta)^{-1} Z_{2,2}=C \int K(x-y) Z_{2,2}(y) d y
$$

For a fixed $x$, we partition the region of the integral into three parts

$$
Q_{1}=\{y:|y| \geq 2|x|\}, \quad Q_{2}=\{y:|y-x| \leq|x| / 2\}, \quad Q_{3}=\left(Q_{1} \cup Q_{2}\right)^{c}
$$

Applying Proposition [6.6] using (6.15) and $\left|\partial_{i} r^{-1}\right| \lesssim C_{l}$ when $|r| \geq \frac{1}{2}$, for $|x| \leq 2 R_{2}$, we obtain

$$
\begin{aligned}
& \left|\partial_{i} Z_{2,2}\right| \lesssim C_{l}^{2}\left(\left|\partial_{i} r^{-1} \phi_{1} \chi_{2}\right|+\left|\partial_{i} \phi_{1} \chi_{2}\right|+\left|\phi_{1} \partial_{i} \chi_{2}\right|\right) \\
& \quad \lesssim M_{1} C_{l}^{2} \mathbf{1}_{|x| \leq 2 R_{2}}\left(C_{l}|x|^{2} \wedge|x|^{2-\beta}+|x| \wedge|x|^{1-\beta}+\left(|x|^{2} \wedge|x|^{2-\beta}\right) R_{2}^{-1}\right) \lesssim M_{1} C_{l}^{2} R_{2}^{1-\beta} \lesssim C_{l}^{1+\beta} M_{1}
\end{aligned}
$$

We also have the pointwise estimate

$$
\begin{equation*}
\left|Z_{2,2}\right| \lesssim C_{l}^{2}\left|\phi_{1} \chi_{2}\right| \lesssim C_{l}^{2} \min \left(|x|^{2},|x|^{2-\beta}\right) \mathbf{1}_{|x| \lesssim R_{2}} M_{1} \lesssim M_{1} C_{l}^{\beta} \tag{6.29}
\end{equation*}
$$

Using the above pointwise estimate, for $|x| \leq R_{3}$, we can obtain

$$
\begin{aligned}
\left|\int_{Q_{1}} K(x-y) Z_{2,2}(y) d y\right| & \lesssim M_{1} \int_{2|x| \lesssim|y| \lesssim R_{2}}|y|^{-2} C_{l}^{2} \min \left(|y|^{2},|y|^{2-\beta}\right) d y \lesssim C_{l}^{2} R_{2}^{2-\beta} M_{1} \lesssim M_{1} C_{l}^{\beta} \\
\left|\int_{|x-y| \leq|x| / 2} K(x-y) Z_{2,2}(y) d y\right| & \lesssim \max _{|x-y| \leq|x| / 2}\left|\nabla Z_{2,2}(y)\right| \int_{|x-y| \leq|x| / 2}|x-y|^{-1} d y \lesssim C_{l}^{1+\beta} M_{1}|x| \lesssim M_{1} C_{l}^{\beta},
\end{aligned}
$$

Since $Q_{3} \subset\{y:|x| / 2 \leq|x-y| \leq 3|x|\}$, we get

$$
\left|\int_{Q_{3}} K(x-y) Z_{2,2}(y) d y\right| \lesssim|x|^{-2}\left|\int_{Q_{3}} Z_{2,2}(y) d y\right| \lesssim|x|^{-2} M_{1} C_{l}^{\beta} \int_{|y| \leq 4|x|} d y \lesssim M_{1} C_{l}^{\beta}
$$

Combining the above estimates, we prove the desired result.
To obtain (6.27), we simply use $r=1$ when $x=y=0$ and

$$
\partial_{x y}\left(\phi_{1}\right)=\partial_{x y}\left(\phi r^{1 / 2}\right)(0)=\partial_{x y} \phi(0)
$$

6.2.5. Weighted $L^{\infty}$ and Hölder estimate. Based on Propositions 6.6 and Proposition 6.7 we show that $Z_{1, i}, Z_{2, i}$ in (6.17) are small in the energy norm. Recall the weights $\varphi_{1}, \varphi_{g, 1}, \psi_{1}$ and $g_{1}$ from (C.1), (C.3), (C.4). Denote

$$
\begin{equation*}
\|\Omega\|_{X} \triangleq \max \left(\left\|\Omega \varphi_{1}\right\|_{L^{\infty}},\left\|\Omega \varphi_{g, 1}\right\|_{L^{\infty}},\left\|\Omega \psi_{1}\right\|_{C_{g_{1}}^{1 / 2}}\right) \tag{6.30}
\end{equation*}
$$

The energy (5.70), (5.54) also includes the norm $\left\|\Omega_{1} \psi_{1}\left|x_{1}\right|^{-1 / 2}\right\|_{L^{\infty}}$, which can be bounded by $\left\|\Omega \psi_{1}\right\|_{C_{g_{1}}^{1 / 2}}$ up to some absolute constant. Thus, we do not include it in the above norm.

Recall that $\varphi_{1, g} \sim c|x|^{1 / 16}$ for large $|x|$. We will fix

$$
\begin{equation*}
\beta=\frac{1}{16} \tag{6.31}
\end{equation*}
$$

in the following estimate.

We want to show that the $Z$ term in (6.17) is small in $X$. However, $Z_{2,4}$ only vanishes to order $O\left(|x|^{2}\right)$ near $x=0$ and is not in the space $X$ since space $X$ involves singular weights of order $|x|^{-\gamma}$ with $\gamma \in(2,3]$. We need to subtract a rank one correction near $x=0$. In the following estimates, the sizes of $Z_{1, i}, Z_{2, i}$ are very small. The reader can mainly pay attention to the vanishing order of these terms near $x=0$.

Proposition 6.8. Let $\phi_{1}$ be the solution to (6.17). Suppose that $\Omega_{1} \in X$ and $C_{l} S<\nu_{2}$, where $\nu_{2}$ is the constant in Proposition 6.7. For $|x| \leq 2, \alpha, \beta>0, \alpha<2$, we have

$$
\begin{aligned}
& \left|\nabla\left(\phi_{1}-x_{1} x_{2} \partial_{x y} \phi_{1}(0)\right)\right| \lesssim \alpha, \beta|x|^{1+\alpha}| | \Omega_{1}|x|^{-\alpha}+|x|^{\beta} \|_{L^{\infty}}, \\
& \quad\left|\phi_{1}-x_{1} x_{2} \partial_{x y} \phi_{1}(0)\right| \lesssim \alpha, \beta|x|^{2+\alpha}| | \Omega_{1}|x|^{-\alpha}+|x|^{\beta} \|_{L^{\infty}} .
\end{aligned}
$$

If $\Omega_{1} \in X$, the vanishing order can be further improved. The weight $\varphi_{1}$ (C.3) is singular of order $|x|^{-2.9}$ near $x=0$,

Proof. Using (6.17) with $i=4$, we get

$$
\phi_{1} \chi_{4}=(-\Delta)^{-1}\left(\Omega_{1}-Z_{1,4}-Z_{2,4}\right)
$$

Using Lemma 6.5, we only need to prove that

$$
\left\|\left(\Omega_{1}-Z_{1,4}-Z_{2,4}\right)\left(|x|^{-\alpha}+|x|^{\beta}\right)\right\|_{\infty} \lesssim\left\|\Omega_{1}\left(|x|^{-\alpha}+|x|^{\beta}\right)\right\|_{\infty}
$$

Since $\chi_{1}=1$ in the support of $\chi_{4}$, the estimate of $Z_{1,4}, Z_{2,4}$ follows directly from Proposition 6.6 and its proof. We only consider a typical term. For $\partial_{1} \phi_{1} \partial_{1} \chi_{4}$ in $Z_{1,4}$, using Proposition 6.6, we get

$$
\begin{aligned}
\left|\partial_{1} \phi_{1} \partial_{1} \chi_{4}\right|\left(|x|^{-\alpha}+|x|^{\beta}\right) & \lesssim\left\|\Omega_{1}\left(|x|^{-\alpha}+|x|^{\beta}\right)\right\|_{\infty} \min \left(|x|,|x|^{1-\beta}\right) R_{4}^{-1} \mathbf{1}_{|x| \asymp R_{4}}\left(|x|^{-\alpha}+|x|^{\beta}\right) \\
& \lesssim\left\|\Omega_{1}\left(|x|^{-\alpha}+|x|^{\beta}\right)\right\|_{\infty} .
\end{aligned}
$$

We need to require $\alpha<2$ since $Z_{2,4}$ in (6.14) only vanishes to order $|x|^{2}$ near $x=0$.
We are in a position to show that the $Z$ term in (6.17) with a correction is small in space $X$.
Proposition 6.9. Suppose that $\Omega_{1} \in X$ and $C_{l} S<\nu_{2}$, where $\nu_{2}$ is the constant in Proposition 6.7. We have

$$
\begin{align*}
& \|Z\|_{X} \lesssim C_{l} S M, \quad\|\nabla Z\|_{L^{\infty}} \lesssim C_{l} M, \quad Z \triangleq Z_{1,4}+Z_{2,4}-\partial_{x y} Z_{2,4}(0)(-\Delta \kappa) \\
& M \triangleq\left\|\Omega_{1}\left(|x|^{-1}+|x|^{\beta}\right)\right\|_{\infty}+\left\|\chi_{3} \nabla^{2}(-\Delta)^{-1} \Omega_{1}\right\|_{\infty} \tag{6.32}
\end{align*}
$$

where $\kappa=-\frac{x y^{3}}{2} \chi$ and $\chi$ is some cutoff function supported near $x=0$, e.g. (6.15).
We will apply Proposition 6.9 to $\Omega_{1}=\omega r^{3 / 2}$ with $\omega \in X$ or $\omega=\bar{\omega} \chi(x / \nu)$ with $1<\nu<R_{4}$, where $\bar{\omega}$ is the approximate steady state for the 2D Boussinesq equation. In both cases, we can further bound the right hand side as follows

$$
\begin{align*}
& \left\|\Omega_{1}\left(|x|^{-1}+|x|^{\beta}\right)\right\|_{\infty}+\left\|\chi_{3} \nabla^{2}(-\Delta)^{-1} \Omega_{1}\right\|_{\infty} \lesssim\left\|\Omega_{1}\right\|_{X} \\
& \left\|\bar{\omega} \chi_{\nu} r^{3 / 2}\left(|x|^{-1}+|x|^{\beta}\right)\right\|_{\infty}+\left\|\chi_{3} \nabla^{2}(-\Delta)^{-1}\left(\bar{\omega} \chi_{\nu}\right)\right\|_{\infty} \lesssim 1 . \tag{6.33}
\end{align*}
$$

Estimates of $\nabla^{2}(-\Delta)^{-1} \Omega_{1}$ in both inequalities follow from standard interpolation inequalities.
Proof. Let $\phi_{1}$ be the solution of (6.17). Now, using Proposition 6.7, for $|x| \leq R_{3}$, we yield

$$
\left|\nabla^{2} \phi_{1}(x)\right| \lesssim M
$$

where $M$ is defined in (6.32), and the norm of $\Omega_{1}$ in Propositions 6.6, 6.7 with $\beta=\frac{1}{16}$ can be bounded by $M$.

Estimate of $Z_{1,4}$. Firstly, we estimate $Z_{1,4}$ (6.17). For $|x| \asymp R_{4}$, from (6.16), we have $C_{l} R_{4} \asymp 1$. Using Lemma 6.4 for $\phi_{1}$ and (6.22) in Proposition 6.6 for $\nabla \phi_{1}$, we obtain

$$
|\phi(x)| \lesssim S^{2-\beta} M, \quad|\nabla \phi(x)| \lesssim|x|^{1-\beta} M\left(C_{l} S\right)^{2-\beta}
$$

where we have used $S \lesssim|x|, C_{l}|x| \asymp C_{l} R_{4} \asymp 1$ to simplify the upper bound in (6.22). Using the above estimate and $R_{4}^{-1} \asymp C_{l}$, we obtain the pointwise estimate

$$
\begin{aligned}
\left|Z_{1,4}\right| & \lesssim \mathbf{1}_{|x| \asymp R_{4}}\left(R_{4}^{-1}\left|\nabla \phi_{1}\right|+R_{4}^{-2}\left|\phi_{1}\right|\right) \lesssim M \mathbf{1}_{|x| \asymp R_{4}}\left(R_{4}^{-1}|x|^{1-\beta}\left(C_{l} S\right)^{2-\beta}+S^{2-\beta} R_{4}^{-2}\right) \\
& \lesssim C_{l}^{\beta}\left(C_{l} S\right)^{2-\beta} M \mathbf{1}_{|x| \asymp R_{4}} .
\end{aligned}
$$

Using (6.15) for $\nabla^{k} \chi_{4}$, Lemma 6.4 and Propositions 6.6, 6.7 for $\nabla^{k} \phi_{1}$, we obtain

$$
\begin{aligned}
\left|\nabla Z_{1,4}\right| & \lesssim \mathbf{1}_{|x| \asymp R_{4}}\left(\left|\nabla^{2} \phi_{1}\right| \cdot\left|\nabla \chi_{4}\right|+\left|\nabla \phi_{1}\right| \cdot\left|\nabla^{2} \chi_{4}\right|+\left|\phi_{1}\right| \cdot\left|\nabla^{3} \chi_{4}\right|\right) \\
& \lesssim \mathbf{1}_{|x| \asymp R_{4}} M\left(R_{4}^{-3}|x|^{2-\beta}+R_{4}^{-2}|x|^{1-\beta}+R_{4}^{-1}\right) \lesssim \mathbf{1}_{|x| \asymp R_{4}} M R_{4}^{-1} \lesssim \mathbf{1}_{|x| \asymp R_{4}} M C_{l} .
\end{aligned}
$$

Note that the weights $\varphi_{1}, \varphi_{g, 1}$ (C.3), (C.4) involve $\left|x_{1}\right|^{-1 / 2}$ which is singular along $x_{1}=0$. For any power $\gamma \in[-3, \beta]$, we have

$$
\left.\left.\left|Z_{1,4}\right| x\right|^{\gamma}\left|\lesssim C_{l}^{\beta}\right| x\right|^{\gamma}\left(C_{l} S\right)^{2-\beta} \mathbf{1}_{|x| \asymp R_{4}} M \lesssim\left(C_{l} S\right)^{2-\beta} \mathbf{1}_{|x| \asymp R_{4}} M
$$

For any power $\gamma \in[-5 / 2,0)$ and $x_{1} \leq 1$, we have

$$
\left.\left.\left|Z_{1,4}\right| x\right|^{\gamma}\left|x_{1}\right|^{-1 / 2}\left|\lesssim \max _{|z| \leq 1}\right| \nabla Z_{1,4}\left(z, x_{2}\right)|\cdot| x\right|^{\gamma} \lesssim R_{4}^{-1+\gamma} M \lesssim C_{l} M
$$

To estimate the Hölder norm of $Z_{1,4} \psi_{1}$, following similar estimates, we obtain

$$
\left|Z_{1,4} \psi_{1}\right| \lesssim\left(C_{l} S\right)^{2-\beta} M, \quad\left|\nabla\left(Z_{1,4} \psi_{1}\right)\right| \lesssim\left(C_{l}+\left(C_{l} S\right)^{2-\beta}\right) M
$$

We can obtain better estimates due to the decay of $\psi_{1}(x)$ for large $x$, see (C.1). But we do not need this extra smallness. Using the above estimates and embedding inequalities, we obtain

$$
\left|Z_{1,4} \varphi_{1}\right| \lesssim\left|Z_{1,4} \varphi_{g, 1}\right| \lesssim\left(C_{l}+\left(C_{l} S\right)^{2-\beta}\right) M, \quad\left\|Z_{1,4} \psi\right\|_{C_{g_{i}}^{1 / 2}} \lesssim\left(C_{l}+\left(C_{l} S\right)^{2-\beta}\right) M
$$

which along with the above estimate for $\nabla Z_{1,4}$ establish the estimates for $Z_{1,4}$ in (6.32).
Estimate of $Z_{2,4}$. Recall

$$
Z_{2,4}=\frac{a C_{l}^{2}}{r} \phi_{1} \chi_{4}
$$

Clearly, we have $\partial_{x y} Z_{1,4}(0)=a C_{l}^{2} \partial_{x y} \phi_{1}(0)$. We perform the following decomposition
$Z_{2,4}-\partial_{x y} Z_{2,4}(0)(-\Delta \kappa)=\frac{a C_{l}^{2}}{r}\left(\phi_{1}-\partial_{x y} \phi_{1}(0)(-\Delta \kappa)\right)+\partial_{x y} \phi_{1}(0) \frac{a C_{l}^{2}}{r}(1-r)(-\Delta \kappa) \triangleq I+I I$.
From the definition of $\kappa$ in Proposition 6.9, for $|x| \leq 1$, we have

$$
-\Delta \kappa=x_{1} x_{2}, \quad(1-r)=C_{l} x_{2}
$$

Thus $I I=O\left(x_{1} x_{2}^{2}\right)$ near $x=0$. Using Proposition 6.6 for $\left.\left|\nabla^{2} \phi_{1}\right| \lesssim \lim \sup _{x \rightarrow 0}\left|\nabla \phi_{1}\right| x\right|^{-1} \mid$, Proposition 6.7 for $\partial_{x y} \phi(0)$ and the fact that $I I$ is supported near $x=0$, we get

$$
\|I I\|_{X} \lesssim C_{l}^{3} M, \quad|\nabla I I| \lesssim C_{l}^{3} M
$$

Applying Propositions 6.6 and 6.8 with $\alpha=1$ and we obtain

$$
\begin{equation*}
|\nabla I| \lesssim C_{l}^{2} \min \left(|x|^{2},|x|^{1-\beta}\right) M, \quad|I| \lesssim C_{l}^{2} x_{1} \min \left(|x|^{2},|x|^{1-\beta}\right) M \tag{6.34}
\end{equation*}
$$

where to obtain the second bound $C_{l}^{2} x_{1}|x|^{1-\beta}$, we have used Proposition 6.6 and integrated the estimate for $\partial_{1} \phi$ in $x_{1}$ to estimate $\phi_{1}$. Note that if the derivative acts on $r^{-1}$, we get $\left|\nabla r^{-1}\right| \lesssim C_{l}$ and then use $C_{l}|x| \lesssim 1$ to remove a growing power $|x|$. Since $C_{l}|x| \lesssim 1$ in the support of $I$, combining the estimate of $\nabla I, \nabla I I$, we obtain the estimate of $\nabla(I+I I)$ in (6.32).

For large $|x| \geq 8 S$, the correction vanishes $(-\Delta \kappa)=0$. Using Lemma 6.4 and the improved estimate (6.22), for $8 S \leq|x| \leq R_{1} / 2 \lesssim C_{l}^{-1}$, we have $|x|^{1-\beta}\left(C_{l} S\right)^{2-\beta} \lesssim C_{l} S^{2-\beta} \lesssim S^{1-\beta}$ and

$$
\left|\phi_{1}\right| \lesssim\left(1+\left|\log \left(C_{l} S\right)\right|\right) S^{2-\beta} M, \quad\left|\nabla\left(\phi_{1} / r\right)\right| \lesssim\left|\nabla \phi_{1}\right|+C_{l}\left|\phi_{1}\right| \lesssim S^{1-\beta} M
$$

where we have used $C_{l} S^{2-\beta}\left(1+|\log | C_{l} S| |\right) \lesssim S^{1-\beta}$ to absorb the logarithm factor. It follows

$$
\begin{align*}
|I| & \lesssim C_{l}^{2}\left(1+\left|\log \left(C_{l} S\right)\right|\right) S^{2-\beta} M \lesssim\left(C_{l} S\right)^{1-\beta}\left(1+\left|\log \left(C_{l} S\right)\right|\right) C_{l}^{1+\beta} S M \lesssim C_{l}^{1+\beta} S M  \tag{6.35}\\
|\nabla I| & \lesssim C_{l}^{2} S^{1-\beta} M \lesssim C_{l} M
\end{align*}
$$

for $|x| \in\left[8 S, R_{1} / 2\right]$. Therefore, for any $\gamma_{1} \in[-3, \beta]$, combining the above estimates and (6.34) and using $|x| \leq R_{1} \lesssim C_{l}^{-1}$ within the support of $I$, we have

$$
\begin{equation*}
|x|^{\gamma}|I| \lesssim C_{l} S M \tag{6.36}
\end{equation*}
$$

Next, we bound $|I||x|^{\gamma_{2}}\left|x_{1}\right|^{-1 / 2}$ for $\gamma_{2} \in[-5 / 2,0]$. If $\left|x_{1}\right| \geq 1$, it follows from the above bound. If $\left|x_{1}\right| \leq 1$, using (6.34), we obtain

$$
|x|^{\gamma_{2}}\left|x_{1}\right|^{-1 / 2}|I| \lesssim|x|^{\gamma_{2}} C_{l}^{2}\left|x_{1}\right|^{1 / 2} \min \left(|x|^{2},|x|^{1-\beta}\right) M \lesssim C_{l}^{2} \min \left(1,|x|^{1-\beta}\right) M \lesssim C_{l}^{1+\beta} M
$$

The above estimates imply

$$
\left\|I \varphi_{1}\right\|_{\infty} \lesssim\left\|I \varphi_{g, 1}\right\|_{\infty} \lesssim\left(C_{l}+C_{l} S\right) M \lesssim C_{l} S\left\|\Omega_{1}\right\|_{X}, \quad\left\|I \psi_{1}\right\|_{\infty} \lesssim C_{l} S M
$$

Since $\psi_{1} \lesssim|x|^{-2}+1,\left|\nabla \psi_{1}\right| \lesssim|x|^{-3}+|x|^{-1}$, for $I$, using (6.34), (6.36), we get

$$
\begin{aligned}
\left|\nabla\left(I \psi_{1}\right)\right| & \lesssim|\nabla I| \psi_{1}+\left|I \nabla \psi_{1}\right| \lesssim C_{l}^{2} \min \left(|x|^{2},|x|^{1-\beta}\right)\left(|x|^{-2}+1\right)| | \Omega_{1} \|_{X}+C_{l} S M \\
& \lesssim\left(C_{l}^{2}\left(1+|x|^{1-\beta}\right)+C_{l} S\right) M \lesssim C_{l} S M, \quad\left|I \psi_{1}\right| \lesssim C_{l} S M
\end{aligned}
$$

In the last inequality for $\nabla\left(I \psi_{1}\right)$, we have used $C_{l}^{1-\beta}|x|^{1-\beta} \lesssim 1$ within the support of $I$ and $C_{l} \ll 1<S$. Using embedding inequalities, we obtain the Hölder estimate of $I \psi_{1}$. We conclude

$$
\left\|Z_{2,4}-\partial_{x y} Z_{1,4}(0)(-\Delta) \kappa\right\|_{X} \lesssim\|I\|_{X}+\|I I\|_{X} \lesssim C_{l} S M
$$

Since $C_{l} S \lesssim 1, C_{l} \lesssim 1$, combining the estimate of $Z_{1,4}, Z_{2,4}$, we complete the proof.
6.3. Main terms for the stream function and velocity. Based on Proposition 6.9 we rewrite (6.17) with $i=4$ as follows

$$
-\Delta\left(\phi_{1} \chi_{4}+a C_{l}^{2} \phi_{x y}(0) \kappa\right)=\Omega_{1} \chi_{4}-Z_{1,4}-\left(Z_{2,4}-\partial_{x y} Z_{2,4}(0)(-\Delta \kappa)\right)
$$

where we have used (6.27) and

$$
\partial_{x y} Z_{2,4}(0)=a C_{l}^{2} \partial_{x y} \phi_{1}(0)=a C_{l}^{2} \partial_{x y} \phi(0)
$$

Recall the definitions of $\phi_{1}, \Omega_{1}$ from (6.17), and $\kappa$ from Proposition 6.9. We introduce

$$
\begin{align*}
\Psi & =\phi r^{1 / 2} \chi_{4}+a C_{l}^{2} \phi_{x y}(0) \kappa, \quad \Omega=\omega r^{3 / 2} \chi_{4}-Z_{1,4}-\left(Z_{2,4}-\partial_{x y} Z_{2,4}(0)(-\Delta \kappa)\right) \\
\Psi_{2} & =-a C_{l}^{2} \phi_{x y}(0) \kappa \tag{6.37}
\end{align*}
$$

Then we obtain

$$
\begin{equation*}
-\Delta \Psi=\Omega, \quad \phi r^{1 / 2} \chi_{4}=\Psi+\Psi_{2} \tag{6.38}
\end{equation*}
$$

Within the support of $\omega$, we have $r^{-1} \lesssim 1$ and $|r-1| \lesssim C_{l} S$. Using Proposition 6.9, we have

$$
\begin{equation*}
\|(\Omega-\omega) \rho\|_{\infty} \lesssim C C_{l} S\|\omega\|_{X}, \quad\|\Omega-\omega\|_{X} \lesssim C_{l} S\|\omega\|_{X} \tag{6.39}
\end{equation*}
$$

for weight $\rho$ with $\|f \rho\|_{L^{\infty}} \lesssim\|f\|_{X}$, e.g. $\rho=\varphi_{1}, \varphi_{g, 1},\left|x_{1}\right|^{-\frac{1}{2}} \psi$. Thus, $\Omega$ and $\omega$ enjoy almost the same estimates.

From Propositions 6.7 6.9, the term $\phi_{x y}(0)$ satisfies

$$
\begin{equation*}
\left|\phi_{x y}(0)\right| \lesssim\left\|\Omega_{1}\right\|_{X} \lesssim\|\Omega\|_{X}, \quad \Psi_{x y}(0)=\phi_{x y}(0) \tag{6.40}
\end{equation*}
$$

Therefore, the term $a C_{l}^{2} \phi_{x y}(0) \kappa$ is very small and vanishes to the order $|x|^{4}$ near $x=0$.
6.3.1. Main terms for the velocity. Recall $u, v$ from (6.13). Since we will only use the estimate of the velocity within the support of the solution, where $\chi_{i}=1$, in the following derivation, we drop the cutoff functions $\chi_{i}$ to simplify the notation. Firstly, from (6.38), we yield

$$
\phi r^{1 / 2}=\Psi+\Psi_{2}
$$

The term $\Psi_{2}$ is smooth with vanishing order $|x|^{4}$, compactly supported, and small. We treat it as a lower order term and do not expand its derivation below. The velocity depends on the derivatives of $\phi$. Using $\partial_{x} r=0, \partial_{y} r=-C_{l}$ defined in (6.9) (please do not confuse $r$ with $\sqrt{x^{2}+y^{2}}$ here), we rewrite $\nabla \phi$ as follows

$$
\begin{align*}
& \phi_{y}=\left(r^{1 / 2} \phi r^{-1 / 2}\right)_{y}=\left(\Psi r^{-1 / 2}\right)_{y}+\left(\Psi_{2} r^{-1 / 2}\right)_{y}=\Psi_{y} r^{-1 / 2}+\frac{C_{l}}{2 r^{3 / 2}} \Psi+\left(\Psi_{2} r^{-1 / 2}\right)_{y}  \tag{6.41}\\
& \phi_{x}=r^{-1 / 2} \Psi_{x}+r^{-1 / 2} \Psi_{2, x}
\end{align*}
$$

Then using (6.13), we can rewrite $u, v$ as follows

$$
\begin{equation*}
u=-\phi_{y}+\frac{1}{r} C_{l} \phi=-\Psi_{y} r^{-1 / 2}-\frac{C_{l}}{2 r^{3 / 2}} \Psi+\frac{1}{r^{3 / 2}} C_{l} \Psi-\left(\Psi_{2} r^{-1 / 2}\right)_{y}-\frac{1}{r^{3 / 2}} C_{l} \Psi_{2} \triangleq u_{M}+u_{R} \tag{6.42}
\end{equation*}
$$

where the main term and the remainder are given by

$$
\begin{equation*}
u_{M}=-\Psi_{y}, \quad u_{R}=-\Psi_{y}\left(r^{-1 / 2}-1\right)+\frac{C_{l}}{2 r^{3 / 2}} \Psi-\left(\Psi_{2} r^{-1 / 2}\right)_{y}-\frac{1}{r^{3 / 2}} C_{l} \Psi_{2} \tag{6.43}
\end{equation*}
$$

An important observation is that the first and the second terms in $u_{R}$ cancel each other near the origin. To see this, we have

$$
\begin{equation*}
r^{-1 / 2}-1=\frac{1-r^{1 / 2}}{r^{1 / 2}}=\frac{1-r}{r^{1 / 2}\left(1+r^{1 / 2}\right)}=\frac{C_{l} y}{r^{1 / 2}\left(1+r^{1 / 2}\right)} \tag{6.44}
\end{equation*}
$$

It follows

$$
\begin{align*}
& -\Psi_{y}\left(r^{-1 / 2}-1\right)+\frac{C_{l}}{2 r^{3 / 2}} \Psi  \tag{6.45}\\
= & -\left(\Psi_{y}-\Psi_{x y}(0) x\right)\left(r^{-1 / 2}-1\right)+\frac{C_{l}}{2 r^{3 / 2}}\left(\Psi-\Psi_{x y}(0) x y\right)+\Psi_{x y}(0)\left(-x\left(r^{-1 / 2}-1\right)+\frac{C_{l} x y}{2 r^{3 / 2}}\right) \\
= & -\left(\Psi_{y}-\Psi_{x y}(0) x\right)\left(r^{-1 / 2}-1\right)+\frac{C_{l}}{2 r^{3 / 2}}\left(\Psi-\Psi_{x y}(0) x y\right)+\Psi_{x y}(0) \frac{C_{l} x y\left(1+r^{1 / 2}-2 r\right)}{2 r^{3 / 2}\left(1+r^{1 / 2}\right)}
\end{align*}
$$

The last term vanishes to the order $O\left(|x|^{3}\right)$ near $x=0$. We treat $u_{R}$ as the remainder since it vanishes to the order $O\left(|x|^{3}\right)$ near $x=0$ and contain the small factor $C_{l}$. Within the support of the solution, we get $C_{l}|x| \leq C_{l} S$, which is small.

For $v$ in (6.13), using (6.41) we have

$$
\begin{align*}
& v=\phi_{x}=\Psi_{x}+\left(r^{-1 / 2}-1\right) \Psi_{x}+r^{-1 / 2} \Psi_{2, x} \triangleq v_{M}+v_{R} \\
& v_{M}=\Psi_{x}, \quad v_{R}=\left(r^{-1 / 2}-1\right) \Psi_{x}+r^{-1 / 2} \Psi_{2, x} \tag{6.46}
\end{align*}
$$

We treat $v_{R}$ as the remainder since it contains the small factor $C_{l}$ and vanishes to the order $|x|^{2}$ near $x=0$. Within the support, $v_{R}$ has size of order $C_{l} S$. We remark that the vanishing order of $v_{R}$ is less than that of $u_{R}\left(O\left(|x|^{3}\right)\right)$. On the other hand, in (6.13), the coefficients of $v$, e,g. $\theta_{y}, \omega_{y}$, have higher vanishing order than those of $u$, e.g. $\theta_{x}, \omega_{x}$, near $x=0$. The remainder terms $u_{R}, v_{R}$ with coefficients have enough vanishing order near 0 for our weighted estimates.
6.3.2. Main terms for the velocity of the approximate steady state. Following [17], we will construct the approximate steady state (6.55) for the 3D Euler (6.13) by truncating the approximate steady state $(\bar{\omega}, \bar{\theta})$ for the 2D Boussinesq. We need to show that the associated velocity (6.13) is close to that in the 2D Boussinesq equation. For $\nu$ sufficiently large and to be chosen, we define

$$
\bar{\omega}_{\nu}=\chi(x / \nu) \bar{\omega}
$$

where $\chi$ is the cutoff function chosen above (6.15).

To avoid confusion, we denote $\bar{\phi}_{2 D}=(-\Delta)^{-1} \bar{\omega}$. Using (6.37), (6.38) with $\omega=\bar{\omega}_{\nu}$ constructed above, and then subtracting (6.37) by $-\Delta \bar{\phi}_{2 D}=\bar{\omega}$, we yield

$$
(-\Delta)\left(\phi r^{1 / 2} \chi_{4}+a C_{l}^{2} \phi_{x y}(0) \kappa-\bar{\phi}_{2 D}\right)=\bar{\omega} \chi_{\nu} r^{3 / 2} \chi_{4}-\bar{\omega}-Z_{1,4}-\left(Z_{2,4}-\partial_{x y} Z_{2,4}(0)(-\Delta) \kappa\right)
$$

Applying Propositions (6.9) (6.33) with $\Omega_{1}=\bar{\omega}_{\nu}$, we have the following estimates for the $Z$ terms

$$
\left\|Z_{1,4}+\left(Z_{2,4}-\partial_{x y} Z_{2,4}(0)(-\Delta) \kappa\right)\right\|_{X} \lesssim C_{l} S
$$

Note that the source term of the elliptic equation only vanishes to the order $|x|^{2}$ near $x=0$ :

$$
\bar{\omega} \chi_{\nu} r^{3 / 2} \chi_{4}-\bar{\omega}=\bar{\omega}\left(r^{3 / 2}-1\right)=-\frac{3}{2} C_{l} \bar{\omega}_{x}(0) x y+O\left(|x|^{3}\right)
$$

We add a correction $\frac{3}{2} C_{l} \bar{\omega}_{x}(0)(-\Delta \kappa)$ with $-\Delta \kappa=x y+l$. o.t. to the above elliptic equation

$$
(-\Delta)\left(\phi r^{1 / 2} \chi_{4}+\left(a C_{l}^{2} \phi_{x y}(0)+\bar{\omega}_{x}(0) \frac{3}{2} C_{l}\right) \kappa-\bar{\phi}_{2 D}\right)=\Omega_{\nu, R}
$$

where

$$
\begin{align*}
\Omega_{\nu, R} \triangleq & \bar{\omega} \chi_{\nu} \chi_{4}\left(r^{3 / 2}-1\right)+\frac{3}{2} C_{l} \bar{\omega}_{x}(0)(-\Delta \kappa)+\bar{\omega}\left(\chi_{4} \chi_{\nu}-1\right) \\
& -Z_{1,4}-\left(Z_{2,4}-\partial_{x y} Z_{2,4}(0)(-\Delta) \kappa\right)  \tag{6.47}\\
\bar{\Psi}= & \phi r^{1 / 2} \chi_{4}+\left(a C_{l}^{2} \phi_{x y}(0)+\bar{\omega}_{x}(0) \frac{3}{2} C_{l}\right) \kappa, \quad \bar{\Psi}_{2}=-\left(a C_{l}^{2} \phi_{x y}(0)+\bar{\omega}_{x}(0) \frac{3}{2} C_{l}\right) \kappa
\end{align*}
$$

Then the source term $\Omega_{\nu, \mathbb{R}}$ vanishes near $x=0$ to the order $|x|^{3}$. We yield

$$
\begin{equation*}
-\Delta\left(\bar{\Psi}-\bar{\phi}_{2 D}\right)=\Omega_{\nu, R}, \quad-\Delta \bar{\Psi}=\Omega_{\nu, R}+\bar{\omega} \tag{6.48}
\end{equation*}
$$

Since $\bar{\omega} \in C^{1}$ and $|\bar{\omega}| \lesssim \min \left(\left|x_{1}\right|,|x|^{-1 / 6}\right),|\nabla \bar{\omega}| \lesssim \min \left(1,|x|^{-7 / 6}\right)$, using Proposition 6.9 and (6.33), we obtain

$$
\begin{equation*}
\left\|\Omega_{\nu, R}\right\|_{X} \lesssim\left\|\bar{\omega}\left(\chi_{4} \chi_{\nu}-1\right)\right\|_{X}+C_{l} S, \quad\left\|\Omega_{\nu, R}\right\|_{C^{1}} \lesssim\left\|\bar{\omega}\left(\chi_{4} \chi_{\nu}-1\right)\right\|_{C^{1}}+C_{l} S \tag{6.49}
\end{equation*}
$$

By choosing $\nu$ sufficiently large and $C_{l} S$ to be small, we can obtain that $\bar{\Psi}-\bar{\phi}_{2 D}$ is very small. Similar to (6.43) and (6.46), based on $\bar{\Psi}, \bar{\Psi}_{2}$ in (6.47) and

$$
\phi r^{1 / 2}=\bar{\Psi}+\bar{\Psi}_{2}
$$

we decompose the velocity in (6.13) associated with $\bar{\omega}_{\nu}$ as follows

$$
\begin{equation*}
\bar{u}=-\bar{\Psi}_{y}+\bar{u}_{R}, \quad \bar{v}=\bar{\Psi}_{x}+\bar{v}_{R} \tag{6.50}
\end{equation*}
$$

for $|x| \leq R_{4}$. The formulas of $\bar{u}_{R}, \bar{v}_{R}$ are similar to those in (6.43), (6.46) with $\Psi$, $\Psi_{2}$ replaced by $\bar{\Psi}, \bar{\Psi}_{2}$. The remaining terms vanish near 0 with order $\bar{u}_{R}=O\left(|x|^{3}\right), \bar{v}_{R}=O\left(|x|^{2}\right)$.

Since $\|f\|_{L^{p}} \lesssim\left\|f \varphi_{g, 1}\right\|_{\infty} \lesssim\|f\|_{X}$ for $p>100$, using embedding (6.75), (6.48), (6.49), we get

$$
\begin{align*}
& \left\|\nabla^{2}\left(\bar{\Psi}-\bar{\phi}_{2 D}\right)\right\|_{L^{\infty}}+\left\|\nabla^{2}\left(\bar{\Psi}-\bar{\phi}_{2 D}\right)\right\|_{C^{1 / 2}} \lesssim\left\|\Omega_{\nu, R}\right\|_{X}+\left\|\Omega_{\nu, R}\right\|_{C^{1}}  \tag{6.51}\\
& \lesssim\left\|\bar{\omega}\left(\chi_{4} \chi_{\nu}-1\right)\right\|_{C^{1}}+\left\|\bar{\omega}\left(\chi_{4} \chi_{\nu}-1\right)\right\|_{X}+C_{l} S
\end{align*}
$$

For the remaining terms $\overline{\mathbf{u}}_{R}=\left(\bar{u}_{R}, \bar{v}_{R}\right)$, for $|x| \leq 2 S$, using $|r-1|, C_{l}|x| \lesssim C_{l} S$, the elliptic equation for $\bar{\Psi}$ (6.48), and embeding inequalities, we get

$$
\begin{equation*}
\left\|\nabla^{2} \bar{\Psi}\right\|_{L^{\infty}}+\left\|\nabla^{2} \bar{\Psi}\right\|_{C^{1 / 2}} \lesssim 1, \sup _{|x| \leq 2 S}\left|\nabla \overline{\mathbf{u}}_{R}(x)\right|+\sup _{|x|,|z| \leq 2 S} \frac{\left|\nabla \overline{\mathbf{u}}_{R}(x)-\nabla \overline{\mathbf{u}}_{R}(z)\right|}{|x-z|^{1 / 2}} \lesssim C_{l} S \tag{6.52}
\end{equation*}
$$

6.3.3. Estimate of the velocity. We need several weighted estimate of $(-\Delta)^{-1} \Omega$ for the main terms in the velocity. We will have some small parameters to absorb the implicit constants.
Lemma 6.10. Suppose that $\Omega \in X(\sqrt{6.30})$ is odd and $-\Delta \Psi=\Omega$. We have

$$
\begin{equation*}
\left|\nabla^{2}\left(\Psi-\Psi_{x y}(0) x y-\frac{1}{6} \partial_{1112} \Psi(0)\left(x_{1}^{3} x_{2}-x_{1} x_{2}^{3}\right)\right)\right| \lesssim|x|^{2.5}\|\Omega\|_{X} \tag{6.53}
\end{equation*}
$$

for $|x| \leq 1$ and

$$
\left|\nabla^{2}\left(\Psi-\Psi_{x y}(0) x y\right)\right| \lesssim \min \left(|x|^{2}, 1\right)\|\Omega\|_{X}, \quad\left|\Psi_{x y}(0)\right|,\left|\partial_{1112} \Psi(0)\right| \lesssim\left\|\Omega \varphi_{g, 1}\right\|_{\infty}
$$

The formula of $\partial_{1112} \Psi(0)$ can be written as an integral of $\Omega$ and is given in (4.25). Note that in Section 4 of Part II [15], we develop the sharp version of the above estimates with better constants. In Appendix C.3.3 we present the proof, which also helps to illustrate the ideas for Section 4 of Part II 15.

In Section 3, Section 4 of Part II [15], for $\mathbf{u}=\nabla^{\perp}(-\Delta)^{-1} \omega$ with $\omega \in X$, we develop weighted estimate for $u_{x}-\hat{u}_{x}, u_{y}-\hat{u}_{y}, v_{x}-\hat{v}_{x}$ with approximations $\widehat{\nabla \mathbf{u}}$ constructed in Section 4.3. In particular, we obtain

$$
|x-z|^{-1 / 2}\left|(f-\hat{f}) \psi_{1}(x)-(f-\hat{f}) \psi_{1}(z)\right| \lesssim\left\|\omega \psi_{1}\right\|_{C_{g_{1}}^{1 / 2}}+\left\|\omega \varphi_{1}\right\|_{\infty} \lesssim\|\omega\|_{X}, \quad f=u_{x}, u_{y}, v_{x}
$$

for $x, z$ with $x_{1}=z_{1}$ or $x_{2}=z_{2}$ and $|x| \leq|z| \leq(1+\mu)|x|$ for some $\mu \in(0,1)$. The estimate up to some absolute constant can be established following the decomposition and argument in Section 4 of Part II 15 and using the asymptotics of the weights $\varphi_{1}, \psi_{1}$ (C.1), (C.3). When $|z|>(1+\mu)|x|$, the estimate follows from the above Lemma, the triangle inequality, $|x|^{-1 / 2} \psi_{1} \lesssim$ $\varphi_{1},\left\|\Omega \varphi_{1}\right\|_{\infty} \lesssim\|\Omega\|_{X}$. Since near $x=0, f-\hat{f}$ agree with the left hand side of (6.53) (see (4.26) and (4.29) , we obtain

$$
\left\|(f-\hat{f}) \psi_{1}\right\|_{C^{1 / 2}} \lesssim\|\omega\|_{X}
$$

Note that the approximations in (4.29), (4.37) except $C_{f 0}(x)\left(-\partial_{12}(-\Delta)^{-1} \omega\right)(0), C_{f}(x) \mathcal{K}_{00} \chi_{0}$ are supported away from 0 with smooth coefficients. Moreover, the functionals in (4.29), (4.37), e.g. $\hat{f}_{N S}\left(x_{i}, 0\right), \mathcal{K}_{00}$, can be bounded by $\|\omega\|_{X}$. Using triangle inequality, we yield

$$
\left\|\left(f-C_{f 0}(x)\left(-\partial_{12}(-\Delta)^{-1} \omega\right)(0)-C_{f}(x) \mathcal{K}_{00} \chi_{0}\right) \psi_{1}\right\|_{C^{1 / 2}} \lesssim\|\omega\|_{X}
$$

where $\chi_{0}$ is defined in Section4.3.2, $\chi_{0}=1$ near 0, and supported near 0. In summary, we have
Lemma 6.11. Suppose that $\Omega \in X$ (6.30) is odd and $-\Delta \Psi=\Omega$. We have

$$
\left\|\psi_{1}\left(\partial_{i j}\left(\Psi-\Psi_{x y}(0) x_{1} x_{2}\right)-\chi_{0} \partial_{1112} \Psi(0) \partial_{i j} G(x)\right)\right\|_{C^{1 / 2}} \lesssim\|\Omega\|_{X}, \quad G(x) \triangleq \frac{1}{6}\left(x_{1}^{3} x_{2}-x_{1} x_{2}^{3}\right)
$$

Using the above estimates for $\nabla^{2}(-\Delta)^{-1} \Omega$, we can obtain the estimate for $\nabla(-\Delta)^{-1} \Omega(x)$ by integration from 0 to $x$, which is more regular.
6.4. Nonlinear stability. In Section 6.4.1, we impose the bootstrap assumption on the support size. In Section 6.4.2, we construct the approximate steady state and impose the normalization conditions, which are small perturbations to those in the 2D Boussinesq. Then we generalize the nonlinear stability analysis of the 2D Boussinesq equations to prove Theorem 4.
6.4.1. Bootstrap assumption on the support size. We fix the exponents $\alpha=1$ or $\alpha=2.9, \beta=$ $\frac{1}{16}$ in Propositions 6.6, 6.7, 6.9, These exponents are related to the singular weights we use. Then the constants $\nu_{1}, \nu_{2}$ in these propositions are determined. We impose the first bootstrap assumption: for $t \geq 0$, we have

$$
\begin{equation*}
C_{l}(t)(1+\max (S(t), S(0)))<\min \left(\nu_{2}, 4^{-6}\right) \tag{6.54}
\end{equation*}
$$

Under the above Bootstrap assumption, the support of $\omega, \theta$ in $D_{1}$ does not touch the symmetry axis and $z= \pm 1$, the cutoff functions (6.16) satisfy $\chi_{i}=1, i \leq 5$ for $x$ in the support, and the assumptions in Propositions 6.6, 6.7, 6.9. We will choose $C_{l}(0)$ at the final step, which guarantees the smallness in (6.54).
6.4.2. Approximate steady state and the normalization condition. Since the rescaled domain $\tilde{D}_{1}$ (6.12) is bounded, we construct the approximate steady state with bounded support. We localize the approximate steady state $\bar{\omega}, \bar{\theta}$ for the 2 D Boussinesq constructed in Section 7 to construct the approximate steady state for (6.13)

$$
\begin{equation*}
\bar{\omega}_{0} \triangleq \chi_{\nu} \bar{\omega}, \quad \bar{\theta}_{0} \triangleq \chi_{\nu}(1+\bar{\theta}) \tag{6.55}
\end{equation*}
$$

where $\nu \geq 1, \chi_{\nu}(x)=\chi_{1}(|x| / \nu)$ is some cutoff function, and $\chi_{1}(y): \mathbb{R} \rightarrow[0,1]$ is even in $y$, $\chi_{1}=1$ for $|y| \leq 1$, and $\chi_{1}(y)=0$ for $|y| \geq 2$. We can choose $\chi_{1}=\tilde{\chi}_{1}^{2}$ for another smooth cutoff function $\tilde{\chi}_{1}$ such that $\chi_{1}^{1 / 2}=\tilde{\chi}_{1}$ is smooth. Clearly, from Definition 6.2, the support size
of $\bar{\omega}_{0}, \bar{\theta}_{0}$ is $2 \nu$. We truncate $1+\bar{\theta}$ rather than $\bar{\theta}$ so that $1+\bar{\theta} \gtrsim 1$ and $(1+\bar{\theta})^{1 / 2}$ has the same regularity as $\bar{\theta}$. This idea follows [17].

Denote $\beta=\arctan \frac{y}{x}, \rho=\sqrt{x^{2}+y^{2}}$. Recall the formula in the polar coordinate:

$$
\begin{equation*}
\partial_{x} g=\left(\cos \beta \partial_{r}-\frac{\sin \beta}{r} \partial_{\beta}\right) g, \quad \partial_{y} g=\left(\sin \beta \partial_{r}+\frac{\cos \beta}{r} \partial_{\beta}\right) g \tag{6.56}
\end{equation*}
$$

We have

$$
\begin{equation*}
\partial_{x} \bar{\theta}_{0}=\chi_{\nu} \bar{\theta}_{x}+\frac{1}{\nu} \cos \beta \chi_{1}^{\prime}(|x| / \nu)(1+\bar{\theta}), \quad \partial_{y} \bar{\theta}_{0}=\chi_{\nu} \bar{\theta}_{y}+\frac{1}{\nu} \sin \beta \chi_{1}^{\prime}(|x| / \nu)(1+\bar{\theta}) \tag{6.57}
\end{equation*}
$$

To show that $\bar{\omega}-\bar{\omega}_{0}, \bar{\theta}-\bar{\theta}_{0}$ is small, from (7.2) we have $\bar{\omega}, \bar{\theta} \in C^{4,1}$, and for $k \leq 3$

$$
\begin{equation*}
|\bar{\omega}| \lesssim \min \left(\left|x_{1}\right|,|x|^{-1 / 4}\right), \quad\left|\nabla^{k} \bar{\omega}\right| \lesssim \min \left(1,|x|^{-1 / 4-k}\right), \quad\left|\nabla^{k+1} \bar{\theta}\right| \lesssim \min \left(1,|x|^{-3 / 5-k}\right) \tag{6.58}
\end{equation*}
$$

To distinguish the notations between the 3D Euler and the 2D Boussinesq equations, we write

$$
\begin{equation*}
\bar{\phi}_{2 D}=(-\Delta)^{-1} \bar{\omega}, \quad \overline{\mathbf{u}}_{2 D}=\nabla^{\perp} \bar{\phi}_{2 D} \tag{6.59}
\end{equation*}
$$

for the 2D Boussinesq. Let $\bar{\phi}$ and $\overline{\mathbf{u}}$ be the stream function and velocity in (6.13) associated with $\bar{\omega}_{0}$. We have the leading order terms for $\mathbf{u}$ (6.50). See more discussions in Section 6.3.2,

We need to adjust the time-dependent normalization condition for $c_{\omega}(t), c_{l}(t)$. We impose the following conditions

$$
\begin{equation*}
\bar{c}_{l}=2 \frac{\bar{\theta}_{x x}(0)}{\bar{\omega}_{x}(0)}, \quad \bar{c}_{\omega}(t)=\frac{1}{2} \bar{c}_{l}+\bar{u}_{x}(0), \quad \bar{c}_{\theta}(t)=\bar{c}_{l}+2 \bar{c}_{\omega}(t) \tag{6.60}
\end{equation*}
$$

for the approximate steady state $\bar{\omega}_{0}, \bar{\theta}_{0}$, and

$$
\begin{equation*}
c_{l}(t)=0, \quad c_{\omega}(t)=u_{x}(t, 0) \tag{6.61}
\end{equation*}
$$

for the perturbations, where $u(t, 0)$ is the velocity in (6.13) and is different from $-\partial_{y}(-\Delta)^{-1} \omega$. The above conditions are the same as (2.11) and (2.26), and play the same role of enforcing (2.12). As a result, the perturbation $\omega, \nabla \theta$ satisfies the vanishing condition (2.29)

$$
\omega=O\left(|x|^{2}\right), \quad \nabla \theta=O\left(|x|^{2}\right)
$$

near $x=0$. Since $\nabla \bar{\theta}_{0}=\nabla \bar{\theta}, \bar{\omega}_{0}=\bar{\omega}$ near $x=0$, the factor $\bar{c}_{l}$ is the same as that for the 2 D Boussinesq.

We remark that $\bar{c}_{\omega}(t)$ is time-dependent since it depends on $\bar{u}_{x}(0)$ and the elliptic equation in (6.13) depends on the rescaling factor $C_{l}$. From the estimate in Proposition (6.7), $\bar{u}_{x}(0)$ is very close to $-\partial_{x y}(-\Delta)^{-1} \bar{\omega}_{0}$. For $\nu$ sufficiently large, comparing the above conditions and (2.11), $\bar{c}_{\omega}$ is very close to $\bar{c}_{\omega, 2 D}$ (2.23) used for the 2D Boussinesq equations in Section 2. From (6.60) and (2.11), we yield

$$
\begin{equation*}
\bar{c}_{\omega}-\bar{c}_{\omega, 2 D}=\bar{u}_{x}(0)-\bar{u}_{x, 2 D}(0) \tag{6.62}
\end{equation*}
$$

Remark 6.12 . We will choose $\nu$ to be very large relatively to 1 . Therefore, we treat $\bar{\omega}_{0} \approx \bar{\omega}, \bar{\theta}_{0} \approx$ $\bar{\theta}$. Due to these small factors and using (6.47) and (6.50), we can treat $\overline{\mathbf{u}} \approx \overline{\mathbf{u}}_{2 D}$. From Remark 6.3 and the bootstrap assumption (6.54), we also have $C_{l} \approx 0, C_{l} S \approx 0, r \approx 1$. We treat the error terms in these approximations as perturbation.
6.4.3. Linearized equations. The equations (6.13) are slightly different from (2.10) for the Boussinesq systems. Denote

$$
\eta=\theta_{x}, \quad \xi=\theta_{y}
$$

Linearizing (6.13) around the approximate steady state $\left(\bar{\omega}_{0}, \bar{\theta}_{0}, \bar{c}_{l}, \bar{c}_{\omega}\right.$ ) (6.55), (6.60), we obtain the equations for the perturbation $(\omega, \eta, \xi)$, which are similar to (2.25), (2.28)

$$
\begin{align*}
\partial_{t} \omega & =-\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \omega+\frac{1}{r^{4}} \eta+\bar{c}_{\omega} \omega-\mathbf{u} \cdot \nabla \bar{\omega}_{0}+c_{\omega} \bar{\omega}_{0}+\overline{\mathcal{F}}_{1}+\mathcal{N}_{1} \\
& \triangleq \mathcal{L}_{1}(\omega, \eta, \xi)+\overline{\mathcal{F}}_{1}+\mathcal{N}_{1} \\
\partial_{t} \eta & =-\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \eta+\left(2 \bar{c}_{\omega}-\bar{u}_{x}\right) \eta-\bar{v}_{x} \xi-\partial_{x}\left(\mathbf{u} \cdot \nabla \bar{\theta}_{0}\right)+2 c_{\omega} \bar{\theta}_{0, x}+\mathcal{N}_{2}+\overline{\mathcal{F}}_{2}  \tag{6.63}\\
& \triangleq \mathcal{L}_{2}(\omega, \eta, \xi)+\overline{\mathcal{F}}_{2}+\mathcal{N}_{2} \\
\partial_{t} \xi & =-\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \xi+\left(2 \bar{c}_{\omega}-\bar{v}_{y}\right) \xi-\bar{u}_{y} \eta-\partial_{y}\left(\mathbf{u} \cdot \nabla \bar{\theta}_{0}\right)+2 c_{\omega} \bar{\theta}_{0, y}+\mathcal{N}_{3}+\overline{\mathcal{F}}_{3} \\
& \triangleq \mathcal{L}_{3}(\omega, \eta, \xi)+\mathcal{N}_{3}+\overline{\mathcal{F}}_{3}
\end{align*}
$$

where

$$
\overline{\mathcal{F}}_{1}=-\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \bar{\omega}_{0}+\frac{1}{r^{4}} \bar{\theta}_{0, x}+\bar{c}_{\omega} \bar{\omega}_{0}
$$

and we adopt similar notations $\mathcal{N}_{i}, \overline{\mathcal{F}}_{i}$ for other nonlinear terms and the error terms from (2.18), (2.19). The $\omega$ equation is different from the corresponding equation in (2.25) since we have $\frac{1}{r^{4}} \theta_{x}$ in (6.13). The $\xi$ equation is also different from the corresponding equation in (2.28) since we do not have the same incompressible conditions $\bar{u}_{x}+\bar{v}_{y}=0, u_{x}+v_{y}=0$. We remark that the velocity $\mathbf{u}, \nabla \mathbf{u}$ in the above system are determined by the elliptic equation in (6.13).

To generalize the analysis of the 2D Boussinesq equations to the 3D Euler equations, we derive the different terms, which are all of lower orders. In the following derivations, we use $f_{2 D}$ to denote the quantity $f$ used in the 2 D Boussinesq. For example, $\overline{\mathbf{u}}_{2 D}$ denote the approximate steady state for the velocity for 2D Boussinesq (6.59). It satisfies $\overline{\mathbf{u}}_{2 D}=$ $\nabla^{\perp}(-\Delta)^{-1} \bar{\omega}$. We introduce the norm $X_{i}$ related to the energy (5.70)

$$
\begin{equation*}
\|f\|_{X_{i}} \triangleq\left\|f \varphi_{i}\right\|_{\infty}+\left\|f \varphi_{g, i}\right\|_{\infty}+\left\|f \psi_{i}\right\|_{C_{g_{i}}^{1 / 2}} \tag{6.64}
\end{equation*}
$$

Lower order terms in the linearized and nonlinear operator. Using (6.43), we get

$$
\begin{align*}
\mathcal{L}_{i} & =\mathcal{L}_{M, i}+\mathcal{L}_{R, i} \\
\mathcal{L}_{M, 1} & =-\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \omega+\eta-\nabla^{\perp} \Psi \cdot \nabla \bar{\omega}_{0}+\Psi_{x y}(0) \bar{\omega}_{0}+\bar{c}_{\omega} \omega \\
\mathcal{L}_{M, 2} & =-\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \eta+\left(2 \bar{c}_{\omega}-\bar{u}_{x}\right) \eta-\bar{v}_{x} \xi-\partial_{x}\left(\nabla^{\perp} \Psi \cdot \nabla \bar{\theta}_{0}\right)+2 \Psi_{x y}(0) \bar{\theta}_{0, x}  \tag{6.65}\\
\mathcal{L}_{M, 3} & =-\left(\bar{c}_{l} x+\overline{\mathbf{u}}\right) \cdot \nabla \xi+\left(2 \bar{c}_{\omega}-\bar{v}_{y}\right) \xi-\bar{u}_{y} \eta-\partial_{y}\left(\nabla^{\perp} \Psi \cdot \nabla \bar{\theta}_{0}\right)+2 \Psi_{x y}(0) \bar{\theta}_{0, y} \\
\mathcal{L}_{R, 1} & =\left(\frac{1}{r^{4}}-1\right) \eta-\mathbf{u}_{R} \cdot \nabla \bar{\omega}_{0}, \quad \mathcal{L}_{R, 2}=-\partial_{x}\left(\mathbf{u}_{R} \cdot \nabla \bar{\theta}_{0}\right), \quad \mathcal{L}_{R, 3}=-\partial_{y}\left(\mathbf{u}_{R} \cdot \nabla \bar{\theta}_{0}\right)
\end{align*}
$$

Note that from (6.40) and (6.43), we have

$$
\begin{equation*}
u_{R}=O\left(|x|^{3}\right), \quad u_{R, x}(0)=0, \quad \Psi_{x y}(0)=\phi_{x y}(0)=u_{x}(0)=c_{\omega} \tag{6.66}
\end{equation*}
$$

We will estimate $\mathcal{L}_{R, i}$ in Section 6.4.4 and show that it can be bounded by $\left(C_{l} S+C_{l}^{\beta}\right) E_{4}(t)$, where $E_{4}(t)$ is the energy norm (5.70) for the 2D Boussinesq.

For the nonlinear terms $\mathcal{N}_{i}$, we decompose the velocity $\mathbf{u}$ into $\mathbf{u}_{M}$ and $\mathbf{u}_{R}$ similarly. We only focus on $\mathcal{N}_{2}$ since other terms are decomposed similarly. Using (6.43), (6.46), we have

$$
\begin{align*}
\mathcal{N}_{2} & =\mathcal{N}_{M, 2}+\mathcal{N}_{R, 2}, \quad \mathcal{N}_{M, 2}=-\mathbf{u} \cdot \nabla \eta-u_{M, x} \eta-v_{M, x} \xi+2 u_{M, x}(0) \eta,  \tag{6.67}\\
\mathcal{N}_{R, 2} & =-u_{R, x} \eta-v_{R, x} \xi
\end{align*}
$$

where we have used $u_{R, x}(0)=0, u_{M, x}(0)=u_{x, 0}=c_{\omega}$ (6.66). The lower order terms $\mathcal{N}_{R, 2}$ have vanishing order $O\left(|x|^{4}\right)$ near $x=0$, and its estimate follows the estimates of $\mathbf{u}_{R, x}$ in Section 6.4.4, and the nonlinear terms in Section 5.9. We do not decompose the transport term since we need to apply the weighted $L^{\infty}$ and $C^{1 / 2}$ estimate. In the weighted estimate, it leads to the nonlinear term $d_{i}(\rho) W_{1, i} \rho$ in (5.6). The estimate of the lower order terms in $d_{i}(\rho) W_{1, i} \rho$ follows the estimate of $\mathcal{N}_{R, 2}$.

Lower order terms in the residual error. Denote

$$
\delta f=f-f_{2 D}, \quad \delta \overline{\mathbf{u}}=\overline{\mathbf{u}}-\overline{\mathbf{u}}_{2 D}, \quad \delta \bar{c}_{\omega}=\bar{c}_{\omega}-\bar{c}_{\omega, 2 D} .
$$

For the residual error, using (6.50) and (2.19), we obtain

$$
\begin{align*}
\overline{\mathcal{F}}_{i} & =\overline{\mathcal{F}}_{M, i}+\overline{\mathcal{F}}_{R, i}  \tag{6.68}\\
\overline{\mathcal{F}}_{M, 1} & =-\left(\bar{c}_{l} x+\overline{\mathbf{u}}_{2 D}\right) \cdot \nabla \bar{\omega}_{0}+\bar{c}_{\omega, 2 D} \bar{\omega}_{0}, \quad \overline{\mathcal{F}}_{R, 1}=-\delta \overline{\mathbf{u}} \cdot \nabla \bar{\omega}_{0}+\delta \bar{c}_{\omega} \cdot \bar{\omega}_{0}+\frac{1-r^{4}}{r^{4}} \bar{\theta}_{0, x} \\
\overline{\mathcal{F}}_{M, i+1} & =\partial_{i}\left(-\left(\bar{c}_{l} x+\overline{\mathbf{u}}_{2 D}\right) \cdot \nabla \bar{\theta}_{0}+\left(\bar{c}_{l}+2 \bar{c}_{\omega, 2 D}\right) \bar{\theta}_{0}\right), \quad \overline{\mathcal{F}}_{R, i}=\partial_{i}\left(-\delta \overline{\mathbf{u}} \cdot \nabla \bar{\theta}_{0}+2 \delta \bar{c}_{\omega} \cdot \bar{\theta}_{0}\right) .
\end{align*}
$$

Note that the profiles $\bar{\omega}, \nabla \bar{\theta}$ decay and we choose the weights (C.4), (C.3), such that

$$
\left|\bar{\omega} \varphi_{g, 1}\right| \lesssim|x|^{-\gamma_{1}}, \quad\left|\nabla \bar{\theta} \varphi_{g, 2}\right| \lesssim|x|^{-\gamma_{2}}
$$

for some $\gamma_{2}>0$, e.g. $\gamma_{2}=\frac{1}{8}$. Since $\overline{\mathcal{F}}_{M, i}$ agrees with the residual error $\overline{\mathcal{F}}_{i}(2.19)$ for the 2D Boussinesq for $|x| \leq \nu$, where $\nu$ is the size of the cutoff function in (6.55), we have

$$
\begin{equation*}
\left\|\overline{\mathcal{F}}_{M, i}-\overline{\mathcal{F}}_{i, 2 D}\right\|_{X_{i}} \lesssim \nu^{-\gamma} \tag{6.69}
\end{equation*}
$$

for some $\gamma>0$, e.g. $\gamma=\frac{1}{8}$. The Hölder estimate of the tail is even smaller since $\psi_{1}$ and $\left||x|^{k} \nabla^{k} \overline{\mathcal{F}}_{i, 2 D}\right|,\left||x|^{k} \nabla^{k} \overline{\mathcal{F}}_{M, i}\right|, k=0,1$ decay, which can be derived using the regularity and asymptotics of the profile (7.2).

