On the Finite-Time Blowup of a One-Dimensional Model for the Three-Dimensional Axisymmetric Euler Equations

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Abstract

In connection with the recent proposal for possible singularity formation at the boundary for solutions of three-dimensional axisymmetric incompressible Euler's equations (Luo and Hou, *Proc. Natl. Acad. Sci. USA* (2014)), we study models for the dynamics at the boundary and show that they exhibit a finite-time blowup from smooth data. © 2017 Wiley Periodicals, Inc.

1 Introduction

In this paper we study a one-dimensional model introduced in [20] (also see [21]) in connection with incompressible Euler equations. The study of one-dimensional models for hydrodynamical equations has a long history, going back to the works of Burgers [4] and Hopf [17]. A slightly different type of one-dimensional model, originating, as far as we know, with the paper of [8], was introduced to illustrate effects of vortex stretching. The equations studied in this paper are related to these works, and we will link some of the models to the boundary behavior of fluid flows. The link is not perfect but useful.

Important modifications of the original model of [8] were proposed by [12, 13] and somewhat differently motivated work of [10]. All these works include modeling of two important features of incompressible flow:

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(i) the vorticity transport (either as a scalar, as in two-dimensional Euler, or as a vector field, as in three-dimensional Euler, with the vector field transport also covering the vortex stretching), and
(ii) the Biot-Savart law, which expresses the velocity field that transports the vorticity in terms of the vorticity itself.

In the present paper the "vorticity" ω will be mainly considered as a scalar function $\omega(x,t)$ of time and a one-dimensional variable x on either the real line $\mathbb{R}$ or the one-dimensional circle $\mathbb{S}^1$, the latter case corresponding to periodic boundary conditions. The Biot-Savart law will be taken as

$$u_x = H\omega, \tag{1.1}$$

where H is the Hilbert transform. In this setting one can consider the equation

$$\omega_t + u\omega_x = 0, \tag{1.2}$$

which has many properties similar to the two-dimensional Euler. We will see below that a natural (approximate) interpretation of this model is in terms of the dynamics at the boundary for the full two-dimensional Euler flows in smooth domains with boundaries.

Model (1.1)–(1.2) is not studied in the works mentioned above. However, one can prove that it shares many properties with the two-dimensional Euler equation, including the global existence and uniqueness of solutions (in suitable classes) for L^∞ initial data, as in [28], and the double exponential growth of ω_x for certain smooth data, similar to [19]. These topics will be addressed elsewhere. Here we will focus on singularity formation for natural extensions of the "two-dimensional Euler model," which in some sense take us from two-dimensional Euler to three-dimensional axisymmetric Euler with swirl or two-dimensional Boussinesq.

In [10] the Biot-Savart law is taken as

$$u = H\omega.$$

With this Biot-Savart law, equation (1.2) is more akin to the surface quasi-geostrophic (SQG), which was proposed as a model for three-dimensional Euler in [9] under slightly different terminology. While the singularity formation for the SQG equation remains open, it was shown in [10] that the one-dimensional model can develop a singularity from smooth initial data in finite time.

In [12,13], the Biot-Savart law is taken as in (1.1), but the vorticity is considered as a vector field $\omega(x,t)\frac{\partial}{\partial x}$ and transported by the velocity field $u(x,t)$ as such (similarly to what we have for three-dimensional Euler), with the transport equation given by

$$\omega_t + u\omega_x = u_x\omega. \tag{1.3}$$

We can also write it in terms of the usual Lie bracket for vector fields as

$$\omega_t + [u,\omega] = 0, \tag{1.4}$$

just like the three-dimensional Euler. The question of global existence of smooth solutions for smooth initial data for the one-dimensional model (1.4) with (1.1) appears to be open. A generalization of the model (1.3) was studied in [23] and also in [5]; see the table below.

As pointed out in [12], the model considered in [8] can be written as

$$\omega_t = u_x \omega,$$

with the Biot-Savart law (1.1), which is of course the same as (1.3) without the "transport term" (in the scalar sense) $u\omega_x$. As shown by [8], this model can blow up in finite time from smooth data.

In this paper we will mostly study the model obtained from (1.1) and (1.2) by adding an additional variable $\theta = \theta(x,t)$ (which can be thought of as temperature in the two-dimensional Boussinesq context or the square of the swirl component u^θ of the velocity field in the three-dimensional axisymmetric case) and considering equations

$$\omega_t + u\omega_x = \theta_x\,, \tag{1.5a}$$

$$\theta_t + u\theta_x = 0. \tag{1.5b}$$

A discussion of connections to three-dimensional axisymmetric Euler flows with swirl or two-dimensional Boussinesq flows is included in the next section. Aspects of the model have been discussed in [18], and we will refer to the model as the HL model. One of our main results is that this model can exhibit finite-time blowup from smooth initial data.

In addition to this model, we will also discuss its variant where the Biot-Savart law (1.1) is replaced by a simplified version that still captures important features of (1.1) in the situation when $\omega(x,t)$ is odd in x:

$$u(x) \sim -x \int_x^\infty \frac{\omega(y)}{y}\, dy.$$

This model was studied by [6], and we will refer to it as the CKY model.

Our proof of finite-time blowup for the CKY model relies on the local aspect still present in the simplified Biot-Savart law, namely, that $(u(x)/x)_x = \omega(x)/x$. This observation allows us to construct an entropy-like functional to capture the blowup. The proof of finite time blowup for the full HL model is subtler, as the Biot-Savart law is now fully nonlocal. The proof is based, at heart, on a nontrivial and not readily evident sign definite estimate for certain quadratic forms associated with the solution. These estimates are novel and intriguing to us, as they provide a blueprint of what one might have to look for in order to understand the full three-dimensional dynamics.

We summarize the above discussion in a table.

Model	Biot-Savart law	Dynamical equations	Regularity of solutions of the model
2d Euler analogy	$u_x = H\omega$	$\omega_t + u\omega_x = 0$	unique global solutions
[12], 3d Euler analogy	$u_x = H\omega$	$\omega_t + u\omega_x = u_x\omega$	global existence and regularity unknown
[8], analogy of vortex stretching without transport term	$u_x = H\omega$	$\omega_t = u_x\omega$	finite-time blowup from smooth data
[23], a generalized model	$u_x = H\omega$	$\omega_t + au\omega_x = u_x\omega$	finite-time blowup when $a < 0$, [5]
[10], SQG analogy	$u = H\omega$	$\omega_t + u\omega_x = 0$	finite-time blowup from smooth data
HL model [18], 2d Boussinesq / 3d axisymmetric Euler analogy	$u_x = H\omega$	$\omega_t + u\omega_x = \theta_x$, $\theta_t + u\theta_x = 0$	finite-time blowup from smooth data, the main new result of this paper
CKY model [6], simplified HL model	$u(x) = -x\int_x^\infty \frac{\omega(y)}{y}\,dy$	$\omega_t + u\omega_x = \theta_x$, $\theta_t + u\theta_x = 0$	finite-time blowup from smooth data

2 Motivation for the Models

In [20] the authors study three-dimensional axisymmetric flow of the incompressible Euler's equation with a configuration roughly as in Figure 2.1

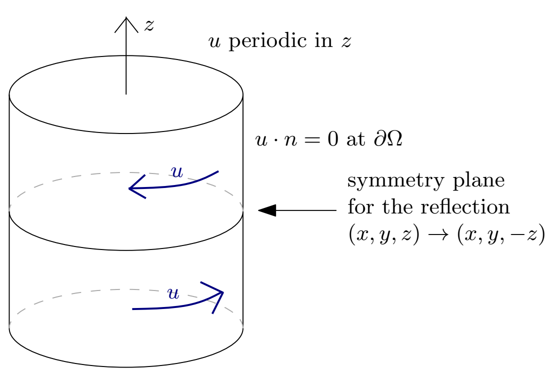

FIGURE 2.1.

