

On the stability of blowup solutions to the complex Ginzburg-Landau equation in \mathbb{R}^d

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Abstract

Building upon the idea in [37], we establish stability of the type-I blowup with log correction for the complex Ginzburg-Landau equation. In the amplitude-phase representation, a generalized dynamic rescaling formulation is introduced, with modulation parameters capturing the spatial translation and rotation symmetries of the equation and novel additional modulation parameters perturbing the scaling symmetry. This new formulation provides enough degrees of freedom to impose normalization conditions on the rescaled solution, completely eliminating the unstable and neutrally stable modes of the linearized operator around the blowup profile. It enables us to establish the full stability of the blowup by enforcing vanishing conditions via the choice of normalization and using weighted energy estimates, without relying on a topological argument or a spectrum analysis. The log correction for the blowup rate is captured by the energy estimates and refined estimates of the modulation parameters.

1 Introduction

We consider the complex Ginzburg-Landau equation

$$\psi_t = (1 + i\beta)\Delta\psi + (1 + i\delta)|\psi|^{p-1}\psi - \gamma\psi, \quad (\text{CGL})$$

where $\psi(t) : \mathbb{R}^d \rightarrow \mathbb{C}$, β, δ, γ are real constants and $p > 1$. The model equation (CGL) was first derived by Stewardson and Stuart in [56] (see also [19], [20]) to examine afresh the problem of plane Poiseuille flow in a wave system. The equation is also used to describe various phenomena in many fields, among which are nonlinear optics with dissipation [47], turbulent behavior [3], Rayleigh-Bénard convection or Taylor-Couette flow in hydrodynamics [21], [50], [49], reaction-diffusion systems [33], [39], [54], [55], the theory of superconductivity [2], [9], [22], [32], etc. For further details on the physical background and derivation of the complex Ginzburg-Landau equation, we refer to the surveys [1], [46], and the references therein.

The local Cauchy problem has been well established through a semigroup approach in the works [29, 30, 31]. A solution to (CGL) blows up in finite time if $\lim_{t \rightarrow T} \|\psi(t)\|_{L^\infty(\mathbb{R}^d)} = +\infty$ for some $T < +\infty$. Singularity formation has been intensively studied for the two limiting models of (CGL): the classical nonlinear heat equation in the limit $\beta, \delta, \gamma \rightarrow 0$,

$$\partial_t \psi = \Delta\psi + |\psi|^{p-1}\psi, \quad \psi(t) : x \in \mathbb{R}^d \rightarrow \mathbb{R}, \quad (\text{NLH})$$

and the nonlinear Schrödinger equation in the limit $\beta, |\delta| \rightarrow \infty$,

$$i\partial_t \psi + \Delta\psi + \mu|\psi|^{p-1}\psi = 0, \quad \mu = \pm 1. \quad (\text{NLS})$$

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We refer to [53] and [26] for intensive lists of references from the early 1960s concerning blowup results of these two equations. However, singularities in (CGL) (collapse, chaotic, or blowup) are much less understood in comparison with what have been established for (NLH) and (NLS). The study of singularity in (CGL) is a challenging problem due to the lack of variational structure, no maximum principle, non-self-adjoint linearized operator, etc. Nevertheless, singularity in (CGL) was experimentally reported in [40], [41] where the authors described an extensive series of experiments on traveling-wave convection in an ethanol/water mixture, and collapse solutions were observed. We have a sharp sufficient criteria for collapse in (CGL) for the case of subcritical bifurcation described in [58]. In [27], the authors used the modulation theory and numerical observations to show that the collapse dynamic is governed in the (CGL) limit of the L^2 -critical cubic (NLS). For the existence of blowup, there are the results of [7] and [8] in which the authors studied (CGL) for the case $\beta = \delta$. In [6] and [52], the authors gave some evidence for the existence of a radial solution that blows up in a self-similar way, their arguments were based on the combination of rigorous analysis and numerical computations. In [60] and [43], the authors rigorously constructed particular examples of initial data for which the solutions of (CGL) blow up in finite time for (β, δ) in the *subcritical* range

$$b_* := p - \delta^2 - \beta\delta(p + 1), \quad b_* > 0 \quad (\text{subcritical range}). \quad (1.1)$$

The constructed blowup solution in the subcritical case admits the asymptotic behavior

$$\psi(x, t) \sim |\log(T - t)|^{\mu} \left[(T - t)(p - 1 + c_p |Z|^2) \right]^{-\frac{1+\delta}{p-1}}, \quad Z = \frac{x}{\sqrt{(T - t)|\log(T - t)|}}, \quad (1.2)$$

where the constants c_p and μ are given by

$$c_p = \frac{(p - 1)^2}{4b_*} > 0, \quad \mu = -\frac{\beta(1 + \delta^2)}{2b_*}. \quad (1.3)$$

The spectral analysis for a non-self-adjoint operator developed in [43] can be implemented for other problems where an energy-type method is not applicable, see for example [28]. The blowup for the critical range, $b_* = 0$, has been recently solved in [51], [24] following the approach of [43]. The blowup for (CGL) in the supercritical range, $b_* < 0$, has recently been solved in [23] for the special choice $\beta = 0$. We remark that in the mentioned works ([60], [43], [51], [24], [23]), the authors focused on the case of dimension $d = 1$, and briefly described the stability properties of constructed blowup solutions through a spectral approach in a restricted (well-prepared) class of initial data.

In this paper, we aim to develop a new approach based on the dynamical rescaling formulation and simple vanishing conditions to study blowup solutions to (CGL). This new approach allows us to establish asymptotically self-similar blowup and a clear notion of stability capturing the logarithm correction (1.2) in the subcritical case for a large class of initial data in all dimensions $d \geq 1$. Throughout this paper, we use the amplitude-phase representation,

$$\psi(x, t) = u(x, t)e^{i\theta(x, t)}, \quad (1.4)$$

where u and θ are real-valued functions of time and space solving the coupled system

$$\partial_t u = [\Delta - |\nabla\theta|^2]u - \beta(2\nabla u \cdot \nabla\theta + u\Delta\theta) + u^p - \gamma u, \quad (1.5a)$$

$$u\partial_t\theta = \beta[\Delta - |\nabla\theta|^2]u + 2\nabla u \cdot \nabla\theta + u\Delta\theta + \delta u^p. \quad (1.5b)$$

The case $\beta = 0$ is related to a class of reaction-diffusion equations appearing in the study of pattern formation, see for example [33] and references therein.

1.1 Main result

For any $k \geq 1$, we introduce the functional spaces \mathfrak{E}_k and \mathfrak{F}_k

$$\mathfrak{E}_k = \left\{ w : \|w\|_{\mathfrak{E}_k} = \sum_{j=0}^k \|\nabla^j w\|_{\rho_j} < +\infty \right\}, \quad \mathfrak{F}_k = \left\{ \phi, \|\phi\|_{\mathfrak{F}_k} = \sum_{j=1}^k \|\nabla^j \phi\|_{\tilde{\rho}_j} < +\infty \right\}, \quad (1.6)$$

where $\|\cdot\|_{\rho_k}$ and $\|\cdot\|_{\hat{\rho}_k}$ stand for the standard weighted L^2 -norm with ρ_k and $\hat{\rho}_k$ being defined as in (1.28). Let \bar{U} be the universal profile

$$\bar{U}(z) = \left(p - 1 + \frac{(p-1)^2}{4b_*} |z|^2 \right)^{-\frac{1}{p-1}}, \quad \forall z \in \mathbb{R}^d, \quad (1.7)$$

and V_0 be a non-degenerate global maximizer of u_0 defined by

$$V_0 = \arg \max u_0(z), \quad u_0(V_0) > 0, \quad -\nabla^2 u_0(V_0) \succ 0, \quad (1.8)$$

where $A \succ 0$ means that A is a positive definite matrix. The main result of this paper is the following theorem.

Theorem 1 (Existence and stability of blowup solutions to (CGL)). *Consider β, δ in the sub-critical range (1.1), i.e. $b_* > 0$, $p > 1$ and $d \geq 1$. Let $K = K(d, p) \in \mathbb{N}$ be defined as in (1.29). There exists an open set $\mathcal{O} \subset \mathfrak{E}_K \times \mathfrak{F}_K$ of initial data $\psi_0 = u_0 e^{\theta_0}$ with the property (1.8) such that the corresponding solution $\psi = u e^{\theta}$ to (CGL) blows up in finite time T and the following asymptotic behaviors hold.*

(i) (The amplitude-phase decomposition)

$$\left\| H(t)u(\mathbf{R}(t)z + V(t), t) - \bar{U}(z) \right\|_{\mathfrak{E}_K} + \left\| \theta(\mathbf{R}(t)z + V(t), t) - \mu(t) - \delta \log \bar{U}(z) \right\|_{\mathfrak{F}_K} \leq \frac{C}{1 + |\log(T-t)|}, \quad (1.9)$$

where $H(t)$ and $\mu(t)$ are scalar functions, $\mathbf{R}(t)$ is an upper triangular matrix and $V(t)$ is a vector in \mathbb{R}^d ,

$$\lim_{t \rightarrow T} \frac{H(t)^{p-1}}{T-t} = 1, \quad \lim_{t \rightarrow T} \frac{\mathbf{R}(t)}{\sqrt{(T-t)|\log(T-t)|}} = I_d, \quad \lim_{t \rightarrow T} V(t) = V_T, \quad (1.10)$$

for some $V_T \in \mathbb{R}^d$, and $\mu(t)$ ¹ admits the expansion

$$\mu(t) = -\frac{\delta}{p-1} \log(T-t) - \frac{d\beta(1+\delta^2)}{2b_*} \log|\log(T-t)| + \hat{\mu}(t), \quad \lim_{t \rightarrow T} \hat{\mu}(t) = \hat{\mu}_T, \quad (1.11)$$

for some scalar function $\hat{\mu}(t)$ and $\hat{\mu}_T \in \mathbb{R}$.

(ii) (L^∞ asymptotic behavior)

$$\left\| \left| \log(T-t) \right|^{\frac{d\beta(1+\delta^2)}{2b_*}} (T-t)^{\frac{1+\delta}{p-1}} e^{-i\hat{\mu}(t)} \psi(\mathbf{R}(t)z + V(t), t) - \bar{U}^{1+i\delta} \right\|_{L^\infty} \leq \frac{C}{1 + |\log(T-t)|^{\sigma'}}, \quad (1.12)$$

where $\sigma' = \min\left\{1, \frac{4}{p-1}\right\}$ and $C = C(u_0, \theta_0) > 0$.

Remark 1 (Description of the set \mathcal{O} of initial data). *For initial data u_0 satisfies the property (1.8), we can define an upper triangular matrix \mathcal{M}_0 with $\mathcal{M}_{0,ii} > 0$ ² and the rescaled variables (U_0, Θ_0)*

$$\begin{aligned} H_0 &= \frac{\kappa_0}{u_0(V_0)}, \quad \mathcal{M}_0^T \mathcal{M}_0 = -\frac{\kappa_0 \nabla^2 u_0(V_0)}{\kappa_2 u_0(V_0)} = H_0 \frac{\nabla^2 u_0(V_0)}{\kappa_2}, \quad \kappa_0 = \bar{U}(0), \quad \kappa_2 = \partial_1^2 \bar{U}(0), \\ U_0(z) &= H_0 u_0(\mathcal{M}_0^{-1}z + V_0), \quad \Theta_0(z) = \theta_0(\mathcal{M}_0^{-1}z + V_0), \end{aligned} \quad (1.13)$$

where \bar{U} is defined in (1.7). Since $\nabla u(V_0, 0) = 0$, by definition, (1.13) implies the following normalization

$$U_0(0) = \kappa_0 = \bar{U}(0), \quad \nabla U_0(0) = 0, \quad \nabla^2 U_0(0) = \nabla^2 \bar{U}(0) = \kappa_2 I_d. \quad (1.14)$$

¹While the \mathfrak{F}_K norm in (1.9) only involves $\nabla^i \phi, i \geq 1$ and $\mu(t)$ does not play a role in (1.9), we keep $\mu(t)$ in (1.9) to indicate that it captures the phase of ψ . See (1.11), (1.12).

²Simple linear algebra shows that \mathcal{M} is uniquely determined.

Let $\nu > 0$ be small, ϵ_2 and C_b be defined in (1.30) and (3.7), the set of initial data in Theorem 1 consists of initial data (u_0, θ_0) satisfying (1.8) and its rescaled variable (U_0, Θ_0) satisfies

$$U_0 \bar{U}^{-1-\epsilon_2} > 2C_b, \quad H_0^{p-1} < \nu, \quad u_0(V_0)^{-p} \text{tr}(\nabla^2 u_0(V_0)) < \nu, \quad (1.15)$$

and

$$\|W_0\|_{\mathfrak{E}_K} = \|U_0 - \bar{U}\|_{\mathfrak{E}_K} < \nu, \quad \|\Phi_0\|_{\mathfrak{F}_K} := \|\Theta_0 - \delta \log \bar{U}\|_{\mathfrak{F}_{K-1}} + \|\langle z \rangle^{K-\frac{d}{2}} \nabla^K (\Theta_0 - \delta \log \bar{U})\|_{L^2} < \nu. \quad (1.16)$$

The last quantity in (1.15) is invariant under the parabolic rescaling: $u_{0,l}(z) = l^{1/(p-1)} u_0(l^{1/2}z)$. We will use its smallness to show that the viscous terms are small compared to the nonlinear terms. The lower bound $U_0 \bar{U}^{-1-\epsilon} > 2C_b$ in (1.15) ensures that $U_0(z) \neq 0$ for any z , without which can lead to low regularity of rescaled velocity $|U|^p$ of u^p (1.5a).

Remark 2 (Positive definiteness of the Hessian of the initial data). While the limiting blowup profile \bar{U} (1.7) is isotropic near $z = 0$, we do not need to assume that the initial data u_0 is isotropic near V_0 , i.e., $\nabla^2 u_0(V_0)$ is close to cI_d for some $c \neq 0$. By introducing the upper triangular matrix $\mathbf{R}(t)$ in the rescaling (see (2.1)), we can handle a much larger class of initial data with non-degenerate global maximizer (1.8).

Remark 3. The asymptotics of the blowup solution (1.12) recovers the constructed result of Masmoudi-Zaag [43] for the case $d = 1$. The set of initial data leading to the blowup solution described in Theorem 1 is larger than the one in [43] which is only a subset of $L^\infty(\mathbb{R})$. We note that there is a free phase-shift $\hat{\mu}$ in (1.12) corresponding to the phase invariant of (CGL) that was fixed to be $\hat{\mu}(t) = 0$ in [43] by a specific choice of initial data through a topological argument. We remark that the relaxing asymptotics (1.10) and (1.11) are natural for rigorous stability analysis in all dimension $d \geq 1$ treated in this present paper.

Note that the asymptotic behavior (1.9), (1.12) involves the 4 parameter functions H, \mathbf{R}, V and μ (or $\hat{\mu}$) which are responsible for all the symmetries of (CGL).³ Theorem 1 is stated in terms of the rescaled profiles, with a singular weight at the origin. We note that the condition (1.8) and parameters \mathcal{M}_0, H_0, V_0 in (1.13) are C^2 -stable if the global maximizer is unique. We can therefore simplify the assumptions in Theorem 1 to obtain the following stability results with a more explicit description of the open set of initial data.

Theorem 2 (Stability of blowup solutions to (CGL)). Let $K = K(d, p) \in \mathbb{N}$ be defined as in (1.29), and $\mathcal{H}^K, \mathfrak{F}_{K-1}$ be the norms defined in (3.9), (1.6). Suppose that (u_0, θ_0) satisfies the assumptions (1.8), (1.15), (1.16) and V_0 is the unique global maximizer: $u_0(V_0) > u_0(z)$ for all $z \neq V_0$. There exists $\epsilon_0 = \epsilon_0(u_0) > 0$ such that if

$$\|\tilde{u}_0 - u_0\|_{\mathcal{H}^K} + \|(\tilde{u}_0 - u_0)\bar{U}^{-1-\epsilon_2}\|_{L^\infty} + \|\tilde{\theta}_0 - \theta_0\|_{\mathfrak{F}_{K-1}} + \|\langle z \rangle^{K-\frac{d}{2}} \nabla^K (\tilde{\theta}_0 - \theta_0)\|_{L^2} < \epsilon_0, \quad (1.17)$$

the solution $\tilde{\psi} = \tilde{u}e^{\tilde{\theta}}$ to (CGL) with the initial data $\tilde{\psi}_0 = \tilde{u}_0 e^{\tilde{\theta}_0}$ blows up in finite time \tilde{T} . Moreover, there exists $H(t), \mathbf{R}(t), V(t), \mu(t)$ satisfying (1.10) and (1.11) such that (1.9) and (1.12) holds for $(\tilde{u}(t), \tilde{\theta}(t))$ with T being replaced by \tilde{T} .

From the proof of Theorem 2, it can be shown that ϵ_0 depends on u_0 through its certain norms. We do not state the dependence explicitly for simplicity.

Remark 4. The assumptions in Theorems 1, 2 are satisfied, e.g. for $u_0 = C\bar{U}$, $\theta_0 = \bar{\Theta}_0$ with C sufficiently large. The weighted norms $\|\cdot\|_{\mathcal{H}^K}$ (3.9) and $\|\cdot\|_{\mathcal{F}_K}$ (1.6) are well-defined for sufficiently smooth functions with fast decay. We do not require that $u_0 - \tilde{u}_0$ agrees up to $\mathcal{O}(|z - V_0|^k)$, $k > 0$ near the maximizer V_0 of u_0 .

1.2 Dynamic rescaling formulation with extra modulation parameters

The dynamic rescaling formulation or the modulation technique was developed to study singularity formulation in the nonlinear Schrödinger equation [44], [42] numerically and various nonlinear PDEs; see the comprehensive

³Although $H(t)$ is absent in (1.12), we can replace the factor $(T - t)^{1/(p-1)}$ by $H(t)$ using (1.10).

references on (NLH), (NLS), and related models in [53], [26]. Recently, researchers also generalized this technique for fluid mechanics [12, 13, 14], [25]. We can establish singularity in two steps. Firstly, one constructs an approximate steady state of the dynamic rescaling equation (analytically or numerically). Secondly, one performs linear and nonlinear stability analysis for perturbation around the approximate steady state with appropriate normalization conditions. The law of blowup will then be prescribed by the normalizing constants. One can establish stability using a L^2 -based [16, 17] or L^∞ -based [13] argument. In these arguments, one of the crucial steps is to design appropriate singular weights depending on the profile, and then use the weights to derive damping terms for the energy estimates. The approach does not require an explicit profile and is robust to small perturbation, which makes it possible to combine weighted energy estimates for stability analysis, a numerical *implicit* profile, and computer-assisted proofs to construct blowup solutions. See for example, [13, 14] for applications in 3D incompressible Euler equations with smooth data and [16, 17], [10], [38] for related 1D models.

In [37], the authors generalized the L^2 -based methodology to establish a type-I⁴ blowup for the semilinear heat equation beyond the self-similar setting where there is a logarithm correction in the self-similar scaling for the spatial variable. Compared with the mentioned works [4], [34], [45], [43] where the authors heavily relied on a spectral analysis with detailed properties of the associated linearized operator to establish the existence and stability, we simply suppresses unstable directions and neutral modes via a clear characterization of weighted Sobolev spaces, without using Brouwer's fixed-point theorem or a topological argument. The correct Type I blowup rate is automatically inferred by enforcing proper vanishing conditions of the perturbation. We remark that for Type I blowup, there is indeed a link between the vanishing conditions at the origin and the vanishing coefficients projected onto the unstable and neutral spectral eigenfunctions; see for example [37] and [45] in the case of the nonlinear heat equation, where the eigenfunctions correspond to Hermite polynomials. A further explanation of this connection is discussed in Step 2 in Section 1.3.

A generalized dynamical rescaling formulation with extra modulation parameters⁵. In this article, we further generalize the above framework and the ideas in [37]. In particular, it consists of three main steps:

Step 1 (Renormalization with extra modulation parameters): The renormalization is an essential step in the study of nonlinear PDEs with symmetries including incompressible/compressible fluids equations. For the equation (CGL), there are trivial symmetries (see Section 2.1) from which we introduce the following renormalization in terms of the amplitude-phase representation $\psi(x, t) = u(x, t)e^{i\theta(x, t)}$,

$$U(z, \tau) = H(\tau)u(\mathbf{R}(\tau)z + V(\tau), t(\tau)), \quad \Theta(z, \tau) = \theta(\mathbf{R}(\tau)z + V(\tau), t(\tau)), \quad t(\tau) = \int_0^\tau H^{p-1}(s)ds, \quad (1.18)$$

where $\mathbf{R}(\tau) \in \mathbb{R}^{d \times d}$ is a upper triangular matrix, $V(\tau) \in \mathbb{R}^d$ and $H(\tau) \in \mathbb{R}_+$. Here, H is responsible for time, V for spatial translation.

The key novelty is that in addition to modulation parameters corresponding to the symmetries, we introduce *extra modulation parameters*. Instead of applying the same rescaling to z_i , we rescale z_i with *different but similar* scalings following the ideas [37]. See also a recent work [36] on the generalized Navier-Stokes equations, in which the author developed a generalized dynamic rescaling formulation by using different rescalings for the r and z directions respectively and a self-similar blowup was observed numerically. In the case of (CGL), we rescale z_i with scaling slightly perturbed from the parabolic scaling. We remark that the choices of different scalings for z_i violate the scaling symmetries. Yet, in the case of (CGL), we will show that the violation is asymptotically small and $\mathbf{R}(\tau)$ converges to $c(\tau)I_d$ asymptotically for some scalar function $c(\tau)$. Therefore, the above renormalization (1.18) asymptotically agrees with the classical dynamical rescaling formulation [13, 14], [16, 17]. These extra parameters provide us extra $d - 1$ degrees of freedom, and we have crucial $1 + d + \frac{d(d+1)}{2}$ degrees of freedom in total in choosing the dynamic variables $H(\tau), V(\tau), \mathbf{R}(\tau)$. A natural idea to represent the scaling in z_i and capture the rotation symmetries is to choose $\mathbf{R}(\tau) = D(\tau)Q(\tau)$

⁴A blowup solution to (CGL) is of Type I if it satisfies the bound $\lim_{t \rightarrow T} (T - t)^{-\frac{1}{p-1}} \|u(t)\|_\infty < \infty$, otherwise, blowup is of Type II.