The error term $\overline{\mathcal{F}}_{R, i}$ does not vanish to the order $|x|^{3}$ near 0 . We use correction $D_{i}^{2} \overline{\mathcal{F}}_{R, i}(0) \cdot$ $f_{\chi, i}, D^{2}=\left(\partial_{x y}, \partial_{x y}, \partial_{x x}\right)$ similar to (4.11) and then incorporate it in the correction (4.11).
6.4.4. Estimate the lower order terms in the linearized operator. In this section, under the bootstrap assumption (6.54), we estimate $\mathcal{L}_{R, i}$ and show that

$$
\begin{equation*}
\left\|\mathcal{L}_{R, i} \varphi_{i}\right\|_{\infty}+\left\|\mathcal{L}_{R, i} \varphi_{g, i}\right\|_{\infty}+\left\|\mathcal{L}_{R, i} \psi_{g, i}\right\|_{C_{g_{i}}^{1 / 2}}+\left\|\mathcal{L}_{R, 1}\left|x_{1}\right|^{-1 / 2} \psi_{1}\right\|_{\infty} \lesssim\left(C_{l} S+C_{l}^{\beta}\right) E_{4}(t) \tag{6.70}
\end{equation*}
$$

for $i=1,2,3$. For $\omega \in X$, by definition of $E_{4}$ and (6.39), we have

$$
\|\Omega\|_{X} \lesssim E_{4}(t)
$$

The estimate of $\left(r^{-4}-1\right) \eta$ follows from $\left|r^{-4}-1\right| \lesssim C_{l} x_{2} \lesssim C_{l} S$ and the bound for $\eta$. Other terms in $\mathcal{L}_{R, i}$ are nonlocal, involving $u_{R}, v_{R}$. We estimate a typical term $\partial_{x} u_{R} \bar{\theta}_{0, x}$.

Estimate of $\Psi_{2}$. Recall the formulas of $u_{R}$ from (6.43) and $\Psi, \Psi_{2}$ from (6.37). The estimates of the terms involving $\Psi_{2}$ are simple since

$$
\Psi_{2}=-\alpha C_{l}^{2} \phi_{x y}(0) \kappa
$$

Recall the definition of $\kappa$ from Proposition 6.9, Using (6.40), we yield

$$
\begin{equation*}
\left|\nabla^{k} \Psi_{2}\right| \lesssim C_{l}^{2}|x|^{4-k} E_{4} \tag{6.71}
\end{equation*}
$$

For $|x| \leq 2$, we have

$$
\left|\nabla \bar{\theta}_{0}\right| \lesssim\left|x_{1}\right|, \quad\left|\nabla^{2} \bar{\theta}_{0}\right| \lesssim 1
$$

We consider a typical term related to $\Psi_{2}$ in $\partial_{x} u_{R} \bar{\theta}_{0, x}$, e.g. $\partial_{x y} \Psi_{2} r^{-1 / 2} \bar{\theta}_{0, x}$ (6.43). We can bound it by

$$
\left|\partial_{x y} \Psi_{2} r^{-1 / 2} \bar{\theta}_{0, x} \varphi_{2}\right| \lesssim C_{l}^{2}|x|^{2}\left|x_{1}\right||x|^{-5 / 2}\left|x_{1}\right|^{-1 / 2} \mathbf{1}_{|x| \leq 2} E_{4} \lesssim C_{l}^{2} E_{4}
$$

Note that for $|x| \leq 1$, we have $\chi=1, \kappa=-\frac{x y^{3}}{2}$ (see Proposition 6.9),

$$
\partial_{x y} \Psi_{2} r^{-1 / 2} \bar{\theta}_{0, x} \psi_{2}=C C_{l}^{2} r^{-1 / 2} y^{2} \bar{\theta}_{0, x} \psi_{2}
$$

for some absolute constant $C, \psi_{2} \sim c|x|^{-5 / 2}$ near $x=0$, and $f=r^{-1 / 2} y^{2} \bar{\theta}_{0, x} \psi_{2}$ vanishes to the order of $|x|^{1 / 2}$ near $x=0$. For $|x|>1 / 2, f$ is smooth and is supported near $x=0$. Hence, we obtain that $f$ is in $C^{1 / 2}$ and

$$
\left\|\partial_{x y} \Psi_{2} r^{-1 / 2} \bar{\theta}_{0, x} \psi_{2}\right\|_{C_{g_{2}}^{1 / 2}} \lesssim C_{l}^{2} E_{4}
$$

The estimates of other terms related to $\Psi_{2}$ or $\bar{\Psi}_{2}(6.50)$ in the residual error, nonlinear terms, or linear parts related to $\Psi_{2}$ follow similar estimates since $\Psi_{2}, \bar{\Psi}_{2}=O\left(|x|^{4}\right)$ near $x=0$ and contain the small factor $C_{l}^{2}$. We treat them as lower order terms.
Estimate of $\Psi-\Psi_{x y}(0) x y$. Next, we estimate other terms in $u_{R}$ related to $\Psi$. Recall the decomposition (6.45). The third term in (6.45) follows an estimate similar to that of $\Psi_{2}$ performed above. The first two terms vanish to a higher order near $x=0$. We estimate a typical term in $\partial_{x} u_{R} \bar{\theta}_{0, x}$ related to $\Psi$ :

$$
\begin{equation*}
I \triangleq \partial_{x}\left(\Psi_{y}-\Psi_{x y}(0) x\right)\left(r^{-1 / 2}-1\right) \bar{\theta}_{0, x}=\left(\Psi_{x y}-\Psi_{x y}(0)\right)\left(r^{-1 / 2}-1\right) \bar{\theta}_{0, x} \tag{6.72}
\end{equation*}
$$

Note that $\left|r^{-1 / 2}-1\right| \lesssim C_{l}\left|x_{2}\right| \lesssim C_{l} S$ within the support of the solution. The weighted $L^{\infty}$ estimate is simple and follows from Lemma 6.10. For example, using $\bar{\theta}_{0, x} \lesssim \min \left(\left|x_{1}\right|,|x|^{-3 / 5}\right)$, we have

$$
\left|I \varphi_{2}\right| \lesssim\left|I \varphi_{g, 2}\right| \lesssim \mathbf{1}_{|x| \leq S} C_{l}\left|x_{2}\right| \min \left(|x|^{2}, 1\right) \min \left(\left|x_{1}\right|,|x|^{-3 / 5}\right) \varphi_{g, 2}| | \Omega \|_{X} \lesssim C_{l} S E_{4}
$$

where the weights $\varphi_{2}, \varphi_{g, 2}$ are defined in (C.4), (C.3) with $\varphi_{g, 2} \lesssim|x|^{\frac{1}{2}}+\left|x_{1}\right|^{-\frac{1}{2}}\left(|x|^{-1 / 6}+|x|^{-5 / 2}\right)$.
For the Hölder estimate, we use Lemma 6.11. Recall $G(x)$ defined in Lemma 6.11. Firstly, we rewrite $I$ as follows
$I=\left(\Psi_{x y}-\Psi_{x y}(0)-\chi_{0} \partial_{x y} G(x) \Psi_{x x x y}(0)\right)\left(r^{-1 / 2}-1\right) \bar{\theta}_{0, x}+\chi_{0} \Psi_{x x x y}(0) \partial_{x y} G(x)\left(r^{-1 / 2}-1\right) \bar{\theta}_{0, x} \triangleq I_{1}+I_{2}$.
Denote

$$
\Psi_{x y, A} \triangleq \Psi_{x y}-\Psi_{x y}(0)-\chi_{0} \partial_{x y} G(x) \Psi_{x x x y}(0)
$$

The estimate of $I_{2}$ is simple since the coefficient vanishes to $O\left(|x|^{4}\right)$ and we obtain a small factor: $\left|r^{-1 / 2}-1\right| \lesssim C_{l} y \lesssim C_{l} S$ within the support of the solution. In particular, we have

$$
\left\|I_{2} \psi_{2}\right\|_{C^{1 / 2}} \lesssim C_{l} S\left|\Psi_{x x x y}(0)\right| \lesssim C_{l} S \cdot E_{4}
$$

For $I_{1}$, we first rewrite $I_{1} \psi_{2}$ as follows

$$
I_{1} \psi_{2}=\left(\Psi_{x y, A} \psi_{1}\right) \frac{\psi_{2}}{\psi_{1}}\left(r^{-1 / 2}-1\right) \bar{\theta}_{0, x}
$$

From Lemmas 6.10 6.11 we have $\Psi_{x y, A} \psi_{1} \in L^{\infty} \cap C^{1 / 2}$. The coefficient $\frac{\psi_{2}}{\psi_{1}}\left(r^{-1 / 2}-1\right) \bar{\theta}_{0, x}$ vanishes to the order $|x|^{3 / 2}$ near $x=0$. Since

$$
\left|\frac{\psi_{2}}{\psi_{1}}\left(r^{-1 / 2}-1\right) \bar{\theta}_{0, x}\right| \lesssim C_{l} S, \quad\left|\nabla\left(\frac{\psi_{2}}{\psi_{1}}\left(r^{-1 / 2}-1\right) \bar{\theta}_{0, x}\right)\right| \lesssim C_{l} S
$$

we obtain the Hölder estimate for $I_{1} \psi_{2}$. The estimates of other terms in $u_{R}$ are similar. We establish (6.70).
6.4.5. Estimates of the lower order terms in the residual error. In this section, we estimate the lower order terms in the residual error (6.68) and show that

$$
\begin{equation*}
\left\|\overline{\mathcal{F}}_{R, i}-c_{i} f_{\chi, i}\right\|_{X_{i}} \lesssim C_{l} S+C_{l}^{\beta}+\left\|\Omega_{\nu, R}\right\|_{X} \tag{6.73}
\end{equation*}
$$

where the norm $X_{i}, f_{\chi, i}, \Omega_{\nu, R}$ are defined in (6.64), (4.11), (6.47), respectively, and $c_{i}=\partial_{x y} \overline{\mathcal{F}}_{R, i}$ for $i=1,2$ and $c_{3}=\partial_{x x} \overline{\mathcal{F}}_{R, 3}(0)$. Using (6.49), we can further bound $\Omega_{\nu, R}$ as follows

$$
\left\|\Omega_{\nu, R}\right\|_{X} \lesssim\left\|\bar{\omega}\left(\chi_{4} \chi_{\nu}-1\right)\right\|_{X}+C_{l} S=\left\|\bar{\omega}\left(\chi_{\nu}-1\right)\right\|_{X}+C_{l} S
$$

where we have used $\nu C_{l} \leq C_{l} S \leq 4^{-5}$ in the last inequality to simplify $\chi_{4} \chi_{\nu}=\chi_{\nu}$, which can be done by choosing $C_{l}$ sufficiently small later.

We focus on the case $i=2$, i.e. the estimate of $\overline{\mathcal{F}}_{R, 2}-c_{2} f_{\chi, 2}$. Firstly, from (6.62), we have

$$
\delta \bar{c}_{\omega}=\bar{c}_{\omega}-\bar{c}_{\omega, 2 D}=\bar{u}_{x}(0)-\bar{u}_{x, 2 D}(0)=\delta \bar{u}_{x}(0)
$$

Note that $\bar{u}_{x}(0)+\bar{v}_{y}(0)=0$ for the velocity (6.50). A direct computation yields

$$
c_{1}=\partial_{x y} \overline{\mathcal{F}}_{R, 1}(0)=\delta \bar{c}_{\omega} \bar{\omega}_{x y}(0)+4 C_{l} \bar{\theta}_{x x}(0), \quad c_{2}=\delta \bar{c}_{\omega} \bar{\theta}_{x x y}(0), \quad c_{3}=\delta \bar{c}_{\omega} \bar{\theta}_{x x y}(0)
$$

We can rewrite $\mathcal{F}_{R, i}(\boxed{6.68})$ as follows

$$
\begin{aligned}
I= & \partial_{x}\left(-\delta \overline{\mathbf{u}} \cdot \nabla \bar{\theta}_{0}+2 \delta \bar{u}_{x}(0) \bar{\theta}_{0}\right)-c_{2} f_{\chi, 2} \\
= & -\left(\delta \bar{u}-\delta \bar{u}_{x}(0) x\right) \bar{\theta}_{0, x x}-\partial_{x}\left(\delta \bar{u}-\delta \bar{u}_{x}(0) x\right) \bar{\theta}_{0, x}-\left(\delta \bar{v}-\delta \bar{v}_{y}(0) y\right) \bar{\theta}_{0, x y} \\
& -\partial_{x}\left(\delta \bar{v}-\delta \bar{v}_{y}(0) y\right) \bar{\theta}_{0, y}+\delta \bar{u}_{x}(0)\left(\bar{\theta}_{0, x}-x \bar{\theta}_{0, x x}+y \bar{\theta}_{0, x y}-\bar{\theta}_{x x y}(0) f_{\chi, 2}\right) \\
\triangleq & I_{1}+I_{2}+I_{3}+I_{4}+I_{5}
\end{aligned}
$$

For $I_{5}$, the coefficient is $C^{2}$ and has sufficiently fast decay. Moreover, using (6.47), (6.48), (6.50), and Proposition 6.7, we have

$$
\left|\delta \bar{c}_{\omega}\right|=\left|\delta \bar{u}_{x}(0)\right| \lesssim\left\|\Omega_{\nu, R}\right\|_{X}
$$

Thus, we can obtain

$$
\left\|I_{5}\right\|_{X_{2}} \lesssim\left\|\Omega_{\nu, R}\right\|_{X}
$$

The estimates of $I_{j}, 1 \leq j \leq 4$ are similar. We focus on the typical terms in $I_{2}$

$$
\begin{equation*}
I_{2}=-\partial_{x}\left(\delta \bar{u}-\delta \bar{u}_{x}(0) x\right) \bar{\theta}_{0, x} \tag{6.74}
\end{equation*}
$$

Recall $\bar{\Psi}$ from (6.47) and $\bar{\phi}_{2 D}=(-\Delta)^{-1} \bar{\omega}$. Denote

$$
\tilde{\Psi}=\bar{\Psi}-\bar{\phi}_{2 D}
$$

Recall the formula of $\overline{\mathbf{u}}$ from (6.50). We have

$$
\delta \bar{u}-\delta \bar{u}_{x}(0) x=-\left(\partial_{y}\left(\bar{\Psi}-\bar{\phi}_{2 D}\right)-\partial_{x y}\left(\bar{\Psi}-\bar{\phi}_{2 D}\right)(0) x\right)+\bar{u}_{R}=-\left(\partial_{y} \tilde{\Psi}-\partial_{x y} \tilde{\Psi}(0) x\right)+\bar{u}_{R}
$$

The formula of the remainder $\bar{u}_{R}$ is given by (6.43) with $\Psi, \Psi_{2}$ replaced by $\bar{\Psi}, \bar{\Psi}_{2}$. From (6.47), we have

$$
-\Delta \tilde{\Psi}=\Omega_{\nu, R}, \quad \Omega_{\nu, R} \in X
$$

Then the estimate of

$$
\partial_{x}\left(\partial_{y} \tilde{\Psi}-\partial_{x y} \tilde{\Psi}(0) x\right) \bar{\theta}_{0, x}
$$

in $I_{2}$ follows from the estimate of $\Psi-\Psi_{x y}(0) x y$ at the end of Section 6.4.4. In particular, we can obtain

$$
\left\|\partial_{x}\left(\partial_{y} \tilde{\Psi}-\partial_{x y} \tilde{\Psi}(0) x\right) \bar{\theta}_{0, x}\right\|_{X_{2}} \lesssim\left\|\Omega_{\nu, R}\right\|_{X}
$$

Other terms in $I_{j}, 1 \leq j \leq 4$ related to $\tilde{\Psi}$ can be estimated similarly.
For the term in $I_{2}$ (6.74) related to $\bar{u}_{R}$, we have several terms due to the formula (6.43), (6.45). The term involving $\bar{\Psi}_{2}$ is simple and its estimate follows from the estimate of $\Psi_{2}$ in Section 6.4.4. For other terms, we estimate a typical term

$$
J=\partial_{x}\left(\bar{\Psi}_{y}-\bar{\Psi}_{x y}(0) x\right) \cdot\left(r^{-1 / 2}-1\right) \bar{\theta}_{0, x}
$$

Since $\bar{\Psi}$ is close to $\bar{\phi}_{2 D}$, we use the decomposition $\bar{\Phi}=\tilde{\Phi}+\bar{\phi}_{2 D}$ and

$$
J=\partial_{x}\left(\tilde{\Psi}_{y}-\tilde{\Psi}_{x y}(0) x\right) \cdot\left(r^{-1 / 2}-1\right) \bar{\theta}_{0, x}+\partial_{x}\left(\bar{\phi}_{2 D, y}-\bar{\phi}_{2 D, x y}(0) x\right) \cdot\left(r^{-1 / 2}-1\right) \bar{\theta}_{0, x} \triangleq J_{1}+J_{2}
$$

The term $J_{1}$ follows from the above estimate. For $J_{2}$, we note that $\bar{\phi}_{2 D}$ satisfies the elliptic equation $-\Delta \bar{\phi}_{2 D}=\bar{\omega}$. From the construction of $\bar{\omega}$ in Section 7 , we have $\bar{\omega} \in C^{2}$ with decays (6.58). To control $\bar{\phi}$, we use embedding inequalities

$$
\begin{equation*}
\left\|\nabla^{2}(-\Delta) \omega\right\|_{L^{\infty}} \lesssim_{\alpha, p}\|\omega\|_{C^{\alpha}}+\|\omega\|_{L^{p}}, \alpha \in(0,1), p \in(1, \infty) \tag{6.75}
\end{equation*}
$$

which can be proved by decomposing the domain of the singular integral into the region near the singularity and away from the singularity, and estimating them by the $C^{\alpha}$ norm of $\omega$ and the $L^{p}$ norm of $\omega$ separately. In particular, from $-\Delta \bar{\phi}_{2 D, x}=\bar{\omega}_{x},-\Delta \bar{\phi}_{2 D}=\bar{\omega}$, we obtain

$$
\left|\nabla^{2} \bar{\phi}_{2 D, x}\right| \lesssim 1, \quad\left|\nabla^{2} \bar{\phi}_{2 D}\right| \lesssim 1
$$

Using $\bar{\phi}_{2 D, y y y}=\bar{\omega}_{y}-\bar{\phi}_{2 D, x x y}$ and the above estimate, we yield $\left|\bar{\phi}_{2 D, y y y}\right| \lesssim 1$, and thus $\left|\nabla^{3} \bar{\phi}_{2 D}\right| \lesssim 1$. Now, using the estimate of $\bar{\phi},\left|\bar{\theta}_{0, x}\right| \lesssim \min \left(\left|x_{1}\right|,|x|^{-3 / 5}\right)$, and the smallness of $\left|r^{-1 / 2}-1\right|(6.44)$ within the support of the solution, we yield

$$
\left|J_{2}\right| \lesssim C_{l}|x| \min (|x|, 1) \min \left(|x|^{-3 / 5},\left|x_{1}\right|\right) \mathbf{1}_{|x| \leq S}
$$

which vanishes to the order $\left|x_{1}\right| \cdot|x|^{2}$ near $x=0$. It follows the weighted $L^{\infty}$ estimate

$$
\left|J_{2} \varphi_{2}\right| \lesssim C_{l} S, \quad\left|J_{2} \varphi_{g, 2}\right| \lesssim C_{l} S, \quad\left|J_{2} \psi_{2}\right| \lesssim C_{l} S \min \left(|x|^{1 / 2}, 1\right)
$$

Recall the weight $\psi_{2} \asymp|x|^{5 / 2}+|x|^{1 / 6}$ from (C.1). We have

$$
\left|\nabla\left(J_{2} \psi_{2}\right)\right| \lesssim C_{l} S\left(|x|^{-1 / 2}+1\right)
$$

Combining the $L^{\infty}$ and $C^{1}$ estimates of $J_{2} \psi_{2}$, we obtain the $C^{1 / 2}$ estimate of $J_{2} \psi_{2}$. Other terms follow similar estimates. We prove (6.73).
6.4.6. Modified finite rank perturbation. Due to the difference of the operators between the 3D Euler (6.63) and the 2D Boussinesq (4.10), we modify the decomposition (4.21) and nonlinear perturbation $N F_{i}$ as follows

$$
\begin{align*}
\partial_{t} W_{1, i}= & \left(\mathcal{L}_{i}-\mathcal{K}_{1 i}-\mathcal{K}_{2 i}\right)\left(W_{1}\right)+\left(\mathcal{L}_{i}-\mathcal{L}_{2 D, i}\right) \hat{W}_{2}+\mathcal{N}_{i}\left(W_{1}+\hat{W}_{2}\right)+\overline{\mathcal{F}}_{i} \\
& -N F_{i}\left(W_{1}, \hat{W}_{2}\right)-\mathcal{R}_{i}\left(W_{1}, \hat{W}_{2}\right) \\
\partial_{t} \hat{W}_{2, i}= & \mathcal{L}_{2 D, i} \hat{W}_{2}+\mathcal{K}_{1 i}\left(W_{1}\right)+\mathcal{K}_{2 i}\left(W_{1}\right)+N F_{i}\left(W_{1}, \hat{W}_{2}\right)+\mathcal{R}_{i}\left(W_{1}, \hat{W}_{2}\right)  \tag{6.76}\\
N F_{i}= & \left(c_{\omega} D_{i}^{2}\left(W_{1}+\hat{W}_{2}\right)(0)+D_{i}^{2}\left(\overline{\mathcal{F}}_{i}(0)+\left(\mathcal{L}_{i}-\mathcal{L}_{2 D, i}\right) \hat{W}_{2}\right)(0)\right) f_{\chi, i}
\end{align*}
$$

where $D^{2}=\left(\partial_{x y}, \partial_{x y}, \partial_{x x}\right)$. Since the stream function $\Psi$ in (6.65) is obtained from a modified source term $\Omega$ (6.37), we also modify the finite rank operator $\mathcal{K}_{2 i}$ (4.13), (4.29), (4.37)

$$
\mathcal{K}_{2 i}\left(W_{1}\right) \triangleq \mathcal{K}_{2 D, 2 i}(\Omega)
$$

Note that we can still represent $\mathcal{K}_{i}\left(W_{1}\right)$ as follows

$$
\begin{equation*}
\mathcal{K}_{2 j}\left(W_{1}(t)\right)=\sum_{i} a_{i}(\Omega(t)) \bar{f}_{i j}, \quad j=1,2,3 \tag{6.77}
\end{equation*}
$$

for some functions $\bar{f}_{i}$, and $a_{i}(\Omega(t))$ independent of space similar to (4.16). Thus, we can apply the same constructions of $\hat{W}_{2}$ and $\mathcal{R}$ in Section4.2.4, and use the same approximate space-time solution $\hat{F}_{i}, \hat{F}_{\chi, i}$ in (4.19), (4.20). Due to (6.39), the linear modes $a_{i}(\Omega(t))$ and $a_{i}(\omega(t))$ satisfy almost the same estimate up to $C_{l} S\|\omega\|_{X}$. To control the nonlinear mode $a_{n l, i}$ in (4.19), (4.20), we modify the bootstrap condition (5.72)

$$
\begin{equation*}
\left|c_{\omega} D_{i}^{2}\left(W_{1}+\hat{W}_{2}\right)(0)+D_{i}^{2}\left(\overline{\mathcal{F}}_{i}(0)+\left(\mathcal{L}_{i}-\mathcal{L}_{2 D, i}\right) \hat{W}_{2}\right)(0)\right|<c_{i} \mu_{6} E_{*}, \quad c_{1}=5, c_{2}=10 \tag{6.78}
\end{equation*}
$$

6.4.7. Comparison between the operators. In this section, we show that the difference between the main parts of the operators in (6.65), (6.67), (6.76) and the operators in (4.10), (4.21) are small. We have estimated the lower order operators in Section 6.4.4, 6.4.5. Here, we only focus on the main terms. We will choose $C_{l}$ very small at the end such that $\nu \ll C_{l}^{-1}$. From (6.16), we get $\chi_{4} \chi_{\nu}=\chi_{\nu}$. Recall that we perform energy estimate on $W_{1}$ with energy $E_{4}$ (5.70).

There are three differences between $\mathcal{L}_{M, i}$ in (6.65), (6.76) and $\mathcal{L}_{i, 2 D}$ in (4.10), (4.21). Firstly, we use $\overline{\mathbf{u}}$ in the transport term instead of $\overline{\mathbf{u}}_{2 D}=\nabla^{\perp} \bar{\phi}_{2 D}$. We estimate the difference $\overline{\mathbf{u}}-\overline{\mathbf{u}}_{2 D}$ using (6.51), (6.52), (6.48), (6.50), and bound $\bar{\omega}\left(\chi_{4} \chi_{\nu}-1\right)=\bar{\omega}\left(\chi_{\nu}-1\right)$ using the decay (6.58)

$$
\begin{equation*}
\left\|\bar{\omega}\left(\chi_{4} \chi_{\nu}-1\right)\right\|_{X}+\left\|\bar{\omega}\left(\chi_{4} \chi_{\nu}-1\right)\right\|_{C^{1}} \lesssim \nu^{-\gamma} \tag{6.79}
\end{equation*}
$$

for some $\gamma>0$, e.g. $\gamma=\frac{1}{8}$. Thus, this difference in the linear stability analysis is bounded by

$$
C\left(C_{l} S+\nu^{-\gamma}\right) E_{4}
$$

where $E_{4}$ is the energy (5.70) for the perturbation
The second difference is that we use the truncated profile $\bar{\theta}_{0}, \bar{\omega}_{0}$ in (6.65) rather than the $(\bar{\theta}, \bar{\omega})$ in (4.10). We estimate it using the decay of the profiles (6.58), the asymptotics of the weights, and the elliptic estimates in Lemmas 6.10, 6.11. For example, in $\mathcal{L}_{M, 1}-\mathcal{L}_{1,2 D}$, we have

$$
\begin{equation*}
\left|\partial_{y} \Psi \partial_{x}\left(\bar{\omega}-\bar{\omega}_{0}\right) \varphi_{g, 1}\right| \lesssim\left|x_{1}\right||x|^{-5 / 4}\left(|x|^{1 / 16}+\left|x_{1}\right|^{-1 / 2}\right) \mathbf{1}_{|x| \geq \nu} \lesssim|x|^{-1 / 8} \tag{6.80}
\end{equation*}
$$

This difference in the linear stability analysis bounded by

$$
C \nu^{-\gamma} E_{4}
$$

for some $\gamma>0$, e.g. $\gamma=\frac{1}{8}$.

Thirdly, the main term in the velocity $\mathbf{u}_{M}=\nabla^{\perp} \Psi$ depends on the modified stream function $\Psi$ obtained from $\Omega$ (6.37), (6.38) rather than $\omega$. The same argument applies to $\mathcal{K}_{2 i}$ (6.77). Due to the equivalence (6.39), this leads to a difference in the linear stability analysis bounded by

$$
C C_{l} S E_{4} .
$$

We also refer to Section 6.4.4 for the estimate of the lower order part $\mathbf{u}_{R}$, which is small.
The nonlinear terms in (2.18), (6.67) all involve the nonlocal terms determined by u. Recall that in the energy estimate of the 2D Boussinesq, we treat the nonlocal term $\mathbf{u}_{2 D}$ as a bad term. Using the estimate of the lower order part $\mathbf{u}_{R}$ in Section 6.4.4 and the above argument to estimate $\mathbf{u}_{M}$, we have a difference in the nonlinear stability bounded by

$$
\left(C C_{l} S+C_{l}^{\beta}\right) E_{4}^{2}
$$

Difference between operators for $\widehat{W}_{2}$. Comparing (4.21) and (6.76), we have extra terms

$$
I=\left(\mathcal{L}_{i}-\mathcal{L}_{2 D, i}\right) \hat{W}_{2}-D_{i}^{2}\left(\mathcal{L}_{i}-\mathcal{L}_{2 D, i}\right) \hat{W}_{2}(0) f_{\chi, i} .
$$

Due to the correction, $I$ vanishes $O\left(|x|^{3}\right)$ near 0. In Section 3 of Part II [15], for each approximate space-time solution $f=\hat{F}_{i}, \hat{F}_{\chi, i}$, we represent it as $f(t, x)=f_{1}(t, x)+a(t) f_{2}(x)$ for $f_{1}(t, \cdot), f_{2}(x) \in C^{4,1}$ with compactly supported both in space $[-D, D] \times[0, D], D \leq 10^{20}$ and time, and $a(t)$ decays exponentially fast. Under the bootstrap assumption (6.76), $\widehat{W}_{2}(t)$ satisfies $C^{4,1}$ estimate uniform in time $\left\|\widehat{W}_{2}(t)\right\|_{C^{4,1}} \lesssim E_{4}\left(W_{1}\right)$. For both the local terms and nonlocal terms in $I$, e.g.

$$
\begin{aligned}
& J_{a}-D_{i}^{2} J_{a}(0) f_{\chi, i}, \quad J_{l o c}=\overline{\mathbf{u}} \cdot \nabla \hat{W}_{2, i}-\overline{\mathbf{u}}_{2 D} \cdot \nabla \hat{W}_{2, i}=\delta \overline{\mathbf{u}} \cdot \nabla \widehat{W}_{2, i}, \\
& J_{\text {nloc }}=\mathbf{u}\left(\hat{\omega}_{2}\right) \cdot \nabla \bar{\omega}_{0}-\mathbf{u}_{2 D}\left(\hat{\omega}_{2}\right) \cdot \nabla \bar{\omega}=\delta \mathbf{u}\left(\hat{\omega}_{2}\right) \cdot \bar{\omega}_{0}-\mathbf{u}_{2 D}\left(\hat{\omega}_{2}\right) \cdot \nabla\left(\bar{\omega}-\bar{\omega}_{0}\right)
\end{aligned}
$$

for $a=l o c, n l o c$, we apply the same estimates as those of the lower order part of residual error in Section 6.4.5 by replacing ( $\bar{\omega}_{0}, \bar{\theta}_{0, x}, \bar{\theta}_{0, y}$ ) by $\widehat{W}_{2, i}$. To estimate $\delta \mathbf{u}\left(\hat{\omega}_{2}\right)$, we apply (6.48), (6.49), (6.50). Since $\hat{\omega}_{2}$ has compact support in $[-D, D]^{2}$ and $\nu, R_{4}$ will be chosen to sufficiently large, instead of (6.49), we have $\hat{\omega}_{2}\left(\chi_{4} \chi_{\nu}-1\right)=0$ and

$$
\left\|\Omega\left(\hat{\omega}_{2}\right)_{\nu, R}\right\|_{X}+\left\|\Omega\left(\hat{\omega}_{2}\right)_{\nu, R}\right\|_{C^{1}} \lesssim C_{l} S \cdot E_{4}\left(W_{1}\right),
$$

where we define $\Omega\left(\hat{\omega}_{2}\right)$ following (6.37), (6.38). For the error due to cutoff $\bar{\omega}_{0}-\bar{\omega}$, we use the decay (6.58) and estimate similar to (6.80). In summary, we have smallness from the difference between two nonlocal operators $\mathbf{u}-\mathbf{u}_{2 D}$ or the decay of the profile in the estimate of this difference, and can bound it by

$$
C\left(C_{l} S+\nu^{-\gamma}+C_{l}^{\beta}\right) E_{4} .
$$

6.5. Nonlinear stability and finite time blowup. For initial perturbation $\omega, \eta, \xi$ in the energy class $E_{4}$ (5.70) with $E_{4}(\omega, \eta, \xi)<E_{*}$, under the bootstrap assumption (6.54), we can perform nonlinear energy estimates similar to those for the 2D Boussinesq equations in Section 5. Combining the estimate (6.69), the estimates in Section 6.4.4 6.4.5 and the discussion in Section (6.4.7), we can bound the additonal terms due to the difference between two energy estimates, including weighted $L^{\infty}$ and weighted Hölder estimate, and the differences between the nonlinear modes (5.72), (6.78) (the coefficients of in $N F_{i}$ (4.11), (6.76)) by

$$
|J| \leq C\left(C_{l} S+\nu^{-\gamma}+C_{l}^{\beta}+\left\|\Omega_{\nu, R}\right\|_{X}\right)\left(1+E_{4}+E_{4}^{2}\right) \leq C_{1, *}\left(C_{l} S+\nu^{-\gamma}+C_{l}^{\beta}\right)\left(1+E_{4}+E_{4}^{2}\right)
$$

for $C_{1, *}$ independent of $C_{l}, S, \nu$, where $\gamma=\frac{1}{16}$, and we further bound $\left\|\Omega_{\nu, R}\right\|_{X}$ using (6.49), (6.79). Recall $c_{\omega}, \bar{c}_{\omega}$ from (6.60), (6.61). From the energy estimate and the definition of $E_{4}$ (5.70), we have

$$
\begin{equation*}
\left|c_{\omega}+\bar{c}_{\omega}-\bar{c}_{\omega, 2 D}\right| \leq 100 E_{4}+C_{2, *}\left(\nu^{-\gamma}+C_{l} S+C_{l}^{\beta}\right) . \tag{6.81}
\end{equation*}
$$

Note that the energy estimates for the 2D Boussinesq equations satisfy the nonlinear stability conditions (A.11) with some $\varepsilon_{0}>0$, and the second inequalities in (5.73) are strict with some gap $\varepsilon_{1}>0$. Now, we choose $\nu>\nu_{*}$ with $\nu_{*}$ large enough and a small $\delta$ such that

$$
\begin{equation*}
\left(C_{1, *}+C_{2, *}\right)\left(\nu_{*}^{-\gamma}+\delta+\delta^{\beta}\right)\left(1+E_{*}+E_{*}^{2}\right)<\min \left(\varepsilon_{0} / 4, \varepsilon_{1} / 4,10^{-4}\right) \tag{6.82}
\end{equation*}
$$

We impose a stronger bootstrap assumption than (6.54)

$$
\begin{equation*}
C_{l}(t)(1+S(t))<\min \left(\delta, \nu_{2}, 4^{-6}\right) \tag{6.83}
\end{equation*}
$$

Under the above bootstrap assumption, (6.78), and the energy assumption for the $W_{1}$ part of the solution (see (4.21))

$$
\begin{equation*}
E_{4}(t)<E_{*} \tag{6.84}
\end{equation*}
$$

using the nonlinear stability estimate for the 2 D Boussiesq equations, (5.73), and (6.82), we can continue the bootstrap assumption for the energy inequality and 6.78). Moreover, using $|\omega| \lesssim|x|^{-\gamma} E_{4}$ from the $L^{\infty}\left(\varphi_{g, 1}\right)$ estimate and (6.81), we have

$$
|\mathbf{u}+\overline{\mathbf{u}}| \lesssim|x|^{1-\gamma / 2}, \quad \bar{c}_{l}>2, \quad \bar{c}_{\omega}+c_{\omega}<-\frac{1}{2}
$$

which means that the whole velocity grows sublinearly and the blowup is focusing $\left(\bar{c}_{l}>2\right)$.
Following the argument in [17], under the bootstrap assumption, we can control the support

$$
C_{l}(t)(1+S(t)) \leq C(S(0)) C_{l}(0)
$$

for some constant $C$ depending on $S(0)$. Thus, for any $S(0)<+\infty$, by choosing $C_{l}(0)$ sufficiently small, the assumption (6.83) is also satisfied, and the bootstrap assumption can be continued.

Passing from nonlinear stability to finite time blowup with smooth data $\omega^{\theta}, u^{\theta}$ compactly supported near $(r, z)=(1,0)$ follows the argument in [17]. We conclude the proof of Theorems 4. 2.

## 7. Construction of an approximate steady state

Following our previous works with Huang on the De Gregorio model 19 and the Hou-Luo model [20], we construct the approximate steady state to the dynamic rescaling equations (2.10) with the normalization conditions (2.11) by solving (2.10) numerically for a long enough time. The residual error is estimated a-posteriori and incorporated in the energy estimate as a small error term. It is extremely challenging to obtain an approximate steady state with a sufficiently small residual error in the weighted energy space (5.70), e.g. of order $10^{-7}$, since the weight is singular of order $|x|^{-\beta}, \beta \geq 2.9$ near 0 and the solution is supported on the whole $\mathbb{R}_{+}^{2}$ with a slowly decaying tail in the far-field, e.g., $\omega(t, x) \sim|x|^{-1 / 3}$ for large $x$. See (7.1). If we solve (2.10) in a very large domain to capture the far-field behavior of the solution, we have to deal with the relatively large round-off errors in the computation. To overcome these difficulties, we follow [20] to use a combination of numerical computation and a semi-analytic construction.
7.1. Far-field asymptotics. Let $(r, \beta)$ be the polar coordinate in $\mathbb{R}_{2}^{+}: r=\left(x^{2}+y^{2}\right)^{1 / 2}, \beta=$ $\arctan (y / x)$. It has be observed in 20 that the approximate steady state (2.10) enjoys the following asymptotics

$$
\begin{equation*}
\omega(r, \beta) \sim g_{1}(\beta) r^{\alpha}, \quad \theta(r, \beta) \sim g_{2}(\beta) r^{1+2 \alpha}, \quad \alpha=\frac{c_{\omega}}{c_{l}}<0, \alpha \approx-\frac{1}{3} \tag{7.1}
\end{equation*}
$$

in the far-field for some angular profiles $g_{1}(\beta), g_{2}(\beta)$, under the mild assumption that $\omega$ decays for large $|x|, c_{l}>0$, and $c_{\omega}<0$. These conditions are satisfied by the blowup solutions [65, 66].

In fact, if $\omega$ decays for large $|x|$, the velocity $\mathbf{u}=\nabla^{\perp}(-\Delta)^{-1} \omega$ has a sublinear growth: $\frac{u(x)}{r} \rightarrow$ 0 as $r \rightarrow \infty$. Note that $x \cdot \nabla=r \partial_{r}$. Passing to the polar coordinate $(r, \beta), r=|x|, \beta=\arctan \frac{x_{2}}{x_{1}}$ and dropping the lower order terms, we yield

$$
c_{l} r \partial_{r} \omega(r, \beta)=c_{\omega} \omega+\theta_{x}+\text { l.o.t., } \quad c_{l} r \partial_{r} \theta(r, \beta)=\left(2 c_{\omega}+c_{l}\right) \theta+\text { l.o.t.. }
$$

Assume that $\omega(r, b)=r^{k} g_{1}(\beta), \theta(r, \beta)=r^{l} g_{2}(\beta)$. Using the above equations and matching the power, we obtain the asymptotic relation (7.1). Thus, we represent the approximate steady state as follows

$$
\begin{equation*}
\bar{\omega}=\bar{\omega}_{1}+\bar{\omega}_{2}, \quad \bar{\theta}=\bar{\theta}_{1}+\bar{\theta}_{2}, \quad \bar{\omega}_{1}=\chi(r) r^{\alpha} \bar{g}_{1}(\beta), \quad \bar{\theta}_{1}=\chi(r) r^{1+2 \alpha} \bar{g}_{2}(\beta) \tag{7.2}
\end{equation*}
$$

where $\chi(r)$ is the radial cut-off function defined in (C.8). The crucial first part is constructed semi-analytically, and it captures the far-field asymptotic behavior of the approximate steady
state. The second part has a much faster decaying rate, and we construct it using numerical computation with a piecewise sixth order B-spline.
7.2. Angular profiles and the representation. Due to symmetry in $x$, we compute (2.10) in a domain $[0, L]^{2}$ with $L \approx 10^{13}$ with stream function supported in a larger domain about $D_{l g}=$ $\left[0, L_{2}\right]^{2}, L_{2} \approx 10^{15}$. We partition $\left[0, L_{2}\right]$ using adaptive mesh $0=y_{0}<y_{1}<. .<y_{N-1}=L_{2}$. See Appendix C. 1 in Part II [15] for construction of $y_{i}$. Since $\theta(t, x, y)$ vanishes quadratically on $x=0$, instead of using $\theta$ in our computation, we consider $\zeta(t, x, y)=\frac{1}{x} \theta(t, x, y)$. Then $\zeta$ is odd in $x$, and its equation can be derived by dividing the $\theta$ equation by $x$.

In the case without semi-analytic part, we represent the numerical solution $(\omega, \rho)$ using a piecewise 6 th order B -spline in $x$ and $y$, e.g.

$$
\begin{equation*}
\omega(t, x, y)=\sum_{i, j} a_{i, j} B_{i}(x) B_{j}(y) \tag{7.3}
\end{equation*}
$$

where $B_{i}(x)$ is the B-spline basis (see Appendix C. 1 of part 2 [15]). For $\psi$, we represent it using a piecewise B-spline with additional weight $\rho_{p}(y)$ vanishing on the boundary $y=0$ to enforce the no-flow boundary condition $\psi(x, 0)=0$. See more details about the representation in Appendix C. 1 of Part II 15. Note that similar representations based on piecewise B-splines have been used in 65. Given the grid point values of $\omega$, we obtain the coefficients of the variable $\omega$ by solving the linear equations (7.3) for $(x, y)$ on the grid and using suitable extrapolation in the far-field. After we obtain the coefficients $a_{i, j}$, we compute the derivatives of $\omega$ using the basis functions

$$
\partial_{x}^{i} \partial_{y}^{j} \omega(t, x, y)=\sum_{i, j} a_{i, j} \partial_{x}^{i} B_{i}(x) \partial_{y}^{j} B_{j}(y)
$$

Similar consideration applies to $\zeta$. We solve the Poisson equations

$$
\begin{equation*}
-\Delta \phi=\omega \tag{7.4}
\end{equation*}
$$

using B-spline based finite element method. After we obtain the B-spline coefficients for $\phi$, we compute its derivatives by taking derivatives on the basis functions. We refer more details of representation to Appendix C. 1 in Part II 15.

In the temporal variable, we use a second order Runge-Kutta method to update the PDE.
To construct the decomposition in (7.2), firstly, we obtain the exponent $\alpha_{1}$ and construct the angular profile and the semi-analytic part $\bar{\omega}_{1}, \bar{\theta}_{1}$ in (7.2). Then, using $\bar{\omega}_{1}, \bar{\theta}_{1}$, we refine the construction $\bar{\omega}_{2}, \bar{\theta}_{2}$ in (7.2).
7.2.1. Fitting the angular profile and the exponent. We need to find the angular profiles in the semi-analytic parts in (7.2). Firstly, we solve (2.10) numerically using the above method without the semi-analytic part, i.e. $\bar{\omega}_{1}=0, \bar{\theta}_{1}=0$, to obtain an approximate steady state $(\bar{\omega}, \bar{\zeta}), \bar{\theta}=x \bar{\zeta}$. Using the ansatz in (7.2) and fitting the angular part of the far-field of $r^{-\bar{\alpha}_{1}} \omega_{1}, r^{-1-2 \bar{\alpha}_{1}} \bar{\theta}=$ $r^{-2 \bar{\alpha}_{1}} \cos \beta \cdot \bar{\zeta}$ with exponent $\bar{\alpha}_{1}=\frac{\bar{c}_{\omega}}{\bar{c}_{l}}$ (7.1), we find the following approximate profiles
$g_{10}(\beta)=\frac{a_{11} \tilde{\beta}\left(1+a_{15} \tilde{\beta}^{2}\right)}{\left(\tilde{\beta}^{2}+a_{12}\right)^{2 / 3}+a_{13} \tilde{\beta}^{2}+a_{14}}, \tilde{\beta}=\frac{\pi}{2}-\beta, \quad g_{20}(\beta)=\frac{a_{21} \cos ^{2} \beta\left(1+a_{25} \sin \beta\right)}{\left(\cos ^{2} \beta+a_{22}\right)^{2 / 3}+a_{23}+a_{24} \cos ^{2} \beta}$,
for some parameters $a_{i j}$. We have the factor $\frac{\pi}{2}-\beta$ since $\omega$ is odd in $x$ and $g_{10}(\beta)$ is odd with respect to $\beta=\pi / 2$. Similarly, we add the factor $\cos ^{2} \beta$ in $g_{20}(\beta)$ since $\theta(x, 0)=0$ and $\theta_{x}(x, 0)$ is odd in $x$. After we find the above analytic formulas, we further approximate the above profiles by piecewise 8 th order B-splines (see Appendix C. 1 of Part II [15]) $B_{i}$ with $k=8$

$$
g_{1}(\beta)=\sum_{1 \leq i \leq n} b_{1 i} B_{i}(x), \quad g_{2}(\beta)=\cos \beta \cdot \sum_{1 \leq i \leq n} b_{2 i} B_{i}(x)
$$

for some coefficients $b_{j i}$. We factor out $\cos \beta$ in $g_{2}(\beta)$ such that both B-splines are odd with respect to $\beta=\pi / 2$. We further use the B -spline to represent the angular profiles for the following reason. To verify that the approximate steady state $(\bar{\omega}, \bar{\theta})$ has a small residual error, we need to estimate the high order derivatives of $\bar{\omega}, \bar{\theta}$, e.g. 6 -th order. However, the high order derivatives of the above analytic forms are very complicated, and are difficult to estimate. On the other
hand, we have a systematic approach to estimate piecewise polynomials. Once we obtain $g_{i}(\beta)$, we construct the semi-analytic part

$$
\begin{equation*}
\bar{\omega}_{10}=\chi(r) r^{\bar{\alpha}_{1}} g_{1}(\beta), \quad \bar{\theta}_{10}=\chi(r) r^{1+2 \bar{\alpha}_{1}} g_{2}(\beta), \quad \bar{\zeta}_{10}=\bar{\theta}_{10} x^{-1} \tag{7.5}
\end{equation*}
$$

To compute the semi-analytic part of the stream function, we follow the ideas outlined in [20]. Given the asymptotic behavior of $\bar{\omega}$ in (7.5), the far-field asymptotic behavior of $\phi=(-\Delta)^{-1} \bar{\omega}_{10}$ is $r^{2+\bar{\alpha}_{1}} f(\beta)$ for some profile $f(\beta)$. We construct $f(\beta)$ by solving

$$
-\Delta\left(r^{2+\bar{\alpha}_{1}} f(\beta)\right)=r^{\bar{\alpha}_{1}} g_{1}(\beta)
$$

with boundary condition $f(0)=f(\pi / 2)=0$ due to the Dirichlet boundary condition $\phi(x, 0)=0$ and the odd symmetry for the solution $\omega$. In the polar coordinate, the above equation is equivalent to

$$
\begin{equation*}
\left(-\partial_{\beta}^{2}-\left(2+\bar{\alpha}_{1}\right)^{2}\right) f(\beta)=g_{1}(\beta), \quad f(0)=f(\pi / 2)=0 \tag{7.6}
\end{equation*}
$$

We represent $f(\beta)$ using a weighted 8th order B-spline and solve the above elliptic equations using the finite element method. Then, we construct the semi-analytic part for $\phi$ as follows

$$
\begin{equation*}
\bar{\phi}_{10}=\chi(r) r^{2+\bar{\alpha}_{1}} f(\beta) \tag{7.7}
\end{equation*}
$$

7.2.2. Refinement. We use the semi-analytic profile (7.5) to capture the far-field contribution of $\bar{\omega}, \bar{\zeta}$. Note that in this step, we do not update the angular profile nor the exponent in (7.5).

Given the grid point values of $\omega(t, x, y)$, we first update the constant $c(t)$ such that $c(t) \bar{\omega}_{10}$ best approximate $\omega(t, x, y)$ in the far-field. Then we represent $\omega_{2}(t, x, y)=\omega(t, x, y)-c(t) \bar{\omega}_{10}$ using the B-spline (7.3). In other words, we interpolate the grid point values using the representation $c(t) \bar{\omega}_{10}+\omega_{2}(t, x, y)$, where $\omega_{2}$ is a piecewise polynomial in the compact domain. Similar consideration applies to $\zeta$. To update the stream functions $\phi$, we use $c(t) \bar{\phi}_{10}$ to capture the far-field of $\phi$ and then construct the near-field part by solving

$$
\begin{equation*}
-\Delta\left(\phi_{2}+c(t) \bar{\phi}_{10}\right)=\omega_{2}+c(t) \bar{\omega}_{10}, \quad \text { or } \quad-\Delta \phi_{2}=\omega_{2}+c(t)\left(\bar{\omega}_{10}+\Delta \bar{\phi}_{10}\right) \tag{7.8}
\end{equation*}
$$

Then the stream function is represented as $\phi_{2}+c(t) \bar{\phi}_{10}$.
Let us motivate the above decomposition to construct the stream function over (7.4). If we use (7.4), the source term $\omega$ has a slow decay $r^{\alpha_{1}} \approx r^{-1 / 3}$. Since the domain is very large, we have to use an adaptive mesh to discretize the domain, which leads to a poor condition number of the stiffness matrix in (7.4). Thus, solving (7.4) can have a significant round-off error. In (7.4), since the semi-analytic part $c(t) \bar{\omega}_{10}$ captures the asymptotic behavior of $\omega(t, x, y), \omega_{2}$ is much smaller than $\omega$ in the far-field. By definition of $\bar{\omega}_{10}, \bar{\phi}_{10}$ (7.5)- (7.7), the far-field of $\bar{\omega}_{10}+\Delta \bar{\phi}_{10}$ is about $\varepsilon r^{-1 / 3}$ with a small constant $\varepsilon$. Hence, the far-field of the source term in (7.8) is much smaller than $\omega(t, x)$, which enables us to overcome the significant round-off error. We remark that similar technique has been used in the Hou-Luo model 20 to overcome the significant round-off errors. The above decomposition is a generalization of the method in [20] to 2D. We refer to [20] for the more motivations and the difficulties caused by the round-off error.

After we obtain the stream function, we can update the PDE using the second order RungeKutta method. We stop the computation at time $t_{*}$ if the residual error on the grid points is about the round-off error. Then we finalize the semi-analytic part in (7.2) as

$$
\begin{align*}
& \bar{\omega}_{1}=\bar{c}_{1} \bar{\omega}_{10}=\chi(r) r^{\alpha}\left(\bar{c}_{1} g_{1}(\beta)\right), \quad \bar{\theta}_{1}=\bar{c}_{2} \bar{\theta}_{10}=\chi(r) r^{1+2 \alpha_{1}}\left(\bar{c}_{2} g_{2}(\beta)\right) \\
& \bar{\phi}_{1}=\bar{c}_{1} \bar{\phi}_{10}=\chi(r) r^{2+\bar{\alpha}_{1}}\left(\bar{c}_{1} f(\beta)\right) \triangleq \chi(r) r^{2+\bar{\alpha}_{1}} \bar{f}(\beta) \tag{7.9}
\end{align*}
$$

where $\bar{c}_{1} \bar{\omega}_{10}, \bar{c}_{2} \bar{\zeta}_{10}$ best approximate $\omega\left(t_{*}, x, y\right), \zeta\left(t_{*}, x, y\right)$ in the far-field, respectively. We construct $\bar{\omega}_{2}, \bar{\theta}_{2}=x \bar{\zeta}_{2}$ in (7.2) by interpolating the grid point values of $\omega-\bar{\omega}_{1}, \theta-\bar{\theta}_{1}$ and applying a low-pass filter to the solution to reduce the round off error.

In Appendix C of Part II [15], we estimate the derivatives of the approximate steady state rigorously, which will be used to verify the residual error.
7.2.3. A multi-level representation. To design the B-spline in the representation (7.2) of the second part of the solution, we use adaptive mesh. Near the origin, the mesh size is small, e.g. $h \leq \frac{1}{256}$ in our computation. In the computation of high order derivatives, e.g. $\nabla^{4} \bar{\omega}$, the round-off error may not be relatively small. To construct the approximate steady state and the approximate solution to the linearized equations in Section 3 of Part II [15], we only need to use lower order derivatives $\nabla \omega, \nabla \eta, \nabla \xi, \nabla^{2} \phi, \nabla \theta$, and the round-off error is neglible. However, to verify the smallness of the weighted norm of the error, e.g. $\overline{\mathcal{F}}_{i}$ (2.19), since we use a weight $\varphi$ singular about order $|x|^{-3}$ near the origin, we need to estimate the piecewise $C^{3}$ bounds for the error and evaluate $\nabla^{3} \overline{\mathcal{F}}_{i}, \nabla^{4} \omega, \nabla^{4} \eta, \nabla^{4} \xi, \nabla^{5} \theta$ on some grid points based on the estimates in Appendix E of Part II [15]. To obtain rigorous bound, we use interval arithmetic. For each operation, e.g. $a \in\left[a_{l}, a_{u}\right], b \in\left[b_{l}, b_{u}\right]$, the interval bound for $a b$ is obtained by considering the worse case. If we use interval arithmetic with a lower order precision, e.g. the double precision which has a machine error about $10^{-16}$, the size of the interval bounding $\nabla^{3} \overline{\mathcal{F}}_{i}$ can be much larger than the actual round off error. One natural way to overcome this problem is using higher order precision, e.g. interval arithmetic with quadruple precision.

To save the computational cost, we refine the B-spline representation of the solution $f$ so that $\nabla^{k} f$ has a much smaller round off error. Note that the round off error of $\nabla^{k} f$ is about $C \varepsilon h^{-k}$, where $C$ is the size of the B-spline coefficient for $f, \varepsilon$ is the machine precision, and $h$ is the mesh near 0 . To reduce it, we either reduce $C$ or increase $h$. We use a multi-level B-spline representation $f=f_{1}+f_{2}+\ldots+f_{n}$. In the coarse level, we use a larger mesh size for $f_{1}$ near 0 , e.g. $h_{1}=24 h_{n}, h_{n}=\frac{1}{256}$ and $h_{n}$ is the mesh size for the finest level near 0 . Since the profile $\bar{f}$ is quite smooth, we use the first level representation $f_{1}$ to interpolate $\bar{f}$ and the round off error for $\nabla^{k} f_{1}$ is very small since $h_{1}$ is much larger. In the next level, we use smaller mesh, e.g. $h_{2}=6 h_{n}$, and use $f_{2}$ to interpolate $\bar{f}-\bar{f}_{1}$. Since $\bar{f}-\bar{f}_{1}$ is much smaller than $\bar{f}$, the coefficients for the B-spline $f_{2}$ are small and the round off error is small. The same procedure and ideas apply to other levels. We choose the supporting points of the B-splines $f_{1}, . ., f_{n-1}$ from the grid points for $f_{n}$, so that the overall representation $f=f_{1}+f_{2}+. .,+f_{n}$ is still piecewise polynomials on the mesh for $f_{n}$. Then we can estimate the piecewise derivatives of $f$ using the method in Appendix B.5.2 in in Part II [15].

For the B-spline part of the stream function $\bar{\phi}_{2}$, near $x=0$, since $\bar{\phi}_{2}=\partial_{x y} \bar{\phi}_{2}(0) x y+O\left(|x|^{3}\right)$, we approximate it using an analytic profile

$$
\begin{equation*}
\bar{\phi}_{3}=a \chi_{\phi, 2 D}, \quad \chi_{\phi, 2 D}=-x y \chi_{\phi}(x) \chi_{\phi}(y), \tag{7.10}
\end{equation*}
$$

where $a$ is chosen to approximate $-\partial_{x y} \bar{\phi}_{2}(0)=-\partial_{x y} \bar{\phi}$, and $\chi_{\phi}$ is some cutoff function with $\chi_{\phi}(x)=1+O\left(|x|^{2}\right)$ near 0 and is constructed in (C.10). We add the negative sign to normalize $u_{x}\left((-\Delta) \chi_{\phi, 2 D}\right)(0)=-\partial_{x y}(-\Delta)^{-1}(-\Delta) \chi_{\phi, 2 D}(0)=-\partial_{x y} \chi_{\phi, 2 D}(0)=1$. In solving the approximate steady state, in the $n-t h$ step, $a$ is determined by the $\partial_{x y} \phi_{n-1}(0)$ in the previous step. Then we use the multi-level B-spline representation $\phi_{2}=\phi_{2,1}+\ldots+\phi_{2, n}$ by solving a modification of (7.8)

$$
-\Delta \phi_{2}=\omega_{2}+c(t)\left(\bar{\omega}_{10}+\Delta \bar{\phi}_{10}\right)+\Delta \phi_{3} \triangleq S .
$$

The approximation term allows us to obtain smaller spline coefficients for $\phi_{2}$ and reduce the round off error. To obtain the top-level B-spline $\phi_{2,1}$ on the coarse mesh $y^{(1)}$, we first restrict $S$ on the mesh $y^{(1)}$ and interpolate $S\left(y_{i}^{(1)}, y_{j}^{(1)}\right)$ using the single level B-spline $S^{(1)}$ with supporting points on $y^{(1)}$. Then we use the B-spline based finite element method to solve $-\Delta \phi_{2,1}=S^{(1)}$. We evaluate $\Delta \phi_{2,1}$ on the fine mesh and further solve $\phi_{2,2}, \phi_{2,3} \ldots$ recursively from the remaining source part $S+\Delta \phi_{2,1}$.

After we obtain the above stream function, we further add a rank-one corretion $\bar{\phi}_{\text {cor }}$ near 0

$$
\bar{\phi}^{N}=\bar{\phi}_{1}+\bar{\phi}_{2}+\bar{\phi}_{3}+\bar{\phi}_{c o r}, \quad \bar{\phi}_{c o r}=-c \cdot \frac{x y^{2}}{2} \kappa_{*}(x) \kappa_{*}(y), \quad c=\partial_{x}\left(\bar{\omega}+\Delta\left(\bar{\phi}_{1}+\bar{\phi}_{2}+\bar{\phi}_{3}\right)\right)(0)
$$

where $\kappa_{*}(x)=1+O\left(|x|^{4}\right)$ is defined in (C.10) and $\bar{\phi}_{c o r}$ satisfies $\partial_{x}(-\Delta) \bar{\phi}_{c o r}(0)=c$. By choosing the above $c$, we get that the error of solving the Poisson equations satisfies $\bar{\varepsilon}=\bar{\omega}+\Delta \bar{\phi}^{N}=$ $O\left(|x|^{2}\right)$. We note that $|c|<10^{-10}$ is very small. Since the stream function $\bar{\phi}=(-\Delta)^{-1} \bar{\omega}$ depends
on $\bar{\omega}$ nonlocally, we cannot construct it exactly and use $\bar{\phi}^{N}$ to approximate it numerically, where $N$ is short for numerics. The nonlocal error $\bar{\varepsilon}$ is decomposed and estimated in Section5.8.

## Appendix A. Some Lemmas for stability estimates

We use the following Lemma for the linear stability analysis.
Lemma A.1. Suppose that $f_{i}(x, z, t): \mathbb{R}_{++}^{2} \times \mathbb{R}_{++}^{2} \times[0, T] \rightarrow \mathbb{R}, 1 \leq i \leq n$, satisfies

$$
\begin{equation*}
\partial_{t} f_{i}+v_{i}(x, z) \cdot \nabla_{x, z} f_{i}=-a_{i i}(x, z, t) f_{i}+B_{i}(x, z, t) \tag{A.1}
\end{equation*}
$$

where $v_{i}(x, z, t)$ are some vector fields Lipschitz in $x, z$ with $\left.v_{i}\right|_{x_{1}=0}=0,\left.v_{i}\right|_{z_{1}=0}=0$, and $B_{i}$ satisfies the following estimate

$$
\begin{equation*}
\left|B_{i}(x, z, t)\right| \leq \sum_{j \neq i}\left|a_{i j}(x, z, t)\right| \cdot\left\|f_{j}\right\|_{L^{\infty}} \tag{A.2}
\end{equation*}
$$

If there exists some constants $M, \lambda, \mu_{i}>0$ such that for all $(x, z)$, we have

$$
\begin{equation*}
a_{i i}(x, z, t)-\sum_{j \neq i}\left|a_{i j}\right| \mu_{i} \mu_{j}^{-1} \geq \lambda, \quad \sum_{j \neq i} \mu_{i} \mu_{j}^{-1}\left|a_{i j}\right| \leq M \tag{A.3}
\end{equation*}
$$

Then for $E(t)=\max _{i}\left(\mu_{i}\left\|f_{i}(t)\right\|_{\infty}\right)$, which is Lipschitz, and $0 \leq t_{0}<t \leq T$, we have

$$
E(t) \leq e^{-\lambda\left(t-t_{0}\right)} E_{0}, \quad E_{0}=E\left(t_{0}\right)
$$

The condition (A.2) means that the damping term is stronger than the bad terms, which further leads to the stability. We apply $f(x, z, t)=\left(\left(S_{i} \psi_{i}\right)(x)-\left(S_{i} \psi_{i}\right)(z)\right) g_{i}(x, z)$ in the weighted Hölder estimate, and $f_{i}(x, z, t)=\left(S_{i} \varphi_{i}\right)(x)$ in the weighted $L^{\infty}$ estimate, $S_{1}=\omega, S_{2}=\eta, S_{3}=\xi$. In the weighted $L^{\infty}$ estimate, we do not need the extra variable $z$ and $f_{i}$ is constant in $z$. For the Boussinesq equations (5.1), we choose

$$
b(x, t)=\bar{c}_{l} x+\overline{\mathbf{u}}(x)+\mathbf{u}(\omega)(x, t), \quad v_{i}(x, t)=b(x, t), \quad \text { or } \quad v_{i}(x, z, t)=(b(x, t), b(z, t))
$$

We will also perform energy estimates on some scalars $a_{i}(t)$ and choose $f_{i}(x, z, t)=a_{i}(t)$ in the above Lemma. In this case, advection term is 0 , and $a_{i i}, a_{i j}, B_{i}$ only depend on $t$.

