Dynamic Stability of the Three-Dimensional Axisymmetric Navier-Stokes Equations with Swirl

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Abstract

In this paper, we study the dynamic stability of the three-dimensional axisymmetric Navier-Stokes Equations with swirl. To this purpose, we propose a new one-dimensional model that approximates the Navier-Stokes equations along the symmetry axis. An important property of this one-dimensional model is that one can construct from its solutions a family of exact solutions of the three-dimensionaFinal Navier-Stokes equations. The nonlinear structure of the one-dimensional model has some very interesting properties. On one hand, it can lead to tremendous dynamic growth of the solution within a short time. On the other hand, it has a surprising dynamic depletion mechanism that prevents the solution from blowing up in finite time. By exploiting this special nonlinear structure, we prove the global regularity of the three-dimensional Navier-Stokes equations for a family of initial data, whose solutions can lead to large dynamic growth, but yet have global smooth solutions. © 2007 Wiley Periodicals, Inc.

1 Introduction

Despite a great deal of effort by many mathematicians and physicists, the question of whether the solution of the three-dimensional Navier-Stokes equations can develop a finite-time singularity from a smooth initial condition with finite energy remains one of the most outstanding open problems [12]. A main difficulty in obtaining the global regularity of the three-dimensional Navier-Stokes equations is due to the presence of vortex stretching, which is absent for the two-dimensional problem. Under a suitable smallness assumption on the initial condition, global existence and regularity results have been obtained for some time [8, 17, 20, 21]. But these methods, based on energy estimates, do not generalize to the three-dimensional Navier-Stokes with large data. Energy estimates seem to be too crude to give a definite answer to whether diffusion is strong enough to control the nonlinear growth due to vortex stretching. A more refined analysis that takes into account the special nature of the nonlinearities and their local interactions seems to be needed.

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In this paper, we study the dynamic stability property of the three-dimensional axisymmetric Navier-Stokes equations with swirl. We show that there is a very subtle dynamic depletion mechanism of vortex stretching in the three-dimensional Navier-Stokes equations. On one hand, the nonlinear vortex stretching term is responsible for producing a large dynamic growth in vorticity in early times. On the other hand, the special structure of the nonlinearity can also lead to dynamic depletion and cancellation of vortex stretching, thus avoiding the finite-time blowup of the Navier-Stokes equations.

This subtle nonlinear stability property can be best illustrated by a new one-dimensional model that we introduce in this paper. This one-dimensional model approximates the three-dimensional axisymmetric Navier-Stokes equations along the symmetry axis. By the well-known Caffarelli-Kohn-Nirenberg theory [3] (see also [18]), the singularity set of any suitable weak solution of the three-dimensional Navier-Stokes equations has one-dimensional Hausdorff measure 0. In the case of axisymmetric three-dimensional Navier-Stokes equations with swirl, if there is any singularity, it must be along the symmetry axis. Thus it makes sense to focus our effort on understanding the possible singular behavior of the three-dimensional Navier-Stokes equations near the symmetry axis at $r = 0$. By expanding the angular velocity (u^θ), the angular vorticity (ω^θ), and the angular stream function (ψ^θ) around $r = 0$, we obtain the following coupled nonlinear partial differential equations (see Section 2 for detailed derivations):

$$(u_1)_t + 2\psi_1(u_1)_z = \nu(u_1)_{zz} + 2(\psi_1)_z u_1, \tag{1.1}$$

$$(\omega_1)_t + 2\psi_1(\omega_1)_z = \nu(\omega_1)_{zz} + (u_1^2)_z, \tag{1.2}$$

$$-(\psi_1)_{zz} = \omega_1, \tag{1.3}$$

where $u_1(z,t) \approx (u^\theta)_r|_{r=0}$, $\omega_1(z,t) \approx (\omega^\theta)_r|_{r=0}$, and $\psi_1(z,t) \approx (\psi^\theta)_r|_{r=0}$.

What we find most surprising is that one can construct a family of *exact solutions* from the above one-dimensional model. Specifically, if (u_1, ω_1, ψ_1) is a solution of the one-dimensional model (1.1)–(1.3), then

$$u^\theta(r,z,t) = r u_1(z,t), \quad \omega^\theta(r,z,t) = r\omega_1(z,t), \quad \psi^\theta(r,z,t) = r\psi_1(z,t),$$

is an exact solution of the three-dimensional axisymmetric Navier-Stokes equations. Thus the one-dimensional model captures some essential nonlinear features of the three-dimensional Navier-Stokes equations. Further, if we let $\tilde{u} = u_1$, $\tilde{v} = -(\psi_1)_z$, and $\tilde{\psi} = \psi_1$, then the one-dimensional model can be rewritten as

$$(\tilde{u})_t + 2\tilde{\psi}(\tilde{u})_z = \nu(\tilde{u})_{zz} - 2\tilde{v}\tilde{u}, \tag{1.4}$$

$$(\tilde{v})_t + 2\tilde{\psi}(\tilde{v})_z = \nu(\tilde{v})_{zz} + (\tilde{u})^2 - (\tilde{v})^2 + c(t), \tag{1.5}$$

where $\tilde{\psi}_z = -\tilde{v}$ and $c(t)$ is an integration constant to ensure that $\int \tilde{v}\, dz = 0$. We will show that if the initial value of $\tilde{u}$ is small but $\tilde{v}$ is large and negative, then the solution of the one-dimensional model can experience large growth. On the other hand, we also find a surprising dynamic depletion mechanism of nonlinearities that prevents the solution from blowing up in finite time. This subtle nonlinear cancellation is partly due to the special nature of the nonlinearities, i.e., $-2\tilde{u}\tilde{v}$ in (1.4) and $\tilde{u}^2 - \tilde{v}^2$ in (1.5). If one modifies the sign of the nonlinear term from $-2\tilde{u}\tilde{v}$ to $2\tilde{u}\tilde{v}$ or changes $\tilde{u}^2 - \tilde{v}^2$ to $\tilde{u}^2 + \tilde{v}^2$ or even modifies the coefficient from $-2\tilde{u}\tilde{v}$ to $-0.9\tilde{u}\tilde{v}$, the dynamic depletion mechanism can be changed completely.

Another interesting fact is that the convection term also helps to stabilize the solution. It cancels some of the destabilizing terms from the right-hand side when we estimate the solution in a high-order norm. Specifically, we find that there is a miraculous cancellation of nonlinear terms in the equation that governs the nonlinear quantity, $\tilde{u}_z^2 + \tilde{v}_z^2$; i.e.,

$$(1.6)\qquad (\tilde{u}_z^2 + \tilde{v}_z^2)_t + 2\tilde{\psi}(\tilde{u}_z^2 + \tilde{v}_z^2)_z = \nu(\tilde{u}_z^2 + \tilde{v}_z^2)_{zz} - 2\nu[(\tilde{u}_{zz})^2 + (\tilde{v}_{zz})^2].$$

Therefore, $\tilde{u}_z^2 + \tilde{v}_z^2$ has a maximum principle.

This pointwise a priori estimate plays an essential role in obtaining global regularity of the one-dimensional model with or without viscosity. If one attempts to prove global regularity of the one-dimensional model using energy estimates, one cannot take full advantage of this local cancellation of nonlinearities and runs into similar difficulties to those we encounter for the three-dimensional Navier-Stokes equations.

Finally, we construct a family of globally smooth solutions of the three-dimensional Navier-Stokes equations with large initial data of finite energy by using the solution of the one-dimensional model. Specifically, we look for a solution of the form

$$(1.7)\qquad \tilde{u}^\theta = r(\bar{u}_1(z,t)\phi(r) + u_1(r,z,t)),$$

$$(1.8)\qquad \tilde{\omega}^\theta = r(\bar{\omega}_1(z,t)\phi(r) + \omega_1(r,z,t)),$$

$$(1.9)\qquad \tilde{\psi}^\theta = r(\bar{\psi}_1(z,t)\phi(r) + \psi_1(r,z,t)),$$

where $\bar{u}_1$, $\bar{\omega}_1$, and $\bar{\psi}_1$ are solutions of the one-dimensional model, and $\phi(r)$ is a cutoff function to ensure that the solution has finite energy. By using the a priori estimate of the solution of the one-dimensional model and using a delicate analysis, we prove that there exists a family of globally smooth functions $u_1(r,z,t)$, $\omega_1(r,z,t)$, and $\psi_1(r,z,t)$ such that $\tilde{u}^\theta$, $\tilde{\omega}^\theta$, and $\tilde{\psi}^\theta$ are solutions of the three-dimensional axisymmetric Navier-Stokes equations. Unlike the other known global solutions with small data, the solutions that we construct above using the one-dimensional model can have large dynamic growth for early times, which is induced by the dynamic growth of the corresponding solution of the one-dimensional model, but yet the solution remains smooth for all times.

There have been some interesting developments in the study of three-dimensional incompressible Navier-Stokes equations and related models. In particular, by exploiting the special structure of the governing equations, Cao and Titi [4] proved the global well-posedness of the three-dimensional viscous primitive equations that model large-scale ocean and atmosphere dynamics. By taking advantage of the limiting property of some rapidly oscillating operators and using nonlinear averaging, Babin, Mahalov, and Nicolaenko [1] prove existence on infinite time intervals of regular solutions to the three-dimensional Navier-Stokes equations for some initial data characterized by uniformly large vorticity.

Some interesting progress has also been made on the regularity of the axisymmetric solutions of the Navier-Stokes equations; see, e.g., [6] and the references cited there. The two-dimensional Boussinesq equations are closely related to the three-dimensional axisymmetric Navier-Stokes equations with swirl (away from the symmetry axis). Recently Chae [5] and Hou and Li [14] proved independently the global existence of the two-dimensional viscous Boussinesq equations with viscosity entering only in the fluid equation, while the density equation remains inviscid. Recent studies by Constantin, Fefferman, and Majda [7] and Deng, Hou, and Yu [10, 11] show that the local geometric regularity of the unit vorticity vector can play an important role in depleting vortex stretching dynamically.

Motivated by these theoretical results, Hou and R. Li [15] recently reinvestigated the well-known computations by Kerr [16] for two antiparallel vortex tubes, in which a finite-time singularity of the three-dimensional incompressible Euler equations was reported. The results of Hou and Li show that there is tremendous dynamic cancellation in the vortex stretching term due to local geometric regularity of the vortex lines. Moreover, they show that the vorticity does not grow faster than double exponential in time, and the velocity field remains bounded up to $T = 19$, beyond the singularity time alleged in [16]. Finally, we mention the recent work of Gibbon et al. (see [13] and the references therein), where they reveal some interesting geometric properties of the Euler equations in quaternion frames.

The rest of the paper is organized as follows. In Section 2, we will derive the one-dimensional model for the three-dimensional axisymmetric Navier-Stokes equations. We discuss some of the properties of the one-dimensional model in Section 3 and prove the global existence of the inviscid one-dimensional model using the Lagrangian coordinate. Section 4 is devoted to proving the global regularity of the full one-dimensional model in the Eulerian coordinate. Finally, in Section 5, we use the solutions of the one-dimensional model to construct a family of solutions of the three-dimensional Navier-Stokes equations and prove that they remain smooth for all times.

2 Derivation of the One-Dimensional Model

We now consider the three-dimensional axisymmetric incompressible Navier-Stokes equations with swirl:

$$\begin{cases} u_t + (u\cdot\nabla)u = -\nabla p + \nu\Delta u, \\ \nabla\cdot u = 0, \\ u|_{t=0} = u_0(\vec{x}), \quad \vec{x} = (x,y,z). \end{cases} \tag{2.1}$$

Let

$$e_r = \left(\frac{x}{r}, \frac{y}{r}, 0\right), \quad e_\theta = \left(-\frac{y}{r}, \frac{x}{r}, 0\right), \quad e_z = (0,0,1),$$

be three unit vectors along the radial, the angular, and the z-directions, respectively, $r = \sqrt{x^2+y^2}$. We will decompose the velocity field as follows:

$$\vec{u} = v^r(r,z,t)e_r + u^\theta(r,z,t)e_\theta + v^z(r,z,t)e_z. \tag{2.2}$$

In the above expression, u^θ is called the *swirl* component of the velocity field $\vec{u}$. The vorticity field can be expressed similarly,

$$\vec{\omega} = -(u^\theta)_z(r,z,t)e_r + \omega^\theta(r,z,t)e_\theta + \frac{1}{r}(ru^\theta)_r(r,z,t)e_z, \tag{2.3}$$

where $\omega^\theta = v^r_z - v^z_r$.

To simplify our notation, we will use u and ω to denote the angular velocity and vorticity components, respectively, and drop the θ superscript in the rest of the paper. One can derive evolution equations for u and ω as follows (see, e.g., [6, 20]).

$$u_t + v^r u_r + v^z u_z = \nu\left(\nabla^2 - \frac{1}{r^2}\right)u - \frac{1}{r}v^r u, \tag{2.4}$$

$$\omega_t + v^r\omega_r + v^z\omega_z = \nu\left(\nabla^2 - \frac{1}{r^2}\right)\omega + \frac{1}{r}(u^2)_z + \frac{1}{r}v^r\omega, \tag{2.5}$$

$$-\left(\nabla^2 - \frac{1}{r^2}\right)\psi = \omega, \tag{2.6}$$

where ψ is the angular component of the stream function; v^r and v^z can be expressed in terms of the angular stream function ψ as follows:

$$v^r = -\frac{\partial\psi}{\partial z}, \quad v^z = \frac{1}{r}\frac{\partial}{\partial r}(r\psi), \tag{2.7}$$

and ∇^2 is defined as

$$\nabla^2 = \partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2. \tag{2.8}$$

Note that equations (2.4)–(2.6) completely determine the evolution of the three-dimensional axisymmetric Navier-Stokes equations once the initial condition is given.