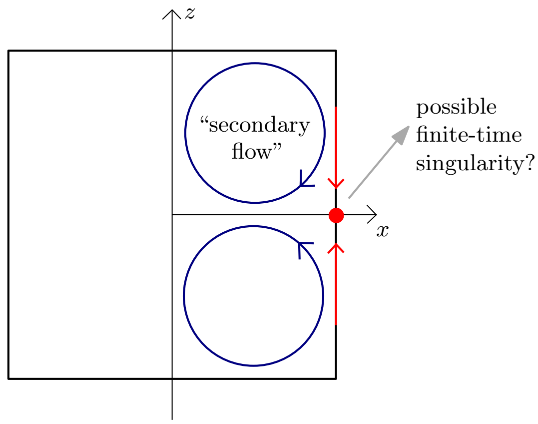

FIGURE 2.2.

In accordance with principles governing the so-called secondary flows [24] (see also the paper by [14]), the initial condition leads a (schematic) picture in the xz-plane shown in Figure 2.2, in which we also indicate the point where a conceivable finite-time singularity (or at least an extremely strong growth of vorticity) is observed numerically. The singularity was not predicted by the classics, who were mostly concerned with slightly viscous flows. In the presence of viscosity the scenario of singularity formation discussed here can be ruled out, due to well-known regularity criteria for the Navier-Stokes equations near boundaries, such as [25] or [16].

One of the standard forms of the axisymmetric Euler equations in the usual cylindrical coordinates (r, θ, z) is

$$\left(\frac{\omega^\theta}{r}\right)_t + u^r\left(\frac{\omega^\theta}{r}\right)_r + u^z\left(\frac{\omega^\theta}{r}\right)_z = \left(\frac{(u^\theta)^2}{r^2}\right)_z, \tag{2.1a}$$

$$(ru^\theta)_t + u^r(ru^\theta)_r + u^z(ru^\theta)_z = 0, \tag{2.1b}$$

with the understanding that u^r, u^z are given from ω^θ via the Biot-Savart law (which follows from the equations curl $u = \omega$, div $u = 0$, together with the boundary condition $u \cdot n = 0$ and suitable decay at ∞ or, respectively, periodicity in z). See, e.g., [22] for more details. We will link these equations to the system (1.5).

A somewhat similar scenario can be considered for the two-dimensional Boussinesq system in a half-space $\Omega = \{(x, y) \in \mathbb{R} \times (0, \infty)\}$ (or in a flat half-cylinder $\Omega = \mathbb{S}^1 \times (0, \infty)$), which we will write in the vorticity form:

$$\omega_t + u^x\omega_x + u^y\omega_y = \theta_x, \tag{2.2a}$$

$$\theta_t + u^x\theta_x + u^y\theta_y = 0. \tag{2.2b}$$

Here $u = (u^x, u^y)$ is obtained from ω by the usual Biot-Savart law (which follows from the equations $\operatorname{curl} u = \omega$ and $\operatorname{div} u = 0$, the boundary condition $u \cdot n = 0$ at $\partial\Omega$, and/or suitable decay at ∞), and θ represents the fluid temperature.

It is well-known that this system has properties similar to the axisymmetric Euler, at least away from the rotation axis, with θ playing the role of the square of the swirl component u^θ (in standard cylindrical coordinates) of the velocity field u. For the purpose of comparison with the axisymmetric flow, Figure 2.2 should be rotated by $\frac{\pi}{2}$, after which it resembles the situation in Figure 2.3.

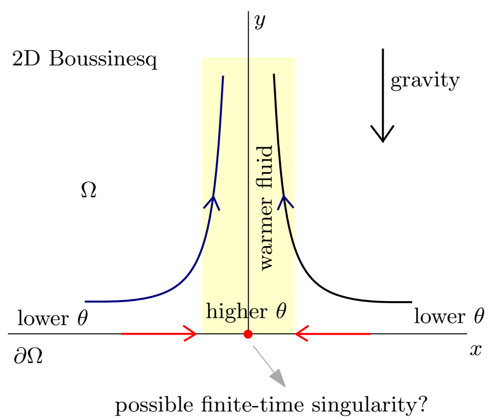

FIGURE 2.3.

We see that both in the three-dimensional axisymmetric case and the two-dimensional Boussinesq the best chance for possible singularity formation seems to be at the points of symmetry at the boundary, and this is where we will focus our attention.

Note that (2.2) restricts well to the boundary $\{(x, y) : y = 0\}$, which can be identified with the real line. At the boundary we can write with a slight abuse of notation $u \sim (u(x), 0)$, and we see that the system would close (as a system on the real line) if we could express the boundary velocity $u(x)$ via the restriction of ω to the boundary.

Similar considerations apply to (2.1) when we restrict the equations to the boundary of Ω given by, say, $r = 1$. The restricted equations will look the same as the restricted Boussinesq equations, after the identifications $x \leftrightarrow z$, $\theta \leftrightarrow (u^\theta)^2$, $\omega \leftrightarrow \omega^\theta$, and $u^x \leftrightarrow u^z$, with the understanding that the Biot-Savart is not quite identical, although the leading terms at the boundary are similar.

To obtain a closed one-dimensional model, we need a way to model the Biot-Savart law. The most natural way to do so is probably to assume that ω is (nearly) constant in y close to the boundary, and discount the influence from the rest of the fluid. Let us assume the thickness of the layer where ω is constant in y is $a > 0$.

We adopt a convention that positive vorticity generates clockwise rotation.[1] Under these assumptions it is easily seen that a reasonable one-dimensional model of the Biot-Savart law (in the case when Ω is the upper half-plane) is given by

$$u(x) = \int_{-\infty}^{\infty} \widetilde{k}(x-y)\omega(y)dy,$$

with the kernel $\widetilde{k}$ determined by

$$\widetilde{k}(x_1) = \int_0^a \frac{\partial}{\partial x_2}\bigg|_{x_2=0} G((x_1, x_2), (0, y_2))dy_2,$$

where

$$G(z, w) = \frac{1}{2\pi}\log|z-w| - \frac{1}{2\pi}\log|z-w^*|, \quad w^* = (w_1, -w_2),$$

is the Green function of the Laplacian in the upper half-plane. A simple calculation gives

$$\widetilde{k}(x) = \frac{1}{\pi}\log\frac{|x|}{\sqrt{x^2+a^2}}.$$

One could work with this kernel, but we will simplify it to

$$k(x) = \frac{1}{\pi}\log|x|. \tag{2.3}$$

This kernel has the same singularity at 0 and gives exactly the Biot-Savart law (1.1). The somewhat unnatural behavior of k for large x will be alleviated by our symmetry assumptions. We see that one-dimensional models discussed in the previous section can be interpreted, to some degree, as the boundary dynamics for two-dimensional flows or three-dimensional axisymmetric flows with swirl. A similar (although somewhat less straightforward) calculation can be carried out in the axisymmetric case, leading again to kernel (2.3), as can be expected from the fact that the leading-order terms in the corresponding elliptic operators are the same.