Proof. For simplicity, we assume that the condition (A.3) holds for $\mu_{i}=1$. Otherwise, we can estimate the variables $\mu_{i} f_{i}$ and introduce $\tilde{a}_{i j}=a_{i j} \mu_{i} \mu_{j}^{-1}$. Then the equations and estimates (A.1), (A.2) become

$$
\begin{aligned}
& \partial_{i} \mu_{i} f_{i}+v_{i}(x, z) \cdot \nabla_{x, z}\left(\mu_{i} f_{i}\right)=-a_{i i}\left(\mu_{i} f_{i}\right)+\mu_{i} B_{i}(x, z, t) \\
& \mu_{i}\left|B_{i}(x, z, t)\right| \leq \sum_{j \neq i} \mu_{i} \mu_{j}^{-1} a_{i j}(x, z, t) \cdot \mu_{j}\left\|f_{j}\right\|_{L^{\infty}}=\sum_{j \neq i} \tilde{a}_{i j}(x, z, t) \cdot \mu_{j}\left\|f_{j}\right\|_{L^{\infty}}
\end{aligned}
$$

The condition A.3) for $a_{i j}$ becomes the condition for $\tilde{a}_{i j}$ with equal weights. Thus, it suffices to consider the case $\mu_{i}=1, i=1,2, . ., n$.

Formally, we can perform $L^{\infty}$ estimate on (A.1) and then evaluate (A.1) at the maximizer to obtain the desired result. To justify it rigorously, we use the characteristics, Duhamel's principle, and a bootstrap argument. We define the characteristics associated with $v_{i}$

$$
\begin{equation*}
\frac{d}{d t}\left(X_{i}(t), Z_{i}(t)\right)=v_{i}\left(X_{i}, Z_{i}, t\right), \quad X_{i}(0)=x_{0}, \quad Z_{i}(0)=z_{0}, \quad F_{i}(t)=f_{i}\left(X_{i}(t), Z_{i}(t), t\right) \tag{A.4}
\end{equation*}
$$

To simplify the notation, we drop $x_{0}, z_{0}$. Denote

$$
\begin{equation*}
A_{i}(t)=a_{i i}\left(X_{i}(t), Z_{i}(t)\right), \quad C_{i}(t) \triangleq \sum_{j \neq i}\left|a_{i j}\left(X_{i}(t), Z_{i}(t), t\right)\right| \tag{A.5}
\end{equation*}
$$

It suffices to prove that for small $\varepsilon>0$, we have

$$
\begin{equation*}
E(t) \leq(1+c(M) \varepsilon) e^{-\lambda_{\varepsilon}\left(t-t_{0}\right)} E_{0}, \quad \lambda_{\varepsilon}=\lambda-\varepsilon, \quad c(M)=\frac{2}{M} \tag{A.6}
\end{equation*}
$$

where $M$ is the upper bound in (A.3). Then taking $\varepsilon \rightarrow 0$ completes the proof.

We want to use a bootstrap argument to prove A.6). Firstly, since $E(0)=E_{0}$ and $E(t)$ is Lipschitz, the above condition holds for $t \in\left[t_{0}, t_{0}+T_{1}\right]$ with some $T_{1}>0$. Now, we want to show that under (A.6), we can obtain

$$
\begin{equation*}
E(t) \leq(1+c(M) \varepsilon / 2) e^{-\lambda_{\varepsilon}\left(t-t_{0}\right)} E_{0} \tag{A.7}
\end{equation*}
$$

By definition, along the characteristics, we get

$$
\frac{d}{d t} F_{i}(t)=-A_{i}(t) F_{i}(t)+B_{i}\left(X_{i}(t), Z_{i}(t), t\right)
$$

Using the estimates (A.2) and definition (A.5), we yield

$$
\left|B_{i}\left(X_{i}(t), Z_{i}(t), t\right)\right| \leq \sum_{j \neq i}\left|a_{i j}\right| E(t) \leq C_{i}(t) E(t)
$$

Using Duhamel's principle and the above estimate, we obtain

$$
\begin{gather*}
F_{i}(t)=e^{\int_{t_{0}}^{t}-A_{i}(s) d s} F_{i}\left(t_{0}\right)+\int_{t_{0}}^{t} B_{i}\left(X_{i}(s), Z_{i}(s), s\right) e^{\int_{s}^{t}-A_{i}(\tau) d \tau} d s \\
\left|F_{i}(t)\right| \leq e^{\int_{t_{0}}^{t}-A_{i}(s) d s} F_{i}\left(t_{0}\right)+\int_{t_{0}}^{t} C_{i}(s) E(s) e^{\int_{s}^{t}-A_{i}(\tau) d \tau} d s \triangleq I+I I \tag{A.8}
\end{gather*}
$$

For the second term, using the bootstrap assumptions (A.6), we yield

$$
|I I| \leq \int_{t_{0}}^{t} C_{i}(s)(1+c(M) \varepsilon) e^{-\lambda_{\varepsilon}\left(s-t_{0}\right)} E_{0} e^{-\lambda_{\varepsilon}(t-s)-\int_{s}^{t}\left(A_{i}(s)-\lambda_{\varepsilon}\right) d s}
$$

Using $C_{i}(s) \leq M, C_{i}(s) \leq A_{i}(s)-\lambda(\mathrm{A} .3)$ and the definition of $c(M)$ A.6), we get

$$
\frac{C_{i}(s)}{C_{i}(s)+\varepsilon} \leq \frac{M}{M+\varepsilon} \leq \frac{1+\varepsilon / M}{1+2 \varepsilon / M}=\frac{1+c(M) \varepsilon / 2}{1+c(M) \varepsilon}, \quad C_{i}(s)+\varepsilon \leq A_{i}(s)-\lambda+\varepsilon=A_{i}(s)-\lambda_{\varepsilon}
$$

which implies

$$
C_{i}(s)(1+c(M) \varepsilon) \leq\left(C_{i}(s)+\varepsilon\right)(1+c(M) \varepsilon / 2) \leq\left(A_{i}(s)-\lambda_{\varepsilon}\right)(1+c(M) \varepsilon / 2)
$$

Note that we choose $c(M)$ in (A.6) small enough such that the above inequality holds. Hence, we can simplify the bound of $I I$ as follows

$$
\begin{aligned}
|I I| & \leq(1+c(M) \varepsilon / 2) e^{-\lambda_{\varepsilon}\left(t-t_{0}\right)} E_{0} \int_{t_{0}}^{t}\left(A_{i}(s)-\lambda_{\varepsilon}\right) e^{-\int_{s}^{t}\left(A_{i}(\tau)-\lambda_{\varepsilon}\right) d \tau} d s \\
& =(1+c(M) \varepsilon / 2) e^{-\lambda_{\varepsilon}\left(t-t_{0}\right)} E_{0}\left(1-e^{-\int_{t_{0}}^{t}\left(A_{i}(\tau)-\lambda_{\varepsilon}\right) d \tau}\right)
\end{aligned}
$$

The estimate of $I$ is trivial. Since $\left|F_{i}(0)\right| \leq E_{0}$, we have

$$
|I|=\left|F_{i}\left(t_{0}\right)\right| e^{-\lambda_{\varepsilon}\left(t-t_{0}\right)} e^{-\int_{t_{0}}^{t}\left(A_{i}(\tau)-\lambda_{\varepsilon}\right) d \tau} \leq E_{0} e^{-\lambda_{\varepsilon}\left(t-t_{0}\right)} e^{-\int_{t_{0}}^{t}\left(A_{i}(\tau)-\lambda_{\varepsilon}\right) d \tau}
$$

which along with the estimate of $I I$ yields

$$
\left|F_{i}(t)\right| \leq|I|+|I I| \leq(1+c(M) \varepsilon / 2) e^{-\lambda_{\varepsilon}\left(t-t_{0}\right)} E_{0}
$$

Since the above estimate holds for any initial data $x_{0}, z_{0}$ and $i$, taking the supremum, we prove (A.7). Then the standard bootstrap argument implies the desired estimate (A.6).

We can generalize the previous linear stability Lemma to the nonlinear stability estimates.
Lemma A.2. Suppose that $f_{i}(x, z, t): \mathbb{R}_{++}^{2} \times \mathbb{R}_{++}^{2} \times[0, T] \rightarrow \mathbb{R}, 1 \leq i \leq n$, satisfies

$$
\begin{equation*}
\partial_{t} f_{i}+v_{i}(x, z) \cdot \nabla_{x, z} f_{i}=-a_{i i}(x, z, t) f_{i}+B_{i}(x, z, t)+N_{i}(x, z, t)+\bar{\varepsilon}_{i} \tag{A.9}
\end{equation*}
$$

where $v_{i}(x, z, t)$ are some vector fields Lipschitz in $x, z$ with $\left.v_{i}\right|_{x_{1}=0}=0,\left.v_{i}\right|_{z_{1}=0}=0$. For some $\mu_{i}>0$, we define the energy

$$
E(t)=\max _{1 \leq i \leq n}\left(\mu_{i}\left\|f_{i}\right\|_{L^{\infty}}\right)
$$

Suppose that $B_{i}, N_{i}$ and $\bar{\varepsilon}_{i}$ satisfy the following estimate
$\mu_{i}\left(\left|B_{i}(x, z, t)\right|+\left|N_{i}(x, z, t)\right|+\left|\bar{e}_{i}\right|\right) \leq \sum_{j \neq i}\left(\left|a_{i j}(x, z, t)\right| E(t)+\left|a_{i j, 2}(x, z, t)\right| E^{2}(t)+\left|a_{i j, 3}(x, z, t)\right|\right)$.

If there exists some $E_{*}, \varepsilon_{0}, M>0$ such that

$$
\begin{align*}
& a_{i i}(x, z, t) E_{*}-\sum_{j \neq i}\left(\left|a_{i j}\right| E_{*}+\left|a_{i j, 2}\right| E_{*}^{2}+\left|a_{i j, 3}(x, z, t)\right|\right)>\varepsilon_{0} \\
& \sum_{j \neq i}\left(\left|a_{i j}\right| E_{*}+\left|a_{i j, 2}\right| E_{*}^{2}+\left|a_{i j, 3}(x, z, t)\right|\right)<M \tag{A.11}
\end{align*}
$$

for all $x, z$ and $t \in[0, T]$. Then for $E(0)<E_{*}$, we have $E(t)<E_{*}$ for $t \in[0, T]$.
The second inequality in (A.11) is only qualitative. Note that the factor $a_{i j}$ (A.10) for linear terms is different from that in (A.2). We have combine the weight $\mu_{j}$ with $a_{i j}, j \neq i$ in (A.10).

Proof. The proof is very similar to that of Lemma A.1. We fix $E(0)$. Without loss of generality, we assume $\mu_{i}=1$. Otherwise, we rewrite the (A.9) in terms of $\mu_{i} f_{i}$. It suffices to prove that under the bootstrap assumption

$$
\begin{equation*}
E(t)<E_{*} \tag{A.12}
\end{equation*}
$$

on $\left[0, T_{1}\right]$, there exists $\varepsilon$ that depends on $E(0), \varepsilon_{0}, M, E_{*}$, such that we can obtain

$$
\begin{equation*}
E(t) \leq(1-\varepsilon) E_{*}, \quad t \in\left[0, T_{1}\right] \tag{A.13}
\end{equation*}
$$

Since $E(0)<E_{*}$ and $E(t)$ is Lipschitz, we know that the bootstrap assumption holds for some short time $T_{1}$.

We adopt most notations from the proof of Lemma A. 1 but use

$$
C_{i}(t) \triangleq \sum_{j \neq i}\left(\left|a_{i j}\left(X_{i}(t), Z_{i}(t), t\right)\right| E(t)+\left|a_{i j, 2}\left(X_{i}(t), Z_{i}(t), t\right)\right| E^{2}(t)+\left|a_{i j, 3}\left(X_{i}(t), Z_{i}(t), t\right)\right|\right)
$$

Using these notations, derivations and estimates similar to those in the proof of Lemma A. 1 , we obtain

$$
\left|F_{i}(t)\right| \leq e^{-\int_{0}^{t}-A_{i}(s) d s} F_{i}(0)+\int_{0}^{t} C_{i}(s) e^{-\int_{s}^{t} A_{i}(\tau) d \tau} d s
$$

Using the bootstrap assumption and (A.11), we obtain

$$
C_{i}(s)<\min \left(M, A_{i}(t) E_{*}-\varepsilon_{0}\right)<(1-\delta) A_{i}(t) E_{*}
$$

for some small $\delta$ depending on $\varepsilon_{0}, M, E_{*}$. Note that if $A_{i}(t) E_{*}<2 M$, we pick $\delta$ such that $A_{i}(t) E_{*}-\varepsilon_{0}<(1-\delta) A_{i}(t) E_{*}$. If $A_{i}(t) E_{*}>2 M$, we require $\delta<1 / 2$. Now, we obtain

$$
\begin{aligned}
\left|F_{i}(t)\right| & \leq e^{-\int_{0}^{t}-A_{i}(s) d s}\left|F_{i}(0)\right|+\int_{0}^{t}\left(A_{i}(s) E_{*}-\varepsilon_{0}\right) e^{-\int_{s}^{t} A_{i}(\tau) d \tau} d s \\
& \leq e^{-\int_{0}^{t}-A_{i}(s) d s}\left|F_{i}(0)\right|+(1-\delta) \int_{0}^{t} A_{i}(s) E_{*} e^{-\int_{s}^{t} A_{i}(\tau) d \tau} d s \\
& =e^{-\int_{0}^{t}-A_{i}(s) d s}\left|F_{i}(0)\right|+(1-\delta) E_{*}\left(1-e^{-\int_{0}^{t}-A_{i}(s) d s}\right) \\
& \leq \max \left(\left|F_{i}(0)\right|,(1-\delta) E_{*}\right) \leq \max \left(|E(0)|,(1-\delta) E_{*}\right)
\end{aligned}
$$

Taking the supremum over the initial data of the trajectory and $i$, we get

$$
E(t) \leq \max \left(|E(0)|,(1-\delta) E_{*}\right)
$$

Since we fix $E(0)$ and $E(0)<E_{*}$, we can pick small $\delta$ to obtain

$$
E(0)<(1-\delta) E_{*}, \quad E(t)<(1-\delta) E_{*}
$$

which is A.13). Using the bootstrap argument, we complete the proof.
A.1. Proof of Lemma 2.5, We prove Lemma 2.5 related to the Hölder estimates.

Proof. Using (2.35), we first derive the equation for $f \varphi$

$$
\partial_{t}(f \varphi)+b(x) \cdot \nabla(f \varphi)=c(x) f \varphi+(b \cdot \nabla \varphi) f+\mathcal{R} \varphi=d(x) f \varphi+\mathcal{R} \varphi
$$

where $d(x)=c(x)+\frac{b \cdot \nabla \varphi}{\varphi}$ is defined in Lemma [2.5] For $x, z \in \mathbb{R}_{2}^{+}$, we derive the equation of $\delta(f \varphi)(x, z)=f \varphi(x)-f \varphi(z):$

$$
\partial_{t} \delta(x, z, t)+b(x) \cdot \nabla_{x}(f \varphi)(x)-b(z) \cdot \nabla_{z}(f \varphi)(z)=(d f \varphi)(x)-(d \eta \varphi)(z)+\delta(\mathcal{R} \psi)
$$

Since
$\nabla_{x}(f \varphi)(x)=\nabla_{x}((f \varphi)(x)-(f \varphi)(z))=\nabla_{x} \delta(f \varphi), \quad \nabla_{z}(f \varphi)(z)=-\nabla_{z}(\delta(f \varphi))$,
$d f \varphi(x)-d f \varphi(z)=d(x)(f \varphi(x)-f \varphi(z))+(d(x)-d(z)) f \varphi(z)=d(x) \delta(f \varphi)(x, z)+(d(x)-d(z)) f \varphi(z)$,
we obtain

$$
\begin{equation*}
\partial_{t} \delta(f \varphi)+\left(b(x) \cdot \nabla_{x}+b(z) \cdot \nabla_{z}\right) \delta(f \varphi)=d(x) \delta(f \varphi)(x, z)+(d(x)-d(z))(f \varphi)(z)+\delta(R \varphi) \tag{A.14}
\end{equation*}
$$

Since $g(h)$ is even in $h_{1}, h_{2}, \partial_{i} g$ is odd in $h_{i}$ and we have

$$
\begin{aligned}
& \left(b(x) \cdot \nabla_{x}+b(z) \cdot \nabla_{z}\right)(\delta(f \varphi) g(x-z)) \\
= & g(x-z) \cdot\left(b(x) \cdot \nabla_{x}+b(z) \cdot \nabla_{z}\right) \delta(f \varphi)+\delta(f \varphi) \cdot\left(b(x) \cdot \nabla_{x}+b(z) \cdot \nabla_{z}\right) g(x-z) \\
= & g(x-z) \cdot\left(b(x) \cdot \nabla_{x}+b(z) \cdot \nabla_{z}\right) \delta(f \varphi)+\delta(f \varphi) \cdot(b(x)-b(z)) \cdot(\nabla g)(x-z) .
\end{aligned}
$$

We further multiply both sides of (A.14) by $g(x-z)$ and use $F(x, z, t)=\delta(f \varphi)(x, z) g(x-z)$ and the above identity to yield
$\partial_{t} F+\left(b(x) \cdot \nabla_{x}+b(z) \cdot \nabla_{z}\right) F=\left(d(x)+\frac{(b(x)-b(z)) \cdot(\nabla g)(x-z)}{g(x-z)}\right) F+((d(x)-d(z))(f \varphi)(z)+\delta(R \varphi)) g(x-z)$,
which concludes the proof of (2.36).

## Appendix B. Proof of Sharp Hölder estimates

In this Appendix, we prove the sharp Hölder estimates in Section 3 and derive the explicit upper bounds given by some explicit integrals. We have proved Lemmas 3.1, 3.3 in Section 3 , In Appendix B.3, we provide some explicit formulas for the functions and the transportation maps for these upper bounds. In Section 5 of the supplementary material II in Part II [16, we will estimate these explicit integrals using some integral formulas and numerical quadrature with computer assistance, and obtain rigorous upper bounds. The codes can be found in [13].
B.1. $C_{x}^{1 / 2}$ estimates of $v_{x}$ and $u_{y}$. We follow the ideas and argument in Section 3.2 to estimate the Hölder seminorm of $u_{y}, v_{x}$. Recall the kernel $K_{2}=\frac{1}{2} \frac{y_{1}^{2}-y_{2}^{2}}{|y|^{4}}$ for $u_{y}, v_{x}$. Firstly, we need the following Lemma for the principle value of the integral.
Lemma B.1. Suppose that $f \in L^{\infty}$, is Hölder continuous near 0. For $0<a, b<\infty$ and $Q=[0, a] \times[0, b],[0, a] \times[-b, 0],[-a, 0] \times[0, b]$, or $[-a, 0] \times[-b, 0]$, we have
P.V. $\int_{Q} K_{2}(y) f(y) d y=\lim _{\varepsilon \rightarrow 0} \int_{Q \cap\left|y_{1}\right| \geq \varepsilon} K_{2}(y) f(y) d y-\frac{\pi}{8} f(0)=\lim _{\varepsilon \rightarrow 0} \int_{Q \cap\left|y_{2}\right| \geq \varepsilon} K_{2}(y) f(y) d y+\frac{\pi}{8} f(0)$.

In the strip $\left|y_{1}\right| \leq \varepsilon, K_{2}(y)<0$ if $\left|y_{2}\right|>\varepsilon$. It contributes to $-\frac{\pi}{8} f(0)$ in the first identity. In the strip $\left|y_{2}\right| \leq \varepsilon, K_{2}(y)>0$ if $\left|y_{1}\right|>\varepsilon$. It contributes to $\frac{\pi}{8} f(0)$ in the second identity.

Proof. Since $K_{2}$ is even in $y_{1}, y_{2}$, we focus on $Q=[0, a] \times[0, b]$ without loss of generality. By definition, we have

$$
\text { P.V. } \int_{Q} K_{2}(y) f(y) d y=\lim _{\varepsilon \rightarrow 0}\left(\int_{\varepsilon}^{a} \int_{0}^{b}+\int_{0}^{\varepsilon} \int_{\varepsilon}^{b}\right) K_{2}(y) f(y) d y \triangleq \lim _{\varepsilon \rightarrow 0}\left(I_{\varepsilon}+I I_{\varepsilon}\right)
$$

We just need to compute $I I_{\varepsilon}$. Since $f$ is Hölder continuous near 0 , we get

$$
\lim _{\varepsilon \rightarrow 0} \int_{0}^{\varepsilon} \int_{\varepsilon}^{b} K_{2}(y)(f(y)-f(0)) d y=0
$$

The first identity follows from

$$
\begin{aligned}
\lim _{\varepsilon \rightarrow 0} \int_{0}^{\varepsilon} \int_{\varepsilon}^{b} K_{2}(y) f(0) d y & =\left.f(0) \lim _{\varepsilon \rightarrow 0} \int_{0}^{\varepsilon} \frac{1}{2} \frac{y_{2}}{y_{2}^{2}+y_{1}^{2}}\right|_{\varepsilon} ^{b} d y_{1} \\
& =\frac{f(0)}{2} \lim _{\varepsilon \rightarrow 0} \int_{0}^{\varepsilon}\left(\frac{b}{b^{2}+y_{1}^{2}}-\frac{\varepsilon}{\varepsilon^{2}+y_{1}^{2}}\right) d y_{1}=-\frac{\pi}{8} f(0)
\end{aligned}
$$

The second identity follows from the same argument.
Next, we perform the sharp Hölder estimates for $u_{y}, v_{x}$. Without loss of generality, we assume $z_{1}=-1, x_{1}=1$ and $z_{2}=x_{2}>0$. Due to the boundary, we do not have translation symmetry of the kernel $K_{2}(y)$ in $y_{2}$ and cannot assume $x_{2}=0$. We are going to estimate
(B.1) $v_{x}(z)-v_{x}(x)=\frac{1}{\pi}\left(\int\left(K_{2, B}\left(y_{1}+1, y_{2}\right)-K_{2, B}\left(y_{1}-1, y_{2}\right)\right) W\left(y_{1}, x_{2}-y_{2}\right) d y-\frac{1}{2}(\omega(z)-\omega(x))\right.$,
where $K_{2, B}(y)=K_{2}(y) \mathbf{1}_{\left|y_{1}\right| \leq B,\left|y_{2}\right| \leq B}$ is the localized version of $K_{2}$ over $[-B, B]^{2}$, and $W$ is the odd extension of $\omega$ from $\mathbb{R}_{+}^{2}$ to $\mathbb{R}^{2}$ (3.3). Denote
(B.2) $A=\min \left(B, x_{2}\right), \quad K^{+} \triangleq K_{2, B}\left(y_{1}+1, y_{2}\right), \quad K_{-} \triangleq K_{2, B}\left(y_{1}-1, y_{2}\right), \quad \Delta(y)=K^{+}-K^{-}$.

We focus on $B \geq 4$. It is easy to see that $\Delta$ is odd in $y_{1}$. Since the transportation cost in the $y$ direction is cheaper (we will choose $\tau<1$ in Lemma 3.4 to capture the property that $[\omega]_{C_{y}^{1 / 2}}$ enjoys better energy estimate than $[\omega]_{C_{x}^{1 / 2}}$ ), we shall use the $Y$-transportation as much as possible to obtain a sharp estimate. Due to the presence of the boundary and the discontinuity of $W$ across the boundary, we partition the domain into the inner part and the outer part

$$
\Omega_{i n} \triangleq\left\{y_{2} \in[-A, A]\right\}, \quad \Omega_{o u t} \triangleq\left\{y_{2} \notin[-A, A]\right\}
$$

Then we have $\omega\left(\cdot, x_{2}-\cdot\right) \in C^{1 / 2}\left(\Omega_{i n}\right)$. We add the parameter $A$ in these domains due to the localization of the kernel. Define

$$
\begin{equation*}
\Delta_{1 D}\left(y_{1}\right)=\int_{-A}^{A} \Delta\left(y_{1}, y_{2}\right) d y_{2} \tag{B.3}
\end{equation*}
$$

Remark that for a fixed $y_{1}, \Delta\left(y_{1}, y_{2}\right)$ may not have a fixed sign over $y_{2}$.
Denote the vertical line $(v l)$ and the horizontal line $(h l)$

$$
v l_{y_{1}} \triangleq\left\{\left(y_{1}, y_{2}\right): y_{2} \in \mathbb{R}\right\}, \quad h l_{x_{2}} \triangleq\left\{\left(y_{1}, x_{2}\right): y_{1} \in \mathbb{R}\right\}
$$

The estimates consist of three steps. In the first two steps, we estimate the integral in $\Omega_{i n}$. In the first step, we fix $y_{1}$ and consider the 1D transportation problem on the vertical line $v l_{y_{1}}$ by moving the positive part of $\Delta$ to its negative part. If $\left|y_{1}\right| \leq 9$, we move the remaining part with total mass $\Delta_{1 D}$ to the horizontal line $h l_{x_{2}}$. In this step, the estimate is bounded by $C[\omega]_{C_{y}^{1 / 2}}$. See the blue arrows and the bluw line in the left figure in Figure 13 for an illustration of the moving direction on $v l_{y_{1}}$.

In the second step, we study the transportation problem on $h l_{x_{2}}$. We also move the remaining part with total mass $\Delta_{1 D}\left(y_{1}\right)$ for $\left|y_{1}\right| \geq 9$ in the first step horizontally. The estimate will be bounded by $C[\omega]_{C_{x}^{1 / 2}}$ for some constant $C$. In the third step, we estimate the integral in the outer domain $\Omega_{\text {out }}$. The estimate will be bounded by $C[\omega]_{C_{y}^{1 / 2}}$ for some constant $C$.

We focus on $\left|y_{2}-x_{2}\right| \leq B$ since otherwise $\Delta=0$. We assume $x_{2}>0$. The case $x_{2}=0$ can be obtained by taking limit $x_{2} \rightarrow 0$.
B.1.1. Sign of $\Delta$ and $\Delta_{1 D}$. Due to the odd symmetry of $\Delta\left(y_{1}, y_{2}\right)$ in $y_{1}$, we focus on $y_{1} \geq 0$. Solving $K_{2}\left(y_{1}+1, y_{2}\right)-K_{2}\left(y_{1}-1, y_{2}\right)=0$, we get $y_{2}=s_{c}\left(y_{1}\right)$ (B.56). It is easy to show that

$$
\begin{align*}
& K_{2}\left(y_{1}+1, y_{2}\right)-K_{2}\left(y_{1}-1, y_{2}\right) \geq 0,\left|y_{2}\right| \geq s_{c}\left(y_{1}\right) \\
& K_{2}\left(y_{1}+1, y_{2}\right)-K_{2}\left(y_{1}-1, y_{2}\right) \leq 0,\left|y_{2}\right| \leq s_{c}\left(y_{1}\right) \tag{B.4}
\end{align*}
$$

See the left subplot in Figure 13 for an illustration of sign of $\Delta(y)$ in different regions. The black curve represents $y_{2}=s_{c}\left(y_{1}\right)$. For $\left|y_{1}\right| \leq B-1$, we get

$$
\begin{equation*}
\Delta(y)=K_{2}\left(y_{1}+1, y_{2}\right)-K_{2}\left(y_{1}-1, y_{2}\right) \triangleq \Delta_{\text {all }}(y) \tag{B.5}
\end{equation*}
$$



Figure 13. Illustration of the sign of the kernel $\Delta(y)$ and transportation plan. The sign of $\Delta(y)$ in different regions is indicated by $\pm$. The blue arrows indicate the direction of 1D transportation plan. Left for $C_{x}^{1 / 2}$ estimate: The black curve and the red curve represent $y_{2}= \pm s_{c}\left(y_{1}\right), y_{2}= \pm T\left(y_{1}, A\right)$ for $y_{1} \geq 0$, respectively. Right for $C_{y}^{1 / 2}$ estimate: The black curve is for $y_{2}= \pm s_{c}\left(y_{1}\right)$, or equivalently $y_{1}=h_{c}^{-}\left(\left|y_{2}\right|\right)$ (two left black curves) and $y_{1}=h_{c}^{+}\left(y_{2}\right)$ (two right black curves). The red curve represents $y_{1}=T\left(m,\left|y_{2}\right|\right)$. Note that these curves do not agree with the actual functions.

The sign of $\Delta\left(y_{1}, \cdot\right)$ is given above. For $\left|y_{1}\right| \in[B-1, B+1]$, we have

$$
\begin{equation*}
\Delta(y)=-K_{2}\left(y_{1}-1, y_{2}\right)=-\frac{1}{2} \frac{\left(y_{1}-1\right)^{2}-y_{2}^{2}}{\left(\left(y_{1}-1\right)^{2}+y_{2}^{2}\right)^{2}} . \tag{B.6}
\end{equation*}
$$

Since $B-1 \geq 1$, it satisfies

$$
\begin{equation*}
\Delta(y) \geq 0,\left|y_{2}\right| \geq s_{c}\left(y_{1}\right) \triangleq\left|y_{1}\right|+1, \quad \Delta(y) \leq 0,\left|y_{2}\right| \leq s_{c}\left(y_{1}\right)=\left|y_{1}\right|+1 \tag{B.7}
\end{equation*}
$$

For $y_{1} \geq B+1$, we have $\Delta(y)=0$. Next, we compute $\Delta_{1 D}$ defined in (B.3). Since $\Delta$ is singular at $y=( \pm 1,0)$ and $B>2$, the singularity is in $J_{1}$

$$
\begin{equation*}
J_{1} \triangleq[-9,9], \quad J_{1}^{+} \triangleq J_{1} \cap \mathbb{R}_{+} \tag{B.8}
\end{equation*}
$$

In the inner part, we have
$S_{\text {in }} \triangleq \int_{\Omega_{i n}} \Delta\left(y_{1}, y_{2}\right) W\left(y_{1}, x_{2}-y_{2}\right) d y=\left(\int_{y_{1} \in J_{1}}+\int_{y_{1} \notin J_{1}}\right) \Delta\left(y_{1}, y_{2}\right) W\left(y_{1}, x_{2}-y_{2}\right) d y \triangleq S_{1}+S_{2}$.
By definition, we yield (B.10)
$S_{1}=\int_{y_{1} \in J_{1}} \int_{-A}^{A} \Delta\left(y_{1}, y_{2}\right)\left(W\left(y_{1}, x_{2}-y_{2}\right)-W\left(y_{1}, x_{2}\right)\right) d y+\int_{y_{1} \in J_{1}} \int_{-A}^{A} \Delta\left(y_{1}, y_{2}\right) W\left(y_{1}, x_{2}\right) d y \triangleq S_{11}+S_{12}$.
For $S_{11}$, since $\left|W\left(y_{1}, x_{2}-y_{2}\right)-W\left(y_{1}, x_{2}\right)\right| \lesssim y_{2}^{1 / 2}$, the integrand is locally integrable. We will estimate $S_{11}$ and $S_{2}$ in Section B.1.2.

We should pay attention to the principle value in the singular integral in $S_{12}$ near the singularity $( \pm 1,0)$. Since $\Delta(y)=K_{2}\left(y_{1}+1, y_{2}\right)-K_{2}\left(y_{1}-1, y_{2}\right)$ near $y_{1}=1$, applying Lemma B. 1 four times to $-K_{2}\left(y_{1}-1, y_{2}\right)$, which leads to $4 \cdot(-1) \cdot\left(-\frac{\pi}{8}\right) W\left(1, x_{2}\right)=\frac{\pi}{2} W\left(1, x_{2}\right)$, we yield

$$
\begin{align*}
S_{12}^{+} & \triangleq \int_{y_{1} \in J_{1}^{+}} \Delta\left(y_{1}, y_{2}\right) W\left(y_{1}, x_{2}\right) d y  \tag{B.11}\\
& =\frac{\pi}{2} W\left(1, x_{2}\right)+\lim _{\varepsilon \rightarrow 0} \int_{y_{1} \in J_{1}^{+} \backslash[1-\varepsilon, 1+\varepsilon]} \Delta\left(y_{1}, y_{2}\right) W\left(y_{1}, x_{2}\right) d y
\end{align*}
$$

Recall the definition of $\Delta$ from (3.1), (B.5), (B.6). Denote

$$
\begin{equation*}
g_{b}(y)=\frac{b}{y^{2}+b^{2}}, \quad \Delta_{1 D}\left(y_{1}\right)=\frac{A}{\left(y_{1}+1\right)^{2}+A^{2}} \mathbf{1}_{\left|y_{1}+1\right| \leq B}-\frac{A}{\left(y_{1}-1\right)^{2}+A^{2}} \mathbf{1}_{\left|y_{1}-1\right| \leq B} \tag{B.12}
\end{equation*}
$$

Recall $\Delta$ and $A \leq B$ from (B.2). For $y_{1} \notin[1-\varepsilon, 1+\varepsilon],\left|y_{2}\right| \leq A$, we have $K_{2, B}\left(y_{1}, y_{2}\right)=$ $K_{2}(y) \mathbf{1}_{\left|y_{1}\right| \leq B}, K_{2}(y)=\frac{1}{2} \partial_{y_{2}} \frac{y_{2}}{|y|^{2}}$ and

$$
\begin{aligned}
\int_{-A}^{A} \Delta\left(y_{1}, y_{2}\right) d y_{2} & =\left.\frac{1}{2}\left(\frac{y_{2}}{\left(y_{1}+1\right)^{2}+y_{2}^{2}} \mathbf{1}_{\left|y_{1}+1\right| \leq B}-\frac{y_{2}}{\left(y_{1}-1\right)^{2}+y_{2}^{2}} \mathbf{1}_{\left|y_{1}-1\right| \leq B}\right)\right|_{-A} ^{A} \\
& =g_{A}\left(y_{1}+1\right) \mathbf{1}_{\left|y_{1}+1\right| \leq B}-g_{A}\left(y_{1}-1\right) \mathbf{1}_{\left|y_{1}-1\right| \leq B}=\Delta_{1 D}\left(y_{1}\right)
\end{aligned}
$$

Plugging the above computation to the P.V. integral yields

$$
S_{12}^{+}=\int_{J_{1}^{+}} \Delta_{1 D}\left(y_{1}\right) W\left(y_{1}, x_{2}\right) d y_{1}+\frac{\pi}{2} \omega\left(1, x_{2}\right)
$$

The computation of the integral over $\mathbb{R}_{-}$is similar due to symmetry. We yield

$$
\begin{equation*}
S_{12}=\int_{J_{1}} \Delta_{1 D}\left(y_{1}\right) W\left(y_{1}, x_{2}\right) d y_{1}+\frac{\pi}{2}\left(\omega\left(1, x_{2}\right)-\omega\left(-1, x_{2}\right)\right) \tag{B.13}
\end{equation*}
$$

B.1.2. First step. We are in a position to estimate $S_{2}$ (B.9) and $S_{11}$ (B.10). Recall the sign of $\Delta$ from (B.4), (B.7)

$$
\Delta(y) \geq 0,\left|y_{2}\right| \geq s_{c}\left(y_{1}\right), \quad \Delta(y) \leq 0,\left|y_{2}\right| \leq s_{c}\left(y_{1}\right)
$$

Since $\Delta$ is even in $y_{2}$ in $\Omega_{i n}$ and odd in $y_{1}$, we focus on the first quadrant.
For a fixed $y_{1} \geq 0$, we transport the positive part of $\Delta$ to its negative part on the line $v l_{y_{1}}$ in the first quadrant. We construct the transportation map $T(y)>0$ by solving

$$
\int_{T(y)}^{y_{2}} \Delta\left(y_{1}, s_{2}\right) d s_{2}=0
$$

For $y_{1} \leq B-1, \Delta=K_{2}\left(y_{1}+1, y_{2}\right)-K_{2}\left(y_{1}-1, y_{2}\right)$. The map $T$ can be obtained from the cubic equation (B.60). For $y_{1} \in[B-1, B+1], \Delta=-K_{2}\left(y_{1}-1, y_{2}\right)$ and we get

$$
\begin{equation*}
0=\int_{T(y)}^{y_{2}} K_{2}\left(y_{1}-1, s_{2}\right) d s_{2}=\left.\frac{1}{2} \frac{s_{2}}{\left(y_{1}-1\right)^{2}+s_{2}^{2}}\right|_{T} ^{y_{2}}, \quad T(y)=\frac{\left(y_{1}-1\right)^{2}}{y_{2}} \tag{B.14}
\end{equation*}
$$

Denote

$$
\tilde{W}(y)=W\left(y_{1}, x_{2}-y_{2}\right)-W\left(y_{1}, x_{2}\right)
$$

Using the above map, the estimates below,

$$
\begin{align*}
& \left|\tilde{W}\left(y_{1}, y_{2}\right)-\tilde{W}\left(y_{1}, T(y)\right)\right|=\left|W\left(y_{1}, y_{2}\right)-W\left(y_{1}, T(y)\right)\right| \leq\left|y_{2}-T(y)\right|^{1 / 2}[\omega]_{C_{y}^{1 / 2}}  \tag{B.15}\\
& \left|\tilde{W}\left(y_{1}, y_{2}\right)\right| \leq\left|y_{2}\right|^{1 / 2}[\omega]_{C_{y}^{1 / 2}}
\end{align*}
$$

and applying Lemma 3.6 to the integral on $\left[T\left(y_{1}, A\right), A\right]$, we yield

$$
\begin{align*}
& \left|\int_{0}^{A} \Delta(y) \tilde{W}(y) d y_{2}\right|=\left|\left(\int_{T\left(y_{1}, A\right)}^{A}+\int_{0}^{T\left(y_{1}, A\right)}\right) \Delta(y) \tilde{W}(y) d y_{2}\right| \\
\leq & \left(\int_{s_{c}\left(y_{1}\right)}^{A}\left|\Delta(y) \| y_{2}-T(y)\right|^{1 / 2} d y_{2}+\int_{0}^{T\left(y_{1}, A\right)}|\Delta(y)|\left|y_{2}\right|^{1 / 2} d y_{2}\right)[\omega]_{C_{y}^{1 / 2}} \tag{B.16}
\end{align*}
$$

See the blue arrows in the left subplot in Figure 13 for an illustration of this transportation plan.

Due to the symmetry of $\Delta$ in $y_{1}, y_{2}$, we can estimate $S_{11}$ (B.10) as follows
(B.17) $S_{11} \leq 4 \int_{y_{1} \in J_{1}^{+}}\left(\int_{s_{c}\left(y_{1}\right)}^{A}\left|y_{2}-T(y)\right|^{1 / 2}|\Delta(y)| d y_{2}+\int_{0}^{T\left(y_{1}, A\right)}\left|y_{2}\right|^{1 / 2}|\Delta(y)| d y_{2}\right) d y_{1}[\omega]_{C_{y}^{1 / 2}}$,
where $J_{1}$ is defined in $(\overline{\mathrm{B} .8})$ and the factor 4 is due to the fact that we have 4 quadrants.

The estimate of $S_{2}(\overline{\mathrm{~B} .9})$ is similar except that we do not further transport the remaining negative part of $\Delta$ to the location $\left(y_{1}, x_{2}\right)$

$$
\begin{equation*}
S_{2}=\int_{y_{1} \notin J_{1}}\left(\int_{T\left(y_{1}, A\right) \leq\left|y_{2}\right| \leq A}+\int_{\left|y_{2}\right| \leq T\left(y_{1}, A\right)}\right) \Delta(y) W\left(y_{1}, x_{2}-y_{2}\right) d y \triangleq I+I I \tag{B.18}
\end{equation*}
$$

For $I$, we obtain
$|I| \leq 2 \int_{y_{1} \notin J_{1}} \int_{s_{c}\left(y_{1}\right)}^{A}\left|T(y)-y_{2}\right|^{1 / 2}|\Delta(y)| d y[\omega]_{C_{y}^{1 / 2}}=4 \int_{y_{1} \notin J_{1}, y_{1} \geq 0} \int_{s_{c}\left(y_{1}\right)}^{A}\left|T(y)-y_{2}\right|^{1 / 2}|\Delta(y)| d y[\omega]_{C_{y}^{1 / 2}}$
For $I I$, we use the odd symmetry of $\Delta\left(y_{1}, y_{2}\right)$ in $y_{1}$ to get

$$
\begin{align*}
|I I| & \leq\left|\int_{y_{1} \notin J_{1}, y_{1} \geq 0} \int_{\left|y_{2}\right| \leq T\left(y_{1}, A\right)} \Delta(y)\left(W(y)-W\left(-y_{1}, y_{2}\right)\right)\right|  \tag{B.20}\\
& \leq \int_{y_{1} \notin J_{1}, y_{1} \geq 0} \int_{\left|y_{2}\right| \leq T\left(y_{1}, A\right)} \sqrt{2 y_{1}|\Delta(y)| d y[\omega]_{C_{x}^{1 / 2}}=\int_{y_{1} \notin J_{1}, y_{1} \geq 0} \sqrt{2 y_{1} \mid} \Delta_{1 D}\left(y_{1}\right) \mid d y_{1}[\omega]_{C_{x}^{1 / 2}},}
\end{align*}
$$

where we have used $\int_{T\left(y_{1}, A\right)}^{A} \Delta(y) d y_{2}=0, \Delta(y) \leq 0$ for $\left|y_{2}\right| \leq T\left(y_{1}, A\right) \leq s_{c}\left(y_{1}\right)$ (B.4), (B.7), and

$$
\begin{equation*}
\int_{\left|y_{2}\right| \leq T\left(y_{1}, A\right)}|\Delta(y)| d y_{2}=\int_{\left|y_{2}\right| \leq T\left(y_{1}, A\right)}-\Delta(y) d y_{2}=\int_{\left|y_{2}\right| \leq A}-\Delta(y) d y_{2}=-\Delta_{1 D}\left(y_{1}\right) \tag{B.21}
\end{equation*}
$$

B.1.3. Second step: Estimate $S_{12}$. We combine the estimate of $S_{12}(\overline{\mathrm{~B} .13)}$ and the local part of $v_{x}$, e.g. $-\frac{\pi}{2}\left(\omega(z)-\omega(x)\right.$ B.1). For $v_{x}$, since $\omega(z)-\omega(x)=-\omega\left(1, x_{2}\right)+\omega\left(-1, x_{2}\right)$, we obtain

$$
\begin{equation*}
I \triangleq S_{12}-\frac{\pi}{2}(\omega(z)-\omega(x))=\int_{J_{1}} \Delta_{1 D}\left(y_{1}\right) W\left(y_{1}, x_{2}\right)+\pi\left(\omega\left(1, x_{2}\right)-\omega\left(-1, x_{2}\right)\right) \tag{B.22}
\end{equation*}
$$

Recall the definition of $\Delta_{1 D}\left(y_{1}\right)$ (B.12). Clearly, $\Delta_{1 D}$ is odd and $\Delta_{1 D}<0$ for $y_{1}>0$. Note that for $k \in[0,9]$, we have

$$
\begin{equation*}
P(k) \triangleq \int_{k}^{9} \Delta_{1 D} d y_{1} \geq \int_{\mathbb{R}_{+}} \Delta_{1 D} d y_{1}=-\int_{-1}^{1} \frac{A}{y_{1}^{2}+A^{2}} d y_{1}=-2 \arctan \left(\frac{1}{A}\right) \geq-\pi \tag{B.23}
\end{equation*}
$$

We transport all the negative part of $\Delta_{1 D}$ on $[1 / 9,9]$ to 1 . Similarly, we transport all the positive part of $\Delta_{1 D}$ on $(-9,-1 / 9]$ to $(-1)$. For $y_{1} \in[-1 / 9,0] \cup[0,1 / 9]$, we move $y_{1}$ to $-y_{1}$. We do not move these parts to $\pm 1$ since $\sqrt{2\left|y_{1}\right|}+\sqrt{2} \leq 2\left|y_{1}-1\right|^{1 / 2}$ for $y_{1} \leq 1 / 9$. Denote $J_{2}=[1 / 9,9] \subset J_{1}$. We derive the following estimate

$$
\begin{aligned}
|I|= & \mid \int_{J_{2} \cap \mathbb{R}_{+}} \Delta_{1 D}\left(y_{1}\right)\left(W\left(y_{1}, x_{2}\right)-W\left(1, x_{2}\right)\right) d y_{1}+\int_{\left(-J_{2}\right) \cap \mathbb{R}_{-}} \Delta_{1 D}\left(y_{1}\right)\left(W\left(y_{1}, x_{2}\right)-W\left(-1, x_{2}\right)\right) d y_{1} \\
& \left.+\int_{0}^{1 / 9} \Delta_{1 D}\left(y_{1}\right)\left(W\left(y_{1}, x_{2}\right)-W\left(-y_{1}, x_{2}\right)\right)+\left(\pi+P\left(\frac{1}{9}\right)\right)\left(W\left(1, x_{2}\right)-W\left(-1, x_{2}\right)\right) \right\rvert\, \\
\leq & \left(2 \int_{1 / 9}^{9}\left|\Delta_{1 D}\right| y_{1}-\left.1\right|^{1 / 2} d y_{1}+\int_{0}^{1 / 9}\left|\Delta_{1 D}\right| \sqrt{2 y_{1}} d y_{1}+\left(\pi+P\left(\frac{1}{9}\right)\right) \sqrt{2}\right)[\omega]_{C_{x}^{1 / 2}}, \quad J_{2}=[1 / 9,9]
\end{aligned}
$$

where we have used the symmetry of $\Delta_{1 D}$ to get the factor 2 .
Remark B.2. The reason why we do not further transport the negative part in $I I$ in $S_{2}$ (B.18) to ( $y_{1}, x_{2}$ ) is the following. The integral in $S_{2}, S_{12}$ that remains to estimate is similar to

$$
M=\int\left(-\delta_{z}+\delta_{\left(1, x_{2}\right)}+\delta_{\left(-z_{1}, z_{2}\right)}-\delta_{\left(-1, x_{2}\right)}\right) f(y) d y, \quad f(y)=W\left(y_{1}, x_{2}-y_{2}\right)
$$

for some $z_{1} \geq 9,\left|z_{2}\right| \leq T\left(z_{1}, A\right)$. If we do so, we will obtain the following estimate

$$
\begin{aligned}
M & \leq\left|f(z)-f\left(z_{1}, x_{2}\right)+f\left(z_{1}, x_{2}\right)-f\left(1, x_{2}\right)\right|+\left|f\left(-z_{1}, z_{2}\right)-f\left(-z_{1}, x_{2}\right)+f\left(-z_{1}, x_{2}\right)-f\left(-1, x_{2}\right)\right| \\
& \leq 2\left|z_{2}-x_{2}\right|^{1 / 2}[f]_{C_{y}^{1 / 2}}+2\left|z_{1}-1\right|^{1 / 2}[f]_{C_{x}^{1 / 2}} \triangleq M_{1}
\end{aligned}
$$

We have another simple estimate without using $[f]_{C_{y}^{1 / 2}}$

$$
M \leq\left|f(z)-f\left(-z_{1}, z_{2}\right)\right|+\left|f\left(1, x_{2}\right)-f\left(-1, x_{2}\right)\right| \leq[f]_{C_{x}^{1 / 2}}\left(\left|2 z_{1}\right|^{1 / 2}+\sqrt{2}\right) \triangleq M_{2}
$$

For $z_{1} \geq 9$ or $z_{1} \in[0,1 / 9]$, we get $2\left|z_{1}-1\right|^{1 / 2} \geq \sqrt{2}+\sqrt{2 z_{1}}$ with equality for $z_{1}=9,1 / 9$. Thus, both the $x$ and $y$-transportation costs in the first estimate are larger. We use the second estimate in the above estimates of $S_{12}, S_{2}$. This estimate also motivatives the choice of $J_{1}$ (B.13).
B.1.4. Third step. It remains to estimate the integral in the outer part. If $B<x_{2}$, since $\Delta\left(y_{1}, y_{2}\right)$ is localized to $\left|y_{2}\right| \leq B=A$, the contribution from outer part $\left|y_{2}\right|>B$ is 0 . If $B>x_{2}=A$, using the odd symmetry of $W(y)$ in $y_{2}$ (3.3) and the even symmetry of $\Delta$ in $y_{2}$, we yields

$$
\begin{aligned}
S_{\text {out }} & =\int_{\Omega_{\text {out }}} \Delta(y) W\left(y_{1}, x_{2}-y_{2}\right) d y=\int_{\mathbb{R}} d y_{1}\left(\int_{x_{2}}^{B}+\int_{-B}^{-x_{2}}\right) \Delta(y) W\left(y_{1}, x_{2}-y_{2}\right) d y_{2} \\
& =-\int_{\mathbb{R}} d y_{1} \int_{x_{2}}^{B} \Delta(y) \omega\left(y_{1}, y_{2}-x_{2}\right) d y_{2}+\int_{\mathbb{R}} d y_{1} \int_{x_{2}}^{B} \Delta(y) \omega\left(y_{1}, x_{2}+y_{2}\right) d y_{2}
\end{aligned}
$$

It follows

$$
\begin{equation*}
\left|S_{\text {out }}\right| \leq \int_{\mathbb{R}} \int_{x_{2}}^{B}|\Delta(y)| \cdot\left|\omega\left(y_{1}, x_{2}+y_{2}\right)-\omega\left(y_{1}, y_{2}-x_{2}\right)\right| d y \leq \sqrt{2 x_{2}} \int_{\mathbb{R}} \int_{x_{2}}^{B}|\Delta(y)| d y[\omega]_{C_{y}^{1 / 2}} \tag{B.25}
\end{equation*}
$$

B.1.5. $C_{x}^{1 / 2}$ Estimate of $u_{y}$. The estimates of $u_{y}$ in step 1 and 2 are similar to that of $v_{x}$ except that we do not transport the remaining negative part of $\Delta\left(y_{1}, y_{2}\right)$ with $\left|y_{2}\right| \leq T\left(y_{1}, A\right)$ to $\left(y_{1}, x_{2}\right)$ for any $y_{1}>0$. See Remark B.3. The estimate of the outer part in the third step is the same as that of $v_{x}$ in Section B.1.4.

Denote $J_{\varepsilon}=[-1-\varepsilon,-1+\varepsilon] \cup[1-\varepsilon, 1+\varepsilon]$. Note that $\Delta=K_{2}\left(y_{1}+1, y_{2}\right)-K_{2}\left(y_{1}-1, y_{2}\right)$ has singularities at $( \pm 1,0)$. Applying Lemma B. 1 four times to $K_{2}\left(y_{1}+1, y_{2}\right)$ and $-K_{2}\left(y_{1}-1, y_{2}\right)$, respectively, we can rewrite $S_{i n}(\overline{\mathrm{~B} .9})$ as follows

$$
\begin{align*}
& S_{\text {in }} \triangleq \int_{\mathbb{R}} \int_{-A}^{A} \Delta(y) W\left(y_{1}, x_{2}-y_{2}\right) d y \\
& =\frac{\pi}{2}\left(\omega\left(1, x_{2}\right)-\omega\left(-1, x_{2}\right)\right)+\lim _{\varepsilon \rightarrow 0} \int_{J_{\varepsilon}^{c}} d y_{1} \int_{-A}^{A} \Delta(y) W\left(y_{1}, x_{2}-y_{2}\right) d y_{2} \triangleq I+\lim _{\varepsilon \rightarrow 0} I I_{\varepsilon} \tag{B.26}
\end{align*}
$$

For $y_{1} \notin J_{\varepsilon}$, we perform a decomposition

$$
\begin{align*}
f\left(y_{1}\right) & \triangleq \int_{-A}^{A} \Delta(y) W\left(y_{1}, x_{2}-y_{2}\right) d y_{2}=\left(\int_{T\left(y_{1}, A\right)<\left|y_{2}\right| \leq A}+\int_{\left|y_{2}\right| \leq T\left(y_{1}, A\right)}\right) \Delta(y) W\left(y_{1}, x_{2}-y_{2}\right) d y_{2}  \tag{B.27}\\
& \triangleq f_{1}\left(y_{1}\right)+f_{2}\left(y_{1}\right)
\end{align*}
$$

For $f_{1}\left(y_{1}\right)$, we estimate it using Lemma 3.6

$$
\begin{equation*}
\left|f_{1}\left(y_{1}\right)\right| \leq 2 \int_{s_{c}\left(y_{1}\right)}^{A}|\Delta(y)|\left|y_{2}-T(y)\right|^{1 / 2} d y[\omega]_{C_{y}^{1 / 2}} \tag{B.28}
\end{equation*}
$$

The part $f_{2}\left(y_{1}\right)$ denotes the purely negative part. If $T\left(y_{1}, A\right)<A$, we get $T\left(y_{1}, A\right)<s_{c}\left(y_{1}\right)<$ $A, \Delta(y)<0$, and (B.21). Using this estimate and the fact that $\Delta$ is odd in $y_{1}$, we get

$$
\begin{align*}
\left|f_{2}\left(y_{1}\right)+f_{2}\left(-y_{1}\right)\right| & =\left|\int_{\left|y_{2}\right| \leq T\left(y_{1}, A\right)} \Delta(y)\left(W\left(y_{1}, x_{2}-y_{2}\right)-W\left(-y_{1}, x_{2}-y_{2}\right)\right) d y_{2}\right|  \tag{B.29}\\
& \leq \int_{\left|y_{2}\right| \leq T\left(y_{1}, A\right)} \sqrt{2 y_{1}}|\Delta(y)| d y_{2}[\omega]_{C_{x}^{12}}=\left|\Delta_{1 D}\left(y_{1}\right)\right| \sqrt{2 y_{1}}[\omega]_{C_{x}^{1 / 2}}
\end{align*}
$$

Integrating (B.28), (B.29) over $y_{1} \notin J_{\varepsilon}$, we establish

$$
\left|I I_{\varepsilon}\right| \leq 4 \int_{J_{\varepsilon}^{c} \cap \mathbb{R}_{+}} \int_{s_{c}\left(y_{1}\right)}^{A}|\Delta(y)|\left|y_{2}-T\right|^{1 / 2} d y[\omega]_{C_{y}^{1 / 2}}+\int_{J_{\varepsilon}^{c} \cap \mathbb{R}_{+}}\left|\Delta_{1 D}\left(y_{1}\right)\right| \sqrt{2 y_{1}}[\omega]_{C_{x}^{1 / 2}}
$$

Near the singularity $(1,0)$ of $\Delta(y),\left|y_{2}-T\right|^{1 / 2} \lesssim\left|y_{2}\right|^{1 / 2}+\left|y_{1}-1\right|^{1 / 2}$. Thus, the integrand in the first integral is locally integrable. Plugging the the above estimate in ( $\overline{\mathrm{B} .26}$ ), we derive

$$
\begin{align*}
& \left|S_{i n}-\frac{\pi}{2}\left(\omega\left(1, x_{2}\right)-\omega\left(-1, x_{2}\right)\right)\right| \leq \lim \sup _{\varepsilon \rightarrow 0}\left|I-\frac{\pi}{2}\left(\omega\left(1, x_{2}\right)-\omega\left(-1, x_{2}\right)\right)\right|+I I_{\varepsilon} \\
\leq & 4 \int_{\mathbb{R}_{+}} \int_{s_{c}\left(y_{1}\right)}^{A}|\Delta(y)|\left|y_{2}-T\right|^{1 / 2} d y[\omega]_{C_{y}^{1 / 2}}+\int_{\mathbb{R}_{+}}\left|\Delta_{1 D}\left(y_{1}\right)\right| \sqrt{2 y_{1}}[\omega]_{C_{x}^{1 / 2}} . \tag{B.30}
\end{align*}
$$

Recall the definition of localized $u_{y}$ (3.4). The term $\frac{\pi}{2}\left(\omega\left(1, x_{2}\right)-\omega\left(-1, x_{2}\right)\right)$ in $S_{i n}$ cancel the local term $\frac{\pi}{2}(\omega(z)-\omega(x))$ in $u_{y}(z)-u_{y}(x)$. Combining the above estimate and the estimate of $S_{\text {out }}$ in (B.25), we prove the estimate of $u_{y}$.
Remark B.3. We do not further transport the remaining negative part $\Delta\left(y_{1}, y_{2}\right)$ for $\left|y_{2}\right| \leq$ $T\left(y_{1}, A\right)$ to ( $y_{1}, x_{2}$ ) on the line $h l_{x_{2}}$ since the remaining integrals in $S_{2}, S_{i n}$ are similar to

$$
\int\left(\delta_{y}-\delta_{\left(-y_{1}, y_{2}\right)}\right) W\left(z_{1}, x_{2}-z_{2}\right) d y, \quad\left|y_{2}\right| \leq T\left(\left|y_{1}\right|, A\right)
$$

which has an optimal bound $\left|2 y_{1}\right|^{1 / 2}[\omega]_{C_{x}^{1 / 2}}$. We apply this estimate in (B.29).
B.1.6. Modification near the singularity. Near the singularity $s_{*}=(1,0)$, the integrand in the estimate in $S_{11}(\overline{\mathrm{~B} .10})$, (B.17) is singular of order $|x|^{-3 / 2}$ and quite complicated. To ease our computation of the integral in Part II, we use a simpler estimate in $y \in[1-\delta, 1+\delta] \times[0, A], \delta<1$ close to $s_{*}$. Since $B \geq 3$, we have $y_{1} \leq B-1, y_{1} \leq 9, \Delta(s)=\Delta_{\text {all }}(s)$ (B.5), and $-K_{2}\left(y_{1}-1, y_{2}\right)$ is the main term. Instead of using ( $(\overline{\mathrm{B} .16})$, we separate two kernels and estimate

$$
S_{11, \delta}^{+}=I_{+}(\delta)-I_{-}(\delta), \quad I_{ \pm} \triangleq \int_{1-\delta}^{1+\delta} \int_{0}^{A} K_{2}\left(y_{1} \pm 1, y_{2}\right) \tilde{W}(y) d y
$$

For $I_{+}$, the integrand is away from the singularity. Using (B.15), we get

$$
\begin{equation*}
\left|I_{+}(\delta)\right| \leq \int_{1-\delta}^{1+\delta} \int_{0}^{A}\left|K_{2}\left(y_{1}+1, y_{2}\right)\right| y_{2}^{1 / 2} d y[\omega]_{C_{y}^{1 / 2}} \triangleq S_{i n, y, \delta,+} \cdot[\omega]_{C_{y}^{1 / 2}} \tag{B.31}
\end{equation*}
$$

In Part II, using an estimate similar to (B.16), we obtain

$$
\begin{equation*}
\left|I_{-}(\delta)\right| \leq\left(2 \int_{[0, \delta] \times[0, A] \backslash Q_{a_{\delta}}^{+}}\left|K_{2}(y)\right| y_{2}^{1 / 2} d y+2 a_{\delta}^{1 / 2} C_{K_{2}}\right)[\omega]_{C_{y}^{1 / 2}} \triangleq S_{i n, y, \delta,-} \cdot[\omega]_{C_{y}^{1 / 2}} \tag{B.32}
\end{equation*}
$$

where

$$
\begin{array}{ll}
a_{\delta}=\min (A, \delta), \quad Q_{a} \triangleq[-a, a] \times[0, a], \quad Q_{a}^{+} \triangleq[0, a]^{2}, \quad C_{K_{2}} \triangleq C_{K_{2}, u p}+C_{K_{2}, \text { low }} \\
C_{K_{2}, u p} \triangleq \int_{0}^{1} \int_{y_{1}}^{1}\left|K_{2}(y)\right| \frac{y_{1}^{2}}{y_{2}}-\left.y_{2}\right|^{1 / 2} d y, \quad C_{K_{2}, \text { low }} \triangleq \int_{0}^{1} \int_{0}^{y_{1}^{2}}\left|K_{2}(y)\right|\left|y_{2}\right|^{1 / 2} d y_{2} \tag{B.33}
\end{array}
$$

We apply a similar modification in the estimate of $\left[u_{y}\right]_{C_{x}^{1 / 2}}$ (B.28)-(B.30) in the region $y \in$ $[1-\delta, 1+\varepsilon] \times[-A, A]$. Using $\int_{T\left(y_{1}, A\right)}^{A} \Delta(y) d y_{2}=0$, we modify the decomposition (B.27)

$$
\begin{aligned}
& \int_{-A}^{A} \Delta(y) W\left(y_{1}, x_{2}-y_{2}\right) d y_{2}=\int_{-A}^{A}\left(\Delta(y)\left(W\left(y_{1}, x_{2}-y_{2}\right)-W\left(y_{1}, x_{2}\right)\right)\right) d y_{2} \\
& +\int_{|y| \leq T\left(y_{1}, A\right)} \Delta(y) W\left(y_{1}, x_{2}\right) d y_{2} \triangleq \tilde{f}_{1}\left(y_{1}\right)+\tilde{f}_{2}\left(y_{1}\right), \quad I I_{i, \delta} \triangleq \int_{1-\delta}^{1+\delta} \tilde{f}_{i}\left(y_{1}\right) d y_{1}
\end{aligned}
$$

We apply the above estimates of $S_{11, \delta}^{+}$to $I I_{1, \delta}$. For $\tilde{f}_{2}\left(y_{2}\right)$, using $\left|W\left(y_{1}, x_{2}\right)-W\left(-y_{1}, x_{2}\right)\right| \leq$ $\sqrt{2 y_{1}}[\omega]_{C_{x}^{1 / 2}}$, we obtain the same estimate as in (B.29).

Note that the above modification only leads to a tiny change of order $\varepsilon^{1 / 2}$ to the estimate, and we choose $\varepsilon=2^{-14}$. We refer the estimate to Section 5 in the supplementary material II [16] (contained in [15]) in Part II.