We will derive the one-dimensional model for the three-dimensional axisymmetric Navier-Stokes equations. By the well-known Caffarelli-Kohn-Nirenberg theory [3], the singularity set of any suitable weak solution of the three-dimensional Navier-Stokes equations has one-dimensional Hausdorff measure 0. Thus, in the case of axisymmetric three-dimensional Navier-Stokes equations with swirl, if there is any singularity, it must be along the symmetry axis, i.e., the z-axis. Therefore, we should focus on understanding the possible singular behavior of the three-dimensional Navier-Stokes equations near the symmetry axis at $r = 0$.

As observed by Liu and Wang in [19], any smooth solution of the three-dimensional axisymmetric Navier-Stokes equations must satisfy the following compatibility condition at $r = 0$:

$$u(0,z,t) = \omega(0,z,t) = \psi(0,z,t) = 0. \tag{2.9}$$

Moreover, all the even-order derivatives of u, ω, and ψ with respect to r at $r = 0$ must vanish. Therefore, we expand the solution u, ω, and ψ around $r = 0$ as follows:

$$u(r,z,t) = ru_1(z,t) + \frac{r^3}{3!}u_3(z,t) + \frac{r^5}{5!}u_5(z,t) + \cdots, \tag{2.10}$$

$$\omega(r,z,t) = r\omega_1(z,t) + \frac{r^3}{3!}\omega_3(z,t) + \frac{r^5}{5!}\omega_5(z,t) + \cdots, \tag{2.11}$$

$$\psi(r,z,t) = r\psi_1(z,t) + \frac{r^3}{3!}\psi_3(z,t) + \frac{r^5}{5!}\psi_5(z,t) + \cdots. \tag{2.12}$$

Substituting the above expansions into (2.4)–(2.6), we obtain to the leading order the following system of equations:

$$r(u_1)_t - r(\psi_1)_z u_1 + 2\psi_1 r(u_1)_z = \nu\left(\frac{4}{3}ru_3 + r(u_1)_{zz}\right) + r(\psi_1)_z u_1 + O(r^3)$$

$$\begin{aligned} r(\omega_1)_t + 2\psi_1 r(\omega_1)_z &= \nu\left(\frac{4}{3}r\omega_3 + r(\omega_1)_{zz}\right) \\ &\quad + 2ru_1(u_1)_z + O(r^3) \\ &\quad - \left(\frac{4}{3}r\psi_3 + r(\psi_1)_{zz} + O(r^3)\right) \\ &= r\omega_1 + O(r^3). \end{aligned}$$

By canceling r from both sides and neglecting the higher-order terms in r, we obtain

$$(u_1)_t + 2\psi_1(u_1)_z = \nu\left(\frac{4}{3}u_3 + (u_1)_{zz}\right) + 2(\psi_1)_z u_1,$$
$$(\omega_1)_t + 2\psi_1(\omega_1)_z = \nu\left(\frac{4}{3}\omega_3 + (\omega_1)_{zz}\right) + (u_1^2)_z,$$
$$-\left(\frac{4}{3}\psi_3 + (\psi_1)_{zz}\right) = \omega_1.$$

Note that $u_3 = u_{rrr}(0, z, t)$ and $(u_1)_{zz} = u_{rzz}(0, z, t)$. If we further make the assumption that the second partial derivative of u_1, ω_1, and ψ_1 with respect to z is much larger than the second partial derivative of these functions with respect to r, then we can ignore the coupling in the Laplacian operator to u_3, ω_3, and ψ_3 in the above equations. Thus, we obtain our one-dimensional model as follows:

$$(u_1)_t + 2\psi_1(u_1)_z = \nu(u_1)_{zz} + 2(\psi_1)_z u_1, \tag{2.13}$$
$$(\omega_1)_t + 2\psi_1(\omega_1)_z = \nu(\omega_1)_{zz} + (u_1^2)_z, \tag{2.14}$$
$$-(\psi_1)_{zz} = \omega_1. \tag{2.15}$$

We remark that the above assumption implies that the solution has an anisotropic scaling; i.e., the solution is more singular along the z-direction than along the r-direction. A possible scenario is that the solution has a pancakelike structure perpendicular to the z-axis.

Let $\tilde{u} = u_1$, $\tilde{v} = -(\psi_1)_z$, $\tilde{\omega} = \omega_1$, and $\tilde{\psi} = \psi_1$. By integrating the ω_1 equation with respect to z and using the relationship $-\frac{\partial^2}{\partial z^2}\psi_1 = \omega_1$, we can obtain an evolution equation for $\tilde{v}$. Now the complete set of evolution equations for $\tilde{u}$, $\tilde{v}$, and $\tilde{\omega}$ are given by

$$(\tilde{u})_t + 2\tilde{\psi}(\tilde{u})_z = \nu(\tilde{u})_{zz} - 2\tilde{v}\tilde{u}, \tag{2.16}$$
$$(\tilde{\omega})_t + 2\tilde{\psi}(\tilde{\omega})_z = \nu(\tilde{\omega})_{zz} + (\tilde{u}^2)_z, \tag{2.17}$$
$$(\tilde{v})_t + 2\tilde{\psi}(\tilde{v})_z = \nu(\tilde{v})_{zz} + (\tilde{u})^2 - (\tilde{v})^2 + c(t), \tag{2.18}$$
$$-(\tilde{\psi})_{zz} = \tilde{\omega}, \tag{2.19}$$

where the constant $c(t)$ is an integration constant that is determined by enforcing the mean of $\tilde{v}$ equal to 0. For example, if $\tilde{\psi}$ is periodic with period 1 in z, then $c(t)$ is given by

$$c(t) = 3\int_0^1 \tilde{v}^2\, dz - \int_0^1 \tilde{u}^2\, dz. \tag{2.20}$$

Note that the equation for $\tilde{\omega}$ is equivalent to that for $\tilde{v}$. So it is sufficient to consider the coupled system for $\tilde{u}$ and $\tilde{v}$:

$$(\tilde{u})_t + 2\tilde{\psi}(\tilde{u})_z = \nu(\tilde{u})_{zz} - 2\tilde{v}\tilde{u} \tag{2.21}$$

$$(\tilde{v})_t + 2\tilde{\psi}(\tilde{v})_z = \nu(\tilde{v})_{zz} + (\tilde{u})^2 - (\tilde{v})^2 + c(t), \tag{2.22}$$

where $\tilde{\psi}$ is related to $\tilde{v}$ by $\tilde{v} = -(\tilde{\psi})_z$. By (2.19), we have $\tilde{v}_z = \tilde{\omega}$.

A surprising result is that one can use the above one-dimensional model to construct a family of *exact solutions* for the three-dimensional axisymmetric Navier-Stokes equations. This is described by the following theorem, which can be verified directly by substituting (2.23) into the three-dimensional axisymmetric Navier-Stokes equations and using the model equation (2.13)–(2.15).

THEOREM 2.1 *Let u_1, ψ_1, and ω_1 be the solution of the one-dimensional model* (2.13)–(2.15) *and define*

$$u(r,z,t) = ru_1(z,t), \quad \omega(r,z,t) = r\omega_1(z,t), \quad \psi(r,z,t) = r\psi_1(z,t). \tag{2.23}$$

Then $(u(r,z,t),\ \omega(r,z,t),\ \psi(r,z,t))$ is an exact solution of the three-dimensional Navier-Stokes equations.

Theorem 2.1 tells us that the one-dimensional model (2.13)–(2.15) preserves some essential nonlinear structure of the original three-dimensional axisymmetric Navier-Stokes equations. As we will see later, the nonlinear structure of the one-dimensional model plays a critical role in stabilizing the solution for large times, although the same nonlinearity can lead to large dynamic growth for early times.

3 Properties of the Model Equation

In this section, we will study some properties of the one-dimensional model equations. We will first consider the properties of some further simplified models obtained from these equations. Both numerical and analytical studies are presented for these simplified models. Based on the understanding of the simplified models, we prove the global existence of the inviscid Lagrangian model, which sheds useful light into our global existence analysis for the full one-dimensional model with or without viscosity.

3.1 The ODE Model

To start with, we consider an ODE model by ignoring the convection and diffusion terms:

$$(\tilde{u})_t = -2\tilde{v}\tilde{u}, \tag{3.1}$$

$$(\tilde{v})_t = (\tilde{u})^2 - (\tilde{v})^2, \tag{3.2}$$

with initial condition $\tilde{u}(0) = \tilde{u}_0$ and $\tilde{v}(0) = \tilde{v}_0$.

Clearly, if $\tilde{u}_0 = 0$, then $\tilde{u}(t) = 0$ for all $t > 0$. In this case, the equation for $\tilde{v}$ is decoupled from $\tilde{u}$ completely and will blow up in finite time if $\tilde{v}_0 < 0$. In fact, if $\tilde{v}_0 < 0$ and $\tilde{u}_0$ is very small, then the solution can experience very large growth dynamically. The growth can be made arbitrarily large if we choose $\tilde{u}_0$ to be arbitrarily small. However, the special nonlinear structure of the ODE system has an interesting cancellation property that has a stabilizing effect of the solution for large times. This is described by the following theorem:

THEOREM 3.1 *Assume that $\tilde{u}_0 \neq 0$. Then the solution $(\tilde{u}(t), \tilde{v}(t))$ of the ODE system* (3.1)–(3.2) *exists for all times. Moreover, we have*

$$\lim_{t\to+\infty} \tilde{u}(t) = 0, \quad \lim_{t\to+\infty} \tilde{v}(t) = 0. \tag{3.3}$$

PROOF: There are several ways to prove this theorem. The simplest way is to reformulate the problem in terms of complex variables.[1] Let

$$w = \tilde{u} + i\tilde{v}.$$

Then the ODE system (3.1)–(3.2) is reduced to the following complex nonlinear ODE:

$$\frac{dw}{dt} = iw^2, \quad w(0) = w_0, \tag{3.4}$$

which can be solved analytically. The solution has the form

$$w(t) = \frac{w_0}{1 - iw_0 t}. \tag{3.5}$$

In terms of the original variables, we have

$$\tilde{u}(t) = \frac{\tilde{u}_0(1 + \tilde{v}_0 t) - \tilde{u}_0\tilde{v}_0 t}{(1 + \tilde{v}_0 t)^2 + (\tilde{u}_0 t)^2}, \tag{3.6}$$

$$\tilde{v}(t) = \frac{\tilde{v}_0(1 + \tilde{v}_0 t) + \tilde{u}_0^2 t}{(1 + \tilde{v}_0 t)^2 + (\tilde{u}_0 t)^2}. \tag{3.7}$$

It is clear from (3.6)–(3.7) that the solution of the ODE system (3.1)–(3.2) exists for all times and decays to zero as $t \to +\infty$ as long as $\tilde{u}_0 \neq 0$. This completes the proof of Theorem 3.1. □

Remark 3.2. Note that the ODE model (3.4) has some similarity with the Constantin-Lax-Majda model [9], which has the form $u_t = uH(u)$, where H is the Hilbert transform. By letting $w = H(u) + iu$ and using the property of the Hilbert transform, Constantin, Lax, and Majda show that their model can be written as the imaginary part of the complex ODE, $w_t = \frac{1}{2}w^2$. It is interesting to note that both models ignore the convection term and have solutions that blow up at a finite time for initial condition satisfying $u(z_0) = 0$ and $H(u)(z_0) > 0$ for some z_0. However, as we will show later, the convection term plays an important role in stabilizing the one-dimensional model and should not be neglected in our study of the Euler equations. By including the convection term in the one-dimensional

[1] We thank Prof. Tai-Ping Liu for suggesting the use of complex variables.

model, we will show in Section 3.3 and Section 4 that no finite-time blowup can occur from smooth initial data.

As we can see from (3.6)–(3.7), the solution can grow very fast in a very short time if $\tilde{u}_0$ is small, but $\tilde{v}_0$ is large and negative. For example, if we let $\tilde{v}_0 = -1/\epsilon$ and $\tilde{u}_0 = \epsilon$ for $\epsilon > 0$ small, we obtain at $t = \epsilon$

$$\tilde{u}(\epsilon) = \frac{1}{\epsilon^3}, \quad \tilde{v}(\epsilon) = \frac{1}{\epsilon}.$$

We can see that within ϵ time, $\tilde{u}$ grows from its initial value of order ϵ to $O(\epsilon^{-3})$, a factor of ϵ^{-4} amplification.