We acknowledge that the assumptions made in the above derivation of the model do not perfectly capture the situation near the boundary. For example, in the axisymmetric case with the boundary at $r = 1$, instead of being roughly constant in r for r slightly below 1, the solution ω^θ obtained from numerical simulations was observed to exhibit nontrivial variations [20]. Nonetheless, according to the preliminary numerical evidence reported in [21, sec. 5], the solution of the one-dimensional model (1.5) on $\mathbb{S}^1$ appears to develop a singularity in finite time *in much the same way* as the full simulation of the axisymmetric flow. This supports the relevance of the one-dimensional model for the finite-time blowup of the full three-dimensional problem.

[1] This is more convenient for our purposes here than the more usual convention with the opposite sign.

2.1 Statement of the Main Results

The HL model (1.5) has the scaling invariance

$$\omega(x,t) \to \omega(\lambda x,t), \quad u(x,t) \to \frac{1}{\lambda} u(\lambda x,t), \quad \theta(x,t) \to \frac{1}{\lambda} \theta(\lambda x,t), \tag{2.4}$$

and is invariant under the translation $\theta \to \theta + \text{const}$. This suggests that the critical space for the local well-posedness of the model should be

$$(\theta_{0x}, \omega_0) \in L^\infty \times L^\infty \quad \text{or perhaps} \quad (\theta_{0x}, \omega_0) \in \dot{H}^{\frac{1}{2}} \times \dot{H}^{\frac{1}{2}}.$$

It is not clear if the equation is locally well-posed in these spaces.[2] From the well-posedness proofs of the Euler equation in the slightly subcritical cases it should be clear that our system is locally well-posed for

$$(\theta_{0x}, \omega_0) \in H^s \times H^s$$

for any $s > \frac{1}{2}$ (see, e.g., [2] for discussion and further references on subcritical local existence results). We will not study these relatively standard issues here, as our interest is in the breakdown of smooth solutions. Our main result is the following:

THEOREM 2.1. *Both in the periodic case ($x \in \mathbb{S}^1$) and in the real-line case ($x \in \mathbb{R}$) with compactly supported initial data (θ_{0x}, ω_0), one can find smooth initial conditions such that the HL model cannot have a smooth global solution starting from those data.*

Our proof of this statement is indirect, using an argument by contradiction, somewhat similar in spirit to the classical proofs of the blowup in the nonlinear Schrödinger equation [15, 27], based on the virial identity. We do not get any detailed information about the nature of blowup. For the periodic HL model, the main quantity used in our proof is essentially

$$\int_0^{x_0} \frac{\theta(x,t)}{x}\, dx.$$

In view of the methods used in [10] the quantity

$$\int_0^{x_0} \frac{\omega(x,t)}{x}\, dx$$

might look more natural, but we did not find a proof based on this quantity. A key component of our proof are the monotonicity properties of the Biot-Savart kernel, which are important in Lemma 3.4 and Lemma 4.2. Under some assumptions, a good quantity for a relatively simple proof of the blowup for the CKY model is an entropy-type integral defined in (3.7).

Some information about the nature of the blowup can be obtained from Beale-Kato-Majda (BKM) type criteria [1], which one can be adapted to our case. For

[2] The answer might also depend on technical details of the definitions. In this context we refer the reader to the recent works of [2, 3] on the ill-posedness of the Euler equation in critical spaces.

example, for a smooth solution (with appropriate decay, in the case of $\mathbb{R}$) defined on a time interval $[0, T)$, any of the following conditions imply that the solution can be smoothly continued beyond T:

$$\int_0^T \|u_x(\cdot,t)\|_{L^\infty}\,dt < +\infty, \qquad \int_0^T \|\theta_x(\cdot,t)\|_{L^\infty}\,dt < +\infty. \tag{2.5}$$

See, for example, [11] for the proof of an analogous result for the two-dimensional inviscid Boussinesq system that can be adapted to our case in a straightforward way.

The BKM condition for the three-dimensional Euler is of course

$$\int \|\omega(\cdot,t)\|_{L^\infty}\,dt < \infty, \quad \omega = (\omega_1,\omega_2,\omega_3) = \operatorname{curl} u.$$

For the three-dimensional axisymmetric flow considered in [20] this becomes

$$\int_0^T \left(\|\omega^\theta(\cdot,t)\|_{L^\infty} + \|(u^\theta)_z(\cdot,t)\|_{L^\infty} + \left\| \frac{1}{r}(ru^\theta)_r(\cdot,t) \right\|_{L^\infty} \right) dt < +\infty,$$

which, when applied in the context of the one-dimensional model, is a stronger condition than either of the conditions in (2.5). It is natural to ask whether in the context of the HL model

$$\int_0^T \|\omega(\cdot,t)\|_{L^\infty}\,dt < +\infty$$

is a good BKM-type condition. It is possible to prove that the above quantity $\int_0^T \|\omega(\cdot,t)\|_{L^\infty}\,dt$ cannot stay finite if the main quantity used in our proof becomes infinite, but one could conceivably have some loss of smoothness while both our main quantity in the proof and $\int_0^T \|\omega(\cdot,t)\|_{L^\infty}\,dt$ remain finite. This is unlikely, but, strictly speaking, our proofs do not rule that out. A similar question also seems to be open for the two-dimensional Boussinesq system (2.2): it is not clear whether the condition

$$\int_0^T \|\omega(\cdot,t)\|_{L^\infty}\,dt < \infty$$

is a good BKM-type condition for the system.

Another natural question one can ask about the nature of blowup is whether the analogue of the kinetic energy $\int_{\mathbb{S}^1} |u|^2\,dx$ stays finite at blowup time. Note that for the two-dimensional Boussinesq system (2.2) there is a natural energy

$$E = \frac{1}{2}\int_\Omega |u(x,y)|^2\,dx\,dy - \int_\Omega y\theta(x,y)dx\,dy \tag{2.6}$$

that is conserved for reasonable classes of solutions. This conservation implies that the kinetic part of the energy should remain finite for the solutions of the two-dimensional Boussinesq system at the blowup time (if any). In the one-dimensional model we do not expect an exact conservation of some quantity analogous to (2.6) since it only models a boundary layer and not a closed system. However, if the kinetic energy became infinite at blowup time, it would clearly indicate a divergence between the one-dimensional model and the original equation. In addition, finite-time loss of regularity in solutions to Euler equations with infinite energy is well documented in the literature (see [7] for examples and further references). In the Appendix, we will provide a simple argument showing that for a class of initial data leading to finite-time blowup in the HL model, the kinetic energy stays finite. Moreover, more is true: any L^p norm $\|u\|_{L^p}$ with $p < \infty$ and the BMO norm $\|u\|_{\mathrm{BMO}}$ remain finite at blowup time. The key to these results is the control of the L^1 norm of the vorticity.

3 The CKY and HL Models on the Real Line

In this section, we discuss the CKY and HL models on the real line. Our primary reason is to outline in the most transparent setting the ideas that will be used later in the rigorous and more technical proof of the finite-time blowup for the HL model in the periodic case. The initial data for which we will prove the blowup will not be in the most general class. We will focus on the main underlying structure and ideas that will be fully developed for the periodic case in a later section—but will also look more technical there. Our main goal is the proof of finite-time blowup for the HL model in the periodic setting, and this section can be thought of as a preview of the ideas and connections employed in this proof in a situation without technical distractions that one has to deal with in the periodic case.

To start with, let us study more carefully the Biot-Savart laws for both models. Motivated by the structure of the Biot-Savart laws, a change of variable is introduced so that the velocities for both models become convolutions. Using the new variables, we will prove the finite-time blowup of the CKY model using an "entropy" functional, and then prove the finite-time blowup for the HL model using another natural functional.

3.1 Comparison of Velocities

Let us first look at the velocity field u in the HL model and the CKY model, respectively.