Summary of the estimates of $\left[v_{x}\right]_{C_{x}^{1 / 2}},\left[u_{y}\right]_{C_{x}^{1 / 2}}$. For $\left[v_{x}\right]_{C_{x}^{1 / 2}}$, combining (B.9), (B.17), (B.19), (B.20), (B.25), (B.24), (B.31), (B.32), we establish

$$
\begin{align*}
& \frac{1}{\sqrt{2}}\left|v_{x}\left(-1, x_{2}\right)-v_{x}\left(1, x_{2}\right)\right| \leq \frac{1}{\pi \sqrt{2}}\left(\left(S_{\text {in, }}+S_{1 D}\right)[\omega]_{C_{x}^{1 / 2}}+\left(S_{\text {in,y }}+S_{\text {out }}\right)[\omega]_{C_{y}^{1 / 2}}\right), \\
& S_{\text {in }, x}=\int_{y_{1} \notin J_{1}, y_{1} \geq 0} \sqrt{2 y_{1} \mid}\left|\Delta_{1 D}\left(y_{1}\right)\right| d y_{1}, \quad S_{\text {out }}=\sqrt{2 x_{2}} \int_{\mathbb{R}} \int_{x_{2}}^{B}|\Delta(y)| d y, \\
& S_{\text {in,y }}=4 \int_{J_{1}^{+} \backslash[1-\delta, 1+\delta]}\left(\int_{s_{c}\left(y_{1}\right)}^{A}\left|y_{2}-T(y)\right|^{\frac{1}{2}}|\Delta(y)| d y_{2}+\int_{0}^{T\left(y_{1}, A\right)}\left|y_{2}\right|^{\frac{1}{2}}|\Delta(y)| d y_{2}\right) d y_{1}  \tag{B.34}\\
& \quad+4\left(S_{\text {in,y, } \delta,+}+S_{\text {in,y, }, \delta,-}\right)+4 \int_{y_{1} \notin J_{1}, y_{1} \geq 0} \int_{s_{c}\left(y_{1}\right)}^{A}\left|T(y)-y_{2}\right|^{\frac{1}{2}}|\Delta(y)| d y, \\
& S_{1 D}=2 \int_{\frac{1}{9}}^{9}\left|\Delta_{1 D}\right| y_{1}-\left.1\right|^{1 / 2} d y_{1}+\int_{0}^{\frac{1}{9}}\left|\Delta_{1 D}\right| \sqrt{2 y_{1}} d y_{1}+\left(\pi+P\left(\frac{1}{9}\right)\right) \sqrt{2},
\end{align*}
$$

where $P(\cdot)$ is defined in (B.23), and the factor $\frac{1}{\sqrt{2}}$ comes from $\frac{1}{|x-z|}=\frac{1}{\sqrt{2}}$ in this case.
For $u_{y}$, combining (B.30) and (B.25), we yield

$$
\begin{align*}
& \left.\frac{1}{\sqrt{2}}\left|u_{y}\left(-1, x_{2}\right)-u_{y}\left(1, x_{2}\right)\right| \leq \frac{1}{\pi \sqrt{2}}\left(\tilde{S}_{1 D}[\omega]_{C_{x}^{1 / 2}}+S_{u p}+S_{o u t}\right)[\omega]_{C_{y}^{1 / 2}}\right), \\
& S_{u p}=4 \int_{y_{1} \in \mathbb{R}_{+} \backslash[1-\delta, 1+\delta]} \int_{s_{c}\left(y_{1}\right)}^{A}\left|T(y)-y_{2}\right|^{\frac{1}{2}}|\Delta(y)| d y_{2} d y_{1}+4\left(S_{i n, y, \delta,+}+S_{i n, y, \delta,-}\right),  \tag{B.35}\\
& \tilde{S}_{1 D}=\int_{\mathbb{R}_{+}}\left|\Delta_{1 D}\left(y_{1}\right)\right| \sqrt{2 y_{1}} d y_{1} .
\end{align*}
$$

The above upper bounds depend on $(A, B), A=\min \left(x_{2}, B\right), A>0, B \geq 2: C_{1}\left(x_{2}, B\right)[\omega]_{C_{x}^{1 / 2}}+$ $C_{2}\left(x_{2}, B\right)[\omega]_{C_{y}^{1 / 2}}$. For any $\tau>0$, we further bound it by

$$
C_{1}\left(x_{2}, B\right)[\omega]_{C_{x}^{1 / 2}}+C_{2}\left(x_{2}, B\right)[\omega]_{C_{y}^{1 / 2}} \leq\left(C_{1}\left(x_{2}, B\right)+C_{2}\left(x_{2}, B\right) \tau\right) \max \left([\omega]_{C_{x}^{1 / 2}}, \tau^{-1}[\omega]_{C_{y}^{1 / 2}}\right) .
$$

We partition the domain of these parameters and use monotonicity of the integrals in $A, B$ to obtain the uniform bound. We refer the details to Section 5 in the supplementary material of Part II 16.
B.2. $C_{y}^{1 / 2}$ estimate of $v_{x}$. Since $W$ is not continuous across the boundary $y_{2}=0$, the localized $v_{x}$ or $u_{y}$ is not in $C_{y}^{1 / 2}$. Therefore, we study the estimate without localization. Without loss of generality, we assume $z_{2}=m+1, x_{2}=m-1, x_{1}=z_{1}=0$ with $m>1$. The case $m=1$ can be obtained by taking limit. The difference $v_{x}(z)-v_{x}(x)$ or $u_{y}(z)-u_{y}(x)$ is given by

$$
I \triangleq \frac{1}{\pi} \int K_{2}\left(y_{1}, 1+y_{2}\right)-K_{2}\left(y_{1}, y_{2}-1\right) W\left(y_{1}, m-y_{2}\right) d y+s(\omega(z)-\omega(x)),
$$

where $s=\frac{1}{2}$ for $u_{y}$ and $s=-\frac{1}{2}$ for $v_{x}$. Denote

$$
\begin{equation*}
\eta\left(y_{1}, y_{2}\right)=W\left(y_{2}, y_{1}\right), \quad \eta_{m}\left(y_{1}, y_{2}\right)=\eta\left(m-y_{1}, y_{2}\right) . \tag{B.36}
\end{equation*}
$$

By definition, $\eta$ is odd in $y_{1}$, discontinuous across $y_{1}=m$, and satisfies

$$
\begin{equation*}
[\eta]_{C_{x}^{1 / 2}(\Omega)}=[\omega]_{C_{y}^{1 / 2}}, \quad[\eta]_{C_{y}^{1 / 2}(\Omega)}=[\omega]_{C_{x}^{1 / 2}}, \tag{B.37}
\end{equation*}
$$

for $\Omega=\left\{y: y_{1} \geq 0\right\}$ or $\Omega=\left\{y: y_{1} \leq 0\right\}$.
Swapping the dummy variables $y_{1}, y_{2}$ and then using $K_{2}\left(y_{1}, y_{2}\right)=-K_{2}\left(y_{2}, y_{1}\right), \eta_{m}(y)=$ $\eta\left(m-y_{1}, y_{2}\right)=W\left(y_{2}, m-y_{1}\right)$, we yield

$$
\begin{align*}
I & =-\frac{1}{\pi} \int\left(K_{2}\left(y_{1}+1, y_{2}\right)-K_{2}\left(y_{1}-1, y_{2}\right)\right) \eta\left(m-y_{1}, y_{2}\right) d y+s(\omega(z)-\omega(x))  \tag{B.38}\\
& =-\frac{1}{\pi} \int \Delta(y) \eta_{m}(y) d y+s\left(\eta_{m}(-1,0)-\eta_{m}(1,0)\right), \quad s=-\frac{1}{2} \text { for } v_{x}, s=\frac{1}{2} \text { for } u_{y} .
\end{align*}
$$

We perform the above reformulation so that we can adopt the analysis of

$$
\begin{equation*}
\Delta(y)=K_{2}\left(y_{1}+1, y_{2}\right)-K_{2}\left(y_{1}-1, y_{2}\right) \tag{B.39}
\end{equation*}
$$

in (B.4) and Section B.1. Since $\eta$ is discontinuous across $y_{1}=0$, which relates to $y_{1}=m$ in the integral in (B.38), and the singularity of $\Delta$ is at $y=( \pm 1,0)$, we decompose the integral into the inner region, the middle region, and the outer region

$$
\begin{align*}
& \Omega_{\text {in }} \triangleq\left\{y:\left|y_{1}\right| \leq 1\right\}, \quad \Omega_{\text {mid }} \triangleq\left\{y:\left|y_{1}\right| \in[1, m]\right\}, \quad \Omega_{\text {out }} \triangleq\left\{y:\left|y_{1}\right|>m\right\} \\
& S_{\alpha} \triangleq \int_{\Omega_{\alpha}} \Delta(y) \eta_{m}(y) d y, \quad \alpha \in\{\text { in, mid }, \text { out }\} \tag{B.40}
\end{align*}
$$

See the right figure in Figure 13 for different regions in $\left\{y_{1} \geq 0\right\}$. In each region, $\eta_{m}$ is Hölder continuous. Since we can obtain a smaller factor from $[\omega]_{C_{y}^{1 / 2}}$ than $[\omega]_{C_{x}^{1 / 2}}$, and we have the relation (B.37), to obtain a sharp estimate of (B.38), we should use the $X$ transportation as much as possible.

Firstly, we analyze the sign of $\Delta(y)$. Since $\Delta$ is odd in $y_{1}$ and even in $y_{2}$, we can focus on $y_{1}, y_{2} \geq 0$. For a fixed $y_{2}$, we have

$$
\begin{align*}
& \Delta(y)<0, y_{1}<h_{c}^{-}\left(y_{2}\right) \leq 1, \quad \Delta(y)>0, y_{1} \in\left(h_{c}^{-}\left(y_{2}\right), 1\right), \quad y_{2} \leq y_{c} \triangleq 3^{-1 / 2},  \tag{B.41}\\
& \Delta(y)<0, y_{1}>h_{c}^{+}\left(y_{2}\right) \geq 1, \quad \Delta(y)>0, y_{1} \in\left(1, h_{c}^{+}\left(y_{2}\right)\right),
\end{align*}
$$

where $h_{c}^{ \pm}\left(y_{2}\right)$ solves $\Delta\left(h_{c}^{ \pm}\left(y_{2}\right), y_{2}\right)=0$ and is given explicitly in (B.56). The factor $y_{c}=3^{-1 / 2}$ comes from solving $\Delta\left(0, y_{c}\right)=0$. See the right figure in Figure 13 for $\operatorname{sgn}(\Delta(y))$ in different regions. Denote $Q_{\varepsilon}=[1-\varepsilon, 1+\varepsilon] \times[-\varepsilon, \varepsilon]$ and $Q_{i}$ is the four quadrants with center at $(1,0)$, e.g. $Q_{1}=\left\{y_{1} \geq 1, y_{2} \geq 0\right\}$. For the P.V. integral, since the kernel $\frac{y_{1}^{2}-y_{2}^{2}}{|y|^{4}}$ has mean 0 in each quadrant $\mathbb{R}^{ \pm} \times \mathbb{R}^{ \pm}$, it is not difficult to show that

$$
\lim _{\varepsilon \rightarrow 0} \int_{Q_{\varepsilon}^{c}} \Delta(y) \eta\left(m-y_{1}, y_{2}\right) \mathbf{1}_{y_{1} \geq 0} d y=\sum_{i=1}^{4} \lim _{\varepsilon_{i} \rightarrow 0} \int_{Q_{\varepsilon}^{c} \cap Q_{i}} \Delta(y) \eta\left(m-y_{1}, y_{2}\right) \mathbf{1}_{y_{1} \geq 0} d y
$$

Thus, we can estimate the P.V. integral separately in each $Q_{i}$.
B.2.1. Inner region $\Omega_{i n}$. In $\Omega_{i n}$, we have $\left|y_{1}\right| \leq 1$. Denote $y_{c}=3^{-1 / 2}$. Note that $\Delta=K_{2}\left(y_{1}+\right.$ $\left.1, y_{2}\right)-K_{2}\left(y_{1}-1, y_{2}\right)$ is singular at $(1,0)$. Applying Lemma B. 1 to $-K_{2}\left(y_{1}-1, y_{2}\right)$ yields

$$
\begin{equation*}
\int_{0}^{1} \int_{0}^{y_{c}} \Delta(y) \eta_{m}(y) d y=\lim _{\varepsilon \rightarrow 0} \int_{0}^{1} \int_{\varepsilon}^{y_{c}} \Delta(y) \eta_{m}(y) d y-\frac{\pi}{8} \eta_{m}(1,0) \triangleq \lim _{\varepsilon \rightarrow 0} I_{\varepsilon}-\frac{\pi}{8} \eta_{m}(1,0) \tag{B.42}
\end{equation*}
$$

Let $T_{1}(y) \geq 0$ be the map that solves

$$
\int_{y_{1}}^{T_{1}(y)} \Delta\left(s, y_{2}\right) d s=0, \quad y_{1} \in\left[0, h_{c}^{-}\left(y_{2}\right)\right]
$$

The formula is given in (B.61). Using the sign inequality (B.41) and applying Lemma 3.6 in the $y_{1}$ direction, we yield

$$
\left|\int_{\varepsilon}^{y_{c}} d y_{2} \int_{0}^{T_{1}\left(0, y_{2}\right)} \Delta(y) \eta_{m}(y) d y_{1}\right| \leq \int_{\varepsilon}^{y_{c}} d y_{2} \int_{0}^{h_{c}^{-}\left(y_{2}\right)}|\Delta(y)|\left|y_{1}-T_{1}(y)\right|^{1 / 2} d y_{1}[\eta]_{C_{x}^{1 / 2}} \triangleq I_{i n,++}
$$

See the blue arrows in $\Omega_{i n}$ in the right figure of Figure 13 for an illustration of this transportation estimate. Using the symmetry of $\Delta$ in $y_{1}, y_{2}$, we generalize the above estimate of the integral in the region $\left\{y: \varepsilon \leq\left|y_{2}\right| \leq y_{c},\left|y_{1}\right| \leq T_{1}\left(0,\left|y_{2}\right|\right)\right\}$, which is bounded by $4 I_{i n,++}$.

The remaining part of the integral in $\Omega_{i n}$ is in the following region

$$
\begin{align*}
& R_{i n, \varepsilon} \triangleq\left\{\left|y_{1}\right| \leq 1,\left|y_{2}\right| \geq y_{c}\right\} \cup\left\{T_{1}\left(0,\left|y_{2}\right|\right) \leq y_{1} \leq 1, \varepsilon \leq\left|y_{2}\right| \leq y_{c}\right\}  \tag{B.43}\\
& R_{i n, \varepsilon}^{+}=R_{i n, \varepsilon} \cap[0,1] \times \mathbb{R}, \quad R_{i n, \varepsilon}^{++}=R_{i n, \varepsilon} \cap[0,1] \times \mathbb{R}^{+}
\end{align*}
$$

Since $\Delta>0$ in $\mathbb{R}_{i n, \varepsilon}^{+}$, we use the odd symmetry of $\Delta$ in $y_{1}$ and even symmetry in $y_{2}$ to obtain

$$
\left|\int_{R_{i n, \varepsilon}} \Delta(y) \eta_{m}(y) d y\right|=\left|\int_{R_{i n, \varepsilon}^{+}} \Delta(y)\left(\eta_{m}\left(y_{1}, y_{2}\right)-\eta_{m}\left(-y_{1}, y_{2}\right)\right) d y\right| \leq 2 \int_{R_{i n, \varepsilon}^{++}}|\Delta(y)| \sqrt{2 y_{1}} d y[\eta]_{C_{x}^{1 / 2}}
$$

where we have the factor 2 since the estimates in $y_{2} \geq 0$ and $y_{2} \leq 0$ are the same.
Plugging the above estimate in (B.42) and using the symmetry of $\Delta$ in $y_{1}, y_{2}$, we derive (B.44)

$$
\begin{aligned}
& \left|\int_{\Omega_{i n}} \Delta(y) \eta_{m}(y)+\frac{\pi}{4}\left(\eta_{m}(1,0)-\eta_{m}(-1,0)\right)\right| \\
\leq & \left(4 \int_{0}^{y_{c}} d y_{2} \int_{0}^{h_{c}^{-}\left(y_{2}\right)}|\Delta(y)|\left|y_{1}-T_{1}(y)\right|^{1 / 2} d y_{1}+2 \int_{R_{i n, 0}^{++}}|\Delta(y)| \sqrt{2 y_{1}} d y\right)[\eta]_{C_{x}^{1 / 2}} \triangleq C_{i n}[\eta]_{C_{x}^{1 / 2}}
\end{aligned}
$$

B.2.2. Estimate in $\Omega_{\text {mid }}$. We develop two estimates for the integral (B.40)

$$
S_{m i d} \triangleq \int_{\Omega_{m i d}} \Delta(y) \eta_{m}(y) d y
$$

First estimate. The first estimate is similar to that in Section B.2.1 Notice that the singularities of $\Delta(\overline{\mathrm{B} .39})$ are $( \pm 1,0)$. We first rewrite $S_{\text {mid }}$ as follows using Lemma B. 1 twice with $Q_{1, \pm}=( \pm[1, m]) \times[0,1]$ and $Q_{2, \pm}=( \pm[1, m]) \times[-1,0]$

$$
\begin{equation*}
S_{m i d}=\lim _{\varepsilon \rightarrow 0} \int_{\Omega_{m i d} \cap\left|y_{2}\right| \geq \varepsilon} \Delta(y) \eta_{m}(y) d y-\frac{\pi}{4}\left(\eta_{m}(1,0)-\eta_{m}(-1,0)\right) . \tag{B.45}
\end{equation*}
$$

To estimate the integral, we first study the sign of $\Delta(y)$. For $y_{1} \in[1, m]$, we have

$$
\Delta(y)>0,\left|y_{2}\right|>s_{c}(m)
$$

where $s_{c}$ is given in (B.56). For $\left|y_{2}\right|<s_{c}(m)$, the sign of $\Delta(y)$ is given in (B.41). Denote

$$
\begin{align*}
& R_{m i d} \triangleq\left\{\left|y_{1}\right| \in[1, m],\left|y_{2}\right| \geq s_{c}(m)\right\} \cup\left\{\left|y_{2}\right|<s_{c}(m), 1 \leq\left|y_{1}\right| \leq T\left(m,\left|y_{2}\right|\right)\right\}  \tag{B.46}\\
& R_{\text {mid }}^{+} \triangleq R_{\text {out }} \cap\left\{y_{1} \geq 0\right\}, \quad R_{\text {mid }}^{++} \triangleq R_{\text {mid }} \cap \mathbb{R}_{2}^{++}
\end{align*}
$$

In $\Omega_{\text {mid }} \backslash R_{\text {mid }}$, using $\operatorname{sgn}(\Delta)\left(\right.$ B.41) and applying Lemma 3.6 in the $y_{1}$ direction, we yield

$$
\left|\int_{\varepsilon}^{s_{c}(m)} d y_{2} \int_{T\left(m, y_{2}\right)}^{m} \Delta(y) \eta_{m}(y) d y_{1}\right| \leq \int_{\varepsilon}^{s_{c}(m)} d y_{2} \int_{h_{c}^{+}\left(y_{2}\right)}^{m}\left|\Delta(y) \| y_{1}-T_{1}(y)\right|^{1 / 2} d y_{1} \triangleq I_{m i d,++}
$$

where $T_{1}$ is given in (B.61). See the blue arrows in $\Omega_{m i d}$ in Figure 13 for an illustration of this transportation estimate. We generalize the above estimate of integral in the region $\left\{y: \varepsilon \leq\left|y_{2}\right| \leq s_{c}(m), T\left(m,\left|y_{2}\right|\right) \leq\left|y_{1}\right| \leq m\right\}$ using symmetry of $\Delta$, which is bounded by $4 I_{\text {mid }}$.

For the integral in $R_{o u t}, \Delta(y)$ is positive if $y_{1}>0$. We use the odd symmetry of $\Delta$ in $y_{1}$ and

$$
\left|\eta_{m}\left(y_{1}, y_{2}\right)-\eta_{m}\left(-y_{1}, y_{2}\right)\right| \leq \sqrt{2 y_{1}}[\eta]_{C_{x}^{1 / 2}}
$$

In particular, we obtain an estimate similar to (B.44)

$$
\begin{align*}
\left|S_{m i d}+\frac{\pi}{4}\left(\eta_{m}(1,0)-\eta_{m}(-1,0)\right)\right| & \leq\left(4 \int_{0}^{s_{c}(m)} d y_{2} \int_{h_{c}^{+}\left(y_{2}\right)}^{m}|\Delta(y)|\left|y_{1}-T_{1}(y)\right|^{1 / 2} d y_{1}\right.  \tag{B.47}\\
& \left.+2 \int_{R_{m i d}^{++}}|\Delta(y)| \sqrt{2 y_{1}} d y\right)[\eta]_{C_{x}^{1 / 2}} \triangleq C_{m i d, 1}(m)[\eta]_{C_{x}^{1 / 2}}
\end{align*}
$$

Second estimate. In the second estimate, instead of using transportation in the $y_{1}$ direction, we use transportation in the $y_{2}$ direction. This estimate will be very useful for $v_{x}$. We also combine the estimate of $S_{\text {mid }}, S_{\text {out }}$ (B.40). Recall $\Omega_{\text {in }} \cup \Omega_{\text {out }}=\left\{\left|y_{1}\right| \geq 1\right\}$. Firstly, applying Lemma B. 1 twice to $K_{2}\left(y_{1}+1, y_{2}\right)$ and $-K_{2}\left(y_{1}-1, y_{2}\right)$, respectively, we yield

$$
\begin{equation*}
S_{\text {mid }}+S_{\text {out }}=\frac{\pi}{4}\left(\eta_{m}(1,0)-\eta_{m}(-1,0)\right)+\lim _{\varepsilon \rightarrow 0} \int_{\left|y_{1}\right| \geq 1+\varepsilon} \Delta(y) \eta_{m}(y) d y \triangleq I+\lim _{\varepsilon \rightarrow 0} I I_{\varepsilon} \tag{B.48}
\end{equation*}
$$

We remark that the above decomposition and the sign of $\left(\eta_{m}(1,0)-\eta_{m}(-1,0)\right)$ are different from those in (B.45) since we take the limit in different variables. The above integral is similar to (B.9) and is simpler since we do not localized the kernel $\Delta$. We apply the same argument as that in Section B.1.1, B.1.2, Recall that $\Delta$ satisfies the sign condition (B.5). Note that

$$
\begin{equation*}
\int_{0}^{\infty} \Delta\left(y_{1}, y_{2}\right) d y_{2}=\frac{y_{2}}{y_{2}^{2}+\left(y_{1}+1\right)^{2}}-\left.\frac{y_{2}}{y_{2}^{2}+\left(y_{1}-1\right)^{2}}\right|_{0} ^{\infty}=0 \tag{B.49}
\end{equation*}
$$

and $\Delta$ is even in $y_{2}$ and odd in $y_{1}$. Though $\eta_{m}(y)$ has a jump across $y_{1}=m$ due to the boundary (B.36), $\eta_{m} \in C^{1 / 2}\left(\Omega_{\text {in }}\right), C^{1 / 2}\left(\Omega_{\text {out }}\right)$. In particular, for a fixed $y_{1}, \eta_{m}\left(y_{1}, \cdot\right) \in C_{y_{2}}^{1 / 2}$. For $\delta>0$, applying Lemma 3.6 in the $y_{2}$ direction, we obtain

$$
\begin{equation*}
I I_{\delta} \leq 4 \int_{1+\delta}^{\infty} d y_{1} \int_{s_{c}\left(y_{1}\right)}^{\infty}\left|\Delta(y) \| y_{2}-T(y)\right|^{1 / 2} d y_{2}[\eta]_{C_{y}^{1 / 2}} \triangleq C_{m i d, 2}(\delta)[\eta]_{C_{y}^{1 / 2}} \tag{B.50}
\end{equation*}
$$

where $s_{c}\left(y_{1}\right)$ and $T$ are the same as those in Section B.1.1, B.1.2 and are given in (B.56), (B.60). We have a factor 4 since the same estimate applies to integral in each quadrant, $\pm[1+\delta, \infty] \times \mathbb{R}_{ \pm}$.

Recall the discussion in Section B.1.6. For $m>1$ close to 1 , to ease the computation, we seek a simpler estimate. Using the transport map $T(y)=\frac{y_{1}^{2}}{y_{2}}(\underline{\mathrm{~B} .14})$ for $K_{2}(y)$ in the $y_{2}$-direction and Lemma 3.6, we have

$$
\begin{equation*}
\left|\lim _{\varepsilon \rightarrow 0} \int_{[a, b] \backslash[-\varepsilon, \varepsilon]} d y_{1} \int_{0}^{\infty} K_{2}(y) f(y) d y\right| \leq \int_{a}^{b}\left|K_{2}(y) \| \frac{y_{1}^{2}}{y_{2}}-y_{2}\right|^{1 / 2} d y[f]_{C_{y}^{1 / 2}} \triangleq I_{K_{2}, \infty}(a, b)[f]_{C_{y}^{1 / 2}} \tag{B.51}
\end{equation*}
$$

for $a<b$. Clearly, $I_{K_{2}, \infty}(a, b)=I_{K_{2}, \infty}(-b,-a)$ for $0<a<b$. Recall $\Delta(y)=K_{2}\left(y_{1}+1, y_{2}\right)-$ $K_{2}\left(y_{1}-1, y_{2}\right)$ B.5). Using a change of variable $\left(y_{1} \pm 1, y_{2}\right) \rightarrow y$ and (B.51), we get

$$
\left|\lim _{\varepsilon \rightarrow 0} \int_{[a, b] \backslash[-\varepsilon, \varepsilon]} \int_{\mathbb{R}} K_{2}\left(y_{1} \pm 1, y_{2}\right) \eta_{m}(y) d y\right| \leq 2 I_{K_{2}, \infty}(a \pm 1, b \pm 1)[\eta]_{C_{y}^{1 / 2}}
$$

For $\varepsilon \rightarrow 0$, choosing $[a, b]=[\varepsilon+1, \delta+1],[-1-\delta,-1-\varepsilon]$, we estimate the remaining part of the integral in $y_{1} \in[a, b]$ as follows

$$
\begin{align*}
& \left|S_{m i d}-\frac{\pi}{4}\left(\eta_{m}(1,0)-\eta_{m}(-1,0)\right)\right|+S_{\text {out }} \leq \lim _{\varepsilon \rightarrow 0}\left|I I_{\varepsilon}\right| \leq C_{\text {out }}(1, \delta)[\eta]_{C_{y}^{1 / 2}}  \tag{B.52}\\
& C_{\text {out }}(1, \delta) \triangleq 4\left(I_{K_{2}, \infty}(0, \delta)+I_{K_{2}, \infty}(2,2+\delta)\right)+C_{m i d, 2}(\delta)
\end{align*}
$$

When we combine different estimates to estimate $\left[v_{x}\right]_{C_{y}^{1 / 2}}$, the factor $-\frac{\pi}{4}\left(\eta_{m}(1)-\eta_{m}(-1)\right)$ allows us to cancel the local term $-\frac{\pi}{2} \omega$ in $v_{x}$ (3.4). The factor $\frac{\pi}{4}\left(\eta_{m}(1)-\eta_{m}(-1)\right)$ in (B.47) has the opposite sign and does not offer such cancellation.
B.2.3. Estimate in the outer region. Recall the integral (B.40) in the outer region

$$
S_{o u t} \triangleq \int_{\Omega_{o u t}} \Delta(y) \eta_{m}(y) d y
$$

In $\Omega_{\text {out }}$, we have $\eta_{m} \in C_{y}^{1 / 2}([m, \infty) \times \mathbb{R})$ and $\eta_{m} \in C_{y}^{1 / 2}((-\infty,-m] \times \mathbb{R})$. For $\delta>0$ and $\delta_{m}=\max (m, \delta+1)$, following the second estimate of $S_{m i d}$ in Section B.2.2 applying estimates (B.50) to $\left|y_{1}\right| \geq \delta_{m}$, and (B.51) to the region $\left|y_{1}\right| \in\left[m \pm 1, \delta_{m} \pm 1\right]$, we yield

$$
\begin{align*}
\left|S_{o u t}\right| & \leq C_{m i d, 2}\left(\delta_{m}-1\right)+4\left(I_{K_{2}, \infty}\left(m+1, \delta_{m}+1\right)+I_{K_{2}, \infty}\left(m-1, \delta_{m}-1\right)\right) \\
& \triangleq C_{\text {out }}(m, \delta), \quad \delta_{m}=\max (m, \delta+1) \tag{B.53}
\end{align*}
$$

We have a factor 4 since the same estimate applies to the region $\pm[m, \infty] \times \mathbb{R}_{ \pm}$. The above notation $C_{\text {out }}(m, \delta)$ is consistent with $C_{o u t}(1, \delta)$ in (B.52), where $\delta_{m}=1+\delta$.
B.2.4. Modification near the singularity. Similar to Section B.1.6 near the singularity ( 1,0 ), for $\left|y_{2}\right| \leq \delta$, we modify the estimate of (B.42), ( (B.47) by separating two kernels (B.5)

$$
S_{i n, \delta} \triangleq \lim _{\varepsilon \rightarrow 0} \int_{\varepsilon \leq\left|y_{2}\right| \leq \delta,\left|y_{1}\right| \leq 1} \Delta(y) \eta_{m}(y) d y, \quad S_{m i d, 1, \delta} \triangleq \lim _{\varepsilon \rightarrow 0} \int_{\varepsilon \leq\left|y_{2}\right| \leq \delta, 1 \leq\left|y_{1}\right| \leq m} \Delta(y) \eta_{m}(y) d y
$$

In Section 5 in the supplementary material II [16 (contained in [15]) in Part II, we establish

$$
\begin{aligned}
& \left|S_{i n, \delta}\right| \leq\left\{\int_{0}^{1} \frac{\sqrt{2} y_{1}^{1 / 2} \delta}{\left(y_{1}+1\right)^{2}+\delta^{2}} d y_{1}+2\left(\sqrt{2} \frac{1}{2} \arctan (\delta)+2 \delta^{1 / 2}\left(f_{s}(\sqrt{1 / \delta})-f_{s}(1)+C_{K_{2}, u p}\right)\right)\right\}[\eta]_{C_{x}^{1 / 2}} \\
& \quad \triangleq C_{i n, \delta}[\eta]_{C_{x}^{1 / 2}} \\
& \left|S_{m i d, 1, \delta}\right| \leq\left\{\int_{\left|y_{2}\right| \leq \delta} \int_{1}^{m} K_{2}\left(y_{1}+1, y_{2}\right)\left|2 y_{1}\right|^{1 / 2} d y_{1}\right. \\
& \left.\quad+2\left(\sqrt{2 m} \frac{1}{2} \arctan \left(\frac{\delta}{m-1}\right)+2 \varepsilon^{1 / 2}\left(f_{s}(\sqrt{(m-1) / \delta})-f_{s}(1)+C_{K_{2}, u p}\right)\right)\right\}[\eta]_{C_{x}^{1 / 2}} \triangleq C_{m i d, 1, \delta}[\eta]_{C_{x}^{1 / 2}}
\end{aligned}
$$

where $C_{K_{2}, u p}$ is defined in (B.33), and $f_{s}$ is given by

$$
f_{s}(t) \triangleq \int_{0}^{t} \frac{s^{2}}{1+s^{4}} d s=\frac{1}{2} \int_{0}^{t^{1 / 2}} \frac{1}{1+s^{2}} s^{1 / 2} d s
$$

We obtain modified estimates of (B.44), (B.47) with bounds $C_{\alpha}$ replaced by $\tilde{C}_{\alpha}, \alpha=$ in, mid
$\tilde{C}_{i n}(\delta) \triangleq 4 \int_{\delta}^{y_{c}} d y_{2} \int_{0}^{h_{c}^{-}\left(y_{2}\right)}|\Delta(y)|\left|y_{1}-T_{1}(y)\right|^{1 / 2} d y_{1}+2 \int_{R_{i n, \delta}^{++}}|\Delta(y)| \sqrt{2 y_{1}} d y+C_{i n, \delta}$,
$\tilde{C}_{m i d, 1}(\delta) \triangleq 4 \int_{\delta}^{s_{c}(m)} d y_{2} \int_{h_{c}^{+}\left(y_{2}\right)}^{m}|\Delta(y)|\left|y_{1}-T_{1}(y)\right|^{1 / 2} d y_{1}+2 \int_{R_{m i d}^{++},\left|y_{2}\right| \geq \delta}|\Delta(y)| \sqrt{2 y_{1}} d y+C_{m i d, 1, \delta}$.
For $m>1$ very close to 1 , we have an additional estimate for $S_{m i d}$ (B.47)

$$
\begin{align*}
& \left|S_{m i d}+\frac{\pi}{4}\left(\eta_{m}(1,0)-\eta_{m}(-1,0)\right)\right|=\left|\lim _{\varepsilon \rightarrow 0} \int_{\left|y_{2}\right| \geq \varepsilon,\left|y_{1}\right| \in[1, m]} \Delta(y) \eta_{m}(y) d y\right| \leq C_{m i d, 3}[\eta]_{C_{x}^{\frac{1}{2}}}  \tag{B.55}\\
& C_{m i d, 3} \triangleq \frac{\pi}{4} \sqrt{2 m}+\frac{1}{4}(m-1)^{2} \sqrt{2 m}+2\left(\sqrt{2 m} \frac{\pi}{8}+2(m-1)^{1 / 2} C_{K_{2}, u p}\right)
\end{align*}
$$

The above modifications and the refinements in (B.52), (B.53) are very tiny and of order $\delta^{1 / 2},|m-1|^{1 / 2}$. We choose a small $\delta$, e.g. $\delta=2^{-14}$, and use them to ease the computation of the integral near $(1,0)$. If $\delta=0$, we recover the previous estimates.
B.2.5. Summarize the estimates. Recall from (B.38)

$$
\begin{aligned}
v_{x}(z)-v_{x}(x) & =-\frac{1}{\pi} \int \Delta(y) \eta_{m} d y-\frac{1}{2}\left(\eta_{m}(-1,0)-\eta_{m}(1,0)\right) \\
& =-\frac{1}{\pi}\left(S_{i n}+S_{m i d}+S_{o u t}-\frac{\pi}{2}\left(\eta_{m}(1,0)-\eta_{m}(-1,0)\right)\right) \\
u_{y}(z)-u_{y}(x) & =-\frac{1}{\pi}\left(S_{i n}+S_{m i d}+S_{o u t}+\frac{\pi}{2}\left(\eta_{m}(1,0)-\eta_{m}(-1,0)\right)\right)
\end{aligned}
$$

Note that $\left|\left(\eta_{m}(1,0)-\eta_{m}(-1,0)\right)\right| \leq \sqrt{2}[\eta]_{C_{x}^{1 / 2}}$. Using the relation (B.37) $[\eta]_{C_{x}^{1 / 2}}=[\omega]_{C_{y}^{1 / 2}},[\eta]_{C_{y}^{1 / 2}}=$ $[\omega]_{C_{x}^{1 / 2}}$, and applying (B.44), (B.47), (B.53), (B.55), (B.54) to $u_{y}$, we prove

$$
\begin{aligned}
\frac{\left|u_{y}(z)-u_{y}(x)\right|}{\sqrt{2}} & \leq \frac{1}{\pi \sqrt{2}}\left(\left(\tilde{C}_{i n}(\delta)+\min \left(\tilde{C}_{m i d, 1}(\delta), C_{m i d, 3}\right)\right)[\eta]_{C_{x}^{1 / 2}}+C_{o u t}(m, \delta)[\eta]_{C_{y}^{1 / 2}}\right) \\
& \leq \frac{1}{\pi \sqrt{2}}\left(\left(\tilde{C}_{i n}(\delta)+\min \left(\tilde{C}_{m i d, 1}(\delta), C_{m i d, 3}\right)\right)[\omega]_{C_{y}^{1 / 2}}+C_{o u t}(m, \delta)[\omega]_{C_{x}^{1 / 2}}\right)
\end{aligned}
$$

for $\delta>0$. Applying ( $\overline{\mathrm{B} .44)}$, (B.52) to $v_{x}$, we prove

$$
\begin{aligned}
\frac{\left|v_{x}(z)-v_{x}(x)\right|}{\sqrt{2}} & \leq \frac{1}{\pi \sqrt{2}}\left(\tilde{C}_{i n}(\delta)[\eta]_{C_{x}^{1 / 2}}+C_{o u t}(1, \delta)[\eta]_{C_{y}^{1 / 2}}+\frac{\pi}{2} \sqrt{2}[\eta]_{C_{x}^{1 / 2}}\right) \\
& \leq \frac{1}{\pi \sqrt{2}}\left(\tilde{C}_{\text {in }}(\delta)[\omega]_{C_{y}^{1 / 2}}+C_{o u t}(1, \delta)[\omega]_{C_{x}^{1 / 2}}+\frac{\pi}{2} \sqrt{2}[\omega]_{C_{y}^{1 / 2}}\right)
\end{aligned}
$$

for $\delta>0$, where the factor $\frac{1}{\sqrt{2}}$ comes from $|x-z|^{-1 / 2}=2^{-1 / 2}$.
B.3. Functions and transportation maps. We present the formulas of the transportation maps and the functions related to the sign of the kernels in the sharp Hölder estimate. Recall

$$
K_{1}=\frac{y_{1} y_{2}}{|y|^{4}}, \quad K_{2}=\frac{1}{2} \frac{y_{1}^{2}-y_{2}^{2}}{|y|^{4}}
$$

B.3.1. Sign functions. Solving $K_{1}\left(y_{1}+1 / 2, y_{2}\right)-K_{1}\left(y_{1}-1 / 2, y_{2}\right)=0$ for $y_{1} \geq 0$, we yield

$$
y_{1}=\left(\frac{1 / 2-2 y_{2}^{2}+\sqrt{16 y_{2}^{4}+4 y_{2}^{2}+1}}{6}\right)^{1 / 2}
$$

See also (3.10). Solving $K_{2}\left(y_{1}+1, y_{2}\right)-K_{2}\left(y_{1}-1, y_{2}\right)=0$ for $y_{2} \geq 0$, we yield

$$
\begin{align*}
& y_{1}=h_{c}^{ \pm}\left(y_{2}\right) \triangleq\left(y_{2}^{2}+1 \pm 2 y_{2} \sqrt{y_{2}^{2}+1}\right)^{1 / 2}  \tag{B.56}\\
& y_{2}=s_{c}\left(y_{1}\right) \triangleq\left(\frac{-\left(y_{1}^{2}+1\right)+2\left(y_{1}^{4}-y_{1}^{2}+1\right)^{1 / 2}}{3}\right)^{1 / 2}
\end{align*}
$$

B.3.2. Transportation maps.

Map for $u_{x}$. For a fixed $s_{2} \neq 0$ and $s_{1}>0$, solving

$$
\int_{T(s)}^{s_{1}}\left(K_{1}\left(s_{1}+1 / 2, s_{2}\right)-K_{1}\left(s_{1}-1 / 2, s_{2}\right)\right) d s_{1}=0
$$

yields the equation of the transportation map in $x$ direction

$$
\begin{equation*}
T^{3}+T^{2} s_{1}+T\left(s_{1}^{2}-\frac{1}{2}+2 s_{2}^{2}\right)-\frac{\left(4 s_{2}^{2}+1\right)^{2}}{16 s_{1}}=0 \tag{B.57}
\end{equation*}
$$

We rewrite the above equation as an equation for $Z=T+\frac{s_{1}}{3}$
$\left.0=Z^{3}+Z\left(\frac{2}{3} s_{1}^{2}-\frac{1}{2}+2 s_{2}^{2}\right)\right)-\left(\frac{\left(4 s_{2}^{2}+1\right)^{2}}{16 s_{1}}+\frac{7}{27} s_{1}^{3}+\frac{s_{1}}{3}\left(2 s_{2}^{2}-\frac{1}{2}\right)\right) \triangleq Z^{3}+p\left(s_{1}, s_{2}\right) Z+q\left(s_{1}, s_{2}\right)$,
The discriminant is given by

$$
\begin{equation*}
\Delta_{Z}\left(s_{1}, s_{2}\right)=-\left(27 q\left(s_{1}, s_{2}\right)^{2}+4 p\left(s_{1}, s_{2}\right)^{3}\right) \tag{B.58}
\end{equation*}
$$

Note that

$$
-q \geq \frac{7}{27} s_{1}^{3}-\frac{s_{1}}{6}+\frac{1}{16 s_{1}} \geq\left(2 \sqrt{\frac{7}{27} \cdot \frac{1}{16}}-\frac{1}{6}\right) s_{1} \geq 0
$$

and $-q, p$ are increasing in $\left|s_{2}\right|$. We yield

$$
-\Delta_{Z}\left(s_{1}, s_{2}\right) \geq-D_{Z}\left(s_{1}, 0\right)=\frac{\left(1-4 s_{1}^{2}\right)^{2}\left(27-56 s_{1}^{2}+48 s_{1}^{4}\right)}{256 s_{1}^{2}} \geq 0
$$

When $s \neq\left(\frac{1}{2}, 0\right)$, the above inequality is strict, and we have a unique real root. Using the solution formula for a cubic equation, we obtain the formula for the real root

$$
\begin{equation*}
Z=r_{1}-\frac{p}{3 r_{1}}, \quad r_{1}=\left(\frac{-q+\sqrt{q^{2}+\frac{4}{27} p^{3}}}{2}\right)^{1 / 3}, \quad T=Z-\frac{s_{1}}{3} \tag{B.59}
\end{equation*}
$$

Map for $\left[u_{y}\right]_{C_{x}^{1 / 2}}$. For a fixed $y_{1} \geq 0$, solving

$$
\int_{T(y)}^{y_{2}}\left(K_{2}\left(y_{1}+1, y_{2}\right)-K_{2}\left(y_{1}-1, y_{2}\right)\right) d y_{2}=0
$$

yields the equation of the transportation map in $y$ direction

$$
\begin{equation*}
T^{3}+T^{2} y_{2}+T\left(y_{2}^{2}+2+2 y_{1}^{2}\right)-\frac{\left(y_{1}^{2}-1\right)^{2}}{y_{2}}=0 \tag{B.60}
\end{equation*}
$$

We rewrite the above equation as an equation for $W=T+\frac{y_{2}}{3}$
$0=W^{3}+W\left(\frac{2 y_{2}^{2}}{3}+2+2 y_{1}^{2}\right)-\left(\frac{\left(y_{1}^{2}-1\right)^{2}}{y_{2}}+\frac{y_{2}^{3}}{27}+\frac{y_{2}}{3}\left(\frac{2 y_{2}^{2}}{3}+2+2 y_{1}\right)\right) \triangleq W^{3}+p_{2}(y) W+q_{2}(y)$,

Since $p_{2}>0$, using the discriminant (B.58), we obtain $-\Delta_{W}(y)>0$. Thus the cubic equation of $T$ or $W$ has a unique real root that can be obtained by the formula similar to (B.59).

Map for $\left[u_{y}\right]_{C_{y}^{1 / 2}}$. For a fixed $y_{1}, y_{2} \geq 0$, solving

$$
\int_{y_{1}}^{T_{1}(y)} \Delta\left(s, y_{2}\right) d s=0
$$

with $T \geq 0$ yields the equation of the transportation map in $x$ direction

$$
1-T^{2}-y_{1}^{2}+T^{2} y_{1}^{2}-2 y_{2}^{2}-T^{2} y_{2}^{2}-y_{1}^{2} y_{2}^{2}-3 y_{2}^{4}=0
$$

or equivalently

$$
\begin{equation*}
T^{2}=\frac{y_{1}^{2}+2 y_{2}^{2}+y_{1}^{2} y_{2}^{2}+3 y_{2}^{4}-1}{y_{1}^{2}-y_{2}^{2}-1} . \tag{B.61}
\end{equation*}
$$

We apply the above map to the following two regions separately

$$
y_{1} \in[0,1], y_{1} \leq h_{c}^{-}\left(y_{2}\right), \quad y_{1} \in[1, \infty], y_{1} \geq h_{c}^{+}\left(y_{2}\right)
$$

## Appendix C. Weights and parameters

C.1. Parameters for the weights. In our energy estimates and the estimates of the nonlocal terms, we need various weights. Below, we present the parameters for the weights. In practice, we use the double floating point values of these parameters which can differ from the values below by the machine precision, e.g. $10^{-16}$. For the Hölder estimates, we use the following weights

$$
\begin{align*}
& \psi_{1}=|x|^{-2}+0.5|x|^{-1}+0.2|x|^{-1 / 6} \\
& \psi_{2}=p_{2,1}|x|^{-5 / 2}+p_{2,2}|x|^{-1}+p_{2,3}|x|^{-1 / 2}+p_{2,4}|x|^{1 / 6} \\
& \psi_{3}=\psi_{2}, \quad \vec{p}_{2, \cdot}=(0.46,0.245,0.3,0.112)  \tag{C.1}\\
& g_{i}(h)=g_{i 0}(h) g_{i 0}(1,0)^{-1}, \quad g_{i 0}(h)=\left(\sqrt{h_{1}+q_{i 1} h_{2}}+q_{i 3} \sqrt{h_{2}+q_{i 2} h_{1}}\right)^{-1}, \\
& \vec{q}_{1,}=(0.12,0.01,0.25), \quad \overrightarrow{q_{2}},=(0.14,0.005,0.27), \quad \overrightarrow{q_{3}},=\vec{q}_{2,} .
\end{align*}
$$

To estimate the weighted $L^{\infty}$ norm of the error of solving the Poisson equation, the weighted $L^{\infty}$ norm of $\mathbf{u}_{A}, \nabla \mathbf{u}_{A}$, and the Hölder estimate of $\psi_{u} \mathbf{u}_{A}, \psi_{d u}(\nabla \mathbf{u})_{A}$, we use

$$
\begin{align*}
& \varphi_{e l l i}=\left|x_{1}\right|^{-1 / 2}\left(|x|^{-2}+0.6|x|^{-1 / 2}\right)+0.3|x|^{-1 / 6}, \quad \psi_{d u}=\psi_{1}, \quad \psi_{u}=|x|^{5 / 2}+0.2|x|^{-7 / 6},  \tag{C.2}\\
& \rho_{10}=\rho_{01}=|x|^{-3}+|x|^{-7 / 6}, \quad \rho_{i j}=\psi_{1}, i+j=2 \\
& \rho_{3}=|x|^{-1}+|x|^{-1 / 6}, \quad \rho_{4}=\left|x_{1}\right|^{-1 / 2}\left(|x|^{-2.5}+0.6|x|^{-1 / 2}\right)+0.3|x|^{-1 / 6}
\end{align*}
$$

where $\psi_{1}$ is given above. The weight $\varphi_{\text {elli }}$ is similar to $\varphi_{1}$ except that we choose a less singular power for the first term. We use $\rho_{3}$ to capture the vanishing order of $\partial_{i} \nabla \mathbf{U}_{\text {app }} \sim|x|^{-1}$ near $x=0(5.82)$ and estimate $\rho_{3} \partial_{i}(\nabla \mathbf{U})_{\text {app }}$. The weight $\rho_{4}$ singular along $x_{1}=0$ is used for another estimate of $u_{A} \rho_{4}$ using energy $\left\|\omega \varphi_{1}\right\|_{\infty}$. See Appendix B. 4 in Part II [15].

For the weighted $L^{\infty}$ estimates with decaying weights, we use the following weights

$$
\begin{aligned}
\varphi_{1} & =x^{-1 / 2}\left(|x|^{-2.4}+0.6|x|^{-1 / 2}\right)+0.3|x|^{-1 / 6} \\
\varphi_{2} & =x^{-1 / 2}\left(p_{5,1}|x|^{-5 / 2}+p_{5,2}|x|^{-3 / 2}+p_{5,3}|x|^{-1 / 6}\right)+p_{5,4}|x|^{-1 / 4}+p_{5,5}|x|^{1 / 7}, \\
\varphi_{3} & =x^{-1 / 2}\left(p_{6,1}|x|^{-5 / 2}+p_{6,2}|x|^{-3 / 2}+p_{6,3}|x|^{-1 / 6}\right)+p_{6,4}|x|^{-1 / 4}+p_{6,5}|x|^{1 / 7}, \\
p_{5, \cdot} & =(0.42,0.144,0.198,0.172,0.0383) \cdot \mu_{0}, \quad \mu_{0}=0.917, \\
p_{6, \cdot} & =\left(2.5 \cdot p_{5,1}, 2 \cdot p_{5,2}, 3.8 \cdot p_{5,3}, 1.71 \cdot p_{5,4}, 2.39 \cdot p_{5,5}\right) .
\end{aligned}
$$

We write the parameter $p_{6, i}$ as the form of $a_{i} p_{5, i}$ since we first determine $p_{5, i}$. for the weight $\varphi_{2}$ of $\eta$ and then determine the parameter $p_{6, i}$ for weights $\varphi_{3}$ relatively to $\varphi_{2}$.

For the weighted $L^{\infty}$ estimates with the growing weights, we use the following weights (C.4)

$$
\begin{aligned}
& \varphi_{g 1}=\varphi_{1}+p_{7,1}|x|^{1 / 16}, \quad \varphi_{g 2}=\varphi_{2}+p_{8,1}|x|^{1 / 4}+p_{8,2}|x|^{\alpha_{n}}, \quad \varphi_{g 3}=\varphi_{3}+p_{9,1}|x|^{1 / 4}+p_{9,2}|x|^{\alpha_{n}} \\
& p_{7,1}=1, p_{8,1}=0.07, \quad p_{8,2}=0.002, \quad p_{9,1}=0.154, \quad p_{9,2}=p_{8,2} p_{9,1} / p_{8,1}, \quad \alpha_{g, n}=1 / 3+10^{-8}
\end{aligned}
$$

We choose $p_{9,2}$ in a way that $p_{9, \text {. }}$ are proportional to $p_{8,}$.
Parameters in the energy. We choose the following parameters in our energy (5.21), (5.27), (5.70)

$$
\begin{align*}
& \tau_{1}=5, \quad \mu_{1}=0.668, \quad \mu_{2}=2 \mu_{1}=1.336, \quad \mu_{4}=0.065, \quad \tau_{2}=0.23 \\
& \mu_{5}=76, \quad \mu_{51}=61, \quad \mu_{52}=15, \quad \mu_{6}=61, \quad \mu_{62}=35.88, \quad \mu_{7}=9.5, \quad \mu_{8}=4.5 \tag{C.5}
\end{align*}
$$

C.1.1. Parameters for the cutoff functions. Recall the following cutoff functions constructed and estimated in Appendix D. 2 in Part II [15]
(C.6) $\chi_{e}(x)=\left(1+\exp \left(\frac{1}{x}+\frac{1}{x-1}\right)\right)^{-1}, \quad \kappa\left(x ; \nu_{1}, \nu_{2}\right)=\kappa_{1}\left(\frac{x}{\nu_{1}}\right)\left(1-\chi_{e}\left(\frac{x}{\nu_{2}}\right)\right), \quad \kappa_{1}(x)=\frac{1}{1+x^{4}}$,
where $e$ is short for exponential. We will mostly use the cutoff function $\kappa_{*}$

$$
\begin{equation*}
\kappa_{*}(x)=\kappa(x, 1 / 3,3 / 2) \tag{C.7}
\end{equation*}
$$

We construct the radial cutoff functions for the far-field approximations of $\omega$ and $\psi$ as follows

$$
\begin{align*}
\chi(r) & =\chi_{1}\left(1-\chi_{2}\right)+\chi_{2}, \chi_{1}(r)=\chi_{r a t i}\left(\frac{x-a_{1}}{l_{1}^{1 / 2}}\right), \chi_{r a t i}(x)=\frac{x^{7}}{\left(1+x^{2}\right)^{7 / 2}}  \tag{C.8}\\
\chi_{2}(r) & =\chi_{e}\left(\frac{x-a_{2}}{9 a_{2}}\right), a_{1}=10, \quad l_{1}=50000, \quad a_{2}=10^{5}
\end{align*}
$$

For the cutoff functions in $\chi_{j, 2}, \chi_{N F}$ in (4.11), $\chi_{\bar{\varepsilon}}, \chi_{\hat{\varepsilon}}$ in (5.81), and $\chi_{j, 2}$ in (5.75), we choose

$$
\begin{align*}
& \chi_{\bar{\varepsilon}}(x, y)=\kappa\left(x ; \nu_{\bar{\varepsilon}, 1}, \nu_{\bar{\varepsilon}, 2}\right) \kappa\left(y ; \nu_{\bar{\varepsilon}, 1}, \nu_{\bar{\varepsilon}, 2}\right), \quad \nu_{\bar{\varepsilon}, 1}=1 / 192, \quad \nu_{\bar{\varepsilon}, 2}=3 / 2 \\
& \chi_{\hat{\varepsilon}}(x, y)=\kappa_{*}(x) \kappa_{*}(y), \quad \chi_{N F}(x, y)=\kappa(x ; 2,10) \kappa(y ; 2,10), \\
& \chi_{12}=-\Delta \phi_{2}, \quad \phi_{2}=-\frac{x y^{3}}{6} \kappa_{*}(x) \kappa_{*}(y), \quad \chi_{22}=x y \kappa_{*}(x) \kappa_{*}(y), \quad \chi_{32}=\frac{x^{2}}{2} \kappa_{*}(x) \kappa_{*}(y),  \tag{C.9}\\
& f_{\chi, 1}=\Delta\left(\frac{x y^{3}}{6} \chi_{N F}(x, y)\right), \quad f_{\chi, 2}=x y \chi_{N F}(x, y), \quad f_{\chi, 3}=\frac{x^{2}}{2} \chi_{N F}(x, y) \\
& \chi_{\text {ode }}(x, y)=1-\chi_{e}\left(\left(x-\nu_{31}\right) / \nu_{31}\right) \chi_{e}\left(\left(y-\nu_{32}\right) / \nu_{32}\right), \quad \nu_{31}=80, \nu_{32}=1200 .
\end{align*}
$$

For the cutoff function in (7.10), we choose

$$
\begin{equation*}
\chi_{\phi}=\kappa_{2}\left(\frac{x}{\nu_{4,1}}\right)\left(1-\chi_{e}\left(\frac{x}{\nu_{4,2}}\right)\right), \quad \kappa_{2}(x)=\frac{1}{1+x^{2}}, \quad \nu_{4,1}=2, \quad \nu_{4,2}=128 . \tag{C.10}
\end{equation*}
$$

C.2. Parameters for approximating the velocity. We choose the following parameters $x_{i}, t_{i}, i \geq 1$ in the first approximation of velocity $u, u_{x}$ in (4.28), (4.29), (4.38) in Section 4.3,

$$
u_{x}: x=(1,2,3,4,6,8,11,16,22,32,48) \cdot 64 h_{x}, \quad t=(16,16,20,24,32,40,56,72,96,128,256) h
$$

$$
u: x=(1,2,4,8,12,16,22,32,64) \cdot 64 h_{x}, \quad t=(8,8,24,40,56,72,96,128,256) h
$$

$$
h_{x}=13 \cdot 2^{-12}, \quad h=13 \cdot 2^{-11}
$$

For the parameters $y_{0}, x_{0}$ in $\mathcal{K}_{00} \chi_{0}$ (4.29), we choose

$$
y_{0}=256 h_{x}, \quad u_{x}: x_{0}=32 h_{x}, \quad u: x_{0}=16 h_{x}, \quad u_{y}, v_{x}, v: x_{0}=128 h_{x}
$$

Remark that we choose the same $y_{0}$ for all cases in the cutoff function $\tilde{\chi}\left(\frac{y-y_{0}}{y_{0}}\right)$ (4.28). For $u, v, u_{x}$, we choose the following parameters $R_{i}$ in the second approximation (4.37)

$$
\begin{equation*}
R=(8,16,32,64,128,256,512,1024) \cdot 64 h_{x} \tag{C.11}
\end{equation*}
$$

C.2.1. Initial conditions for the linearized equations. Recall the formula (4.19) and the approximation terms (4.11) near the origin, (4.13) for the finite rank perturbation. We use the formulas (4.39), (4.40) to approximate the velocity. Denote by $n_{u_{x}}, n_{u}$ the number of the terms in (4.39) for $f=u_{x}$ and $f=u$, respectively, and $n_{R}=m$ the number of terms in (4.40). See Appendix C.2. We label the rank-one term as follows

$$
a_{1}\left(W_{1}\right) \bar{F}_{1}(0)=c_{\omega}\left(W_{1}\right) \bar{f}_{c_{\omega}}, \quad a_{n_{l i n}+i}(s) \bar{F}_{n_{l i n}+i}=a_{n l, i}(W) \bar{F}_{\chi, i}, \quad n_{l i n}=n_{R}+n_{u}+n_{u_{x}}+2
$$

for $i=1,2,3$, where $n_{\text {lin }}$ is the number of total rank for approximating the linearized operator, and $a_{n l, i}(W) \bar{F}_{\chi, i}$ approximates the nonlinear and error terms (4.17).

We denote by $a_{i}\left(W_{1}\right) \bar{F}_{i}(0)$ the rank-one term generated by the approximation (4.40) for $i=2,3, . ., n_{R}+1$, by the approximation term $\mathcal{K}_{00}$ of $v, v_{x}, u_{y}$ for $i=n_{R}+2$ (see the discussion above (4.29) ), by (4.39) with $f=u_{x}$ for $i=n_{R}+3, . ., n_{R}+n_{u_{x}}+2$, and by (4.39) with $f=u$ for $i=n_{R}+n_{u_{x}}+3, . ., n_{R}+n_{u_{x}}+n_{u}+2=n_{\text {lin }}$. For example, for $i=2$, we have

$$
\begin{aligned}
& a_{2}\left(W_{1}\right)=\left(I_{1}-u_{x}(0)\right)\left(W_{1}\right), \quad \bar{F}_{2}(0)=\left(\hat{\tilde{\mathbf{u}}} \cdot \nabla \bar{\omega}, \hat{\tilde{\mathbf{u}}}_{x} \cdot \nabla \bar{\theta}+\hat{\tilde{\mathbf{u}}} \cdot \bar{\nabla} \theta_{x}, \quad \hat{\tilde{\mathbf{u}}}_{y} \cdot \nabla \bar{\theta}+\hat{\tilde{\mathbf{u}}} \cdot \nabla \bar{\theta}_{y}\right), \\
& \hat{\tilde{f}}=C_{f 0}\left(1-\chi_{t o t}\right) S_{1}^{R}, f=u, v, u_{x}, u_{y}, v_{x}, v_{y}
\end{aligned}
$$

where $I_{1}-u_{x}(0)$ is the integral in (4.40). Note that we have changed the sign in (4.13), which does not change the estimate of the solution, e.g. $\left|\hat{F}_{i}(t)\right|$ and error $\left|\left(\partial_{t}-\mathcal{L}\right) F_{i}\right|$.

Note that $\eta_{x y}(0)=\xi_{x x}(0)=\theta_{x x y}(0)$ (2.27), $\partial_{x y} \overline{\mathcal{F}}_{2}(0)=\partial_{x x y} F_{\bar{\theta}}(0)=\partial_{x x} \overline{\mathcal{F}}_{3}(0)$ (2.19), we get $a_{n l, 2}(W)=a_{n l, 3}(W)$. Thus, we can combine $\left(0, a_{n l, 2}(W) \mathcal{F}_{\chi, 2}, 0\right),\left(0,0, a_{n l, 3}(W) \mathcal{F}_{\chi, 3}\right)$ as a rank-one term $a_{n l, 2}(W)\left(0, \overline{\mathcal{F}}_{\chi, 2}, \overline{\mathcal{F}}_{\chi, 3}\right)$, and construct the approximate space time solution from initial data $\left(0, \overline{\mathcal{F}}_{\chi, 2}, \overline{\mathcal{F}}_{\chi, 3}\right)$.