Remark 3.3. The key ingredient in obtaining the global existence in Theorem 3.1 is that the coefficient on the right-hand side of (3.1) is less than -1. For this ODE system, there are two distinguished phases. In the first phase, if $\tilde{v}$ is negative and large in magnitude but $\tilde{u}$ is small, then $\tilde{v}$ can experience tremendous dynamic growth, which is essentially governed by

$$\tilde{v}_t = -\tilde{v}^2.$$

However, as $\tilde{v}$ becomes very large and negative, it will induce a rapid growth in $\tilde{u}$. The nonlinear structure of the ODE system is such that $\tilde{u}$ will eventually grow even faster than $\tilde{v}$ and force $(\tilde{u})^2 - (\tilde{v})^2 < 0$ in the second phase. From this time on, $\tilde{v}$ will increase in time and eventually become positive. Once $\tilde{v}$ becomes positive, the nonlinear term, $-\tilde{v}^2$, becomes stabilizing for $\tilde{v}$. Similarly, the nonlinear term, $-2\tilde{u}\tilde{v}$, becomes stabilizing for $\tilde{u}$. This subtle dynamic stability property of the ODE system can best be illustrated by the phase diagram in Figure 3.1.

In Appendix A, we prove the same result for a more general ODE system of the following form:

$$(\tilde{u})_t = -d\tilde{v}\tilde{u}, \tag{3.8}$$

$$(\tilde{v})_t = (\tilde{u})^2 - (\tilde{v})^2, \tag{3.9}$$

for any constant $d \geq 1$. However, if $d < 1$, it is possible to construct a family of solutions for the ODE systems (3.8)–(3.9) that blow up in a finite time.

3.2 The Reaction-Diffusion Model

In this subsection, we consider the reaction-diffusion system

$$(\tilde{u})_t = \nu\tilde{u}_{zz} - 2\tilde{v}\tilde{u}, \tag{3.10}$$

$$(\tilde{v})_t = \nu\tilde{v}_{zz} + (\tilde{u})^2 - (\tilde{v})^2. \tag{3.11}$$

As we can see for the corresponding ODE system, the structure of the nonlinearity plays an essential role in obtaining global existence. Intuitively, one may think that the diffusion term would help to stabilize the dynamic growth induced by the nonlinear terms. However, because the nonlinear ODE system in the absence of viscosity is very unstable, the diffusion term can actually have a destabilizing

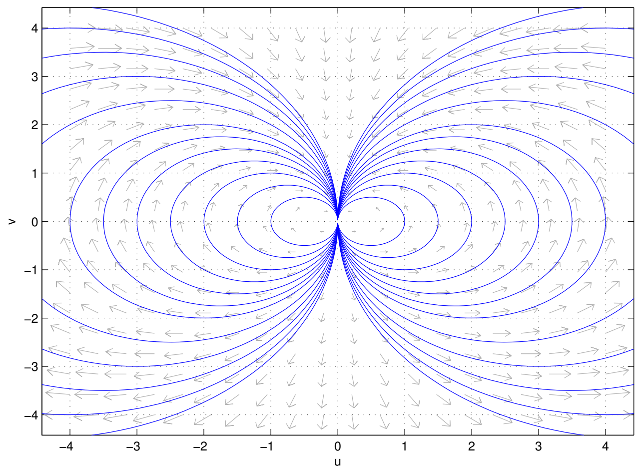

FIGURE 3.1. The phase diagram for the ODE system.

effect. Below we will demonstrate this somewhat surprising fact through careful numerical experiments.

In Figures 3.2 through 3.4, we plot a time sequence of solutions for the above reaction diffusion system with the following initial data:

$$\tilde{u}_0(z) = \epsilon(2 + \sin(2\pi z)), \quad \tilde{v}_0(z) = -\frac{1}{\epsilon} - \sin(2\pi z),$$

where $\epsilon = 0.001$. For this initial condition, the solution is periodic in z with period 1. We use a pseudospectral method to discretize the coupled system (3.10)–(3.11) in space and use the simple forward Euler discretization for the nonlinear terms and the backward Euler discretization for the diffusion term. In order to resolve the nearly singular solution structure, we use $N = 32\,768$ grid points with an adaptive time step satisfying

$$\Delta t_n(|\max\{\tilde{u}^n\}| + |\min\{\tilde{u}^n\}| + |\max\{\tilde{v}^n\}| + |\min\{\tilde{v}^n\}|) \le 0.01,$$

where $\tilde{u}^n$ and $\tilde{v}^n$ are the numerical solution at time t_n and $t_n = t_{n-1} + \Delta t_{n-1}$ with the initial time step size $\Delta t_0 = 0.01\epsilon$. During the time iterations, the smallest time step is as small as $O(10^{-10})$.

From Figure 3.2, we can see that the magnitude of the solution $\tilde{v}$ increases rapidly by a factor of 150 within a very short time ($t = 0.000\,998\,17$). As the solution $\tilde{v}$ becomes large and negative, the solution $\tilde{u}$ increases much more rapidly than $\tilde{v}$. By time $t = 0.001\,004\,2$, $\tilde{u}$ has increased to about 2.5×10^8 from its initial condition, which is of magnitude 10^{-3}. This is an increase by a factor of 2.5×10^{11}. At this time, the minimum of $\tilde{v}$ has reached -2×10^8.

Note that since $\tilde{u}$ has outgrown $\tilde{v}$ in magnitude, the nonlinear term, $\tilde{u}^2 - \tilde{v}^2$, on the right-hand side of the $\tilde{v}$-equation has changed sign. This causes the solution $\tilde{v}$ to split. By the time $t = 0.001\,004\,314$ (see Figure 3.3), both $\tilde{u}$ and $\tilde{v}$ have split and settled down to two relatively stable traveling-wave solutions. The wave on the left will travel to the left while the wave on the right will travel to the right. Due to the periodicity in z, the two traveling waves approach each other from the right side of the domain. The "collision" of these two traveling waves tends to annihilate each other. In particular, the negative part of $\tilde{v}$ is effectively eliminated during this nonlinear interaction. By the time $t = 0.001\,006\,03$ (see Figure 3.4), the solution $\tilde{v}$ becomes all positive. Once $\tilde{v}$ becomes positive, the effect of nonlinearity becomes stabilizing for both $\tilde{u}$ and $\tilde{v}$, as in the case of the ODE system. From then on, the solution decays rapidly. By $t = 0.2007$, the magnitude of $\tilde{u}$ is as small as 5.2×10^{-8}, and $\tilde{v}$ becomes almost a constant function with value close to 5. From this time on, $\tilde{u}$ is essentially decoupled from $\tilde{v}$ and will decay like $O(1/t)$.

3.3 The Lagrangian Convection Model

Next, we consider the one-dimensional model equations in the absence of viscosity. The corresponding equations are given as follows:

$$\tilde{u}_t + 2\tilde{\psi}\tilde{u}_z = -2\tilde{v}\tilde{u}, \tag{3.12}$$

$$\tilde{v}_t + 2\tilde{\psi}\tilde{v}_z = \tilde{u}^2 - \tilde{v}^2 + c(t), \tag{3.13}$$

where $\tilde{v} = -\tilde{\psi}_z$, and $c(t)$ is defined in (2.20) to ensure that $\int_0^1 \tilde{v}\,dz = 0$.

Introduce the Lagrangian flow map

$$\frac{\partial z(\alpha,t)}{\partial t} = 2\tilde{\psi}(z(\alpha,t),t), \tag{3.14}$$

$$z(\alpha,0) = \alpha. \tag{3.15}$$

Differentiating (3.14) with respect to α, we get

$$\frac{\partial z_\alpha}{\partial t} = 2z_\alpha \frac{\partial\tilde{\psi}}{\partial z}(z(\alpha,t),t) = -2z_\alpha\tilde{v}(z(\alpha,t),t).$$

Denote $v(\alpha,t) = \tilde{v}(z(\alpha,t),t)$, $u(\alpha,t) = \tilde{u}(z(\alpha,t),t)$, and $J(\alpha,t) = z_\alpha(\alpha,t)$. Then we can show that J, u, and v satisfy the following system of equations:

$$\frac{\partial J(\alpha,t)}{\partial t} = -2J(\alpha,t)v(\alpha,t), \tag{3.16}$$

$$\frac{\partial u(\alpha,t)}{\partial t} = -2u(\alpha,t)v(\alpha,t), \tag{3.17}$$

$$\frac{\partial v(\alpha,t)}{\partial t} = u^2 - v^2 + 3\int_0^1 v^2 J\,d\alpha - \int_0^1 u^2 J\,d\alpha, \tag{3.18}$$

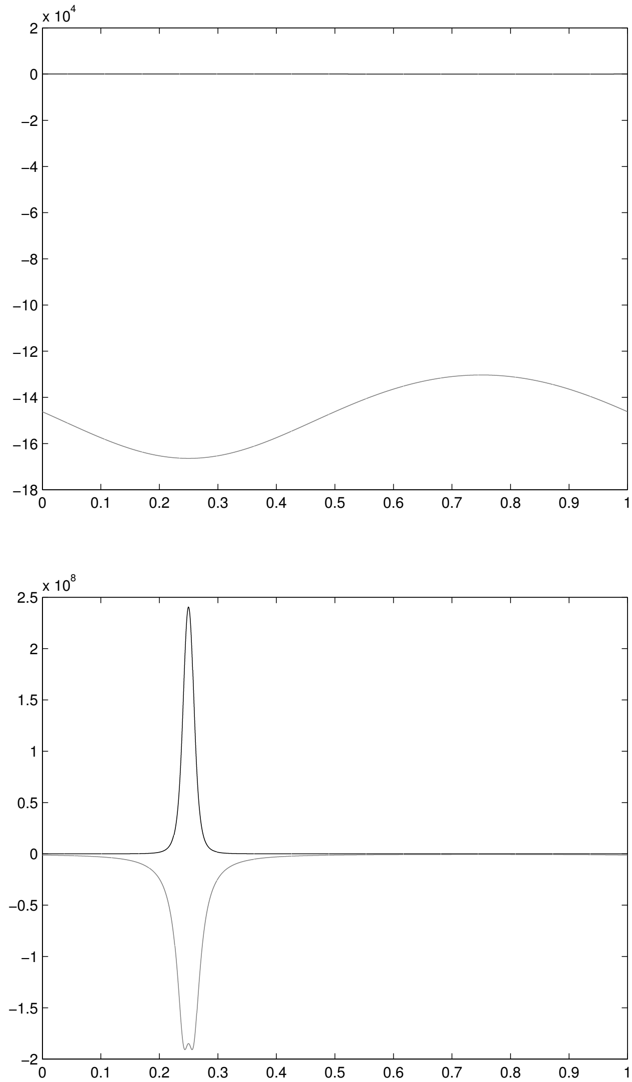

FIGURE 3.2. The solutions u (dark curve at the top) and v (light curve at the bottom) at $t = 0.000\,998\,17$ (top figure) and $t = 0.001\,004\,2$ (bottom figure); $N = 32\,768$, $\nu = 1$.

with initial data $J(\alpha,0) = 1$, $u(\alpha,0) = \tilde{u}_0(\alpha)$, and $v(\alpha,0) = \tilde{v}_0(\alpha)$. Since $\int_0^1 \tilde{v}(z,t)dz = 0$, we have

$$\int_0^1 v(\alpha,t)J(\alpha,t)d\alpha = 0, \tag{3.19}$$

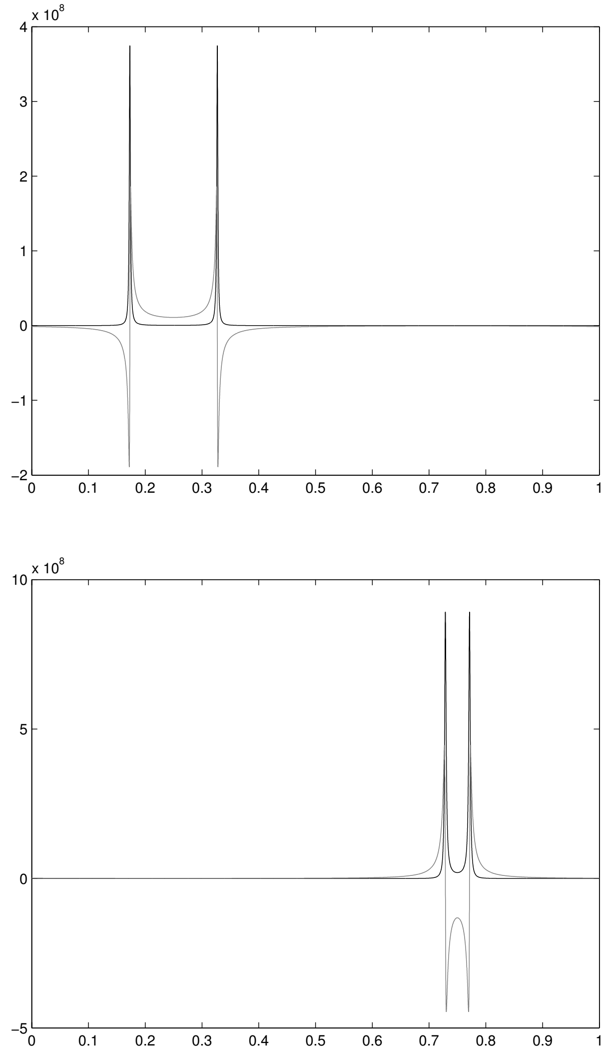

FIGURE 3.3. The solutions u (dark) and v (light) at $t = 0.001\,004\,314$ (top figure) and $t = 0.001\,005\,862$ (bottom figure); $N = 32\,768$, $\nu = 1$.

which implies that

$$\int_0^1 J(\alpha, t)d\alpha \equiv \int_0^1 J(\alpha, 0)d\alpha = 1. \tag{3.20}$$

It is interesting to note that the one-dimensional model formulated in the Lagrangian coordinate retains some of the essential properties of the ODE system. In