⁵The modulation parameters are also known as normalization constants in the dynamic rescaling formulation.

with a diagonal matrix D and an orthogonal matrix Q . Yet, it is challenging to parametrize a time-dependent orthogonal matrix in $\mathbb{R}^{d \times d}$. Instead, we use the upper triangular matrix $\mathbf{R}(\tau)$ with $\frac{d(d+1)}{2}$ parameters.

To determine these modulation parameters, we impose normalization conditions on $\nabla^i U(0), i = 0, 1, 2$, so that the perturbation of U vanishes $O(|z|^3)$ near $z = 0$ and we can perform weighted energy estimates mentioned above. Note that the number of (different) equations and that of the degrees of freedom are *exactly* the same. For (CGL), these conditions allow us to completely eliminate the unstable and neutrally stable modes of the linearized operator. See Step 2 in Section 1.3 for more details.

Step 2 (Equations of the profiles and modulation parameters): We derive the equations of $F = (U, \Theta)$ and matrix (or vectors) $\mathcal{Q}(\tau)$ governing the modulation parameters $\mathbf{R}(\tau), V(\tau), H(\tau)$,

$$\partial_\tau F = \mathcal{N}_F(F, \mathcal{Q}), \quad \frac{d}{d\tau} \mathcal{Q} = \mathcal{N}_\mathcal{Q}(F, \mathcal{Q}), \quad (1.19)$$

where \mathcal{N}_F is a nonlinear function and $\mathcal{N}_\mathcal{Q}$ is a matrix. Then the rescaling system is completely determined, and we further construct the approximate steady state $(\bar{F}, \bar{\mathcal{Q}})$ analytically or numerically.

Step 3 (Stability analysis and the log correction): In general, we *do not* know *a-priori* that the approximate steady state $(\bar{F}, \bar{\mathcal{Q}})$ is stable in some suitable topology. Nevertheless, if we can establish stability of $(\bar{F}, \bar{\mathcal{Q}})$ following the strategy mentioned above and $H(\tau)^{p-1}$ is integrable, then we can obtain finite time blowup using (1.18) and the law of blowup will then be prescribed by the normalizing constants.

For (CGL), we will use energy method with an energy E for the perturbation $F - \bar{F}$ to establish

$$\frac{d}{d\tau} E \leq -c_1 \cdot E + C \text{tr}(\mathcal{Q}) + l.o.t., \quad \frac{d}{d\tau} \text{tr}(\mathcal{Q}) \leq -c_2 \cdot \text{tr}(\mathcal{Q})^2 + l.o.t.,$$

for some $c_1, c_2, C > 0$, where \mathcal{Q} further satisfies that it is a positive definite matrix and *l.o.t.* denotes some terms that are very small. The second ODE of $\text{tr}(\mathcal{Q})$ further implies that $|\text{tr}(\mathcal{Q})| \lesssim (1 + \tau)^{-1}$. A further refinement of this algebraic decay in the self-similar time implies a log correction $\log(T - t)$ in the blowup rate. We will elaborate more in Step 2 in Section 1.3.

One can thus hope to combine the above method for a log correction and the framework [13, 14] to problems with numerical steady states, while spectral analysis heavily hinges on a simple and analytical approximate steady state with explicit (nonlinear heat [45]) or at least asymptotical spectral information of the linearized operator (Keller-Segel [18]). For example, constructing a smooth (approximate) steady state analytically for 3D incompressible Euler or Navier-Stokes equations (NSE) is challenging and remains an open problem. On the other hand, constructing a numerical approximate steady state with computer-assistance is much more feasible. See [13, 14], [59] for the construction in 3D Euler equations. For NSE, self-similar blowup with a perfect self-similar scaling has been ruled out [57], [48]. Yet, one can construct a blowup violating these non-blowup results by adding a log correction in the spatial variable. See numerical evidence on the singular behavior of NSE with a potential logarithm correction in the potential blowup by the second author [35].

1.3 Ideas of the blowup analysis

We first discuss some of difficulties in the study of singularity formation in (CGL). Then we follow the generalized dynamic rescaling framework to establish the existence and stability of asymptotically self-similar blowup solutions to (CGL) by briefly discussing the strategy and main ideas of our analysis.

Difficulties: Compared with (NLH) or (NLS), the analysis for the complex Ginzburg-Landau equation (CGL) has the following additional challenges.

1. The complex Ginzburg-Landau equation (CGL) is not of a gradient form, rendering energy estimates hard. To overcome this challenge, we use the amplitude-phase representation (1.4), (1.5) to analyze (CGL).
2. We remove the even symmetry assumption of the perturbation required in [37] to recover full stability. Without the even symmetry assumption, we have more potentially unstable modes for the linearized operator.

We control these unstable modes using the generalized dynamic rescaling formulation in Step 1 in Section 1.2.

3. We consider the whole range of the nonlinearity $p > 1$. For the analysis of the phase equation (1.5b) and the nonlinearity u^p (1.5a), (1.5b), we need to bound the rescaled amplitude U from below, which we establish using the maximal principle and a weighted L^∞ estimate. Due to the non-integer power p to control $\nabla^K(U^p)$ in the H^K estimate, which leads to terms like $U^{p-K}(\nabla U)^K$, we need to obtain sharp decay estimates for $\nabla^i U$. This is done by choosing an almost tight power in the far field of the weight for the weighted H^k energy estimates and using interpolation and embedding inequalities following [11]. An additional difficulty comes from the coupling between u, θ in the viscous terms in (1.5a), (1.5b). We design the top order energy with a special algebraic structure to cancel out the top order terms and show that the viscous terms have a good sign. See Step 3(b) in Section 1.3.

Ideas and strategy: We briefly discuss the strategy and main ideas of our analysis.

Step 1 (Dynamical rescaling formulation): We follow Step 1 in Section 1.2 to perform the rescaling (1.18). Then, we introduce the following factors governing the evolution of these parameters

$$\frac{H_\tau}{H} = c_U, \quad c_U = -\frac{1}{p-1} + c_W, \quad \mathcal{M} = e^{-\frac{\tau}{2}} \mathbf{R}^{-1}, \quad \mathcal{V} = -\mathbf{R}^{-1} V_\tau, \quad \mathcal{P} = \mathcal{M}_\tau \mathcal{M}^{-1}. \quad (1.20)$$

Step 2 (Normalization and vanishing conditions): Let \bar{U} be the profile defined in (1.7). To determine the law for the parameter functions $H(\tau), V(\tau), \mathbf{R}(\tau)$, we enforce the following normalization conditions on the amplitude U :

$$k = 0, 1, 2, \quad \nabla^k U(0, \tau) = \nabla^k \bar{U}(0). \quad (1.21)$$

Since $\nabla^2 U \in \mathbb{R}^{d \times d}$ is symmetric, we have $1 + d + \frac{d(d+1)}{2}$ different equations, which match the degrees of freedom of the dynamic variables exactly. The above conditions determine the initial modulation parameters $H(0), \mathbf{R}(0), V(0)$ and the leading order system of $c_W, \mathcal{P}, \mathcal{V}$ (1.20)

$$c_W = \frac{2(1-\beta\delta)}{4b_*} \text{tr}(\mathcal{Q}) + O(\mathcal{E}_0), \quad \mathcal{V} = O(\mathcal{E}_0), \quad \mathcal{P} = O(|\mathcal{Q}| + \mathcal{E}_0), \quad \text{where } \mathcal{Q} = H^{p-1} e^\tau \mathcal{M} \mathcal{M}^T, \quad (1.22)$$

where \mathcal{E}_0 tracks some lower order terms depending on the perturbation (W, Φ) (1.24a), (1.24b) and \mathcal{Q} . The main unknown $\mathcal{Q} \in \mathbb{R}^{d \times d}$ (a positive definite matrix) solves the following ODE

$$\frac{d}{d\tau} \text{tr}(\mathcal{Q}) = -\text{tr}(\mathcal{Q}^2) + O(\mathcal{E}_0 |\mathcal{Q}|) \leq -\frac{1}{d} (\text{tr}(\mathcal{Q}))^2 + O(\mathcal{E}_0 |\mathcal{Q}|). \quad (1.23)$$

From (1.23), (1.22), (1.20), we can control all the modulation parameters. A refined estimate using $\text{tr}(\mathcal{Q})$ and $\text{tr}(\mathcal{Q}^{-1})$ yields $\mathcal{Q} = \frac{1}{\tau} \mathbf{I}_d + O(\tau^{-3/2+})$, together with an asymptotic refinement of the phase yield the asymptotics in Theorem 1. See Section 2.3 for deriving (1.22), Section 4 for the estimates of \mathcal{Q} and Proposition 5 for the refinement of the phase.

Roughly speaking, imposing (1.21) for U is equivalent to imposing local orthogonality conditions for the perturbation $W = U - \bar{U}$ to $1, z_i, z_i z_j, 1 \leq i, j \leq d$. These functions are all the neutrally stable and unstable modes of $L = \text{Id} - \frac{1}{2} z \cdot \nabla$, which behaves similarly to the main linearized operator $\mathcal{L}_{\bar{U}}$ in (1.24a) for $|z|$ small. We then get a damping in the weighted L^2 energy estimate.

Step 3 (Stability analysis): We linearize (U, Θ) around the approximate steady state $(\bar{U}, \bar{\Theta})$ defined in (1.7) and (1.27) and obtain the equations for the perturbation $W = U - \bar{U}, \Phi = \Theta - \bar{\Theta}$,

$$W_\tau = \mathcal{L}_{\bar{U}} W + \mathcal{F}_U + \mathcal{N}_U + \mathcal{D}_U, \quad \mathcal{L}_{\bar{U}} W = \left(-\frac{1}{p-1} + p\bar{U}^{p-1} - \frac{1}{2} z \cdot \nabla \right) W \quad (1.24a)$$

$$\Phi_\tau = -\frac{1}{2} z \cdot \nabla \Phi + \mathcal{F}_\Theta + \mathcal{N}_\Theta + \mathcal{D}_\Theta, \quad (1.24b)$$

where $\mathcal{F}_U, \mathcal{N}_U, \mathcal{D}_U, \mathcal{F}_\Theta, \mathcal{N}_\Theta, \mathcal{D}_\Theta$ are small residue, nonlinear, and viscous terms (3.3), and will be treated perturbatively. Below, we outline the stability estimates and focus on the linear part and the viscous terms.

(a) *Estimates of the (W, Φ) :* $\mathcal{L}_{\bar{U}}$ to the leading order is the linearized equation for Riccati equation or semilinear heat equation. With the vanishing conditions $\nabla^k W = 0, k = 0, 1, 2$ (1.21), we obtain its stability using weighted H^k estimates with singular weights. See [37] in the case of $p = 2$ and [15], [16, 17], [10].

The right-hand side of (1.24) only involves $\nabla\Phi$, which enjoys better stability. We estimate $\nabla\Phi$ by performing a similarly weighted H^k estimate (starting from $k = 1$) on (1.24b) and exploiting the term $-\frac{1}{2}z \cdot \nabla\Phi$.

In the nonlinear estimates and the estimates of the phase, we need to control $1/U$. We use the maximal principle and a weighted L^∞ estimate to obtain a lower bound of U .

(b) *Estimates of the viscous terms:* For the viscous terms $\mathcal{D}_U, \mathcal{D}_\Theta$ (2.9), the main difficulty is the coupling between U and Θ . At the top H^K estimate, the highest order derivative terms read

$$\begin{aligned}\nabla^K \mathcal{D}_U &= \Delta_{\mathcal{Q}} \nabla^K U - \beta U \Delta_{\mathcal{Q}} \nabla^K \Theta + l.o.t. := I_1 + I_2 + l.o.t., \\ \nabla^K \mathcal{D}_\Theta &= \beta \frac{\Delta_{\Theta} \nabla^K U}{U} + \Delta_{\mathcal{Q}} \nabla^K \Theta + l.o.t. := I_3 + I_4 + l.o.t.,\end{aligned}$$

where $\Delta_{\mathcal{Q}} F$ is a weighted elliptic operator defined in (2.6). The terms I_1, I_4 lead to damping terms of $\nabla^{K+1}U$ and $\nabla^{K+1}\Theta$ via integration by parts. To control I_2, I_3 , we exploit their cancellation using the energy J_1 below with some weight ρ_K independent of U, Θ and couple their estimates in J_2

$$J_1 = \int (|\nabla^K W|^2 + U^2 |\nabla^K \Phi|^2) \rho_K, \quad J_2 = \int ((-\beta U \Delta_{\mathcal{Q}} \nabla^K \Theta) \cdot \nabla^K U + (\beta \frac{\Delta_{\Theta} \nabla^K U}{U}) \cdot U^2 \nabla^K \Phi) \rho_K. \quad (1.25)$$

Applying integration by parts, J_2 reduces to some lower order terms and we can close the viscous estimates. For estimates of intermediate-order terms, we use interpolation inequalities following [11].

(c) *Choosing the weights:* To extract damping in the energy estimates, we need to design various suitable weights. The weights (1.28) for W are very similar to those of the semilinear heat equation [37]. They are singular near $z = 0$ for the lower order energy estimates and regular for the top order energy estimates so that the viscous terms will have a good sign. In addition, we choose an almost optimal rate at the infinity for these weights to obtain a sharp decay estimate for (W, Φ) using interpolation and embedding following [11].

Organization of the paper: The rest of the paper is organized as follows. In Section 2, we introduce the generalized dynamic rescaling formulation using the symmetries of (CGL) and derive the ODEs governing the modulation parameters. Section 3 is devoted to the stability analysis of the profile. In Section 4, we establish the asymptotics of the blowup rate. In Section 4.3, we prove Theorem 1 and Theorem 2.

Notations: We use i to denote the imaginary number, \bar{f} to denote approximate profiles for the variable f , e.g., \bar{U} , rather than conjugates, and (\cdot, \cdot) to denote the inner product on \mathbb{R}^d : $(f, g) = \int_{\mathbb{R}^d} fg$. For a weight ρ , we denote $\|f\|_\rho = (|f|^2, \rho)^{1/2}$. For matrix notations, we use $\text{tr}(R)$ to denote the trace of a matrix R , $\mathbf{T}^u(R)$ to denote the upper triangular part of R ; namely $(\mathbf{T}^u(R))_{ij} = R_{ij} \mathbf{1}_{i \leq j}$. We use $\delta_{ij} = \mathbf{1}_{i=j}$ to denote the Kronecker delta function, and $|\mathbf{T}| := (\sum_i \mathbf{T}_i^2)^{1/2}$ with summation over all entries \mathbf{T}_i to denote the tensor norm of a tensor \mathbf{T} , e.g., higher-order derivatives $\nabla^k f$. We use C to denote an absolute constant only dependent on the constants p, β, δ, γ and the dimension d , which may vary from line to line. $C(\mu)$ denotes a constant depending on μ . We denote $A = O(B)$ or $A \lesssim B$ if there exists an absolute constant $C > 0$, such that $|A| \leq CB$, and denote $A \approx B$ if $A \lesssim B$ and $B \lesssim A$. Furthermore, we denote

$$\Lambda = z \cdot \nabla, \quad \langle z \rangle = \sqrt{1 + |z|^2}. \quad (1.26)$$

Parameters and special functions: We introduce

$$\bar{\Theta} = \frac{\delta}{p-1} \tau + \delta \log \bar{U}, \quad c_p = \frac{(p-1)^2}{4b_*}, \quad \sigma = -\frac{2}{p-1}. \quad (1.27)$$

We choose the weights for the H^k estimates as follows:

$$\begin{aligned} \rho_k &= |z|^{-6+\epsilon-d+2k} + c_0 |z|^{-2\sigma-\epsilon-d+2k}, \quad 0 \leq k \leq \frac{d+5}{2}, \quad \rho_k = 1 + c_1 |z|^{-2\sigma-\epsilon-d+2k}, \quad \frac{d+5}{2} < k, \\ \dot{\rho}_k &= |z|^{2k-1-d}, \quad 0 < k \leq \frac{d}{2}, \quad \dot{\rho}_k = 1 + |z|^{2k-1-d}, \quad \frac{d}{2} < k < K, \quad \dot{\rho}_K = U^2 \rho_K, \end{aligned} \quad (1.28)$$

where we determine the constants in the following order:

$$K = 2d + 4 + 2 \left\lceil \frac{p+1}{\min\{p-1, c_p\}} \right\rceil, \quad (1.29)$$

$$\epsilon = \min\left\{ \frac{p-1}{5(p+3)(K+p)}, \frac{4}{5(p+3)(K+p)} \right\}, \quad \epsilon_2 = \frac{(p-1)\epsilon}{4}. \quad (1.30)$$

For c_0, c_1 used in (1.28), we determine c_0 via (3.26) and c_1 via (3.47). Note that c_0, c_1 only depends on K, ϵ, p, δ . Hence, they are considered as fixed constants throughout the paper.

2 Generalized dynamical rescaling formulation

In this section, we introduce a generalized dynamic rescaling formulation and decompose the complex Ginzburg-Landau equation into the equation of the phase and the amplitude. We will consider a linearization around the approximate profiles and choose the modulation parameters based on the vanishing conditions. Finally, we will estimate the ODE of the modulation parameters.

2.1 Symmetries and renormalization

We exploit the following symmetries of equation (CGL) to study stability for general perturbation, which will motivate our choice of rescaling. If $\psi(x, t)$ solves (CGL), all of the following also solves (CGL):

1. Phase shift: $\psi_a(x, t) := e^{ia}\psi(x, t)$, for $a \in \mathbb{R}$.
2. Parabolic scaling for $\gamma = 0$: $\psi^l(x, t) := l^{1/(p-1)}\psi(l^{1/2}x, lt)$, for $l \in \mathbb{R}$.
3. Translation: $\psi_b(x, t) := \psi(x - b, t)$, for $b \in \mathbb{R}^d$.
4. Rotation: $\psi^R(x, t) := \psi(Rx, t)$, for orthogonal matrix $RR^T = I_d$.

We use the symmetry groups of the parabolic scaling via a rescale in amplitude, of the translation via a shift in space, and of the rotation via a rotation and rescaling in the spatial variable parametrized by an upper triangular matrix. In sum, we can exploit modulation with $1 + d + \frac{(d+1)d}{2}$ degree of freedom. The phase shift invariance corresponding to a constant addition in Θ is taken care of in (1.11) and (1.12). It is irrelevant to the dynamical modulation of stability since the right-hand sides of (1.5a) and (1.5b) only involve the derivatives of Θ . We remark that the modulation of symmetries due to Galilean transformations, including the rotation symmetry, has been used successfully to obtain shock formation in compressible Euler equations with fine characterization [5]. Below, we will use a general upper triangle matrix, which simplifies the parametrization of the time-dependent orthogonal matrix.

For solution ψ to (CGL), we consider the amplitude-phase form $\psi(x, t) = u(x, t)e^{i\theta(x, t)}$, where $u(t) : x \in \mathbb{R}^d \rightarrow \mathbb{R}_+$ and $\theta(t) : x \in \mathbb{R}^d \rightarrow \mathbb{R}$. For the amplitude u and phase ψ , we introduce the generalized dynamic rescaling formulation

$$U(z, \tau) = H(\tau)u(\mathbf{R}(\tau)z + V(\tau), t(\tau)), \quad \Theta(z, \tau) = \theta(\mathbf{R}(\tau)z + V(\tau), t(\tau)), \quad (2.1)$$

where the main unknown parameter functions are $\mathbf{R} \in \mathcal{C}^1([\tau_0, +\infty), \mathbb{R}^{d \times d})$ a non degenerate upper triangular matrix, $V \in \mathcal{C}^1([\tau_0, +\infty), \mathbb{R}^d)$ and H is given by

$$H = H(0) \exp\left(\int_0^\tau c_U(s) ds\right), \quad t(\tau) = \int_0^\tau H^{p-1}(s) ds. \quad (2.2)$$

In a compact form, we have

$$U(z, \tau)e^{i\Theta(z, \tau)} = H(\tau)(ue^{i\theta})(\mathbf{R}(\tau)z + V(\tau), t(\tau)), \quad \psi = ue^{i\theta}. \quad (2.3)$$

We decompose the solution into the approximate steady states (1.27), with perturbations W, Φ :

$$U = \bar{U} + W, \quad \Theta = \bar{\Theta} + \Phi, \quad c_U = -\frac{1}{p-1} + c_W, \quad H = e^{-\frac{\tau}{p-1}} C_W. \quad (2.4)$$

If $\mathbf{R}(\tau)$ is a scalar factor and $V = 0$, (2.3) reduces to the standard dynamic rescaling formulation, see e.g., [17, 16, 12]. If $\mathbf{R}(\tau)$ is a diagonal matrix and $V = 0$, it reduces to a formulation similar to [37]. We will show that \mathbf{R} is close to some identity matrix and V is some lower order term.