## C.3. Estimate of the stream function in 3D Euler.

## C.3.1. Proof of Lemma 6.4

Proof. Recall $r=1-C_{l} y$ in (6.9) and $\tilde{\omega}(r, z)=\omega^{\theta}(r, z) / r$ in (6.6). Since the support size satisfies $C_{l}(\tau) S(\tau)<1 / 4$, within the support of $\omega^{\theta}(r, z)$, we have $r \geq 1 / 2$. Hence, $|\tilde{\omega}| \asymp\left|\omega^{\theta}\right|$ and $\left|r \omega^{\theta}(r, z)\right| \lesssim|\tilde{\omega}(r, z)|$. We can apply Lemma 6.1 and (6.10) to get

$$
\begin{aligned}
|\phi(x, y)| & \lesssim C_{\omega} C_{l}^{-2} \int\left|\tilde{\omega}\left(r_{1}, z_{1}\right)\right|\left(1+\left|\log \left(\left(r_{1}-\left(1-C_{l} y\right)\right)^{2}+\left(z_{1}-C_{l} x\right)^{2}\right)\right|\right) d r_{1} d z_{1} \\
& =C_{\omega} \int\left|\tilde{\omega}\left(1-C_{l} y_{1}, C_{l} x_{1}\right)\right|\left(1+\mid \log \left(C_{l}^{2}\left(\left(y_{1}-y\right)^{2}+\left(x_{1}-x\right)^{2}\right) \mid\right) d y_{1} d x_{1}\right.
\end{aligned}
$$

where we have used Lemma 6.1 and $r_{1} \leq 1$ in the first inequality, and used change of variables $r_{1}=1-C_{l} y_{1}, z_{1}=C_{l} x_{1}$ in the second identity. Since $C_{\omega}\left|\tilde{\omega}\left(1-C_{l} y_{1}, C_{l} x_{1}\right)\right|=\left|\omega\left(x_{1}, y_{1}\right)\right|$ and $|(x, y)|>2 S$, within the support of $\omega$, we get $\left|(x, y)-\left(x_{1}, y_{1}\right)\right|^{2} \asymp|(x, y)|^{2}$. It follows

$$
\begin{aligned}
|\phi(x, y)| & \lesssim\left(1+\left|\log \left(C_{l}^{2}|(x, y)|^{2}\right)\right|\right) \int|\omega(z)| d z \lesssim\left(1+\left|\log \left(C_{l}|(x, y)|\right)\right|\right)\left|\mid \omega(1+|z|)^{\beta} \|_{\infty} \int_{|z| \leq S}(1+|z|)^{-\beta} d z\right. \\
& \lesssim\left\|\omega(1+|z|)^{\beta}\right\|_{\infty} S^{2-\beta}\left(1+\left|\log \left(C_{l}|(x, y)|\right)\right|\right)
\end{aligned}
$$

This proves the desired result.
Next, we prove the estimates in Lemmas 6.5, 6.10, Denote by $K_{s}(x, y)$ the symmetrized kernel of $K$ (3.2). For $G=\log |y|, K_{1}=\frac{y_{1} y_{2}}{|y|^{4}}, K_{2}=\frac{y_{1}^{2}-y_{2}^{2}}{2|y|^{4}}$, we have the following symmetrized estimate for $|y| \geq 2|x|$, which are proved in Part II [15] using ideas similar to Taylor expansion (C.12)

$$
\begin{aligned}
& \left\lvert\, \nabla\left(\left.G_{s}(x, y)-\frac{8 y_{1} y_{2} x_{1} x_{2}}{|y|^{4}}\left|\lesssim \frac{|x|^{3}}{|y|^{4}}, \quad\right| y|\geq 2| x|, \quad| K_{1, s}(x, y)-\frac{4 y_{1} y_{2}}{|y|^{4}} \right\rvert\, \leq \frac{6|x|^{2}}{\operatorname{Den}^{2}(x, y)}\right.\right. \\
& \left|\partial_{x_{1}} K_{1, s}(x, y)\right| \leq \frac{12 x_{1}}{\operatorname{Den}^{2}(\mathrm{x}, \mathrm{y})}, \quad\left|K_{1, s}(x, y)-\frac{4 y_{1} y_{2}}{|y|^{4}}-2\left(x_{1}^{2}-x_{2}^{2}\right) \partial_{y_{1}}^{2} K_{1}(y)\right| \leq \frac{10 \sqrt{2}|x|^{4}}{\operatorname{Den}^{3}(x, y)}
\end{aligned}
$$

where Den is defined below and satisfies $\operatorname{Den}(x, y) \asymp|y|^{2}$ for $|y| \geq 2|x|$

$$
\operatorname{Den}(x, y)=\sum_{i=1,2} \min _{\left|z_{i}\right| \leq x_{i}}\left|y_{i}-z_{i}\right|^{2}=\sum_{i=1,2}\left(\max \left(y_{i}-x_{i}, 0\right)\right)^{2}
$$

Similarly, we can obtain the following estimates for $K_{2, s}$
(C.13)
$\left|K_{2, s}\right| \leq \frac{12 x_{1} x_{2}}{\operatorname{Den}^{2}}, \quad\left|\partial_{x_{i}} K_{2, s}(x, y)\right| \leq \frac{12 x_{3-i}}{\operatorname{Den}^{2}}, \quad\left|K_{2, s}-4 x_{1} x_{2} \cdot \frac{12 y_{1} y_{2}\left(y_{1}^{2}-y_{2}^{2}\right)}{|y|^{8}}\right| \leq \frac{40 x_{1} x_{2}|x|^{2}}{\operatorname{Den}^{3}}$,
C.3.2. Proof of other estimates in Lemma 6.5. In this subsection, we prove the second and the third estimate in Lemma 6.5. Using the Green function, we have

$$
\phi=-\frac{1}{2 \pi} \int \log |x-y| W(y) d y, \quad \phi_{x y}(0)=-\frac{4}{\pi} \int_{\mathbb{R}_{2}^{+}} \frac{y_{1} y_{2}}{|y|^{4}} W(y) d y,
$$

where $W$ is the odd extension of $\omega$ from $\mathbb{R}_{2}^{+}$to $\mathbb{R}_{2}$. We only need to prove the estimate for the derivatives. The estimate for $\phi$ can be obtained by integration from 0 to $|x|$. Without loss of generality, we estimate $\partial_{1}\left(\phi-x_{1} x_{2} \partial_{12} \phi(0)\right)$.

Similar to the proof of the first estimate in Lemma 6.5 we use the partition

$$
\begin{equation*}
Q_{1}=\{y:|y| \geq 2|x|\}, \quad Q_{2}=\{y:|y-x| \leq|x| / 2\}, \quad Q_{3}=\left(Q_{1} \cup Q_{2}\right)^{c} . \tag{C.14}
\end{equation*}
$$

Denote $M=\left\|\omega\left(|x|^{-\alpha}+|x|^{\beta}\right)\right\|_{\infty}$. We have

$$
|\omega| \leq M \min \left(|x|^{\alpha},|x|^{-\beta}\right) .
$$

In $Q_{1}$, we combine the estimate of $\phi$ and $-\psi_{x y}(0) x y$. Symmetrizing the kernel, we get
$\phi-\phi_{x y}(0) x y=-\frac{1}{2 \pi} \int_{\mathbb{R}_{2}^{++}} K(x, y) W(y) d y$,
$K(x, y)=\log |x-y|+\log |x+y|-\log \left|\left(x_{1}-y_{1}, x_{2}+y_{2}\right)\right|-\log \left|\left(x_{1}+y_{1}, x_{2}-y_{2}\right)\right|-\frac{8 y_{1} y_{2} x_{1} x_{2}}{|y|^{4}}$.
For $\alpha<2$, applying (C.12) $\left|\partial_{x_{1}} K(x, y)\right| \lesssim \frac{|x|^{3}}{|y|^{4}}$ for $|y| \geq 2|x|$, we obtain

$$
\begin{align*}
\left|\int_{Q_{1}} \partial_{x_{1}} K(x, y) W(y) d y\right| & \lesssim M|x|^{3} \int_{|y| \geq 2|x|}|y|^{-4} \min \left(|y|^{\alpha},|y|^{-\beta}\right) d y  \tag{C.15}\\
& =M|x|^{3} \min \left(|x|^{-2+\alpha},|x|^{-2-\beta}\right)=M \min \left(|x|^{1+\alpha},|x|^{1-\beta}\right) .
\end{align*}
$$

Note that if $\alpha=2$, the integral for small $y, \int_{2|x| \leq|y| \leq 1}|y|^{-2} d y$, leads to a log factor.
In $Q_{2}, Q_{3}$, we estimate two integrals separately. For $\phi$ in $Q_{2}$, we have $|x-y| \leq|x| / 2$ and $|x| \asymp|y|$ in $|x-y| \leq|x| / 2$. Thus, we get

$$
\begin{aligned}
\int_{|x-y| \leq|x| / 2}\left|\partial_{x_{1}} \log \right| x-y|W(y)| d y & \lesssim M \min \left(|x|^{\alpha},|x|^{-\beta}\right) \int_{|x-y| \leq|x| / 2}|x-y|^{-1} d y \\
& \lesssim M \min \left(|x|^{\alpha},|x|^{-\beta}\right)|x|,
\end{aligned}
$$

In $Q_{3}$, we get $|x| / 2 \leq|x-y| \leq 3|x|,|y| \leq 2|x|$. It follows

$$
\begin{aligned}
\int_{Q_{3}}\left|\partial_{x_{1}} \log \right| x-y|W(y)| d y & \lesssim M|x|^{-1} \int_{|y| \leq 2|x|} \min \left(|y|^{\alpha},|y|^{-\beta}\right) d y \\
& \lesssim M|x|^{-1} \min \left(|x|^{2+\alpha},|x|^{2-\beta}\right) \lesssim M \min \left(|x|^{1+\alpha},|x|^{1-\beta}\right) .
\end{aligned}
$$

For $\phi_{x y}(0) x_{1} x_{2}$, we have

$$
\int_{|y| \leq 2|x|}\left|\partial_{x_{1}} \frac{x_{1} x_{2} y_{1} y_{2}}{|y|^{4}} W(y)\right| d y \lesssim M|x| \int_{|y| \leq 2|x|}|y|^{-2} \min \left(|y|^{\alpha},|y|^{-\beta}\right)=M|x| \min \left(|x|^{\alpha},|x|^{-\beta}\right) .
$$

Combining the above estimates, we prove the second estimate in Lemma 6.5
For the last estimate in Lemma 6.5, if $\alpha \in(2,3]$, the integrand in (C.15) is integrable near $|x|=0$. We yield

$$
\left|\int_{Q_{1}} \partial_{x_{1}} K(x, y) W(y) d y\right| \lesssim M|x|^{3} .
$$

For other terms, since $\alpha>2, \beta>0$, the desired estimate follows from

$$
\min \left(|x|^{1+\alpha},|x|^{1-\beta}\right) \lesssim|x|^{3} .
$$

C.3.3. Proof of Lemma 6.10. Firstly, we consider the second estimate in Lemma 6.10. The bound by $\|\Omega\|_{X}$ follows from embedding. We focus on the bound by $|x|^{2}\|\Omega\|_{X}$ and assume that $|x| \leq 1$. We consider the estimate for $\partial_{12} \Psi-\Psi_{12}(0)$. Firstly, we have

$$
\partial_{12} \Psi=\frac{1}{\pi} \int K(x-y) \Omega(y) d y, K(z)=\frac{z_{1} z_{2}}{|z|^{4}}, \quad \partial_{12} \Psi(0)=\frac{4}{\pi} \int_{\mathbb{R}_{2}^{++}} K(y) W(y) d y
$$

where we extend $\Omega$ from $\mathbb{R}_{2}^{+}$to $\mathbb{R}_{2}$ by natural odd extension. From (C.1)-(C.4), we get

$$
\begin{equation*}
\max \left(\varphi_{1}, \varphi_{g, 1}\right) \gtrsim|x|^{-\alpha},|x|^{\beta}, \quad \alpha=2.9, \beta=1 / 16, \quad \psi_{1} \asymp|x|^{-2}, \text { for }|x| \lesssim 1 \tag{C.16}
\end{equation*}
$$

Following the standard partition ( $\mathbf{C . 1 4}$ ), the symmetrization argument similar to that in the proof of Lemma 6.5 or Section C.3.2, and the estimate (C.12), we obtain

$$
\begin{equation*}
\left|\int_{|y| \geq 2|x|} K(x-y) W(y)-4 \int_{|y| \geq 2|x|, \mathbb{R}_{2}^{++}} K(y) W(y) d y\right| \lesssim|x|^{2}| | \Omega \|_{X} \tag{C.17}
\end{equation*}
$$

For $|y| \leq 2|x|$, we have

$$
\int_{|y| \leq 2|x|} K(y) \Omega(y) d y \lesssim \int_{|y| \lesssim 2|x|} \min \left(|y|^{\alpha},|y|^{-\beta}\right)|y|^{-2} d y| | \Omega\left\|_{X} \lesssim|x|^{\alpha}| | \Omega\right\|_{X} \lesssim|x|^{2}| | \Omega \|_{X}
$$

In region $Q_{3} \subset\{y:|x-y| \asymp|x|\}$, we have

$$
\begin{aligned}
\left|\int_{Q_{3}} K(x-y) \Omega(y) d y\right| & \lesssim|x|^{-2} \int_{|y| \leq 3|x|}|\Omega(y)| d y \lesssim|x|^{-2} \int_{|y| \leq 3|x|} \min \left(|y|^{\alpha},|y|^{-\beta}\right) d y\|\Omega\|_{X} \\
& \lesssim\|\Omega\|_{X} \min \left(|x|^{\alpha},|x|^{-\beta}\right)
\end{aligned}
$$

In the singular region $Q_{2}=\{y:|x-y| \leq|x| / 2\}$, for any $|s|,|t| \leq|x| / 2$, and $|x| \leq 1$, we have

$$
\begin{aligned}
& \Omega(x+s)-\Omega(x+t)=\left(\Omega \psi_{1} \psi_{1}^{-1}\right)(x+s)-\left(\Omega \psi_{1} \psi_{1}^{-1}\right)(x+t) \\
\leq & \left|\Omega \psi_{1}(x+s)-\Omega \psi_{1}(x+t)\right| \psi_{1}^{-1}(x+s)+\left|\psi_{1}^{-1}(x+s)-\psi_{1}^{-1}(x+t)\right| \cdot\left|\Omega \psi_{1}(x+t)\right| \triangleq I_{1}+I_{2}
\end{aligned}
$$

Since $\left|\nabla \psi^{-1}(x)\right| \lesssim|x|$, for $|s|,|t| \lesssim|x| / 2$, we get

$$
\left|\psi_{1}^{-1}(x+s)-\psi_{1}^{-1}(x+t)\right| \lesssim|x||s-t| .
$$

Using the above estimate for the weight $\psi_{1} \Omega \lesssim \frac{\psi_{1}}{\varphi_{1}}\left\|\Omega \varphi_{1}\right\|_{\infty} \lesssim \frac{\psi_{1}}{\varphi_{1}}\|\Omega\|_{X}$, we yield $\left|I_{1}\right| \lesssim\left|\left|\Omega \|_{X}\right| x\right|^{2}|s-t|^{1 / 2}, \quad I_{2} \lesssim|s-t||x| \cdot \psi_{1}(x+t) / \varphi_{1}(x+t) \lesssim|s-t||x| \cdot|x|^{\alpha-2}=|s-t||x|^{\alpha-1}$.

Using the symmetry of the kernel that $K(s)$ is odd in $s_{1}, s_{2}$, we get

$$
\begin{aligned}
&\left|\int_{|x-y| \leq|x| / 2} K(x-y) W(y) d y\right|=\int_{|s| \leq|x| / 2, s_{1} \geq 0} K(s)\left(W(x+s)-W\left(x+\left(-s_{1}, s_{2}\right)\right)\right) d s \mid \\
& \lesssim|x|^{2} \int_{|s| \leq|x| / 2}|s|^{-3 / 2} d s+|x|^{\alpha-1} \int_{|s| \leq|x| / 2}|s|^{-1} d s \lesssim|x|^{5 / 2}+|x|^{\alpha} \lesssim|x|^{2} .
\end{aligned}
$$

For small $|x|$, we can improve the above estimate by optimizing the window $a(x)$ for the singular region $|x-y| \leq a(x)$. We complete the estimate of the second inequality in Lemma 6.10

Next, we consider the first estimate in Lemma 6.10. For $\Omega \in X$, using (4.25), we have

$$
\partial_{1112} \Psi(0)=\frac{4}{\pi} \int_{\mathbb{R}_{2}^{++}} \partial_{11} K(y) \Omega(y) d y, \quad \partial_{11} K(y)=\frac{12 y_{1} y_{2}\left(y_{1}^{2}-y_{2}^{2}\right)}{|y|^{8}}
$$

The proof is completely similar. We use the above partiton of the domains. In $Q_{1}$, we use the symmetrizing estimates (C.12), (C.13) to yield

$$
\left|\int_{|y| \geq 2|x|} K(x-y) W(y) d y-\int_{|y| \geq 2|x|, \mathbb{R}_{2}^{++}}\left(4 K(y)+2\left(x_{1}^{2}-x_{2}^{2}\right) \partial_{11} K(y)\right) W(y) d y\right| \lesssim|x|^{5 / 2}| | \Omega \|_{X}
$$

In $Q_{2}, Q_{3}$, we estimate the integrals separately, and use the above estimates and

$$
\left.|x|^{2} \int_{|y| \lesssim|x|}\left|\frac{24 y_{1} y_{2}\left(y_{1}^{2}-y_{2}^{2}\right)}{|y|^{8}}\right| \Omega(y) d y \lesssim|x|^{2} \int_{|y| \lesssim|x|}|y|^{-4} \min \left(|y|^{\alpha},|y|^{-\beta}\right) d y| | \Omega\right|_{X} \lesssim \min \left(|x|^{\alpha},|x|^{-\beta}\right)
$$

Using the pointwise estimate (C.16) and the integral formulas of $\partial_{12} \Psi(0), \partial_{1112} \Psi(0)$, we get

$$
\left|\partial_{12} \Psi(0)\right|+\left|\partial_{1112} \Psi(0)\right| \lesssim\left\|W \varphi_{g, 1}\right\|_{L^{\infty}} \lesssim\|W\|_{X}
$$

## Appendix D. Inequalities for nonlinear stability

D.1. Inequalities for nonlinear stability. In this Section, we present all the inequalities for nonlinear stability in Lemma A.2 with the final energy $E_{4}$ (5.70). We have performed energy estimates in Section 5. The estimates of other nonlinear terms in the Hölder estimates are similar, and we refer them to Section 8.5 in the supplementary material I [18].

We verify these inequalities for nonlinear stability with computer assistance, and the codes can be found in [13]. The codes are implemented in MatLab with package INTLAB 82 for interval arithmetic. The estimates of the constants in Lemma 2.2, and the constructions and estimates of the approximate space-time solutions in Lemma 2.3 are performed in parallel using the Caltech High Performance Computing 1 . Other computer-assisted estimates and main part of the verifications are done in Mac Pro (Rack,2019) with 2.5 GHz 28 -core Intel Xeon W processor and $768 \mathrm{~GB}(6 \times 128 \mathrm{~GB})$ of DDR4 ECC memory.

Variables and Notations. Recall $W_{1}$ from (5.3), $\mathbf{U}, \tilde{\mathbf{U}}, \mathbf{U}_{A}$ from (5.82), the energies $E_{i}$ (5.21), (5.27), (5.54), (5.70), the weights $\rho_{10}=\rho_{01}, \rho_{i j}=\psi_{1}, i+j=2$ (C.2) for the $L^{\infty}$ estimates of $\mathbf{u}_{A},(\nabla \mathbf{u})_{A}$, and $\psi_{u}, \psi_{1}\left(\mathbb{C . 2 )}\right.$ ) for the Hölder estimate of $\psi_{u} \mathbf{u}_{A}, \psi_{1}(\nabla \mathbf{u})_{A}$. The notation $f_{A}(\omega)=$ $\tilde{f}(\omega)-\tilde{\hat{f}}(\omega)$ is introduced in (5.2), where $\tilde{\hat{f}}$ is the finite rank approximation of $f(\omega)$ (4.38). Note that in general $\nabla\left(\mathbf{U}_{A}\right) \neq(\nabla \mathbf{U})_{A}$. The weights $\psi_{i}, \varphi_{i}, \varphi_{g, i}$ are defined in (C.1), (C.3), (C.4).

Below, we use $\mathcal{T}$ to denote some functions related to the transport terms. We introduce $\mathcal{T}$ to bound $b \cdot \nabla f$

$$
\begin{equation*}
\mathcal{T}(b, f)(x) \triangleq\left|b_{1}\right| \cdot\left|f_{x}\right|+\left|b_{2}\right| \cdot\left|f_{y}\right|=\frac{1}{\rho_{10}}\left(\left|b_{1} \rho_{10}\right| \cdot\left|f_{x}\right|+\left|b_{2} \rho_{10}\right| \cdot\left|f_{y}\right|\right) \tag{D.1}
\end{equation*}
$$

For $b=\mathbf{u}_{A}=\left(u_{A}, v_{A}\right), \rho_{10} \mathcal{T}\left(\mathbf{u}_{A}, f\right)$ agrees with $\mathcal{T}_{u}(f)$ defined in Section 5.3. If $b=\mathbf{U}$ and $f=\rho$ is weight, we estimate piecewise $L^{\infty}$ norm of $U / x, V / y, \frac{x_{i} \partial_{i} \rho}{\rho}$ and denote

$$
\begin{equation*}
\mathcal{T}_{w g}(\mathbf{U}, \rho)(x) \triangleq\left|\frac{U}{x}\right| \cdot\left|\frac{x \partial_{x} \rho}{\rho}\right|+\left|\frac{V}{y}\right| \cdot\left|\frac{y \partial_{y} \rho}{\rho}\right| \tag{D.2}
\end{equation*}
$$

where $w g$ is short for weight and we use it to emphasize that the second component is a weight.
We derive piecewise weighted estimate $\rho_{i j} u_{i j}$ for a singular weight $\rho_{i j}$ associated with $u_{i j}$ and unweighted estimate $u_{i j}$. Then, we apply two estimates to bound $\rho u_{i j}$

$$
\rho u_{i j}=\left(\rho / \rho_{i j}\right) \cdot\left(\rho_{i j} u_{i j}\right)
$$

where $u_{i j}$ denotes $u, v, u_{x}, v_{x}, u_{y}$ for $(i, j)=(0,1),(1,0),(1,1),(2,0),(0,2)$ and $\rho_{10}, \rho_{20}$ are given in (C.2). The above two bounds are slightly different since $\max _{x \in Q}\left|f^{-1}(x)\right| \cdot \max _{x \in Q}|f(x)| \neq 1$. The second bound is useful near $x=0$ since both terms are regular. Note that we further establish weighted estimate for $\rho_{4} u_{A}$ using $\|\omega \varphi\|_{\infty}$.

Recall that we have modified the decomposition of linear and nonlinear terms in (5.90) and discussions therein to simplify the nonlocal error estimates. Below, the estimates are based on the decompositions in (5.90) and $\overline{\mathbf{u}}^{N}, \bar{c}_{\omega}^{N}$ etc. We modify (5.4) below and num is short for numerics

$$
\begin{align*}
& \mathcal{T}_{d}^{\text {num }}(\rho)=\rho^{-1}\left(\left(\bar{c}_{l} x+\overline{\mathbf{u}}^{N}\right) \cdot \nabla \rho\right), \quad d_{1, L}^{\text {num }}(\rho)=\mathcal{T}_{d}^{\text {num }}(\rho)+\bar{c}_{\omega}^{N}  \tag{D.3}\\
& d_{2, L}^{\text {num }}(\rho)=\mathcal{T}_{d}^{\text {num }}(\rho)+2 \bar{c}_{\omega}^{N}-\bar{u}_{x}^{N}, \quad d_{3, L}^{\text {num }}(\rho)=\mathcal{T}_{d}^{\text {num }}(\rho)+2 \bar{c}_{\omega}^{N}+\bar{u}_{x}^{N}
\end{align*}
$$

In the linear $L^{\infty}(\varphi)$ estimates in Section 5.3, and $L^{\infty}\left(\varphi_{4}\right)$ estimate in Section 5.3.4 we only use the energy $E_{1}$ (5.21). In the linear Hölder estimate in Section 5.4, we only use $E_{2}$ (5.27). In the $L^{\infty}\left(\varphi_{g, i}\right)$ estimates in Section 5.5, we only use $E_{3}$ (5.54). In the remaining energy estimates for functionals and nonlinear terms in Sections 5.7.5.9 we use the full energy $E_{4}$ (5.70).

[^0]D.1.1. Weighted $L^{\infty}$ estimate. In Sections 5.3, 5.5 and Section 5.9, we establish the following weighted $L^{\infty}$ estimates for (5.90).

Weighted $L^{\infty}\left(\varphi_{i}\right)$ estimate. We establish the following linear weighted $L^{\infty}\left(\varphi_{i}\right)$ estimates for the bad terms $B_{\text {modi, } i}$ (5.89) in Section 5.3

$$
\begin{align*}
& L_{1}\left(\varphi_{1}\right) \leq \frac{\varphi_{1}}{\varphi_{2}} E_{1}+\varphi_{1} \mathcal{T}\left(\mathbf{U}_{A}, \bar{\omega}\right) \\
& L_{2}\left(\varphi_{2}\right) \leq \frac{\varphi_{2}}{\varphi_{3}}\left|\bar{v}_{x}^{N}\right| E_{1}+\varphi_{2} \mathcal{T}\left(\mathbf{U}_{A}, \bar{\theta}_{x}\right)+\frac{\varphi_{2}}{\rho_{20}}\left(\left|U_{x, A} \rho_{20} \bar{\theta}_{x}\right|+\left|V_{x, A} \rho_{20} \bar{\theta}_{y}\right|\right)  \tag{D.4}\\
& L_{3}\left(\varphi_{3}\right) \leq \frac{\varphi_{3}}{\varphi_{2}}\left|\bar{u}_{y}^{N}\right| E_{1}+\varphi_{3} \mathcal{T}\left(\mathbf{U}_{A}, \bar{\theta}_{y}\right)+\frac{\varphi_{3}}{\rho_{20}}\left(\left|U_{y, A} \rho_{20} \bar{\theta}_{x}\right|+\left|U_{x, A} \rho_{20} \bar{\theta}_{y}\right|\right)
\end{align*}
$$

where $\psi_{1}=\rho_{i j}, i+j=2$. Here, we keep $\mathbf{U}_{A},(\nabla \mathbf{U})_{A}$ terms in the above estimates to simplify the notations. The terms involving $\mathbf{u}_{A}\left(\omega_{1}\right),(\nabla \mathbf{u})_{A}\left(\omega_{1}\right)$ can be bounded by $C(\mathbf{u}, a) E_{1}, C(\nabla \mathbf{u}, a) E_{1}$ with some weight $a$ introduced in (5.18) in Section 5.3. See discussion between (5.82) and (5.83) for the estimate of $\mathbf{U}_{A},(\nabla \mathbf{U})_{A}$. For (5.92) and the error (5.87), (5.88), we have nonlinear energy estimates

$$
\begin{align*}
& \mathcal{N} \mathcal{F}_{i}\left(\varphi_{i}\right) \leq \mathcal{T}_{w g}\left(\mathbf{U}, \varphi_{i}\right) E_{4}+N_{n l o c, i}\left(\varphi_{i}\right)+N_{\hat{W}_{2}, i}\left(\varphi_{i}\right)+\left|\overline{\mathcal{F}}_{l o c, i} \varphi_{i}\right|+\left|\mathcal{R}_{l o c, i} \varphi_{i}\right|, \\
& N_{n l o c, 1}\left(\varphi_{1}\right) \leq\left|U_{x}(0)\right| E_{4}, \quad N_{\hat{W}_{2}, 1}\left(\varphi_{1}\right) \leq \varphi_{1} \mathcal{T}\left(\tilde{\mathbf{U}}, \hat{\omega}_{2}\right)+\left|U_{x}(0)\right| \cdot\left|\hat{W}_{2,1, M} \varphi_{1}\right| \\
& N_{n l o c, 2}\left(\varphi_{2}\right) \leq\left(\left|U_{x}(0)\right|+\left|\tilde{U}_{x}\right|+\frac{\varphi_{2}}{\varphi_{3}}\left|\tilde{V}_{x}\right|\right) E_{4} \\
& N_{\hat{W}_{2}, 2}\left(\varphi_{2}\right) \leq \varphi_{2} \mathcal{T}\left(\tilde{U}, \hat{\eta}_{2}\right)+\left|U_{x}(0)\right| \cdot\left|\hat{W}_{2,2, M} \varphi_{2}\right|+\frac{\varphi_{2}}{\rho_{20}}\left(\left|\tilde{U}_{x} \rho_{20}\right| \cdot\left|\hat{\eta}_{2}\right|+\left|\tilde{V}_{x} \rho_{20}\right| \cdot\left|\hat{\xi}_{2}\right|\right),  \tag{D.5}\\
& N_{n l o c, 3}\left(\varphi_{3}\right) \leq\left(3\left|U_{x}(0)\right|+\left|\tilde{U}_{y}\right| \frac{\varphi_{3}}{\varphi_{2}}+\left|\tilde{U}_{x}\right|\right) E_{4}, \\
& N_{\hat{W}_{2}, 3}\left(\varphi_{3}\right) \leq \varphi_{3} \mathcal{T}\left(\tilde{U}, \hat{\xi}_{2}\right)+\left|U_{x}(0)\right| \cdot\left|\hat{W}_{2,3, M} \varphi_{3}\right|+\frac{\varphi_{3}}{\rho_{20}}\left(\left|\tilde{U}_{y} \rho_{20}\right| \cdot\left|\hat{\eta}_{2}\right|+\left|\tilde{U}_{x} \rho_{20}\right| \cdot\left|\hat{\xi}_{2}\right|\right),
\end{align*}
$$

where we have used (D.2) to simplify (5.96), and $\widehat{W}_{2,, M}=\left(\hat{\omega}_{2, M}, \hat{\eta}_{2, M}, \hat{\xi}_{2, M}\right)$ are defined in (5.93). The notation $L_{i}\left(\varphi_{i}\right), \mathcal{N} \mathcal{F}\left(\varphi_{i}\right)$ are only used to indicate the weighted $L^{\infty}\left(\varphi_{i}\right)$ estimate of linear and nonlinear terms. We have used $V_{y, A}=-U_{x, A}, \tilde{V}_{y}=-\tilde{U}_{x}$. In (D.4), (D.5), we do not multiply the terms related to $\mathcal{T}_{u}(\cdot)$ and $(\nabla \mathbf{U})_{A}$ by $E_{4}$ since we will further bound it using $\left|\mathbf{u}_{A}\left(\omega_{1}\right)\right| \leq C(x) E_{1},\left|\left(\nabla \mathbf{u}\left(\omega_{1}\right)\right)_{A}\right| \leq C(x) E_{1}, E_{1} \leq E_{4} \leq E_{*}$. Under the bootstrap assumption, we can combine the estimate of $\mathbf{u}_{A}$ and the error part in $\mathbf{U}_{A}$ (5.82) in (5.83). The same reasoning applies to the $\hat{W}_{2}, U$ terms. See Sections 5.8 and 5.7 .

Substituting the estimates of $\mathbf{U}_{A}, \mathbf{U}, \hat{W}_{2}$ (see also (5.83)), bounding $E_{4}$ by $E_{*}=5 \cdot 10^{-6}$ in (D.4), (D.5), and applying Lemma A.2, we obtain the nonlinear stability conditions for the $L^{\infty}\left(\varphi_{1}\right)$ estimates

$$
\begin{equation*}
-d_{i, L}^{n u m}\left(\varphi_{i}\right) E_{*}-L_{i}\left(\varphi_{i}\right)-\mathcal{N} \mathcal{F}_{i}\left(\varphi_{i}\right) \geq \lambda, \quad \forall x \in \mathbb{R}_{2}^{++}, \lambda>0 \tag{D.6}
\end{equation*}
$$

We do the same substitution in the following stability conditions.
$L^{\infty}\left(\varphi_{4}\right)$ estimate. Recall $\varphi_{4}=\psi_{1}\left|x_{1}\right|^{-\frac{1}{2}}$, the weight $\frac{\sqrt{2}}{\tau_{1}}$ in $E_{4}$ (5.21). We have established $L^{\infty}\left(\varphi_{4}\right)$ linear estimates in Section 5.3.4 and nonlinear estimates similar to $\mathcal{N} \mathcal{F}_{1}\left(\varphi_{1}\right)$ (D.5)

$$
\begin{aligned}
& \frac{\sqrt{2}}{\tau_{1}} L_{1}\left(\varphi_{4}\right) \leq \frac{\sqrt{2}}{\tau_{1}}\left(\frac{\varphi_{4}}{\varphi_{2}} E_{1}+\varphi_{4} \mathcal{T}\left(\mathbf{U}_{A}, \bar{\omega}\right)\right) \\
& \frac{\sqrt{2}}{\tau_{1}} \mathcal{N} \mathcal{F}_{1}\left(\varphi_{4}\right) \leq\left(\mathcal{T}_{w g}\left(\mathbf{U}, \varphi_{4}\right)+\left|U_{x}(0)\right|\right) E_{1}+\frac{\sqrt{2}}{\tau_{1}} \cdot \frac{\varphi_{4}}{\varphi_{1}}\left(N_{\hat{W}_{2}, 1}\left(\varphi_{1}\right)+\left|\overline{\mathcal{F}}_{l o c, 1} \varphi_{1}\right|+\left|\mathcal{R}_{l o c, 1} \varphi_{1}\right|\right) .
\end{aligned}
$$

We do not multiply $\mathcal{T}_{w g}\left(\mathbf{U}, \varphi_{4}\right) E_{4}$ by $\frac{\sqrt{2}}{\tau_{1}}$. See (5.96). Since we have weighted $L^{\infty}\left(\varphi_{1}\right)$ estimates of similar terms $f \varphi_{1}$ in (D.4), (D.5), e.g. $\mathcal{T}\left(\mathbf{U}_{A}, \bar{\omega}\right) \varphi_{1}, \overline{\mathcal{F}}_{l o c, 1} \varphi_{1}, \mathcal{R}_{l o c, 1} \varphi_{1}, N_{\hat{W}_{2}, 1}$, we can use such estimates and estimate $\frac{\varphi_{4}}{\varphi_{1}}$ to bound $f \varphi_{4}$. Similar to (D.6), the stability conditions
read

$$
\begin{equation*}
-d_{i, L}^{n u m}\left(\varphi_{4}\right) E_{*}-\frac{\sqrt{2}}{\tau_{1}}\left(L_{1}\left(\varphi_{4}\right)+\mathcal{N} \mathcal{F}_{1}\left(\varphi_{4}\right)\right) \geq \lambda, \quad \forall x \in \mathbb{R}_{2}^{++}, \lambda>0 \tag{D.7}
\end{equation*}
$$

$L^{\infty}\left(\varphi_{g, i}\right)$ estimate. In the $L^{\infty}\left(\varphi_{g, i}\right)$ estimate, we estimate $\mu_{g, i}\left\|W_{1, i} \varphi_{g, i}\right\|_{L^{\infty}}$. Recall $\mu_{g}=$ $\tau_{2}\left(\mu_{4}, 1,1\right)$ from (5.54), (C.5). We have established linear stability estimate in Section 5.5 (D.8)
$\mu_{g, 1} L_{1}\left(\varphi_{g, 1}\right) \leq \mu_{g, 1}\left(\varphi_{g 1}\left(\varphi_{2}^{-1} \wedge \mu_{g, 2}^{-1} \varphi_{g 2}^{-1}\right) E_{3}+\varphi_{g 1} \mathcal{T}\left(\mathbf{U}_{A}, \bar{\omega}\right)\right), \quad \mu_{g}=\tau_{2}\left(\mu_{4}, 1,1\right)$,
$\mu_{g, 2} L_{2}\left(\varphi_{g, 2}\right) \leq \mu_{g, 2}\left(\varphi_{g 2}\left|\bar{v}_{x}^{N}\right|\left(\varphi_{3}^{-1} \wedge \mu_{g, 3}^{-1} \varphi_{g 3}^{-1}\right) E_{3}+\varphi_{g 2} \mathcal{T}\left(\mathbf{U}_{A}, \bar{\theta}_{x}\right)+\frac{\varphi_{g 2}}{\psi_{1}}\left(\left|U_{x, A} \psi_{1}\right| \cdot\left|\bar{\theta}_{x}\right|+\left|V_{x, A} \psi_{1}\right| \cdot\left|\bar{\theta}_{y}\right|\right)\right)$,
$\mu_{g, 3} L_{3}\left(\varphi_{g, 3}\right) \leq \mu_{g, 3}\left(\varphi_{g 3}\left|\bar{u}_{y}^{N}\right|\left(\varphi_{2}^{-1} \wedge \mu_{g, 2}^{-1} \varphi_{g 2}^{-1}\right) E_{3}+\varphi_{g 3} \mathcal{T}\left(\mathbf{U}_{A}, \bar{\theta}_{y}\right)+\frac{\varphi_{g 3}}{\psi_{1}}\left(\left|U_{y, A} \psi_{1}\right| \cdot\left|\bar{\theta}_{x}\right|+\left|U_{x, A} \psi_{1}\right| \cdot\left|\bar{\theta}_{y}\right|\right)\right)$.
We have used $\mu_{g, i}$ to rewrite the parameters in the $L^{\infty}\left(\varphi_{g, i}\right)$ estimate in Section 5.5 equivalently, so that the form of estimates is more symmetric in the parameters. We also use $\psi_{1}=\rho_{20}$ and keep $u_{i j, A}$ in the estimates, which can be bounded by $C_{g i j}(x)$ defined in Section 5.5. Denote

$$
W_{i, g, j}=\min \left(\frac{\varphi_{g, j}}{\varphi_{i}}, \frac{\varphi_{g, j}}{\mu_{g, i} \varphi_{g, i}}\right)=\varphi_{g, j}\left(\varphi_{i}^{-1} \wedge\left(\mu_{g, i} \varphi_{g, i}\right)^{-1}\right)
$$

Using $\left|W_{1, i} \varphi_{i}\right|, \mu_{g, i}\left|W_{1, i} \varphi_{g, i}\right| \leq E_{3}$, we have $\left|W_{1, i} \varphi_{g, j}\right| \leq W_{i, g, j} E_{3}$, which motivates the above notation. Note that $\mu_{g, i} W_{i, g, i} \leq 1$. We can simplify some terms in the above estimate using $W_{i, g, j}$, e.g. $\varphi_{g 1}\left(\varphi_{2}^{-1} \wedge \mu_{g, 2}^{-1} \varphi_{g 2}^{-1}\right)$. We have the following nonlinear estimates similar to (D.5) (D.9)
$\mu_{g, i} \mathcal{N} \mathcal{F}_{i}\left(\varphi_{g, i}\right) \leq \mathcal{T}_{w g}\left(\mathbf{U}, \varphi_{g, i}\right) E_{4}+\mu_{g, i} N_{n l o c, i}\left(\varphi_{g, i}\right)+\frac{\mu_{g, i} \varphi_{g, i}}{\varphi_{i}}\left(N_{\hat{W}_{2}, i}\left(\varphi_{i}\right)+\left|\overline{\mathcal{F}}_{l o c, i} \varphi_{i}\right|+\left|\mathcal{R}_{l o c, i} \varphi_{i}\right|\right)$,
$\mu_{g, 1} N_{n l o c, 1}\left(\varphi_{g, 1}\right) \leq\left|U_{x}(0)\right| E_{4}, \quad \mu_{g, 2} N_{n l o c, 2}\left(\varphi_{g, 2}\right) \leq \mu_{g, 2}\left(\left(\left|U_{x}(0)\right|+\left|\tilde{U}_{x}\right|\right) W_{2, g, 2}+\left|\tilde{V}_{x}\right| W_{3, g, 2}\right) E_{4}$,
$\mu_{g, 3} N_{n l o c, 3}\left(\varphi_{g, 3}\right) \leq \mu_{g, 3}\left(\left|\tilde{U}_{y}\right| W_{2, g, 3}+\left(3\left|U_{x}(0)\right|+\left|\tilde{U}_{x}\right|\right) W_{3, g, 3}\right) E_{4}$.
Note that the terms $\mathcal{T}_{w g}\left(\mathbf{U}, \varphi_{g, i}\right) E_{4}$ do not multiply by $\mu_{g, i}$. See (5.96). We use the weighted $L^{\infty}\left(\varphi_{1}\right)$ estimates of similar terms $f \varphi_{i}$ in (D.4), (D.5), e.g. $\mathcal{T}\left(\mathbf{U}_{A}, \bar{\omega}\right) \varphi_{i}, \overline{\mathcal{F}}_{l o c, i} \varphi_{i}, \mathcal{R}_{l o c, i} \varphi_{i}$, and further estimate $\frac{\varphi_{g, i}}{\varphi_{i}}$ to bound $f \varphi_{g, i}=\left(f \varphi_{i}\right) \cdot \frac{\varphi_{g, i}}{\varphi_{i}}$. The stability conditions read

$$
\begin{equation*}
-d_{i, L}^{n u m}\left(\varphi_{g, i}\right) E_{*}-\mu_{g, i} L_{i}\left(\varphi_{g, i}\right)-\mu_{g, i} \mathcal{N} \mathcal{F}_{i}\left(\varphi_{g, i}\right) \geq \lambda, \quad \forall x \in \mathbb{R}_{2}^{++}, \lambda>0 \tag{D.10}
\end{equation*}
$$

D.1.2. Weighted Hölder estimates. For $x, z \in \mathbb{R}_{2}^{++}$, we assume $x_{1} \leq z_{1}$ and have the following linear Hölder estimate from Section 5.4. To simplify the notations, we drop the dependence of $x, z$ in $\delta$. Denote $h=x-z$.
$\mu_{h, i} g_{i}(h) \delta\left(L_{i}\right)=\left(d_{i, L}^{n u m}\left(p_{x, z}\right)+d_{g, i}^{n u m}\right) \cdot \mu_{h, i} g_{i}(h) \delta\left(W_{1, i} \psi_{i}\right)+\delta_{d a m p, i}\left(q_{x, z}\right)+\mu_{h, i} g_{i}(h) \delta\left(B_{m o d i, i}\right)$,
$\left|\delta_{d a m p, i}\left(q_{x, z}\right)\right| \leq \mu_{h, i}\left|\delta\left(d_{i, L}, x, z\right)\right| g_{i}(h) \frac{\psi_{i}}{\varphi_{i}}\left(q_{x, z}\right) E_{2}, \quad\left|\mu_{h, i} g_{i}(h) \delta\left(B_{m o d i, i}\right)\right| \leq \mu_{h, i} g_{i}(h) L_{n l o c, i}+L_{l o c, i}$,
$\left|L_{n l o c, 1}\right| \leq \delta_{\square}\left(U_{A} \psi_{u} \cdot \bar{\omega}_{x} \frac{\psi_{1}}{\psi_{u}}+V_{A} \psi_{u} \cdot \bar{\omega}_{y} \frac{\psi_{1}}{\psi_{u}}, h\right)$,
$\left|L_{n l o c, 2}\right| \leq \delta_{\square}\left(U_{A} \psi_{u} \cdot \bar{\theta}_{x x} \frac{\psi_{2}}{\psi_{u}}+V_{A} \psi_{u} \cdot \bar{\theta}_{x y} \frac{\psi_{2}}{\psi_{u}}+U_{x, A} \psi_{1} \cdot \bar{\theta}_{x} \frac{\psi_{2}}{\psi_{1}}+V_{x, A} \psi_{1} \cdot \bar{\theta}_{y} \frac{\psi_{2}}{\psi_{1}}, h\right)$,
$\left|L_{n l o c, 3}\right| \leq \delta_{\square}\left(U_{A} \psi_{u} \cdot \bar{\theta}_{x y} \frac{\psi_{2}}{\psi_{u}}+V_{A} \psi_{u} \cdot \bar{\theta}_{y y} \frac{\psi_{2}}{\psi_{u}}+U_{y, A} \psi_{1} \cdot \bar{\theta}_{x} \frac{\psi_{2}}{\psi_{1}}-U_{x, A} \psi_{1} \cdot \bar{\theta}_{y} \frac{\psi_{2}}{\psi_{1}}, h\right)$,
$\left|L_{l o c, 1}\right| \leq \min \left(\mu_{h, 1} g_{1}(x-z)\left(\frac{\psi_{1}}{\varphi_{2}}(x)+\frac{\psi_{1}}{\varphi_{2}}(z)\right), \min _{(p, q)=(x, z),(z, x)}\left(\mu_{h, 1} \delta_{\square}\left(\frac{\psi_{1}}{\psi_{2}}, h\right) \frac{\psi_{2}}{\varphi_{2}}(p) g_{1}(h)+\frac{\mu_{h, 1}}{\mu_{h, 2}} \frac{g_{1}(h)}{g_{2}(h)} \frac{\psi_{1}}{\psi_{2}}(q)\right)\right) E_{2}$
$\left|L_{l o c, 2}\right| \leq\left(\mu_{h, 2} g_{2}(h) \delta_{\square}\left(\bar{v}_{x}^{N}, h\right) \frac{\psi_{2}}{\varphi_{3}}(z)+\frac{\mu_{h, 2}}{\mu_{h, 3}}\left|\bar{v}_{x}^{N}(x)\right|\right) E_{2}$,
$\left|L_{l o c, 3}\right| \leq\left(\mu_{h, 3} g_{3}(h) \delta_{\square}\left(\bar{u}_{y}^{N}, h\right) \frac{\psi_{2}}{\varphi_{2}}(z)+\frac{\mu_{h, 3}}{\mu_{h, 2}}\left|\bar{u}_{y}^{N}(x)\right|\right) E_{2}, \quad \mu_{h}=\tau_{1}^{-1}\left(1, \mu_{1}, \mu_{2}\right)$,
where $\mu_{h}, \tau_{1}, \mu_{i}$ are given in (5.27), (C.5), $\delta_{\square}$ is defined in (5.36), $d_{g, i}^{\text {num }}$ is the damping term (5.29) from the Hölder weight $g_{i}$ with $b(x)$ (5.4) replaced by $b^{N}=\bar{c}_{l} x+\overline{\mathbf{u}}^{N}$, $L_{n l o c, i}$ is the estimate of the nonlocal terms involving $\mathbf{U}_{A},(\nabla \mathbf{U})_{A}$ (5.89), and $L_{l o c, i}$ estimate the local terms

$$
\begin{equation*}
\mu_{h, 1} g_{1} \delta\left(\left(\psi_{1} / \psi_{2}\right) \cdot \eta_{1} \psi_{2}\right), \quad \mu_{h, 2} g_{2} \delta\left(\bar{v}_{x}^{N} \cdot \xi_{1} \psi_{2}\right), \quad \mu_{h, 3} g_{2}(h) \delta\left(\bar{u}_{y}^{N} \cdot \eta_{1} \psi_{2}\right) \tag{D.12}
\end{equation*}
$$

For $L_{l o c, 2}, L_{l o c, 3}$, we use (5.46) with $(f, i, j)=\left(\bar{v}_{x}^{N}, 3,2\right)$ and $\left(\bar{u}_{y}^{N}, 2,3\right)$. Note that we assume $x_{1} \leq z_{1}$. For the damping terms, the choice of $\left(p_{x, z}, q_{x, z}\right)=(x, z)$ or $(z, x)$ depends on the locations of $x, z$, and we have two estimates of such terms. Instead of expanding the estimates again, we refer it to Section 5.4.3. We have an improved estimate for the damping coefficients $\delta_{d a m p, i}\left(q_{x, z}\right)$, which are explicit functions, in Section 8.4 in the supplementary material I [18]. See remark 5.2. We optimize this improved estimate and the above estimate.

We have used $\psi_{2}=\psi_{3}, g_{2}=g_{3}, v_{y, A}=-u_{x, A}$ and $\mu_{h, \text {. (5.70) }}$ to rewrite the parameters in the estimate in Section 5.4 equivalently. In (D.11), for the terms $\delta_{\square}\left(u_{A} \psi_{u} \cdot f_{1}+v_{A} \psi_{1} \cdot f_{2}+\ldots, s\right)$ with some functions $f_{1}, f_{2}$, we bound it using (5.36) and

$$
\delta_{i}\left(\sum_{m} p_{m} q_{m}, x, z\right) \leq \sum_{m} \delta_{i}\left(p_{m} q_{m}, x, z\right) \leq \sum_{m} \delta_{i}\left(p_{m}, q_{m}, x, z\right), \quad x_{3-i}=z_{3-i}
$$

For example, we have

$$
\delta_{i}\left(U_{A} \psi_{u} \cdot \bar{\omega}_{x} \frac{\psi_{1}}{\psi_{u}}+V_{A} \psi_{u} \cdot \bar{\omega}_{y} \frac{\psi_{1}}{\psi_{u}}, x, z\right) \leq \delta_{i}\left(U_{A} \psi_{u}, \bar{\omega}_{x} \frac{\psi_{1}}{\psi_{u}}, x, z\right)+\delta_{i}\left(V_{A} \psi_{u}, \bar{\omega}_{y} \frac{\psi_{1}}{\psi_{u}}, x, z\right)
$$

and then apply (5.36) and (5.37) to bound $\delta_{\square}(\cdot, x, z)$. For each term, e.g. $U_{A} \psi_{u}, \bar{\omega}_{x} \frac{\psi_{1}}{\psi_{u}}$, we can obtain its piecewise $C_{x_{i}}^{1 / 2}$ and $L^{\infty}$ estimate. It simplifies the notations and estimates. We apply the same convention for other terms and the terms below.

In Sections 5.9, 5.9.3 (see also Section 8.5 in the supplementary material I [18), we establish the nonlinear estimates for (5.90), (5.89) (see also (5.46))

$$
\left.\begin{array}{l}
\begin{array}{rl}
\mu_{h, i} g_{i}(h)\left|\delta\left(\mathcal{N} \mathcal{F}_{i}\left(\psi_{i}\right)\right)\right| \leq & \mathcal{N} \mathcal{T}_{i}+N_{W_{1}, i}+N_{\hat{W}_{2}, i}+\mu_{h, i} g_{i}(h)\left|\delta\left(\mathcal{B}_{o p, j}\left(\mathbf{U}_{A}, \widehat{W_{2}}\right)\right)\right| \\
& +\mu_{h, i} g_{i}(h) \delta_{\square}\left(\overline{\mathcal{F}}_{l o c, i} \psi_{i}+\mathcal{R}_{l o c, i} \psi_{i}, h\right),
\end{array}  \tag{D.13}\\
\mathcal{N} \mathcal{T}_{i} \leq\left\{\left|\mathcal{T}_{u A}\left(\psi_{i}\right)(x)+\mathcal{T}_{u R}\left(\psi_{i}\right)(x)\right|+\mu_{h, i} g_{i}(h) \delta_{\square}\left(\mathcal{T}_{u A}\left(\psi_{i}\right)+\mathcal{T}_{u R}\left(\psi_{i}\right), h\right) \frac{\psi_{i}}{\varphi_{i}}(z)\right.
\end{array}\right] \begin{aligned}
\left.+\max _{p=x, z}\left|\mathcal{T}_{c_{\omega}}(p)\right|+\mu_{h, i} g_{i}(h) \min \left(\left|\delta\left(\mathcal{T}_{c_{\omega}}\left(\psi_{i}\right), x, z\right)\right| \min (|x|,|z|)^{\frac{1}{2}} \max _{p=x, z} \frac{\psi_{i}(p)}{\varphi_{i}(p)|p|^{\frac{1}{2}}}, \delta_{\square}\left(\mathcal{T}_{c_{\omega}}\left(\psi_{i}\right), h\right) \max _{p=x, z} \frac{\psi_{i}(p)}{\varphi_{i}(p)}\right)\right\} E_{4}, \\
\left|N_{W_{1}, 1}\right| \leq\left|U_{x}(0)\right| E_{4}, \\
\left|N_{W_{1}, 2}\right| \leq\left(\left|U_{x}(0)\right|+\mu_{h, 2} g_{2}(h) \delta_{\square}\left(\tilde{U}_{x}, h\right) \frac{\psi_{2}}{\varphi_{2}}(z)+\left|\tilde{U}_{x}(x)\right|+\mu_{h, 2} g_{2}(h) \delta_{\square}\left(\tilde{V}_{x}, h\right) \frac{\psi_{2}}{\varphi_{3}}(z)+\frac{\mu_{h, 2}}{\mu_{h, 3}}\left|\tilde{V}_{x}(x)\right|\right) E_{4}, \\
\left|N_{W_{1}, 3}\right| \leq\left(3\left|U_{x}(0)\right|+\mu_{h, 3} g_{3}(h) \delta_{\square}\left(\tilde{U}_{y}, h\right) \frac{\psi_{2}}{\varphi_{2}}(z)+\frac{\mu_{h, 3}}{\mu_{h, 2}}\left|\tilde{U}_{y}(x)\right|+\mu_{h, 3} g_{3}(h) \delta_{\square}\left(\tilde{U}_{x}, h\right) \frac{\psi_{2}}{\varphi_{3}}(z)+\left|\tilde{U}_{x}(x)\right|\right) E_{4}, \\
\left|\delta\left(\psi_{1} \mathcal{B}_{o p, 1}\left(\mathbf{U}_{A}, \widehat{W}_{2}\right)\right)\right| \leq \delta_{\square}\left(U_{A} \psi_{u} \cdot \frac{\psi_{1} \partial_{x} \widehat{W}_{2,1}}{\psi_{u}}+V_{A} \psi_{u} \cdot \frac{\psi_{1} \partial_{y} \widehat{W}_{2,1}}{\psi_{u}}, h\right), \\
\left|\delta\left(\psi_{2} \mathcal{B}_{o p, 2}\left(\mathbf{U}_{A}, \widehat{W}_{2}\right)\right)\right| \leq \delta_{\square}\left(U_{A} \psi_{u} \cdot \frac{\psi_{2} \partial_{x} \widehat{W}_{2,2}}{\psi_{u}}+V_{A} \psi_{u} \cdot \frac{\psi_{2} \partial_{y} \widehat{W}_{2,2}}{\psi_{u}}+U_{x, A} \psi_{1} \cdot \hat{\eta}_{2} \frac{\psi_{2}}{\psi_{1}}+V_{x, A} \psi_{1} \cdot \hat{\xi}_{2} \frac{\psi_{2}}{\psi_{1}}, h\right), \\
\left|\delta\left(\psi_{2} \mathcal{B}_{o p, 3}\left(\mathbf{U}_{A}, \widehat{W}_{2}\right)\right)\right| \leq \delta_{\square}\left(U_{A} \psi_{u} \cdot \frac{\psi_{2} \partial_{x} \widehat{W}_{2,3}}{\psi_{u}}+V_{A} \psi_{u} \cdot \frac{\psi_{2} \partial_{y} \widehat{W}_{2,3}}{\psi_{u}}+U_{y, A} \psi_{1} \cdot \hat{\eta}_{2} \frac{\psi_{2}}{\psi_{1}}-U_{x, A} \psi_{1} \cdot \hat{\xi}_{2} \frac{\psi_{2}}{\psi_{1}}, h\right), \\
\left|N_{\hat{W}_{2}, i}\right| \leq \mu_{h, i} g_{i}(h)\left(\left|U_{x}(0)\right| \delta_{\square}\left(\hat{W}_{2, i, M} \psi_{1}, h\right)+\delta_{\square}\left(\mathcal{B}_{o p, j}\left(\left(\mathbf{U}_{a p p},(\nabla \mathbf{U})_{a p p}\right), \widehat{W}_{2}\right), h\right)\right),
\end{aligned}
$$

where we have used $\psi_{2}=\psi_{3}, g_{2}=g_{3}, \mathcal{N} \mathcal{T}_{i}$ denotes the estimate of the nonlinear transport part $\mathcal{T}_{u A}, \mathcal{T}_{u R}, \mathcal{T}_{c_{\omega}}$ (5.91), $N_{W_{1}, i}$ for terms involving $W_{1}$ other than the transport part in (5.92), $\mathcal{B}_{o p, j}\left(\mathbf{U}_{A}, \hat{W}_{2}\right)$ is short for $\mathcal{B}_{o p, j}\left(\mathbf{U}_{A},(\nabla \mathbf{U})_{j}, \hat{W}_{2}\right)(2.16), N_{\hat{W}_{2}, i}$ for $\left(\mathbf{U}_{\text {app }}, \hat{W}_{2}\right)$ in (5.95) and the term $\hat{W}_{2, i, M}$ (5.93). The term $\delta\left(\mathcal{T}_{c_{\omega}}, x, z\right) g_{i}(h) \min (|x|,|z|)^{1 / 2}$ is further estimated in Section 8.5.2. See Section 5.9.3 and Remark 5.2 for motivations.

For $|x-z|$ not small, using the weighted $L^{\infty}\left(\varphi_{i}\right)$ estimates of the linear terms $\left.L_{i}\left(\varphi_{i}\right)\right)$ (D.4) and nonlinear terms (D.5), and bounding $\frac{\psi_{i}}{\varphi_{i}}$, we have a simple $L^{\infty}$ estimate

$$
\begin{align*}
& \left|\mu_{h, i} g_{i}(x-z) \delta\left(B_{\text {modi }, i}\right)\right| \leq \mu_{h, i} g_{i}(x-z)\left(\frac{\psi_{i}}{\varphi_{i}}\left|L_{i}\left(\varphi_{i}\right)\right|(x)+\frac{\psi_{i}}{\varphi_{i}}\left|L_{i}\left(\varphi_{i}\right)\right|(z)\right)  \tag{D.14}\\
& \left\lvert\, \delta\left(\mathcal{N} \mathcal{F}_{i}\left(\psi_{i}\right) \left\lvert\, \leq \sum_{y=x, z} \frac{\psi_{i}}{\varphi_{i}}(y)\left(\mathcal{T}_{w g}\left(\mathbf{U}, \psi_{i}\right) E_{4}+N_{n l o c, i}\left(\varphi_{i}\right)+N_{\hat{W}_{2}, i}\left(\varphi_{i}\right)+\left|\overline{\mathcal{F}}_{l o c, i} \varphi_{i}\right|+\left|\mathcal{R}_{l o c, i} \varphi_{i}\right|\right)(y)\right.\right.\right. \\
& \left|\mathcal{T}_{w g}\left(\mathbf{U}, \psi_{i}\right)\right| \leq\left|\mathcal{T}_{c w}\right|+\left|\mathcal{T}_{u A}\right|+\left|\mathcal{T}_{u R}\right|, \quad\left|W_{1, i} \psi_{i}\right| \leq \psi_{i} / \varphi_{i} E_{4}
\end{align*}
$$

where we replace the transport part $\mathcal{T}_{w g}\left(\mathbf{U}, \varphi_{i}\right)\left(\overline{\mathrm{D} .5)}\right.$ by $\mathcal{T}_{w g}\left(\mathbf{U}, \psi_{i}\right)$ since we use weight $\psi_{i}$ (5.92). For the local terms (D.12), we optimize the $C^{1 / 2}$ estimate of $L_{l o c, i}$ in (D.11), and the $L^{\infty}$ estimate in (D.4) with weight $\frac{\psi_{i}}{\varphi_{i}}$ similar to the above.