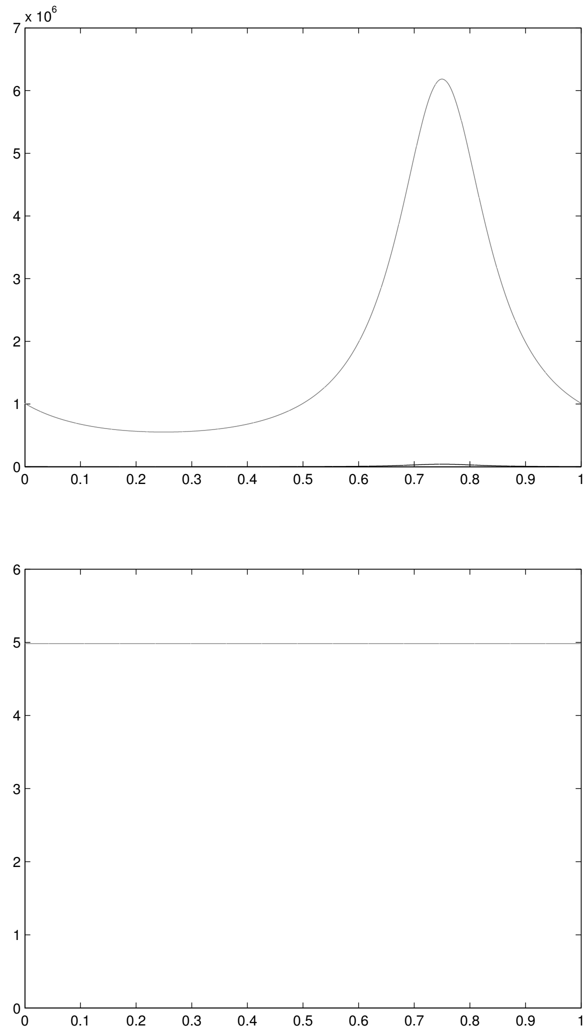

FIGURE 3.4. The solutions u (dark, bottom curve) and v (light, top curve) at $t = 0.001\,006\,03$ (top figure) and $t = 0.2007$ (bottom figure); $N = 32\,768$, $\nu = 1$. Note that at $t = 0.001\,006\,03$, the value of u becomes quite small and is very close to the x-axis (see top figure). By $t = 0.2007$, the value of u is of the order 5.2×10^{-8} and is almost invisible from the bottom figure.

the following, we will explore the special nonlinear structure of the model equation to prove the global well-posedness of the one-dimensional model in the Lagrangian form. As we will see, the understanding of the one-dimensional model in the Lagrangian form gives critical insight into our understanding of the full one-dimensional model.

THEOREM 3.4 *Assume that* $\tilde{u}(z,0)$ *and* $\tilde{v}(z,0)$ *are in* $C^m[0,1]$ *with* $m \geq 1$ *and periodic with period* 1. *Then the solution* $(\tilde{u},\tilde{v})$ *of the one-dimensional inviscid model will be in* $C^m[0,1]$ *for all times.*

PROOF: Differentiating the $\tilde{u}$- and $\tilde{v}$-equations with respect to α, we get

$$\frac{d\tilde{u}_\alpha}{dt} = -2\tilde{v}\tilde{u}_\alpha - 2\tilde{u}\tilde{v}_\alpha, \tag{3.21}$$

$$\frac{d\tilde{v}_\alpha}{dt} = 2\tilde{u}\tilde{u}_\alpha - 2\tilde{v}\tilde{v}_\alpha. \tag{3.22}$$

Multiplying (3.21) by $\tilde{u}_\alpha$ and (3.22) by $\tilde{v}_\alpha$, and adding the resulting equations, we have

$$\frac{1}{2}\frac{d}{dt}(\tilde{u}_\alpha^2 + \tilde{v}_\alpha^2) = -2\tilde{v}(\tilde{u}_\alpha^2 + \tilde{v}_\alpha^2). \tag{3.23}$$

Therefore, we obtain

$$\frac{1}{2}\frac{d}{dt}\log(\tilde{u}_\alpha^2 + \tilde{v}_\alpha^2) = -2\tilde{v}. \tag{3.24}$$

Integrating from 0 to t, we get

$$\begin{aligned}\left(\sqrt{\tilde{u}_\alpha^2 + \tilde{v}_\alpha^2}\right)(\alpha,t) &= \left(\sqrt{(\tilde{u}_0)_\alpha^2 + (\tilde{v}_0)_\alpha^2}\right)e^{-2\int_0^t \tilde{v}(\alpha,s)ds} \\ &= \left(\sqrt{(\tilde{u}_0)_\alpha^2 + (\tilde{v}_0)_\alpha^2}\right)J(\alpha,t),\end{aligned} \tag{3.25}$$

where we have used

$$J(\alpha,t) = e^{-2\int_0^t \tilde{v}(\alpha,s)ds},$$

which follows from (3.16) and $J(\alpha,0) \equiv 1$.

Using (3.20), we further obtain

$$\begin{aligned}\int_0^1 \sqrt{\tilde{u}_\alpha^2 + \tilde{v}_\alpha^2}\, d\alpha &\leq \left\| \sqrt{(\tilde{u}_0)_\alpha^2 + (\tilde{v}_0)_\alpha^2} \right\|_{L^\infty} \int_0^1 J(\alpha,t)d\alpha \\ &= \left\| \sqrt{(\tilde{u}_0)_\alpha^2 + (\tilde{v}_0)_\alpha^2} \right\|_{L^\infty}.\end{aligned} \tag{3.26}$$

In particular, we have

$$\int_0^1 |\tilde{v}_\alpha| d\alpha \leq \int_0^1 \sqrt{\tilde{u}_\alpha^2 + \tilde{v}_\alpha^2}\, d\alpha \leq \left\| \sqrt{(\tilde{u}_0)_\alpha^2 + (\tilde{v}_0)_\alpha^2} \right\|_{L^\infty}. \tag{3.27}$$

Since $\int_0^1 \tilde{v} J\, d\alpha = 0$ and $J > 0$, there exists $\alpha_0(t) \in [0,1]$ such that $\tilde{v}(\alpha_0(t),t) = 0$. Therefore, we get

$$|\tilde{v}(\alpha,t)| = \left| \int_{\alpha_0}^{\alpha} \tilde{v}_\alpha\, d\alpha' \right| \leq \int_0^1 |\tilde{v}(\alpha',t)| d\alpha' \leq \left\| \sqrt{(\tilde{u}_0)_\alpha^2 + (\tilde{v}_0)_\alpha^2} \right\|_{L^\infty}. \tag{3.28}$$

This proves that

$$\|\tilde{v}\|_{L^\infty} \le \left\| \sqrt{(\tilde{u}_0)^2_\alpha + (\tilde{v}_0)^2_\alpha} \right\|_{L^\infty}. \tag{3.29}$$

Using the equations for J and $\tilde{u}$, we also obtain

$$e^{-2tC_0} \le J(\alpha, t) \le e^{2tC_0}, \tag{3.30}$$

$$\|\tilde{u}\|_{L^\infty} \le \|\tilde{u}_0\|_{L^\infty} e^{2tC_0}, \tag{3.31}$$

where $C_0 = \| \sqrt{(\tilde{u}_0)^2_\alpha + (\tilde{v}_0)^2_\alpha} \|_{L^\infty}$.

The bound on $J(\alpha, t)$ in turn gives a bound on $\tilde{u}^2_\alpha + \tilde{v}^2_\alpha$ through (3.25). We can then bootstrap to obtain regularity of the solution in higher-order norms. This completes the proof of Theorem 3.4. □

Next, we illustrate the behavior of the solution through numerical computations. We use a pseudospectral method to discretize in space and a second-order Runge-Kutta discretization in time with an adaptive time stepping. In Figures 3.5 and 3.6, we plot a sequence of snapshots of the solution for the inviscid model (3.16)–(3.18) in the Lagrangian coordinate using the initial data

$$u(\alpha, 0) = 1, \quad v(\alpha, 0) = 1 - \frac{1}{\delta} \exp^{-(x-0.5)^2/\epsilon},$$

with $\epsilon = 0.0001$ and $\delta = \sqrt{\epsilon\pi}$.

We can see that the solution experiences a similar splitting process as in the reaction-diffusion model. In Figure 3.7, we perform a similar computation in the Eulerian coordinate with $\epsilon = 0.000\,01$. We can see that as the solution $\tilde{v}$ grows large and negative, the initial sharp profile of $\tilde{v}$ becomes wider and smoother. This is a consequence of the incompressibility of the fluid flow. If we change the sign of the convection velocity from $2\tilde{\psi}$ to $-2\tilde{\psi}$, the profile of $\tilde{v}$ becomes focused dynamically and develops an unphysical "shocklike" solution, which seems to evolve into a finite-time blowup; see Figure 3.8.

4 Global Well-Posedness of the One-Dimensional Viscous Model

Based on the understanding we have gained from the previous sections, we are ready to present a complete proof of the global well-posedness of the full one-dimensional model. It is not easy to obtain global regularity of the one-dimensional model by using an energy type of estimates. If we multiply the $\tilde{u}$-equation by $\tilde{u}$, and the $\tilde{v}$-equation by $\tilde{v}$, and integrate over z, we arrive at

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \tilde{u}^2\,dz = -3\int_0^1 (\tilde{u})^2\tilde{v}\,dz - \nu\int_0^1 \tilde{u}_z^2\,dz, \tag{4.1}$$

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \tilde{v}^2\,dz = \int_0^1 \tilde{u}^2\tilde{v}\,dz - 3\int_0^1 (\tilde{v})^3\,dz - \nu\int_0^1 \tilde{v}_z^2\,dz. \tag{4.2}$$

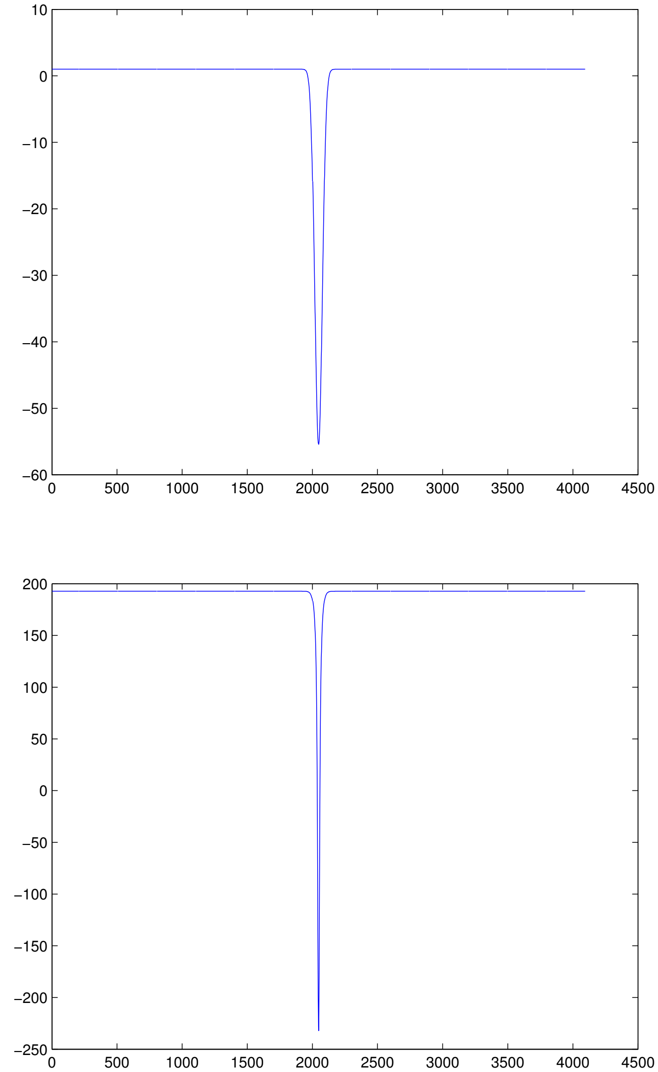

FIGURE 3.5. The Lagrangian solution v at $t = 0$ (top figure) and $t = 0.0188$ (bottom figure); $N = 4096$, $\nu = 0$. The solution is plotted against the number of grid points corresponding to the range $[0, 1]$ in physical space.

Even for this one-dimensional model, the energy estimate shares the same essential difficulty as the three-dimensional Navier-Stokes equations. It is not clear how to control the nonlinear vortex-stretching-like terms by the diffusion terms.