We first compute the spatial derivative

$$\nabla U = He^{-i\Theta} \nabla \psi \mathbf{R} - iH\psi e^{-i\Theta} \nabla \Theta,$$

$$\nabla^2 U = He^{-i\Theta} \mathbf{R}^T \nabla^2 \psi \mathbf{R} - iHe^{-i\Theta} (\nabla \psi \mathbf{R} \nabla \Theta^T + \nabla \Theta \mathbf{R}^T \nabla \psi^T) - H\psi e^{-i\Theta} \nabla \Theta \nabla \Theta^T - iH\psi e^{-i\Theta} \nabla^2 \Theta.$$

We then write from (CGL) the equation for U ,

$$\begin{aligned} U_\tau = & -i\Theta_\tau U + c_U U - \left(\frac{1}{2}z + \mathcal{P}z + \mathcal{V}\right) \cdot \nabla U - iU \left(\frac{1}{2}z + \mathcal{P}z + \mathcal{V}\right) \cdot \nabla \Theta + (1 + i\delta)U^p - C_U^{p-1} \gamma U \\ & + (1 + i\beta)(\Delta_{\mathcal{Q}} U + 2i\langle \nabla U, \nabla \Theta \rangle_{\mathcal{Q}} - U\langle \nabla \Theta, \nabla \Theta \rangle_{\mathcal{Q}} + iU\Delta_{\mathcal{Q}} \Theta), \end{aligned}$$

where \mathcal{P}, \mathcal{V} are related to the matrix \mathbf{R} as

$$\mathcal{M}^{-1} = e^{-\tau/2} \mathbf{R}, \quad \mathcal{V} = -\mathbf{R}^{-1} \dot{V}, \quad \mathcal{P} = \dot{\mathcal{M}} \mathcal{M}^{-1}, \quad \mathcal{Q} := C_W^{p-1} \mathcal{M} \mathcal{M}^T, \quad (2.5)$$

and we use the notation

$$\Delta_{\mathcal{Q}} f := \text{tr}(\mathcal{Q} \nabla^2 f), \quad \langle x, y \rangle_{\mathcal{Q}} := x^T \mathcal{Q} y, \quad \forall x, y \in \mathbb{R}^d. \quad (2.6)$$

Taking the real and imaginary parts we arrive at the following equations for U and Θ :

$$U_\tau = c_U U - \left(\frac{1}{2}z + \mathcal{P}z + \mathcal{V}\right) \cdot \nabla U + U^p - C_U^{p-1} \gamma U + \mathcal{D}_U, \quad (2.7)$$

$$\Theta_\tau = -\left(\frac{1}{2}z + \mathcal{P}z + \mathcal{V}\right) \cdot \nabla \Theta + \delta U^{p-1} + \mathcal{D}_\Theta, \quad (2.8)$$

where \mathcal{D}_U and \mathcal{D}_Θ consists of the viscous terms and the nonlinear quadratic term we define the viscous terms as follows:

$$\mathcal{D}_U = \Delta_{\mathcal{Q}} U - 2\beta \langle \nabla U, \nabla \Theta \rangle_{\mathcal{Q}} - U \langle \nabla \Theta, \nabla \Theta \rangle_{\mathcal{Q}} - \beta U \Delta_{\mathcal{Q}} \Theta, \quad (2.9a)$$

$$\mathcal{D}_\Theta = \beta \frac{\Delta_{\mathcal{Q}} U}{U} + 2 \frac{\langle \nabla U, \nabla \Theta \rangle_{\mathcal{Q}}}{U} - \beta \langle \nabla \Theta, \nabla \Theta \rangle_{\mathcal{Q}} + \Delta_{\mathcal{Q}} \Theta. \quad (2.9b)$$

We will show that the diffusion and $\mathcal{Q}, c_W, \mathcal{V}, \mathcal{P}, H^{p-1}$ are lower order terms. See Remark 5. Dropping these terms and setting $\partial_\tau U = 0$, we obtain the leading order parts of (2.7) and (2.8):

$$-\frac{1}{p-1} \bar{U} - \frac{1}{2} \Lambda \bar{U} + \bar{U}^p = 0, \quad \bar{\Theta}_\tau = -\frac{1}{2} \Lambda \bar{\Theta} + \delta \bar{U}^{p-1},$$

whose solution are given by the approximate profiles $(\bar{U}, \bar{\Theta})$ defined in (1.7), (1.27).

2.2 Initial rescaling and normalization conditions

We will choose some proper initial modulation parameters $H(0), V(0), \mathcal{M}(0)$ and the dynamic variables $c_W, \mathcal{V}, \mathcal{P}$ such that the perturbation W vanishes to the third order. We denote the following constants

$$\kappa_0 = \bar{U}(0) = (p-1)^{-\frac{1}{p-1}}, \quad \kappa_2 = \partial_1^2 \bar{U}(0) = -\frac{2c_p \kappa_0}{(p-1)^2}, \quad \kappa_4 = \partial_1^4 \bar{U}(0) = \frac{12pc_p^2 \kappa_0}{(p-1)^4}. \quad (2.10)$$

Given initial data (u, θ) (2.3) satisfying (1.8), we first define $\mathcal{M}_0, V_0, H(0), \Theta_0, U_0$ using (1.13). Then we determine other initial rescalings and initial data using

$$V(0) = V_0, \quad \mathcal{M}(0) = \mathcal{M}_0, \quad \mathbf{R}(0) = \mathcal{M}_0^{-1}, \quad \Theta(z, 0) = \Theta_0(z), \quad U(z, 0) = U_0(z).$$

We impose the following normalization conditions in time as

$$U(0, \tau) = \bar{U}(0) = \kappa_0, \quad \nabla U(0, \tau) = \nabla \bar{U}(0) = 0, \quad \nabla^2 U(0, \tau) = \nabla^2 \bar{U}(0) = \kappa_2 I_d.$$

From (1.14), the above holds for $\tau = 0$. By the ansatz (2.3), it reduces a dynamical condition in time

$$\partial_\tau \nabla^k U(0, \tau) = 0, \quad k = 0, 1, 2,$$

which we can use (2.7) to simplify as

$$\begin{aligned} c_U + \kappa_0^{p-1} - H^{p-1} \gamma + \frac{\mathcal{D}_U(0)}{\kappa_0} &= 0, \\ \kappa_2 \mathcal{V} &= \nabla \mathcal{D}_U(0), \\ (c_U - 1 + p\kappa_0^{p-1} - H^{p-1} \gamma) \delta_{ij} - (\mathcal{P}_{ij} + \mathcal{P}_{ji}) + \frac{\partial_{ij} \mathcal{D}_U(0) - \mathcal{V} \cdot \nabla \partial_{ij} U(0)}{\kappa_2} &= 0, \end{aligned} \quad (2.11)$$

for any indices i, j . Notice that the inverse of an upper-triangular matrix is still upper-triangular, and as a consequence $\mathcal{P} = \mathcal{M}_\tau \mathcal{M}^{-1}$ is upper-triangular. We can further simplify the equations for c_U and \mathcal{P} by the ansatz (2.4) as follows:

$$c_W = -\frac{\mathcal{D}_U(0)}{\kappa_0} + H^{p-1} \gamma, \quad (1 + \delta_{ij}) \mathcal{P}_{ij} = -\frac{\mathcal{D}_U(0)}{\kappa_0} \delta_{ij} + \frac{\partial_{ij} \mathcal{D}_U(0) - \mathcal{V} \cdot \nabla \partial_{ij} W(0)}{\kappa_2}, \quad (2.12)$$

for any $i \leq j$. We will estimate equations (2.12) and (2.11) in the next subsection.

2.3 ODE for the modulation parameters

In this subsection, we simplify equations (2.12) and (2.11) to derive a leading order ODE. We will treat the perturbations W, Φ as low-order terms and estimate them in Section 3. Denote

$$\Gamma = \max_{0 \leq k \leq 5, 1 \leq l \leq 5} (\|\nabla^k W\|_\infty, \|\nabla^l \Phi\|_\infty), \quad \mathcal{E}_0 = |\mathcal{Q}|(\Gamma + \Gamma^4) + H^{p-1}. \quad (2.13)$$

Clearly, we have $\Gamma^i |\mathcal{Q}| \lesssim \mathcal{E}_0, 1 \leq i \leq 4$. We use \mathcal{E}_0 to track some lower order terms.

Lemma 1. *We have the following estimates for the modulation parameters:*

$$c_W = \frac{2(1 - \beta\delta)}{(p-1)^2} c_p \text{tr}(\mathcal{Q}) + \mathcal{O}(\mathcal{E}_0), \quad \mathcal{V} = \mathcal{O}(\mathcal{E}_0), \quad (2.14)$$

and

$$\mathcal{P} = \mathcal{O}(|\mathcal{Q}|(1 + \Gamma^4) + H^{p-1}), \quad \mathcal{Q}_\tau = -(\mathcal{Q}_u + \frac{1}{2} \mathcal{Q}_d) \mathcal{Q} - \mathcal{Q}(\mathcal{Q}_u^T + \frac{1}{2} \mathcal{Q}_d) + \mathcal{O}(\mathcal{E}_0 |\mathcal{Q}|), \quad (2.15)$$

where $\mathcal{Q}_u, \mathcal{Q}_d$ are the strictly upper part and diagonal part of \mathcal{Q} .

Remark 5. Formally to the leading order when we take the trace, we have

$$\mathrm{tr}(\mathcal{Q})_\tau \approx -\mathrm{tr}(\mathcal{Q}^2), \quad \mathrm{tr}(\mathcal{Q}^{-1})_\tau \approx -d.$$

Recall that $\mathcal{Q} = C_W^{p-1} \mathcal{M} \mathcal{M}^T$ is positive. We can estimate $\mathrm{tr}(\mathcal{Q})$, $\mathrm{tr}(\mathcal{Q}^{-1})$ and obtain $\mathcal{Q} \approx \tau^{-1} \mathbf{I}_d$. Therefore $\mathcal{Q}, c_W, \mathcal{V}, \mathcal{P}$ are indeed small and the viscous terms can be treated perturbatively. We will make this heuristic rigorous by choosing $H(0) = C_W(0)$ small; see Section 3.6. Although we allow anisotropy in the initial data, the profile will converge to a isotropic one, i.e. \mathcal{Q} will converge to a diagonal matrix.

Proof. Notice that by (2.9a), we have

$$\mathcal{D}_U = \mathrm{tr}(\mathcal{Q} S_1), \quad S_1 = \nabla^2 U - 2\beta \nabla U \nabla \Theta^T - U \nabla \Theta \nabla \Theta^T - \beta U \nabla^2 \Theta. \quad (2.16)$$

We can simplify (2.12) by an asymptotic expansion of S_1 near the origin up to the second order.

$$\begin{aligned} \bar{U} &= \kappa_0 + \frac{\kappa_2}{2} |z|^2 + \frac{\kappa_4}{24} |z|^4 + O(|z|^6), \quad \nabla \bar{\Theta} = \frac{\delta}{\bar{U}} \nabla \bar{U} = \delta \frac{\kappa_2}{\kappa_0} z + O(|z|^3), \\ \nabla^2 \bar{U} &= \kappa_2 I_d + \frac{\kappa_4}{6} |z|^2 I_d + \frac{\kappa_4}{3} z z^T + O(|z|^4), \\ \nabla^2 \bar{\Theta} &= \frac{\delta}{\bar{U}} \nabla^2 \bar{U} - \frac{\delta}{\bar{U}^2} \nabla \bar{U} \nabla \bar{U}^T = \delta \left(\frac{\kappa_2}{\kappa_0} I_d + \left(\frac{\kappa_4}{6\kappa_0} - \frac{\kappa_2^2}{2\kappa_0^2} \right) (|z|^2 I_d + 2z z^T) \right) + O(|z|^4). \end{aligned}$$

Decomposing $U = \bar{U} + W$, $\Theta = \bar{\Theta} + \Phi$ (2.4) and using Γ (2.13) to control the perturbation W, Φ , we expand

$$\begin{aligned} S_1 &= \kappa_2 (1 - \beta \delta) I_d + \left(\frac{\kappa_4}{3} + (-2\beta - \delta) \delta \frac{\kappa_2^2}{\kappa_0} - \beta \delta \left(\frac{\kappa_4}{3} - \frac{\kappa_2^2}{\kappa_0} \right) \right) z z^T \\ &\quad + \left(\frac{\kappa_4}{6} - \beta \delta \left(\frac{\kappa_2^2}{2\kappa_0} + \frac{\kappa_4}{6} - \frac{\kappa_2^2}{2\kappa_0} \right) \right) |z|^2 I_d + O(|z|^4 + \Gamma + \Gamma^3). \end{aligned}$$

The estimates for $\nabla^i S_1$ are similar. We have chosen Γ (2.13) to control $\nabla^k W, \nabla^{k+1} \Phi$ with high enough order k . In particular, for an error term I in S_1 bounded by $|z|^4 + \Gamma + \Gamma^3$, e.g., $(U - \bar{U}) \nabla^2 (\Theta - \bar{\Theta})$, we have

$$|\nabla^i I| \lesssim |z|^{4-i} + \Gamma + \Gamma^3, \quad i = 1, 2.$$

Since we only need the expression at $z = 0$, the error term $O(|z|^j)$, $j \geq 1$ vanishes in the following derivations. For this reason, we do not track the constant associated with $|z|^j$.

As a consequence, we have the expressions for derivatives of \mathcal{D}_U as

$$\begin{aligned} \mathcal{D}_U(0) &= \kappa_2 (1 - \beta \delta) \mathrm{tr}(\mathcal{Q}) + O(\mathcal{E}_0), \quad \nabla \mathcal{D}_U(0) = O(\mathcal{E}_0), \\ \partial_{ij} \mathcal{D}_U(0) &= \delta_{ij} (1 - \beta \delta) \frac{\kappa_4}{3} \mathrm{tr}(\mathcal{Q}) + 2 \left((1 - \beta \delta) \frac{\kappa_4}{3} - (\beta + \delta) \delta \frac{\kappa_2^2}{\kappa_0} \right) \mathcal{Q}_{ij} + O(\mathcal{E}_0). \end{aligned}$$

We plug the estimates into (2.12) and (2.11) and get

$$\begin{aligned} c_W &= -\frac{\kappa_2}{\kappa_0} (1 - \beta \delta) \mathrm{tr}(\mathcal{Q}) + O(\mathcal{E}_0), \quad v = O(\mathcal{E}_0), \\ \mathcal{P} &= \mathbf{T}^u \left[\frac{\kappa_4}{6\kappa_2} (1 - \beta \delta) \mathrm{tr}(\mathcal{Q}) I_d + \left((1 - \beta \delta) \frac{\kappa_4}{3\kappa_2} - (\beta + \delta) \delta \frac{\kappa_2}{\kappa_0} \right) (2\mathcal{Q} - \mathrm{diag}(\mathcal{Q})) \right] \\ &\quad - \mathbf{T}^u \left(\frac{\kappa_2}{2\kappa_0} (1 - \beta \delta) \mathrm{tr}(\mathcal{Q}) I_d + O(\mathcal{E}_0) \right), \end{aligned}$$

where we recall that \mathbf{T}^u is the upper-triangular part of the matrix. Notice that by (2.10), we have the relationship

$$\frac{\kappa_4}{6\kappa_2} = p \frac{\kappa_2}{2\kappa_0}, \quad (1 - \beta \delta) \frac{\kappa_4}{3\kappa_2} - (\beta + \delta) \delta \frac{\kappa_2}{\kappa_0} = (p - \delta^2 - \beta \delta (1 + p)) \frac{\kappa_2}{\kappa_0} = -\frac{1}{2}.$$

Therefore, we collect

$$\mathcal{P} + \frac{p-1}{2}c_W I_d = -\mathbf{T}^u(\mathcal{Q} - \frac{1}{2}\text{diag}(\mathcal{Q})) + O(\mathcal{E}_0). \quad (2.17)$$

Recall that $\mathcal{M}_\tau = \mathcal{P}\mathcal{M}$ by definition, and we can compute

$$\mathcal{Q}_\tau = (p-1)c_W \mathcal{Q} + C_W^{p-1}(\mathcal{P}\mathcal{M}\mathcal{M}^T + \mathcal{M}\mathcal{M}^T\mathcal{P}^T) = (\mathcal{P} + \frac{p-1}{2}c_W I_d)\mathcal{Q} + \mathcal{Q}(\mathcal{P} + \frac{p-1}{2}c_W I_d)^T.$$

If we decompose \mathcal{Q} into the strictly upper, lower, and diagonal parts as $\mathcal{Q} = \mathcal{Q}_u + \mathcal{Q}_u^T + \mathcal{Q}_d$, then we can simplify

$$\mathcal{Q}_\tau = -(\mathcal{Q}_u + \frac{1}{2}\mathcal{Q}_d)\mathcal{Q} - \mathcal{Q}(\mathcal{Q}_u^T + \frac{1}{2}\mathcal{Q}_d) + \mathcal{E}_\mathcal{Q}\mathcal{Q} + \mathcal{Q}\mathcal{E}_\mathcal{Q}^T, \quad \mathcal{E}_\mathcal{Q} = O(\mathcal{E}_0).$$

and we conclude the proof of the lemma. \square

Remark 6. In the above ODE of \mathcal{Q} , we get -1 for the coefficient of \mathcal{Q}^2 since we have normalized the profile (1.27). The factor $p - \delta^2 - \beta\delta(1+p) > 0$ (corresponding to the subcritical case) appears in the constant c_p . For the critical case

$$p - \delta^2 - \beta\delta(1+p) = 0,$$

without such a normalization, we can see that if we do something similar, the coefficient of \mathcal{Q}^2 will be zero. We could keep track of the next order terms of size Γ to derive a system of size $|\mathcal{Q}|^3$. This can potentially help us establish a result similar to [24] but we do not pursue it here.

3 Stability analysis and finite time blowup

In this section, we perform stability analysis and establish nonlinear stability of the perturbation around the approximate steady state following the ideas and strategy outlined in Section 1.3.

We linearize (2.7) and (2.8) around the approximate profile as in ansatz (2.4) and obtain the equations of the perturbations as follows:

$$W_\tau = \mathcal{L}_{\bar{U}}W + \mathcal{F}_U + \mathcal{N}_U + \mathcal{D}_U, \quad (3.1)$$

$$\Phi_\tau = -\frac{1}{2}\Lambda\Phi + \mathcal{F}_\Theta + \mathcal{N}_\Theta + \mathcal{D}_\Theta, \quad (3.2)$$

where we recall from (2.9a) and (2.9b) the definition of \mathcal{D}_U and \mathcal{D}_Θ , and define the linear, residue, and nonlinear parts respectively as

$$\begin{aligned} \mathcal{L}_{\bar{U}}W &= c_{\bar{v}}W - \frac{1}{2}\Lambda W + p\bar{U}^{p-1}W, \quad \Lambda = z \cdot \nabla, \\ \mathcal{F}_U &= c_W U - (\mathcal{P}z + \mathcal{V}) \cdot \nabla U - C_U^{p-1}\gamma U, \quad \mathcal{N}_U = (\bar{U} + W)^p - \bar{U}^p - p\bar{U}^{p-1}W, \\ \mathcal{F}_\Theta &= -(\mathcal{P}z + \mathcal{V}) \cdot \nabla \Theta, \quad \mathcal{N}_\Theta = \delta((\bar{U} + W)^{p-1} - \bar{U}^{p-1}). \end{aligned} \quad (3.3)$$

We will group the terms by integrability: \mathcal{L}_U and \mathcal{N}_U vanish to the third order at the origin. Recall that by Lemma 1 and Remark 5, we know that $\mathcal{Q}, c_W, \mathcal{V}, \mathcal{P}$ are small. The viscous terms $\mathcal{D}_U, \mathcal{D}_\Theta$ are small of order $|\mathcal{Q}|$ with a typical size of $1/\tau$. Also obviously $H = C_W e^{-\tau/(p-1)}$ is small.

We define the weighted H^k energy as follows

$$E_k^2 = (|\nabla^k W|^2, \rho_k), \quad 0 \leq k \leq K, \quad F_k^2 = (|\nabla^k \Phi|^2, \rho_k), \quad 0 < k \leq K. \quad (3.4)$$

Our goal is to prove the following nonlinear stability results.

Theorem 3. Denote $E_\mathcal{Q} = \text{tr}(\mathcal{Q})$. There exists $0 < E_* < 1$ sufficiently small and $\mu_3 > 0$, such that for any initial perturbation satisfying

$$U\rho > 2C_b, \quad E(0) < E_*, \quad E_\mathcal{Q} < E_*, \quad H^{p-1}(0) < E_*, \quad (3.5)$$

we have the following estimates for all $\tau > 0$

$$\begin{aligned} E_{\mathcal{Q}}(\tau) &\leq \min(2E_{\mathcal{Q}}(0), 4d/\tau), \quad U(\tau)\rho > C_b, \quad H^{p-1}(\tau) < H^{p-1}(0)e^{-\tau/2}, \\ E(\tau) &\leq e^{-\lambda\tau/2}(E(0) + \mu_3 H^{p-1}(0) + \mu_3 E_{\mathcal{Q}}(0)) + \mu_3 \min(E_{\mathcal{Q}}(0), 1/\tau). \end{aligned} \quad (3.6)$$

Moreover, the bootstrap assumptions 1, 2, 3 (introduced below) hold for all $\tau > 0$.

Note that the parameters μ_1, μ_2 will be introduced in the proof in Section 3.6.

We impose the following weak bootstrap assumptions for nonlinear estimates. To control $1/U$, which will appear in the estimate of $\mathcal{N}_U, \mathcal{N}_{\Theta}$, we impose a lower bound on U , which is almost comparable with \bar{U} , up to a small power. To simplify the notations in the nonlinear estimates and to ensure that \mathbf{R}, \mathcal{M} are invertible (2.5), (2.4), we impose the following weak assumptions on the energy E_i, F_j and $\det(\mathcal{Q})$.

Assumption 1. Let ϵ_2 be defined in (1.30). We impose the following bootstrap assumptions

$$U \geq C_b \bar{U}^{1+\epsilon_2}, \quad C_b = \min_{|z| \leq 1} \bar{U}^{-\epsilon_2}(z)/4 > 0, \quad (3.7)$$

$$\max(E_K, E_0, F_K, F_1) \leq 1, \quad \det(\mathcal{Q}) > 0. \quad (3.8)$$

Below, we will first establish some functional inequalities in Section 3.1. We will start with the L^2 analysis of perturbations W and $\nabla\Phi$ in Sections 3.2, 3.3, and then build higher-order estimates in Section 3.4. We obtain a lower bound of the amplitude U via the maximal principle in Section 3.5, inspired by [17]. Then we close the nonlinear estimates and prove Theorem 3 via a bootstrap argument in Section 3.6.