Combining the above estimates, we obtain the the stability conditions for the weighted Hölder estimate

$$
\begin{equation*}
-\left(d_{i, L}^{n u m}\left(p_{x, z}\right)+d_{g, i, \mathbf{U}}\right) E_{*}-\mu_{h, i} g_{i}(h)\left(\left|\delta\left(d_{i, L}, x, z\right)\right| \frac{\psi_{i}}{\varphi_{i}}\left(q_{x, z}\right)+L_{n l o c, i}+\delta\left(N_{i} \psi_{i}\right)\right)-L_{l o c, i} \geq \lambda \tag{D.15}
\end{equation*}
$$

for some $\lambda>0$, uniformly for any $x, z \in \mathbb{R}_{2}^{++}, x_{1} \leq z_{1}$, where $\left(p_{x, z}, q_{x, z}\right)=(x, z)$ or $(z, x)$ depends on the locations of $x, z$. See Section 5.4.3 and remark 5.2, and Section 8.4. Here, $d_{g, i, \mathbf{U}}$ is the nonlinear damping factors (5.29), (5.4) by adding $\mathbf{U}$

$$
d_{g, i, \mathbf{U}} \triangleq\left(b_{U}(x)-b_{U}(z)\right) \cdot\left(\nabla g_{i}\right)(x-z) g_{i}^{-1}(x-z), \quad b_{U}=\bar{c}_{l} x+\overline{\mathbf{u}}^{N}+\mathbf{U}
$$

To verify the above inequalities, we follow Section 5.4.5.
D.1.3. ODEs for $c_{\omega}$ and $\omega_{x y}(0), \theta_{x x y}(0)$. Recall the estimate (5.59) and $f_{*}$ (5.60)
$c_{\omega}(p)=-\frac{4}{\pi} \int_{\mathbb{R}_{2}^{++}} f_{*}(y) p(y) d y=-\frac{4}{\pi}\left\langle p, f_{*}\right\rangle, f_{*}(y)=\frac{y_{1} y_{2}}{|y|^{4}}, \varphi_{M, i}=\max \left(\varphi_{i}, \mu_{g, i} \varphi_{g, i}\right),\left|W_{1, i}\right| \leq \varphi_{M, i}^{-1} E_{4}$.
For (5.61) and $q=1, \chi_{\text {ode }}$ (C.9), following Section5.6.1) we have the following linear estimates

$$
\begin{aligned}
\left|\left\langle\Gamma_{1, M}, q f_{*}\right\rangle\right| \leq & \langle | \nabla \cdot\left(\left(\bar{c}_{l} x+\overline{\mathbf{u}}^{N}\right) f_{*} q\right)\left|, \varphi_{M, 1}^{-1}\right\rangle E_{4}+\langle | U_{A} \bar{\omega}_{x}+V_{A} \bar{\omega}_{y}\left|, f_{*} q\right\rangle \\
\left|\left\langle\Gamma_{2, M}, q f_{*}\right\rangle\right| \leq & \langle | \nabla \cdot\left(\left(\bar{c}_{l} x+\overline{\mathbf{u}}^{N}\right) f_{*} q\right)-\bar{u}_{x}^{N} f_{*} q\left|, \varphi_{M, 2}^{-1}\right\rangle E_{4}+\langle | U_{A} \bar{\theta}_{x x}+V_{A} \bar{\theta}_{x y}\left|, f_{*} q\right\rangle \\
& +\left(\langle | U_{x, A}| | \bar{\theta}_{x}\left|+\left|V_{x, A}\right|\right| \bar{\theta}_{y}\left|, f_{*} q\right\rangle+\langle | \bar{v}_{x}^{N}\left|, \varphi_{M, 3}^{-1} f_{*} q\right\rangle\right.
\end{aligned}
$$

and nonlinear estimates

$$
\begin{aligned}
N\left(c_{\omega}\left(\omega_{1} q\right)\right) \triangleq & \frac{4}{\pi}\left\langle\tilde{\mathcal{N}}_{1}+\overline{\mathcal{F}}_{l o c, 1}-\mathcal{R}_{l o c, 1}, f_{*} q\right\rangle \leq \gamma_{1}\left|U_{x}(0)\right| E_{4}+\frac{4}{\pi}\langle | \overline{\mathcal{F}}_{l o c, 1}\left|+\left|\mathcal{R}_{l o c, 1}\right|+N_{\hat{W}_{2}, 1}, f_{*} q\right\rangle \\
& +\frac{4}{\pi}\langle | U \partial_{x}\left(f_{*} q\right)+V \partial_{y}\left(f_{*} q\right)\left|, \varphi_{M, i}^{-1}\right\rangle E_{4} \\
N\left(c_{\omega}\left(\eta_{1} q\right)\right) \triangleq & \frac{4}{\pi}\left|\left\langle\tilde{\mathcal{N}}_{2}+\overline{\mathcal{F}}_{l o c, 2}-\mathcal{R}_{l o c, 2}, f_{*} q\right\rangle\right| \leq \frac{4}{\pi}\langle | \overline{\mathcal{F}}_{l o c, 1}\left|+\left|\mathcal{R}_{l o c, 1}\right|+N_{\hat{W}_{2}, 1}+\left|\tilde{U}_{x}\right| \varphi_{M, 2}^{-1}+\left|V_{x}\right| \varphi_{M, 3}^{-1}, f_{*} q\right\rangle \\
& +\gamma_{2}\left|U_{x}(0)\right| E_{4}+\frac{4}{\pi}\langle | U \partial_{x}\left(f_{*} q\right)+V \partial_{y}\left(f_{*} q\right)\left|, \varphi_{M, i}^{-1}\right\rangle E_{4}
\end{aligned}
$$

where $\gamma=\left(\mu_{51}, \mu_{52}\right)$ for $q=\chi_{\text {ode }}$ (C.9), and $\gamma=\left(\mu_{5}, \mu_{62}\right)$ for $q=1$. We can use the weighted $L^{\infty}\left(\varphi_{i}\right)$ estimate for $\overline{\mathcal{F}}, \mathcal{R}, N_{\hat{W}_{2}}$ from (D.5). For $q=1, q f_{*}$ is singular near 0 . For $\mathbf{u}=\mathbf{U}, \overline{\mathbf{u}}^{N}$, we decompose
$u f_{*, x}+v f_{*, y}=\left(\frac{u}{x}+\frac{v}{y}\right)\left(x f_{*, x}+y f_{*, y}\right)+\left(\frac{u}{x}-\frac{v}{y}\right)\left(x f_{*, x}-y f_{*, y}\right)=2\left(\frac{u}{x}+\frac{v}{y}\right) f_{*}+\left(\frac{u}{x}-\frac{v}{y}\right) \frac{4 x y\left(x^{2}-y^{2}\right)}{|(x, y)|^{6}}$
In the first term, we exploit the cancellation near 0

$$
u / x+v / y=\tilde{u} / x+\tilde{v} / y, \quad \tilde{u}=u-u_{x}(0) x, \tilde{v}=v+u_{x}(0)
$$

which vanishes $O(|x|)$ near 0 . Then we apply the piecewise bounds of $\mathbf{U}, \overline{\mathbf{u}}^{N}$ to estimate the integrals. Moreover, from (5.62), if $q \equiv 1$ or $q=\chi_{o d e}=1$, we can simplify the integrand

$$
\nabla \cdot\left(x f_{*}(x)\right)=0, \quad \nabla \cdot\left(\left(\bar{c}_{l} x+\overline{\mathbf{u}}^{N}\right) q f_{*}\right)=\overline{\mathbf{u}}^{N} \cdot \nabla f_{*}, \quad q=1 .
$$

Recall $\chi_{o d e}$ from (C.9), the damping terms in the ODEs $\bar{c}_{\omega}^{N}\left\langle\omega_{1}, f_{*} q\right\rangle, 2 \bar{c}_{\omega}^{N}\left\langle\eta_{1}, f_{*} q\right\rangle$ (5.61) and Section 5.6. The stability conditions for $c_{\omega}\left(\omega_{1} \chi_{o d e}\right), c_{\omega}\left(\eta_{1} \chi_{o d e}\right), c_{\omega}\left(\eta_{1}\right)$ reads

$$
\begin{align*}
& -\bar{c}_{\omega}^{N} E_{*}-\mu_{51}^{-1}\left(\frac{4}{\pi}\left|\left\langle\Gamma_{1, M}, \chi_{\text {ode }} f_{*}\right\rangle\right|+N\left(c_{\omega}\left(\omega_{1} \chi_{o d e}\right)\right)+\mu_{52} E_{*}\right)>0, \\
& -2 \bar{c}_{\omega}^{N} E_{*}-\mu_{52}^{-1}\left(\frac{4}{\pi}\left|\left\langle\Gamma_{2, M}, \chi_{\text {ode }} f_{*}\right\rangle\right|+N\left(c_{\omega}\left(\eta_{1} \chi_{\text {ode }}\right)\right)>0,\right.  \tag{D.16}\\
& -2 \bar{c}_{\omega}^{N} E_{*}-\mu_{62}^{-1}\left(\frac{4}{\pi}\left|\left\langle\Gamma_{2, M}, f_{*}\right\rangle\right|+N\left(c_{\omega}\left(\eta_{1}\right)\right)>0 .\right.
\end{align*}
$$

For the estimate of $c_{\omega}(\omega)$ (5.68), using the estimate in Section 5.6 and the above estimates, we obtain the stability conditions,

$$
-\bar{\lambda}_{c_{\omega}} E_{*}-\mu_{6}^{-1}\left(\frac{4}{\pi}\left|\left\langle\Gamma_{1, M}, f_{*}\right\rangle\right|+N\left(c_{\omega}\left(\omega_{1}\right)\right)+\mu_{62} E_{*}+N_{\hat{W}_{2}, \text { ode }}\right)>0
$$

where $N_{\hat{W}_{2}, \text { ode }}$ bounds

$$
\begin{aligned}
N_{\hat{W}_{2}, o d e} \leq & \sum_{i \geq 2}\left|c_{\omega}\left(\hat{F}_{i}(0)\right) a_{i}\left(W_{1}, \hat{W}_{2}\right)(t)\right|+\sum_{i \geq 1}\left|a_{i}\left(W_{1}\left(t-T_{i}\right), \hat{W}_{2}\left(t-T_{i}\right)\right) \cdot c_{\omega}\left(\hat{F}_{i}\left(T_{i}\right)\right)\right| \\
& +\int_{0}^{t \wedge T_{i}}\left|a_{i}\left(W_{1}(t-s), \hat{W}_{2}(t-s)\right)\right| \cdot\left|\partial_{s} c_{\omega}\left(\hat{F}_{i}(s)\right)-\bar{\lambda}_{c_{\omega}} c_{\omega}\left(\hat{F}_{i}(s)\right)\right| d s
\end{aligned}
$$

and we have used $\mathbf{1}_{t \geq T_{i}} \leq 1$ (5.68). Under the bootstrap assumptions, all the terms $b_{i}=$ $a_{i}\left(W_{1}, \hat{W}_{2}\right)$ can be bounded by $c_{i} E_{4}$ for some constant $c_{i}$ (5.78). See Section 5.6. For linear modes, $a_{i}$ only depends on $W_{1}$.

For $\omega_{x y}(0), \theta_{x x y}(0)$ in (5.69), since $\left|c_{\omega}(\omega)\right|<\mu_{6} E_{4},\left|\omega_{x y}(0)<\mu_{8} E_{4}, \theta_{x x y}(0)<\mu_{6} E_{4}\right|$ (5.70), the stability conditions read

$$
\begin{aligned}
& \left(2 \bar{c}_{l}-\bar{c}_{\omega}\right) E_{*}-\mu_{8}^{-1}\left(\left(\mu_{7}+\mu_{6}\left|\bar{\omega}_{x y}(0)\right|\right) E_{*}+\mu_{6} \mu_{8} E_{*}^{2}+\left|\partial_{x y} \overline{\mathcal{F}}_{1}(0)\right|\right)>0, \\
& \left(3 \bar{c}_{l} / 2-\bar{c}_{\omega}\right) E_{*}-\mu_{7}^{-1}\left(\mu_{6}\left|\bar{\theta}_{x x y}(0)\right| E_{*}+\mu_{6} \mu_{7} E_{*}^{2}+\left|\partial_{x y} \overline{\mathcal{F}}_{2}(0)\right|\right)>0 .
\end{aligned}
$$

where we have used $2 \bar{c}_{l}-2 \bar{c}_{\omega}+\bar{u}_{x}(0)=\frac{3}{2} \bar{c}_{l}-\bar{c}_{\omega}$ (2.11). We check the stronger condition (5.73)

$$
\mu_{8} \mu_{6} E_{*}+\left|\partial_{x y} \overline{\mathcal{F}}_{1}(0)\right|<5 \mu_{6} E_{*}, \quad \mu_{7} \mu_{6} E_{*}+\left|\partial_{x y} \overline{\mathcal{F}}_{2}(0)\right|<10 \mu_{6} E_{*}
$$

To obtain (5.102), under the bootstrap assumption, we verify

$$
\begin{equation*}
\left|\widehat{W}_{2, i}\right|<100 E_{4}<100 E_{*} \tag{D.17}
\end{equation*}
$$

Plots of the nonlinear weighted $L^{\infty}$ estimate. In Figure 14 we plot the rigorous piecewise lower bounds of $\min \left(L H S_{i} / E_{*}, 0.1\right)$ in a region covering $D=\left[0,10^{15}\right]^{2}$, where $L H S_{i}$ denotes the left hand side (LHS) of (D.6) in the $i$-th equation. We normalize $L H S_{i}$ by $1 / E_{*}$ and take the minimum with a threshold to highlight the region with small linear damping factors.

In Figure 15, we plot $\min \left(L H S_{i} / E_{*}, c_{i}\right)$ with $c=(0.4,0.1,0.4)$ with $L H S$ being the left hand side of (D.10). All of these bounds are positive. We only use 7 approximation terms for the velocity in $[0,200]^{2}$ away from the boundary (see (4.37), (4.28)). Thus the stability factor is weaker for $x$ not very large and in the bulk. We can get a better stability factor by using few more approximations. As we can close the full estimates, we do not need such an improvement.

The weighted $L^{\infty}\left(\varphi_{4}\right)$ estimate has a much larger stability factor $\geq 1.5 E_{*}$, and thus we do not plot it. Beyond $D$, we have much larger damping factors and use the $L^{\infty}$ estimate in Section 8.6 in the supplementary material I 18 .

We cannot visualize the Hölder estimate conditions (D.15) and refer them to the codes 13 .


Figure 14. Estimates of nonlinear $L^{\infty}\left(\varphi_{i}\right)$ stability conditions.


Figure 15. Estimates of nonlinear $L^{\infty}\left(\varphi_{g, i}\right)$ stability conditions.
D.2. Estimate the Hölder weights. The Hölder weights $g=g_{i}(\mathbb{C} .1)$ are $-1 / 2$ homogeneous. In our energy estimates, we estimate several 0 -homogeneous quantities related to $g$ for $h_{i} \geq 0$

$$
f(h)=h_{k}^{1 / 2} g(h), h_{k} \frac{\left(\partial_{j} g\right)}{g}(h),|h| \frac{\left(\partial_{j} g\right)}{g}(h), k, j=1,2, \quad \frac{g_{i_{1}}(h)}{g_{i_{2}}(h)}, 1 \leq i_{1}, i_{2} \leq 3
$$

Since $f(h)=f\left(\frac{h_{1}}{h_{2}}, 1\right)$ for $h_{2} \neq 0$ and $f(h)=f\left(1, \frac{h_{2}}{h_{1}}\right), h_{1} \neq q 0$, we can estimate it by partitioning $\left(h_{1}, h_{2}\right) \in[0,1] \times\{1\},\{1\} \times[0,1]$ and using the monotonicities of $g, \partial_{j} g$. From $g=g_{i}$ (C.1), we have

$$
g(s)=\frac{1}{A_{1}(s)^{1 / 2}+A_{2}(s)^{1 / 2}}, \partial_{i} g=-\frac{1}{2} \frac{a_{1 i} A_{1}^{-1 / 2}+a_{2 i} A_{2}^{-1 / 2}}{\left(A_{1}^{1 / 2}+A_{2}^{1 / 2}\right)^{2}}, \frac{\partial_{i} g}{g}=-\frac{1}{2} \frac{a_{1 i} A_{1}^{-1 / 2}+a_{2 i} A_{2}^{-1 / 2}}{A_{1}^{1 / 2}+A_{2}^{1 / 2}} .
$$

for $A_{i}=a_{i 1} s_{1}+a_{i 2} s_{2}$ with $a_{i j}>0$. Clearly, $g$ is decreasing in $\left|s_{i}\right|$. For $s_{1}, s_{2}>0$, since $A_{i}$ is increasing in $s_{1}, s_{2}, \partial_{i} g, \frac{\partial_{i} g}{g}$ are negative and increasing in $s_{1}, s_{2}$. It follows that $\left|\frac{\partial_{i} g}{g}\right|$ is decreasing in $s_{1}, s_{2}$.

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# SUPPLEMENTARY MATERIAL FOR "STABLE NEARLY SELF-SIMILAR BLOWUP OF THE 2D BOUSSINESQ AND 3D EULER EQUATIONS WITH SMOOTH DATA I: ANALYSIS" 

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#### Abstract

In Part I [3], we have established linear energy estimates and nonlinear $L^{\infty}$ energy estimate and estimated typical terms in the nonlinear Hölder energy estimate. In this supplementary material, we estimate each nonlinear terms in $C^{1 / 2}$ estimate, and estimate the piecewise bounds of the damping terms near 0 , which is not straightforward due to their low regularities and the singular weights near 0 . Moreover, we perform the energy estimate in the very far-field using the asymptotics of the estimates.


## 8. Estimates of nonlinear and error terms and piecewise bounds

Firstly, we recall some notations from [1,3]. Recall from [3] the following variables
(8.1) $W_{1,1}=\omega_{1}, W_{1,2}=\eta_{1}, W_{1,3}=\xi_{1}, \quad \widehat{W}_{2,1}=\hat{\omega}_{2}, \widehat{W}_{2,2}=\hat{\eta}_{2}, \widehat{W}_{2,3}=\hat{\xi}_{2}, \quad \bar{W}=\left(\bar{\omega}, \bar{\theta}_{x}, \bar{\theta}_{y}\right)$, and the full energy $E_{4}$ for $W_{1}, W$ satisfying

$$
\begin{align*}
& E_{4}(t) \geq \max \left(\max _{i}\left\|W_{1, i} \varphi_{i}\right\|_{\infty}, \sqrt{2} \tau_{1}^{-1}\left\|\omega_{1}\left|x_{1}\right|^{-1 / 2} \psi_{1}\right\|_{\infty}, \max _{i} \mu_{h, i}\left[W_{1, i} \psi_{i}\right]_{C_{g_{i}}^{1 / 2}}\right.  \tag{8.2}\\
& \left.\max _{i} \mu_{g, i}\left\|W_{1, i} \varphi_{g, i}\right\|_{\infty}, \mu_{6}^{-1}\left|c_{\omega}(\omega)\right|\right), \quad \mu_{h}=\tau_{1}^{-1}\left(1, \mu_{1}, \mu_{2}\right), \quad \mu_{g}=\tau_{2}\left(\mu_{4}, 1,1\right),
\end{align*}
$$

where the parameters $\tau_{i}, \mu_{i j}$ are given in (A.3), and the weights $\psi, \varphi$ are chosen in Appendix C. 1 in [3]. We use $\psi$ to denote radial weights, and $\varphi$ to denote mixed weights in Appendix C. 1 in [3]. We do not write the full $E_{4}$ since some norms are not used in the supplementary material. Under the bootstrap assumption, for $\mu_{5}$ is given in (A.3), we get

$$
\begin{equation*}
\left|c_{\omega}\left(\omega_{1}\right)\right|<\mu_{5} E_{4} \tag{8.3}
\end{equation*}
$$

Denote the mesh for computing the approximate steady state (see Appendix C. 1 in Part II [1]) and various domain

$$
\begin{align*}
& y_{1}<y_{2}<.<y_{N}, \quad Q_{i j}=\left[y_{i}, y_{i+1}\right] \times\left[y_{j}, y_{j+1}\right], \quad N=748, \quad N_{1}=707 \\
& N_{2}=730<N, \quad R_{1}=5000, R_{2}=y_{N_{2}}, \quad \Omega_{n e a r} \triangleq\left[0, y_{N_{1}}\right]^{2} \tag{8.4}
\end{align*}
$$

We have $y_{N}>10^{15}$.
We decompose the weighted $L^{\infty}$ estimate into the region $\left[0, y_{N}\right]^{2}$ and the far-field $\left(\left[0, y_{N-1}\right]^{2}\right)^{c}$.
We decompose the Hölder estimate with $x, z$ into the five parts. In case (a.1)-(a.3), we assume that $x \in Q_{i, j}, z \in Q_{p, q}$.
(a.1) $x$ or $z$ in the near-field and the bulk $\Omega_{\text {near }}$ with $|x-z| \leq R_{1}$. Since $\Omega_{n e a r}$ is the union of $Q_{i j}, \max (i, j) \leq N_{1}-1$, we check the stability inequality for $\max (\min (i, p), \min (j, q))<$ $N_{1},|i-p|,|j-q| \leq n_{R_{1}}$, where $n_{R_{1}}$ satisfies $y_{n_{R_{1}}}>R_{1}$. For $\max (|i-p|,|j-q|)>n_{R_{1}}$, since the spacing of mesh $y_{i+1}-y_{i}$ is increasing, we get

$$
|x-z| \geq \operatorname{dis}\left(Q_{i j}, Q_{p q}\right) \geq y_{n_{R_{1}}}>R_{1}
$$

(a.2) $x$ or $z$ in $\left[1, y_{N-2}\right]^{2} \backslash \Omega_{n e a r}$ with $|i-p|,|j-q| \leq 1$. We check the stability for $|i-p|,|j-q| \leq$ 1 similar to the case of (a.1).
(a.3) $x, z \in\left[1, y_{N}\right]^{2} \backslash\left[1, y_{N_{1}-2}\right]^{2}$ with $|i-p|,|j-q| \geq 2$. In this case, $x, z$ are in the far-field and not very close, the Hölder estimate follows from the weighted $L^{\infty}$ estimate.
(b) $x, z$ in the far-field with $|z|_{\infty},|x|_{\infty} \geq y_{N_{2}}>10^{12}$ and any $|x-z| \geq 0$. We use the asymptotic estimates.
(c) $x \in \Omega_{\text {near }}$ and $z \in \mathbb{R}_{2}^{++}$with $|x-z| \geq R_{1}$. In this case, $x$ and $z$ are far apart, the Hölder estimate follows from the weighted $L^{\infty}$ estimate and triangle inequality.

Since $y_{i+1}-y_{i}>R_{1}$ for $i \geq 700$, cases (a.1), (a.2), (a.3), (c) cover the estimate for $x$ or $z$ in $\left[1, y_{N-2}\right]^{2}$. The remaining case $x, z \notin\left[1, y_{N-2}\right]^{2}$ is covered by case (b).

The most important case for the Hölder estimate is case (a.1) and has been discussed in Section 5 in [3], and we have outlined the method for the verification. Estimates in cases (a.2),(a.3),(b), (c) are relatively easy since the bad term in the estimates are very small due to large $|x-z|$, or the decay of the coefficients of the nonlocal terms in the far-field. Similarly, the $L^{\infty}$ stability estimate in the far-field is simple since we have large damping factors.

Organization. The remaining sections are organized as follows. In Section 8.1, we recall the modified decompositions of linear and nonlinear terms from Section 5.8 in [3]. In Sections 8.2 . 8.5, we estimate the piecewise bounds for the damping terms near 0 and estimate the modified nonlinear terms. In Sections 8.6, 8.7, we discuss the energy estimate for $x$ outside the computational domain, e.g. $|x|>10^{12}$, using the asymptotics of the coefficients in the energy estimates. In Section 8.8, we discuss the $C^{1 / 2}$ estimate with large $|x-z|$ related to case (c).
8.1. Decompositions of the nonlinear and nonlocal error terms. In this section, we recall the nonlocal error terms and the modified linearization equations from Section 5.8 in [3]. Then we will estimate the nonlinear terms with more details in Section 8.5.
8.1.1. Decomposition and estimates of the velocity. Recall that $\mathbf{u}_{A}(f)$ is the velocity after subtracting the approximation term $\tilde{\hat{\mathbf{u}}}$ defined in Section 5 3]

$$
\mathbf{u}_{A}(f) \triangleq \tilde{\mathbf{u}}(f)-\tilde{\hat{\mathbf{u}}}(f)=\mathbf{u}-\hat{\mathbf{u}}, \quad \mathbf{u}_{x, A} \triangleq \tilde{\mathbf{u}}_{x}(f)-\tilde{\hat{\mathbf{u}}}_{x}(f), \quad \mathbf{u}_{y, A} \triangleq \tilde{\mathbf{u}}_{y}(f)-\tilde{\hat{\mathbf{u}}}_{y}(f)
$$

and the notation $\tilde{u}=u-u_{x}(0) x, \tilde{v}=v+u_{x}(0) y$.
Firstly, we recall the following decomposition of error from Section 5.8 in 3]. Denote by $\bar{\phi}^{N}, \hat{\phi}^{N}$ the stream function constructed numerically for the Poisson equation $-\Delta \bar{\phi}=\bar{\omega},-\Delta \hat{\phi}=$ $\hat{\omega}_{2}, \overline{\mathbf{u}}^{N}, \hat{\mathbf{u}}^{N}$ the associated velocity, and $\bar{\varepsilon}, \hat{\varepsilon}$ the errors

$$
\begin{align*}
& \overline{\mathbf{u}}^{N}=\mathbf{u}\left(-\Delta \bar{\phi}^{N}\right)=\nabla^{\perp} \bar{\phi}^{N}, \quad \bar{\varepsilon}=\bar{\omega}-(-\Delta) \bar{\phi}^{N} \\
& \hat{\varepsilon}=\hat{\omega}-(-\Delta) \hat{\phi}^{N}, \quad \mathbf{u}(\bar{\omega})=\overline{\mathbf{u}}^{N}+\mathbf{u}(\bar{\varepsilon}), \quad \mathbf{u}\left(\hat{\omega}_{2}\right)=\mathbf{u}\left(-(-\Delta) \hat{\phi}^{N}\right)+\mathbf{u}(\hat{\varepsilon}) \tag{8.5}
\end{align*}
$$

The error $\varepsilon$ only vanishes $O\left(|x|^{2}\right)$ and we perform a correction near 0 . For $\varepsilon=\bar{\varepsilon}$ or $\hat{\varepsilon}$ and some cutoff function $\chi_{\varepsilon}=1+O\left(|x|^{4}\right)$ near 0 , we decompose

$$
\begin{align*}
& \varepsilon=\varepsilon_{1}+\varepsilon_{2}, \quad \varepsilon_{2}=\varepsilon_{x y}(0) \Delta\left(\frac{x^{3} y}{2} \chi_{\varepsilon}\right), \quad \mathbf{u}\left(\varepsilon_{2}\right)=\nabla^{\perp}(-\Delta)^{-1} \varepsilon_{2}=\frac{1}{2} \varepsilon_{x y}(0) \nabla^{\perp}\left(x^{3} y \chi_{\varepsilon}\right)  \tag{8.6}\\
& \mathbf{u}(\varepsilon)=\mathbf{u}\left(\varepsilon_{1}\right)+\mathbf{u}\left(\varepsilon_{2}\right)=\mathbf{u}_{A}\left(\varepsilon_{1}\right)+\hat{\mathbf{u}}\left(\varepsilon_{1}\right)+\mathbf{u}\left(\varepsilon_{2}\right), \quad u_{x}(\varepsilon)(0)=u_{x}\left(\varepsilon_{1}\right)(0)
\end{align*}
$$

where $u_{x}\left(\varepsilon_{2}\right)(0)=-\varepsilon_{x y}(0) /\left.2 \cdot \partial_{y}\left(x^{3} y \chi_{\varepsilon}\right)\right|_{x, y=0}=0$. We perform a similar decomposition for $\nabla \mathbf{u}(\varepsilon)$. We choose $\chi_{\varepsilon}$ in the above form such that $\varepsilon_{2}=\chi_{x y}(0) x y+O\left(|x|^{4}\right)$ and we can obtain $\mathbf{u}\left(\chi_{\varepsilon}\right)$ explicitly. We choose $\chi_{\bar{\varepsilon}}$ for $\bar{\varepsilon}$ and $\chi_{\hat{\varepsilon}}$ for $\hat{\varepsilon}$ in (A.2) and they have different parameters.

Let $\omega$ be the perturbation without decomposition. We combine these errors and perturbations and perform the following decompositions

$$
\begin{align*}
& \mathbf{U}=(U, V)=\mathbf{u}(\omega+\bar{\varepsilon}), \quad \mathbf{U}_{A}=\mathbf{u}_{A}\left(\omega_{1}+\bar{\varepsilon}_{1}+\hat{\varepsilon}_{1}\right), \quad \mathbf{U}=\mathbf{U}_{A}+U_{x}(0)(x,-y)+\mathbf{U}_{a p p}, \\
& \mathbf{U}_{a p p}=\tilde{\hat{\mathbf{u}}}\left(\omega_{1}+\bar{\varepsilon}_{1}+\hat{\varepsilon}_{1}\right)+\mathbf{u}\left(\bar{\varepsilon}_{2}+\hat{\varepsilon}_{2}\right)+\tilde{\mathbf{u}}\left(-\Delta \hat{\phi}^{N}\right) \tag{8.7}
\end{align*}
$$

Similarly, we decompose $\nabla \mathbf{U}$ and define $(\nabla \mathbf{U})_{A},(\nabla \mathbf{U})_{\text {app }}$.

Estimates of the velocity. We use the method in Section 4 in [1] to establish the same type of weighted $L^{\infty}$ and $C^{1 / 2}$ estimates for $\mathbf{u}_{A}\left(\hat{\varepsilon}_{1}\right), \mathbf{u}_{A}\left(\bar{\varepsilon}_{1}\right)$ as those for $u_{A}\left(\omega_{1}\right)$, i.e. the weighted $L^{\infty}$ estimate for $\mathbf{u}_{A}, \nabla \mathbf{u}_{A}$, and the Hölder estimate for $\psi_{u} \mathbf{u}_{A}, \psi_{d u}(\nabla \mathbf{u})_{A}$. For the perturbation $\mathbf{u}_{A}\left(\omega_{1}\right)$, we use the norm $\left\|\omega_{1} \varphi_{1}\right\|_{\infty},\left\|\omega_{1} \varphi_{g, 1}\right\|_{\infty},\left[\omega_{1} \psi_{1}\right]_{C_{x_{i}}^{1 / 2}}$ and estimate various quantities involving the velocity, e.g.

$$
\left|u_{i j, A}(x)\right| \leq C_{i j, 1}(x, \rho)\left\|\omega_{1} \rho\right\|_{\infty}+C_{i j, 2}(x, \rho)\left[\omega_{1} \psi \|_{C_{x}^{1 / 2}}+C_{i j, 3}(x, \rho)\left[\omega_{1} \psi \|_{C_{y}^{1 / 2}}, \quad \rho=\varphi_{1}, \varphi_{g, 1}\right.\right.
$$

For the error $\mathbf{u}_{A}(f), f=\bar{\varepsilon}_{1}, \hat{\varepsilon}_{1}$, we use the norm $\left\|f \varphi_{\text {elli }}\right\|_{\infty},\left[f \psi_{1}\right]_{C_{x_{i}}^{1 / 2}}$. We will combine the estimate of the nonlocal error term and the perturbation in the energy estimate in Section 8.1.3.

The terms $\tilde{\mathbf{u}}\left((-\Delta) \hat{\phi}^{N}\right)=\nabla^{\perp} \phi^{N}+\hat{\phi}_{x y}^{N}(0)(x,-y)$ vanishes like $O\left(|x|^{3}\right)$ near 0 , and we estimate its $C^{3}$ bounds folllowing Section 8.1.2. The terms $I_{1}=\tilde{\hat{\mathbf{u}}}\left(\omega_{1}+\bar{\varepsilon}_{1}+\hat{\varepsilon}_{1}\right), I_{2}=\mathbf{u}\left(\bar{\varepsilon}_{2}+\hat{\varepsilon}_{2}\right)$ consist of finite rank operators on $\omega_{1}, \bar{\varepsilon}_{1}, \bar{\varepsilon}_{2}, \hat{\varepsilon}_{1}, \hat{\varepsilon}_{2}$ with smooth coefficients. Moreover, by definition, $I_{1}, I_{2}, \tilde{\mathbf{u}}\left((-\Delta) \hat{\phi}^{N}\right)=O\left(|x|^{3}\right),(\nabla \mathbf{u})_{a p p}=O\left(|x|^{2}\right)$ near $x=0$. We establish their piecewise $C^{3}$ estimate near the origin and $C^{1}$ estimate away from the origin, and combine the estimates of these terms together. See nonlinear estimates in Section 5.6 in 3 . This allows us to estimate the $C^{3}$ bounds of $\mathbf{U}_{a p p}$ and $\mathbf{U}_{a p p}=O\left(|x|^{3}\right)$. Similarly, we estimate $(\nabla \mathbf{U})_{a p p}$. We factor out the term $u_{x}(0) x$ in the above decomposition since the constant in our estimates for such a term is larger than others, and control it using the energy $E_{4}$. Using these $C^{k}$ estimates of $\mathbf{U}_{\text {app }}$, we can further obtain its $C^{1 / 2}$ estimate.

We can further estimate the $C^{1 / 2}$ norm of $\mathbf{U}, \nabla \mathbf{U}, \tilde{\mathbf{U}}, \nabla \tilde{\mathbf{U}}$. For example, we use

$$
\tilde{U}_{x}=\left(U_{x, A} \psi_{1}\right) \psi_{1}^{-1}+U_{x, a p p}
$$

and the $C^{1 / 2}$ estimates of $U_{x, A} \psi_{1}, \psi_{1}^{-1}, U_{x, \text { app }}$ to obtain the $C^{1 / 2}$ estimate of $\tilde{U}_{x}$. Similarly, using the $C^{1}$ estimates of $\psi_{1}^{-1}, \psi_{u}^{-1}, \mathbf{U}_{a p p}$ and the $C^{1 / 2}$ estimate of $\mathbf{U}_{A} \psi_{u}, \mathbf{U}_{x, A} \psi_{1}$, we can obtain the $C^{1 / 2}$ estimate of $\tilde{\mathbf{U}}, \overline{\nabla \mathbf{U}}$. See Section 8.2 for basic Hölder estimates of a product.
8.1.2. Estimate of local and nonlocal terms. Recall from Sections 3, 3.7 in Part II 1 that we represent $\hat{W}_{2}, \hat{\phi}^{N}$ using Duhamel's principle and piecewise fifth order polynomials and an analytic basis. For variables that depend locally on $\hat{W}_{2}$ (8.1) and $\hat{\phi}^{N}$, under the bootstrap assumption $E_{4}(t)<E_{*}$, we can estimate piecewise $C^{k}, k \leq 4$ bounds of $\nabla^{\perp} \hat{\phi}^{N}, \hat{W}_{2}, \hat{W}_{2}-(-\Delta) \hat{\phi}^{N}$ following Section 5.7 [3] using the energy $E_{4}$, e.g.

$$
\left|\nabla^{k} \hat{W}_{2, i}(x)\right| \leq C_{k, i}(x) E_{4}
$$

8.1.3. Decomposition of the nonlinear and error terms. We introduce the bilinear operator $B_{o p, i}$ for $(\mathbf{u}, M), G=\left(G_{1}, G_{2}, G_{3}\right)$

$$
\begin{align*}
& \mathcal{B}_{o p, 1}=-\mathbf{u} \cdot \nabla G_{1}+M_{11}(0) G_{1}, \quad \mathcal{B}_{o p, 2}=-\mathbf{u} \cdot \nabla G_{2}+2 M_{11}(0) G_{2}-M_{11} G_{2}-M_{21} G_{3} \\
& \mathcal{B}_{o p, 3}=-\mathbf{u} \cdot \nabla G_{3}+2 M_{11}(0) G_{3}-M_{12} G_{2}-M_{22} G_{3} \tag{8.8}
\end{align*}
$$

If $M=\nabla \mathbf{u}, M_{11}=u_{x}, M_{12}=u_{y}, M_{21}=v_{x}, M_{22}=v_{y}$, then we drop $M$ to simplify the notation

$$
\begin{align*}
& \mathcal{B}_{o p, 1}(\mathbf{u}, G)=-\mathbf{u} \cdot \nabla G_{1}+u_{x}(0) G_{1}, \mathcal{B}_{o p, 2}(\mathbf{u}, G)=-\mathbf{u} \cdot \nabla G_{2}+2 u_{x}(0) G_{2}-u_{x} G_{2}-v_{x} G_{3},  \tag{8.9}\\
& \mathcal{B}_{o p, 3}(\mathbf{u}, G)=-\mathbf{u} \cdot \nabla G_{3}+2 u_{x}(0) G_{3}-u_{y} G_{2}-v_{y} G_{3} .
\end{align*}
$$

We abuse the notation to write

$$
\begin{equation*}
\mathcal{B}_{o p, i}\left(\mathbf{u}_{A}, G\right)=\mathcal{B}_{i}\left(\left(\mathbf{u}_{A},(\nabla \mathbf{u})_{A}\right), G\right), \quad \mathcal{B}_{o p, i}\left(\mathbf{u}_{a p p}, G\right)=\mathcal{B}_{i}\left(\left(\mathbf{u}_{a p p},(\nabla \mathbf{u})_{a p p}\right), G\right) \tag{8.10}
\end{equation*}
$$

In $\mathcal{B}_{o p, i}\left(\mathbf{u}_{A}, G\right), \mathcal{B}_{o p, i}\left(\mathbf{u}_{a p p}, G\right)$, we just replace $\nabla \mathbf{u}$ in 8.9$)$ by $(\nabla \mathbf{u})_{A},(\nabla \mathbf{u})_{a p p}$. Note that $\mathbf{u}_{A}$ does not satisfy the differential relation among $\mathbf{u}_{A}$ and $(\nabla \mathbf{u})_{A}$. We apply the operator $\mathcal{B}_{o p, i}$ to $\mathbf{u}, \tilde{\mathbf{u}}, \mathbf{u}_{A}, \mathbf{u}_{\text {app }}$.

Recall $W_{1}$ from (8.1) and the following modified linearized equations from Section 5.8 in 3 (8.11)

$$
\begin{aligned}
\partial_{t}\left(W_{1, i} \rho_{i}\right)+\left(\bar{c}_{l} x+\overline{\mathbf{u}}^{N}+\mathbf{U}\right) \cdot \nabla\left(W_{1, i} \rho_{i}\right) & =C_{i}(x) W_{1, i} \rho_{i}+\tilde{N}_{i}\left(\rho_{i}\right)+\left(B_{\text {modi }, i}+\overline{\mathcal{F}}_{l o c, i}+\mathcal{R}_{l o c, i}\right) \rho_{i}, \\
C \cdot(x) & =\left(\bar{c}_{\omega}^{N}, 2 \bar{c}_{\omega}^{N}-\bar{u}_{x}^{N}, 2 \bar{c}_{\omega}^{N}+\bar{u}_{x}^{N}\right),
\end{aligned}
$$

where $(\bar{\omega}, \bar{\theta})$ is the approximate steady state, $\overline{\mathbf{u}}^{N}$ is given in 8.6), and $\bar{c}_{l}, \bar{c}_{\omega}$ are determined by

$$
\begin{equation*}
\bar{c}_{l}=2 \frac{\bar{\theta}_{x x}(0)}{\bar{\omega}_{x}(0)}, \quad \bar{c}_{\omega}^{N}=\frac{1}{2} \bar{c}_{l}+\bar{u}_{x}^{N}(0) \tag{8.12}
\end{equation*}
$$

Here $B_{\text {modi, } i}$ denotes the linear bad terms

$$
\begin{align*}
& B_{\text {modi }, 1}(x) \triangleq \eta_{1}-\mathbf{U}_{A} \cdot \nabla \bar{\omega}, \quad B_{\text {modi }, 2}(x) \triangleq-\bar{v}_{x}^{N} \xi_{1}-\mathbf{U}_{A} \cdot \nabla \bar{\theta}_{x}-\mathbf{U}_{x, A} \cdot \nabla \bar{\theta} \\
& B_{\text {modi }, 3}(x) \triangleq-\bar{u}_{y}^{N} \eta_{1}-\mathbf{U}_{A} \cdot \nabla \bar{\theta}_{y}-\mathbf{U}_{y, A} \cdot \nabla \bar{\theta} \tag{8.13}
\end{align*}
$$

and $\tilde{\mathcal{N}}_{i}$ denote the modified nonlinear terms in the estimate of $W_{1, i} \rho_{i}$

$$
\begin{align*}
& \tilde{\mathcal{N}}_{1}\left(\rho_{1}\right)=\left(\mathbf{U} \cdot \nabla \rho_{1}\right) \cdot \omega_{1}+U_{x}(0) \omega_{1} \rho_{1}+\mathcal{B}_{o p, 3}\left(\tilde{\mathbf{U}}, \hat{W}_{2}\right) \rho_{1}+U_{x}(0) \hat{W}_{2,1, M} \\
& \tilde{\mathcal{N}}_{2}\left(\rho_{2}\right)=\left(\mathbf{U} \cdot \nabla \rho_{2}\right) \cdot \eta_{1}+\left(U_{x}(0) \eta_{1}-\tilde{U}_{x} \eta_{1}-\tilde{V}_{x} \xi_{1}\right) \rho_{2}+\mathcal{B}_{o p, 2}\left(\tilde{\mathbf{U}}, \widehat{W}_{2}\right) \rho_{2}+U_{x}(0) \hat{W}_{2,2, M}  \tag{8.14}\\
& \tilde{\mathcal{N}}_{3}\left(\rho_{3}\right)=\left(\mathbf{U} \cdot \nabla \rho_{3}\right) \cdot \xi_{1}+\left(3 U_{x}(0) \xi_{1}-\tilde{U}_{y} \eta_{1}-\tilde{V}_{y} \xi_{1}\right) \rho_{3}+\mathcal{B}_{o p, 3}\left(\tilde{\mathbf{U}}, \widehat{W}_{2}\right) \rho_{3}+U_{x}(0) \hat{W}_{2,3, M}
\end{align*}
$$

where $\mathcal{B}_{o p, i}$ is given in (8.9), $\hat{W}_{2, i, M}$ is given by

$$
\begin{align*}
& \hat{W}_{2,,, M}=\left(\hat{\omega}_{2}, \hat{\eta}_{2}, \hat{\xi}_{2}\right), \quad \hat{\omega}_{2, M} \triangleq \hat{\omega}_{2}-x \partial_{x} \hat{\omega}_{2}+y \partial_{y} \hat{\omega}_{2}-\hat{\omega}_{2, x y}(0) f_{\chi, 1}  \tag{8.15}\\
& \hat{\eta}_{2, M} \triangleq \hat{\eta}_{2}-x \partial_{x} \hat{\eta}_{2}+y \partial_{y} \hat{\eta}_{2}-\hat{\eta}_{2, x y}(0) f_{\chi, 2}, \quad \hat{\xi}_{2, M} \triangleq 3 \hat{\xi}_{2}-x \partial_{x} \hat{\xi}_{2}+y \partial_{y} \hat{\xi}_{2}-\hat{\xi}_{2, x x}(0) f_{\chi, 3}
\end{align*}
$$

with $f_{\chi, i}$ defined in (A.2). The terms $\overline{\mathcal{F}}_{l o c, i}, \mathcal{R}_{l o c, i}$ are the essential local part of the residual error and residual operators. Since we have estimated $\overline{\mathcal{F}}_{l o c, i}, \mathcal{R}_{l o c, i}$ in Part I [3] and Part II [1], we do not present their formulas and refer them to Section 5.8 in 3 .

We introduce notations for the damping terms $\mathcal{T}_{d}(\rho), d_{i}(\varphi)$, the advection $b(x)$, and the damping factor $d_{g_{i}}$ in the Hölder estimate derived in Section 5.1 [3]

$$
\begin{align*}
& b(x)=\bar{c}_{l} x+\overline{\mathbf{u}}^{N}+\mathbf{U}, \quad d_{g, i} \triangleq \frac{(b(x)-b(z)) \cdot\left(\nabla g_{i}\right)(x-z)}{g_{i}(x-z)}, \mathcal{T}_{d, N}(\rho)=\rho^{-1}(\mathbf{U} \cdot \nabla \rho) \\
& \mathcal{T}_{d}(\rho)=\rho^{-1}\left(\left(\bar{c}_{l} x+\overline{\mathbf{u}}^{N}+\mathbf{U}\right) \cdot \nabla \rho\right)=\rho^{-1}(b \cdot \nabla \rho)=\mathcal{T}_{d, L}+\mathcal{T}_{d, N}  \tag{8.16}\\
& \mathcal{T}_{d, L}(\rho)=\rho^{-1}\left(\left(\bar{c}_{l} x+\overline{\mathbf{u}}^{N}\right) \cdot \nabla \rho\right), \quad d_{l o c, 1}=\bar{c}_{\omega}^{N}, d_{l o c, 2}=2 \bar{c}_{\omega}^{N}-\bar{u}_{x}^{N}, d_{l o c, 3}=2 \bar{c}_{\omega}^{N}+\bar{u}_{x}^{N} \\
& d_{i}(\rho) \triangleq \mathcal{T}_{d}(\rho)+d_{l o c, i}, \quad d_{i, L}(\rho) \triangleq \mathcal{T}_{d, L}(\rho)+d_{l o c, i}, \quad d_{i}(\rho)=d_{i, L}(\rho)+\mathcal{T}_{d, N}(\rho)
\end{align*}
$$

The subscripts $d, L$ are short for damping, linear. Denote by $B_{a d, i}\left(\rho_{i}\right)$ all the bad terms in the weighted estimates

$$
\begin{equation*}
B_{a d, i}\left(\rho_{i}\right) \triangleq B_{\text {modi,i }} \cdot \rho_{i}+\mathcal{N} \mathcal{F}_{i}\left(\rho_{i}\right), \quad \mathcal{N} \mathcal{F}_{i}\left(\rho_{i}\right) \triangleq \tilde{\mathcal{N}}_{i}\left(\rho_{i}\right)+\overline{\mathcal{F}}_{l o c, i} \rho_{i}-\mathcal{R}_{l o c, i} \rho_{i} \tag{8.17}
\end{equation*}
$$

8.2. Basic Hölder estimates. We use the following notations for the Hölder estimate

$$
\begin{align*}
& \delta_{i}(f, x, z) \triangleq \frac{|f(x)-f(z)|}{|x-z|^{1 / 2}}, \quad \Delta_{i}(f, x, z) \triangleq \frac{f(z)-f(x)}{z_{i}-x_{i}}, \quad z_{i}>x_{i}, z_{3-i}=x_{3-i}  \tag{8.18}\\
& \delta(f)(x, z) \triangleq f(x)-f(z)
\end{align*}
$$

By definition, we have $\delta_{i}(f, x, z)=\delta_{i}(f, z, x), \Delta_{i}(f, x, z)=\Delta_{i}(f, z, x)$. By abusing the notations of $\delta_{i}$, we denote by $\delta_{i}(f, g, x, z)$ a basic $C^{1 / 2}$ estimate for product

$$
\begin{equation*}
\delta_{i}(f, g, x, z)=\min _{(p, q)=(x, z),(z, x)} \delta_{i}(f, x, z)|g(p)|+\delta_{i}(g, x, z)|f(q)|, \quad x_{3-i}=z_{3-i} \tag{8.19}
\end{equation*}
$$

Clearly, if $g=1$, we get $\delta_{i}(f, g, x, z)=\delta_{i}(f, x, z)$. We have the following basic estimates

$$
\begin{align*}
& \delta(f g)(x, z)=\delta(f)(x, z) g(p)+\delta(g)(x, z) f(q) \\
& \delta_{i}(f g, x, z) \leq \delta_{i}(f, g, x, z), \quad x_{3-i}=z_{3-i} \tag{8.20}
\end{align*}
$$

Given the piecewise $C_{x}^{1 / 2}, C_{y}^{1 / 2}$ estimates of $f$, we use the following method for the piecewise Hölder estimate of $f$ with Hölder weight $g$ and two points $(x, z)$

$$
\begin{align*}
& |(f(z)-f(x)) g(h)|=|(f(z)-f(w)+f(w)-f(x)) g(h)| \\
\leq & \left(\delta_{1}(f, w, x)\left|h_{1}\right|^{1 / 2}+\delta_{2}(f, z, w)\left|h_{2}\right|^{1 / 2}\right) g(h), \quad h=z-x, \quad w=\left(z_{1}, x_{2}\right) \tag{8.21}
\end{align*}
$$



Figure 1. Left, right figures correspond to the locations of $(x, z)$ in cases (1) $s \geq 0$, (2) $s<0, s=\left(z_{1}-x_{1}\right)\left(z_{2}-x_{2}\right)$. The red line and blue line represents two choices of the $C_{x}^{k}, C_{y}^{k}, k=\frac{1}{2}$ or 1 estimates used to estimate $f(x)-f(z)$.

The function $\left|h_{i}\right|^{1 / 2} g(h)$ is 0-homogeneous and we apply the method in Section 8.6.1 to estimate it. For $\tilde{w}=\left(x_{1}, z_{2}\right)$, we derive another estimate. We optimize two estimates for $|\delta(f)(x, z)| g(x-z)$. In Figure 1 we illustrate the locations of $x, z$ and the $C_{x_{i}}^{1 / 2}\left(\left(\delta_{i}(f, p, q)\right)\right)$ estimates and the triangle inequality used to estimate $\delta(f, x, z) g(h)$. We introduce $\delta_{\square}$ to denote this estimate and similar estimate for the product

$$
\begin{gather*}
\delta_{\square}(f, x, z, s) \triangleq \min \left(\delta_{1}\left(f, x,\left(z_{1}, x_{2}\right)\right) s_{1}^{1 / 2}+\delta_{2}\left(f,\left(z_{1}, x_{2}\right), z\right) s_{2}^{1 / 2},\right. \\
\\
\left.\delta_{2}\left(f, x,\left(x_{1}, z_{2}\right)\right) s_{2}^{1 / 2}+\delta_{1}\left(f,\left(x_{1}, z_{2}\right), z\right) s_{1}^{1 / 2}\right),  \tag{8.22}\\
\delta_{\square}(f, g, x, z, s) \triangleq \min \left(\delta_{1}\left(f, g, x,\left(z_{1}, x_{2}\right)\right) s_{1}^{1 / 2}+\delta_{2}\left(f, g,\left(z_{1}, x_{2}\right), z\right) s_{2}^{1 / 2},\right. \\
\\
\left.\delta_{2}\left(f, g, x,\left(x_{1}, z_{2}\right)\right) s_{2}^{1 / 2}+\delta_{1}\left(f, g,\left(x_{1}, z_{2}\right), z\right) s_{1}^{1 / 2}\right),
\end{gather*}
$$

where $\delta_{i}(f, g, x, z)$ is defined in (8.19). We use the notation $\square$ since it mimics Figure 1 and indicates that we use $C_{x}^{1 / 2}, C_{y}^{1 / 2}$ estimates to obtain the $C^{1 / 2}$ estimate. By definition, we get

$$
\begin{equation*}
|\delta(f, x, z)| \leq \delta_{\square}(f, x, z, x-z), \quad \delta(f, g, x, z) \leq \delta_{\square}(f, g, x, z, x-z) \tag{8.23}
\end{equation*}
$$

Note that $\delta_{i}(f, x, z), \delta_{i}(f, g, x, z), \delta_{\square}(f, x, z, s), \delta_{\square}(f, g, x, z, s)$ are symmetric in $x, z$. We introduce an extra variable $s$ to reduce bounding $\delta_{\square}(f, x, z, x-z) g(x-z)$ to estimating $\delta_{i}(f, x, z)$ and $\left|x_{j}-z_{j}\right|^{1 / 2} g_{i}(x-z)$ separately, and $\delta_{\square}(f, x, z, s) g_{i}(s)$ is 0 -homogeneous in $s$.

Hölder estimate of $W_{1, i}$ terms. For $x_{1} \leq z_{1}$ and $f W_{1, i} \psi_{i}$ with $f \in C^{1 / 2}$, using the energy $\left\|W_{1, i} \varphi_{i}\right\|_{\infty}, \mu_{h, i}\left[W_{1, i} \psi_{i}\right]_{C_{g_{i}}^{1 / 2}} \leq E_{4}$ (8.2), (8.20), (8.23), we perform its $C^{1 / 2}$ estimate as follows

$$
\begin{align*}
& \left|\mu_{h, j} g_{j}(h) \delta\left(f W_{1, i} \psi_{i}, x, z\right)\right| \leq \mu_{h, j} g_{j}(h)\left(\left|\delta(f) W_{1, i} \psi_{i}(z)\right|+\left|f(x) \delta\left(W_{1, i} \psi_{i}\right)\right|\right) \\
& \leq \frac{\mu_{h, j} g_{j}(h)}{\mu_{h, i} g_{i}(h)}|f(x)| E_{4}+\mu_{h, j} g_{j}(h) \delta_{\square}(f, x, z, h) \frac{\psi_{i}}{\varphi_{i}}(z) E_{4}, \quad h=x-z \tag{8.24}
\end{align*}
$$

If $i=j$, the first term reduces to $|f(x)| E_{4}$. We only pick one decomposition in 8.20) with the coefficient of $\delta\left(W_{1, i} \psi_{i}\right)$ evaluating at $x$, i.e. $f(x)$, to simplify the estimates. Note that $x_{1} \leq z_{1}$.
8.3. Additional estimates of weights. Before we perform the $L^{\infty}$ and $C^{1 / 2}$ estimate, we need several asymptotic estimates of the weight, and the estimates of the ratio among weights. The reader who is not interested in rigorous verification can skip to Section 8.5 for the estimates of nonlinear terms and error terms.
8.3.1. Asymptotics of the weights. Consider a radial weights $\rho(x)=\rho(r)=\sum_{i \leq n} p_{i} r^{a_{i}}$, with $p_{i}>0$ and $a_{i}$ is increasing with $i$. Firstly, we have

$$
\partial_{i} r^{\alpha}=\alpha \frac{x_{i}}{r} r^{\alpha-1}, \quad \partial_{i} \partial_{j} r^{\alpha}=\alpha \partial_{j}\left(x_{i} r^{\alpha-2}\right)=\left(\alpha(\alpha-2) \frac{x_{i}}{r} \frac{x_{j}}{r}+\alpha \delta_{i j}\right) r^{\alpha-2} \triangleq C_{i, j}(\beta, \alpha) r^{\alpha-2}
$$

where $\beta=\arctan \left(x_{2} / x_{1}\right)$, and $C_{i, j}(\beta, \alpha)$ is obtained by simplifying $x_{1} / r=\cos \beta, x_{2} / r=\sin \beta$. For $r \geq r_{*}$, since $r^{a_{i}-a_{n}} \leq r_{*}^{a_{i}-a_{n}}$, it is easy to obtain

$$
\begin{align*}
& p_{n} r^{a_{n}} \leq \rho(x) \leq r^{a_{n}}\left(\sum_{i \leq n} p_{i} r_{*}^{a_{i}-a_{n}}\right),\left|\partial_{x_{i}} \rho(x)\right|=\left|\sum_{i \leq n} p_{i} a_{i} x_{i} r^{a_{i}-2}\right| \leq \frac{x_{i}}{r} r^{a_{n}-1}\left(\sum_{l \leq n}\left|p_{l} a_{l}\right| r_{*}^{a_{l}-a_{n}}\right)  \tag{8.25}\\
& \left|\partial_{1}^{i} \partial_{2}^{j} \rho(r)\right| \leq r^{a_{n}-2} \sum_{l \leq n}\left|p_{l}\right|\left|C_{i, j}\left(\beta, a_{l}\right)\right| r_{*}^{a_{l}-a_{n}} \triangleq \rho_{i j, u}(\beta) r^{a_{n}-2}, i+j=2
\end{align*}
$$

We can bound the angular function $\rho_{i j, u}(\beta)$ by partitioning $\beta \in[0, \pi / 2]$. In particular, we obtain

$$
\begin{equation*}
\left|\partial_{x_{1}}^{i} \partial_{x_{2}}^{j} \rho\right| \leq \rho_{i j, u}(\beta) r^{a_{n}-i-j} \cos \beta^{i} \sin \beta^{j}, \quad i+j \leq 1 \tag{8.26}
\end{equation*}
$$

for some constants $\rho_{i j, l}, \rho_{i j, u}$ depending on $r_{*}, a_{i}, p_{i}$.
For two radial weights $f, g$ and $\alpha$, applying the above estimates and

$$
\begin{equation*}
\left|\partial_{i}\left(\frac{f}{g} r^{\alpha}\right)\right| \leq\left|\alpha \frac{x_{i}}{r} r^{\alpha-1} \frac{f}{g}\right|+r^{\alpha}\left(\left|\frac{\partial_{i} f}{g}\right|+\left|\frac{f \partial_{i} g}{g^{2}}\right|\right) \tag{8.27}
\end{equation*}
$$

we can derive the asymptotics of $r^{\alpha} \frac{f}{g}$ and its derivatives. Note that we can extract the common angular factor $\frac{x_{i}}{r}$ in the above estimate. Similarly, using the triangle inequality (8.26)- (8.25), we can obtain the asymptotics of

$$
\begin{equation*}
\left|\partial_{i} \rho / \rho\right| \leq C(\rho, i)(\beta) r^{-1}, \quad\left|\partial_{j}\left(\partial_{i} \rho / \rho\right)\right| \leq C(\rho, i, j)(\beta) r^{-2} \tag{8.28}
\end{equation*}
$$

8.3.2. Piecewise bounds of the ratio of two weights. We have estimated the piecewise bounds for the radial weights $\rho(r)$ and the mixed weight in Appendix A.2, A. 3 in Part II [1]. Denote

$$
\begin{equation*}
\varphi=x^{-1 / 2} P(r)+Q(r), \quad P(r)=\sum_{i \leq n_{p}} p_{i} r^{a_{i}}, \quad Q(r)=\sum_{i \leq n_{q}} q_{i} r^{b_{i}}, \quad \rho(r)=\sum_{i \leq n_{s}} s_{i} r^{c_{i}} \tag{8.29}
\end{equation*}
$$

with $p_{i}, q_{i}, s_{i}>0$ and $a_{i}, b_{i}, c_{i}$ are increasing with $i$. In all of our choices of mixed weights, $P(r)$ is more singular than $Q$ near $r=0$ and decays faster than $Q$ for large $R$, i.e. $a_{1}<b_{1}, a_{n_{p}}<b_{n_{q}}$. In the energy estimates, we need several piecewise bounds of the ratio of these weights in $\left[x_{l}, x_{u}\right] \times\left[y_{l}, y_{u}\right]$. Since these weights are singular near $(0,0)$ and along $x=0$, we need to factorize out these singularities. Due to symmetry, we focus on $x \geq 0$ and use $r=\left(x^{2}+y^{2}\right)^{1 / 2}$. Consider the following decomposition for a radial weight and the mixed weight

$$
\begin{align*}
& P_{m}=\sum_{i \leq n_{p}} p_{i} r^{a_{i}-a_{1}}, P=r^{a_{1}} P_{m}, \varphi=x^{-1 / 2} \varphi_{m, h}=x^{-1 / 2} r^{a_{1}} \varphi_{m}  \tag{8.30}\\
& \varphi_{m, h} \triangleq P+x^{1 / 2} Q, \quad \varphi_{m}=r^{a_{1}} \varphi_{m, h}=P_{m}+r^{a_{1}} x^{1 / 2} Q(r)
\end{align*}
$$

Since $r^{a_{1}} Q(r)$ is not singular, $\varphi_{m}$ is not singular, and $\varphi_{m, h}$ is only singular along $x=0$. Using the piecewise bounds for the radial weights $P, P_{m}, r^{a_{1}} Q(r)$, we can obtain the piecewise bounds for the above modified weights.

We introduce $D_{r}=r \partial_{r}$. Using $D_{r} \rho_{m}=\sum s_{l}\left(c_{l}-c_{1}\right) r^{c_{l}-c_{1}}$, we get
$x_{i} \partial_{i} \rho=x_{i} \sum s_{l} c_{l} x_{i} r^{c_{l}-2}=\frac{x_{i}^{2}}{r^{2}} \sum s_{l} c_{l} r^{c_{l}}, \frac{x_{i} \partial_{i} \rho}{\rho}=\frac{x_{i}^{2}}{r^{2}}\left(c_{1}+\frac{\sum s_{l}\left(c_{l}-c_{1}\right) r^{c_{l}-c_{1}}}{\sum s_{l} r^{c_{l}-c_{1}}}\right)=\frac{x_{i}^{2}}{r^{2}}\left(c_{1}+\frac{D_{r} \rho_{m}}{\rho_{m}}\right)$.
Since $c_{i}$ is increasing with $i$, the leading power in $D_{r} \rho_{m}$ is $r^{c_{2}-c_{1}}$, which vanishes near $r=0$. For all radial weights we use, $c_{2}-c_{1}>1$ and thus $x_{i} \partial_{i} \rho / \rho \in C^{1}$ locally, and we can estimate its piecewise derivatives. We will use (8.31) in Section 8.4.3,

For $(x, y) \in\left[x_{l}, x_{u}\right] \times\left[y_{l}, y_{u}\right] \subset \mathbb{R}_{2}^{++}$, we have the following simple estimates

$$
\begin{equation*}
\frac{x_{l}^{2}}{x_{l}^{2}+y_{u}^{2}} \leq \frac{x^{2}}{r^{2}} \leq \frac{x_{u}^{2}}{x_{u}^{2}+y_{l}^{2}}, \quad \frac{y_{l}^{2}}{x_{u}^{2}+y_{l}^{2}} \leq \frac{y^{2}}{r^{2}} \leq \frac{y_{u}^{2}}{y_{u}^{2}+x_{l}^{2}} \tag{8.32}
\end{equation*}
$$

Ratio between mixed weights. Let $\varphi_{i}=x^{-1 / 2} P_{i}(r)+Q_{i}(r)$ be two mixed weights with the leading power $r^{\alpha_{i}}$ for $P_{i}(r)$. For $\alpha_{1} \geq \alpha_{2}$, we use (8.30), the decompositions

$$
\frac{\varphi_{1}}{\varphi_{2}}=\frac{\varphi_{1, m, h}}{\varphi_{2, m, h}}=r^{\alpha_{1}-\alpha_{2}} \frac{\varphi_{1, m}}{\varphi_{2, m}}
$$

and piecewise bounds of each part to obtain three estimates and then optimize them. We use the second identity to overcome the singularity on $x=0$ and the third near $r=0$.

Similarly, for two radial weights with leading power $r^{\alpha_{i}}$, we estimate $P_{1} / P_{2}$ using the following decompositions (8.30)

$$
P_{1} / P_{2}=r^{\alpha_{1}-\alpha_{2}} P_{1, m} / P_{2, m}
$$

Ratio between the radial and the mixed weight. Let $\rho, \varphi$ be a radial weight and a mixed weight (8.29). We use the following decomposition to estimate $\rho / \varphi, \rho /\left(\varphi x^{1 / 2}\right), \rho /\left(\varphi r^{1 / 2}\right)$

$$
\begin{equation*}
\frac{\rho}{\varphi}=r^{c_{1}-a_{1}} \frac{\rho_{m} x^{1 / 2}}{\varphi_{m}}, \quad \frac{\rho}{\varphi x^{1 / 2}}=\frac{\rho}{\varphi_{m, h}}=r^{c_{1}-a_{1}} \frac{\rho_{m}}{\varphi_{m}}, \quad \frac{\rho / r^{1 / 2}}{\varphi}=r^{c_{1}-a_{1}} \frac{\rho_{m}}{\varphi_{m}}\left(\frac{x}{r}\right)^{1 / 2} \tag{8.33}
\end{equation*}
$$

The factor $x / r$ is estimated using (8.32).
We remark that to obtain sharp piecewise bounds of the ratio, we can evaluate the ratio on fine grids, and then use the derivative bounds of the radio and apply the estimate (A.8).
8.3.3. Ratio among radial weights. In the Hölder estimate in Section 8.5.1 we need to estimate $\frac{\partial_{i} P}{P Q}$ for radial weights $P, Q$ (8.29). Using the piecewise bounds of $P, P^{-1}, Q^{-1}$, (A.7), and following Appendix E. 6 in Part II [1], we can obtain piecewise $C^{1}, C^{1 / 2}$ estimates of $\frac{\partial_{i} P}{P Q}$. We need more careful estimates near $r=0$. Consider the decomposition (8.29), for $P, Q$. We get

$$
\begin{align*}
\frac{\partial_{i} P}{P Q} & =\left(\frac{\partial_{i} P_{m}}{P_{m}}+\frac{\partial_{i} r^{a_{1}}}{r^{a_{1}}}\right) \frac{r^{-b_{1}}}{Q_{m}}=x_{i} \frac{\sum_{2 \leq j \leq n_{p}}\left(a_{j}-a_{1}\right) p_{j} r^{a_{j}-a_{1}-2-b_{1}}}{P_{m} Q_{m}}+\frac{a_{1} x_{i}}{r^{2}} \cdot \frac{r^{-b_{1}}}{Q_{m}}  \tag{8.34}\\
& \triangleq I \cdot x_{i}+I I / Q_{m} .
\end{align*}
$$

Since $I$ is the ratio among three radial weights non-singular near $r=0$, we can estimate their $C^{1}$ bounds. Using (A.7), we further obtain the estimate of $I x_{i}$.

For $I I$, since $b_{1} \leq-2$, a direct calculation yields

$$
|I I| \leq\left|a_{1}\right| x_{i}^{u} r_{u}^{-b_{1}-2}, \quad\left|\partial_{x_{j}} I I\right| \leq\left|a_{1}\right|\left(\delta_{i j}+\left|b_{1}+2\right| \frac{x_{i}}{r} \frac{x_{j}}{r}\right) r^{-b_{1}-2}
$$

Applying (8.32) for $x_{j} / r$ and $r^{-b_{1}-2} \leq r_{u}^{-b_{1}-2}$, we obtain the $C^{1}$ estimates for $I I$. Using the $C^{1}$ estimates of $Q_{m}^{-1}, I I$ and (A.7) and following Appendix E. 6 in Part II [1], we obtain the $C^{1}$ and $C^{1 / 2}$ estimates of $I I / Q_{m}$. Combining two parts, we obtain the refined estimate near $r=0$.