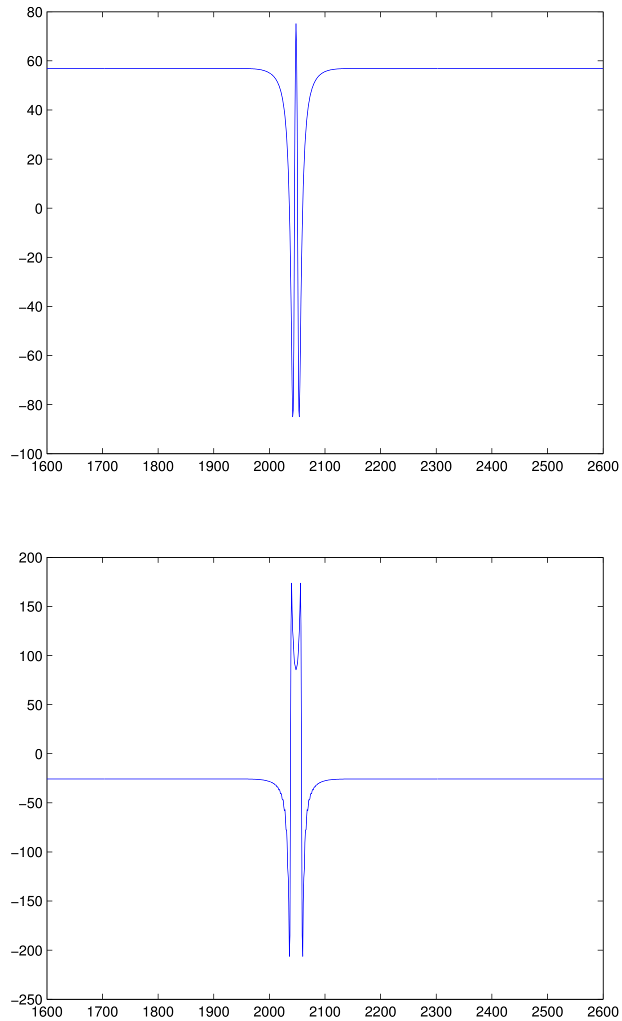

FIGURE 3.6. The Lagrangian solution v at $t = 0.023$ (top figure) and $t = 0.0337$ (bottom figure); $N = 4096$, $\nu = 0$. The solutions are plotted against the number of grid points corresponding to the range $[0.391, 0.635]$ in physical space.

On the other hand, if we assume that

$$\int_0^T \|\tilde{v}\|_{L^\infty}\, dt < \infty,$$

similar to the Beale-Kato-Majda nonblowup condition for vorticity [2], then one can easily show that there is no blowup up to $t = T$.

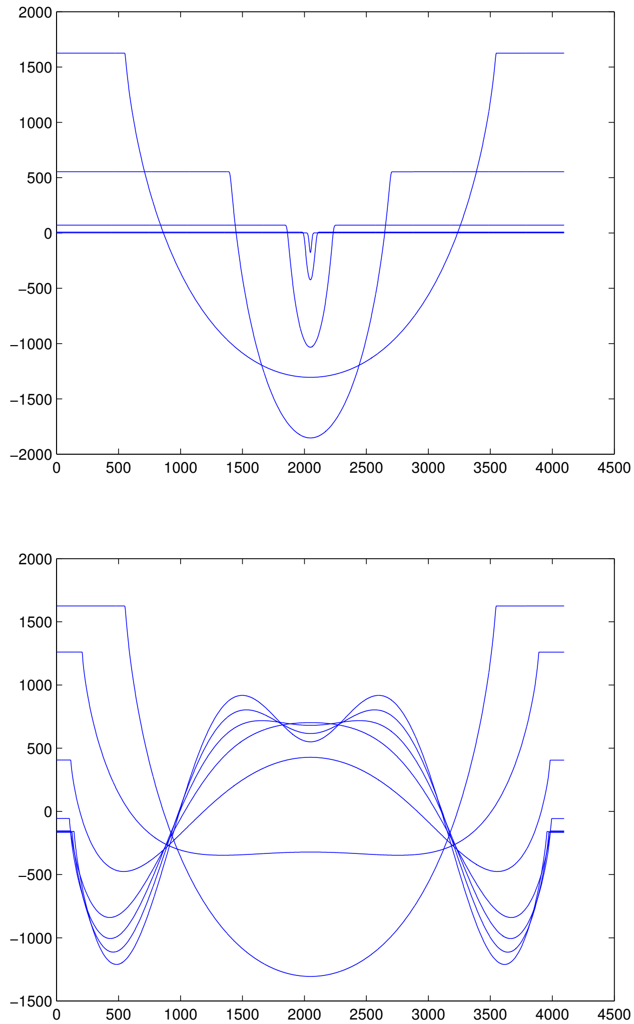

FIGURE 3.7. The time sequence of v in the Eulerian coordinate, $N = 4096$, $\nu = 0$. The top figure corresponds to $t = 0$, $0.0033, 0.0048, 0.0055, 0.0059$. The bottom figure corresponds to $t = 0.0059, 0.0062, 0.0066, 0.007, 0.0074, 0.0078, 0.0081$. The solutions are plotted against the number of grid points corresponding to the range $[0, 1]$ in physical space.

In order to obtain the global regularity of the one-dimensional model, we need to use a local estimate. We will prove that if the initial conditions for $\tilde{u}$ and $\tilde{v}$ are in C^m with $m \geq 1$, then the solution will remain in C^m for all times.

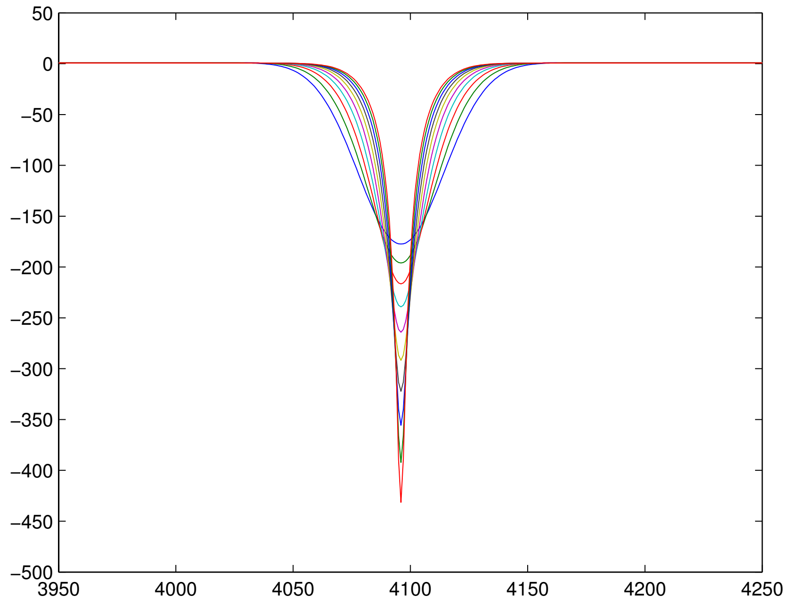

FIGURE 3.8. The time sequence of solution v in the Lagrangian coordinate by solving the model equation with the wrong sign; $N = 8192$, $\nu = 0$. The time sequence is from $t = 0$ to 0.0033, corresponding to a sequence of curves from the top to the bottom. The solutions are plotted against the number of grid points corresponding to the range $[0.482, 0.519]$ in physical space.

THEOREM 4.1 *Assume that* $\tilde{u}(z,0)$ *and* $\tilde{v}(z,0)$ *are in* $C^m[0,1]$ *with* $m \geq 1$ *and periodic with period* 1. *Then the solution* $(\tilde{u},\tilde{v})$ *of the one-dimensional model will be in* $C^m[0,1]$ *for all times.*

PROOF: Motivated by our analysis for the inviscid Lagrangian model, we will try to obtain an a priori estimate for the nonlinear term $\tilde{u}_z^2 + \tilde{v}_z^2$. Differentiating the $\tilde{u}$-equation and the $\tilde{v}$-equation with respect to z, we get

$$(\tilde{u}_z)_t + 2\tilde{\psi}(\tilde{u}_z)_z - 2\tilde{v}\tilde{u}_z = -2\tilde{u}_z\tilde{v} - 2\tilde{u}\tilde{v}_z + \nu(\tilde{u}_z)_{zz}, \tag{4.3}$$

$$(\tilde{v}_z)_t + 2\tilde{\psi}(\tilde{v}_z)_z - 2\tilde{v}\tilde{v}_z = 2\tilde{u}\tilde{u}_z - 2\tilde{v}\tilde{v}_z + \nu(\tilde{v}_z)_{zz}. \tag{4.4}$$

Note that one of the nonlinear terms resulting from differentiating the convection term cancels one of the nonlinear terms on the right-hand side. After canceling the same nonlinear term from both sides, we obtain

$$(\tilde{u}_z)_t + 2\tilde{\psi}(\tilde{u}_z)_z = -2\tilde{u}\tilde{v}_z + \nu(\tilde{u}_z)_{zz}, \tag{4.5}$$

$$(\tilde{v}_z)_t + 2\tilde{\psi}(\tilde{v}_z)_z = 2\tilde{u}\tilde{u}_z + \nu(\tilde{v}_z)_{zz}. \tag{4.6}$$

Multiplying (4.5) by $\tilde{u}_z$ and (4.6) by $\tilde{v}_z$, we have

$$\frac{1}{2}(\tilde{u}_z^2)_t + \tilde{\psi}(\tilde{u}_z^2)_z = -2\tilde{u}\tilde{u}_z\tilde{v}_z + \nu\tilde{u}_z(\tilde{u}_z)_{zz}, \tag{4.7}$$

$$\frac{1}{2}(\tilde{v}_z^2)_t + \tilde{\psi}(\tilde{v}_z^2)_z = 2\tilde{u}\tilde{u}_z\tilde{v}_z + \nu\tilde{v}_z(\tilde{v}_z)_{zz}. \tag{4.8}$$

Now, we add (4.7) to (4.8). Surprisingly, the nonlinear vortex-stretching-like terms cancel each other. We get

$$(\tilde{u}_z^2 + \tilde{v}_z^2)_t + 2\tilde{\psi}(\tilde{u}_z^2 + \tilde{v}_z^2)_z = 2\nu(\tilde{u}_z(\tilde{u}_z)_{zz} + \tilde{v}_z(\tilde{v}_z)_{zz}). \tag{4.9}$$

Further, we note that

$$\begin{aligned}(\tilde{u}_z^2 + \tilde{v}_z^2)_{zz} &= (2\tilde{u}_z\tilde{u}_{zz} + 2\tilde{v}_z\tilde{v}_{zz})_z \\ &= 2(\tilde{u}_z(\tilde{u}_z)_{zz} + \tilde{v}_z(\tilde{v}_z)_{zz}) + 2[(\tilde{u}_{zz})^2 + (\tilde{v}_{zz})^2].\end{aligned}$$

Therefore, equation (4.9) can be rewritten as

$$(\tilde{u}_z^2 + \tilde{v}_z^2)_t + 2\tilde{\psi}(\tilde{u}_z^2 + \tilde{v}_z^2)_z = \nu(\tilde{u}_z^2 + \tilde{v}_z^2)_{zz} - 2\nu[(\tilde{u}_{zz})^2 + (\tilde{v}_{zz})^2]. \tag{4.10}$$

Thus, the nonlinear quantity $(\tilde{u}_z^2 + \tilde{v}_z^2)$ satisfies a maximum principle that holds for both $\nu = 0$ and $\nu > 0$:

$$\|\tilde{u}_z^2 + \tilde{v}_z^2\|_{L^\infty} \le \|(\tilde{u}_0)_z^2 + (\tilde{v}_0)_z^2\|_{L^\infty}. \tag{4.11}$$

Since $\tilde{v}$ has zero mean, the Poincaré inequality implies that $\|\tilde{v}\|_{L^\infty} \le C_0$, with C_0 defined by

$$C_0 = \|((\tilde{u}_0)_z^2 + (\tilde{v}_0)_z^2)^{1/2}\|_{L^\infty}.$$

The boundedness of $\tilde{u}$ follows from the bound on $\tilde{v}$:

$$\|\tilde{u}(t)\|_{L^\infty} \le \|\tilde{u}_0\|_{L^\infty} \exp(2C_0 t).$$

The higher-order regularity follows from the standard estimates. This proves Theorem 4.1. □

5 Construction of a Family of Globally Smooth Solutions

In this section, we will use the solution from the one-dimensional model to construct a family of globally smooth solutions for the three-dimensional axisymmetric Navier-Stokes equations with smooth initial data of finite energy. We remark that a special feature of this family of globally smooth solutions is that the solution can potentially develop very large dynamic growth, and it violates the so-called smallness condition required by classical global existence results [8, 21].

Let $\bar{u}_1(z,t)$, $\bar{\omega}_1(z,t)$, and $\bar{\psi}$ be the solution of the one-dimensional model problem. We will construct a family of globally smooth solutions of the three-dimensional Navier-Stokes equations from the solution of the one-dimensional model problem. Denote by $\tilde{u}(r,z,t)$, $\tilde{\omega}(r,z,t)$, and $\tilde{\psi}(r,z,t)$ the solution of the corresponding three-dimensional Navier-Stokes equations. Further, we define

$$\tilde{u}_1 = \frac{\tilde{u}}{r}, \quad \tilde{\omega}_1 = \frac{\tilde{\omega}}{r}, \quad \tilde{\psi}_1 = \frac{\tilde{\psi}}{r}. \tag{5.1}$$

Let $\phi(r) = \phi_0(r/R_0)$ be a smooth cutoff function, where $\phi_0(r)$ satisfies $\phi_0(r) = 1$ if $0 \le r \le \frac{1}{2}$ and and $\phi_0(r) = 0$ if $r \ge 1$. Our strategy is to construct a family of globally smooth functions u_1, ω_1, and ψ_1 that are periodic in z and such that

$$\tilde{u} = r(\bar{u}_1(z,t)\phi(r) + u_1(r,z,t)) = \bar{u} + u, \tag{5.2}$$

$$\tilde{\omega} = r(\bar{\omega}_1(z,t)\phi(r) + \omega_1(r,z,t)) = \bar{\psi} + \psi, \tag{5.3}$$

$$\tilde{\psi} = r(\bar{\psi}_1(z,t)\phi(r) + \psi_1(r,z,t)) = \bar{\omega} + \omega, \tag{5.4}$$

is a solution of the three-dimensional Navier-Stokes equations.