For the HL model on $\mathbb{R}$, we use the the following representations:

$$u_{\mathrm{HL}}(x) := \frac{1}{\pi} \int_{\mathbb{R}} \omega(y) \log|x-y| dy,$$
$$\partial_x u_{\mathrm{HL}}(x) = \mathrm{P.V.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{\omega(y)}{x-y} \, dy = (H\omega)(x).$$

We will study the situation when ω is odd and θ is even, i.e.,

$$\omega(-x,t) = -\omega(x,t), \quad \theta(-x,t) = \theta(x,t).$$

In this situation one can restrict the attention to $x \in (0, \infty)$, and the expression for u in terms of ω can be written as

$$\frac{u_{\mathrm{HL}}(x)}{x} = -\frac{1}{\pi}\int_0^\infty \frac{y}{x}\log\left|\frac{x+y}{x-y}\right|\omega(y)\frac{dy}{y}.$$

We note that the last integral is of the form

$$\int_0^\infty M\left(\frac{x}{y}\right)\omega(y)\frac{dy}{y}, \tag{3.1}$$

which represents convolution in the multiplicative group $\mathbb{R}_+$ taken with respect to the natural invariant measure $\frac{dy}{y}$. The kernel M is given by

$$M(s) = \frac{1}{s}\log\left|\frac{s+1}{s-1}\right|, \quad s > 0.$$

We will use the following decomposition of M:

$$\begin{aligned} M(s) &= \frac{1}{2}\left(\frac{1}{s}+s\right)\log\left|\frac{s+1}{s-1}\right| + \frac{1}{2}\left(\frac{1}{s}-s\right)\log\left|\frac{s+1}{s-1}\right| \\ &= M_{\mathrm{sym}}(s) + M_a(s). \end{aligned} \tag{3.2}$$

We have

$$M_{\mathrm{sym}}\left(\frac{1}{s}\right) = M_{\mathrm{sym}}(s), \quad M_a\left(\frac{1}{s}\right) = -M_a(s).$$

We collect some properties of the function M in the following lemma. The proof is elementary and will be omitted here; for the periodic case the corresponding lemma is the more technical Lemma 4.1, for which we will provide a complete proof.

LEMMA 3.1. *The function M has the following properties*:

(i) *M is increasing on $(0,1)$ and decreasing on $(1,\infty)$.*
(ii) $\lim_{s\to 0_+} M(s) = 2$ *and* $\lim_{s\to 0_+} M'(s) = 0$.
(iii) *M_a is continuous and decreasing in $(0,\infty)$, with* $\lim_{s\to 0_+} M_a(s) = 1$.
(iv) $M(s) = 2/s^2 + O(1/s^3)$, $s \to \infty$.

Note that the velocity of the CKY model can be written in a form similar to (3.1) (if we assume ω is compactly supported in $(0,1)$). The Biot-Savart law here is given by

$$\frac{u_{\mathrm{CKY}}(x)}{x} = -\frac{1}{\pi}\int_0^\infty \widetilde{M}\left(\frac{x}{y}\right)\omega(y)\frac{dy}{y} \quad \text{with } \widetilde{M}(s) = 2\chi_{[0,1]}(s),$$

where we use the notation χ_A for the characteristic function of the set A. In some sense, we can think of $\widetilde{M}$ as the most natural, crudest approximation of M. Moreover, note that the velocity for the HL model is "stronger" than for the CKY model in the following sense:

$$0 \le \widetilde{M} \le M, \quad \int_0^\infty (M(s) - \widetilde{M}(s))\frac{ds}{s} < \infty.$$

As a result, when $\omega(y) \geq 0$ for $y \geq 0$, we have $u_{\mathrm{HL}}(x) \leq u_{\mathrm{CKY}} \leq 0$. This suggests that the finite-time blowup for the CKY model should imply the finite-time blowup for the HL model. However, there is no straightforward comparison principle that would directly substantiate such a claim. The Biot-Savart law of the CKY model is simpler and can be made local after differentiation. This makes the CKY model easier to deal with. For the HL model more sophisticated arguments are needed, due to the combination of its nonlocal and nonlinear nature.

3.2 A Change of Coordinates

In view of formulae from the last subsection and the scaling symmetry (2.4), it seems natural to work with the variables $\xi, U(\xi), \Omega(\xi), \Theta(\xi)$ defined by

$$x = e^{-\xi}, \quad U(\xi) = -\frac{u(x)}{x}, \quad \Omega(\xi) = \omega(x), \quad \Theta(\xi) = -\theta(x) + \theta(0). \tag{3.3}$$

In addition, we will use the notation

$$\rho(\xi) = \Theta_\xi(\xi).$$

We note that

$$\Theta(\xi) = -\int_\xi^\infty \rho(\eta)d\eta.$$

In the coordinates (3.3) the HL model and the CKY model can both be written as

$$\begin{aligned} \Omega_t + U\Omega_\xi &= e^\xi \Theta_\xi, \\ \Theta_t + U\Theta_\xi &= 0, \end{aligned} \tag{3.4}$$

where U is given by U_{HL} and U_{CKY}, respectively. For the HL model, its Biot-Savart law (1.1) becomes [3]

$$U_{\mathrm{HL}} = K * \Omega \quad \text{with } K(\xi) = \frac{1}{\pi} M(e^{-\xi}).$$

By Lemma 3.1 the function K is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$ with $\lim_{\xi\to\infty} K(\xi) = \frac{2}{\pi}$. For the CKY model, the Biot-Savart law becomes

$$U_{\mathrm{CKY}} = \widetilde{K} * \Omega \tag{3.5}$$

where

$$\widetilde{K} = \frac{2}{\pi} \chi_{[0,\infty)}.$$

Hence in the CKY model we have

$$U_{\mathrm{CKY}}(\xi) = \frac{2}{\pi} \int_{-\infty}^{\xi} \Omega(\xi')d\xi'.$$

The decomposition (3.2) corresponds in the new coordinates to

$$K(\xi) = K_{\mathrm{sym}}(\xi) + K_a(\xi),$$

[3] We use the usual notation $f * g$ for convolution.

with

$$K_{\mathrm{sym}}(\xi) = \frac{1}{2}(K(\xi) + K(-\xi)), \quad K_a(\xi) = \frac{1}{2}(K(\xi) - K(-\xi)).$$

The function $K_a(\xi)$ is continuous and increasing, with limits $\pm\frac{1}{\pi}$ at $\pm\infty$, respectively. The function K_{sym} is everywhere above $\frac{1}{\pi}$, with the difference $K_{\mathrm{sym}} - \frac{1}{\pi}$ being integrable.

3.3 Blowup in the CKY Model—The Entropy Functional Approach

Let us assume that

$\rho_0(\xi) = \Theta_\xi(\xi, 0)$ is nonnegative and compactly supported with $\int_{\mathbb{R}} \rho_0(\xi) d\xi = 1$.

In fact, we will assume (without loss of generality) that the support of ρ is in $(0,\infty)$. This can be achieved for many compactly supported initial data $\theta_0(x)$; one only needs θ_0 to be nondecreasing, constant near 0, and vanishing at $x \geq 1$. Observe that $\Theta_0(\xi)$ in this case is nondecreasing and approaches 0 as $\xi \to \infty$, so in particular $\Theta_0(\xi) \leq 0$ for all ξ. The normalization $\int_{\mathbb{R}} \rho_0 \, d\xi = 1$ is not important, but the condition $\rho_0 \geq 0$ will play an important role in our argument. It can be relaxed to the condition that $\rho_0(\xi)$ has a "bump" of this form, but we will not pursue this here.