In Section 8.5.1, we need to estimate $S=\frac{x_{1} \partial_{1} P-x_{2} \partial_{2} P}{P}$ for a radial weight $P$. For $r$ away from 0 , the estimates follow from the previous method. Near $r=0$, using the above computation, we get
$S=\left(x_{1}^{2}-x_{2}^{2}\right)\left(\frac{\sum_{2 \leq j \leq n_{p}}\left(a_{j}-a_{1}\right) p_{j} r^{a_{j}-a_{1}-2}}{P_{m}}+\frac{a_{1}}{r^{2}}\right)=\frac{x_{1}^{2}-x_{2}^{2}}{r^{2}}\left(\frac{\tilde{P}_{m}}{P_{m}}+a_{1}\right), \tilde{P}_{m}=\sum_{2 \leq j \leq n_{p}}\left(a_{j}-a_{1}\right) p_{j} r^{a_{j}-a_{1}}$.
Note that $\tilde{P}_{m}, P_{m}$ are radial weights and non-singular near $r=0$, we can estimate their $C^{1}$ bounds and the $C^{1}$ bounds of $\frac{\tilde{P}_{m}}{P_{m}}+a_{1}$. Denote $f(x)=\frac{x_{2}^{2}}{r^{2}}$. We have $\frac{x_{1}^{2}-x_{2}^{2}}{r^{2}}=1-2 f(x), f(x)=\frac{2 x_{2}^{2}}{r^{2}}$ and this singular factor is not $C^{1 / 2}$. We estimate $\delta_{i}(f, x, z)|w|^{1 / 2}$ for $w=x, z, i=1,2$ in Section 8.9.2. Combining these estimates, we can estimate

$$
\begin{equation*}
\delta_{i}(S, x, z)|w|^{1 / 2}, \quad w=x, z, \quad i=1,2, \quad S=\left(x \partial_{x} P-y \partial_{y} P\right) P^{-1} \tag{8.35}
\end{equation*}
$$

8.4. Piecewise estimates of the damping coefficients $d_{i}$. Recall the damping terms $d_{i}, d_{g, i}$ from (8.16). In this section, we derive the piecewise upper bounds of $d_{i}, d_{g, i}$. The technicalities mainly come from the weights that are singular near 0 . The reader who is not interested in rigorous verification can skip to Section 8.5 for the estimates of nonlinear terms and error terms.
8.4.1. Piecewise bounds of $d_{i}$ from the singular weights. Let $\Omega=\left[x_{l}, x_{u}\right] \times\left[y_{l}, y_{u}\right] \subset \mathbb{R}_{2}^{++}$. We estimate the upper bound of the following in $\Omega$

$$
\mathcal{T}_{d, L}(\varphi)=\left(\left(\bar{c}_{l} x+\overline{\mathbf{u}}^{N}\right) \cdot \nabla \varphi\right) / \varphi, \quad \varphi=|x|^{-1 / 2} P(r)+Q(r)
$$

where $P, Q$ are radial weights. We assume that $\varphi$ has the form (8.29). Using the piecewise bounds of $\partial_{x}^{i} \partial_{y}^{j} P, \partial_{x}^{i} \partial_{y}^{j} Q$,

$$
\left|\partial_{x}^{i} x^{-1 / 2}\right| \leq\left|\frac{(2 i-1)!!}{2^{i}} x_{l}^{-1 / 2-i}\right|
$$

and (A.7), we obtain piecewise bounds for $\varphi$. We have several estimates for $\mathcal{T}(\varphi)$.
(1) We evaluate $\mathcal{T}_{d, L}(\varphi)$ on the grid point $\left(x_{\alpha}, y_{\beta}\right), \alpha, \beta=l, u$ and estimate $\partial_{i}^{2} \mathcal{T}_{d, L}(\varphi)$ using the piecewise bounds for $\overline{\mathbf{u}}^{N}, \varphi, \varphi^{-1}$ and A.7). Then we use (A.8) to estimate $\mathcal{T}_{d, L}(\varphi)$.
(2) We estimate the upper bound of the numerator $\left(\bar{c}_{l} x+\overline{\mathbf{u}}^{N}\right) \cdot \nabla \varphi$ using its grid point values, derivative bounds, and (A.8). Then we estimate $\mathcal{T}_{d, L}(\varphi)$ using this upper bound, the piecewise bounds for $\varphi$, and (A.6) for the ratio $f / g$.

Since the weight $\varphi$ is singular along $x=0$ and near $(x, y)=0$, we need more careful estimates. We introduce $B_{u, i}$ below and rewrite $\mathcal{T}_{d, L}(\varphi)$ as follows

$$
\begin{equation*}
B_{u, 1}=\bar{c}_{l}+\frac{\bar{u}^{N}}{x}, \quad B_{u, 2}=\bar{c}_{l}+\frac{\bar{v}^{N}}{y}, \quad \mathcal{T}_{d, L}(\varphi)=B_{u, 1} \frac{x \partial_{x} \varphi}{\varphi}+B_{u, 2} \frac{y \partial_{y} \varphi}{\varphi}=I_{1}+I_{2} \tag{8.36}
\end{equation*}
$$

For $\varphi$ (8.29), using $r \partial_{r}=x \partial_{x}+y \partial_{y}$, we have
$x \partial_{x} \varphi=-\frac{1}{2} x^{-1 / 2} P+x^{1 / 2} \partial_{x} P+x \partial_{x} Q, \quad y^{k} \partial_{y} \varphi=x^{-1 / 2} y^{k} \partial_{y} P+y^{k} \partial_{y} Q, k \geq 0$,
$x \cdot \nabla \varphi=-\frac{1}{2} x^{-1 / 2} P+x^{-1 / 2} r \partial_{r} P+r \partial_{r} Q, \frac{x \partial_{x} \varphi}{\varphi}=\frac{-P / 2+x \partial_{x} P+x^{3 / 2} \partial_{x} Q}{P+x^{1 / 2} Q}, \frac{y \partial_{y} \varphi}{\varphi}=\frac{y P_{y}+x^{1 / 2} y Q_{y}}{P+x^{1 / 2} Q}$.
To overcome the singularity $|x|^{-1 / 2}$, we rewrite $I_{1}$ as follows

$$
I_{1}=B_{u, 1} I_{3}, \quad I_{3}=\frac{x \partial_{x} \varphi}{\varphi}=\frac{-P / 2+x \partial_{x} P}{P+x^{1 / 2} Q}+\frac{x \partial_{x} Q}{x^{-1 / 2} P+Q}=\frac{-P / 2+x \partial_{x} P}{P+x^{1 / 2} Q}+\frac{x^{3 / 2} \partial_{x} Q}{P+x^{1 / 2} Q}
$$

We estimate the upper and lower bounds of $B_{1} /\left(P+x^{1 / 2} Q\right)$ and $B_{1} /\left(x^{-1 / 2} P+Q\right)$, which are positive, the upper bound of $-P / 2+x \partial_{x} P, x \partial_{x} Q, x^{3 / 2} \partial_{x} Q$, and then apply (A.6). We use the third identity of $I_{3}$ to remove the singular weight $|x|^{-1 / 2}$, and the second identity to control $I_{3}$ for large $r$ and $x$ close to 0 . If we use the third identity for such an estimate, since $Q$ decays slower than $P$, the upper bound is $\frac{\left(x^{3 / 2} \partial_{x} Q\right)_{u}}{P}$, which grows to $\infty$. The same argument applies to estimate the upper and lower bounds of $x \partial_{x} \varphi / \varphi$. For $x$ close to 0 and large $r$, we use the second identity of $I_{3}$ for an improved estimate. We optimize the estimates based on these two identities. The estimates of $I_{2}$ (8.36) , $y \partial_{y} \varphi / \varphi$ are similar and easier.

To overcome the singularity near $(x, y)=0$, we have an additional estimate. Using the decomposition (8.30) for $\varphi=r^{\alpha}\left(x^{-1 / 2} P_{m}+r^{-\alpha} Q\right) \triangleq r^{\alpha} \tilde{\varphi}$, we get

$$
\frac{\partial \varphi}{\varphi}=\frac{\partial \tilde{\varphi}}{\tilde{\varphi}}+\frac{\partial r^{\alpha}}{r^{\alpha}}, \quad \frac{D \varphi}{\varphi}=\frac{D \tilde{\varphi}}{\tilde{\varphi}}+\frac{D r^{\alpha}}{r^{\alpha}}, \quad \frac{x \partial_{x} r^{\alpha}}{r^{\alpha}}=\alpha \frac{x^{2}}{r^{2}}, \frac{y \partial_{y} r^{\alpha}}{r^{\alpha}}=\alpha \frac{y^{2}}{r^{2}}, \quad D=x \partial_{x}, y \partial_{y}
$$

We rewrite $\mathcal{T}_{d, L}(\varphi)$ as follows

$$
\mathcal{T}_{d, L}(\varphi)=\frac{\left(\bar{c}_{l} x+\overline{\mathbf{u}}^{N}\right) \cdot \nabla\left(r^{\alpha} \tilde{\varphi}\right)}{r^{\alpha} \tilde{\varphi}}=\frac{\left(\bar{c}_{l} x+\overline{\mathbf{u}}^{N}\right) \cdot \nabla \tilde{\varphi}}{\tilde{\varphi}}+\alpha \frac{B_{u, 1} x^{2}+B_{u, 2} y^{2}}{r^{2}} \triangleq J_{1}+J_{2}
$$

where we have used $B_{u, i}$ from (8.36). The first part in the above formula of $D \varphi / \varphi$ or $\mathcal{T}_{d, L}$ is only singular along $x=0$ and we apply the above estimates again. For $J_{2}$, our weights satisfy $\alpha<0$. Let $B_{u, i, l}$ be the uniform lower bounds of $B_{u, i}$ in $\Omega$. Since $\alpha<0$, we yield

$$
J_{2} \leq \alpha f, \quad f(x, y)=\frac{B_{u, 1, l} x^{2}+B_{u, 2, l} y^{2}}{r^{2}}
$$

If $B_{u, 1, l}<B_{u, 2, l}, f$ is increasing in $y$ and decreasing in $x$, and we get $f(x, y) \geq f\left(x_{u}, y_{l}\right)$. If $B_{u, 1, l} \geq B_{u, 2, l}$, we get $f(x, y) \geq f\left(x_{l}, y_{u}\right)$ similarly.

Estimate of $d_{i}$ for radial weights. We apply the above arguments to derive the piecewise upper and lower bounds for $\mathcal{T}_{d, L}, d_{i}(8.16)$ with a radial weight $\rho$, which is simpler. We apply the above method (1) to estimate the piecewise bounds for $\partial_{j} \mathcal{T}_{d, L}(\rho), \partial_{j} d_{i, L}$. We will use these estimates in the weighted Hölder estimate.
Estimate of $d_{i}$ for $\varphi_{4}$. In our nonlinear $L^{\infty}\left(\varphi_{4}\right)$ estimate, we need to estimate $J=\left|\frac{U \phi_{4, x}}{\phi_{4}}\right|+$ $\left|\frac{V \phi_{4, y}}{\phi_{4}}\right|$ for $\varphi_{4}=x_{1}^{-1 / 2} \psi_{1}, \psi_{1}=|x|^{-2}+0.6|x|^{-1}+0.3|x|^{-1 / 6}=\sum p_{i} r^{a_{i}}$. Using (8.37) with $(P, Q)=\left(\psi_{1}, 0\right)$ we get

$$
|J|=\left|-\frac{1}{2} \frac{U}{x}\right|+\left|\frac{U \psi_{1, x}}{\psi_{1}}\right|+\left|\frac{V \psi_{1, y}}{\psi_{1}}\right| \leq \frac{1}{2}\left|\frac{U}{x}\right|+\max \left(\left|\frac{U}{x}\right|,\left|\frac{V}{y}\right|\right)\left(\left|\frac{x \psi_{1, x}}{\psi_{1}}\right|+\left|\frac{y \psi_{1, y}}{\psi_{1}}\right|\right) .
$$

Since $a_{i}<0$, for $(x, y) \in \mathbb{R}_{2}^{++}$, we get $\psi_{1, x}, \psi_{1, y}<0$, which along with (8.31) gives
$0 \leq J_{2}=\left|\frac{x \psi_{1, x}}{\psi_{1}}\right|+\left|\frac{y \psi_{1, y}}{\psi_{1}}\right|=-\frac{x \psi_{1, x}}{\psi_{1}}-\frac{y \psi_{1, y}}{\psi_{1}}=-\frac{x_{1}^{2}+x_{2}^{2}}{r^{2}} \frac{\sum p_{i} a_{i} r^{a_{i}}}{\sum p_{i} r^{a_{i}}}=-\frac{\sum p_{i} a_{i} r^{a_{i}}}{\sum p_{i} r^{a_{i}}} \leq \max _{i}\left|a_{i}\right|$.
Similarly, using (8.37), we get

$$
\frac{\left(\bar{c}_{l} x+\overline{\mathbf{u}}^{N}\right) \cdot \nabla \varphi_{4}}{\varphi_{4}}=-\frac{1}{2}\left(\bar{c}_{l}+\frac{\bar{u}^{N}}{x}\right)+\frac{\left(\bar{c}_{l} x+\overline{\mathbf{u}}^{N}\right) \cdot \nabla \psi_{1}}{\psi_{1}} .
$$

The second term can be estimated using the above method.
8.4.2. Piecewise bounds of $d_{g, i}$ from the Hölder weights. In this section, we estimate the damping term (8.16) $d_{g, i}$. Using (8.54), we only need to estimate $\frac{\left.\left(\overline{\mathbf{u}}^{N}+\mathbf{U}\right)(x)-\left(\overline{\mathbf{u}}^{N}+\mathbf{U}\right)(z)\right) \cdot(\nabla g)(x-z)}{g(x-z)}$.

Recall the notation $\Delta_{i}(f)$ from (8.18). We have two estimates. In the first estimate, we estimate piecewise upper and lower bounds of the Lipschitz constant in $x$
$C(f, 1, l)_{i, k, j} \leq \frac{f(z)-f(x)}{z_{1}-x_{1}}=D_{1}(f, x, z) \leq C(f, 1, u)_{i, k, j}, \quad x_{2}=z_{2}, x_{1} \leq z_{1} x \in Q_{i j}, z \in Q_{k j}$
for $f=\bar{u}^{N}+U, \bar{v}^{N}+V$ using the piecewise bounds of $\nabla \overline{\mathbf{u}}^{N}+\nabla \mathbf{U}$ and the method in Appendix E. 7 in Part II [1]. The second index in $Q_{i j}, Q_{k j}$ is the same since $x_{2}=z_{2}$. Similarly, we estimate the piecewise bounds for the Lipschitz constant in $y$. Since we treat $\mathbf{U}$ as a perturbation, we get

$$
\partial_{i} \bar{u}^{N}-\left|\partial_{i} U\right| \leq \partial_{i}\left(\bar{u}^{N}+U\right) \leq \partial_{i} \bar{u}^{N}+\left|\partial_{i} U\right|
$$

and similar estimate for $\bar{v}^{N}+V$. We assume $x_{1} \leq z_{1}$ without loss of generality. Denote

$$
r=z-x, \quad w=\left(z_{1}, x_{2}\right), \quad s=\operatorname{sgn}\left(r_{1}, r_{2}\right) .
$$

We have two cases for the configuration of $x, z: r_{1} r_{2} \geq 0$ and $r_{1} r_{2}<0$, Note that $\left(\partial_{i} g\right)(h)$ is odd in $h_{1}$ and even in $h_{2}$. We write $J_{1}=(u(z)-u(x))\left(\partial_{1} g\right)(z-x) / g(z-x)$ as follows
$J_{1}=(u(w)-u(x)+u(z)-u(w)) \frac{\left(\partial_{1} g\right)(z-x)}{g(z-x)}=\frac{u(w)-u(x)}{z_{1}-x_{1}} r_{1} \frac{\left(\partial_{1} g\right)(z-x)}{g(z-x)}+\frac{u(z)-u(w)}{z_{2}-x_{2}} r_{2} \frac{\left(\partial_{1} g\right)(z-x)}{g(z-x)}$.
Since $r_{1} \geq 0$ and $r_{1} \partial_{1} g\left(r_{1}, r_{2}\right)=\left|r_{1}\right| \partial_{1} g\left(\left|r_{1}\right|,\left|r_{2}\right|\right), r_{2}\left(\partial_{1} g\right)\left(r_{1}, r_{2}\right)=\left|r_{2}\right| s \partial_{1} g\left(\left|r_{1}\right|,\left|r_{2}\right|\right)$, we get

$$
J_{1}=\Delta_{1}(u, x, w)\left|r_{1}\right| \frac{\left(\partial_{1} g\right)\left(\left|r_{1}\right|,\left|r_{2}\right|\right)}{g\left(\left|r_{1}\right|,\left|r_{2}\right|\right)}+\Delta_{2}(u, w, z)\left|r_{2}\right| s \frac{\left(\partial_{1} g\right)\left(\left|r_{1}\right|,\left|r_{2}\right|\right)}{g\left(\left|r_{1}\right|,\left|r_{2}\right|\right)} .
$$

The function $\left|r_{1}\right| \frac{\left(\partial_{1} g\right)\left(\left|r_{1}\right|,\left|r_{2}\right|\right)}{g\left(\left|r_{1}\right|,\left|r_{2}\right|\right)}$ is 0-homogeneous, and we use the method in Section 8.6.1 to estimate it by partitioning the range of $\left|r_{1}\right| /\left|r_{2}\right|,\left|r_{2}\right| /\left|r_{1}\right|$. Similarly, we have

$$
J_{2}=\frac{(v(z)-v(x))\left(\partial_{2} g\right)(z-x)}{g(z-x)}=\Delta_{1}(v, x, w) s\left|r_{1}\right| \frac{\partial_{2} g\left(\left|r_{1}\right|,\left|r_{2}\right|\right)}{g\left(\left|r_{1}\right|,\left|r_{2}\right|\right)}+\Delta_{2}(v, w, z)\left|r_{2}\right| \frac{\partial_{2} g\left(\left|r_{1}\right|,\left|r_{2}\right|\right)}{g\left(\left|r_{1}\right|,\left|r_{2}\right|\right)}
$$

Using the estimates of $\left|r_{1}\right| \frac{\left(\partial_{1} g\right)\left(\left|r_{1}\right|\left|\left|r_{2}\right|\right)\right.}{g\left(\left|r_{1}\right|,\left|r_{2}\right|\right)}$ in Section 8.6.1 piecewise bounds of $\Delta_{i}(f, x, z)$, we obtain piecewise estimates of $d_{g, i}$. See the red path $\left(w=\left(z_{1}, x_{2}\right)\right.$ in Figure 1 for an illustration of using $C_{x}^{1 / 2}, C_{y}^{1 / 2}$ estimates of $f$ to estimate $\delta(f, x, z)$.

Note that we can also choose $w=\left(x_{1}, z_{2}\right)$ and derive another decomposition

$$
f(z)-f(x)=f(z)-f(w)+f(w)-f(x)=\Delta_{1}(f, z, w) r_{1}+\Delta_{2}(f, w, x) r_{2}
$$

for $f=\bar{u}^{N}+U, \bar{v}^{N}+V$. Using this decomposition and the above argument, we derive another estimate of $J_{1}+J_{2}$. We optimize both estimates to bound $d_{g, i}$ from above.
8.4.3. Piecewise Hölder estimates of the damping terms. In this section, we estimate the piecewise bounds $\delta_{j}\left(d_{i, L} W_{1, i} \psi_{i}, x, z\right)$ (8.18) for $x \in Q_{i j}, z \in Q_{p q}$. We have

$$
\begin{align*}
& \delta\left(d_{i, L} W_{1, i} \psi_{i}, x, z\right) g_{i}(x-z)=d_{i, L}(p) \delta\left(W_{1, i} \psi_{i}, x, z\right) g_{i}(x-z)+I \\
& I=\left(W_{1, i} \psi_{i}\right)(q) \delta\left(d_{i, L}, x, z\right) g_{i}(x-z),|I| \leq\left\|W_{1, i} \varphi_{i}\right\|_{\infty}\left|\delta\left(d_{i, L}, x, z\right) g_{i}(x-z)\right| \frac{\psi_{i}(q)}{\varphi_{i}(q)} \tag{8.38}
\end{align*}
$$

where $(p, q)=(x, z)$ or $(z, x)$. The first term is the damping term, and we need to estimate the weighted difference in $|I|$.

For $x, z$ away from 0 , we use the piecewise $C^{1}$ bounds of $d_{i, L}$ established in Section 8.4.1, and the method in Appendix E. 6 in Part II [1] to estimate $\delta_{j}\left(d_{i, L}, x, z\right)$. For $|x-z|$ not small, we have another estimate of $\delta_{j}\left(d_{i, L}, x, z\right)$ using the piecewise upper and lower bounds of $d_{i, L}$.

The main difficulty is the estimate near $x=0$ since $d_{i, L} \notin C^{1 / 2}$ near $r=0$. We use $\psi_{i} / \varphi_{i} \lesssim r^{1 / 2}$ near $r=0$ to compensate the low regularity of $d_{i, L}$ near 0 . We focus on the estimate of $\mathcal{T}_{d, L}$ in $d_{i, L} 8.16$ ). The remaining parts $d_{l o c, i}$ in $d_{i, L}$, i.e. $B_{u, 4}$ below, are locally $C^{1}$ and their estimates follow from the previous methods.

Recall $B_{u, 1}, B_{u, 2}$ from (8.36) and we further introduce $B_{u, 3}, B_{u, 4}$

$$
\begin{aligned}
& B_{u, 1}=\bar{c}_{l}+\bar{u}^{N} / x, B_{u, 2}=\bar{c}_{l}+\bar{v}^{N} / y, B_{u, 3}=\bar{v}^{N} / y-\bar{u}^{N} / x=B_{u, 2}-B_{u, 1} \\
& d_{i, L}(\rho)=\mathcal{T}_{d, L}(\rho)+B_{u, 4}, \quad B_{u, 4}=\bar{c}_{\omega}^{N}(i=1), 2 \bar{c}_{\omega}^{N}-\bar{u}_{x}^{N}(i=2), 2 \bar{c}_{\omega}^{N}+\bar{u}_{x}^{N}(i=3)
\end{aligned}
$$

Suppose that $\rho$ has the representation (8.29). Using the decompositions (8.36), (8.31), we get

$$
\begin{align*}
\mathcal{T}_{d, L}(\rho) & =B_{u, 1} \frac{x \partial_{x} \rho}{\rho}+B_{u, 2} \frac{y \partial_{y} \rho}{\rho}=\left(B_{u, 1} \frac{x^{2}}{r^{2}}+B_{u, 2} \frac{y^{2}}{r^{2}}\right)\left(c_{1}+\frac{D_{r} \rho_{m}}{\rho_{m}}\right)=\left(B_{u, 1}+B_{u, 3} \frac{y^{2}}{r^{2}}\right)\left(c_{1}+R_{\rho}\right)  \tag{8.39}\\
& \triangleq \mathcal{T}_{R}(\rho)+\mathcal{T}_{S}(\rho) \frac{y^{2}}{r^{2}}, \quad \mathcal{T}_{R}=B_{u, 1}\left(c_{1}+R_{\rho}\right), \mathcal{T}_{S}=B_{u, 3}\left(c_{1}+R_{\rho}\right), \quad R_{\rho} \triangleq \frac{D_{r} \rho_{m}}{\rho_{m}}
\end{align*}
$$

The functions $\mathcal{T}_{R}, \mathcal{T}_{S}, R_{\rho}$ are locally Lipschitz, and we derive their piecewise bounds below.
From the discussion around (8.31), we can estimate the piecewise $C^{1}$ bounds of $\rho_{m}, D_{r} \rho_{m}$. Using (A.7), we can estimate the derivatives of $R_{\rho}$ and the piecewise bounds $\delta_{i}\left(R_{\rho}, x, z\right)$. Using $\bar{u}^{N}(0, y)=0$, the piecewise estimates in Section 8.9.1 and

$$
\partial_{x} B_{u, 1}=\partial_{x} \frac{\bar{u}^{N}}{x}=\frac{\bar{u}_{x}^{N} x-\bar{u}^{N}}{x^{2}}=\frac{1}{x^{2}} \int_{0}^{x} \bar{u}_{x x}^{N}(z, y) z d z, \quad \partial_{y} B_{u, 1}=\frac{\bar{u}_{y}^{N}}{x}
$$

we obtain piecewise $C^{1}$ estimates of $B_{u, 1}$. The same argument applies to the estimates of $B_{2}$. Using these $C^{1}$ bounds, the Leibniz rule (A.7), the method in Appendix E. 6 in Part II [1], we obtain the piecewise $C^{1}$ and the Hölder estimate of the regular part $\mathcal{T}_{R}, \mathcal{T}_{S}$.
Weighted estimate of the singular part. The difficult term is $\mathcal{T}_{S} \frac{y^{2}}{r^{2}}$, which is not $C^{1 / 2}$ near $r=0$. We estimate $\delta_{i}\left(\mathcal{T}_{S} \frac{y^{2}}{r^{2}}, x, z\right)$ with $\psi_{i}(x) / \varphi_{i}(x) \lesssim r^{1 / 2}$ (8.38). The factor $B_{3} c_{1} y^{2} / r^{2}$ captures the anisotropic structure of the advection in $x, y$ directions near $r=0$. See Section 2.7.2 3]. In $Q_{i j}$ (8.4) near $(0,0)$, the ratio $y^{2} / r^{2}$ can vary a lot. Thus, we introduce the angle $\beta$ below to control this factor and establish the following estimate

$$
\begin{align*}
& \delta_{1}\left(\mathcal{T}_{R}+\mathcal{T}_{S} f, x, z\right)|w|^{1 / 2} \leq C_{11}(x, z)+C_{12}(x, z) \sin ^{2}(\beta), x_{1} \leq z_{1}, x_{2}=z_{2} \\
& \delta_{2}\left(\mathcal{T}_{R}+\mathcal{T}_{S} f, x, z\right)|w|^{1 / 2} \leq C_{21}(x, z)+C_{22}(x, z) \sin ^{3 / 2}(\beta), x_{1}=z_{1}, x_{2} \leq z_{2}  \tag{8.40}\\
& \beta=\max \left(\arctan \left(x_{2} / x_{1}\right), \arctan \left(z_{2} / z_{1}\right)\right), \quad f(x, y)=y^{2} / r^{2}
\end{align*}
$$

for $w=x$ and $w=z$. We have estimated $\mathcal{T}_{R}, \mathcal{T}_{S}$ in the above. For $x \in Q_{k_{1} l_{1}, z \in Q_{k_{2} l_{2}} \text {, we }}$ can estimate the piecewise bounds for $C_{i j}(x, z)$. For $f(z)$, we use (8.32) to obtain its piecewise bound. We defer the estimate $\delta_{i}(f, x, z)|w|^{1 / 2}$ to Section 8.9.2. Using

$$
\delta_{i}\left(\mathcal{T}_{S} f, x, z\right)|w|^{1 / 2} \leq \delta_{i}\left(\mathcal{T}_{S}, x, z\right) f(z)|w|^{1 / 2}+\delta_{i}(f, x, z) \mathcal{T}_{S}(x)|w|^{1 / 2}
$$



Figure 2. Left, mid, right figure correspond to the locations of $(x, z)$ in cases (1.a), (1.b), (2). The red line represents the triangle inequalities and $C_{x_{i}}^{1 / 2}$ estimate used to estimate $\left|\delta\left(d_{i, L}, x, z\right)\right|$.
the piecewise estimates of $\delta_{i}(f, x, z)|w|^{1 / 2}$ in (8.65), (8.67), we obtain the estimate of $\delta_{i}\left(\mathcal{T}_{S} f, x, z\right)|w|^{1 / 2}$ depending on $\beta$ in (8.40). Using the estimate in Section 8.9.2 we also obtain the estimate of $\delta_{i}(f, x, z)|w|^{1 / 2}, \delta_{1}\left(\mathcal{T}_{R}+\mathcal{T}_{S} f, x, z\right)|w|^{1 / 2}$ independent of $\beta$. Using (8.40) and (8.39), we get

$$
\begin{equation*}
\mathcal{T}_{d, L}(\rho)=\mathcal{T}_{R}+\mathcal{T}_{S}(\sin \beta)^{2} \tag{8.41}
\end{equation*}
$$

Near $r=0$, we have $\mathcal{T}_{S} \approx B_{u, 1} c_{1}, B_{u, 1} \approx 5, c_{1}=-2,-5 / 2$ (8.39) in our estimates and thus $B_{u, 3} c_{1}, \mathcal{T}_{S}<0$. Although the upper bound (8.40) becomes larger for larger $\beta$, we gain a larger damping factor from $\mathcal{T}_{d, L}$ (8.38), which dominates the upper bounds obtained in (8.40).

Improved estimate of the damping terms near 0 . We combine (8.39), (8.40), and (8.41) to obtain an improved estimate for $\delta\left(d_{i, L}, x, z\right) W_{1, i} g_{i}(x-z)$ (8.38) with any $x, z \in \mathbb{R}_{2}^{++}, z \neq x$ near 0 . We assume $x_{1} \leq z_{1}$. There are two cases: (1) $z_{2} \geq x_{2}$, (2) $z_{2} \leq x_{2}$. Denote

$$
\lambda_{x}=\frac{x_{2}}{|x|}, \quad \lambda_{z}=\frac{z_{2}}{|z|}, \quad \lambda=\max \left(\lambda_{x}, \lambda_{z}\right)
$$

For $\beta \in\left[\beta_{i}, \beta_{i+1}\right]$, we get $\lambda \in\left[\sin \left(\beta_{i}\right), \sin \beta_{i+1}\right]$ and then derive the upper bound of (8.41), (8.40). We choose $p=x$ or $p=z$ in (8.38) such that $\lambda_{p}=\lambda$. From (8.41), this choice provides a larger damping term $\mathcal{T}_{d, L}(\rho)$. Then we choose $q=\{x, z\} \backslash p$. Denote

$$
h_{1}=\left|x_{1}-z_{1}\right|, \quad h_{2}=\left|x_{2}-z_{2}\right|, \quad R_{i}=\frac{\psi_{i}}{\varphi_{i}}
$$

Recall the notations $\delta, \delta_{i}$ from (8.18).
Case (2): $z_{2} \leq x_{2}$. Since $\frac{y_{2}}{|y|}$ is increasing in $y_{2}$ (8.32), we have $\lambda_{z} \leq \frac{x_{2}}{\left|\left(z_{1}, x_{2}\right)\right|} \leq \frac{x_{2}}{|x|}=\lambda_{x}$. Thus, we get $(p, q)=(x, z), \lambda=\lambda_{x}$. Denote $w=\left(z_{1}, x_{2}\right)$. We estimate

$$
\begin{aligned}
& \left|\delta\left(d_{i, L}, x, z\right)\right| g_{i}(h) R_{i}(z) \leq\left(\left|\delta\left(d_{i, L}, x, w\right)\right|+\left|\delta\left(d_{i, L}, z, w\right)\right|\right) g_{i}(h) R_{i}(z) \\
\leq & \left(\delta_{1}\left(d_{i, L}, x, w\right)|z|^{1 / 2} \cdot R_{i}(z)|z|^{-1 / 2} \cdot h_{1}^{1 / 2}+\delta_{2}\left(d_{i, L}, z, w\right)|z|^{1 / 2} \cdot R_{i}(z)|z|^{-1 / 2} \cdot h_{2}^{1 / 2}\right) g_{i}(h)
\end{aligned}
$$

Since $|z| \leq\left|\left(z_{1}, x_{2}\right)\right|=|w|$, we use (8.40) to bound

$$
\delta_{1}\left(d_{i, L}, x, w\right)|z|^{1 / 2} \leq \delta_{1}\left(d_{i, L}, x, w\right)|w|^{1 / 2}, \quad \delta_{2}\left(d_{i, L}, z, w\right)|z|^{1 / 2}
$$

Using $R_{i}(y) \sim\left|y_{1}\right|^{1 / 2}|x|_{i}^{a}, a=(0.9,0,0)$ near 0 , we obtain piecewise bounds for $R_{i}(z)\left|z_{1}\right|^{-1 / 2}$, $R_{i}(z)|z|^{-1 / 2}$. To estimate $h_{j}^{1 / 2} g_{i}$, we follow Section 8.6.1.

For (1) $x_{2} \leq z_{2}$, we have two cases (1.a) $\lambda_{x}<\lambda_{z}$ and (1.b) $\lambda_{z}<\lambda_{x}$. In both cases, we choose $w=\left(z_{1}, x_{2}\right)$. In Figure 2 we illustrate the locations of $(x, z)$ and the triangle inequalities used to bound $\mid \delta\left(d_{i, L}, x, z\right)$ in different cases.
Case (1.a): $x_{2} \leq z_{2}, \lambda_{x}<\lambda_{z}$. We get $(p, q)=(z, x)$. Since $|x| \leq|w| \leq|z|$, we yield

$$
\begin{aligned}
& \left|\delta\left(d_{i, L}, x, z\right)\right| g_{i}(h) R_{i}(x) \leq\left(\left|\delta\left(d_{i, L}, x, w\right)\right|+\left|\delta\left(d_{i, L}, z, w\right)\right|\right) g_{i}(h) R_{i}(x) \\
\leq & \left(\delta_{1}\left(d_{i, L}, x, w\right)|x|^{1 / 2} \cdot h_{1}^{1 / 2}+\delta_{2}\left(d_{i, L}, w, z\right)|w|^{1 / 2} \cdot h_{2}^{1 / 2}\right) g_{i}(h) \cdot R_{i}(x)|x|^{-1 / 2}
\end{aligned}
$$

We use (8.40) to estimate $\delta_{1}\left(d_{i, L}, x, w\right)|x|^{1 / 2}, \delta_{2}\left(d_{i, L}, w, z\right)|w|^{1 / 2}$ and estimate $R_{i}(x)|x|^{-\frac{1}{2}}, h_{j} g_{i}(h)$ similarly.

Case (1.b): $x_{2} \leq z_{2}, \lambda_{x}>\lambda_{z}$. We get $(p, q)=(x, z)$ and

$$
\begin{aligned}
& \left|\delta\left(d_{i, L}, x, z\right)\right| g_{i}(h) R_{i}(z) \leq\left(\left|\delta\left(d_{i, L}, x, w\right)\right|+\left|\delta\left(d_{i, L}, w, z\right)\right|\right) g_{i}(h) R_{i}(z) \\
\leq & \left(\delta_{1}\left(d_{i, L}, x, w\right)\left|z_{1}\right|^{1 / 2} \cdot R_{i}\left|z_{1}\right|^{-1 / 2} \cdot h_{1}^{1 / 2}+\delta_{2}\left(d_{i, L}, w, z\right)|z|^{1 / 2} \cdot R_{i}(z)|z|^{-1 / 2} \cdot h_{2}^{1 / 2}\right) g_{i}(h) .
\end{aligned}
$$

We estimate $R_{i}(z)|z|^{-\frac{1}{2}}, R_{i}(z)\left|z_{1}\right|^{-\frac{1}{2}}, h_{j} g_{i}(h)$ similarly. Since $\left|z_{1}\right| \leq|w|$, we use (8.40) to bound

$$
\delta_{1}\left(d_{i, L}, x, w\right)\left|z_{1}\right|^{1 / 2} \leq \delta_{1}\left(d_{i, L}, x, w\right)|w|^{1 / 2}, \quad \delta_{2}\left(d_{i, L}, w, z\right)|z|^{1 / 2}
$$

Four parameters. In summary, in the improved estimate near 0, we introduce four parameters $x, z, \lambda=\left(\lambda_{x}, \lambda_{z}\right), h_{i}=\left|x_{i}-z_{i}\right|$ and derive the bound

$$
\begin{aligned}
& d_{i, L}(p) \leq C_{31}(x, z)+C_{32}(x, z) \lambda^{2} \\
& \delta\left(d_{i, L}, x, z\right) g_{i}(x-z)\left|W_{1, i} \psi_{i}(q)\right| \leq \sum_{l} C_{l}(x, z) h_{1}^{a_{l 1}} h_{2}^{1 / 2-a_{l 1}} g_{i}(h) \lambda^{a_{l 3}} E_{4}
\end{aligned}
$$

For each $x, z \in Q_{i_{1} j_{1}}, Q_{i_{2} j_{2}}$, we first derive the piecewise bounds for $C_{l}(x, z)$. By partitioning $\lambda=\sin (\beta) \in[0,1]$ (or $\beta \in[0, \pi / 2]$ ) into small intervals and using monotonicity in $\lambda$, we estimate $\lambda^{a}$. The function in $h$ is 0-homogeneous, which can be estimated following Section 8.6.1.
8.5. Estimate of nonlinear terms. We estimate the nonlinear terms in 8.11). The estimate of each term follows from direct $L^{\infty}$ or $C^{1 / 2}$ estimates (8.19) (8.22). The main technicality comes from the singular weights near $r=0$ or $x=0$. We have discussed the $C^{k}$ estimates of $\hat{W}_{2}$ in Section 8.1.2,
8.5.1. Weighted Hölder estimate. In the Hölder estimate, we use radial weights $\rho_{i}=\psi_{i}$ and estimate $\mu_{h, i}\left[W_{1, i} \psi_{i}\right]_{C_{g_{i}}^{1 / 2}}$ in the energy $E_{4}$ (8.2).
8.5.2. Piecewise estimates of the damping terms. The main technical term is the coefficient $\mathcal{T}_{d, N}\left(\psi_{i}\right)$ from the first term $\mathcal{T}_{d, N}\left(\psi_{i}\right)\left(W_{1, i} \psi_{i}\right)$ in (8.14). See (8.16) for $\mathcal{T}_{d, N}$. Since we have piecewise $C^{1 / 2}$ estimates of $W_{1, i} \psi_{i}$ using the energy, we only need to control $\mathcal{T}_{d, N}\left(\psi_{i}\right)$ in order to estimate $\mathcal{T}_{d, N}\left(\psi_{i}\right)\left(W_{1, i} \psi_{i}\right)$. Using (8.7), we perform the decomposition
$\mathcal{T}_{d, N}\left(\psi_{i}\right)=\frac{\mathbf{U} \cdot \nabla \psi_{i}}{\psi_{i}}=U_{x}(0) \frac{\left(x \partial_{x} \psi_{i}-y \partial_{y} \psi_{i}\right)}{\psi_{i}}+\frac{\mathbf{U}_{A}\left(\omega_{1}\right) \cdot \nabla \psi_{i}}{\psi_{i}}+\frac{\mathbf{U}_{a p p} \cdot \nabla \psi_{i}}{\psi_{i}} \triangleq \mathcal{T}_{c_{\omega}}+\mathcal{T}_{u A}+\mathcal{T}_{u R}$.
For $\mathcal{T}_{u A}$, since we have piecewise $C^{1 / 2}$ estimates of $\mathbf{U}_{A} \psi_{u}$, we decompose it as follows

$$
\begin{equation*}
\mathcal{T}_{u A}=\left(U_{A} \psi_{u}\right) \frac{\partial_{x} \psi_{i}}{\psi_{i} \psi_{u}}+\left(V_{A} \psi_{u}\right) \frac{\partial_{y} \psi_{i}}{\psi_{i} \psi_{u}}, \quad \mathcal{T}_{u R}=\frac{\mathbf{U}_{a p p}}{|x|^{2}} \cdot|x|^{2} \frac{\nabla \psi_{i}}{\psi_{i}} \tag{8.43}
\end{equation*}
$$

Since $\psi_{i}, \psi_{u}$ are radial weights, we can estimate piecewise derivatives of $\psi_{i}, \psi_{i}^{-1}, \psi_{u}^{-1}$ following Appendix A. 2 in Part II [1. See also Sections 8.3.2, 8.3.3 for the estimates of the ratio among the weights. Using these estimates, (A.7) and following Appendix E. 6 in Part II [1], we can obtain piecewise $C^{1}, C^{1 / 2}$ estimates of

$$
\begin{equation*}
\frac{\partial_{j} \psi_{i}}{\psi_{i}}, \quad \frac{x \partial_{x} \psi_{i}-y \partial_{y} \psi_{i}}{\psi_{i}}, \quad \frac{\partial_{j} \psi_{i}}{\psi_{i} \psi_{u}}, \quad|x|^{2} \frac{\partial_{j} \psi_{i}}{\psi_{i}} \tag{8.44}
\end{equation*}
$$

The second ratio is estimated in Section 8.3.3. Using these estimates, A.8), and evaluating these functions on a fine mesh, we can refine the estimate. We use the estimates in (8.34) in Section 8.3 .3 with $(P, Q)=\left(\psi_{i}, \psi_{u}\right),\left(\psi_{i},|x|^{-2}\right)$ to refine the estimate of $\left(\partial_{j} \psi_{i}\right)\left(\psi_{i} \psi_{u}\right)^{-1},|x|^{2}\left(\partial_{j} \psi_{i}\right) \psi_{i}^{-1}$ near $r=0$. We remark that the $C^{1}$ estimates of these two terms are not singular near $r=0$, while the $C^{1}$ estimates of the first two terms can be singular.

Using the $C^{1 / 2}$ estimates of $\frac{\partial_{j} \psi_{i}}{\psi_{i} \psi_{u}}$ and of $u_{A} \psi_{u}, v_{A} \psi_{u}$ from the functional inequalities, (8.20), we obtain the $C^{1 / 2}$ estimate of $\mathcal{T}_{d, N}$. For $\mathcal{T}_{u, R}$, we can estimate

$$
|x|^{-k} \mathbf{U}_{a p p}, \quad k=2,3, \quad|x|^{-2} \partial_{i} \mathbf{U}_{a p p}
$$

and yield the $C^{1}$ estimates of $\frac{\mathbf{U}_{a p p}}{|x|^{2}}$, which along with the $C^{1}$ estimates of $|x|^{2} \frac{\partial_{j} \psi_{i}}{\psi_{i}}$ imply the $C^{1}, C^{1 / 2}$ estimates of $\mathcal{T}_{u R}$. We have another estimate of $\mathcal{T}_{u R}$ using the $C^{1}$ bounds of $\mathbf{U}_{\text {app }}, \partial_{j} \psi_{i} / \psi_{i}$ (8.44). We optimize these two estimates for $\mathcal{T}_{u R}$.


Figure 3. Left, right figures correspond to the locations of $(x, z)$ in cases (1), (2). The red line and blue line represents the $C_{x_{i}}^{1 / 2}$ estimate used to estimate $\left|\delta\left(d_{i, L}, x, z\right)\right|$. In case (1), we have two estimates.

Using the above (weighted) $C^{1 / 2}$ estimates for $\mathcal{T}_{u A}, \mathcal{T}_{u R}$, and (8.24), we get

$$
\begin{align*}
& \mu_{h, i} g_{i}(x-z) \delta\left(\mathcal{T}_{u A}\left(\psi_{i}\right)+\mathcal{T}_{u R}\left(\psi_{i}\right)\left(W_{1, i} \psi_{i}\right), x, z\right) \\
\leq & \left(\left|\mathcal{T}_{u A}(x)+\mathcal{T}_{u R}(x)\right|+\mu_{h, i} g_{i}(x-z) \delta_{\square}\left(\mathcal{T}_{u A}+\mathcal{T}_{u R}, x, z, x-z\right)\left(\psi_{i} / \varphi_{i}\right)(z)\right) E_{4} \tag{8.45}
\end{align*}
$$

For $\mathcal{T}_{c_{\omega}}$ away from $r=0$, we use the $C^{1}$ estimate of $\mathcal{T}_{c_{\omega}}$ (8.44). Near $r=0$, we use the estimates in (8.35) and Section 8.3.3 to estimate $\delta_{i}\left(\mathcal{T}_{c_{\omega}}, x, z\right) \min (|x|,|z|)^{1 / 2}$. We have

$$
\begin{align*}
& \mu_{h, i}\left|\delta\left(\mathcal{T}_{c_{\omega}} W_{1, i} \psi_{i}, x, z\right) g_{i}(x-z)\right| \leq \mu_{h, i}\left|\delta\left(W_{1, i} \psi_{i}, x, z\right) g_{i}(x-z)\right| \max _{p=x, z}\left|\mathcal{T}_{c_{\omega}}(p)\right|+\mu_{h, i} I \\
& \quad \leq E_{4} \max _{p=x, z}\left|\mathcal{T}_{c_{\omega}}(p)\right|+\mu_{h, i} I \\
& I \leq\left|\delta\left(\mathcal{T}_{c_{\omega}}, x, z\right) g_{i}(x-z)\right| \min \left(\left|W_{1, i} \psi_{i}(x)\right|,\left|W_{1, i} \psi_{i}(z)\right|\right)  \tag{8.46}\\
& \leq\left|\delta\left(\mathcal{T}_{c_{\omega}}, x, z\right)\right| g_{i}(x-z) \min (|x|,|z|)^{1 / 2} \max _{p=x, z} \frac{\psi_{i}(p)}{\varphi_{i}(p)|p|^{1 / 2}} E_{4}
\end{align*}
$$

The ratio $\frac{\psi_{i}}{\varphi_{i} \mid x^{1 / 2}}$ is estimated in (8.33). The weighted Hölder estimate of $\mathcal{T}_{c_{\omega}}$ is similar to that in Section 8.4.3, Denote $h_{i}=\left|x_{i}-z_{i}\right|$. We assume that $x_{1} \leq z_{1}$ and consider two cases (1) $x_{2} \leq z_{2}$ and (2) $x_{2} \geq z_{2}$. Denote $M=\min (|x|,|z|)^{1 / 2}$. Recall $\delta, \delta_{i}$ from (8.18).

In case (1) of $x_{2} \leq z_{2}$, for $w=\left(z_{1}, x_{2}\right)$, we have $|x| \leq|w| \leq|z|$, and

$$
\begin{aligned}
I I & \triangleq\left|\delta\left(\mathcal{T}_{c_{\omega}}, x, z\right)\right| g_{i}(x-z) M^{1 / 2} \leq\left|\delta\left(\mathcal{T}_{c_{\omega}}, x, w\right)+\delta\left(\mathcal{T}_{c_{\omega}}, w, z\right)\right| g_{i}(x-z) M^{1 / 2} \\
& \leq\left(\delta_{1}\left(\mathcal{T}_{c_{\omega}}, x, w\right)|x|^{1 / 2} h_{1}^{1 / 2}+\delta_{2}\left(\mathcal{T}_{c_{\omega}}, w, z\right)|w|^{1 / 2} h_{2}^{1 / 2}\right) g_{i}(h)
\end{aligned}
$$

and we then estimate $h_{j}^{1 / 2} g_{i}(h)$ following the methods in Section 8.6.1. We can also choose $w=\left(x_{1}, z_{2}\right)$ and obtain another estimate.

In case (2) $x_{2} \geq z_{2}$, we choose $w=\left(z_{1}, x_{2}\right)$. Then we have $|x|,|z| \leq|w|$ and $I I \leq\left|\delta\left(\mathcal{T}_{c_{\omega}}, x, w\right)+\delta\left(\mathcal{T}_{c_{\omega}}, z, w\right)\right| g_{i}(h) M^{1 / 2} \leq\left(\delta_{1}\left(\mathcal{T}_{c_{\omega}}, x, w\right)|x|^{1 / 2} h_{1}^{1 / 2}+\delta_{2}\left(\mathcal{T}_{c_{\omega}}, z, w\right)|z|^{1 / 2} h_{2}^{1 / 2}\right) g_{i}(h)$.

In the above estimates of $I I$, we further use the estimate of $\delta_{j}\left(\mathcal{T}_{c_{\omega}}, p, q\right) \min (|p|,|q|)^{\frac{1}{2}}$. In Figure 3, we illustrate the $C_{x_{i}}^{\frac{1}{2}}$ estimate and the triangle inequalities used to bound $\left|\delta\left(\mathcal{T}_{c_{\omega}}, x, z\right)\right|$ in various cases.
8.5.3. Other nonlinear terms. The estimates of other nonlinear terms are simple. We follow Sections 8.1.1 8.1.2 for the $L^{\infty}$ and $C^{1 / 2}$ estimates of $\nabla \mathbf{U}, \mathbf{U}$ (8.7) and the $C^{k}$ estimates of $\hat{W}_{2}$.

Nonlinear terms involving $W_{1}$. We estimate nonlinear terms in (8.14) involving $\omega_{1}, \eta_{1}, \xi_{1}$ other than $\mathcal{T}_{d, N} W_{1, i} \rho_{i}$. We write down the decomposition of each term below as $A \cdot B$ and estimate $\delta(f, x, z)$ for each $f=A, B$,

$$
\begin{align*}
& U_{x}(0) \cdot \omega_{1} \psi_{1}, \quad\left(U_{x}(0) \eta_{1}-\tilde{U}_{x} \eta_{1}-\tilde{V}_{x} \xi_{1}\right) \psi_{2}=U_{x}(0) \cdot \eta_{1} \psi_{2}-\tilde{U}_{x} \cdot\left(\eta_{1} \psi_{2}\right)-\tilde{V}_{x} \cdot\left(\xi_{1} \psi_{2}\right)  \tag{8.47}\\
& \left(3 U_{x}(0) \xi_{1}-\tilde{U}_{y} \eta_{1}-\tilde{V}_{y} \xi_{1}\right) \psi_{2}=3 U_{x}(0) \cdot\left(\xi_{1} \psi_{2}\right)-\tilde{U}_{y} \cdot\left(\eta_{1} \psi_{2}\right)-\tilde{V}_{y} \cdot\left(\xi_{1} \psi_{2}\right)
\end{align*}
$$

Since each term involves $W_{1, i} \psi_{i}$, we can estimate it using (8.24).
Note that we use the same weight $\psi_{2}=\psi_{3}$ for $\eta_{1}, \xi_{1}$ in the Hölder estimate.

It remains to estimate the $\mathcal{B}_{o p, j}\left(\mathbf{U}, \hat{W}_{2}\right)$ in (8.14). We estimate three terms separately.
Nonlinear terms $\mathcal{B}_{o p, j}\left(\mathbf{U}_{A}, \widehat{W}_{2}\right)$. We estimate $\mathcal{B}_{o p, j}\left(\mathbf{U}_{A}, \widehat{W}_{2}\right) \psi_{i}$ (8.14). Similar to (8.47), we write down the decomposition of each term in $\mathcal{B}_{o p, j}\left(\mathbf{U}_{A}, \widehat{W}_{2}\right) \psi_{i}$ (8.9), (8.10) below as $A \cdot B$ and estimate $\delta_{i}(f, x, z)$ for each $f=A, B$ and then use (8.20), (8.23) to estimate the $A B$. For the advection term of $\widehat{W}_{2, i}$ (8.14), (8.10), we use

$$
-\mathbf{U}_{A} \cdot \nabla \widehat{W}_{2, i} \psi_{i}=-\left(U_{A} \psi_{u}\right) \cdot\left(\frac{\partial_{x} \widehat{W}_{2, i} \psi_{i}}{\psi_{u}}\right)-\left(V_{A} \psi_{u}\right) \cdot\left(\frac{\partial_{y} \widehat{W}_{2, i} \psi_{i}}{\psi_{u}}\right), i=1,2,3
$$

Near $0, \nabla \widehat{W}_{2, i} \frac{\psi_{i}}{\psi_{u}}$ has a vanishing order $O(|x|)$, and we can estimate its $C^{1}$ bound.
For other nonlinear terms in $B_{j}\left(U_{A}, \widehat{W}_{2}\right), j=2,3$ involving $(\nabla \mathbf{U})_{A}$, we use

$$
\left(-\left(\partial_{i} U\right)_{A} \hat{\eta}_{2}-\left(\partial_{i} V\right)_{A} \hat{\xi}_{2}\right) \psi_{2}=-\left(\left(\partial_{i} U\right)_{A} \psi_{1}\right) \cdot\left(\frac{\hat{\eta}_{2} \psi_{2}}{\psi_{1}}\right)-\left(\left(\partial_{i} V\right)_{A} \psi_{1}\right) \cdot\left(\frac{\hat{\xi}_{2} \psi_{2}}{\psi_{1}}\right), \quad i=1,2
$$

We use $i=1$ for $B_{2}$ and $i=2$ for $B_{3}$. The above estimate is similar to the linear Hölder estimate in Section 5 [3] and we treat $\hat{W}_{2}$ similar to $\bar{\omega}, \bar{\theta}_{x}, \bar{\theta}_{y}$.

Remaining nonlinear terms of $\widehat{W}_{2}$. The remaining nonlinear terms of $\widehat{W}_{2}$ in (8.14) are

$$
I_{1}=U_{x}(0) \widehat{W}_{2, j, M}, \quad I_{2}=B_{j}\left(\mathbf{U}_{a p p}, \widehat{W}_{2}\right)
$$

With the second order correction near $x=0, I_{1}$ vanishes like $O\left(|x|^{3}\right)$. By definition, we have

$$
\widehat{W}_{2, i}=O\left(|x|^{2}\right), \quad \mathbf{U}_{a p p}=O\left(|x|^{3}\right)
$$

It follows that $I_{2}$ vanishes to the order $O\left(|x|^{4}\right)$ near $|x|=0$. For $I_{1}$, we estimate $C^{1 / 2}$ of $F_{i} \psi_{i}$ by optimizing the estimate of $C^{1 / 2}$ estimate of $\left(F_{i}|x|^{\alpha_{i}}\right) \cdot \psi_{m, i}$ and $F_{i} \psi_{i}$, where $\psi_{i}=|x|^{\alpha_{i}} \psi_{m, i}$ (8.31). Near $r=0$, we estimate the $C^{1 / 2}$ semi-norm of $F_{i}|x|^{p}, p=-2,-5 / 2$ carefully using the method in Section 8.9.3 to overcome the singularity near $r=0$.

For $I_{2}, I_{2} \psi_{i}=O\left(|x|^{a}\right), a \geq \frac{3}{2}$ near $r=0$, we obtain piecewise $C^{1}$ estimates of $I_{2} \psi_{i}$. We estimate a typical term $T=\partial_{x}^{i} \partial_{y}^{j}\left(U_{a p p} \partial_{x} \hat{\eta}_{2} \psi_{2}\right)$ in $\mathcal{B}_{o p, 2}\left(\mathbf{U}_{a p p}, \hat{W}_{2}\right) \psi_{2}$ (8.14) as follows. Away from 0, we estimate $T$ using (A.7), the estimate of $U_{a p p}, \hat{\eta}_{2}$ and the triangle inequality. Near $r=0$, we have three terms
$D\left(U_{\text {app }}\right) \cdot \partial_{x} \hat{\eta}_{2} \cdot \psi_{2}+U_{\text {app }} \cdot D \partial_{x} \hat{\eta}_{2} \cdot \psi_{2}+U_{\text {app }} \partial_{x} \hat{\eta}_{2} \cdot D \psi_{2}=J_{1}+J_{2}+J_{3}, \quad D=\partial_{x}^{i} \partial_{y}^{j}, i+j \leq 1$.
For $J_{1}$, since $U_{\text {app }}=O\left(|x|^{3}\right), \hat{\eta}_{2}=O\left(|x|^{2}\right)$, we extract these vanishing orders and estimate

$$
\left(D U_{a p p}\right) \cdot\left(\partial_{x} \hat{\eta}_{2} /|x|\right) \cdot\left(\psi_{2}|x|^{3}\right)
$$

The estimate of $J_{2}$ is similar. For $J_{3}$, we use (8.31) for $\psi_{2}=\psi_{m, 2}|x|^{\alpha_{2}}$. We obtain

$$
\left.J_{3}=\left(U_{\text {app }} \partial_{x} \hat{\eta}_{2}\right)|x|^{\alpha_{2}-1}\right) \frac{D \psi}{|x|^{\alpha_{2}-1}}, \quad \frac{D \psi_{2}}{|x|^{\alpha_{2}-1}}=D \psi_{m, 2}|x|+\alpha_{2} \psi_{m, 2} \frac{x_{1}^{i} x_{2}^{j}}{|x|}, i+j=1
$$

With the above $C^{1 / 2}$ estimates, we can estimate $\delta_{\square}\left(I_{k}, x, z, x-z\right), k=1,2$ (8.22).
8.6. $L^{\infty}$ stability analysis in the far-field. Recall parameters $N_{i}, N$ from (8.4). For $|x|_{\infty} \geq$ $y_{N_{2}}$, the piecewise polynomial part in the approximate steady states is 0 . We have

$$
\begin{equation*}
\bar{\omega}(x)=\bar{\omega}_{1}=\bar{g}_{1}(\beta) r^{\bar{\alpha}_{1}}, \quad \bar{\theta}(x)=\bar{\theta}_{1}=\bar{g}_{2}(\beta) r^{1+2 \bar{\alpha}_{1}}, r=\left(x_{1}^{2}+x_{2}^{2}\right)^{1 / 2}, \beta=\arctan \left(x_{2} / x_{1}\right) \tag{8.48}
\end{equation*}
$$

To perform energy estimate in the far-field, we need to estimate the asymptotic behavior of $\partial_{x}^{i} \partial_{y}^{j} f(r, \beta)$. Using the formulas of derivatives,

$$
\partial_{x} g=\left(\cos \beta \partial_{r}-\frac{\sin \beta}{r} \partial_{\beta}\right) g, \quad \partial_{y} g=\left(\sin \beta \partial_{r}+\frac{\cos \beta}{r} \partial_{\beta}\right) g
$$

the triangle inequality, (8.48), and the piecewise bounds of $\partial_{\beta}^{i} \bar{g}_{j}(\beta)$ established using the method in Appendix C.2, C. 3 in Part II [1], we obtain

$$
\begin{equation*}
\left|\partial_{x}^{i} \partial_{y}^{j} \bar{\omega}\right| \leq \bar{\omega}_{a g, i j}(\beta) r^{\bar{\alpha}_{1}-i-j}, \quad\left|\partial_{x}^{i} \partial_{y}^{j} \bar{\theta}\right| \leq \bar{\theta}_{a g, i j}(\beta) r^{1+2 \bar{\alpha}_{1}-i-j}, \quad\left|\partial_{x}^{i} \partial_{y}^{j} \bar{\phi}\right| \leq \bar{\phi}_{a g, i j}(\beta) r^{2+\bar{\alpha}_{1}-i-j} \tag{8.49}
\end{equation*}
$$

for $|x|_{\infty} \geq y_{N_{2}}$, and some piecewise constants $\bar{\omega}_{a g, i j}(\beta), \bar{\theta}_{a g, i j}(\beta), \bar{\phi}_{a g, i j}(\beta)$.
We perform the weighted $L^{\infty}\left(\varphi_{i}\right)$ in the far field $|x|_{\infty} \geq y_{N-1}$. We can decompose the weights $\varphi_{i}$ as follows and have the following estimate uniformly for $|x| \geq y_{N-1}$ from (8.25)

$$
\begin{equation*}
\varphi_{i}=|x|^{-1 / 2} P_{1, i}+P_{2, i}, \quad\left|P_{j, i}(x)\right| \leq c_{j, i} r^{\alpha_{j, i}}, j=1,2, i=1,2,3 \tag{8.50}
\end{equation*}
$$

for some radial weights $P_{j, i}, c_{j, i}$, with $P_{1, i}$ decaying faster than $P_{2, i}$. We estimate the nonlocal terms, the local terms in (8.11), (8.14), and the damping terms $d_{i}$ (8.16) in order.

We use the $C^{1 / 2}$ and $L^{\infty}$ estimate of $\mathbf{u}_{A},(\nabla \mathbf{u})_{A}$ established in Section 7 in the supplementary material for Part II [2].
Estimate of $f /|x|^{\alpha}$. Since the weight involves the power $|x|^{-1 / 2}$ singular along $x=0$, we need to estimate the asymptotic behavior of $f /|x|^{1 / 2}$ for $f=u_{A}, \bar{\omega}_{y}, \nabla \bar{\theta}, \bar{\theta}_{x y}$. We estimate $u_{A} /|x|^{1 / 2} r^{\gamma}$ for suitable power $\gamma$ at the end of Section 7.3 .1 in the supplementary material of Part II [2]. For other functions $f$ depending on the profile, if $f(0, y)=0$, or equivalently $f(r, \pi / 2)=0$, we can estimate $f /|x|^{\alpha}=f /(r \cos \beta)^{\alpha}$ for $\alpha \in[0,1]$. Firstly, using the polar representation (8.48) and the method in Appendix C. 3 in Part II [1, we can estimate $\partial_{\beta} f$.

Using the piecewise derivative bounds discussed in Appendix E. 7 in Part II [1] and

$$
\begin{align*}
& \frac{f(r, \beta)}{\pi / 2-\beta}=-\int_{\beta}^{\pi / 2} \partial_{\beta}(f, \tau) d \tau, x=r \cos \beta, \cos \beta=(\pi / 2-\beta) \sin \xi, \xi \in[\beta, \pi / 2]  \tag{8.51}\\
& \frac{f(r, \beta)}{|x|^{\alpha}}=r^{-\alpha} \frac{f(r, \beta)}{\frac{\pi}{2}-\beta}\left(\frac{\pi}{2}-\beta\right)^{1-\alpha}\left(\frac{\pi / 2-\beta}{\cos \beta}\right)^{\alpha}=r^{-\alpha} \frac{f(r, \beta)}{\frac{\pi}{2}-\beta}\left(\frac{\pi}{2}-\beta\right)^{1-\alpha}(\sin \xi)^{\alpha}
\end{align*}
$$

we can obtain the piecewise bounds of $\frac{f(r, \beta)}{\pi / 2-\beta}$ and $f(r, \beta) /|x|^{\alpha}$.
Suppose that $f(0, y)=0,|f(r, \beta)| r^{b} \leq C_{1},\left|\partial_{x} f r^{b+1}\right| \leq C_{2}$ for all $\max (x, y) \geq R$ and $b+1 \leq 0$. Using $\left|f_{x}(z, y)\right| \leq C_{2} r^{-b-1}$ for $z \in[0, x]$, we have another simple estimate for $f / x^{1 / 2}$
$\left|\frac{f}{x^{1 / 2}}\right|=\left|\frac{1}{x^{1 / 2}} \int_{0}^{x} f_{x}(z, y) d z\right| \leq C_{2} r^{-(b+1)} x^{1 / 2}=C_{2} r^{-b-1 / 2}(\cos \beta)^{1 / 2} \leq C_{2} r^{-b} R^{-1 / 2}(\cos \beta)^{1 / 2}, \beta \in\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$,
$\left|\frac{f}{x^{1 / 2}}\right| \leq \min \left(C_{1}(\cos \beta)^{-1 / 2} r^{-b-1 / 2}, C_{1} r^{-b} R^{-1 / 2}\right) \leq \min \left(C_{1}(\cos \beta)^{-1 / 2} R^{-1 / 2}, C_{1} R^{-1 / 2}\right) r^{-b}, \beta \in\left[0, \frac{\pi}{4}\right]$,
where we have used $x=\max (x, y) \geq R$ in the second case. The above estimates also provide a uniform estimate for $f / x^{1 / 2} r^{b+1 / 2}$.