With the above definition, we can deduce the other two velocity components $\tilde{v}^r$ and $\tilde{v}^3$ as follows:

$$\tilde{v}^r = -\tilde{\psi}_z = -r\phi(r)\bar{\psi}_{1z} + v^r(r,z,t) = \bar{v}^r + v^r, \tag{5.5}$$

$$\tilde{v}^z = \frac{(r\tilde{\psi})_r}{r} = \phi(2\bar{\psi}_1) + r\phi_r\bar{\psi}_1 + v^z(r,z,t) = \bar{v}^z + v^z. \tag{5.6}$$

With the above notation, we can write the velocity vector in two parts as $\tilde{\mathbf{u}} = \bar{\mathbf{u}} + \mathbf{u}$.

We will choose the initial data for the one-dimensional model of the following form:

$$\begin{gathered} \bar{\psi}_1(z,0) = \frac{A}{M^2}\Psi_1(zM), \quad \bar{u}_1(z,0) = \frac{A}{M}U_1(zM), \\ \bar{\omega}_1(z,0) = AW_1(zM), \end{gathered} \tag{5.7}$$

where A and M are some positive constants, and $\Psi_1(y)$ and $U_1(y)$ are smooth periodic functions in y with period 1. Moreover, we assume that Ψ_1 and U_1 are odd functions in y. Clearly, we have $W_1 = -(\Psi_1)_{yy}$, which is also a smooth, periodic, and odd function in y. It is easy to see that this feature of the initial data is preserved by the solution dynamically. In particular, $\bar{\psi}_1(z,t)$, $\bar{u}_1(z,t)$, and $\bar{\omega}_1(z,t)$ are periodic functions in z with period $1/M$ and odd in z within each period. Using this property and the a priori estimate (4.11), we obtain the following estimate for the solution of the one-dimensional model:

$$\|\bar{\psi}_1(t)\|_{L^\infty} \le C_0\,\frac{A}{M^2}, \tag{5.8}$$

$$\|\bar{u}_1(t)\|_{L^\infty} \le C_0\,\frac{A}{M}, \quad \|\bar{\psi}_{1z}(t)\|_{L^\infty} \le C_0\,\frac{A}{M}, \tag{5.9}$$

$$\|\bar{\omega}_1(t)\|_{L^\infty} \le C_0 A, \quad \|\bar{u}_{1z}(t)\|_{L^\infty} \le C_0 A, \tag{5.10}$$

where

$$C_0 = \|(U_{1y}^2 + W_1^2)^{1/2}\|_{L^\infty}. \tag{5.11}$$

Remark 5.1. As we know from the discussions in the previous sections and as indicated by (5.8)–(5.11), if the regularity of the periodic profiles in the initial condition, i.e., U_1 and Ψ_1, is very poor, the solution $\bar{u}_1(z,t)$ and $\bar{\psi}_z(z,t)$ will grow very fast dynamically. The amplification factor is determined by C_0 defined in (5.11).

Let $R_0 = M^{1/4}$. From (5.8)–(5.11) and the definition of $\bar{\mathbf{u}}$, we have

$$\|\bar{\mathbf{u}}\|_{L^2} \approx \frac{AR_0^2}{M} = \frac{A}{\sqrt{M}}, \quad \|\nabla\bar{\mathbf{u}}\|_{L^2} \approx AR_0^2 = A\sqrt{M}. \tag{5.12}$$

We would like to emphasize that the corresponding three-dimensional solution defined by (5.2)–(5.4) in general does not preserve the same special structure in the z-direction of the one-dimensional model problem since the correction terms u_1, ω_1, and ψ_1 are periodic in z with period 1 instead of period $1/M$.

We assume that the initial conditions for u_1, ω_1, and ψ_1 are chosen in such a way that the principal contributions to the energy and the enstrophy come from $\bar{\mathbf{u}}$, the mollified solution of the one-dimensional model. Specifically, we assume that the initial condition for $\tilde{\mathbf{u}}$ satisfies

$$\|\tilde{\mathbf{u}}_0\|_{L^2} \approx \frac{AR_0^2}{M} = \frac{A}{\sqrt{M}}, \quad \|\nabla\tilde{\mathbf{u}}_0\|_{L^2} \approx AR_0^2 = A\sqrt{M}. \tag{5.13}$$

Thus, we have

$$\|\tilde{\mathbf{u}}_0\|_{L^2}\,\|\nabla\tilde{\mathbf{u}}_0\|_{L^2} \approx A^2. \tag{5.14}$$

By choosing A large enough, the above product can be made arbitrarily large. Thus it violates the classical "smallness" condition that guarantees the global existence of the three-dimensional Navier-Stokes equations [21].

Furthermore, we have from the energy inequality that

$$\|\tilde{\mathbf{u}}(t)\|_{L^2} \le \|\tilde{\mathbf{u}}_0\|_{L^2} \le \frac{A}{\sqrt{M}}. \tag{5.15}$$

Using the above bound and (5.12), we obtain an a priori bound for the perturbed velocity field $\mathbf{u}$ in the L^2 norm:

$$\|\mathbf{u}(t)\|_{L^2} \le \frac{A}{\sqrt{M}}. \tag{5.16}$$

Let $f = u_1^2$, and define

$$H^2(t) = \int (f^2 + \omega_1^2) r\,dr\,dz = \int (u_1^4 + \omega_1^2) r\,dr\,dz, \tag{5.17}$$

$$E^2(t) = \int (|\nabla f|^2 + |\nabla\omega_1|^2) r\,dr\,dz, \tag{5.18}$$

where the integration is over $[0,1] \times [0,\infty)$.

If we further assume that the initial conditions for u_1, ω_1, and ψ_1 are odd functions of z, then it is easy to verify that $\tilde{u}_1$, $\tilde{\omega}_1$, and $\tilde{\psi}_1$ are odd functions of

z for all times. Since $\bar{u}_1$, $\bar{\omega}_1$, and $\bar{\psi}_1$ are also odd functions of z, we conclude that u_1, ω_1, and ψ_1 are odd functions of z for all times. It follows by the Poincaré inequality that we have

$$\int f^2 r\,dr\,dz \le \int f_z^2 r\,dr\,dz \le \int |\nabla f|^2 r\,dr\,dz, \tag{5.19}$$

$$\int \omega_1^2 r\,dr\,dz \le \int \omega_{1z}^2 r\,dr\,dz \le \int |\nabla \omega_1|^2 r\,dr\,dz. \tag{5.20}$$

This implies that

$$H \le E. \tag{5.21}$$

Now we can state the main theorem of this section.

THEOREM 5.2 *Assume that the initial conditions for u_1, ω_1, and ψ_1 are smooth functions of compact support and odd in z. For any given $A > 1$, $C_0 > 1$, and $\nu > 0$, there exists $C(A, C_0, \nu) > 0$ such that if $M > C(A, C_0, \nu)$ and $H(0) \le 1$, then the solution of the three-dimensional Navier-Stokes equations given by* (5.2)–(5.4) *remains smooth for all times.*

PROOF: First of all, we can use (2.4)–(2.6) to derive the corresponding evolution equations for $\tilde{u}_1$, $\tilde{\omega}_1$, and $\tilde{\psi}_1$ as follows:

$$(\tilde{u}_1)_t + \tilde{v}^r(\tilde{u}_1)_r + \tilde{v}^z(\tilde{u}_1)_z = 2(\tilde{\psi}_1)_z\tilde{u}_1 + \nu\left(\tilde{u}_{1zz} + \tilde{u}_{1rr} + \frac{3\tilde{u}_{1r}}{r}\right), \tag{5.22}$$

$$(\tilde{\omega}_1)_t + \tilde{v}^r(\tilde{\omega}_1)_r + \tilde{v}^z(\tilde{\omega}_1)_z = (\tilde{u}_1^2)_z + \nu\left(\tilde{\omega}_{1zz} + \tilde{\omega}_{1rr} + \frac{3\tilde{\omega}_{1r}}{r}\right), \tag{5.23}$$

$$-\left(\tilde{\psi}_{1zz} + \tilde{\psi}_{1rr} + \frac{3\tilde{\psi}_{1r}}{r}\right) = \tilde{\omega}_1. \tag{5.24}$$

Substituting (5.2) into (5.22) and using (5.1), we obtain an evolution equation for u_1:

$$\begin{aligned}\frac{\partial u_1}{\partial t} + \tilde{v}^r u_{1r} + \tilde{v}^z u_{1z} &= \nu\Delta u_1 + 2\tilde{\psi}_{1z}\tilde{u}_1 - \bar{u}_{1t}\phi - \tilde{v}^r\bar{u}_1\phi_r \\ &\quad - \phi\tilde{v}^z\bar{u}_{1z} + \nu\Delta(\bar{u}_1\phi),\end{aligned} \tag{5.25}$$

where we have used Δ to denote the modified Laplacian operator defined by

$$\Delta w = w_{zz} + w_{rr} + \frac{3w_r}{r} \equiv w_{zz} + \Delta_r w.$$

On the other hand, we know that $\bar{u}_1$ satisfies the one-dimensional model equation

$$\bar{u}_{1t} + 2\bar{\psi}_1\bar{u}_{1z} = \nu\bar{u}_{1zz} + 2\bar{\psi}_{1z}\bar{u}_1. \tag{5.26}$$

Multiplying (5.26) by ϕ and subtracting the resulting equation from (5.25), we have

$$\begin{aligned} &u_{1t} + \tilde{v}^r u_{1r} + \tilde{v}^z u_{1z} \\ &\quad = \nu \Delta u_1 + 2(\tilde{\psi}_{1z}\tilde{u}_1 - \phi\bar{\psi}_{1z}\bar{u}_1) - \tilde{v}^r \bar{u}_1 \phi_r \\ &\quad - \phi([r\phi_r + 2(\phi - 1)]\bar{\psi}_1 + v^z)\bar{u}_{1z} + \nu \bar{u}_1 \Delta_r \phi. \end{aligned} \tag{5.27}$$

Similarly, we obtain

$$\begin{aligned} &\omega_{1t} + \tilde{v}^r \omega_{1r} + \tilde{v}^z \omega_{1z} \\ &\quad = \nu \Delta \omega_1 + ((u_1 + \bar{u}\phi)_z^2 - \bar{u}_{1z}^2 \phi) - \tilde{v}^r \bar{\omega}_1 \phi_r \\ &\quad - \phi([r\phi_r + 2(\phi - 1)]\bar{\psi}_1 + v^z)\bar{\omega}_{1z} + \nu \bar{\omega}_1 \Delta_r \phi. \end{aligned} \tag{5.28}$$

We divide the analysis into two parts. The first part is devoted to estimates of the velocity equation. The second part is devoted to estimates of the vorticity equation.

Part I: Estimates for the Velocity Equation

First we will present our analysis for the velocity equation.

Multiply (5.27) by u_1^3 and integrate over $[0, 1] \times [0, \infty)$. Using the incompressibility condition

$$(r\tilde{v}^r)_r + (r\tilde{v}^z)_z = 0,$$

we get

$$\begin{aligned} &\frac{1}{4}\frac{d}{dt}\int u_1^4 r\,dr\,dz \\ &\quad \le -\frac{3\nu}{4}\int |\nabla(u_1^2)|^2 r\,dr\,dz \\ &\qquad + 2\int u_1^3(\tilde{\psi}_{1z}\tilde{u}_1 - \phi\bar{\psi}_{1z}\bar{u}_1) r\,dr\,dz - \int \tilde{v}^r \bar{u}_1 \phi_r u_1^3 r\,dr\,dz \\ &\qquad - \int \phi([r\phi_r + 2\phi(\phi - 1)]\bar{\psi}_1 + v^z)\bar{u}_{1z} u_1^3 r\,dr\,dz \\ &\qquad + \nu \int \bar{u}_1 (\Delta_r \phi) u_1^3 r\,dr\,dz \\ &\quad \equiv -\frac{3\nu}{4}\int |\nabla(u_1^2)|^2 r\,dr\,dz + \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}, \end{aligned} \tag{5.29}$$

where we have used the fact that

$$\begin{aligned} \int u_1^3 \Delta u_1 r\,dr\,dz &= \int u_1^3 \left(u_{1zz} + \frac{(ru_{1r})_r}{r} + \frac{2u_{1r}}{r}\right) r\,dr\,dz \\ &= -\frac{3}{2}\int (u_1^2 u_{1z}^2 + u_1^2 u_{1r}^2) r\,dr\,dz + 2\int u_1^3 u_{1r}\,dr\,dz \end{aligned}$$

$$= -\frac{3}{4}\int \left[((u_1^2)_z)^2 + ((u_1^2)_r)^2\right] r\,dr\,dz - \frac{1}{2}\int u_1^4(0,z,t)dz$$

$$\leq -\frac{3}{4}\int \left[((u_1^2)_z)^2 + ((u_1^2)_r)^2\right] r\,dr\,dz. \tag{5.30}$$

In the following, we will estimate the right-hand side of (5.29) term by term.