3.1 Functional inequalities

In this section, we establish a few functional inequalities, which will be used to estimate the decay of the solution and close the nonlinear estimates. We introduce the following norms

$$\|f\|_{\dot{\mathcal{H}}^k} := \|\nabla^k f g_k^{1/2}\|_{L^2}, \quad \|f\|_{\mathcal{H}^k} := \|f\|_{\dot{\mathcal{H}}^k} + \|f\|_{\dot{\mathcal{H}}^0}, \quad g_k = \langle z \rangle^{-2\sigma - \epsilon - d + 2k}. \quad (3.9)$$

By definition of ρ_k (1.28) and (3.4) for E_k , we have

$$g_k \lesssim \rho_k, \quad \|f\|_{\dot{\mathcal{H}}^k} \lesssim E_k, \quad \|f\|_{\mathcal{H}^k} \lesssim E_0 + E_k. \quad (3.10)$$

We define the low-order terms $\mathcal{E}_1, \mathcal{E}_2$ that we later show to be small:

$$\begin{aligned} \mathcal{E}_1 &:= |\mathcal{Q}| + H^{p-1} + \sum_{i \leq K-1} E_i + \sum_{1 \leq j \leq K-1} F_j + (F_K + F_1 + E_K + E_0)^2, \\ \mathcal{E}_2 &:= |\mathcal{Q}| + H^{p-1}. \end{aligned} \quad (3.11)$$

We treat \mathcal{E}_1 as a lower order term since it either contains nonlinear terms or energy with order lower than E_K, F_K . Note that \mathcal{E}_2 is the low-order term of order P in Lemma 1 by assuming (3.8), which implies $\Gamma \lesssim 1$. See (3.15).

Following Lemma C.4 in [11], we have the following weighted interpolation and embedding inequalities.

Proposition 1 (Interpolation). Let $\sigma = -\frac{2}{p-1}$ and ϵ be the constants defined in (2.10), (1.30). For any $\mu > 0$, there exists a constant $C(\mu)$, such that the following interpolation inequalities hold:

$$E_k \leq \mu E_l + C(\mu) E_0, \quad \forall 0 \leq k < l \leq K, \quad (3.12a)$$

$$F_k \leq \mu F_l + C(\mu) F_1, \quad \forall 1 \leq k < l \leq K, \quad (3.12b)$$

$$\|f\|_{\dot{\mathcal{H}}^k} \leq \mu \|f\|_{\dot{\mathcal{H}}^l} + C(\mu) \|f\|_{\mathcal{H}^0}, \quad \forall 0 \leq k < l \leq K. \quad (3.12c)$$

Moreover, we have the following embedding

$$|\nabla^l W| \lesssim \langle z \rangle^{-l+\sigma+\epsilon/2} \min(E_{l+d}, \|W\|_{\mathcal{H}^{l+d}}), \quad \forall 0 \leq l \leq K-d, \quad (3.13a)$$

$$|\nabla^l \Phi| \lesssim \langle z \rangle^{-l+1/2} F_{l+d}, \quad \forall 1 \leq l \leq K-d-1. \quad (3.13b)$$

As a result, for $k < l$, we have the following product rule for $i \leq n, j \leq m$ with $i+d \leq n$, or $j+d \leq m$,

$$\|\nabla^i F \nabla^j G g_{i+j}^{1/2}\|_{L^2} \lesssim \|F\|_{\mathcal{H}^n} \|G\|_{\mathcal{H}^m}. \quad (3.14)$$

Finally, assuming (3.8), for $0 \leq i \leq K, 1 \leq j \leq K$ and Γ defined in (3.58), we have

$$E_i \lesssim 1, \quad F_j \lesssim 1, \quad \Gamma \lesssim 1. \quad (3.15)$$

Since K (1.29) is absolute, $l, k \leq K$, and the parameters c_0, c_1 in the weights (1.28) and norms depend on absolute constants K, ϵ (1.30), we only need to track the constants related to μ .

Proof. (a) Interpolation inequalities. To prove (3.12), we use integration by parts. For (3.12a), we compute for $K > k > 0$ that

$$E_k^2 = - \sum_i \int (\partial_i^2 \nabla^{k-1} W \cdot \nabla^{k-1} W \rho_k + \partial_i \nabla^{k-1} W \cdot \nabla^{k-1} W \partial_i \rho_k).$$

Notice that the weights (1.28) satisfy

$$\rho_k^2 \lesssim \rho_{k+1} \rho_{k-1}, \quad (\partial_i \rho_k)^2 \lesssim \rho_k \rho_{k-1}.$$

Combined with a Cauchy-Schwarz inequality, we obtain

$$E_k^2 \lesssim E_{k-1} (E_k + E_{k+1}).$$

Since ϵ only depends on K (1.30), by a weighted AM-GM inequality, for any $\mu > 0$, we have

$$E_k^2 \leq C(\mu) E_{k-1}^2 + \mu E_{k+1}^2.$$

From here, to conclude the first inequality, since $\mu > 0$ is arbitrary, we only need to show it holds for $k = l-1$, which we can combine the above estimates for $k = 1, 2, \dots, l-1$ to establish.

The proof of (3.12c) follows from the same argument.

For the second inequality (3.12b), we can repeat the same procedure to conclude, provided that the weights $\dot{\rho}_k$ satisfy the same inequalities for $K > k > 1$ (1.28):

$$\dot{\rho}_k^2 \lesssim \dot{\rho}_{k+1} \dot{\rho}_{k-1}, \quad (\partial_i \dot{\rho}_k)^2 \lesssim \dot{\rho}_k \dot{\rho}_{k-1}.$$

When $k+1 < K$, this is obvious. For $k = K-1$, we only need to show

$$\langle z \rangle^{2K-1-d} \lesssim \dot{\rho}_K \approx U^2 \langle z \rangle^{-2\sigma+2K-\epsilon-d},$$

which is true, since by the choice of ϵ, ϵ_2 (1.30) and the bootstrap Assumption 1 we have

$$U \geq C_b \bar{U}^{1+\epsilon_2} \approx \langle z \rangle^{\sigma-\epsilon/2}, \quad 2\epsilon < 1.$$

(b) Embedding (3.13). To prove the L^∞ estimates (3.13), one can proceed as in [37] and invoke the weighted Morrey-type inequality. Below, we present a simpler proof. By a density argument, we can assume that $W \in C_c^\infty$. Without loss of generality, we fix $z \in \mathbb{R}^d$ with $z_i \geq 0$ and estimate $\nabla^l W(z)$. Consider the region $\Omega(z) = \{y \in \mathbb{R}^d, y_i \geq z_i\}$. We have $|y| \geq |z|$ for any $y \in \Omega(z)$. Denote $\delta = -2\sigma - \epsilon - d > -d$ (1.30). We have

$$|\nabla^l W(z)| \lesssim_l \int_{\Omega(z)} |\partial_1 \partial_2 \dots \partial_d \nabla^l W(y)| dy \lesssim_l \| \langle y \rangle^{l+d+\delta/2} \nabla^{l+d} W \|_{L^2} \left(\int_{|y| \geq |z|} \langle y \rangle^{-2l-2d-\delta} dy \right)^{1/2}.$$

For $0 \leq l \leq K - d - 1$, using $\rho_{l+d} \gtrsim \langle y \rangle^{2(l+d)+\delta}$ (1.28), $-2l - d - \delta - 1 < -1$, and the first inequality (3.12a) with $k = l + d < K$, for any $\mu > 0$, we further obtain

$$|\nabla^l W(y)| \lesssim_K E_{l+d} \left(\int_{R \geq |y|} \langle R \rangle^{-2l-2d-\delta} R^{d-1} dR \right)^{1/2} \lesssim_K E_{l+d} \langle y \rangle^{-l-(d+\delta)/2}.$$

Rearranging the power on both sides, we prove (3.13a).

The proof of (3.13b) with $1 \leq l \leq K - d - 1$ is similar by replacing W by $\nabla \Phi$ in the above argument and using $\hat{\rho}_{l+d} \gtrsim \langle y \rangle^{2(l+d)+\delta_2}$ for $\delta_2 = -d - 1$ (1.28) and $-2l - d - \delta_2 - 1 = -2l < -1$ for $l \geq 1$.

(c) Other inequalities. For (3.14), without loss of generality, we assume that $i + d \leq n$. Using $g_{i+j}^{1/2} \lesssim \langle z \rangle^i g_j^{1/2}$, $\sigma + \frac{\epsilon}{2} < 0$ (3.12c), we prove

$$\|\nabla^i F \nabla^j G g_{i+j}^{1/2}\|_{L^2} \lesssim \|F\|_{\mathcal{H}^{i+d}} \|\nabla^j G g_j^{1/2}\|_{L^2} \lesssim \|F\|_{\mathcal{H}^n} \|G\|_{\mathcal{H}^m}.$$

The inequalities (3.15) follow from (3.12a), (3.12b) and the assumption (3.8). \square

Combining Assumption 1 and the decay estimates (3.13a), (3.13b), we have the following estimates.

Corollary 2. Denote $W = \bar{U} - U$. Under the Assumption 1, for any $\alpha \in [0, 1]$ and any $q \in \mathbb{R}$ we have

$$\begin{aligned} \langle z \rangle^{\sigma-\epsilon/2} &\lesssim U \lesssim \langle z \rangle^{\sigma+\epsilon/2}, \\ |(\bar{U} + \alpha W)(z)|^q &\lesssim C(|q|) \langle z \rangle^{\sigma q + |q|/2}, \\ \|\bar{U} + \alpha W\|_{\mathcal{H}^\kappa}, \|U\|_{\mathcal{H}^\kappa} &\lesssim 1, \end{aligned} \quad (3.16)$$

where σ, ϵ are defined in (2.10). As a result, for $i + j \leq K$, we have the following estimates for the weights

$$\begin{aligned} \hat{\rho}_i &\lesssim \rho_i \langle z \rangle^{\sigma+\epsilon/2-1/2}, \quad i \leq K-1, \quad \hat{\rho}_K \lesssim \rho_K \langle z \rangle^{\sigma+\epsilon/2}, \\ \rho_{i+j} &\lesssim \rho_i \langle z \rangle^{2j}, \quad \hat{\rho}_{i+j} \lesssim \hat{\rho}_i \langle z \rangle^{2j}. \end{aligned} \quad (3.17)$$

Proof. (a) Estimate of U . By definition of ϵ_2 (1.30), we get $-\frac{2}{p-1}(1 + \epsilon_2) = -\frac{2}{p-1} - \frac{\epsilon}{2}$. Thus, under Assumption 1, we yield

$$U \gtrsim \bar{U}^{1+\epsilon_2} \gtrsim \langle z \rangle^{-\frac{2}{p-1} - \frac{\epsilon}{2}}.$$

Since $\bar{U} + \alpha W = \bar{U} + \alpha(U - \bar{U}) = \alpha U + (1 - \alpha)\bar{U}$, which is between U, \bar{U} , and $\bar{U}, U > 0$, using (3.13a), $K > d$ (1.29) and the above estimate, we prove

$$|(\bar{U} + \alpha W)| \lesssim \bar{U} + U \lesssim (1 + E_K + E_0) \langle z \rangle^{\sigma+\epsilon/2}, \quad |(\bar{U} + \alpha W)|^{-1} \lesssim \min(\bar{U}, U)^{-1} \lesssim \langle z \rangle^{-\sigma+\epsilon/2}.$$

The first estimate with $\alpha = 1$ implies the upper bound for U in (3.16). Raising the above estimates to $|q|$ -th power proves the second estimate in (3.16).

For the last estimate in (3.16), using triangle inequality, $|\nabla^i \bar{U}| \lesssim \langle z \rangle^{\sigma-i}$, and (3.10), we prove

$$\|\bar{U} + \alpha W\|_{\mathcal{H}^\kappa} + \|\bar{U} + W\|_{\mathcal{H}^\kappa} \lesssim \|\bar{U}\|_{\mathcal{H}^\kappa} + \|W\|_{\mathcal{H}^\kappa} \lesssim 1 + E_0 + E_K \lesssim 1.$$

(b) Estimate of weights. We consider (3.17). Using $U \lesssim \langle z \rangle^{\sigma+\epsilon/2}$, clearly, we have

$$\begin{aligned} |z| \leq 1: \hat{\rho}_i &\lesssim \rho_i \lesssim \rho_i \langle z \rangle^{2\sigma+\epsilon-1}, \\ |z| \geq 1: \hat{\rho}_i &\lesssim |z|^{2k-1-d} \lesssim \rho_i \langle z \rangle^{2\sigma+\epsilon-1}, \quad i \leq K-1, \quad \hat{\rho}_K \lesssim U^2 \rho_K \lesssim \rho_K \langle z \rangle^{2\epsilon+\epsilon}. \end{aligned}$$

For $(f, \tau) = (\rho, -2\sigma - \epsilon)$ or $(\hat{\rho}, -1)$ and $i + j < K$, from the definition of f_i (1.28), we have

$$f_{i+j} \lesssim f_i \lesssim f_i \langle z \rangle^{2j}, \quad |z| \leq 1, \quad f_{i+j} \approx |z|^{2i+2j+\tau-d} \lesssim |z|^{2i+\tau-d} \langle z \rangle^{2j} \lesssim f_i \langle z \rangle^{2j}, \quad |z| \geq 1.$$

For $i + j = K$, the above estimate still holds for $f = \rho$. For $\hat{\rho}_i$ and $\hat{\rho}_K$, using $U \lesssim \langle z \rangle^{\sigma+\epsilon/2}$, we obtain

$$\hat{\rho}_K \lesssim \langle z \rangle^{2\sigma+\epsilon} \rho_K \lesssim \langle z \rangle^{2K-1-d} = \langle z \rangle^{2i-1-d} \langle z \rangle^{2j} \lesssim \hat{\rho}_i \langle z \rangle^{2j}.$$

We complete the proof of (3.17). \square

Proposition 3. Suppose that (3.8) holds true. For $0 \leq k \leq K, j_1 + j_2 \leq k$ and any $\alpha \in [0, 1]$, denote

$$V = \bar{U} + \alpha W, \quad I_{(j_1, j_2)} = \nabla^{j_1} W_1 \cdot \nabla^{j_2} W_2 \nabla^{k-j_1-j_2} V^{p-2}.$$

We have the following product estimates

$$\|I_{(j_1, j_2)}\|_{g_k} \lesssim \|W_1\|_{\mathcal{H}^{\max(j_1, K-1)}} \|W_2\|_{\mathcal{H}^{\max(j_2, K-1)}}, \quad (3.18)$$

and

$$\|W_1 W_2 V^{p-2}\|_{\mathcal{H}^K} \lesssim (\|W_1\|_{\mathcal{H}^{K-1}} \|W_2\|_{\mathcal{H}^K} + \|W_1\|_{\mathcal{H}^K} \|W_2\|_{\mathcal{H}^{K-1}}), \quad (3.19a)$$

$$\|W_1 W_2 V^{p-2}\|_{\rho_0} \lesssim \|W_2\|_{\rho_0} \|W_1\|_{\mathcal{H}^d}. \quad (3.19b)$$

Moreover, we have

$$\|\langle z \rangle^{\sigma+\epsilon/2} \nabla^{l+1} U \nabla^m (U^{-1})\|_{g_k} \lesssim \|U\|_{\mathcal{H}^{\max(l+1, m, k)}}, \quad l+m = k \leq K, \quad (3.20)$$

$$|\nabla^{l+1} U \cdot \nabla^m (U^{-1})| \lesssim \langle z \rangle^{-l-m}, \quad l, m \leq K-1-d, \quad (3.21)$$

Proof. A direct computation yields

$$I_{j_1, j_2} \leq \sum_{2 \leq q \leq k+2} \sum_{\sum_{i=1}^q j_i = k, j_i \geq 0} I_{\vec{j}, q}, \quad I_{\vec{j}, q} = |V|^{p-q} \cdot |\nabla^{j_1} W_1| |\nabla^{j_2} W_2| \prod_{l=3}^q |\nabla^{j_l} V|. \quad (3.22)$$

For a fixed (j, q) , we denote

$$J_1 = W_1, \quad J_2 = W_2, \quad J_l = V, \quad l \geq 3, \quad i = \arg \max_{l \leq q} j_l.$$

If there are more than one indices a with $j_a = \arg \max_{l \leq q} j_l$, we just pick one of them. Clearly, we have $j_i \leq k/2, l \neq i$ (1.29). By Proposition 1 (3.13a), we have

$$|\nabla^{j_l} J_l| \lesssim \langle z \rangle^{-j_l + \sigma + \epsilon/2} \|J_l\|_{\mathcal{H}^{j_l+d}}$$

Applying the above L^∞ estimates to $J_l, l \neq i$, and Corollary 2 for $V = \bar{U} + \alpha W$, we obtain

$$I_{\vec{j}, q} \lesssim |\nabla^{j_i} J_i| \prod_{l \neq i} \langle z \rangle^{-j_l + \sigma + \epsilon/2} \|J_l\|_{\mathcal{H}^{j_l+d}} \langle z \rangle^{(p-q)\sigma + |p-q|\epsilon/2}. \quad (3.23)$$

Combining the exponents of the $\langle z \rangle$ terms, and using the definitions of $\sigma = -\frac{2}{p-q}$ (2.10) and ϵ (1.30), we yield

$$\begin{aligned} \xi &= \sum_{1 \leq l \neq i \leq q} (-j_l + \sigma + \epsilon/2) + (p-q)\sigma + |p-q|\epsilon/2 \\ &= -(k-j_i) + (p-q+q-1)\sigma + (p+q+q-1)\epsilon/2 = -(k-j_i) - 2 + (p+K)\epsilon < -(k-j_i). \end{aligned} \quad (3.24)$$

Since $\rho_k^{1/2} \langle z \rangle^{-(k-j_i)} \lesssim \rho_{k-j_i}$ (1.28), applying weighted L^2 bound to $\nabla^{j_i} J_i$, we further obtain

$$I_{\vec{j}, q} \lesssim \|J_i\|_{\mathcal{H}^{j_i}} \prod_{l \neq i} \|J_l\|_{\mathcal{H}^{j_l+d}}.$$

Since $j_l + d \leq k/2 + d \leq K-1$ for $l \neq i, k \leq K, j_i \leq k$, and $\|V\|_{\mathcal{H}^K} \lesssim 1 + E_0 + E_K$, we obtain $j_l + d \leq \max(j_l, K-1)$ and thus

$$I_{\vec{j}, q} \lesssim \|W_1\|_{\mathcal{H}^{\max(j_1, K-1)}} \|W_2\|_{\mathcal{H}^{\max(j_2, K-1)}}.$$

Using $\max(p - q, 0) + q \leq p + K + q$ and summing the estimates of $I_{\vec{j}, q}$, we conclude the proof of (3.18).

The estimate (3.19a) follows from summing the estimates of $I_{(j_1, j_2)}$ (3.18) over (j_1, j_2) with $j_1 + j_2 \leq K$, and using the fact that we have $j_1 \leq K - 1$ or $j_2 \leq K - 1$. The estimate (3.19b) follows from applying $L^2(\rho_0)$ estimate to W_2 and L^∞ estimate to W_1, V similar to the above.

For the last estimate (3.20), we note that $U = \bar{U} + W$. Applying the Leibniz rule, we obtain

$$|\nabla^{l+1} U \nabla^m (U^{-1})| \lesssim \sum_{1 \leq q \leq k+1} \sum_{\sum_{l=1}^q j_l = k+1, j_1 \geq 1, j_l \geq 0} T_{\vec{j}, q}, \quad T_{\vec{j}, q} = |U|^{-q} \cdot \prod_{l=1}^q |\nabla^{j_l} U|.$$

Denote $i = \arg \max_l j_l$. Applying the above estimates of $I_{\vec{j}, q}$ with $(J_1, \dots, J_l), V, p - q$ replaced by $(U, U, \dots, U), U, -q$ (3.22)-(3.24), and using $\max(-q, 0) = 0, q \leq K + 1$, we obtain

$$|\langle z \rangle^{\sigma + \epsilon/2} T_{\vec{j}, q}| \lesssim |\nabla^{j_i} U| \langle z \rangle^\xi \prod_{l \neq i} \|U\|_{\mathcal{H}^{j_l + d}},$$

where

$$\begin{aligned} \xi &= \sigma + \frac{\epsilon}{2} + \sum_{1 \leq l \neq i \leq q} (-j_l + \sigma + \frac{\epsilon}{2}) + (-q)\sigma + |-q| \frac{\epsilon}{2} \\ &= -(k + 1 - j_i) + (-q + q)\sigma + (q + q)\epsilon/2 = -(k - j_i) - 1 + (1 + K)\epsilon < -(k - j_i). \end{aligned}$$

Using $j_l + d \leq K/2 + d < K, \|U\|_{\mathcal{H}^{j_l + d}} \lesssim \|U\|_{\mathcal{H}^\kappa}$ for $l \neq i, \rho_k^{1/2} \lesssim \rho_{j_i}^{1/2} \langle z \rangle^{k - j_i}, q, j_i \leq k + 1$, and applying weighted L^2 estimate to $\nabla^{j_i} J_i$, we establish

$$\|\langle z \rangle^{\sigma + \epsilon/2} T_{\vec{j}, q}\|_{g_k} \lesssim \|U\|_{\mathcal{H}^{j_i}} (1 + \|U\|_{\mathcal{H}^\kappa})^{q-1} \lesssim \|U\|_{\mathcal{H}^{k+1}}.$$

Combining the estimates for $T_{\vec{j}, q}$ with different \vec{j}, q , we conclude the proof of (3.20).

For (3.21), denote $k = l + m$. Applying L^∞ estimate to each term $\nabla^{j_i} U$ in $T_{\vec{j}, q}$ and noting that $j_i \leq \max(l + 1, m) \leq K - d$, we prove

$$T_{j, q} \lesssim \langle z \rangle^\xi \lesssim \langle z \rangle^{-k-1/2},$$

where we have used

$$\xi = \sum_{1 \leq l \leq q} (-j_l + \sigma + \frac{\epsilon}{2}) + (-q)\sigma + |-q| \frac{\epsilon}{2} = -(k + 1) + q\epsilon < -k - 1/2.$$

We complete the proof. \square

3.2 L^2 stability analysis of the amplitude

In this section, we estimate the weighted L^2 energy $E_0^2 = (W, W\rho_0)$ (3.4). In the following energy estimates, without specification, we will assume that the bootstrap assumption 1 holds true. We will show that the following lemma holds.