Estimate of the nonlocal terms. We focus on the typical terms $u_{x, A} \bar{\theta}_{x}, u_{A} \bar{\theta}_{x x}$ in the $\eta_{1}$ equation. Using (8.49), (8.50), and the estimate in the paragraph above (8.51), we have
$\left.\left|\bar{\theta}_{x}\right| x_{1}\right|^{-1 / 2} P_{1, i}\left|\leq C_{1}(\beta) r^{2 \bar{\alpha}_{1}+\alpha_{1, i}-1 / 2},\left|\bar{\theta}_{x} P_{2, i}\right| \leq C_{2}(\beta) r^{2 \bar{\alpha}_{1}+\alpha_{2, i}},\left|\bar{\theta}_{x x} P_{j, i}\right| \leq C_{2+j}(\beta) r^{2 \bar{\alpha}_{1}-1+\alpha_{j, i}}\right.$,
for some piecewise constant functions $C_{i}(\beta)$. Then we apply the estimates in Section 7.3.1 in the supplementary material of Part II [2] to estimate $u_{x, A} r^{\alpha}$ with $\alpha=2 \bar{\alpha}_{1}+\alpha_{1, i}-1 / 2,2 \bar{\alpha}_{1}+\alpha_{2, i}$, and $u_{A}\left|x_{1}\right|^{-1 / 2} r^{2 \bar{\alpha}_{1}-1+\alpha_{1, i}}, u_{A} r^{2 \bar{\alpha}_{1}-1+\alpha_{2, i}}$. The estimates of other nonlocal terms, e.g. $\mathbf{U}_{A},(\nabla \mathbf{U})_{A}$ (8.7) (8.11) are similar. We remark that the far field estimate of these nonlocal terms are bounded. For example, for the weights $\varphi_{i}$ and large $|x|$, we have

$$
\varphi_{1} \sim|x|^{-1 / 6}, \quad\left|u_{x, A}\right| \lesssim \log |x||x|^{1 / 6}, \quad 2 \bar{\alpha}_{1}+\alpha_{2, i}=2 \bar{\alpha}_{1}+\frac{1}{7} \approx-2 / 3+1 / 7
$$

and the estimate for $u_{x, A} r^{2 \bar{\alpha}_{1}+1 / 7}$ decays for large $|x|$. Similar reasoning applies to other powers.
Estimate of local terms. The estimate of the local terms in (8.11), e.g. $\eta_{1}$ in the $\omega_{1}$ equation, is trivial. We just need to bound $\eta_{1} \varphi_{1} \leq\left\|\eta_{1} \varphi_{2}\right\|_{\infty}\left|\varphi_{1} / \varphi_{2}\right|$ and the ratio $\varphi_{1} / \varphi_{2}$. Since $\varphi_{2}$ decays slower, $\varphi_{1} / \varphi_{2}$ is bounded. More generally, to bound $P=|x|^{-1 / 2} P_{1}+P_{2}, Q=|x|^{-1 / 2} Q_{1}+Q_{2}$ with $P_{i}, Q_{i}$ radial weights and $P_{i} \lesssim Q_{i}$, we use

$$
\begin{equation*}
P / Q \leq \max \left(P_{1} / Q_{1}, P_{2} / Q_{2}\right) \tag{8.52}
\end{equation*}
$$

and the estimate in (8.25). The estimates of $\bar{v}_{x}^{N} \eta_{1}, \bar{u}_{y}^{N} \xi_{1}$ in (8.11) follow the above estimate and (8.49) for $\bar{v}_{x}^{N}$. We also treat $-\bar{u}_{x}^{N} \eta_{1}, \bar{u}_{x}^{N} \xi_{1}$ from the damping terms $d_{2} \eta_{1}, d_{3} \xi_{1}$ as a perturbation using the same method.

Estimate of the damping terms. It remains to estimate $\frac{\left(\bar{c}_{l} x+\overline{\mathbf{u}}^{N}\right) \nabla \varphi_{i}}{\varphi_{i}} W_{1, i} \varphi_{i}$ from $d_{i}$ (8.16). We defer the estimate of terms involving $\mathbf{U}$ to the nonlinear estimates. The main part is given by the $\bar{c}_{l} x$ term since $\overline{\mathbf{u}}^{N}$ grows sublinearly for large $|x|$.

Since $r \partial_{r} P, r \partial_{r} Q$ are radial weights, using (8.37), (8.52), we obtain

$$
\begin{align*}
& \frac{x \cdot \nabla \varphi}{\varphi} \leq \frac{x^{-1 / 2} r \partial_{r} P+r \partial_{r} Q}{x^{-1 / 2} P+Q} \leq \max \left(\frac{r \partial_{r} P}{P}, \frac{r \partial_{r} Q}{Q}\right),\left|\frac{\partial_{y} \varphi}{\varphi}\right| \leq \max \left(\left|\frac{\partial_{y} P}{P}\right|,\left|\frac{\partial_{y} Q}{Q}\right|\right)  \tag{8.53}\\
& \left|\frac{x \partial_{x} \varphi}{\varphi}\right| \leq \max \left(\frac{\left|1 / 2 \cdot x^{-1 / 2} P+x^{1 / 2} \partial_{x} P\right|}{x^{-1 / 2} P}, \frac{\left|x \partial_{x} Q\right|}{Q}\right) \leq \max \left(1 / 2+\frac{\left|x \partial_{x} P\right|}{P}, \frac{\left|x \partial_{x} Q\right|}{Q}\right)
\end{align*}
$$

For radial weight $P(x)=\sum_{i \leq n} p_{i} r^{a_{i}}=r^{a_{n}} \sum_{i \leq n} p_{i} r^{a_{i}-a_{n}} \triangleq r^{a_{n}} P_{m, n}(x)$ with $a_{i}$ increasing in $i$, since $a_{i}-a_{n} \leq 0, P_{m, n}(r)$ is decreasing in $r$, we have

$$
\frac{r \partial_{r} P}{P}=\frac{r \partial_{r} r^{a_{n}}}{r^{a_{n}}}+\frac{r \partial_{r} P_{m}}{P_{m}}=a_{n}+\frac{r \partial_{r} P_{m, n}}{P_{m, n}} \leq a_{n} .
$$

The second term has faster decay, and $a_{n}$ is the main term. Using (8.49), (8.50), we can derive the asymptotic estimates of $\frac{\bar{u}^{N}}{x}$ and $\bar{v}^{N}$, and then estimate $\frac{\overline{\mathbf{u}}^{N} \cdot \nabla \varphi_{i}}{\varphi_{i}}$ as follows

$$
\left|\frac{\overline{\mathbf{u}}^{N} \cdot \nabla \varphi_{i}}{\varphi_{i}}\right| \leq\left|\frac{\bar{u}^{N}}{x}\right|\left|\frac{x \partial_{x} \varphi_{i}}{\varphi_{i}}\right|+\left|\bar{v}^{N}\right|\left|\frac{\partial_{y} \varphi_{i}}{\varphi_{i}}\right| .
$$

We do not estimate $\frac{y \partial_{y} \varphi_{i}}{\varphi_{i}}$ since $\varphi_{i}$ is not singular on $y=0$. Note that the $\bar{c}_{\omega}^{N}$ term in $d_{i}$ (8.16) is a damping term. We keep the following terms as damping terms in this estimate

$$
\left(\left(\bar{c}_{l} x \cdot \nabla \varphi_{i}\right) / \varphi_{i}+c_{i} \bar{c}_{\omega}^{N}\right) W_{1, i} \varphi_{i}, \quad c=(1,2,2)
$$

whose coefficient does not decay for large $|x|$. We further bound $\frac{\bar{c}_{l} x \cdot \nabla \varphi_{i}}{\varphi_{i}}$ from above using the first estimate in (8.53).
8.6.1. Estimate the Hölder weights. The Hölder weights $g=g_{i}$ is $-1 / 2$ homogeneous. Taking derivative on $\lambda$, we yield

$$
\left(x \cdot \nabla_{x} g_{i}\right)\left(\lambda x_{1}, \lambda x_{2}\right)=\partial_{\lambda} g_{i}\left(\lambda x_{1}, \lambda x_{2}\right)=\partial_{\lambda}\left(\lambda^{-1 / 2} g_{i}\left(x_{1}, x_{2}\right)\right)=-\frac{1}{2} \lambda^{-3 / 2} g_{i}\left(x_{1}, x_{2}\right)
$$

Choosing $\lambda=1$, we yield the following useful identity for the damping term $d_{g, i}$ (8.16)

$$
\begin{equation*}
x \cdot \nabla g_{i}(x)=-\frac{1}{2} g_{i}(x) \tag{8.54}
\end{equation*}
$$

In our energy estimates, we estimate several 0 -homogeneous quantities related to $g$ for $h_{i} \geq 0$

$$
f(h)=h_{k}^{1 / 2} g(h), h_{k} \frac{\left(\partial_{j} g\right)}{g}(h),|h| \frac{\left(\partial_{j} g\right)}{g}(h), k, j=1,2, \quad \frac{g_{i_{1}}(h)}{g_{i_{2}}(h)}, 1 \leq i_{1}, i_{2} \leq 3
$$

Since $f(h)=f\left(\frac{h_{1}}{h_{2}}, 1\right)$ for $h_{2} \neq 0$ and $f(h)=f\left(1, \frac{h_{2}}{h_{1}}\right), h_{1} \neq q 0$, we can estimate it by partitioning $\left(h_{1}, h_{2}\right) \in[0,1] \times\{1\},\{1\} \times[0,1]$ and using the monotonicity of $g, \partial_{j} g$. From $g=g_{i}$ (A.4), we have

$$
g(s)=\frac{1}{A_{1}(s)^{1 / 2}+A_{2}(s)^{1 / 2}}, \partial_{i} g=-\frac{1}{2} \frac{a_{1 i} A_{1}^{-1 / 2}+a_{2 i} A_{2}^{-1 / 2}}{\left(A_{1}^{1 / 2}+A_{2}^{1 / 2}\right)^{2}}, \frac{\partial_{i} g}{g}=-\frac{1}{2} \frac{a_{1 i} A_{1}^{-1 / 2}+a_{2 i} A_{2}^{-1 / 2}}{A_{1}^{1 / 2}+A_{2}^{1 / 2}}
$$

for $A_{i}=a_{i 1} s_{1}+a_{i 2} s_{2}$ with $a_{i j}>0$. Clearly, $g$ is decreasing in $\left|s_{i}\right|$. For $s_{1}, s_{2}>0$, since $A_{i}$ is increasing in $s_{1}, s_{2}, \partial_{i} g, \frac{\partial_{i} g}{g}$ are negative and increasing in $s_{1}, s_{2}$. It follows that $\left|\frac{\partial_{i} g}{g}\right|$ is decreasing in $s_{1}, s_{2}$.
8.7. $C^{1 / 2}$ stability analysis in the far-field. We use the $C^{1 / 2}$ estimate of $\mathbf{u}_{A}, \nabla \mathbf{u}_{A}$ established in Section 7 in the supplementary material of Part II [2].

For the $C^{1 / 2}$ estimate of a typical terms in (8.11) in the far-field, e.g. $u_{A} \bar{\omega}_{x} \psi_{1}$, we perform the following decomposition

$$
\begin{equation*}
u_{A} \bar{\omega}_{x} \psi_{1}=u_{A} \psi_{u} \cdot\left(\bar{\omega}_{x} \psi_{1} / \psi_{u}\right) \tag{8.55}
\end{equation*}
$$

We need the asymptotic $L^{\infty}$ and $C^{1 / 2}$ estimates of $u_{A} \psi_{u}$ and $\bar{\omega}_{x} \frac{\psi_{1}}{\psi_{u}}$. More generally, we need to estimate the asymptotic behavior of $\mathbf{u}_{A} \psi_{u},(\nabla \mathbf{u})_{A} \psi_{1}$ and weighted profile $\bar{S} \frac{f}{g}$ for two weights $f, g$.

From Section 7 in the supplementary material of Part II [2], for $|z| \geq|x|_{\infty} \geq R_{2}$, we have the $C_{x_{i}}^{1 / 2}$ and $L^{\infty}$ estimates of $\mathbf{u}_{A}(x), \nabla \mathbf{u}_{A}(x)$ uniformly for $\lambda=|x|_{\infty} / x_{c 2}$

$$
\begin{align*}
& \delta_{i}\left(\left(\partial_{x}^{i} \partial_{y}^{j} \phi\right)_{A} \psi_{u}, x, z\right) \leq C_{u_{h}, i j}, i+j=1, \quad \delta_{i}\left(\left(\partial_{x}^{i} \partial_{y}^{j} \phi\right)_{A} \psi_{1}, x, z\right) \leq C_{u_{h}, i j}, i+j=2 \\
& \frac{\left.\mid \partial_{x}^{i} \partial_{y}^{j} \phi\right)_{A} \psi_{u}(x) \mid}{|x|^{1 / 2}} \leq C_{u, i j}, i+j=1, \quad \frac{\left.\mid \partial_{x}^{i} \partial_{y}^{j} \phi\right)_{A} \psi_{1}(x) \mid}{|x|^{1 / 2}} \leq C_{u, i j}, i+j=2 \tag{8.56}
\end{align*}
$$

for some constant $C_{u_{h}, i j}, C_{u, i j}$. Here, we use $\left(\partial_{x}^{i} \partial_{y}^{j} \phi\right)_{A}$ to denote $\mathbf{u}_{A},(\nabla \mathbf{u})_{A}$, e.g. $-\left(\partial_{y} \phi\right)_{A}=u_{A}$. This notation is consistent with the relation between the velocity and the stream function $u=-\partial_{y} \phi$.
8.7.1. Hölder estimates of a product. Firstly, we estimate $[A B]_{C_{x_{i}}^{1 / 2}}$ with

$$
\left|\partial_{x}^{i} \partial_{y}^{j} A(r, \beta)\right| \leq a_{i j} r^{\alpha-i-j}, \quad\left|\partial_{x}^{i} \partial_{y}^{j} B(r, \beta)\right| \leq b_{i j} r^{\beta-i-j}, \quad i+j \leq 1
$$

for all $|x| \geq R$. Using the Leibniz rule, for $i+j \leq 1$, we get

$$
C=A B, \quad\left|\partial_{x}^{i} \partial_{y}^{j} C\right| \leq c_{i j} r^{\alpha+\beta-i-j}, \quad c_{i j}=\sum_{i_{1} \leq i, j_{1} \leq j} a_{i_{1} j_{1}} b_{i-i_{1}, j-j_{1}}
$$

for $|x| \geq R$. In the $C_{x_{i}}^{1 / 2}$ estimate, for $R \leq|x| \leq|y|$ and $\alpha+\beta \leq 0$, we have

$$
|C(x)-C(y)| \leq 2 c_{00}|x|^{\alpha+\beta}, \quad|C(x)-C(y)| \leq\left|\partial_{i} C(\xi)\right||x-y| \leq c_{10}|x|^{\alpha+\beta-1}|x-y|
$$

for some $|\xi| \geq|x|$. It follows

$$
\begin{equation*}
\frac{|C(x)-C(y)|}{|x-y|^{1 / 2}} \leq \min \left(\frac{c_{00} 2|x|^{\alpha+\beta}}{|x-y|^{1 / 2}}, c_{10}|x|^{\alpha+\beta-1}|x-y|^{1 / 2}\right) \leq \sqrt{2 c_{00} c_{10}}|x|^{\alpha+\beta-1 / 2} \tag{8.57}
\end{equation*}
$$

The $C_{x_{2}}^{1 / 2}$ estimate is similar. Using (8.49), (8.26), (8.27), and the above estimates, we can estimate the asymptotic behavior of the $L^{\infty}$ and $C_{x_{i}}^{1 / 2}$ semi-norm of weighted profile, e.g. $\bar{\omega}_{x} \frac{\psi_{1}}{\psi_{2}}$ (8.55).

Next, we estimate $\delta_{i}\left(u_{A} F, x, y\right)$ or $\delta_{i}\left((\nabla u)_{A} F, x, y\right)$ with

$$
\left|\delta_{i}(F, x, y)\right| \leq f_{i} r^{\alpha_{i}}, \quad|F(x)| \leq f_{0} r^{\alpha_{0}}, \quad|x|,|y| \geq r, \quad \alpha_{0} \leq 0, \alpha_{1}+1 / 2, \alpha_{2}+1 / 2 \leq 0
$$

Recall the estimate of (8.56). We focus on $u_{x, A} F$. The estimates of other terms are similar. For $|x|,|y| \geq R$, using $|a(x) b(x)-a(y) b(y)| \leq|(a(x)-a(y)) b(x)|+|a(y)(b(x)-b(y))|$, we obtain

$$
\begin{aligned}
\left|\delta_{i}\left(u_{x, A} F, x, y\right)\right| & \leq\left|\delta_{i}\left(u_{x, A}, x, y\right) F(z)\right|+\left|u_{x, A}(x) \delta_{i}(F, x, y)\right| \leq C_{u_{h}, 11} f_{0}|x|^{\alpha_{0}}+C_{u, 11}|x|^{1 / 2} f_{i}|x|^{\alpha_{i}} \\
& \leq C_{u_{h}, 11} f_{0} R^{\alpha_{0}}+C_{u, 11} f_{i} R^{\alpha_{i}+1 / 2}, \quad i=1,2
\end{aligned}
$$

Using the above estimate, we can estimate (8.55) and other nonlocal terms in (8.11) in the weighted Hölder estimate in the far-field, e.g. $u_{x, A} \bar{\theta}_{x} \psi_{2}, u_{A} \bar{\theta}_{x x} \psi_{2}$.
8.7.2. Estimates of the local terms. The Hölder estimate of the local terms, e.g. $\bar{v}_{x}^{N} \xi_{1}$ in the $\eta_{1}$ equation, is simple. See (8.24). We have

$$
\left|\delta_{i}\left(\bar{v}_{x}^{N} \xi_{1} \psi_{2}, x, y\right)\right| \leq\left|\bar{v}_{x}^{N} \delta_{i}\left(\xi_{1} \psi_{2}, x, y\right)\right|+\left|\delta_{i}\left(\bar{v}_{x}^{N}, x, y\right) \xi_{1} \psi_{2}\right|
$$

and use (8.57) to estimate $\delta_{i}\left(\bar{v}_{x}^{N}, x, y\right)$, the energy to control $\delta_{i}\left(\xi_{1} \psi_{2}, x, y\right), \xi_{1} \psi_{2}$. Note that to bound $\xi_{1} \psi_{2}$, we use $\varphi_{3} \geq p_{n} r^{a_{n}}$ for some $p_{n}, a_{n}$, where $r^{a_{n}}$ is slowest decay power in $\varphi_{3}$, and

$$
\left|\xi_{1} \psi_{2}\right| \leq\left\|\xi_{1} \varphi_{3}\right\|_{\infty} \frac{\psi_{2}}{\varphi_{3}} \leq E_{4} \frac{\psi_{2}}{p_{n} r^{a_{n}}}
$$

The ratio of the weight is further estimated using (8.27).
8.7.3. Estimates of the damping terms. It remains to estimate the damping terms $d_{g, i}$ and $d_{i}$ (8.16). Since $\overline{\mathbf{u}}^{N}$ decays for large $|x|$, using (8.49) and (8.54), for $|x|,|z| \geq R$, we bound $d_{g, i}$ from above as follows

$$
\begin{aligned}
d_{g, i, l i n} \triangleq & -\frac{\bar{c}_{l}}{2}+\frac{\left(\overline{\mathbf{u}}^{N}(x)-\overline{\mathbf{u}}^{N}(z)\right) \cdot \nabla g_{i}(x-z)}{g_{i}(x-z)} \leq-\frac{\bar{c}_{l}}{2}+R^{\bar{\alpha}_{1}}\left(\left\|\bar{\phi}_{a g, 11}\right\|_{\infty}\left|x_{1}-z_{1}\right|\right. \\
& \left.\left.\left.+\left\|\bar{\phi}_{a g, 02}\right\|_{\infty}\left|x_{2}-z_{2}\right|\right) \frac{\left|\partial_{1} g_{i}(x-z)\right|}{g_{i}(x-z)}+\left\|\bar{\phi}_{a g, 20}\right\|_{\infty}\left|x_{1}-z_{1}\right|+\left\|\bar{\phi}_{a g, 1}\right\|_{\infty}\left|x_{2}-z_{2}\right|\right) \frac{\left|\partial_{1} g_{i}(x-z)\right|}{g_{i}(x-z)}\right)
\end{aligned}
$$

Similarly, using the estimate for $\nabla \mathbf{U}$, we can bound $d_{g, i, n o n}=\frac{\delta(\mathbf{U}, x, z) \cdot \nabla g_{i}}{g_{i}}$. Then, we estimate $d_{g, i}=d_{g, i, l i n}+d_{g, i, n l i n}$ from above. Since $h_{k} \frac{\partial_{l} g_{i}(h)}{g_{i}(h)}$ is 0 -homogeneous, we follow Section 8.6.1 to estimate it.

For $d_{i}(8.16)$, let $a_{n, i}$ be the last power in the radial weight $\psi_{i}$. We decompose $\psi_{i}=r^{a_{n, i}} \psi_{m, i}$, where $\psi_{m, i}(x) \asymp 1$ for large $|x|$. Since $\frac{\partial_{j} \psi_{i}}{\psi_{i}}=\frac{\partial_{j} r^{a_{n, i}}}{r^{a_{n, i}}}+\frac{\partial_{j} \psi_{m, i}}{\psi_{m, i}}$, we decompose $d_{i}$ as follows

$$
d_{i}=\frac{\bar{c}_{l} x \cdot \nabla|x|^{a_{n, i}}}{|x|^{a_{n, i}}}+\frac{\bar{c}_{l} x \cdot \nabla \psi_{m, i}}{\psi_{m, i}}+\frac{\overline{\mathbf{u}}^{N} \cdot \nabla \psi_{i}}{\psi_{i}}+\frac{\mathbf{U} \cdot \nabla \psi_{i}}{\psi_{i}}=a_{n, i} \bar{c}_{l}+I_{i, 2}+I_{i, 3}+N_{i} .
$$

The last term $N_{i}$ is nonlinear. Since we have $C^{1}$ bounds for $\mathbf{U}$, we derive the $C^{1}$ asymptotic bounds for $\mathbf{U}$ and $\frac{\nabla \psi_{i}}{\psi_{i}}$ and use the product rule to bound $N_{i}$. Since $x \cdot \nabla \psi_{m, i}=r \partial_{r} \psi_{m, i}, \psi_{m, i}$ is a radial weight and the last power in $\psi_{m, i}$ is $1, I_{i, 2}$ is the ratio between two radial weights with a denominator decaying faster. We use (8.27) and (8.57) to control $I_{i, 2}$. We use (8.28) and (8.49) to estimate $\overline{\mathbf{u}}^{N} \cdot \nabla \psi_{i} / \psi_{i}$ and its derivatives.

Note that in $d_{i}, i=2,3$, we also have the local terms $\bar{u}_{x}^{N}$ or $-\bar{v}_{y}^{N}$. Since it has the same decay as $I_{i, 3}$, we combine it with $I_{i, 3}$ and apply (8.57) for the Hölder estimate. Then, we use (8.49), the above estimates for $I_{2}, I_{3}$, and the estimate in Section 8.7.2 to estimate

$$
\left(I_{1,2}+I_{1,3}\right) \omega_{1} \psi_{1}, \quad\left(I_{2,2}+I_{2,3}-\bar{u}_{x}^{N}\right) \eta_{1} \psi_{2}, \quad\left(I_{3,2}+I_{3,3}-\bar{v}_{y}^{N}\right) \eta_{1} \psi_{2}
$$

We treat the above terms as perturbation. The main damping term in $d_{i} W_{1, i} \psi_{i}$ is given by

$$
\begin{equation*}
d_{i, M} W_{1, i}, \quad d_{i, M} \triangleq \bar{c}_{l} a_{n, i}+c_{i} \bar{c}_{\omega}^{N}, \quad c=[1,2,2] . \tag{8.58}
\end{equation*}
$$

8.8. Hölder estimates with large distance. We derive the piecewise bounds of the bad terms $B_{a d, i} \psi_{i}$ (8.17) and the damping factor $d_{i, L}\left(\psi_{i}\right)$ (8.16) in the mesh $Q_{i, j}$ (8.4) in the $C^{1 / 2}$ estimate of $W_{1, i} \psi_{i}$. For the Hölder estimate with large $|x-z|$, we further estimate

$$
\tilde{B}_{a d, i} \triangleq\left|B_{a d, i}\left(\psi_{i}\right)\right|+\left|\left(d_{i}\left(\psi_{i}\right)-d_{i, M}\left(\psi_{i}\right)\right) W_{1, i}\right|, \quad \max _{x \in\left[0, y_{N}\right]^{2}}\left|\tilde{B}_{i} \psi_{i}\right| \leq C_{i, 1} E_{4}
$$

where $E_{4}$ is the energy and $d_{i, M}$ is defined in (8.58). Since these terms decay in the far field, for $|z-x|$ large, they are very small and can be treated as a small perturbation. Using these estimates and the triangle inequality, for $|x-z| \geq R_{1}$ with $x \in \Omega_{n e a r}, z \in\left[0, y_{N}\right]^{2}$, we can bound

$$
\begin{equation*}
\left(\tilde{B}_{a d, i}(x)+\tilde{B}_{a d, i}(z)\right) g_{i}(x-z) \leq\left(\left|\tilde{B}_{a d, i}(x)\right|+\left|\tilde{B}_{a d, i}(z)\right|\right) R_{1}^{-1 / 2}|x-z|^{1 / 2} g_{i}(x-z) \tag{8.59}
\end{equation*}
$$

Beyond the mesh $\left[0, y_{N}\right]^{2}$, from Section 8.7, we can estimate $\frac{\tilde{B}_{i}(z)}{|z|^{1 / 2 \mid}}$ uniformly for $|z| \geq y_{N}$. Since $x \in \Omega_{\text {near }}=\left[0, y_{N_{1}}\right]^{2}$, we get $|z-x| \geq|z|(1-|x| /|z|) \geq y_{N}\left(1-\sqrt{2} y_{N_{1}} / y_{N}\right)$ and

$$
\begin{gather*}
\left(\tilde{B}_{a d, i}(x)+\tilde{B}_{a d, i}(z)\right) g_{i}(x-z) \leq\left(\frac{\tilde{B}_{a d, i}(x)}{|z|^{1 / 2}}+\frac{\tilde{B}_{a d, i}(z)}{|z|^{1 / 2}}\right) \frac{|z|^{1 / 2}}{|x-z|^{1 / 2}}|x-z|^{1 / 2} g_{i}(x-z) \\
\leq\left(\frac{\tilde{B}_{a d, i}(x)}{y_{N}^{1 / 2}}+\frac{\tilde{B}_{a d, i}(z)}{|z|^{1 / 2}}\right) \frac{\left|y_{N}\right|^{1 / 2}}{\left(y_{N}-\sqrt{2} y_{N_{1}}\right)^{1 / 2}}|x-z|^{1 / 2} g_{i}(x-z) \tag{8.60}
\end{gather*}
$$

Since $|h|^{1 / 2} g_{i}(h)$ is 0 -homogeneous, we follow Section 8.6.1 to estimate it.
It remains to estimate the damping factor $d_{i, g}$ from the weight $g_{i}$. We use (8.54) and treat $\left(\overline{\mathbf{u}}^{N}(x)-\overline{\mathbf{u}}^{N}(z)\right) \cdot \nabla g(x-z) / g(x-z)$ as perturbation. We estimate the Lipschitz norm of $\overline{\mathbf{u}}^{N}$ for $|x-z| \geq R_{1}$. Denote $f=\bar{u}^{N}$ or $\bar{v}^{N}$. Supose $|x|_{\infty} \leq|z|_{\infty}$. Clearly, we have

$$
|z|_{\infty} \geq \max _{i}\left(|x|_{i},|z|_{i}\right) \geq \max \left(\left|z_{1}-x_{1}\right|,\left|z_{2}-x_{2}\right|\right) \geq|x-z| / \sqrt{2} \geq R_{1} / \sqrt{2}
$$

If $|x|_{\infty} \geq \frac{R_{1}}{2 \sqrt{2}}$, we bound the Lipschitz norm by $\nabla f(z),|z|_{\infty} \geq \frac{R_{1}}{2 \sqrt{2}}$. If $|x|_{\infty}<\frac{R_{1}}{2 \sqrt{2}}$, we get
$|x| \leq \sqrt{2}|x|_{\infty} \leq \frac{R_{1}}{2} \leq \frac{|x-z|}{2}, \quad|z| \geq|x-z|-|x| \geq \frac{|x-z|}{2}, \frac{|f(x)-f(z)|}{|x-z|} \leq \frac{|f(x)|}{R_{1}}+\frac{|f(z)|}{2|z|}$.
By bounding $f(x)$ in $\left[0, R_{1} /(2 \sqrt{2})\right]$, and $f(z) /|z|$ outside $\left[0, R_{1} / \sqrt{2}\right]^{2}$, we obtain the Lipschitz bound. Since $\nabla \overline{\mathbf{u}}^{N}$ and $\overline{\mathbf{u}}^{N}(x) /|x|$ decay for large $|x|$, the above bounds are very small for large $R_{1}$. Denote $h_{i}=x_{i}-z_{i}, f=\left(u^{N}+U, v^{N}+V\right)$ With these estimates, we estimate

$$
\left|\frac{(f(x)-f(z)) \cdot \nabla g_{i}}{g_{i}}\right| \leq \frac{\mid\left(f_{1}(x)-f_{1}(z) \mid\right.}{|h|} \cdot|h|\left|\frac{\partial_{1} g_{i}}{g_{i}}(h)\right|+\frac{\mid\left(f_{2}(x)-f_{2}(z) \mid\right.}{|h|} \cdot|h|\left|\frac{\partial_{2} g_{i}}{g_{i}}(h)\right|
$$

Since $|h| \frac{\partial_{j} g_{i}(h)}{g_{i}(h)}$ is 0 -homogeneous, we follow Section 8.6.1 to estimate it. We treat the above terms as perturbations and we keep the following terms as damping terms

$$
\left(d_{i, M}-\bar{c}_{l} / 2\right) \delta\left(W_{1, i} \psi_{i}\right) g_{i}(x-z), \quad\left(d_{i, M}-\bar{c}_{l} / 2\right)<0
$$

The Hölder estimate in case (a.3). For $x, z \in\left[0, y_{N}\right]^{2} \backslash\left[0, y_{N_{1}-2}\right]^{2}$ with $x, z$ at least 1 grid apart, the Hölder estimate is similar since $x, z$ are in the very far-field and $\tilde{B}_{i}(x)$ is small. In this case, we use $|x-z| \geq \min _{i \geq 1} y_{i+1}-y_{i}=y_{2}$ and follow (8.59) to estimate the bad terms $\tilde{B}_{i}$. Note that we do not need to use (8.60).
8.9. Piecewise bounds. In this section, we assume that the points, e.g. $x, y, z$, are in $\mathbb{R}_{2}^{++}$.
8.9.1. Piecewise bound of integral with power. Given mesh $0=y_{1}<. .<y_{n}$ and the upper and lower bounds of in $I_{i}=\left[y_{i}, y_{i+1}\right]$, we estimate

$$
I_{k}(f)=\frac{1}{x^{k}} \int_{0}^{x} f(s) s^{k-1} d z, k=1,2
$$

We consider the lower bound, and the upper bound is similar. For $x \in I_{i}=\left[x_{i}, x_{i+1}\right]$, we get

$$
I_{k}(f)=\frac{1}{x^{k}}\left(\int_{0}^{x_{i}}+\int_{x_{i}}^{x}\right) f(s) s^{k-1} d s \geq \frac{1}{k x^{k}}\left(\sum_{j \leq i-1} f_{l, j}\left(x_{j+1}^{k}-x_{j}^{k}\right)+\left(x^{k}-x_{i}^{k}\right) f_{l, i}\right) .
$$

Denote $S_{k, i}=\sum_{j \leq i-1} f_{l, j}\left(x_{j+1}^{k}-x_{j}^{k}\right)$. Since $x^{k} \in\left[x_{i}^{k}, x_{i+1}^{k}\right]$, we yield

$$
\begin{equation*}
I_{k}(f) \geq \frac{1}{k} f_{l, i}+\frac{1}{k x^{k}}\left(S_{k, i}-x_{i}^{k} f_{l, i}\right) \geq \frac{1}{k} f_{l, i}+\frac{1}{k} \min \left(\frac{S_{k, i}-x_{i}^{k} f_{l, i}}{x_{i}^{k}}, \frac{S_{k, i}-x_{i}^{k} f_{l, i}}{x_{i+1}^{k}}\right) \tag{8.61}
\end{equation*}
$$

Using a similar argument, for $f(0)=0$ and $x \in\left[x_{k}, x_{k+1}\right]$, we estimate $f(x) / x$ as follows (8.62)
$\frac{f}{x}=\frac{1}{x}\left(f\left(x_{k}\right)+\int_{x_{k}}^{x} f_{x}(z) d z\right) \geq \frac{1}{x}\left(f\left(x_{k}\right)+\left(x-x_{k}\right) f_{x, l, k}\right) \geq \min \left(\frac{f\left(x_{k}\right)}{x_{k}}, \frac{f\left(x_{k}\right)+\left(x_{k+1}-x_{k}\right) f_{x, l, k}}{x_{k+1}}\right)$, where $f_{x, l, k}$ is the lower bound of $f_{x}$ in $\left[x_{k}, x_{k+1}\right]$.

To estimate 2D functions, we first denote by $g_{i, j}^{u}, g_{i, j}^{l}$ the upper and lower bound of $f$ in a grid cell $Q_{i j}=\left[x_{i}, x_{i+1}\right] \times\left[y_{j}, y_{j+1}\right]$. Then for $f(0, y)=0$ with constant power and $(x, y) \in Q_{i j}$, using the Mean-Value Theorem and $\frac{A x+B}{x} \geq A+\min \left(B / x_{i}, B / x_{i+1}\right)$, we have

$$
\begin{equation*}
\frac{f(x, y)}{x} \geq \frac{1}{x}\left(f\left(x_{i}, y\right)+f_{x, i j}^{l}\left(x-x_{i}\right)\right) \geq \min \left(\frac{1}{x_{i}} f\left(x_{i}, y\right), \frac{1}{x_{i+1}}\left(f\left(x_{i}, y\right)+f_{x, i j}^{l}\left(x_{i+1}-x_{i}\right)\right)\right) \tag{8.63}
\end{equation*}
$$

For $f\left(x_{i}, y\right)$, we further bound it from below using

$$
f\left(x_{i}, y\right) \geq \min \left(f\left(x_{i}, y_{j}\right), f\left(x_{i}, y_{j+1}\right)\right)-\left(y_{j+1}-y_{j}\right)^{2}\left\|f_{y y}\right\|_{L^{\infty}\left(Q_{i j}\right)} / 8
$$

In the first interval $x \in\left[x_{1}, x_{2}\right]$ with $x_{1}=0$, we simply use

$$
f(x, y) / x \geq f_{x, 1 j}^{l}=\min _{(x, y) \in Q_{1 j}} f_{x}(x, y)
$$

The estimates of the upper bound of $\frac{f(x, y)}{x}$ and the bounds of $\frac{f(x, y)}{y}$ with $f(x, 0)=0$ are similar.
8.9.2. Hölder estimate of the fractional power. We estimate $\delta_{i}(f, x, z), f=x_{2}^{2} /\left(x_{1}^{2}+x_{2}^{2}\right)$. Denote

$$
\begin{equation*}
\sin \beta=\max \left(x_{2}|x|^{-1}, z_{2}|z|^{-1}\right) \tag{8.64}
\end{equation*}
$$

$C_{x}^{1 / 2}$ estimate of $f$. In this case, since $x_{2}=z_{2}$, we write $\left(x_{1}, x_{2}\right)=(x, y),\left(z_{1}, z_{2}\right)=(z, y), x \leq$ $z$. Then $\sin \beta=y /\left(x^{2}+y^{2}\right)^{1 / 2}$. We have

$$
\begin{aligned}
& \left|\frac{y^{2}}{x^{2}+y^{2}}-\frac{y^{2}}{z^{2}+y^{2}}\right| \frac{|w|^{1 / 2}}{|x-z|^{1 / 2}}=\frac{y^{2}(x+z)|z-x|^{1 / 2}}{\left(x^{2}+y^{2}\right)\left(y^{2}+z^{2}\right)}|w|^{1 / 2} \triangleq M M_{2} \\
& M=\frac{y^{2}|w|^{1 / 2}}{\left(x^{2}+y^{2}\right)\left(y^{2}+z^{2}\right)^{1 / 4}}, M_{2}=\frac{(x+z)|x-z|^{1 / 2}}{\left(y^{2}+z^{2}\right)^{3 / 4}}
\end{aligned}
$$

For $w=(x, y)$ or $(z, y)$, we estimate $M$ using (8.32). Moreover, since $|w| \leq|(y, z)|$, we get

$$
\begin{equation*}
M \leq y^{2} /\left(x^{2}+y^{2}\right) \leq(\sin \beta)^{2}, \quad \delta_{1}(f, x, z)|w|^{1 / 2} \leq M_{2}(\sin \beta)^{2} \tag{8.65}
\end{equation*}
$$

Suppose that $a \in\left[a_{l}, a_{u}\right]$ for $a=x, y, z$. To estimate the upper bound of $M_{2}$, we write $(x+z)|z-x|^{1 / 2}=\left|z^{2}-x^{2}\right|^{1 / 2}|x+z|^{1 / 2}$ and have

$$
M_{2}=\left(\frac{z^{2}-x^{2}}{z^{2}+y^{2}}\right)^{1 / 2}\left(\frac{x}{\left(y^{2}+z^{2}\right)^{1 / 2}}+\frac{z}{\left(y^{2}+z^{2}\right)^{1 / 2}}\right)^{1 / 2} \triangleq A^{1 / 2}(B+C)^{1 / 2}
$$

To estimate $A$, we use the fact that $A$ is increasing in $z^{2}$, and decreasing in $x^{2}, y^{2}$. For $C$, we use (8.32). For $B$, since $x \leq z$, using (8.32), we get

$$
B \leq \frac{x}{\left(y^{2}+x^{2}\right)^{1 / 2}} \leq \frac{x_{u}}{\left(y_{l}^{2}+x_{u}^{2}\right)^{1 / 2}}, \quad B \leq \frac{x_{u}}{\left(y_{l}^{2}+z_{l}^{2}\right)^{1 / 2}}
$$

and optimize two estimates. We have another estimate for $M_{2}$. Denote $\lambda=x / z \in[0,1]$. Using Young's inequality

$$
\begin{equation*}
(1-\lambda)(1+\lambda)^{2}=\frac{1}{2}(2-2 \lambda)(1+\lambda)^{2} \leq \frac{1}{2}\left(\frac{(2-2 \lambda)+2(1+\lambda)}{3}\right)^{3}=\frac{1}{2}\left(\frac{4}{3}\right)^{3}=\frac{32}{27}, \quad c=\left(\frac{32}{27}\right)^{1 / 2} \tag{8.66}
\end{equation*}
$$

for $\lambda \in[0,1]$, we get

$$
M_{2}=\frac{z^{3 / 2}}{\left(y^{2}+z^{2}\right)^{3 / 4}} \cdot \frac{(x+z)(z-x)^{1 / 2}}{z^{3 / 2}}=\frac{z^{3 / 2}}{\left(y^{2}+z^{2}\right)^{3 / 4}}(1+\lambda)(1-\lambda)^{1 / 2} \leq c \frac{z^{3 / 2}}{\left(y^{2}+z^{2}\right)^{3 / 4}}
$$

and estimate the factor using (8.32).
$C_{y}^{1 / 2}$ estimate. In this case, we get $x_{1}=z_{1}$. Since $f\left(x_{2}, x_{1}\right)=1-f\left(x_{1}, x_{2}\right),\left|\left(a_{2}, a_{1}\right)\right|=\left|\left(a_{1}, a_{2}\right)\right|$

$$
\left|f\left(x_{1}, x_{2}\right)-f\left(z_{1}, z_{2}\right)\right||x-z|^{-1 / 2}|w|^{1 / 2}=\left|f\left(x_{2}, x_{1}\right)-f\left(z_{2}, z_{1}\right)\right| \cdot|x-z|^{-1 / 2}|w|^{1 / 2}
$$

the estimate of $\delta_{2}(f, x, z)$ follows from the above $C_{x}^{1 / 2}$ estimate. Note that we need to swap the variables $x_{1}, x_{2}$. To bound $\delta_{2}(f, x, z)$ using $\sin \beta$ (8.64), we first assume $x_{2} \leq z_{2}$. Then we get $\sin \beta=\frac{z_{2}}{\left(x_{1}^{2}+z_{2}^{2}\right)^{1 / 2}}$. Since $z_{1}=x_{1}$ and $|w| \leq|z|$, we have

$$
\begin{align*}
I & =\left|\frac{x_{2}^{2}}{x_{1}^{2}+x_{2}^{2}}-\frac{z_{2}^{2}}{x_{1}^{2}+z_{2}^{2}}\right| \frac{|w|^{1 / 2}}{|x-z|^{1 / 2}}=\frac{x_{1}^{2}}{x_{1}^{2}+x_{2}^{2}} \frac{\left(z_{2}-x_{2}\right)^{1 / 2}\left(z_{2}+x_{2}\right)}{x_{1}^{2}+z_{2}^{2}}|w|^{1 / 2}  \tag{8.67}\\
& \leq \frac{x_{1}^{2}}{x_{1}^{2}+x_{2}^{2}}\left(\frac{z_{2}}{|z|}\right)^{3 / 2} \frac{\left(z_{2}-x_{2}\right)^{1 / 2}\left(z_{2}+x_{2}\right)}{z_{2}^{3 / 2}}=\frac{x_{1}^{2}}{x_{1}^{2}+x_{2}^{2}}(\sin \beta)^{3 / 2}(1-\lambda)^{1 / 2}(1+\lambda), \lambda=\frac{x_{2}}{z_{2}} .
\end{align*}
$$

We estimate $\frac{x_{1}^{2}}{x_{1}^{2}+x_{2}^{2}}$ using (8.32), and $(1-\lambda)^{1 / 2}(\lambda+1)$ using the monotonicity of $1-\lambda, 1+\lambda$, piecewise bounds of $\lambda \leq 1$, and (8.66).
8.9.3. Piecewise Hölder estimates of power. We estimate $\delta_{i}(f, x, y)$ for $f(x)=|x|^{p}, p=1 / 2,1$. Since $f\left(x_{1}, x_{2}\right)=f\left(x_{2}, x_{1}\right)$, we consider $C_{x}^{1 / 2}$ estimate. The $C_{y}^{1 / 2}$ estimate can be obtained by swapping the variables $x_{1}, x_{2}$. Without loss of generality, we assume $y_{1} \geq x_{1}, x_{2}=y_{2}$. For each variable $a=x_{1}, x_{2}, y_{1}$, we assume $a \in\left[a^{l}, a^{u}\right]$.

Using $|y|^{2}-|x|^{2}=y_{1}^{2}-x_{1}^{2}=\left(y_{1}-x_{1}\right)\left(y_{1}+x_{1}\right)$ and a direct computation yields
$\frac{|y|^{1 / 2}-|x|^{1 / 2}}{|x-y|^{1 / 2}}=\frac{|y|^{2}-|x|^{2}}{(|y|+|x|)\left(|y|^{1 / 2}+|x|^{1 / 2}\right)|y-x|^{1 / 2}}=M M_{1}, M \triangleq \frac{y_{1}+x_{1}}{|x|+|y|}, M_{1}=\frac{\left|y_{1}-x_{1}\right|^{1 / 2}}{|x|^{1 / 2}+|y|^{1 / 2}}$,
$\frac{|y|-|x|}{|x-y|^{1 / 2}}=\frac{|y|^{2}-|x|^{2}}{|x-y|^{1 / 2}(|x|+|y|)}=\frac{\left|y_{1}-x_{1}\right|^{1 / 2}\left(x_{1}+y_{1}\right)}{|x|+|y|}=\left|y_{1}-x_{1}\right|^{1 / 2} M \leq\left(y_{1}^{u}-x_{1}^{l}\right)^{1 / 2} M$.
Clearly, $M\left(x_{1}, y_{1}, x_{2}\right)$ is decreasing in $x_{2}$. A direct computation yields

$$
\partial_{x_{1}} M=\frac{|x|+|y|-\left(x_{1}+y_{1}\right) \frac{x_{1}}{|x|}}{(|x|+|y|)^{2}} \geq \frac{|x|+|y|-\left(x_{1}+y_{1}\right)}{(|x|+|y|)^{2}} \geq 0
$$

Similarly, $\partial_{y_{1}} M \geq 0$. We get $M\left(x_{1}, y_{1}, x_{2}\right) \leq M\left(x_{1}^{u}, y_{1}^{u}, x_{2}^{l}\right)$.
Since $y_{1} \geq x_{1}$, for $M_{1}$, we write $M_{1}=\frac{\left(1-x_{1} / y_{1}\right)^{1 / 2}}{\left(|x| / y_{1}\right)^{1 / 2}+\left(|y| / y_{1}\right)^{1 / 2}}$. Since $|y| / y_{1}$ is decreasing in $y_{1}$ (8.32), $M_{1}$ is increasing in $y_{1}$. Clearly, $M_{1}\left(x_{1}, y_{1}, x_{2}\right)$ is decreasing in $x_{2}$, and decreasing in $x_{1}$. We get $M_{1} \leq M_{1}\left(x_{1}^{l}, y_{1}^{u}, x_{2}^{l}\right)$.

For $p \geq 2$ and $b \geq a$, using the convexity of $t^{p-1}$ for $p-1 \geq 1$ and Jensen's inequality, we get

$$
b^{p}-a^{p}=p \int_{a}^{b} t^{p-1} d t \leq p \frac{b-a}{2}\left(b^{p-1}+a^{p-1}\right)
$$

Using the above estimates, we get

$$
\frac{|y|^{p}-|x|^{p}}{|y-x|^{1 / 2}} \leq \frac{|y|-|x|}{|y-x|^{1 / 2}} \cdot \frac{p}{2}\left(|x|^{p-1}+|y|^{p-1}\right)
$$

We have estimated $\frac{|y|-|x|}{|y-x|^{1 / 2}}$ in the above. Clearly, $|x|,|y|$ are increasing in $x_{1}, x_{2}, y_{1}$.
8.9.4. Piecewise Hölder estimates of weighted functions. Given $F$ depending on the profile $\bar{\omega}, \bar{\theta}, \overline{\mathbf{u}}^{N}$ and radial weights $f, g$ with leading power $r^{a}, r^{b}$, respectively, we estimate $\delta_{i}(F f / g, x, z)$ piecewisely. Using the decomposition (8.31), we get

$$
F(x) f(r) / g(r)=F(x) f_{m}(r) / g_{m}(r) r^{a-b}
$$

Using the $C^{1}$ estimates of $f, g, F, f_{m}, g_{m}$, the Leibniz rule (A.7), and (8.20), we obtain $C^{1 / 2}, L^{\infty}$ estimates of $F f / g, F f_{m} / g_{m}$. Since the weights $f_{m}, g_{m}$ are $C^{1}$ in our applications (see the discussions around (8.31)), the $C^{1}, C^{1 / 2}$ estimates of $F f_{m} / g_{m}$ are not singular. For weights $f, g$ singular near $r=0$, we use the second identity and estimate the singular part $r^{a-b}$. If $a-b=0,1 / 2,1$ or $a-b \geq 2$, we use the $C^{1 / 2}$ estimates in Section 8.9.3.

Below, we discuss three special cases $F f_{m} / g_{m}|x|^{-1 / 2}, F /|x|^{p}, p=5 / 2,2$ with $F(0, y)=0$.
Estimate of $F f_{m} / g_{m} /|x|^{1 / 2}$. For $F$ odd in $x$, near $r=0$, we have $F(x) f_{m} / g_{m} r^{a-b} \approx x r^{-1 / 2}$, which is $C^{1 / 2}$. We perform $C^{1 / 2}$ estimate on $F(x) r^{-1 / 2}$, which along with the $C^{1 / 2}$ estimate of $f_{m} / g_{m}$ and (8.20) give the desired estimate. Suppose that $|x| \leq|y|$ with $x_{2}=y_{2}$ for the $C_{x_{1}}^{1 / 2}$ estimate and $x_{1}=y_{1}$ for the $C_{x_{2}}^{1 / 2}$ estimate. A direct calculation yields
$\frac{\left|\frac{F(y)}{|y|^{1 / 2}}-\frac{F(x)}{|x|^{1 / 2}}\right|}{\left|x_{i}-y_{i}\right|^{1 / 2}} \leq \frac{|F(y)-F(x)|}{|y|^{1 / 2}\left|x_{i}-y_{i}\right|^{1 / 2}}+|F(x)| \frac{\left.| | x\right|^{-1 / 2}-|y|^{-1 / 2} \mid}{\left|x_{i}-y_{i}\right|^{1 / 2}}=\frac{|F(y)-F(x)|}{\left|x_{i}-y_{i}\right|} M_{3}^{1 / 2}+\left|\frac{F(x)}{x_{1}}\right| M_{4} M_{5}$,
$M_{3}=\frac{\left|x_{i}-y_{i}\right|}{|y|}, \quad M_{4} \triangleq \frac{x_{1}}{|x|^{1 / 2}|y|^{1 / 2}}, \quad M_{5} \triangleq \frac{\left.| | y\right|^{1 / 2}-|x|^{1 / 2} \mid}{\left|x_{i}-y_{i}\right|^{1 / 2}}, \quad T_{1}=\frac{|F(y)-F(x)|}{\left|x_{i}-y_{i}\right|}, \quad T_{2}=\frac{F(x)}{x_{1}}$.
Since $F \in C^{1}$, we can estimate $\left\lvert\, \frac{F(y)-F(x) \mid}{\left|x_{i}-y_{i}\right|}\right.$ piecewisely using the method in Appendix E. 7 in Part II [1]. We use the estimate for $M_{5}$ in Section 8.9.3 and the method in Section 8.9.1 to estimate $F(x) / x_{1}$ piecewisely. For $M_{3}$ with $i=1$, since $y_{1}-x_{1} \geq 0, x_{2}=y_{2}, M_{3}$ is decreasing in $x_{1}, x_{2}$. Since $M_{3}=\frac{1-x_{1} / y_{1}}{\left(1+\left(y_{2} / y_{1}\right)^{2}\right)^{1 / 2}}, M_{3}$ is increasing in $y_{1}$. Thus $M_{3}(x, y) \leq M_{3}\left(x_{1}^{l}, x_{2}^{l}, y_{1}^{u}, x_{2}^{l}\right)$. The estimate for $M_{3}$ with $i=2$ is similar $M_{3}(x, y) \leq M_{3}\left(x_{1}^{l}, x_{2}^{l}, x_{1}^{l}, y_{2}^{u}\right)$.

In the $C_{x_{1}}^{1 / 2}$ estimate, we get $x_{2}=y_{2}$. For $M_{4}$, using (8.32) and $|y| \geq|x|$, we have two estimates

$$
M_{4} \leq \frac{x_{1}}{|x|} \leq \frac{x_{1}^{u}}{\left|\left(x_{1}^{u}, x_{2}^{l}\right)\right|}, \quad M_{4}=\left(\frac{x_{1}}{|x|}\right)^{1 / 2}\left(\frac{x_{1}}{|y|}\right)^{1 / 2} \leq\left(\frac{x_{1}^{u}}{\left|\left(x_{1}^{u}, x_{2}^{l}\right)\right|}\right)^{1 / 2}\left(\frac{x_{1}^{u}}{\left|\left(y_{1}^{l}, x_{2}^{l}\right)\right|}\right)^{1 / 2}
$$

Near $r=0$, we have a further improvement. Denote $x_{1} / y_{1}=\lambda \in[0,1]$. Since $x_{1} \leq y_{1} \leq$ $|y|,|x| \leq|y|$, from the above estimate and $M M_{1}$ (8.68), we bound the upper bound as follows
$M_{5}=\frac{|y|^{1 / 2}-|x|^{1 / 2}}{|x-y|^{1 / 2}} \leq M M_{1} \leq M_{1} \leq \frac{\left(y_{1}-x_{1}\right)^{1 / 2}}{x_{1}^{1 / 2}+y_{1}^{1 / 2}}=\frac{(1-\lambda)^{1 / 2}}{1+\lambda^{1 / 2}}, M_{3} \leq \frac{y_{1}-x_{1}}{y_{1}}=1-\lambda$,
$T_{1} M_{3}^{1 / 2}+T_{2} M_{4} M_{5} \leq T_{1}(1-\lambda)^{1 / 2}+T_{2} \frac{x_{1}^{1 / 2}}{y_{1}^{1 / 2}} M_{5} \leq(1-\lambda)^{1 / 2}\left(T_{1}+\frac{T_{2} \lambda^{1 / 2}}{1+\lambda^{1 / 2}}\right) \leq\left(1-\lambda_{l}\right)^{1 / 2}\left(T_{1}^{u}+\frac{\lambda_{u}^{1 / 2} T_{2}^{u}}{1+\lambda_{l}^{1 / 2}}\right)$.
We partition $\lambda \in[0,1]$ and use the above estimate to obtain an uniform upper bound for $\lambda \in[0,1]$.

In the $C_{x_{2}}^{1 / 2}$ estimate, we get $x_{1}=y_{1}$ and $M_{4}=\left(\frac{x_{1}}{|x|}\right)^{1 / 2}\left(\frac{y_{1}}{|y|}\right)^{1 / 2}$, which can be estimated using (8.32). We also have a direct estimate for $F(x) / x_{1} M_{4}$ using the piecewise bound of $F(x)$

$$
\left|F(x) / x_{1} M_{4}\right| \leq|F(x)|\left|\left(x_{1}^{l}, x_{2}^{l}\right)\right|^{-1 / 2}\left|\left(y_{1}^{l}, y_{2}^{l}\right)\right|^{-1 / 2}
$$

We bound $M_{5}$ using previous method and then obtain the bound for $F(x) / x_{1} M_{4} M_{5}$.
Estimate of $F /|x|^{p}, p=2,5 / 2$. The power $r^{p}, p=5 / 2,2$ is the leading power in the Hölder weight $\psi_{i}$. In this case, our function satisfies $F=O\left(r^{3}\right)$ near $r=0$ and we have piecewise $C^{3}$ bounds of $F$. To estimate the $C^{1 / 2}$ norm of $F / r^{5 / 2}$, using the above estimate with $g=F / r^{2}$, we only need to estimate the Lipschitz norm of $g$, and the $L^{\infty}$ norm of $g / x$ and $F / r^{5 / 2}$. Using the Taylor expansion and the method in Appendix E. 5 in Part II [1], we can control $F / r^{5 / 2}, F / r^{2} / x_{1}$. To estimate the $C^{1}$ bound, we use

$$
\partial_{x}\left(F / r^{2}\right)=\frac{F_{x}}{r^{2}}-\frac{2 F x}{r^{4}}, \quad \partial_{y}\left(F / r^{2}\right)=\frac{\partial_{y} F}{r^{2}}-\frac{2 F y}{r^{4}},
$$

and estimate two terms separately using the method in Appendix E. 5 in Part II [1].

## Appendix A. Parameters, explicit functions, and basic estimates

A.1. Parameters and some functions. Recall the following cutoff functions constructed and estimated in Appendix D. 2 in Part II [1]

$$
\begin{equation*}
\kappa\left(x ; \nu_{1}, \nu_{2}\right)=\kappa_{1}\left(\frac{x}{\nu_{1}}\right)\left(1-\chi_{e}\left(\frac{x}{\nu_{2}}\right)\right), \quad \kappa_{1}(x)=\frac{1}{1+x^{4}}, \quad \chi_{e}(x)=\left(1+\exp \left(\frac{1}{x}+\frac{1}{x-1}\right)\right)^{-1} \tag{A.1}
\end{equation*}
$$

For the cutoff functions in (8.6) in Section (8.1.1), and $\chi_{j, 2}, f_{\chi, j}$, we choose

$$
\begin{align*}
& \chi_{\bar{\varepsilon}}(x, y)=\kappa\left(x ; \nu_{\bar{\varepsilon}, 1}, \nu_{\bar{\varepsilon}, 2}\right) \kappa\left(y ; \nu_{\bar{\varepsilon}, 1}, \nu_{\bar{\varepsilon}, 2}\right), \quad \nu_{\bar{\varepsilon}, 1}=1 / 192, \quad \nu_{\bar{\varepsilon}, 2}=3 / 2, \\
& \chi_{\hat{\varepsilon}}(x, y)=\kappa_{*}(x) \kappa_{*}(y), \quad \kappa_{*}(x)=\kappa(x, 1 / 3,3 / 2), \quad \chi_{N F}(x, y)=\kappa(x ; 2,10) \kappa(y ; 2,10),  \tag{A.2}\\
& f_{\chi, 1}=\Delta\left(\frac{x y^{3}}{6} \chi_{N F}(x, y)\right), \quad f_{\chi, 2}=x y \chi_{N F}(x, y), \quad f_{\chi, 3}=\frac{x^{2}}{2} \chi_{N F}(x, y) .
\end{align*}
$$

We use the following parameters for the energy (8.2)

$$
\begin{align*}
& \tau_{1}=5, \quad \mu_{1}=0.668, \quad \mu_{2}=1.336, \quad \mu_{4}=0.065, \quad \tau_{2}=0.23 \\
& \mu_{5}=76, \quad \mu_{51}=61, \quad \mu_{52}=15, \quad \mu_{6}=61, \quad \mu_{62}=35.88  \tag{A.3}\\
& \mu_{7}=9.5, \quad \mu_{8}=4.5, \quad E_{*}=5 \cdot 10^{-6}, \quad \mu_{h}=\tau_{1}^{-1}\left(1, \mu_{1}, \mu_{2}\right), \quad \mu_{g}=\tau_{2}\left(\mu_{4}, 1,1\right) .
\end{align*}
$$

Recall from Appendix C. 1 in Part I [3] the following Hölder weight

$$
\begin{align*}
& g_{i}(h)=g_{i 0}(h) g_{i 0}(1,0)^{-1}, \quad g_{i 0}(h)=\left(\sqrt{h_{1}+q_{i 1} h_{2}}+q_{i 3} \sqrt{h_{2}+q_{i 2} h_{1}}\right)^{-1}  \tag{A.4}\\
& \vec{q}_{1},=(0.12,0.01,0.25), \quad \vec{q}_{2},=(0.14,0.005,0.27), \quad \vec{q}_{3},=\vec{q}_{2} .
\end{align*}
$$

A.2. Basic piecewise estimates. Denote by $f_{l}, f_{u}$ the lower and upper bound of $f$. We have

$$
\begin{align*}
& (f-g)_{l}=f_{l}-g_{u}, \quad(f-g)_{u}=f_{u}-g_{l}, \quad(f+g)_{\gamma}=f_{\gamma}+g_{\gamma} \\
& (f g)_{l}=\min \left(f_{l} g_{l}, f_{u} g_{l}, f_{l} g_{u}, f_{u} g_{u}\right), \quad(f g)_{u}=\max \left(f_{l} g_{l}, f_{u} g_{l}, f_{l} g_{u}, f_{u} g_{u}\right) \tag{A.5}
\end{align*}
$$

where $\gamma=l, u$. If $g \geq 0$, we can simplify the formula for the product

$$
\begin{align*}
& (f g)_{l}=\min \left(f_{l} g_{l}, f_{l} g_{u}\right), \quad(f g)_{u}=\max \left(f_{u} g_{l}, f_{u} g_{u}\right),  \tag{A.6}\\
& (f / g)_{l}=\min \left(f_{l} / g_{l}, f_{l} / g_{u}\right), \quad(f / g)_{u}=\max \left(f_{u} / g_{l}, f_{u} / g_{u}\right) .
\end{align*}
$$

Given the piecewise bounds of the derivatives of $f, g$, using the Leibniz rule

$$
\begin{equation*}
\partial_{x}^{i} \partial_{y}^{j}(f g)=\sum_{k \leq i, l \leq j}\binom{i}{k} \partial_{x}^{k} \partial_{y}^{l} f \cdot \partial_{x}^{i-k} \partial_{y}^{j-l} g \tag{A.7}
\end{equation*}
$$

and (A.5), (A.6), we can estimate the piecewise bounds of $\partial_{x}^{i} \partial_{y}^{j}(f g)$.
We have the following simple linear error estimates (see, e.g. Appendix C. 2 in Part II [1])

$$
\begin{align*}
& \max _{x \in\left[x_{l}, x_{u}\right]}|f(x)| \leq \max \left(\left|f\left(x_{l}\right)\right|,\left|f\left(x_{u}\right)\right|\right)+h^{2}\left\|f_{x x}\right\|_{L^{\infty}(I)} / 8, \quad h=x_{u}-x_{l} \\
& \min _{\alpha, \beta=l, u} f\left(x_{\alpha}, y_{\beta}\right)-\operatorname{err}_{i} \leq f(x) \leq \max _{\alpha, \beta=l, u} f\left(x_{\alpha}, y_{\beta}\right)+\operatorname{err}_{i}, \quad i=1,2  \tag{A.8}\\
& \operatorname{err}_{1}=\left(\left\|f_{x}\right\|_{L^{\infty}(Q)} h_{1}+\left\|f_{y}\right\|_{L^{\infty}(Q)} h_{2}\right) / 2, \quad \operatorname{err}_{2}=\left(\left\|f_{x x}\right\|_{L^{\infty}(Q)} h_{1}^{2}+\left\|f_{y y}\right\|_{L^{\infty}(Q)} h_{2}^{2}\right) / 8 .
\end{align*}
$$

## References

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