Estimate for the I Term

Using (5.2)–(5.4), we have

$$\mathrm{I} = 2\int u_1^3(\psi_{1z}u_1 + \phi\bar{\psi}_{1z}u_1 + \phi\bar{u}_1\psi_{1z} + (\phi^2-\phi)\bar{\psi}_{1z}\bar{u}_1) r\,dr\,dz$$

$$\equiv \mathrm{I}_a + \mathrm{I}_b + \mathrm{I}_c + \mathrm{I}_d. \tag{5.31}$$

Using the Hölder inequality, we have

$$\mathrm{I}_a \leq 2\|\psi_{1z}\|_{L^2}\,\|f\|_{L^4}^2.$$

Note that

$$\|\psi_{1z}\|_{L^2}^2 = \int \psi_{1z}^2 r\,dr\,dz = \int \psi_{1z}^2\, d\left(\frac{r^2}{2}\right)dz$$

$$= -\frac{1}{2}\int r\psi_{1z}\psi_{1zr} r\,dr\,dz$$

$$\leq \frac{1}{2}\|r\psi_{1z}\|_{L^2}\,\|\psi_{1zr}\|_{L^2} \leq \frac{A}{M^{1/2}}\|\psi_{1zr}\|_{L^2}, \tag{5.32}$$

where we have used $r\psi_{1z} = \psi_z$ and (5.16) to obtain

$$\|r\psi_{1z}\|_{L^2} = \|\psi_z\|_{L^2} \leq \|\mathbf{u}\|_{L^2} \leq \frac{A}{M^{1/2}}.$$

On the other hand, using the Sobolev interpolation inequality, we have

$$\|f\|_{L^4} \leq \|f\|_{L^2}^{1/4}\|\nabla f\|_{L^2}^{3/4}.$$

This implies that

$$\mathrm{I}_a \leq 2\|\psi_{1z}\|_{L^2}\,\|f\|_{L^4}^2 \leq 2\,\frac{A^{1/2}}{M^{1/4}}\,HE^{3/2} + 2\,\frac{(c_2C_0)^{1/2}A}{M^{11/8}}\,H^{1/2}E^{3/2},$$

where $c_2 = \|\Delta_r\phi_0\|_{L^\infty}$, and we have used

$$\|\psi_{1zr}\|_{L^2} \leq \|\omega_1\|_{L^2} + \frac{c_2C_0A}{M^{9/4}}, \tag{5.33}$$

which we prove in Appendix B.

The estimate for I_b follows from (5.9):

$$\mathrm{I}_b \leq 2C_0\,\frac{A}{M}\int u_1^4 r\,dr\,dz \leq 2C_0\,\frac{A}{M}\,H^2.$$

As for I_c, we use (5.9), (5.32), and the Hölder inequality to obtain

$$
\begin{aligned}
\mathrm{I}_c &\le 2C_0 \frac{A}{M} \|\psi_{1z}\|_{L^2} \|f\|_{L^3}^{3/2} \\
&\le 2C_0 \frac{A}{M} \frac{(A)^{1/2}}{M^{1/4}} \|\psi_{1zr}\|_{L^2}^{1/2} \|f\|_{L^2}^{3/4} \|\nabla f\|_{L^2}^{3/4} \\
&\le \frac{2C_0(A)^{3/2}}{M^{5/4}} H^{5/4} E^{3/4} + \frac{2\sqrt{c_2}C_0^{3/2}A^2}{M^{19/8}} H^{3/4} E^{3/4}, \qquad (5.34)
\end{aligned}
$$

where we have used (5.33) and the Sobolev interpolation inequality

$$
\|f\|_{L^3} \le c_0 \|f\|_{L^2}^{1/2} \|\nabla f\|_{L^2}^{1/2}. \qquad (5.35)
$$

Finally, we use (5.9) and the Hölder inequality that

$$
\begin{aligned}
\mathrm{I}_d &\le 2C_0^2 \frac{A^2}{M^2} \int_{r \le R_0} |u_1|^3 r\,dr\,dz \le 2C_0^2 \frac{A^2}{M^2} \left(\int u_1^4 r\,dr\,dz \right)^{3/4} R_0^{2/4} \\
&\le \frac{2C_0^2 A^2}{M^{2-1/8}} H^{3/2}. \qquad (5.36)
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
\mathrm{I} \le{}& 2\frac{A^{1/2}}{M^{1/4}} H E^{3/2} + 2\frac{(c_2C_0)^{1/2}A}{M^{11/8}} H^{1/2}E^{3/2} \\
&+ 2C_0 \frac{A}{M} H^2 + \frac{2C_0(A)^{3/2}}{M^{5/4}} H^{5/4}E^{3/4} \qquad (5.37) \\
&+ \frac{2\sqrt{c_2}C_0^{3/2}A^2}{M^{19/8}} H^{3/4}E^{3/4} + \frac{2C_0^2A^2}{M^{2-1/8}} H^{3/2}.
\end{aligned}
$$

Estimate for the II Term

Using (5.9), (5.16), and the Hölder inequality, we have

$$
\begin{aligned}
\mathrm{II} \le \frac{c_1C_0A}{MR_0} \int |\tilde{v}^r|\,|u_1^3| r\,dr\,dz &\le \frac{c_1C_0A}{MR_0} \|\tilde{v}^r\|_{L^2} \|f\|_{L^3}^{3/2} \\
&\le \frac{c_1C_0A^2}{M^{3/2}R_0} \|f\|_{L^2}^{3/4} \|\nabla f\|_{L^2}^{3/4} \qquad (5.38) \\
&\le \frac{c_1C_0A^2}{M^{7/4}} H^{3/4}E^{3/4},
\end{aligned}
$$

where $c_1 = \|(\phi_0)_r\|_{L^\infty}$, and we have used the Sobolev interpolation inequality (5.35).

Estimate for the III Term

Using (5.8) and (5.10), and following the same steps as in our estimate for the I_d term and the II term, we get

$$\begin{aligned}
\mathrm{III} &\le (2+c_1)\frac{C_0^2 A^2}{M^2}\int_{r\le R_0} |u_1^3| r\,dr\,dz + C_0 A\int |v^z|\,|u_1^3| r\,dr\,dz \\
&\le (2+c_1)\frac{C_0^2 A^2}{M^{2-1/8}} H^{3/2} + C_0 A\|v^z\|_{L^2}\,\|f\|_{L^3}^{3/2} \\
&\le (2+c_1)\frac{C_0^2 A^2}{M^{2-1/8}} H^{3/2} + \frac{C_0 A^2}{M^{1/2}} H^{3/4} E^{3/4}. \qquad (5.39)
\end{aligned}$$

Estimate for the IV Term

Using (5.9) and the Hölder inequality, we have

$$\begin{aligned}
\mathrm{IV} \le \frac{\nu c_2 C_0 A}{M R_0^2}\int_{r\le R_0} |u_1|^3\, r\,dr\,dz &\le \frac{\nu c_2 C_0 A}{M^{3/2}} H^{3/2} R_0^{2/4} \\
&\le \frac{\nu c_2 C_0 A}{M^{3/2-1/8}} H^{3/2}. \qquad (5.40)
\end{aligned}$$

Part II: Estimates for the Vorticity Equation

Next, we will present our analysis for the vorticity equation. Multiplying (5.28) by ω_1 and integrating over $[0,1]\times[0,\infty)$, we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int \omega_1^2 r\,dr\,dz \le -\nu\int |\nabla\omega_1|^2 r\,dr\,dz &+ \int (\tilde{u}_1^2 - \bar{u}_1^2\phi)_z\omega_1 r\,dr\,dz \\
&- \int \tilde{v}^r\bar{\omega}_1\omega_1\phi_r r\,dr\,dz \\
&- \int \phi([r\phi_r + 2(\phi-1)]\bar{\psi}_1 + v^z)\bar{\omega}_{1z}\omega_1 r\,dr\,dz \\
&+ \nu\int \bar{\omega}_1\Delta_r\phi\omega_1 r\,dr\,dz \\
\equiv -\nu\int |\nabla\omega_1|^2 r\,dr\,dz &+ \bar{\mathrm{I}} + \overline{\mathrm{II}} + \overline{\mathrm{III}} + \overline{\mathrm{IV}}, \qquad (5.41)
\end{aligned}$$

where $\Delta_r\phi = \phi_{rr} + 3\phi_r/r$.

We will estimate the terms $\bar{\mathrm{I}}$ to $\overline{\mathrm{IV}}$ one by one.

Estimate for the $\bar{\mathrm{I}}$ Term

Using (5.2) and (5.9) and integration by parts, we have

$$\begin{aligned}
\bar{\mathrm{I}} &= -\int (\tilde{u}_1^2 - \bar{u}_1^2\phi)\omega_{1z} r\,dr\,dz \\
&= -\int (u_1^2 + 2\bar{u}_1\phi u_1 + (\phi^2-\phi)\bar{u}_1^2)\omega_{1z} r\,dr\,dz \\
&\le \int u_1^2|\omega_{1z}|r\,dr\,dz + \frac{2C_0A}{M}\int_{r\le R_0} |u_1|\,|\omega_{1z}|r\,dr\,dz \\
&\quad + \frac{C_0^2A^2R_0}{M^2}\|\omega_{1z}\|_{L^2} \\
&\le \left(\left(\int u_1^4 r\,dr\,dz\right)^{1/2} \right. \\
&\quad \left. + \frac{2C_0A}{M}\left(\int_{r\le R_0} u_1^2 r\,dr\,dz\right)^{1/2} + \frac{C_0^2A^2}{M^{7/4}}\right)\|\omega_{1z}\|_{L^2}.
\end{aligned} \tag{5.42}$$

Let $\Gamma = r\tilde{u}$. It is easy to show that Γ satisfies the following evolution equation (see also [20]):

$$\Gamma_t + \tilde{v}^r\Gamma_r + \tilde{v}^z\Gamma_z = \nu\left(\Gamma_{zz} + \Gamma_{rr} - \frac{\Gamma_r}{r}\right).$$

Moreover, for $\tilde{u}$ smooth, we have $\Gamma|_{r=0} = 0$. Thus, Γ has a maximum principle, i.e.,

$$\|\Gamma\|_{L^\infty} \le \|\Gamma_0\|_{L^\infty} \le c_0.$$

This implies that

$$|r^2u_1| \le |r\tilde{u}| + r^2|\phi\bar{u}_1| \le c_0 + R_0^2\frac{C_0}{M} \le c_0 + \frac{C_0}{M^{1/2}} \le \tilde{c}_0.$$

Therefore, we obtain

$$\begin{aligned}
\int u_1^4 r\,dr\,dz &= \int (u_1^2)^2\,d\left(\frac{r^2}{2}\right)dz \\
&= -\int r u_1^2 (u_1^2)_r r\,dr\,dz \\
&\le \left(\int r^2u_1^4 r\,dr\,dz\right)^{1/2}\|\nabla f\|_{L^2}
\end{aligned}$$

$$\begin{aligned}&\le \|ru_1\|_{L^2}^{1/2}\left(\int r^2u_1^6 r\,dr\,dz\right)^{1/4}\|\nabla f\|_{L^2}\\&\le \tilde{c}_0^{1/4}\frac{A^{1/2}}{M^{1/4}}\left(\int u_1^5 r\,dr\,dz\right)^{1/4}\|\nabla f\|_{L^2}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\left(\int u_1^5 r\,dr\,dz\right)^{1/4} &\le \left(\int u_1^4 r\,dr\,dz\right)^{1/8}\|f\|_{L^3}^{3/8}\\&\le \left(\int u_1^4 r\,dr\,dz\right)^{1/8}\|f\|_{L^2}^{3/16}\|\nabla f\|_{L^2}^{3/16}.\end{aligned}$$

Combining the above estimates, we obtain

$$\left(\int u_1^4 r\,dr\,dz\right)^{1/2} \le \frac{\tilde{c}_0^{1/7}A^{2/7}}{M^{1/7}}\,H^{3/28}E^{19/28}. \tag{5.43}$$

Thus, we obtain

$$\begin{aligned}\bar{\mathrm{I}} &\le \Bigg(\frac{\tilde{c}_0^{1/7}A^{2/7}}{M^{1/7}}\,H^{3/28}E^{19/28}\\&\qquad + \frac{2C_0A}{M}\,R_0^{1/2}\left(\int u_1^4 r\,dr\,dz\right)^{1/4} + \frac{C_0^2A^2}{M^{7/4}}\Bigg)\|\omega_{1z}\|_{L^2}\\&\le \frac{\tilde{c}_0^{1/7}A^{2/7}}{M^{1/7}}\,H^{3/28}E^{47/28} + \frac{2C_0A}{M^{7/8}}\,H^{1/2}E + \frac{C_0^2A^2}{M^{7/4}}\,E.\end{aligned} \tag{5.44}$$

Estimate for the $\overline{\mathrm{II}}$ Term

Using (5.10), (5.16), and the Hölder inequality, we get

$$\begin{aligned}\overline{\mathrm{II}} &\le \frac{c_1C_0A}{R_0}\|\tilde{v}^r\|_{L^2}\,\|\omega_1\|_{L^2}\\&\le \frac{c_1C_0A}{R_0}\,\frac{A}{M^{1/2}}\|\omega_1\|_{L^2} \le \frac{c_1C_0A^2}{M^{3/4}}\,H.\end{aligned} \tag{5.45}$$

Estimate for the $\overline{\mathrm{III}}$ Term

Integration by parts gives

$$\begin{aligned}\overline{\mathrm{III}} &= \int \phi([r\phi_r + 2(\phi-1)]\bar{\psi}_{1z} + v_z^z)\bar{\omega}_1\omega_1 r\,dr\,dz\\&\quad + \int \phi([r\phi_r + 2(\phi-1)]\bar{\psi}_1 + v^z)\bar{\omega}_1\omega_{1z} r\,dr\,dz.\end{aligned} \tag{5.46}$$

We first study the term $\int \phi v_z^z\bar{\omega}_1\omega_1 r\,dr\,dz$. Note that using (5.10), we have

$$\left|\int \phi v_z^z\bar{\omega}_1\omega_1 r\,dr\,dz\right| \le C_0A\|(\phi v^z)_z\|_{L^2}\,\|\omega_1\|_{L^2}.$$

On the other hand, we have by the Sobolev interpolation inequality that

$$\begin{aligned}\|(\phi v^z)_z\|_{L^2} &\le \|\phi v^z\|_{L^2}^{1/2}\,\|\nabla(\phi v^z)_z\|_{L^2}^{1/2}\\ &\le \frac{\sqrt{A}}{M^{1/4}}\|\nabla(r\phi v_1^z)_z\|_{L^2}^{1/2}\\ &\le 2\,\frac{\sqrt{A}}{M^{1/4}}\,R_0^{1/4}\|\nabla\omega_1\|_{L^2}^{1/2} + 2\,\frac{\sqrt{c_2C_0}A}{M^{7/8-1/16}},\end{aligned}$$

where we have used

$$\|\nabla(v_1^z)_z\|_{L^2} \le \|\nabla\omega_1\|_{L^2} + \frac{c_2C_0A}{M^{5/4}},$$

which we prove in Appendix B.