We also assume that the initial data $\Omega_0(\xi)$ is nonnegative and compactly supported.

The evolution equation for ρ is

$$\rho_t + (U\rho)_\xi = 0. \tag{3.6}$$

This is the usual equation of continuity, and with our assumptions this means that ρ can be considered as a density transported by U.

Let us consider the entropy of ρ, defined by

$$I = I(t) = \int_{\mathbb{R}} -\log(\rho)\rho \, d\xi. \tag{3.7}$$

In this subsection, we use an approach different from [6] to show the finite-time blowup of the CKY model. Namely, we will prove that the entropy functional $I(t)$ blows up in finite time, implying that the solution must have a finite-time singularity as well.

LEMMA 3.2. *Assume that a smooth positive density* ρ (*of unit total mass*) *is compactly supported in* $[0,\infty)$. *Then*

$$\int_{\mathbb{R}} \xi\rho(\xi) d\xi \geq \exp(I - 1).$$

PROOF. Define $f(\cdot)$ by $\rho(\xi) = f(\xi)e^{-\xi/J}$ where $J := \int_{\mathbb{R}} \xi\rho(\xi) d\xi > 0$. Observe

$$I = \int_{\mathbb{R}} -\log(f)\rho \, d\xi + \frac{1}{J}\int_{\mathbb{R}} \xi\rho \, d\xi = \int_{\mathbb{R}} -\log(f)\rho \, d\xi + 1.$$

So, by Jensen's inequality, we obtain

$$e^{I-1} = \exp\left(\int_{\mathbb{R}} -\log(f)\rho\, d\xi\right) \le \int_{\mathbb{R}} \frac{\rho}{f}\, d\xi = J. \qquad \square$$

When ρ evolves by the equation of continuity (3.6), we have

$$\frac{d}{dt} I = \int_{\mathbb{R}} U_\xi \rho\, d\xi.$$

In the CKY model we have $U_\xi = \frac{2}{\pi}\Omega$, and hence

$$\frac{d}{dt} I = \frac{2}{\pi} \int_{\mathbb{R}} \Omega \rho\, d\xi.$$

The equation

$$\Omega_t + U \Omega_\xi = e^\xi \rho,$$

together with the equation of continuity for ρ gives

$$\frac{d}{dt} \int \Omega\rho = \int (e^\xi \rho)\rho\, d\xi = \int e^{\xi + \log\rho} \rho\, d\xi.$$

By Jensen's inequality the last integral is greater than or equal to

$$\exp\left(\int \xi\rho\, d\xi - I\right).$$

Estimating the integral in the last expression from below by Lemma 3.2, we finally obtain

$$\ddot{I} \ge \frac{2}{\pi} e^{(e^{I-1} - I)}.$$

In addition, the above calculations show that I is strictly increasing and convex as a function of t. It is now easy to see that I must always become infinite in finite time.

In the original coordinates, we have

$$I(t) = \int_0^\infty \theta_x \log(x\theta_x) dx.$$

It is not difficult to see, using the blowup criteria (2.5), that $I(t)$ can only become infinite if the solution loses regularity. We will sketch this argument below in the next section when proving Theorem 2.1.

We have proved the following result:

THEOREM 3.3. *For the initial data as above, the CKY model given by* (3.4) *with the Biot-Savart law* (3.5) *develops a singularity in finite time.*

Remark. Due to the local nature of equation $U_\xi = \Omega$ in the CKY model, it is not hard to see that a similar argument applies whenever θ is increasing on some interval and the initial vorticity ω is nonnegative on $(0, \infty)$.

3.4 Blowup in the CKY Model and the HL Model—A New Functional

If one tries to reproduce the above proof for the HL model, it quickly becomes apparent that the condition $\rho_\xi \geq 0$ seems necessary for the proof to go through. This leads to growth at infinity for the initial data, which is not very reasonable. Hence the above proof cannot be directly applied to the HL model. However, a different functional $\int_{\mathbb{R}} \rho\xi \, d\xi$ can be used to prove finite-time blowup for both models, which we will demonstrate below.

Let's assume ρ_0 and Ω_0 satisfy the conditions in the previous subsection. Let $F(t) := \int \rho\xi \, d\xi = -\int_0^\infty \Theta \, d\xi$. Taking the time derivative of F, we have

$$\frac{dF}{dt} = \int U\Theta_\xi \, d\xi$$

for both models. For the CKY model, since $(U_{\mathrm{CKY}})_\xi = \frac{2}{\pi}\Omega$, the right-hand side is equal to

$$G(t) := -\frac{2}{\pi}\int \Omega\Theta \, d\xi.$$

For the HL model, since its velocity field is "stronger" than in the CKY model (see Section 3.1), we have $\int U_{\mathrm{HL}}\Theta_\xi \, d\xi \geq \int U_{\mathrm{CKY}}\Theta_\xi \, d\xi = G(t)$.

Now let us take the time derivative of $G(t)$:

$$\begin{aligned}\frac{dG}{dt} = -\frac{2}{\pi}\int \Omega_t\Theta + \Omega\Theta_t \, d\xi &= \frac{2}{\pi}\int U\Omega_\xi\Theta + U\Omega\Theta_\xi - e^\xi\Theta\Theta_\xi \, d\xi \\ &= -\frac{2}{\pi}\int U_\xi\Omega\Theta \, d\xi + \frac{1}{\pi}\int e^\xi\Theta^2 \, d\xi.\end{aligned}$$

For the CKY model, the first term on the right-hand side is positive, since $U_\xi = \frac{2}{\pi}\Omega$ and $\Theta \leq 0$. For the HL model, we claim that $\int_{-\infty}^\xi U_\xi(\eta)\Omega(\eta)d\eta \geq 0$ for any $\Omega \geq 0$ and for any $\xi \in \mathbb{R}$. The proof of the claim will be given in the next subsection. Once this is proved, due to the fact that $\Theta \leq 0$ and is increasing, we can use integration by parts and the substitution $\Theta(\xi) = s$ to obtain the following estimate for the HL model:

$$-\frac{2}{\pi}\int U_\xi\Omega\Theta \, d\xi = \frac{2}{\pi}\int_{\Theta_{\min}}^0 \int_{-\infty}^{\Theta^{-1}(s)} U_\xi\Omega \, d\xi \, ds \geq 0,$$

where Θ^{-1} is the inverse function of Θ. As a result, for both models, we have

$$\frac{dG}{dt} \geq \frac{1}{\pi}\int e^\xi\Theta^2 \, d\xi \geq \frac{1}{\pi}\left(\int_0^\infty -\Theta e^{\frac{\xi}{2}}e^{-\frac{\xi}{2}} \, d\xi\right)^2 = \frac{1}{\pi}F(t)^2,$$

which gives us the system

$$\frac{dF}{dt} \geq G(t) \geq 0, \qquad \frac{dG}{dt} \geq \frac{1}{\pi}F(t)^2,$$

and one can obtain that $F(t)$ blows up in finite time by a standard ODE argument.

