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On singularity formation of a 3D model for incompressible Navier–Stokes equations

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Abstract

We investigate the singularity formation of a 3D model that was recently proposed by Hou and Lei (2009) in [15] for axisymmetric 3D incompressible Navier–Stokes equations with swirl. The main difference between the 3D model of Hou and Lei and the reformulated 3D Navier–Stokes equations is that the convection term is neglected in the 3D model. This model shares many properties of the 3D incompressible Navier–Stokes equations. One of the main results of this paper is that we prove rigorously the finite time singularity formation of the 3D inviscid model for a class of initial boundary value problems with smooth initial data of finite energy. We also prove the global regularity of the 3D inviscid model for a class of small smooth initial data.

Keywords: Finite time singularities; Nonlinear nonlocal system; Incompressible Navier–Stokes equations

1. Introduction

The question of whether a solution of the 3D incompressible Navier–Stokes equations can develop a finite time singularity from smooth initial data with finite energy is one of the most outstanding mathematical open problems [12,24,28]. Most regularity analysis for the 3D

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Navier–Stokes equations relies on energy estimates. Due to the incompressibility condition, the convection term does not contribute to the energy norm of the velocity field or any L^p $(1 < p \leqslant \infty)$ norm of the vorticity field. As a result, the main effort has been to use the diffusion term to control the nonlinear vortex stretching term without making use of the convection term explicitly.

In a recent paper by Hou and Lei [15], the authors investigated the effect of convection by constructing a new 3D model for axisymmetric 3D incompressible Navier–Stokes equations with swirl. Specifically, their 3D model is given below:

$$\partial_t u_1 = \nu\left(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2\right)u_1 + 2\partial_z\psi_1 u_1, \tag{1}$$

$$\partial_t \omega_1 = \nu\left(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2\right)\omega_1 + \partial_z\big((u_1)^2\big), \tag{2}$$

$$-\left(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2\right)\psi_1 = \omega_1. \tag{3}$$

Note that (1)–(3) is already a closed system. The only difference between this 3D model and the reformulated Navier–Stokes equations is that the convection term is neglected in the model. If one adds the convection term back to the left-hand side of (1) and (2), one would recover the full Navier–Stokes equations. This model preserves almost all the properties of the full 3D Navier–Stokes equations, including the energy identity for smooth solutions of the 3D model and the divergence free property of the reconstructed 3D velocity field given by $u^\theta = r u_1$, $u^r = -\partial_z(r\psi_1)$, $u^z = \frac{1}{r}\partial_r(r^2\psi_1)$. Moreover, they proved the corresponding non-blow-up criterion of Beale–Kato–Majda type [1] as well as a non-blow-up criterion of Prodi–Serrin type [26,27] for the model. In a subsequent paper, they proved a new partial regularity result for the model [16] which is an analogue of the Caffarelli–Kohn–Nirenberg theory [2] for the full Navier–Stokes equations.

Despite the striking similarity at the theoretical level between the 3D model and the Navier–Stokes equations, the former seems to have a very different behavior from the full Navier–Stokes equations. In [15], the authors presented numerical evidence which supports that the 3D model may develop a potential finite time singularity. They further studied the mechanism that leads to these singular events in the 3D model. On the other hand, the Navier–Stokes equations with the same initial data seems to have a completely different behavior.

One of the main results of this paper is that we prove rigorously the finite time singularity formation of this 3D model for a class of initial boundary value problems with smooth initial data of finite energy. In our analysis, we focus on the inviscid version of the 3D model and consider the initial boundary value problem of the generalized 3D model which has the following form [15] (we drop the subscript 1 and substitute (3) into (2)):

$$u_t = 2u\psi_z, \tag{4}$$

$$-\Delta\psi_t = \big(u^2\big)_z, \tag{5}$$

where Δ is an n-dimensional Laplace operator with $(\mathbf{x}, z) \equiv (x_1, x_2, \ldots, x_{n-1}, z)$. Our results in this paper apply to any dimension greater than or equal to two $(n \geqslant 2)$. To simplify our presentation, we only present our analysis for $n = 3$. We consider the generalized 3D model in both a bounded domain and in a semi-infinite domain with a mixed Dirichlet Robin boundary condition. The main result of this paper is the following theorem.

Theorem 1.1. *Let $\Omega_{\mathbf{x}} = (0,a) \times (0,a)$, $\Omega = \Omega_{\mathbf{x}} \times (0,b)$ and $\Gamma = \{(\mathbf{x}, z) \mid \mathbf{x} \in \Omega_{\mathbf{x}},\ z = 0\}$. Assume that the initial condition $u_0 > 0$ for $(\mathbf{x}, z) \in \Omega$, $u_0|_{\partial\Omega} = 0$, $u_0 \in H^2(\Omega)$, $\psi_0 \in H^3(\Omega)$ and satisfies* (6). *Moreover, we assume that ψ satisfies the following mixed Dirichlet Robin boundary condition:*

$$\psi|_{\partial\Omega\setminus\Gamma} = 0, \qquad (\psi_z + \beta\psi)|_\Gamma = 0, \tag{6}$$