Lemma 2 (Weighted L^2 estimate). *Under the bootstrap assumption 1, it holds*

$$\frac{1}{2} \frac{d}{d\tau} E_0^2 \leq \left(-\frac{\epsilon}{8} + C\mathcal{E}_1\right) E_0^2 + C\mathcal{E}_2 E_0. \quad (3.25)$$

or some absolute constant $C > 0$.

Proof. Notice that W vanishes at the origin to the third order so this choice of singular weight induces a well-defined energy. We have by (3.1) that

$$\frac{1}{2} \frac{d}{d\tau} E_0^2 = (\mathcal{L}_{\bar{U}} W, W\rho_0) + (\mathcal{N}_U, W\rho_0) + (\mathcal{F}_U + \mathcal{D}_U, W\rho_0).$$

For the leading order linear term, we have via integration by parts that

$$(\mathcal{L}_{\bar{U}}W, W\rho_0) = (d_0W, W\rho_0),$$

where we calculate the damping

$$\begin{aligned} d_0 &:= c_U + \frac{1}{2\rho_0} \nabla \cdot (d_{\bar{U}}z\rho_0) + p\bar{U}^{p-1} = -\frac{1}{p-1} + p\bar{U}^{p-1} + \frac{1}{4\rho_0} \nabla \cdot (z\rho_0) \\ &= -\frac{1}{p-1} + \frac{p}{p-1+c_p|z|^2} + \frac{(-6+\epsilon)|z|^{-6+\epsilon} + (\frac{4}{p-1}-\epsilon)c_0|z|^{\frac{4}{p-1}-\epsilon}}{4(|z|^{-6+\epsilon} + c_0|z|^{\frac{4}{p-1}-\epsilon})} \leq -\frac{\epsilon}{8}. \end{aligned}$$

The last inequality amounts to

$$\begin{aligned} &4p(|z|^{-6+\epsilon} + c_0|z|^{\frac{4}{p-1}-\epsilon}) + ((-6 - \frac{4}{p-1} + \frac{3\epsilon}{2})|z|^{-6+\epsilon} - \frac{\epsilon}{2}c_0|z|^{\frac{4}{p-1}-\epsilon})(p-1+c_p|z|^2) \\ &\leq -(p-1)|z|^{-6+\epsilon} - \frac{\epsilon}{2}c_0c_p|z|^{2+\frac{4}{p-1}-\epsilon} + 4pc_0|z|^{\frac{4}{p-1}-\epsilon} \leq 0. \end{aligned}$$

The last inequality is implied by a weighted AM-GM inequality provided that

$$\left(\frac{\epsilon c_0 c_p}{2(6 + \frac{4}{p-1} - 2\epsilon)}\right)^{6 + \frac{4}{p-1} - 2\epsilon} \left(\frac{p-1}{2}\right)^2 \geq \left(\frac{4pc_0}{8 + \frac{4}{p-1} - 2\epsilon}\right)^{8 + \frac{4}{p-1} - 2\epsilon}. \quad (3.26)$$

Notice that $\epsilon \leq 1/2$ is fixed to be small. We can choose a sufficiently small constant $c_0 > 0$ such that we can conclude the linear estimate

$$(\mathcal{L}_{\bar{U}}W, W\rho_0) \leq -\frac{\epsilon}{8}E_0^2. \quad (3.27)$$

The nonlinear estimate is more subtle due to the general nonlinearity p . We use Taylor's expansion or Newton-Leibniz's formula twice to derive

$$\mathcal{N}_U = W^2 p(p-1) \int_0^1 (1-\alpha)(\bar{U} + \alpha W)^{p-2} d\alpha. \quad (3.28)$$

Using Proposition 3 (3.19b) with $(W_1, W_2) = (W, W)$ and \mathcal{E}_1 defined in (3.11), we obtain

$$\|W^2(\bar{U} + \alpha W)^{p-2}\|_{\rho_0} \lesssim \|W\|_{\mathcal{H}^d} \|W\|_{\rho_0} \lesssim (E_0 + E_d)E_0 \lesssim \mathcal{E}_1 E_0,$$

which implies

$$|(\mathcal{N}_U, W\rho_0)| \leq \|\mathcal{N}_U\|_{\rho_0} E_0 \lesssim \mathcal{E}_1 \|W\|_{\rho_0} E_0 \lesssim \mathcal{E}_1 E_0^2. \quad (3.29)$$

Finally, we estimate the viscous and residue terms together. We group the terms to make them integrable. Consider a fixed 1D smooth cutoff function χ such that it equals 1 in $[-1, 1]$ and 0 outside of $[-2, 2]$. We use the notation \tilde{f} to denote functions only differing from f near the origin, where they equal the residue of f when expanded until its second-order Taylor's expansion at the origin, via the cutoff function χ . For illustrative purposes, we will explicitly write down \tilde{f} by the expansions

$$f = (f(0) + z^T \nabla f(0) + \frac{1}{2} z^T \nabla^2 f(0) z) \chi(|z|) + \tilde{f},$$

where \tilde{f} vanishes to the third order at the origin. By the choice of the modulation parameters in (2.11), it's easy to see that⁶

$$\mathcal{F}_U + \mathcal{D}_U = c_W \tilde{U} - \mathcal{P}z \cdot \nabla \tilde{U} - v \cdot \nabla \tilde{U} - H^{p-1} \gamma \tilde{U} + \tilde{\mathcal{D}}_U := \tilde{\mathcal{F}}_U + \tilde{\mathcal{D}}_U. \quad (3.30)$$

⁶Note that \tilde{U} does not denote the perturbation.

Each of the terms in $\widetilde{\mathcal{F}}_U$ vanishes to the third order at the origin. Notice that $\rho_k = \rho_0|z|^{2k}$, for $k = 1, 2, 3$. Recall the definition of $\mathcal{E}_1, \mathcal{E}_2$ from (3.11). By Lemma 1, we have that

$$\|\widetilde{\mathcal{F}}_U\|_{\rho_0} \lesssim \mathcal{E}_2(1 + E_0 + E_1 + \|\widetilde{\nabla W}\|_{\rho_0}).$$

We can decompose the integral region into the near field $I = [0, 1]^d$ and the rest of the outer region I^c to estimate

$$\|\widetilde{\nabla W}\|_{\rho_0} \lesssim \left(\int_{z \in I} |z|^{\epsilon-d} \right)^{1/2} \sup_{z \in I} |\widetilde{\nabla W}/|z|^3| + \|\widetilde{\nabla W}|z|\|_{\rho_0} \lesssim \Gamma + E_1.$$

Combined with Proposition 1, we have the residue estimate

$$|(\widetilde{\mathcal{F}}_U, W_{\rho_0})| \lesssim \mathcal{E}_2 E_0. \quad (3.31)$$

For the viscous term, we notice as in (2.16), we can write

$$\widetilde{\mathcal{D}}_U = \text{tr}(\mathcal{Q}\widetilde{S}_1), \quad S_1 = S_{11} + S_{12} + S_{13} + S_{14}.$$

We estimate the four terms respectively. We compute

$$\|\widetilde{S}_{11}\|_{\rho_0} = \|\widetilde{\nabla^2 U}\|_{\rho_0} \lesssim 1 + \|\widetilde{\nabla^2 W}\|_{\rho_0} \lesssim 1 + \Gamma + E_2,$$

where in the last inequality we use again the decomposition of the integral into the near and far fields. For the remaining three viscous terms, we estimate similarly as follows:

$$\begin{aligned} \|\widetilde{S}_{12}\|_{\rho_0} &= 2|\beta| \|\widetilde{\nabla U \nabla \Theta^T}\|_{\rho_0} \lesssim 2|\beta| \|\widetilde{\nabla W \nabla \Phi^T}\|_{\rho_0} + (1 + \Gamma)^2 \lesssim \Gamma(\Gamma + E_1) + (1 + \Gamma)^2, \\ \|\widetilde{S}_{13}\|_{\rho_0} &= \|\widetilde{U \nabla \Theta \nabla \Theta^T}\|_{\rho_0} \lesssim \|\widetilde{W \nabla \Phi \nabla \Phi^T}\|_{\rho_0} + (1 + \Gamma)^3 \lesssim \Gamma^2(\Gamma + E_0) + (1 + \Gamma)^3, \\ \|\widetilde{S}_{14}\|_{\rho_0} &= \|\widetilde{U \nabla^2 \Theta}\|_{\rho_0} \lesssim \|\widetilde{W \nabla^2 \Phi}\|_{\rho_0} + (1 + \Gamma)^2 \lesssim \Gamma(\Gamma + E_0) + (1 + \Gamma)^2. \end{aligned}$$

We can collect the viscous estimate by Proposition 1 and Assumption 1:

$$|(\widetilde{\mathcal{D}}_U, W_{\rho_0})| \lesssim \mathcal{E}_2(1 + E_0 + E_1 + E_2 + \Gamma)^3 E_0 \lesssim \mathcal{E}_2(1 + E_0 + E_K)^3 E_0 \lesssim \mathcal{E}_2 E_0. \quad (3.32)$$

We thereby conclude the proof of Lemma 2 using (3.27), (3.29), (3.31), and (3.32). \square

One sees that we already have leading order damping in the L^2 estimates. However, to close the nonlinear estimates, we will need higher order estimates to control the L^∞ norms.

3.3 H^1 stability analysis of the phase

We consider the weighted H^1 norm of the phase $F_1^2 = (\nabla \Phi, \nabla \Phi \rho_1)$ (3.4). We choose this norm since Φ does not decay at the origin. We will show that the following lemma holds.

Lemma 3 (Weighted H^1 estimate). *Under the bootstrap assumption 1, it holds*

$$\frac{1}{2} \frac{d}{d\tau} F_1^2 \leq -\frac{1}{8} F_1^2 + C(E_{K-1} + E_0)^2 + C\mathcal{E}_2 F_1, \quad (3.33)$$

for some absolute constant $C > 0$.

Proof. We have by (3.2) that

$$\frac{1}{2} \frac{d}{d\tau} F_1^2 = (\nabla(-\frac{1}{2}\Lambda\Phi) + \nabla \cdot \mathcal{N}_\Theta + \nabla \cdot \mathcal{F}_\Theta + \nabla \cdot \mathcal{D}_\Theta, \nabla \Phi \rho_1).$$

For the leading order linear term, we have via integration by parts that

$$(\partial_i(-\frac{1}{2}\Lambda\Phi), \partial_i\Phi\dot{\rho}_1) = -\frac{1}{4}(\partial_i\Phi, \partial_i\Phi\dot{\rho}_1).$$

Therefore we have the linear estimate

$$(\nabla(-\frac{1}{2}\Lambda\Phi), \nabla\Phi\dot{\rho}_1) = -\frac{1}{4}F_1^2. \quad (3.34)$$

For the nonlinear estimate, we again use Newton-Leibniz's formula to get

$$\mathcal{N}_\Theta = \delta((\bar{U} + W)^{p-1} - \bar{U}^{p-1}) = \delta(p-1)W \int_0^1 (\bar{U} + \alpha W)^{p-2} d\alpha. \quad (3.35)$$

It is not difficult to see that $\langle z \rangle^{\sigma + \frac{\epsilon-1}{2}} \in \mathcal{H}^i$ for any $i \geq 0$ since $\epsilon-1 < 0$. Note that $\dot{\rho}_1$ (1.28) is locally integrable and $\dot{\rho}_1 \lesssim \langle z \rangle^{\sigma + \frac{\epsilon-1}{2}} g_1$. Applying L^∞ estimate for $W, \bar{U} + \alpha W$ from Corollary 2, (3.13a) in Proposition 1, and Proposition 3 (3.18) with $W_1 = \langle z \rangle^{\sigma + \frac{\epsilon-1}{2}}, W_2 = W, j_1 = 0, j_2 = 1, k = 1$, we obtain

$$\begin{aligned} \|\nabla\mathcal{N}_\Theta\|_{\dot{\rho}_1} &\lesssim \|\nabla\mathcal{N}_\Theta\|_{L^\infty(|z|\leq 1)} + \|W_1\nabla\mathcal{N}_\Theta\|_{g_1} \lesssim E_0 + E_{K-1} + \|W_1\nabla\mathcal{N}_\Theta\|_{\rho_1} \\ &\lesssim E_0 + E_{K-1} + \|W_2\|_{\mathcal{H}^{K-1}} \lesssim E_0 + E_{K-1}. \end{aligned}$$

We can collect the nonlinear estimate, via an AM-GM inequality as follows:

$$|(\nabla\mathcal{N}_\Theta, \nabla\Phi\dot{\rho}_1)| \leq C\|\nabla\mathcal{N}_\Theta\|_{\dot{\rho}_1} F_1 \leq C(E_0 + E_{K-1})^2 + \frac{1}{8}F_1^2. \quad (3.36)$$

For the residue estimate, we have

$$|(\nabla\mathcal{F}_\Theta, \nabla\Phi\dot{\rho}_1)| \leq \|\nabla\mathcal{F}_\Theta\|_{\dot{\rho}_1} F_1 \lesssim \mathcal{E}_2(1 + F_1 + \|\nabla^2\Phi\|_{\dot{\rho}_1})F_1 \lesssim \mathcal{E}_2 F_1. \quad (3.37)$$

For the viscous estimate, we have

$$\begin{aligned} \|\nabla\mathcal{D}_\Theta\|_{\dot{\rho}_1} &\lesssim \mathcal{E}_2 \left(\left\| \frac{\nabla^3 U}{U} \right\|_{\dot{\rho}_1} + \left\| \frac{\nabla^2 U}{U} \right\|_{\dot{\rho}_1} \left\| \frac{\nabla U}{U} \right\|_\infty + \left\| \frac{\nabla U}{U} \right\|_\infty \|\nabla^2\Theta\|_{\dot{\rho}_1} \right. \\ &\quad \left. + \left(\left\| \frac{\nabla^2 U}{U} \right\|_\infty + \left\| \frac{\nabla U}{U} \right\|_\infty^2 \right) \|\nabla\Theta\|_{\dot{\rho}_1} + 1 + F_1 + \|\nabla^3\Phi\|_{\dot{\rho}_1} \right). \end{aligned} \quad (3.38)$$

To estimate the integral $L^2(\dot{\rho}_1)$, we apply L^∞ estimate in the region $|z| \leq 1$ and (3.20) and $\dot{\rho}_1 \lesssim \langle z \rangle^{\sigma + \epsilon/2} g_1 \lesssim \langle z \rangle^{\sigma + \epsilon/2} g_2$ to the region $|z| \geq 1$:

$$\|\nabla^l U/U\|_{\dot{\rho}_1} \lesssim \|\nabla^l U/U\|_{L^\infty(|z|\leq 1)} + \|\langle z \rangle^{\sigma + \epsilon/2} \nabla^l U/U\|_{g_1} \lesssim 1 + E_0 + E_K \lesssim 1, \quad l = 2, 3.$$

Applying (3.21) with $(l, m) = (1, 0), (0, 0)$, we get

$$|\nabla^{l+1} U/U| \lesssim 1. \quad (3.39)$$

As a consequence, we can simplify the viscous estimate as follows:

$$\|\nabla\mathcal{D}_\Theta\|_{\dot{\rho}_1} \lesssim \mathcal{E}_2(1 + F_1 + \|\nabla^2\Phi\|_{\dot{\rho}_1} + \|\nabla^3\Phi\|_{\dot{\rho}_1}). \quad (3.40)$$

Finally, since $\dot{\rho}_1$ is L^1 integrable and $\dot{\rho}_i \lesssim \langle z \rangle^{2i-2} \dot{\rho}_1$ (3.17), we can decompose the integral region into $I = [0, 1]^d$ and the rest of the outer region I^c as in the L^2 estimate of the amplitude to compute

$$\|\nabla^l \Phi\|_{\dot{\rho}_1} \lesssim \sup_{z \in I} |\nabla^l \Phi| + \|\nabla^l \Phi |z|^{2l-2}\|_{\dot{\rho}_1} \lesssim \Gamma + F_l, \quad \|\nabla^l \bar{\Theta}\|_{\dot{\rho}_1} \lesssim 1, \quad l = 2, 3.$$

We use Proposition 1 and the bootstrap assumption (3.8) to further obtain

$$\|\nabla^l \Phi\|_{\dot{\rho}_1} + \|\nabla \bar{\Theta}\|_{\dot{\rho}_1} \lesssim 1 + F_1 + F_K \lesssim 1.$$

Plugging in the estimate in (3.40) and combined with (3.34), (3.36), and (3.37), we establish Lemma 3. \square

3.4 H^K stability analysis

For the estimate at the highest order, we consider the weighted H^K energies (3.4)

$$E_K^2 = (|\nabla^K W|^2, \rho_K), \quad F_K^2 = (|\nabla^K \Phi|^2, U^2 \rho_K).$$

In this section, we will establish the following lemma.

Lemma 4 (Weighted H^K estimate). *Under the bootstrap assumption 1, we have*

$$\frac{1}{2} \frac{d}{d\tau} (E_K^2 + F_K^2) \leq -\frac{\epsilon}{8} (E_K^2 + F_K^2) + \mu_0 \mathcal{E}_1 (E_K + F_K) \quad (3.41)$$

for some absolute constant $\mu_0 > 0$.

3.4.1 Estimates of the amplitude

Recall the definitions of $\mathcal{L}_{\bar{U}}, \mathcal{N}_U, \mathcal{F}_U$ from (3.3). We have

$$\frac{1}{2} \frac{d}{d\tau} E_K^2 = (\nabla^K (\mathcal{L}_{\bar{U}} W) + \nabla^K \mathcal{N}_U + \nabla^K \mathcal{F}_U + \nabla^K \mathcal{D}_U, \nabla^K W \rho_K).$$

For the leading order linear term, we can calculate the damping similarly as in the L^2 estimates. A direct computation yields $\bar{U}^{p-1} \in \mathcal{H}^i$ (1.27) for any $i \geq 0$. Using the Leibniz rule, the product rule (3.14) in Proposition 1 with $i+j=K, j \leq K-1, m=K-1, n=i+d$, and $g_K \approx \rho_K$ (1.28), (3.9), we yield

$$\|\nabla^K (\bar{U}^{p-1} W) - \bar{U}^{p-1} \nabla^K W\|_{\rho_K} \lesssim \sum_{j \leq K-1} \nabla^{K-j} \|\bar{U}^{p-1}\|_{\mathcal{H}^{i+d}} \|W\|_{\mathcal{H}^j} \lesssim \sum_{j \leq K-1} E_j \lesssim \mathcal{E}_1.$$

Therefore, we can compute

$$\nabla^K (\mathcal{L}_{\bar{U}} W) = c_{\bar{U}} \nabla^K W - \frac{1}{2} \sum_i z_i \nabla^K \partial_i W - \frac{K}{2} \nabla^K W + p \bar{U}^{p-1} \nabla^K W + O(\mathcal{R}_{\mathcal{L}, K}), \quad \|\mathcal{R}_{\mathcal{L}, K}\|_{\rho_K} \lesssim \mathcal{E}_1.$$

We can calculate the damping similar to the L^2 case as follows:

$$\begin{aligned} d_K &:= -\frac{1}{p-1} - \frac{K}{2} + p \bar{U}^{p-1} + \frac{1}{4\rho_K} \nabla \cdot (z \rho_K) \\ &= \frac{p}{p-1 + c_p |z|^2} - \frac{1}{p-1} - \frac{K}{2} + \frac{d + (2K + \frac{4}{p-1} - \epsilon) c_1 |z|^{\frac{4}{p-1} - \epsilon - d + 2K}}{4(1 + c_1 |z|^{\frac{4}{p-1} - \epsilon - d + 2K})} \leq -\frac{\epsilon}{8}, \end{aligned}$$

where the last inequality holds for a sufficiently small c_1 , which we defer till (3.47) where we combine this damping with the estimates of the nonlinear term in the phase equation.

For the nonlinear term, we use Netwon-Leibniz's formula twice as in the L^2 estimate (3.28), to derive

$$|(\nabla^K \mathcal{N}_U, \nabla^K W \rho_K)| \lesssim \sup_{\alpha \in [0, 1]} (1 - \alpha) \|\nabla^K (W^2 (\bar{U} + \alpha W)^{p-2})\|_{\rho_K} E_K.$$

Since $\|f\|_{\rho_K} \lesssim \|f\|_{\mathcal{H}^K}$ (1.28), (3.9), using the product estimate (3.19a) with $(W_1, W_2) = (W, W)$ and $\|f\|_{\mathcal{H}^K} \lesssim E_0 + E_K$ (3.10), we obtain

$$|(\nabla^K \mathcal{N}_U, \nabla^K W \rho_K)| \lesssim E_K \|W\|_{\mathcal{H}^K} \|W\|_{\mathcal{H}^{K-1}} \lesssim E_K (E_K + E_0) (E_0 + E_{K-1}) \lesssim E_K \mathcal{E}_1. \quad (3.42)$$

Recall \mathcal{E}_2 from (3.11). For the residue term, we have via integration by parts that

$$|(\nabla^K \mathcal{F}_U, \nabla^K W \rho_K)| \lesssim \mathcal{E}_2 (E_K^2 + E_K + (|\nabla^K W|^2, |z \cdot \nabla \rho_K| + |\nabla \rho_K|)).$$

Since we have $|\nabla \rho_K| \langle z \rangle \lesssim \rho_K$, we can conclude the residue estimate

$$|(\nabla^K \mathcal{F}_U, \nabla^K W \rho_K)| \lesssim \mathcal{E}_2 (E_K + E_K^2) \lesssim \mathcal{E}_1 E_K. \quad (3.43)$$

3.4.2 Estimates of the phase

We have

$$\begin{aligned} \frac{1}{2} \frac{d}{d\tau} F_K^2 &= \left(\nabla^K \left(-\frac{1}{2} \Lambda \Phi \right) + \nabla^K \mathcal{N}_\Theta + \nabla^K \mathcal{F}_\Theta + \nabla^K \mathcal{D}_\Theta, \nabla^K \Phi U^2 \rho_K \right) \\ &\quad + \left(\mathcal{L}_{\bar{U}} W + \mathcal{N}_U + \mathcal{F}_U + \mathcal{D}_U, U |\nabla^K \Phi|^2 \rho_K \right). \end{aligned}$$

Notice that the weight is time-dependent. We remark that it is essential to pair the two linear terms and the two residue terms together to cancel out the leading order term via integration by parts. For the leading order linear term, we have via integration by parts that

$$\left(\nabla^K \left(-\frac{1}{2} \Lambda \Phi \right), \nabla^K \Phi U^2 \rho_K \right) + \left(\mathcal{L}_{\bar{U}} W + \mathcal{N}_U, U |\nabla^K \Phi|^2 \rho_K \right) = (\dot{d}_K, |\nabla^K \Phi|^2 U^2 \rho_K),$$

where we can calculate the damping

$$\dot{d}_K = \frac{-K}{2} + \frac{1}{4\rho_K} \nabla \cdot (z\rho_K) - \frac{1}{p-1} + \frac{U^p}{U} < d_K + U^{p-1} - \bar{U}^{p-1}.$$

Notice that by (3.28), (3.13a) in Proposition 1 with $l = 0$, and Corollary 2, we can further estimate

$$|U^{p-1} - \bar{U}^{p-1}| \lesssim \sup_{0 \leq \alpha \leq 1} |W(\bar{U} + \alpha W)^{p-2}| \lesssim (E_0 + E_{K-1})(1 + E_K + E_0)^{p+2} \lesssim \mathcal{E}_1. \quad (3.44)$$

For the residue term, similarly via integration by parts, we have

$$\begin{aligned} |(\nabla^K \mathcal{F}_\Theta, \nabla^K \Phi U^2 \rho_K)| + |(\mathcal{F}_U, U |\nabla^K \Phi|^2 \rho_K)| &\lesssim \mathcal{E}_2(F_K + F_K^2 + \left(\frac{\nabla \cdot ((Pz + v)\rho_K)}{2\rho_K}, |\nabla^K \Phi|^2 U^2 \rho_K \right)) \\ &\lesssim \mathcal{E}_2(F_K + F_K^2) \lesssim \mathcal{E}_1 F_K, \end{aligned} \quad (3.45)$$

where the inequality is again by the fact that $|\nabla \rho_K| \langle z \rangle \lesssim \rho_K$.