3.5 Quadratic Forms in the HL Model

Now we prove the claim in the last subsection. Let

$$I(\Omega,\xi) = \int_{-\infty}^{\xi} U_\xi(\eta)\Omega(\eta)d\eta.$$

LEMMA 3.4. *For any smooth, compactly supported $\Omega \geq 0$, we have $I(\Omega,\xi) \geq 0$ for all ξ.*

PROOF. Let us write $\Omega = \Omega_l + \Omega_r$, where

$$\Omega_l = \Omega\, \chi_{(-\infty,\xi]}, \quad \Omega_r = \Omega\, \chi_{(\xi,\infty)}.$$

We have

$$U = U_l + U_r, \quad U_l = K * \Omega_l, \quad U_r = K * \Omega_r,$$

and

$$I = I(\Omega,\xi) = \int_{\mathbb{R}} U_\xi(\eta)\Omega_l(\eta)d\eta = \int_{\mathbb{R}} U_{l\xi}\Omega_l\, d\eta + \int_{\mathbb{R}} U_{r\xi}\Omega_l\, d\eta.$$

We claim that in the last expression both integrals are nonnegative. Denoting by K' the derivative of K (taken in the sense of distributions), we can write for the first integral

$$\begin{aligned}\int_{\mathbb{R}} U_{l\xi}\Omega_l\, d\eta &= \int_{\mathbb{R}}\int_{\mathbb{R}} K'(\eta-\zeta)\Omega_l(\zeta)\Omega_l(\eta)d\eta\, d\zeta \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} K_a'(\eta-\zeta)\Omega_l(\zeta)\Omega_l(\eta)d\eta\, d\zeta \geq 0,\end{aligned}$$

as K_a is increasing. The second integral is equal to

$$\int_{\mathbb{R}}\int_{\mathbb{R}} K'(\eta-\zeta)\Omega_r(\zeta)\Omega_l(\eta)d\zeta\, d\eta,$$

and we note that the integration can be restricted to the domain $\{\eta < \zeta\}$ as the integrand vanishes elsewhere. As $K'(\xi) > 0$ for $\xi < 0$, the result follows. □

4 Finite-Time Blowup for the HL Model in the Periodic Setting

In this section we consider the HL model in $[0, L]$ with periodic condition, and we will prove our main result—Theorem 2.1 in the periodic setting, i.e., $x \in \mathbb{S}^1$. Now u_x is the (periodic) Hilbert transform of ω, namely,

$$u_x(x) = \frac{1}{L}\,\text{P.V.}\int_0^L \omega(y)\cot[\mu(x-y)]dy =: H\omega(x), \quad \mu := \frac{\pi}{L}.$$

As a result, the nonlocal velocity u is defined by

$$u(x) = Q\omega(x) := \frac{1}{\pi}\int_0^L \omega(y)\log|\sin[\mu(x-y)]|dy. \tag{4.1}$$

In this case the change of variables of the previous section is less natural, and we will work in the original coordinates.

In this section, we consider smooth, odd, periodic initial data θ_{0x}, ω_0 with period L. We note that such functions are also odd with respect to the origin taken at $x = \frac{1}{2}L$. In addition, we suppose $\theta_{0x}, \omega_0 \geq 0$ on $[0, \frac{1}{2}L]$. Then the initial velocity u_0 also becomes odd at $x = 0$ and $\frac{1}{2}L$ and $u_0 \leq 0$ on $[0, \frac{1}{2}L]$ (for the proof of the last assertion on u_0, see (4.4)). Lastly, since (1.5) is invariant under a constant shift $\theta \to \theta + \text{const}$, we may assume $\theta_0(0) = 0$.

Thanks to the transport structure of (1.5), the evolution preserves the assumptions as long as the solution exists. That is, θ_x, ω, u stay odd at $x = 0$ and $\frac{1}{2}L$ with period L, $\theta(0) = 0$, and $\theta_x, \omega, (-u) \geq 0$ on $[0, \frac{1}{2}L]$.

The proof will be based on an idea that builds on the insight gained from the whole-line case. It considers the integral

$$I(t) := \int_0^{L/2} \theta(x,t) \cot(\mu x) dx$$

and shows that it must blow up in finite time if $I(0) > 0$. As will be shown below, this implies the blowup of the HL model.

The proof relies on the following two lemmas, which reveal the key properties of the velocity u as defined by the Biot-Savart law (4.1). These lemmas can be viewed as analogues of Lemmas 3.1 and 3.4 from the whole-line case. It is worth noting that both lemmas follow directly from (4.1) and do not depend on the actual dynamics of the flow.

LEMMA 4.1. *Let ω be periodic with period L and odd at $x = 0$, and let $u = Q\omega$ be as defined by* (4.1). *Then for any $x \in [0, \frac{1}{2}L]$,*

$$u(x)\cot(\mu x) = -\frac{1}{\pi}\int_0^{L/2} K(x,y)\omega(y)\cot(\mu y)dy, \tag{4.2}$$

where

$$K(x,y) = s\log\left|\frac{s+1}{s-1}\right| \quad \textit{with } s = s(x,y) = \frac{\tan(\mu y)}{\tan(\mu x)}. \tag{4.3}$$

Furthermore, the kernel $K(x, y)$ has the following properties:

(a) $K(x,y) \geq 0$ *for all* $x,\ y \in (0, \frac{1}{2}L)$ *with* $x \neq y$;
(b) $K(x,y) \geq 2$ *and* $K_x(x,y) \geq 0$ *for all* $0 < x < y < \frac{1}{2}L$;
(c) $K(x,y) \geq 2s^2$ *and* $K_x(x,y) \leq 0$ *for all* $0 < y < x < \frac{1}{2}L$.

PROOF. The velocity $u = Q\omega$ as defined by (4.1) admits the following representation:

$$\begin{aligned} u(x) &= \frac{1}{\pi}\left[\int_0^{L/2} + \int_{L/2}^{L}\right]\omega(y)\log|\sin[\mu(x-y)]|dy \\ &= \frac{1}{\pi}\int_0^{L/2}\omega(y)\big\{\log|\sin[\mu(x-y)]| - \log|\sin[\mu(x+y)]|\big\}dy \qquad (4.4) \\ &= \frac{1}{\pi}\int_0^{L/2}\omega(y)\log\left|\frac{\tan(\mu x)-\tan(\mu y)}{\tan(\mu x)+\tan(\mu y)}\right|dy. \end{aligned}$$

This shows that

$$u(x)\cot(\mu x) = -\frac{1}{\pi}\int_0^{L/2} K(x,y)\omega(y)\cot(\mu y)dy,$$

where $K(x, y)$ is the kernel defined by (4.3); hence the first part of the lemma follows.

It remains to check that $K(x, y)$ satisfies the properties (a)–(c). To prove (a), note that $x,\ y \in (0, \frac{1}{2}L)$ implies $s(x, y) > 0$; hence $|s + 1| > |s - 1|$. The claim follows easily.

For (b), observe that $s(x, y) > 1$ for $0 < x < y < \frac{1}{2}L$. Hence we can express $K(x, y)$ as a Taylor series in terms of s^{-1}:

$$K(x,y) = s\big\{\log(1+s^{-1}) - \log(1-s^{-1})\big\} = 2\sum_{n=0}^{\infty}\frac{s^{-2n}}{2n+1} \ge 2.$$

By taking derivatives of x on both sides and observing that $s_x(x, y) \le 0$, we also deduce $K_x(x, y) \ge 0$ for $0 < x < y < \frac{1}{2}L$. This establishes (b).