Thus we can use (5.9) to show that

$$\begin{aligned}\overline{\text{III}} &\le (2+c_1)\frac{C_0^2A^2}{M}\,R_0\|\omega_1\|_{L^2} + \left(\frac{2C_0A^{3/2}}{M^{1/4-1/16}}\,HE^{1/2} + \frac{2\sqrt{c_2}C_0^{3/2}A^2}{M^{7/8-1/16}}\,H\right)\\ &\quad + (2+c_1)\frac{C_0^2A^2}{M^2}\,R_0\|\omega_{1z}\|_{L^2} + C_0A\|v^z\|_{L^2}\,\|\omega_{1z}\|_{L^2}\\ &\le (2+c_1+2\sqrt{c_2})\frac{C_0^2A^2}{M^{3/4}}\,H + \frac{2C_0A^{3/2}}{M^{1/4-1/16}}\,HE^{1/2} + (3+c_1)\frac{C_0^2A^2}{M^{1/2}}\,E.\end{aligned}\tag{5.47}$$

Estimate for the $\overline{\text{IV}}$ Term

Let $g(z,t) = \int_0^z \bar{\omega}_1(\eta,t)\,d\eta$. Then we have $g_z = \bar{\omega}_1$ and $|g| \le C_0A/M$. Thus we have

$$\begin{aligned}\overline{\text{IV}} = \nu\int \bar{\omega}_1(\Delta_r\phi)\omega_1 r\,dr\,dz &= \nu\int g_z(\Delta_r\phi)\omega_1 r\,dr\,dz\\ &= -\nu\int g(\Delta_r\phi)\omega_{1z} r\,dr\,dz\\ &\le \frac{\nu c_2C_0A}{MR_0^2}\int_{r\le R_0} |\omega_{1z}|r\,dr\,dz\\ &\le \frac{\nu c_2C_0A}{MR_0^2}\,R_0\|\omega_{1z}\|_{L^2} \le \frac{\nu c_2C_0A}{M^{5/4}}\,E.\end{aligned}\tag{5.48}$$

By adding the estimates for $\int u_1^4 r\,dr\,dz$ to those for $\int \omega_1^2 r\,dr\,dz$, we obtain an estimate for $\frac{d}{dt}H^2$. Note that except for the diffusion terms, each term in our estimates from I to $\overline{\text{IV}}$ can be bounded by

$$\frac{\nu}{16}\,E^2 + \frac{\epsilon}{16}\,g(H),$$

where $g(H)$ is a polynomial of H with positive rational exponents and positive coefficients that depend on C_0, A, and ν, and $\epsilon = 1/M^\gamma$ for some $\gamma > 0$. Putting all the estimates together, we get

$$\frac{d}{dt} H^2 \le -\frac{\nu}{2} E^2 + \epsilon g(H) \le -\frac{\nu}{2} H^2 + \epsilon g(H), \tag{5.49}$$

since $H \le E$.

For given $A > 1$, $C_0 > 1$, and $\nu > 0$, we can choose M large enough so that

$$-\frac{\nu}{2} + \epsilon g(1) \le 0.$$

Thus, if the initial condition for u_1, ω_1, and ψ_1 are chosen such that $H(0) \le 1$, then we must have

$$H(t) \le 1 \quad \text{for all } t > 0.$$

Using this a priori estimate on $H(t)$, we can easily follow the standard argument to prove the global regularity of ψ_1, u_1, and ω_1 in higher-order norms. This completes the proof of Theorem 5.2. □

Appendix A

In this appendix, we prove the following result for the generalized ODE system:

THEOREM A.1 *Assume that $\tilde{u}_0 \neq 0$ and $d \ge 1$. Then the solution $(\tilde{u}(t), \tilde{v}(t))$ of the ODE system* (3.8)–(3.9) *exists for all times. Moreover, we have*

$$\lim_{t\to+\infty} \tilde{u}(t) = 0, \quad \lim_{t\to+\infty} \tilde{v}(t) = 0. \tag{A.1}$$

PROOF: We first make a change of variables into the polar coordinates[2]

$$\tilde{v} = r\cos\theta, \quad \tilde{u} = r\sin\theta. \tag{A.2}$$

Substituting the above change of variables into the ODE system, we obtain

$$r'\cos\theta - r(\sin\theta)\theta' = r^2\sin^2\theta - r^2\cos^2\theta, \tag{A.3}$$

$$r'\sin\theta + r(\cos\theta)\theta' = -dr^2\cos\theta\sin\theta. \tag{A.4}$$

From the above equations, we can easily derive

$$r' = -r^2\cos\theta(\cos^2\theta + (d-1)\sin^2\theta), \tag{A.5}$$

$$\theta' = -r\sin\theta((d-1)\cos^2\theta + \sin^2\theta). \tag{A.6}$$

Note that if $\tilde{u}_0 > 0$, then $\tilde{u}(t) > 0$ as long as $|\int_0^t \tilde{v}(s)ds| < \infty$. Similarly, if $\tilde{u}_0 < 0$, then $\tilde{u}(t) < 0$. Thus, if the solution starts from the upper (or lower) half-plane, it will stay in the upper (or lower) half-plane. Without loss of generality, we may consider the solution starting from the upper half-plane. It follows from (A.6) that $\theta' \le 0$ since $d \ge 1$ and $r \ge 0$. Therefore, $\theta(t)$ is monotonically decreasing.

[2] This proof was inspired by a discussion with Mr. Mulin Cheng.

On the other hand, $\theta(t)$ is bounded from below by zero. As a result, the limit of $\theta(t)$ as $t \to \infty$ exists. Let us denote the limiting value as $\overline{\theta}$. Clearly, we must have

$$\lim_{t\to\infty} \theta' = 0, \quad \lim_{t\to\infty} \theta = \overline{\theta}. \tag{A.7}$$

First, we consider the case in which the solution starts from the second quarter (y-axis included). We claim that this solution must cross the y-axis into the first quarter. If the solutions stay in the second quarter forever, then $\overline{\theta}$ must be no less than $\pi/2$. From (A.6) and (A.7), we know that

$$\lim_{t\to\infty} r = 0. \tag{A.8}$$

However, from (A.5), we have $r' \geq 0$, which contradicts (A.8). The contradiction implies that the solution must cross the y-axis at a later time.

Now we only need to consider the case in which the solution starts from the first quarter since the system is autonomous. Since $\theta(t)$ decreases monotonically, we obtain

$$r' \leq -\cos^3(\theta_0) r^2, \tag{A.9}$$

where we have used the fact that $d \geq 1$ and $\cos^2\theta + (d-1)\sin^2\theta \geq \cos^2\theta$. Solving the above ODE inequality gives

$$r(t) \leq \frac{r_0}{1 + r_0(\cos^3\theta_0)t}. \tag{A.10}$$

Thus, we conclude that

$$\lim_{t\to\infty} r(t) = 0. \tag{A.11}$$

To determine the limiting angle $\overline{\theta}$, we use the fact that

$$\tan(\overline{\theta}) = \lim_{t\to\infty} \left(\frac{\tilde{u}'}{\tilde{v}'}\right) = -\frac{d\tan(\overline{\theta})}{\tan^2(\overline{\theta}) - 1}.$$

Since $d \geq 1$, we conclude that $\overline{\theta} = 0$, which implies

$$\lim_{t\to\infty} \theta(t) = 0. \tag{A.12}$$

This completes the proof of Theorem A.1. □

Appendix B

In this appendix, we prove the next two estimates, which relate the L^2 norm of the derivatives of ψ to that of ω_1:

$$\|\psi_{1zz}\|_{L^2} + \|\psi_{1rz}\|_{L^2} + \|\psi_{1rr}\|_{L^2} + \left\|\frac{\psi_{1r}}{r}\right\|_{L^2} \leq \|w_1\|_{L^2} + \frac{c_2 C_0 A}{M^{9/4}} \tag{B.1}$$

and

$$\|\nabla v_{1z}^z\|_{L^2} = \left\|\nabla\left(\frac{2\psi_{1z}}{r} + \psi_{1rz}\right)\right\|_{L^2} \le \|\nabla w_1\|_{L^2} + \frac{c_2 C_0 A}{M^{5/4}}, \tag{B.2}$$

where $c_2 = \|\Delta_r \phi_0\|_{L^\infty}$.

PROOF: From the definition, we have

$$-\Delta\tilde{\psi}_1 = \tilde{w}_1.$$

Using the definition of $\tilde{\psi}_1$ and $\tilde{w}_1$, we can rewrite the above equation as

$$-w_1 = \Delta\psi_1 + (\Delta_r\phi)\bar{\psi}_1. \tag{B.3}$$

Multiplying (B.3) by ψ_{1zz} and integrating over $[0,1]\times[0,\infty)$, we obtain

$$\begin{aligned}
\|w_1\|_{L^2}\|\psi_{1zz}\|_{L^2} &\ge \int (\Delta\psi_1\psi_{1zz} - (\Delta_r\phi)\bar{\psi}_1\psi_{1zz}) r\,dr\,dz \\
&\ge \int (\psi_{1zz}^2 + \psi_{1rz}^2) r\,dr\,dz - 2\int \psi_{1rz}\psi_{1z}\,dr\,dz \\
&\quad - \frac{c_2 C_0 A}{M^2 R_0^2}\int |\psi_{1zz}| r\,dr\,dz \\
&\ge \int (\psi_{1zz}^2 + \psi_{1rz}^2) r\,dr\,dz + \int_0^1 \psi_{1z}^2(0,z,t)dz \\
&\quad - \frac{c_2 C_0 A}{M^2 R_0}\|\psi_{1zz}\|_{L^2},
\end{aligned} \tag{B.4}$$

where we have used (5.8). This implies that

$$\|\psi_{1zz}\|_{L^2} + \|\psi_{1rz}\|_{L^2} \le \|w_1\|_{L^2} + \frac{c_2 C_0 A}{M^{9/4}}. \tag{B.5}$$

Next, we multiply (B.3) by $\Delta_r\psi_1$ and integrate over $[0,1]\times[0,\infty)$. We obtain by using a similar argument that

$$\begin{aligned}
\|w_1\|_{L^2}\,\|\Delta_r\psi_1\|_{L^2} &\ge \int (\Delta\psi_1\Delta_r\psi_1 - (\Delta_r\phi)\bar{\psi}_1\Delta_r\psi_1) r\,dr\,dz \\
&\ge \int [(\Delta_r\psi_1)^2 + \psi_{1rz}^2] r\,dr\,dz - \frac{c_2 C_0 A}{M^2 R_0}\|\Delta_r\psi_1\|_{L^2}.
\end{aligned} \tag{B.6}$$

On the other hand, we note that

$$\begin{aligned}
\int (\Delta_r\psi_1)^2 r\,dr\,dz &= \int \left(\psi_{1rr}^2 + 9\frac{\psi_{1r}}{r^2}\right) r\,dr\,dz + 6\int \psi_{1r}\psi_{1rr}\,dr\,dz \\
&= \int \left(\psi_{1rr}^2 + 9\frac{\psi_{1r}^2}{r^2}\right) r\,dr\,dz - 3\int_0^1 \psi_{1r}^2(0,z,t)dz \\
&= \int \left(\psi_{1rr}^2 + 9\frac{\psi_{1r}^2}{r^2}\right) r\,dr\,dz,
\end{aligned}$$

where we have used the fact that $\psi_{1r}(0,z,t) = 0$ since ψ_{1r} is odd in r. Thus we obtain

$$\|\psi_{1rr}\|_{L^2} + 3\left\|\frac{\psi_{1r}}{r}\right\|_{L^2} \le \|\Delta_r \psi_1\| \le \|w_1\|_{L^2} + \frac{c_2 C_0 A}{M^{9/4}}. \tag{B.7}$$

Combining estimate (B.6) with (B.7) gives the desired estimate (B.1). Similarly, we can prove (B.2). □

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