For the nonlinear term, using Newton-Leibniz's formula (3.35), we obtain

$$|\nabla^K \mathcal{N}_\Theta| \leq I_{0,K} + C \sum_{j \leq K-1} I_{0,j}, \quad I_{i,j} = \delta(p-1) \cdot \nabla^j W \cdot \nabla^{K-j} (U + \alpha \bar{W}).$$

Applying (3.18) in Proposition 3 with $(W_1, W_2, j_1, j_2, k) = (U, W, 0, j, K-j)$ and $\bar{U} \in \mathcal{H}^i$ (3.16), we obtain

$$\|U I_{0,j}\|_{\rho_K} \lesssim \|U\|_{\mathcal{H}^K} \|W\|_{\mathcal{H}^{K-1}} \lesssim E_0 + E_{K-1} \lesssim \mathcal{E}_1.$$

Recall $\dot{\rho}_K = U^2 \rho_K$ (1.28). For $j \leq K-1$, the above estimate implies

$$\|I_{0,j}\|_{\dot{\rho}_K} = \|U I_{0,j}\|_{\rho_K} \lesssim \mathcal{E}_1, \quad |(I_{0,j}, \nabla^K \Phi \dot{\rho}_K)| \lesssim \|I_{0,j}\|_{\dot{\rho}_K} F_K \lesssim \mathcal{E}_1 F_K. \quad (3.46)$$

The term $I_{0,K}$ is trickier and we need to estimate by an AM-GM inequality:

$$((\bar{U} + \alpha W)^{p-2} \nabla^K W, \nabla^K \Phi U^2 \rho_K) \leq \frac{1}{2} \|(U^{\frac{1}{2}} (\bar{U} + \alpha W)^{\frac{p-2}{2}} \nabla^K \Phi)\|_{\dot{\rho}_K}^2 + \frac{1}{2} \|U^{\frac{1}{2}} (\bar{U} + \alpha W)^{\frac{p-2}{2}} \nabla^K W\|_{\rho_K}^2,$$

where we pair one of U in U^2 with $\rho_K^{1/2}$ to get $\dot{\rho}_K^{1/2}$. Applying $U = (1-\alpha)W + \bar{U} + \alpha W$, Newton-Leibniz's rule for $(\bar{U} + \alpha W)^{p-1} - \bar{U}^{p-1}$, Proposition 1 for W , and Corollary 2 for $\bar{U} + sW$, $s \in [0, 1]$, which are similar to the estimate of \mathcal{N}_Θ (3.35), we obtain

$$\begin{aligned} U(\bar{U} + \alpha W)^{p-2} &\leq (\bar{U} + \alpha W)^{p-1} + C|W(\bar{U} + \alpha W)^{p-2}| \leq \bar{U}^{p-1} + C \sup_{s \in [0,1]} |W(\bar{U} + sW)^{p-2}| \\ &\leq \bar{U}^{p-1} + C(E_{K-1} + E_0) \leq (\min\{p-1, c_p\})^{-1} \langle z \rangle^{-2} + C\mathcal{E}_1, \end{aligned}$$

where we extract a decay at the far field for the leading order term. We can calculate the damping

$$\begin{aligned} & \delta(p-1)(\min\{p-1, c_p\})^{-1}\langle z \rangle^{-2} + d_K \\ & \leq \frac{\delta(p-1)(p+1)}{\min\{p-1, c_p\}(1+|z|^2)} - \frac{1}{p-1} - \frac{K}{2} + \frac{d + (2K + \frac{4}{p-1} - \epsilon)c_1|z|^{\frac{4}{p-1} - \epsilon - d + 2K}}{4(1 + c_1|z|^{\frac{4}{p-1} - \epsilon - d + 2K})}. \end{aligned}$$

Recall the definition of K in (1.29) and similar to the L^2 damping, we can use a weighted AM-GM inequality to conclude for a sufficiently small positive c_1 , we have

$$(p-1)\delta(\min\{p-1, c_p\})^{-1/2}\langle z \rangle^{-1/2} + d_K \leq -\frac{\epsilon}{8}. \quad (3.47)$$

As a consequence, we collect the linear and nonlinear estimates of the phase, and the linear estimate of the amplitude together as follows:

$$\begin{aligned} & \left(\nabla^K(\mathcal{L}_{\bar{U}}W), \nabla^K W \rho_K \right) + \left(\nabla^K(-\frac{1}{2}\Lambda\Phi + \mathcal{N}_{\Theta}) + \frac{\mathcal{L}_{\bar{U}}W + \mathcal{N}_{\bar{U}}}{U}, \nabla^K \Phi U^2 \rho_K \right) \\ & \leq -\frac{\epsilon}{8}(E_K^2 + F_K^2) + C\mathcal{E}_1(E_K + F_K). \end{aligned} \quad (3.48)$$

3.4.3 Estimates of the viscous terms

Finally, we estimate the viscous terms. The simpler term can be estimated as follows:

$$(\mathcal{D}_U, U|\nabla^K \Phi|^2 \rho_K) \leq \|\frac{\mathcal{D}_U}{U}\|_{\infty} F_K^2 \lesssim \mathcal{E}_1 F_K. \quad (3.49)$$

The last inequality is derived similarly to the H^1 viscous estimates in (3.38), (3.39).

We group leading order viscous terms as follows and estimate them together:

$$(\nabla^K \mathcal{D}_U, \nabla^K W \rho_K) + (\nabla^K \mathcal{D}_{\Theta}, \nabla^K \Phi U^2 \rho_K),$$

and we will use integration by parts to cancel out the leading order terms and extract damping. Recall the definition of the viscous terms in (2.9). For any tensor f , we define

$$|f|_{\mathcal{Q}}^2 = \sum_i (\nabla f_i)^T \mathcal{Q} \nabla f_i,$$

where we sum over its scalar entry components f_i .

Notice that $|\nabla \rho_K| \lesssim \rho_K$. We compute the damping of the amplitude using integration by parts and the Cauchy-Schwarz inequality as

$$\begin{aligned} & (\nabla^K \Delta_{\mathcal{Q}} U, \nabla^K W \rho_K) \leq C\mathcal{E}_1 E_K - (|\nabla^K W|_{\mathcal{Q}}^2, \rho_K) + C|\mathcal{Q}|^{1/2} \|\nabla^K W\|_{\rho_K} \| |\nabla^K W|_{\mathcal{Q}} \|_{\rho_K} \\ & \leq C\mathcal{E}_1 E_K - \frac{1}{2} (|\nabla^K W|_{\mathcal{Q}}^2, \rho_K). \end{aligned}$$

Using (3.13a) from Proposition 1 and (3.16), we get

$$|\nabla U| \lesssim U, \quad |\nabla(U^2 \rho_K)| \lesssim |\nabla U| U \rho_K + U^2 |\nabla \rho_K| \lesssim U^2 \rho_K.$$

Similarly, we compute the damping of the phase as

$$(\nabla^K \Delta_{\mathcal{Q}} \Theta, \nabla^K \Phi U^2 \rho_K) \leq C\mathcal{E}_1 F_K - \frac{1}{2} (|\nabla^K \Phi|_{\mathcal{Q}}^2, U^2 \rho_K).$$

For the four intermediate terms in the viscous terms

$$I_1 = \langle \nabla U, \nabla \Theta \rangle_{\mathcal{Q}}, \quad I_2 = U \langle \nabla \Theta, \nabla \Theta \rangle_{\mathcal{Q}}, \quad I_3 = \frac{\langle \nabla U, \nabla \Theta \rangle_{\mathcal{Q}}}{U}, \quad I_4 = \langle \nabla \Theta, \nabla \Theta \rangle_{\mathcal{Q}},$$

we can simply control their weighted norms using the diffusion term.

We consider the most challenging term I_3 . Using the Leibniz rule, (3.20) in Proposition 3, we obtain

$$\|I_3\|_{\dot{\rho}_K} \lesssim \sum_{0 \leq i \leq K} \|I_{3,i}\|_{\dot{\rho}_K}, \quad I_{3,i} = (\nabla^i \frac{\nabla U}{U}, \nabla^{K-i+1} \Theta)_{\mathcal{Q}}.$$

For $1 \leq i \leq K-1$, applying (3.21) to $\nabla U/U$ if $i \leq K/2+1 < K-d-1$ and (3.12) to Θ if $i > K/2$, which implies $K-i+1 \leq K/2+1 < K-d-1$, we obtain

$$|I_{3,i}| \lesssim |\mathcal{Q}| (\langle z \rangle^{-i} |\nabla^{K-i+1} \Theta| + \langle z \rangle^{-(K-i)} |\nabla^i (\nabla U/U)|).$$

Since $i \leq K-1, K-i+1 \leq K$, using the estimate (3.17) for weights

$$\dot{\rho}_K^{1/2} \lesssim \langle z \rangle^{i-1} \dot{\rho}_{K-i+1}^{1/2}, \quad \langle z \rangle^{-(K-i)} \dot{\rho}_K^{1/2} \lesssim \dot{\rho}_i^{1/2} \lesssim \langle z \rangle^{\sigma+\epsilon/2} \rho_i^{1/2},$$

and (3.20), we obtain

$$\|I_{3,i}\|_{\dot{\rho}_K} \lesssim |\mathcal{Q}| (\|\nabla^{K-i+1} \Theta\|_{\dot{\rho}_{K-i}} + \|\langle z \rangle^{\sigma+\epsilon/2} \nabla^i (\nabla U/U)\|_{\rho_i}) \lesssim |\mathcal{Q}| \lesssim \mathcal{E}_1.$$

For $I_{3,0}, I_{3,K}$, we use the Cauchy-Schwarz inequality to compute that its ρ_K norm is bounded by

$$\mathcal{E}_1 + |\mathcal{Q}|^{1/2} (\|\nabla \Theta\|_{\infty} \|\nabla^K W\|_{\mathcal{Q}} + \|\frac{\nabla U}{U}\|_{\infty} \|U\|_{\mathcal{Q}} \|\nabla^K \Phi\|_{\mathcal{Q}}).$$

Similarly, we have the estimates for the other three viscous terms I_1, I_2, I_4 . Combined with (3.39), we can use the Cauchy-Schwarz inequality to derive that

$$\begin{aligned} & \left(-2\beta \nabla^K \langle \nabla U, \nabla \Theta \rangle_{\mathcal{Q}} - \nabla^K (U \langle \nabla \Theta, \nabla \Theta \rangle_{\mathcal{Q}}), \nabla^K W \rho_K \right) + \left(2\nabla^K \frac{\langle \nabla U, \nabla \Theta \rangle_{\mathcal{Q}}}{U} - \beta \nabla^K \langle \nabla \Theta, \nabla \Theta \rangle_{\mathcal{Q}}, \nabla^K \Phi U^2 \rho_K \right) \\ & \leq C\mathcal{E}_1 (E_K + F_K) + \frac{1}{8} (|\nabla^K W|_{\mathcal{Q}}^2, \rho_K) + (|\nabla^K \Phi|_{\mathcal{Q}}^2, U^2 \rho_K). \end{aligned}$$

Finally, for the last two viscous terms, we use integration by parts to cancel out the leading order terms. Applying estimates similar to the those for I_3 in the above, we can extract the leading order terms, which involve $\nabla^{K+2} U$ or $\nabla^{K+2} \Theta$,

$$\begin{aligned} -\beta \left(\nabla^K (U \Delta_{\mathcal{Q}} \Theta), \nabla^K W \rho_K \right) &= -\beta (U \Delta_{\mathcal{Q}} \nabla^K \Theta, \nabla^K W \rho_K) + O(\mathcal{E}_1 E_K) + \frac{1}{16} (|\nabla^K \Phi|_{\mathcal{Q}}^2, U^2 \rho_K), \\ \beta \left(\nabla^K \frac{\Delta_{\mathcal{Q}} U}{U}, \nabla^K \Phi U^2 \rho_K \right) &= \beta \left(\frac{\Delta_{\mathcal{Q}} \nabla^K U}{U}, \nabla^K \Phi U^2 \rho_K \right) + O(\mathcal{E}_1 F_K) + \frac{1}{16} (|\nabla^K U|_{\mathcal{Q}}^2, \rho_K). \end{aligned}$$

Now, we use $U = \bar{U} + W, \Theta = \bar{\Theta} + \Phi$ and integration by parts to cancel out the leading order terms.

$$\begin{aligned} & -(\nabla^K \Delta_{\mathcal{Q}} \Theta, \nabla^K W U \rho_K) + (\nabla^K \Delta_{\mathcal{Q}} U, \nabla^K \Phi U \rho_K) \\ &= -(\nabla^K \Delta_{\mathcal{Q}} \Phi, \nabla^K W U \rho_K) + (\nabla^K \Delta_{\mathcal{Q}} W, \nabla^K \Phi U \rho_K) + O(\mathcal{E}_1 (E_K + F_K)) \\ &= \sum_{i,j} \left(\mathcal{Q}_{ij} (-\partial_i (\partial_j \nabla^K \Phi \cdot \nabla^K W) + \partial_j (\partial_i \nabla^K W \cdot \nabla^K \Phi)), U \rho_K \right) \\ &\leq C\mathcal{E}_1 (E_K + F_K) + \frac{1}{16\beta} (|\nabla^K W|_{\mathcal{Q}}^2, \rho_K) + (|\nabla^K \Phi|_{\mathcal{Q}}^2, U^2 \rho_K). \end{aligned}$$

We notice that the remaining terms from integration by parts are controlled since $|\nabla(U \rho_K)| \lesssim U \rho_K$.

Combining the viscous estimates with the estimates (3.42), (3.43), (3.45), and (3.48), we conclude the proof of Lemma 4.

3.4.4 Summary of the H^K estimates

Using (3.12a), (3.12b) in Proposition 1, for any $\mu > 0$, we obtain

$$\mathcal{E}_1 \leq |\mathcal{Q}| + H^{p-1} + C(\mu)(E_0 + F_1) + \mu(E_K + F_K) + (E_0 + F_1 + E_K + F_K)^2.$$

By Lemma 4, choosing $\mu < \frac{\epsilon}{16\mu_0}$ and then collecting (3.41), (3.33), and (3.25), we obtain that there exists a sufficiently small constant $1 > \nu_1 > \nu_2 > 0$, ν_2 determined after ν_1 , such that for the energy

$$E^2 = E_K^2 + F_K^2 + 1/\nu_1 F_1^2 + 1/\nu_2 E_0^2, \quad (3.50)$$

the following estimate holds

$$\frac{1}{2} \frac{d}{d\tau} E^2 \leq -\frac{\epsilon}{16} E^2 + C(|\mathcal{Q}| + H^{p-1})E + CE^3, \iff \frac{d}{d\tau} E \leq -\frac{\epsilon}{16} E + \mu_1(|\mathcal{Q}| + H^{p-1}) + \mu_1 E^2, \quad (3.51)$$

for some absolute constant $\mu_1 > 0$. Here, the constant C would depend on ν_1, ν_2 . Once we fix ν_1, ν_2 , then C becomes a fixed constant μ_1 . The estimate holds provided that Assumption 1 is valid.

3.5 Lower bound of the amplitude

We now prove the bootstrap Assumption 1 by estimating the lower bound of $U\rho$, for the weight $\rho = \bar{U}^{-1-\epsilon_2}$. We will proceed with a maximal principle argument and a barrier argument. Notice that

$$\nabla U = \frac{\nabla(U\rho) - U\nabla\rho}{\rho}, \quad \nabla^2 U = \frac{\nabla^2(U\rho) - U\nabla^2\rho}{\rho} - \frac{\nabla(U\rho)\nabla\rho^T + \nabla\rho\nabla(U\rho)^T - 2U\nabla\rho\nabla\rho^T}{\rho^2}.$$

We compute by (2.7) that

$$\partial_\tau(U\rho) = \mathcal{P}_U(U\rho), \quad \mathcal{P}_U f = A_0 f + A_1 \cdot \nabla f + \text{tr}(Q\nabla^2 f). \quad (3.52)$$

where the coefficients A_0, A_1 of the parabolic operator \mathcal{P}_U are:

$$A_0 = c_U - H^{p-1}\gamma + U^{p-1} + \left(\frac{1}{2}z + \mathcal{P}z + \mathcal{V}\right) \cdot \frac{\nabla\rho}{\rho} - \frac{\Delta_{\mathcal{Q}}\rho - 2\beta\langle\nabla\rho, \nabla\Theta\rangle_{\mathcal{Q}}}{\rho} + 2\frac{\langle\nabla\rho, \nabla\rho\rangle_{\mathcal{Q}}}{\rho^2} - \langle\nabla\Theta, \nabla\Theta\rangle_{\mathcal{Q}} - \beta\Delta_{\mathcal{Q}}\Theta,$$

$$A_1 = -\left(\frac{1}{2}z + \mathcal{P}z + \mathcal{V} + 2\frac{\mathcal{Q}\nabla\rho}{\rho} + 2\beta\mathcal{Q}\right),$$

Notice that \bar{U} is the approximate steady state and $|\nabla\rho|\langle z \rangle \lesssim \rho$. We can calculate the damping using the nonlinear estimate (3.44) and Lemma 1 that:

$$A_0 = O(\mathcal{E}_1) - \epsilon_2 \frac{z \cdot \nabla \bar{U}}{\bar{U}}, \quad A_1 = -\left(\frac{1}{2}z + \mathcal{P}z\right) + O(\mathcal{E}_1), \quad |\mathcal{P}| \lesssim \mathcal{E}_1.$$

Next, we define a barrier function $F = \bar{U}^{-4\epsilon_2}$. Since $|z \cdot \nabla F| \lesssim F, |\nabla^i F| \lesssim F, i = 1, 2$, we get

$$\begin{aligned} \mathcal{P}_U F &= \left(O(\mathcal{E}_1) - \epsilon_2 \frac{z \cdot \nabla \bar{U}}{\bar{U}} + \frac{A_1 \cdot \nabla F}{F}\right)F + \text{tr}(Q\nabla^2 F) \\ &= \left(O(\mathcal{E}_1) - \epsilon_2 \frac{z \cdot \nabla \bar{U}}{\bar{U}} - \frac{1}{2} \frac{z \cdot \nabla F}{F}\right)F = \left(O(\mathcal{E}_1) - (\epsilon_2 - 2\epsilon_2) \frac{z \cdot \nabla \bar{U}}{\bar{U}}\right)F = \left(O(\mathcal{E}_1) + \epsilon_2 \frac{z \cdot \nabla \bar{U}}{\bar{U}}\right)F. \end{aligned}$$

- For $|z| \geq 1$, we derive by the form of \bar{U} in (1.27) the lower bound $-\frac{z \cdot \nabla \bar{U}}{\bar{U}} \geq \mu_{U,2}$ for some positive constant $\mu_{U,2}$. Recall the definition of \mathcal{E}_1 (3.11) and $E \lesssim 1$ from (3.50) and Assumption (3.8). Since $|\mathcal{Q}| \lesssim \text{tr}(\mathcal{Q})$, for some positive constant $\mu_{U,1}$, we have

$$A_0 \geq \mu_{U,2}\epsilon_2 - \mu_{U,1}(\text{tr}(\mathcal{Q}) + H^{p-1} + E), \quad \mathcal{P}_U F \leq (\mu_{U,1}(\text{tr}(\mathcal{Q}) + H^{p-1} + E) - \mu_{U,2}\epsilon_2)F. \quad (3.53)$$

- For $|z| \leq 1$, since ρ is bounded on the interval and we recall the definition of Γ (2.13), we can estimate

$$U\rho = \bar{U}^{-\epsilon_2} + W\rho \geq 4C_b - C\Gamma \geq 4C_b - \mu_{U,3}E, \quad (3.54)$$

for some positive constant $\mu_{U,3}$. Here we use the definition of C_b in Assumption 1.

Hence, by enforcing $E, |\mathcal{Q}| + H^{p-1}$ sufficiently small, we will verify the following bootstrap assumption.