Finally, to prove (c), we proceed as in (b) and note that $0 < s(x, y) < 1$ for $0 < y < x < \frac{1}{2}L$. Thus

$$K(x,y) = s\{\log(1+s) - \log(1-s)\} = 2\sum_{n=0}^{\infty}\frac{s^{2n+2}}{2n+1} \ge 2s^2.$$

In addition, taking derivatives in x and using $s_x(x, y) \le 0$ shows $K_x(x, y) \le 0$ for $0 < y < x < \frac{1}{2}L$. This completes the proof of the lemma. □

Remark. Lemma 4.1 implies that the periodic HL velocity u_{HL} as defined by (4.1) is, up to a constant factor, "stronger" than the CKY velocity u_{CKY} provided that $\omega \ge 0$ on $[0, \frac{1}{2}L]$. Indeed, since $\cot(\mu x) \sim (\mu x)^{-1}$ for small x, the properties (a) and (b) of the kernel $K(x, y)$ imply that, for any $z \in (0, \frac{1}{2}L)$, there exists a positive constant C depending on L and z such that

$$u_{\mathrm{HL}}(x) \le -Cx\int_x^z \frac{1}{y}\,\omega(y)dy \quad \forall x \in [0, z].$$

The second lemma that we shall prove concerns the positivity of a certain quadratic form of the vorticity ω, which plays the central role in the proof of Theorem 2.1.

LEMMA 4.2. *Let the assumptions in Lemma* 4.1 *be satisfied and assume in addition that* $\omega \geq 0$ *on* $[0, \frac{1}{2}L]$. *Then for any* $a \in [0, \frac{1}{2}L]$,

$$\int_a^{L/2} \omega(x)[u(x)\cot(\mu x)]_x \, dx \geq 0. \tag{4.5}$$

PROOF. First, we note that for any $x, a \in [0, \frac{1}{2}L]$, $\omega(x)$ can be decomposed as

$$\omega(x) = \omega(x)\chi_{[0,a]}(x) + \omega(x)\chi_{[a,\frac{1}{2}L]}(x) =: \omega_l(x) + \omega_r(x),$$

where χ_A denotes the characteristic function of the set A. Using this decomposition and Lemma 4.1, we split the integral in (4.5):

$$\begin{aligned}
&\int_a^{L/2} \omega(x)[u(x)\cot(\mu x)]_x \, dx \\
&\quad = -\frac{1}{\pi}\int_0^{L/2} \omega_r(x) \int_0^{L/2} \omega_l(y)\cot(\mu y) K_x(x,y) dy \, dx \\
&\qquad - \frac{1}{\pi}\int_0^{L/2} \omega_r(x) \int_0^{L/2} \omega_r(y)\cot(\mu y) K_x(x,y) dy \, dx =: -I_1 - I_2.
\end{aligned}$$

Clearly, the lemma follows if both I_1 and I_2 are nonpositive. For I_1, we observe from the definition of ω_l and ω_r that the integrand in I_1 is nonzero only when $y \leq x$. Using $K_x \leq 0$ as proved in Lemma 4.1(c) and the assumption that $\omega \geq 0$ on $[0, \frac{1}{2}L]$, we then deduce $I_1 \leq 0$. As for I_2, we write $G := K_x$ and observe that

$$I_2 = \frac{1}{2\pi}\int_0^{L/2}\int_0^{L/2} \omega_r(x)\omega_r(y) T(x,y) dy \, dx, \tag{4.6}$$

where $T(x,y) = \cot(\mu y)G(x,y) + \cot(\mu x)G(y,x)$. We shall show that $T(x,y) \leq 0$ for all $x,\ y \in (0, \frac{1}{2}L)$, which then implies $I_2 \leq 0$. To this end, we compute

$$G(x,y) = -\mu\csc^2(\mu x)\tan(\mu y)\left\{\log\left|\frac{s+1}{s-1}\right| - \frac{2s}{s^2-1}\right\},$$

and then

$$\begin{aligned}
T(x,y) = &-\mu[\csc^2(\mu x) + \csc^2(\mu y)]\log\left|\frac{s+1}{s-1}\right| \\
&+ \mu[\csc^2(\mu x) - \csc^2(\mu y)]\frac{2s}{s^2-1}.
\end{aligned}$$

Thanks to Lemma 4.1(b) and (c), we have

$$K(x,y) \geq \frac{2s^2}{s^2+1} \quad \forall s \geq 0,$$

which implies that

$$\log\left|\frac{s+1}{s-1}\right| \geq \frac{2s}{s^2+1} \quad \forall s \geq 0.$$

It then follows that $\forall x, y \in (0, \frac{1}{2}L)$,

$$T(x,y) \leq -\left\{\frac{4\mu s \csc^2(\mu x)\sec^2(\mu y)}{s^2+1}\right\} \cdot \left\{\frac{\cos^2(\mu x) - \cos^2(\mu y)}{s^2-1}\right\} \leq 0$$

and hence $I_2 \leq 0$. This completes the proof of the lemma. □

Remark. The positivity of the integral in (4.5) corresponds to the positivity of $\int_a^1 \omega(x)[u(x)/x]_x\, dx$ in the CKY model, and so to Lemma 3.4 in the whole-line case. Indeed, thanks to the local nature of the Biot-Savart law for the CKY model, the positivity of the latter integral follows almost immediately from the definition of the CKY velocity u_{CKY}:

$$\int_a^1 \omega(x)[u_{\text{CKY}}(x)/x]_x\, dx = \int_a^1 \frac{1}{x}\,\omega^2(x) dx \geq 0.$$

In this sense, Lemma 4.2 is quite surprising since the HL velocity u_{HL} as defined by (4.1) does not have such a simple structure. It is the careful analysis of the kernel $K(x, y)$ (Lemma 4.1) and the symmetrization technique (4.6) that make the estimate (4.5) possible.

PROOF OF THEOREM 2.1. We are going to show a finite-time blowup of the quantity

$$I(t) := \int_0^{L/2} \theta(x,t)\cot(\mu x) dx. \tag{4.7}$$

We claim that this implies the finite-time blowup of the corresponding solution (θ, ω) of the HL model (1.5). Indeed, applying integration by parts to (4.7), we see

$$I(t) = -\frac{1}{\mu}\int_0^{L/2} \theta_x(x,t)\log|\sin(\mu x)| dx,$$

which can be bounded as follows for some constant $C > 0$:

$$|I(t)| \leq C\|\theta_x(\cdot,t)\|_{L^\infty} \leq C\|\theta_{0x}\|_{L^\infty}\exp\left\{\int_0^t \|u_x(\cdot,s)\|_{L^\infty}\, ds\right\}.$$

Thus if $I(t)$ blows up at a finite time T, for $\int_0^t \|u_x(\cdot,s)\|_{L^\infty}\, ds$ the same must hold true. The finite-time blowup of the HL model then follows from the corresponding Beale-Kato-Majda type condition (2.5) for the HL model.

To prove the finite-time blowup of $I(t)$, we assume $I(0) > 0$ and consider

$$\begin{aligned}\frac{d}{dt} I(t) &= -\int_0^{L/2} u(x)\theta_x(x)\cot(\mu x) dx \\ &= \frac{1}{\pi}\int_0^{L/2} \theta_x(x)\int_0^{L/2} \omega(y)\cot(\mu y) K(x,y) dy\, dx,\end{aligned}$$

where in the second step we have used the representation formula (4.2) from Lemma 4.1. According to our choice of the initial data and the properties of the kernel $K(x, y)$ as proved in Lemma 4.1(a) and (b), we have $\theta_x, \omega \geq 0$ on $[0, \frac{1}{2}L]$, $K \geq 0$ for $y < x$, and $K \geq 2$ for $y > x$. Thus

$$\begin{aligned} \frac{d}{dt}I(t) &\geq \frac{2}{\pi}\int_0^{L/2} \theta_x(x) \int_x^{L/2} \omega(y)\cot(\mu y)dy\, dx \\ &= \frac{2}{\pi}\int_0^{L/2} \theta(x)\omega(x)\cot(\mu x)dx =: J(t). \end{aligned}$$

Taking the time derivative of $J(t)$ gives

$$\begin{aligned} \frac{d}{dt}J(t) &= \frac{2}{\pi}\int_0^{L/2} -(\theta(x)\omega(x))_x u(x)\cot(\mu x) + \theta_x(x)\theta(x)\cot(\mu x)dx \\ &= \frac{2}{\pi}\int_0^{L/2} \theta(x)\omega(x)(u(x)\cot(\mu x))_x\, dx \\ &\quad + \frac{\mu}{\pi}\int_0^{L/2} \theta^2(x)\csc^2(\mu x)dx =: T_1 + T_2 \end{aligned}$$

For T_1, since $\theta(x)$ is a nonnegative increasing function on $[0, L/2]$, one has, after integrating by parts,

$$T_1 = \frac{2}{\pi}\int_0^{L/2} \theta_y(y)\left[\int_y^{L/2} \omega(x)(u(x)\cot(\mu x))_x\, dx\right]dy \geq 0,$$

where we applied Lemma 4.2 to get the inequality.