Assumption 2.

$$A_0 > 0, \mathcal{P}_U F < 0, \quad |z| \geq 1, \quad U\rho > 2C_b, \quad |z| \leq 1. \quad (3.55)$$

Now, we consider $\Omega_c = U\rho + cF$ for $c > 0$. From Corollary 2 and the choice of ϵ_2 , we obtain

$$U\rho \lesssim \langle z \rangle^{\sigma+\epsilon/2} \langle z \rangle^{-\sigma+\epsilon/2} = \langle z \rangle^\epsilon, \quad F = \bar{U}^{-4\epsilon_2} \gtrsim \langle z \rangle^{8\epsilon_2/(p-1)} \gtrsim \langle z \rangle^{2\epsilon}.$$

Under the assumption (3.55) and 1, for any $c > 0$, we have

$$\Omega_c(z) > 2C_b, \quad |z| \leq 1, \quad \lim_{|z| \rightarrow \infty} \Omega_c = \infty.$$

Using the above estimates of \mathcal{P}_U , we get

$$\partial_\tau \Omega_c = \partial_\tau(U\rho) = \mathcal{P}_U(U\rho + cF) - c\mathcal{P}_U F = \mathcal{P}_U \Omega_c - c\mathcal{P}_U F > \mathcal{P}_U \Omega_c.$$

By choosing initial data with $U\rho > 2C_b$ and then applying the maximal principle to the operator \mathcal{P}_U on $|z| \geq 1$, we obtain

$$\Omega_c > 2C_b, \quad U\rho + cF \geq 2C_b, \quad |z| \geq 1.$$

Since c is arbitrary, taking $c \rightarrow 0$, we prove $U\rho > 2C_b$ for $|z| \geq 1$, which along with (3.55) for $U\rho$ concludes $U\rho \geq 2C_b, \forall z \in \mathbb{R}^d$ and strengthens (3.7) in Assumption 1.

In Section 3.6, we prove Assumption 2.

3.6 Bootstrap argument and blowup

In this section, we prove Theorem 3 by combining previous estimates and use a bootstrap argument.

Recall the ODE of Q from Lemma 1 and \mathcal{E}_0, Γ from (2.13)

$$\mathcal{Q}_\tau = -(\mathcal{Q}_u + \frac{1}{2}\mathcal{Q}_d)\mathcal{Q} - \mathcal{Q}(\mathcal{Q}_u^T + \frac{1}{2}\mathcal{Q}_d) + \mathcal{O}(|\mathcal{Q}|\mathcal{E}_0), \quad \mathcal{E}_0 = |\mathcal{Q}|(\Gamma + \Gamma^4) + H^{p-1}. \quad (3.56)$$

Since the parameters ν_i in the energy E (3.50) have been chosen as some absolute constants, under the bootstrap assumption 1, we get

$$E \lesssim 1, \quad \Gamma \lesssim E_0 + E_K \lesssim E \lesssim 1, \quad \mathcal{E}_0 \lesssim |\mathcal{Q}|\Gamma + H^{p-1} \lesssim |\mathcal{Q}|E + H^{p-1}. \quad (3.57)$$

Taking trace on both side of (3.56) and then using $\mathcal{Q} = \mathcal{Q}_u + \mathcal{Q}_u^T + \mathcal{Q}_d$,

$$\begin{aligned} \text{tr}((\mathcal{Q}_u + \frac{1}{2}\mathcal{Q}_d)\mathcal{Q} + \mathcal{Q}(\mathcal{Q}_u + \frac{1}{2}\mathcal{Q}_d)^T) &= \text{tr}((\mathcal{Q}_u + \frac{1}{2}\mathcal{Q}_d + \frac{1}{2}\mathcal{Q}_d + \mathcal{Q}_u^T)\mathcal{Q}) = \text{tr}(\mathcal{Q}^2), \\ |\mathcal{Q}| \approx \text{tr}(\mathcal{Q}), \quad \text{tr}(\mathcal{Q}^2) &= \sum \lambda_{\mathcal{Q},i}^2 \geq \frac{1}{d}(\sum \lambda_{\mathcal{Q},i})^2 = \frac{1}{d}(\text{tr}(\mathcal{Q}))^2, \end{aligned}$$

where $\lambda_{\mathcal{Q},i}$ is the eigenvalue of \mathcal{Q} , and the above estimates, we get for a constant μ_2 :

$$\partial_\tau \text{tr}(\mathcal{Q}) \leq -\text{tr}(\mathcal{Q}^2) + \mu_2(E \text{tr}(\mathcal{Q})^2 + H^{p-1} \text{tr}(\mathcal{Q})) \leq -\frac{1}{d}(\text{tr} \mathcal{Q})^2 + \mu_2(E(\text{tr} \mathcal{Q})^2 + H^{p-1} \text{tr}(\mathcal{Q})). \quad (3.58)$$

Recall c_W from (2.4). To simplify the nonlinear estimates, in addition to bootstrap assumption 1, we impose the following assumption

Assumption 3.

$$|c_W| < \frac{1}{2} \min((p-1)^{-1}, 1), \quad E(\tau) < \min(\frac{1}{4d\mu_1}, \frac{\epsilon}{32\mu_2}), \quad (3.59)$$

where μ_1 is the constant in (3.50). We denote

$$\epsilon_1 = \mu_2 H^{p-1}, \quad a(\tau) = \exp(\mu_2 \int_0^\tau H^{p-1}(s) ds), \quad \lambda = \frac{\epsilon}{32}. \quad (3.60)$$

Consequence of bootstrap assumptions. We perform the energy estimates under the assumptions (3.59) and 1 and show that these estimates can be strengthened.

Using (2.2), (2.4), we obtain $(p-1)c_U(s) = -1 + (p-1)c_W(s) \leq -\frac{1}{2}$ and

$$H^{p-1}(\tau) \leq H^{p-1}(0) \exp\left(\int_0^\tau (p-1)c_U(s)ds\right) \leq H^{p-1}(0) \exp(-\tau/2), \quad -\frac{1}{d} + \mu_2 E(\tau) < -\frac{1}{2d}. \quad (3.61)$$

We can solve the ODE of $(\text{tr}(Q))^{-1}$ using the above estimate and (3.58) to obtain

$$\partial_\tau E_{\mathcal{Q}}^{-1} \geq \frac{1}{2d} - \mu_2 H^{p-1} E_{\mathcal{Q}}^{-1}, \quad E_{\mathcal{Q}} := \text{tr}(Q).$$

By choose $H^{p-1}(0)$ small enough such that $\exp(2\epsilon_1) < 2$, for any $0 \leq s \leq \tau$, we get

$$a(\tau)a(s)^{-1} \leq e^{\epsilon_1 \int_0^\tau \exp(-s/2)ds} \leq e^{2\epsilon_1} < 2, \quad a(\tau)^{-1}a(s) > \frac{1}{2}, \quad a(0) = 1.$$

Solving the above ODE, we yield

$$\begin{aligned} E_{\mathcal{Q}}^{-1}(\tau) &\geq a(\tau)^{-1}E_{\mathcal{Q}}^{-1}(0) + \frac{1}{2d} \int_0^\tau a(\tau)^{-1}a(s)ds \geq \frac{1}{2}(E_{\mathcal{Q}}^{-1}(0) + \frac{1}{2d}\tau), \\ E_{\mathcal{Q}}(\tau) &\leq \min(2E_{\mathcal{Q}}(0), 4d/\tau). \end{aligned} \quad (3.62)$$

Using (3.59), the above estimates, and $-\frac{\epsilon}{16} + \mu_1 E < -\frac{\epsilon}{32} = \lambda$ (3.60), we obtain

$$\frac{d}{d\tau} E \leq -\lambda E + C(E_{\mathcal{Q}} + H^{p-1}(0)e^{-\tau/2}).$$

Solving the ODE and using (3.62), we obtain

$$E(\tau) \leq e^{-\lambda\tau} E(0) + C \int_0^\tau e^{-\lambda(\tau-s)} (\min(E_{\mathcal{Q}}(0), \frac{1}{s}) + H^{p-1}(0)e^{-s/2}) ds,$$

where C is some absolute constant and can depend on ϵ, λ . Since $\lambda < 1/2$, by decomposing the integral into $s < \tau/2$ and $s \geq \tau/2$, we obtain

$$\begin{aligned} E(\tau) &\leq e^{-\lambda\tau} (E(0) + CH^{p-1}(0)) + C \left(E_{\mathcal{Q}}(0) \int_0^{\tau/2} e^{-\lambda(\tau-s)} ds + \int_{\tau/2}^\tau e^{-\lambda(\tau-s)} \min(E_{\mathcal{Q}}(0), 1/\tau) ds \right) \\ &\leq e^{-\lambda\tau/2} (E(0) + \mu_3 H^{p-1}(0) + \mu_3 E_{\mathcal{Q}}(0)) + \mu_3 \min(E_{\mathcal{Q}}(0), 1/\tau) \end{aligned} \quad (3.63)$$

for some absolute constant $\mu_3 > 0$.

Plugging the above estimates and (3.57) into Lemma 1, and using $E \lesssim 1$ (3.59), we get for some $\mu_4 > 0$:

$$|c_W(\tau)| < C(E_{\mathcal{Q}}(\tau) + E_{\mathcal{Q}}(\tau)E(\tau) + H^{p-1}(\tau)) < \mu_4 (\min(E_{\mathcal{Q}}(0), 1/\tau) + H^{p-1}(0)e^{-\tau/2}). \quad (3.64)$$

Continuation of the bootstrap assumptions. For initial data satisfying

$$E(0) < E_*, \quad E_{\mathcal{Q}}(0) < E_*, \quad H^{p-1}(0) < E_*, \quad (3.65)$$

with E_* sufficiently small, we obtain from (3.62), (3.63), (3.64) the following estimates

$$\begin{aligned} E(\tau) &\leq e^{-\lambda\tau/2} E_*(1 + 2\mu_3) + \mu_3 \min(E_*, 1/\tau) < E_*(1 + 3\mu_3), \quad E_{\mathcal{Q}}(\tau) < 2E_*, \quad |c_W| < \mu_4 E_*, \\ H^{p-1}(\tau) &\leq H^{p-1}(0) < E_*, \quad E(\tau) + \text{tr}(Q) + H^{p-1} < (4 + 3\mu_3)E_*. \end{aligned}$$

Therefore, there exists $\nu_3 > 0$ such that for $E_* < \nu_3$, the bootstrap assumption (3.59) can be strengthened and continued. Plugging the above estimates into (3.53), (3.54) we obtain

$$A_0 \geq \mu_{U,2}\epsilon_2 - \mu_{U,3}(3\mu_3 + 4)E_*, \quad \mathcal{P}_U F \leq (\mu_{U,2}(4 + 3\mu_3)E_* - \mu_{U,2}\epsilon_2)F, \quad U\rho \geq 4C_b - \mu_{U,3}(1 + 3\mu_3)E_*.$$

By further requiring E_* to be sufficiently small, the Assumption 2 can be strengthened and continued. The L^∞ estimate in Section 3.5 strengthens (3.7) in the Assumption 1. Using the definition (3.50) and the above estimate for E , we obtain

$$(E_0 + E_K + F_1 + F_K)(t) \leq C(\nu_1, \nu_2)E_*,$$

which strengthens the first inequality in (3.8) in Assumption 1 by further choosing E_* to be small enough.

For the second inequality in (3.8), applying the Jacobi's formula $\frac{d}{d\tau} \det(\mathcal{Q}(\tau)) = \det(\mathcal{Q}(\tau)) \text{tr}(\mathcal{Q}^{-1} \frac{d}{d\tau} \mathcal{Q})$ to (2.15) and using $\text{tr}(AB) = \text{tr}(BA)$, $\mathcal{Q} = \mathcal{Q}_u + \mathcal{Q}_d + \mathcal{Q}_u^T$, we obtain

$$\partial_\tau \det(\mathcal{Q}) = \det(\mathcal{Q}) \cdot \text{tr}(-\mathcal{Q} + O(\mathcal{E}_0)).$$

From the above estimates, $|\mathcal{Q}|$ and \mathcal{E}_0 remain uniformly bounded for all $\tau > 0$. Since $\det(\mathcal{Q}(0)) > 0$, we prove $\det(\mathcal{Q}) \geq \det(\mathcal{Q}(0))e^{-C\tau}$, which strengthens the second inequality in (3.8). This concludes the proof of Theorem 3.

4 Refined asymptotics

In this section, building on Theorem 3, we obtain sharp asymptotics stated in Theorem 1. In Section 4.1, we estimate the sharp blowup rates for the amplitude similarly as in [37]. In Section 4.2, we estimate the asymptotics related to the phase and prove L^∞ convergence. In Section 4.3, we combine Theorem 3, Propositions 4 and 5 to prove Theorem 1.

4.1 Asymptotics of the amplitude and blowup rate

We use O_{in} and C_{in} to track any constant depending on the norm of the initial data $\text{tr}(\mathcal{Q}(0)), \text{tr}(\mathcal{Q}^{-1}(0))$. We have the following results for the asymptotics.

Proposition 4. *Suppose that the initial data $(U, \Theta, \mathcal{Q}, H)$ satisfy the assumption in Theorem 3. We have the following asymptotics for the modulation parameters*

$$\left| \frac{H(\tau)^{p-1}}{T-t(\tau)} - 1 \right| \lesssim C_{in} \langle \tau \rangle^{-1}, \quad \lim_{\tau \rightarrow \infty} \frac{\tau}{|\log(T-t(\tau))|} = 1, \quad \lim_{t \rightarrow T} \frac{\mathbf{R}(t)}{\sqrt{(T-t)|\log(T-t)|}} = I_d. \quad (4.1)$$

We consider $\tau \geq 2$. Note that $E_* < 1$. We focus on the asymptotics as $\tau \rightarrow \infty$ and the decay rate in τ .

Refined estimate of \mathcal{Q} . By inserting (3.63) and (3.61) into (3.58), we get

$$\partial_\tau E_{\mathcal{Q}} \leq -\frac{1}{d} E_{\mathcal{Q}}^2 + C \left(\left(\frac{1}{\tau} + e^{-\lambda\tau/2} \right) E_{\mathcal{Q}}^2 + E_{\mathcal{Q}} e^{-\tau/2} \right),$$

for some absolute constant $C > 0$. Since $E_{\mathcal{Q}} > 0$, we arrive at the ODE

$$\partial_\tau E_{\mathcal{Q}}^{-1} \geq \frac{1}{d} - C \left(\frac{1}{\tau} + e^{-\lambda\tau/2} \right) - C E_{\mathcal{Q}}^{-1} e^{-\tau/2}.$$

By introducing the integrating factor $a(\tau) = \exp(-CE_* \int_1^\tau e^{-s/2} ds)$, and using the fast convergence $|a(\tau)/a(s) - 1| \lesssim E_* e^{-s/2}$, $a(\tau) \geq e^{-CE_*}$ for $1 \leq s < \tau$, we can solve the above ODE and obtain

$$E_{\mathcal{Q}}^{-1} \geq \frac{1}{d} \tau + O(\log \tau) + E_{\mathcal{Q}}^{-1}(2) e^{-CE_*} \geq \frac{1}{d} \tau + O_{in}(\log \tau).$$

Since $\text{tr}(\mathcal{Q}) = \sum \lambda_{\mathcal{Q},i}$, we know that

$$\min(\lambda_{\mathcal{Q},i}) \leq \frac{1}{d} E_{\mathcal{Q}} \leq \frac{1}{\tau} + O_{in} \left(\frac{\log \tau}{\tau^2} \right). \quad (4.2)$$

Next we estimate $\text{tr}(\mathcal{Q}^{-1})$. From (3.56), we have by (3.57) that

$$\partial_\tau \text{tr}(\mathcal{Q}^{-1}) = d - 2\text{tr}(\mathcal{E}_\mathcal{Q} \mathcal{Q}^{-1}) \leq d + \mu_2(EE_\mathcal{Q} + H^{p-1})\text{tr}(\mathcal{Q}^{-1}).$$

By the above estimates of $E_\mathcal{Q}$, and the same estimates of E and H^{p-1} in (3.63) and (3.61), we have that for sufficiently large τ , there exists a μ_5 such that

$$\partial_\tau \text{tr}(\mathcal{Q}^{-1}) \leq d + \frac{C}{\tau^2} \text{tr}(\mathcal{Q}^{-1}).$$

We conclude that

$$\text{tr}(\mathcal{Q}^{-1}) \leq d\tau + O(\log \tau) + \text{tr}(\mathcal{Q}^{-1}(2)) \leq d\tau + O_{in}(\log \tau)$$

Using $\text{tr}(\mathcal{Q}^{-1}) = \sum_i \lambda_{\mathcal{Q},i}^{-1}$, we obtain

$$\max(\lambda_{\mathcal{Q},i}) \geq \frac{d}{\text{tr}(\mathcal{Q}^{-1})} \geq \frac{1}{\tau} + O_{in}\left(\frac{\log \tau}{\tau^2}\right). \quad (4.3)$$

Combining the above estimates, we obtain

$$\text{tr}(\mathcal{Q}^{-1})\text{tr}(\mathcal{Q}) \leq d^2 + O_{in}\left(\frac{\log \tau}{\tau}\right).$$

Using $\text{tr}(\mathcal{Q}^\alpha) = \sum \lambda_{\mathcal{Q},i}^\alpha$, $\alpha = 1, -1$, we derive

$$\text{tr}(\mathcal{Q}^{-1})\text{tr}(\mathcal{Q}) = \sum \lambda_{\mathcal{Q},i} \sum \lambda_{\mathcal{Q},i}^{-1} = d^2 + \sum_{i < j} \left(\sqrt{\frac{\lambda_{\mathcal{Q},i}}{\lambda_{\mathcal{Q},j}}} - \sqrt{\frac{\lambda_{\mathcal{Q},j}}{\lambda_{\mathcal{Q},i}}} \right)^2.$$

It follows

$$\left(\sqrt{\frac{\lambda_{\mathcal{Q},i}}{\lambda_{\mathcal{Q},j}}} - \sqrt{\frac{\lambda_{\mathcal{Q},j}}{\lambda_{\mathcal{Q},i}}} \right)^2 = O_{in}\left(\frac{\log \tau}{\tau}\right), \quad \forall i < j, \quad \frac{\max(\lambda_{\mathcal{Q},i})}{\min(\lambda_{\mathcal{Q},i})} = 1 + O_{in}\left(\left(\frac{\log \tau}{\tau}\right)^{1/2}\right).$$

Combining the above estimate with (4.2) and (4.3), we have that each one of the eigenvalue satisfies

$$\lambda_{\mathcal{Q},i} = \frac{1}{\tau} + O_{in}(\tau^{-3/2} \sqrt{\log \tau}) = \frac{1}{\tau} + O_{in}(a_\tau), \quad a_\tau = \tau^{-3/2+\epsilon_3}, \quad \epsilon_3 = \frac{1}{10}.$$

Since \mathcal{Q} is symmetric and $\mathcal{Q}(\tau) = R(\tau)\Lambda R(\tau)^T$ for $\Lambda = \text{diag}(\lambda_{\mathcal{Q},1}, \dots, \lambda_{\mathcal{Q},d})$ and some orthogonal matrix R , which satisfies $|R(\tau)| \leq C$ for C independent in τ , the above estimates further imply,

$$\mathcal{Q} = R(\tau) \left(\frac{1}{\tau} I_d + O_{in}(a_\tau) \right) R(\tau)^T = \frac{1}{\tau} I_d + O_{in}(a_\tau). \quad (4.4)$$

Estimate of \mathbf{R} and blowup rate. Recall from (2.4), (2.5)

$$\mathcal{M} = e^{-\tau/2} \mathbf{R}^{-1}, \quad \mathcal{Q} = C_W^{p-1} \mathcal{M} \mathcal{M}^T = C_W^{p-1} e^{-\tau/2} \mathbf{R}^{-1} (e^{-\tau/2} \mathbf{R}^{-1})^T = M_Q M_Q^T, \quad M_Q := C_U^{(p-1)/2} \mathbf{R}^{-1}.$$

Note that $\mathbf{R}, \mathcal{M}, M_Q$ are upper triangular matrices. Due to $M_{Q,ii}(0) > 0$ and the non-degeneracy $0 < \det(\mathcal{Q}) = \det(M_Q)^2 = \prod M_{Q,ii}^2$ for all τ from (3.8), by continuity, we have $M_{Q,ii}(\tau) > 0$, which are the eigenvalues of M_Q, M_Q^T . For each real eigenpair (λ, v) of M_Q^T with $\|v\|_2^2 = 1$, we obtain

$$\lambda^2 = \lambda^2 \|v\|_2^2 = v^T M_Q M_Q^T v = v^T \mathcal{Q} v = \tau^{-1} \|v\|_2^2 + O_{in}(a_\tau) = \tau^{-1} + O_{in}(a_\tau).$$

Since $M_{Q,ii} > 0$ is a eigenvalue of M_Q , we obtain

$$M_{Q,ii} = \tau^{-1/2} (1 + \tau O(a_\tau))^{1/2} = \tau^{-1/2} + O(\tau^{1/2} a_\tau) = \tau^{-1/2} + O((\log \tau)^{1/2} / \tau).$$

Next, we estimate the strictly upper part of M_Q : M_Q^u . Taking trace, we get

$$\sum_{i \neq j} M_{Q,ij}^2 = \text{tr}(M_Q M_Q^T) - \sum_i M_{Q,ii}^2 = \text{tr}(\mathcal{Q}) - \sum_i M_{Q,ii}^2 = d/\tau - d/\tau + O(a_\tau) = O(a_\tau)$$

which implies $M_Q^u = O(a_\tau^{1/2})$. Comparing the strictly upper part $\mathcal{Q} = (M_{Q,d} + M_Q^u)(M_{Q,d} + M_Q^u)^T$, we get

$$M_Q^u M_{Q,d} = \mathcal{Q}^u - (M_{Q,u} M_{Q,u}^T)^u = O(a_\tau), \quad M_Q^u = O(a_\tau) M_{Q,d}^{-1} = O(a_\tau \tau^{1/2}).$$

Therefore, we conclude,

$$C_U^{(p-1)/2} \mathbf{R}^{-1} = M_Q = \frac{1}{\sqrt{\tau}} I_d + O_{in}(\tau^{1/2} a_\tau) = \frac{1}{\sqrt{\tau}} I_d + O_{in}(\tau^{-1+\epsilon_3}). \quad (4.5)$$

Using (2.2), we define the blowup time as $T = t(\infty)$. Using (2.2) for $t(\tau)$, Lemma 1 and (3.59) for c_W , and (3.64), (3.6), (4.4) for $\text{tr}(\mathcal{Q})$, \mathcal{E}_0 , we obtain

$$t_\tau = H^{p-1}, \quad (p-1)c_W = \frac{\mu_5}{\tau} + O_{in}(a_\tau), \quad |(p-1)c_W| < \frac{1}{2}, \quad a_\tau = \tau^{-3/2+\epsilon_3}, \quad \mu_5 = \frac{2(1-\beta\delta)dc_p}{p-1}. \quad (4.6)$$

Using (2.2), (2.4), for $\tau \geq 2, s > 0$, we obtain

$$\frac{H^{p-1}(\tau+s)}{H^{p-1}(\tau)} = e^{-s} F(\tau, s), \quad F(\tau, s) := e^{(p-1) \int_\tau^{\tau+s} c_W(z) dz}.$$

Since $|(p-1)c_W(s)| < \min(\frac{1}{2}, C\tau^{-1})$, using $|e^x - 1| \lesssim |x|(e^x + 1)$, $\partial_s F(\tau, s) = (p-1)c_W(\tau+s)F(\tau, s)$, $F(\tau, 0) = 1$, for $0 \leq z \leq s$, we obtain

$$\begin{aligned} |F(\tau, s) - 1| &\lesssim s/\tau(F(\tau, s) + F(\tau, 0)) \lesssim e^{s/2} s/\tau, \\ |c_W(\tau+z) - \mu_5 \tau^{-1}| &\lesssim |(\tau+z)^{-1} - \tau^{-1}| + a_\tau \lesssim z\tau^{-2} + a_\tau, \end{aligned}$$

which implies

$$\begin{aligned} |c_W(\tau+z)F(\tau, z) - \mu_5 \tau^{-1}| &\lesssim |c_W(\tau+z)(F(\tau, z) - 1) + (c_W(\tau+z) - \mu_5 \tau^{-1})| \lesssim a_\tau + e^{z/2} z\tau^{-2} + z\tau^{-2}, \\ |F(\tau, s) - 1 - s\mu_5 \tau^{-1}| &\lesssim s \max_{0 \leq z \leq s} |c_W(\tau+z)F(\tau, z) - \mu_5 \tau^{-1}| \lesssim e^{s/2} s^2 \tau^{-2} + s\tau^{-3/2+\epsilon_3}. \end{aligned}$$

Therefore, integrating $\frac{H^{p-1}(\tau+s)}{H^{p-1}(\tau)}$ for s from 0 to ∞ and using the above estimates, we get

$$\begin{aligned} \frac{T - t(\tau)}{H^{p-1}(\tau)} &= \int_0^\infty \frac{H^{p-1}(\tau+s)}{H^{p-1}(\tau)} ds = \int_0^\infty e^{-s} F(\tau, s) ds = \int_0^\infty (1 + \mu_5 s \tau^{-1}) e^{-s} ds + O_{in}(\tau^{-3/2+\epsilon_1}) \\ &= 1 + \mu_5 \tau^{-1} + O_{in}(\tau^{-3/2+\epsilon_1}), \end{aligned} \quad (4.7)$$

where we use $\int_0^\infty s e^{-s} ds = 1$. Since $(p-1)c_U = (p-1)(\bar{c}_U + c_W) = -1 + O(\tau^{-1})$, we further obtain

$$\log(T - t(\tau)) = (1 + O(\tau)^{-1}) \log(H^{p-1}) = (1 + O(\tau)^{-1})(O_{in}(1) + \int_0^\tau (p-1)c_U) = -\tau + O_{in}(\log(\tau)). \quad (4.8)$$

Combining (4.5), (4.7), (4.8), we prove (4.1) and Proposition 4.

4.2 Asymptotics of phase and L^∞ convergence

In this section, our goal is to prove the following convergence.

Proposition 5. *Suppose that the initial data (U, Θ) satisfy the assumption in Theorem 3. We have*

$$|U(z, \tau)e^{\iota(\Theta - A(\tau))} - \bar{U}^{1+\iota\delta}| \lesssim \min(\langle z \rangle^{\sigma+\epsilon/2} \langle \tau \rangle^{-\epsilon}, \langle \tau \rangle^{\max(-1, 2\sigma)}). \quad (4.9)$$

where $A(\tau)$ satisfies the following estimate for some constant C_{in} depending on the initial data

$$|A - \bar{A}| \leq C_{in}, \quad A(\tau) := \frac{\delta}{p-1}\tau + \Phi(\tau, 0), \quad \bar{A}(\tau) := -\frac{\delta \log(T - t(\tau))}{p-1} - \frac{d\beta(1 + \delta^2) \log |\log(T - t(\tau))|}{2b_*}. \quad (4.10)$$

Proof. We will first estimate $\Phi(0)$ and then prove convergence.

Estimate of $\Phi(0)$. Firstly, we compute A_τ, \bar{A}_τ . Using (3.2), we perform similar computations to the proof of Lemma 1 to compute that

$$\Phi_\tau(0) = -v \cdot \nabla \Phi(0) + \mathcal{D}_\Theta(0) = O(\mathcal{E}_0) + (\beta + \delta) \frac{\kappa_2}{\kappa_0} \text{tr}(Q).$$

Applying (4.4) and the estimates (3.6), (3.57) for \mathcal{E}_0 , we yield

$$A_\tau(\tau) = \frac{\delta}{p-1} - \frac{d(\beta + \delta)}{2b_*\tau} + O_{in}(\tau^{-3/2+\epsilon_1}), \quad \bar{A}_\tau(\tau) = \frac{\delta}{p-1} \frac{t_\tau}{T-t} - \frac{d\beta(1 + \delta^2)t_\tau}{2b_* |\log(T-t)|(T-t)}.$$

Using $t_\tau = H^{p-1}$ (4.6), (4.7), and (4.8), we yield

$$\frac{t_\tau}{T-t(\tau)} = \frac{H^{p-1}}{T-t(\tau)} = 1 - \frac{\mu_5}{\tau} + O_{in}(\tau^{-3/2+\epsilon_1}), \quad \frac{t_\tau}{(T-t)\log(T-t)} = \frac{1}{\tau} + O_{in}(\log(\tau)\tau^{-2}).$$

Using the definition of μ_5 (4.6), c_p (1.27), we conclude

$$\begin{aligned} A_\tau - \bar{A}_\tau &= \left(-\frac{d(\beta + \delta)}{2b_*} + \frac{\mu_5\delta}{p-1} + \frac{d\beta(1 + \delta^2)}{2b_*}\right) \frac{1}{\tau} + O_{in}(\tau^{-\frac{3}{2}+\epsilon_3}) \\ &= \frac{\delta}{\tau} \left(\frac{2(1 - \beta\delta)dc_p}{(p-1)^2} - \frac{d(1 - \beta\delta)}{2b_*}\right) + O_{in}(\tau^{-\frac{3}{2}+\epsilon_3}). \end{aligned}$$

The first term vanishes due to (1.27) for c_p . Since the error term is integrable in $\tau \geq 2$, we conclude the asymptotics of the phase (4.10).

L^∞ convergence. Recall $\Theta = \bar{\Theta} + \Phi$ (2.4). Integrating (3.13b) with $l = 1$, we obtain

$$|\Phi(z) - \Phi(0)| \lesssim \int_0^1 |\nabla \Phi(tz)| dt \lesssim E \int_0^1 \langle tz \rangle^{-1/2} dt \lesssim E \langle z \rangle^{1/2}. \quad (4.11)$$

Using $U = \bar{U} + W$ (2.4), we decompose

$$J := Ue^{\iota(\Theta - \bar{\Theta} - \Phi(0))} - \bar{U} = Ue^{\iota(\Phi - \Phi(0))} - \bar{U} = We^{\iota(\Phi - \Phi(0))} + \bar{U}(e^{\iota(\Phi - \Phi(0))} - 1) = I + II.$$

Applying (3.13a) to I , (4.11) and $|e^{\iota x} - 1| \lesssim \min(|x|, 1)$ to II , and $E \lesssim \langle \tau \rangle^{-1}$ (3.6), we prove

$$\begin{aligned} |J| &\lesssim \langle z \rangle^{\sigma+\epsilon/2} E + \langle z \rangle^\sigma \min(E \langle z \rangle^{1/2}, 1) \lesssim \min(\langle z \rangle^{\sigma+\epsilon/2} (E + E^\epsilon), E + E^{-2\sigma}) \\ &\lesssim \min(\langle z \rangle^{\sigma+\epsilon/2} \langle \tau \rangle^{-\epsilon}, \langle \tau \rangle^{\max(-1, 2\sigma)}). \end{aligned}$$

Since $\Theta - \bar{\Theta} - \Phi(0) + \delta \log \bar{U} = \Theta - A(\tau)$ (1.27) and (4.10), we get $U^{\iota\delta} J = Ue^{\iota(\Theta - A(\tau))} - \bar{U}^{1+\iota\delta}$. Since $|U^{\iota\delta}| = 1$, the above estimate conclude the proof of (4.9). \square

4.3 Proof of Theorem 1

In this section, we prove Theorem 1 with the open set \mathcal{O} prescribed in Remark 1.

Verification of assumptions in Theorem 3. We first choose $\nu < 1$ in (1.15) for Theorem 1. Following the proof of Corollary 2, we obtain $U_0 \lesssim \langle z \rangle^{\sigma+\epsilon/2}$. Using the definitions (3.4) and (1.16) of F_K and $\mathfrak{F}_{\bar{R}}$, and (1.28), we get

$$\dot{\rho}_K = \rho_K U^2 \leq C \langle z \rangle^{2\sigma+\epsilon} \rho_K \geq C(1 + |z|^{-d-2K}), \quad F_K \leq C \|\Phi\|_{\mathfrak{F}_{\bar{R}}}.$$

From (1.6), (3.50), we obtain $E \leq CE_{in}$. By choosing $\nu = cE_*$ in (1.15) with $c > 0$ sufficiently small, the assumptions (1.15) implies the assumptions (3.5) in Theorem 3 except for $E_{\mathcal{Q}} < E_*$. Using the definitions of $\mathcal{M}_0, H(0)$ (1.13) $\mathcal{R}_0, \mathcal{Q}$ (2.5), and $C_W(0) = H(0)$ (2.4), we obtain

$$\text{tr}(\mathcal{Q}(0)) = H(0)^{p-1} \text{tr}(\mathcal{M}\mathcal{M}^T) = H(0)^{p-1} \text{tr}(\mathcal{M}^T \mathcal{M}) = C u_0(V_0)^{-p} \text{tr}(\nabla^2 u_0(V_0))$$

for some absolute constant C . Therefore, by further choosing c small in $\mu = cE_*$, we obtain $|E_{\mathcal{Q}}| < E_*$ from the last assumption in (1.15). We verify the assumptions in Theorem 3 and can use the results in Theorem 3, and Propositions 4 and 5.

For the time t in Theorem 1, we use the change of variables $t = t(\tau)$ (2.2). Then we only need to prove Theorem 1 in terms of the self-similar time τ .

Proof of estimates (1.9), (1.12). Using Theorem 3, we obtain the estimates (3.6), which along with the relation between (U, Θ) , (u, θ) and $W = U - \bar{U}$, $\Phi = \Theta - \bar{\Theta}$ prove (1.9) in Theorem 1.

To obtain (1.12) and (1.11), we choose $\mu(t(\tau)) = A(\tau)$, $\hat{\mu}(t) = A(\tau) - \bar{A}(\tau)$. Using $\Theta - A = \Theta - \bar{A} - \hat{\mu}$ and the formula of \bar{A} (4.10), we obtain

$$\begin{aligned} J &:= |\log(T-t)|^{i \frac{d\beta(1+\delta^2)}{2b_*}} (T-t)^{\frac{1+i\delta}{p-1}} \psi(\mathbf{R}(t)z + \mathcal{V}(t), t) e^{-i\hat{\mu}(t)} = (T-t)^{\frac{1}{p-1}} H^{-1} U(z, \tau) e^{i(\Theta(z, \tau) - \bar{A} - \hat{\mu})} \\ &= (T-t)^{\frac{1}{p-1}} H^{-1} U(z, \tau) e^{i(\Theta(z, \tau) - A(\tau))}. \end{aligned}$$

We denote

$$J_2 = U(z, \tau) e^{i(\Theta(z, \tau) - A(\tau))}.$$

Using the limits $H(\tau)/(T-t)^{1/(p-1)} \rightarrow 1$ and $\tau/|\log(T-t(\tau))| \rightarrow 1$ as $\tau \rightarrow \infty$ (4.1), and the estimate (4.9), we prove

$$\begin{aligned} |J - \bar{U}^{1+i\delta}| &\lesssim |(T-t)^{\frac{1}{p-1}} H^{-1} - 1| \cdot |\bar{U}^{1+i\delta}| + (T-t)^{\frac{1}{p-1}} H^{-1} |J_2 - \bar{U}^{1+i\delta}| \\ &\lesssim C_{in}(1+\tau)^\eta \lesssim C_{in}(1+|\log(T-t(\tau))|)^\eta, \end{aligned}$$

where $\eta = \max(-1, 2\sigma)$.

Proof of rates (1.10), (1.11). Next, we show that $V(\tau)$ converges as $\tau \rightarrow \infty$ in (1.10). From Lemma 1 for \mathcal{V} , the decay estimates for \mathcal{Q}, E in Theorem 3, and $\mathbf{R}(\tau) \rightarrow 0$ as $\tau \rightarrow \infty$ in Proposition 4, we obtain

$$|\mathbf{R}(\tau)\mathcal{V}(\tau)| \lesssim (1+\tau)^{-2}.$$

Since the upper bound is integrable in τ , using $|\dot{V}(\tau)| = |\mathbf{R}(\tau)\mathcal{V}(\tau)|$ (2.5), we prove that $V(\tau)$ converges as $\tau \rightarrow \infty$. The asymptotics (1.11) follows from the definition of μ and (4.1). This ends the proof of Theorem 1.

4.4 Proof of Theorem 2

To prove Theorem 2, we only need to show that for $\epsilon_0 = \epsilon_0(u_0)$, assumption (1.17) implies that \tilde{u}_0 is in the open set \mathcal{O} in Theorem 1.

Estimates of $\tilde{V}_0, \tilde{\mathcal{M}}_0, \tilde{H}_0$. Since V_0 is the unique maximizer and $\nabla^2 u_0(V_0) \succ 0$, for δ_1 sufficiently small, we obtain that \tilde{u}_0 admits a global non-degenerate maximizer \tilde{V}_0 close to V_0 ⁷ with $|V_0 - \tilde{V}_0| \rightarrow 0$ as $\|\tilde{u}_0 - u_0\|_{L^\infty} \rightarrow 0$. Using embedding (3.13a) in Proposition 1, we obtain

$$\|u_0 - \tilde{u}_0\|_{C^2} \lesssim \|u_0 - \tilde{u}_0\|_{\mathcal{H}^\kappa}.$$

Denote

$$\delta_1 := \|u_0 - \tilde{u}_0\|_{\mathcal{H}^\kappa}, \quad \delta_2 := |V_0 - \tilde{V}_0|.$$

Using continuity and the above embedding, we obtain

$$\lim_{\epsilon_0 \rightarrow 0} |\tilde{V}_0 - V_0| = \lim_{\epsilon_0 \rightarrow 0} \delta_2 = 0, \quad |\tilde{u}_0(\tilde{V}_0) - u_0(V_0)| \lesssim \delta_1 + \delta_2, \quad |\nabla^2 \tilde{u}_0(\tilde{V}_0) - \nabla^2 u_0(V_0)| \lesssim \delta_1 + \delta_2. \quad (4.12a)$$

Upon choosing $\epsilon_0 > 0$ small, we can define the initial modulation parameters (1.13) $\tilde{H}_0, \tilde{\mathcal{M}}_0$ associated with \tilde{u}_0 and obtain

$$\lim_{\epsilon_0 \rightarrow 0} |\tilde{\mathcal{M}}_0 - \mathcal{M}_0| + |H_0 - \tilde{H}_0| = 0. \quad (4.12b)$$

We denote by \tilde{U}_0 the rescaled variables for \tilde{u}_0 :

$$\tilde{U}_0 = \tilde{H}_0 \tilde{u}_0(\tilde{\mathcal{M}}_0^{-1} z + \tilde{V}_0). \quad (4.13)$$

Verification of assumptions. We show that $(\tilde{u}_0, \tilde{U}_0)$ satisfies assumptions (1.15). The implicit constants can depend on u_0 .

Firstly, assumptions (1.8), (1.15) for \tilde{u}_0 except for

$$\tilde{U}_0 \bar{U}^{-1-\epsilon_2} > 2C_b \quad (4.14)$$

follow from continuity and choosing δ_1 small. Condition (4.14) follows from the assumption (4.14) for U_0 , (1.17), the triangle inequality, and choosing ϵ_0 small.

Next, we verify (1.16) for \tilde{U}_0 , i.e.

$$\|\tilde{U}_0 - \bar{U}_0\|_{\mathcal{E}_\kappa} < \nu. \quad (4.15)$$

Using the definition (4.13), we decompose $\tilde{U}_0 - \bar{U}$ as follows

$$\tilde{U}_0 - \bar{U} = \tilde{H}_0 \left(\tilde{u}_0(\tilde{\mathcal{M}}_0^{-1} z + \tilde{V}_0) - u_0(\tilde{\mathcal{M}}_0^{-1} z + \tilde{V}_0) \right) + \left(\tilde{H}_0 u_0(\tilde{\mathcal{M}}_0^{-1} z + \tilde{V}_0) - \bar{U}_0 \right) := J_1 + J_2. \quad (4.16)$$

For J_1 , using a change of variable, (4.12a), the assumption (1.17) for $u_0 - \tilde{u}_0$, and the embedding (3.13a), we obtain

$$\|J_1\|_{\mathcal{H}^\kappa} \lesssim \|u_0 - \tilde{u}_0\|_{\mathcal{H}^\kappa} \lesssim \epsilon_0, \quad \lim_{\epsilon_0 \rightarrow 0} \max_{|z| \leq 1} |\nabla^i J_1| = 0, \quad \text{for } i = 0, 1, 2, 3. \quad (4.17)$$

Denote $H_1 = \tilde{H}_0 H_0^{-1}, \mathcal{M}_1 = \mathcal{M}_0 \mathcal{M}_0^{-1}, V_1 = \mathcal{M}_0(\tilde{V}_0 - V_0)$. For J_2 , using $u_0(z) = H_0^{-1} U_0(\mathcal{M}_0(z - V_0))$ (1.13) and a change of variable, we obtain

$$\begin{aligned} J_2 &= \tilde{H}_0 H_0^{-1} U_0(\mathcal{M}_0 \mathcal{M}_0^{-1} z + \mathcal{M}_0(\tilde{V}_0 - V_0)) - \bar{U}_0 = H_1 U_0(\mathcal{M}_1 z + V_1) - \bar{U} \\ &= H_1(U_0 - \bar{U})(\mathcal{M}_1 z + V_1) + \left(H_1 \bar{U}(\mathcal{M}_1 z + V_1) - \bar{U} \right) := J_{21} + J_{22}. \end{aligned}$$

From (4.12b), we obtain that $\mathcal{M}_1 - 1 = o(1), H_1 - 1 = o(1), V_1 = o(1)$. Since $\|U_0 - \bar{U}_0\|_{\mathcal{E}_\kappa} < \nu$, by choosing ϵ_0 small enough, we yield

$$\|J_{21}\|_{\mathcal{E}_\kappa} < \nu. \quad (4.18)$$

Using the smoothness of \bar{U} and the embedding (3.13a), we obtain

$$\lim_{\epsilon_0 \rightarrow 0} \|J_{22}\|_{\mathcal{H}^\kappa} + \max_{|z| \leq 1, i \leq 3} |\nabla^i J_{22}| = 0. \quad (4.19)$$

⁷Since for any $\delta > 0$, there exists a $r > 0$, such that $u_0(V_0) > u_0(V_0 + z) + \delta, |z| > r$, we obtain $|\tilde{V}_0 - V_0| < r$ when $\delta_1 < \delta/2$.

Next, we show that for f with $\nabla^j f(0) = 0, j \leq 2$ and $i \leq K$, we have

$$\|\nabla^i f\|_{\rho_i} \lesssim \|f\|_{\mathcal{H}^K}. \quad (4.20)$$

Using the definition of (1.28) ρ_K, \mathcal{E}_K (3.4), g_k, \mathcal{H}^K (3.9), and embedding (3.13a), for f with $\nabla^j f(0) = 0, j \leq 2$, we have

$$\begin{aligned} \|\nabla^i f\|_{\rho_i} &\lesssim \|\nabla^3 f\|_{L^\infty} \| |x|^{3-i} \mathbf{1}_{|x| \leq 1} \|_{\rho_i} + \|\nabla^i f\|_{g_i} \lesssim \|f\|_{\mathcal{H}^K}, \quad i \leq 3, \\ \|\nabla^i f\|_{\rho_i} &\lesssim \|\nabla^i f\|_{L^\infty} \|\mathbf{1}_{|x| \leq 1}\|_{\rho_i} + \|\nabla^i f\|_{g_i} \lesssim \|f\|_{\mathcal{H}^K}, \quad i \leq (d+5)/2. \end{aligned}$$

For $\frac{d+5}{2} < i \leq K$, ρ_i and g_i are equivalent and (4.20) follows from (3.12c). From the definition of \tilde{U}_0 and J_i , for $l \leq 2, i = 1, 2$, we obtain $\nabla^l J_{21}(0) = 0, \nabla^l (J_1 + J_{22})(0) = 0$. Applying (4.20), we obtain

$$\|J_1 + J_{22}\|_{\mathcal{E}_K} \lesssim \|J_1 + J_{22}\|_{\mathcal{H}^K},$$

which goes to 0 as $\epsilon_0 \rightarrow 0$. Since the inequality (4.18) is strict, for ϵ_0 small enough, we prove (4.15). Condition (1.16) follows from a similar argument, we conclude the proof of Theorem 2.

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