For T_2, we can find a lower bound using the Cauchy-Schwarz inequality:

$$\begin{aligned} T_2 &\geq \frac{\mu}{\pi}\int_0^{L/2} \theta^2(x)\cot^2(\mu x)dx \\ &\geq \frac{\mu}{\pi}\frac{2}{L}\left(\int_0^{L/2} \theta(x)\cot(\mu x)dx\right)^2 \geq \frac{2}{L^2}I(t)^2 \end{aligned}$$

Finally, we have

$$\frac{dJ}{dt} \geq \frac{2}{L^2}I^2,$$

implying

$$\frac{d}{dt}I(t) \geq J(0) + c_0\int_0^t I(t)^2\, dt \geq c_0\int_0^t I(t)^2\, dt, \quad c_0 = \frac{2}{L^2}. \tag{4.8}$$

From this inequality finite-time blowup can be inferred in a standard way. We sketch one such argument below.

The inequality (4.8) can be equivalently written as

$$g'' \geq 2c_0 g(g')^{1/2} \quad \text{where } g(t) = \int_0^t I^2(s)ds.$$

Note that $\alpha := g'(0) > 0$ and $g(0) = 0$. It is not difficult to show that if h satisfies the second-order differential equation $h'' = 2c_0 h(h')^{1/2}$ for the initial data $h'(0) = \alpha$ and $h(0) = 0$, then $g \geq h$ as long as these functions are well-defined.

By substitution $f := h'$, we obtain

$$\sqrt{f}\,\frac{df}{dh} = 2c_0 h.$$

Solving this equation we see that for $t > 0$, h satisfies

$$(h'(t))^{3/2} = \alpha^{3/2} + \frac{3}{2}\,c_0 h^2(t).$$

From the differential inequality

$$h'(t) \geq \left[\frac{3}{2}\,c_0 h^2(t)\right]^{2/3},$$

together with $h(t_0) > 0$ for any $t_0 > 0$, we get finite-time blowup for h. Indeed, for any fixed $t_0 > 0$, we have $h(t_0) \geq \alpha \cdot t_0$ from $h'(t) \geq \alpha$. Then, for $t \geq t_0$, a simple computation leads to

$$h(t) \geq \left((\alpha t_0)^{-1/3} - \frac{1}{3}\left(\frac{3}{2}c_0\right)^{2/3}(t - t_0)\right)^{-3}.$$

It immediately implies the finite-time blowup of $I = (g')^{1/2}$. The proof of Theorem 2.1 is complete. □

Appendix: On the Finite Kinetic Energy at the Blowup Time

In this appendix we prove bounds on the kinetic energy as well as other norms of the velocity u in the periodic HL model, showing that they remain finite at the blowup time. We will consider a class of initial data that is smaller than the entire set for which we prove finite-time blowup. With more technical effort, we can generalize the bounds below to a larger class; however, our main point is to provide blowup examples with finite kinetic energy. We therefore choose to work with the smaller class to reduce technicalities.

In the blowup scenario for the L-periodic HL model, we assumed that ω_0 and θ_{0x} are smooth and odd with respect to 0 and $L/2$, and $\omega_0, \theta_{0x} \geq 0$ on $[0, \frac{1}{2}L]$. Let us assume, in addition, that ω_0 and θ_{0x} vanish on $[L/4, L/2]$. Then Lemma 4.1 implies that this remains true for all times.

PROPOSITION A.3. *Suppose that the initial data to the periodic HL model are as described above. Then while the solution remains regular, the fluid velocity u satisfies*

$$\|u(\cdot, t)\|_{\mathrm{BMO}} \leq C(\|\omega_0\|_{L^1} + \|\theta_0\|_{L^\infty} t). \tag{A.1}$$

In particular, the right-hand side of (A.1) *also bounds any L^p, $p < \infty$, norm $\|u\|_{L^p}$, with a constant dependent on p.*

PROOF. The first observation is that

$$\partial_t \int_0^{L/2} \omega(x,t)dx = \int_0^{L/2} \theta_x \, dx - \int_0^{L/2} u\omega_x \, dx.$$

The first integral on the right-hand side is bounded by $2\|\theta_0\|_{L^\infty}$. In the second integral, while the solution remains smooth, we can integrate by parts to obtain

$$\begin{aligned} -L\int_0^{L/2} u\omega_x \, dx &= L\int_0^{L/2} u_x\omega \, dx \\ &= \int_0^{L/2} \omega(x)\text{P.V.}\int_0^{L} \cot(\mu(x-y))\omega(y)dy\, dx \\ &= \int_0^{L/2} \omega(x)\text{P.V.}\int_0^{L/2} \cot(\mu(x-y))\omega(y)dy\, dx \\ &\quad - \int_0^{L/2} \omega(x)\int_0^{L/2} \cot(\mu(x+y))\omega(y)dy\, dx, \end{aligned}$$

where $\mu = \pi/L$ as before. Here we used $\omega(L-x) = -\omega(x)$ when transforming the last integral. Now

$$\int_0^{L/2} \text{P.V.}\int_0^{L/2} \cot(\mu(x-y))\omega(y)\omega(x)dy\, dx = 0$$

by symmetry. On the other hand,

$$\int_0^{L/2}\int_0^{L/2} \cot(\mu(x+y))\omega(y)\omega(x)dy\, dx \geq 0$$

since the support of ω lies in $[0, L/4]$ for all times by our choice of the initial data and Lemma 4.1. Thus we get

$$\int_0^{L/2} \omega(x,t)dx \leq \int_0^{L/2} \omega_0(x)dx + 2\|\theta_0\|_{L^\infty} t.$$

Due to solution symmetries and the choice of the initial data, this implies the key estimate

$$\|\omega(\cdot,t)\|_{L^1} \leq \|\omega_0\|_{L^1} + 2\|\theta_0\|_{L^\infty} t \tag{A.2}$$

for all t, while the solution remains smooth. Now the bound on $\|u\|_{L^2}$ follows by a simple direct estimate using (4.1):

$$u(x) = \frac{1}{\pi}\int_0^L \omega(y)\log|\sin[\mu(x-y)]|dy.$$

More generally, (4.1) also implies the bound $\|u\|_{\text{BMO}} \leq C\|\omega\|_{L^1}$. Indeed, it is not difficult to see that the function $\log|\sin[\mu(x-y)]|$ belongs to BMO (see, e.g., [26] for the $\log|x|$ case). Then the formula (4.1) above and (A.2) show that u is a convolution of the BMO function with an L^1 density, leading to $\|u\|_{\text{BMO}} \leq$

$C\|\omega\|_{L^1}$ bound and so to (A.1). The bound on the L^p norm of u for any $p < \infty$ follows from the BMO bound. □

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