with $\beta > \frac{\sqrt{2}\pi}{a}(\frac{1+e^{-2\pi b/a}}{1-e^{-2\pi b/a}})$. Define $\phi(x_1, x_2, z) = (\frac{e^{-\alpha(z-b)}+e^{\alpha(z-b)}}{2}) \sin(\frac{\pi x_1}{a}) \sin(\frac{\pi x_2}{a})$ where α satisfies $0 < \alpha < \sqrt{2}\pi/a$ and $2(\frac{\pi}{a})^2 \frac{e^{\alpha b}-e^{-\alpha b}}{\alpha(e^{\alpha b}+e^{-\alpha b})} = \beta$. If u_0 and ψ_0 satisfy the following condition:

$$\int_\Omega (\log u_0)\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z > 0, \qquad \int_\Omega \psi_{0z}\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z > 0, \tag{7}$$

then the solution of the 3D *inviscid model* (4)–(5) *will develop a finite time singularity in the H^2 norm.*

The analysis of the finite time singularity for the 3D model is rather subtle. The main technical difficulty is that this is a multi-dimensional nonlinear nonlocal system. Currently, there is no systematic method of analysis to study singularity formation of a nonlinear nonlocal system. The key issue is under what condition the solution u has a strong alignment with the solution ψ_z dynamically. If u and ψ_z have a strong alignment for long enough time, then the right-hand side of the u-equation would develop a quadratic nonlinearity dynamically, which would lead to a finite time blow-up. Note that ψ is coupled to u in a nonlinear and nonlocal fashion. It is not clear whether u and ψ_z will develop such a nonlinear alignment dynamically. As a matter of fact, not all initial boundary conditions of the 3D model would lead to finite time blow-up. One of the interesting results we obtain in this paper is that we prove the global regularity of the 3D inviscid model for a class of small initial data with an appropriate boundary condition. We would like to point out that since there is no viscosity in the 3D inviscid model, such global regularity result is still interesting even though some smallness condition is imposed on the initial data. We note that there is currently no corresponding global regularity result for the incompressible 3D Euler equation even with small initial data.

One of the main contributions of this paper is that we introduce an effective method of analysis to study singularity formation of this nonlinear nonlocal multi-dimensional system. There are several important steps in our analysis. The first one is that we reformulate the u-equation so that the right-hand side of the reformulated u-equation becomes linear. This is accomplished by dividing both sides of (4) by u and introducing $\log(u)$ as a new variable. This is possible since $u_0 > 0$ in Ω implies that $u > 0$ in Ω as long as the solution remains smooth. The reformulated system now has the form:

$$\big(\log(u)\big)_t = 2\psi_z, \quad (\mathbf{x}, z) \in \Omega, \tag{8}$$

$$-\Delta\psi_t = \big(u^2\big)_z. \tag{9}$$

This idea is similar in spirit to the renormalized Boltzmann equation introduced by DiPerna and Lions in their study of the global renormalized weak solution of the Boltzmann equations [10].

The second step is to work with the weak formulation of the reformulated model (8)–(9) by introducing an appropriately chosen weight function ϕ as our test function. How to choose this weight function ϕ is crucial in obtaining the nonlinear estimate that is required to prove finite time blow-up of the nonlocal system. Guided by our analysis, we look for a smooth and positive eigen-function in Ω that satisfies the following two conditions simultaneously:

$$-\Delta\phi = \lambda_1\phi, \qquad \partial_z^2\phi = \lambda_2\phi, \quad \text{for some } \lambda_1, \lambda_2 > 0,\ (\mathbf{x}, z) \in \Omega. \tag{10}$$

The function ϕ defined in Theorem 1.1 satisfies both of these conditions. We remark that such eigen-function exists only for space dimension greater than or equal to two. In the third step, we multiply ϕ to (8) and ϕ_z to (9), integrate over Ω, and perform integration by parts. We obtain by using (10) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\log u)\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z = 2\int_\Omega \psi_z\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z, \tag{11}$$

$$\lambda_1\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \psi_z\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z = \lambda_2\int_\Omega u^2\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z. \tag{12}$$

All the boundary terms resulting from integration by parts vanish by using the boundary condition of ψ, the fact that $u|_{z=0} = u|_{z=b} = 0$, the property of our eigen-function ϕ, and the specific choice of α defined in Theorem 1.1. Substituting (12) into (11) gives the crucial estimate for our blow-up analysis:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_\Omega (\log u)\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z = \frac{2\lambda_2}{\lambda_1}\int_\Omega u^2\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z. \tag{13}$$

Further, we note that

$$\begin{aligned}\int_\Omega \log(u)\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z &\leqslant \int_\Omega \big(\log(u)\big)^+\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z \leqslant \int_\Omega u\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z \\ &\leqslant \bigg(\int_\Omega \phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z\bigg)^{1/2}\bigg(\int_\Omega \phi u^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}z\bigg)^{1/2} \equiv \frac{2a}{\pi\sqrt{\alpha}}\bigg(\int_\Omega \phi u^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}z\bigg)^{1/2}.\end{aligned} \tag{14}$$

Integrating (13) twice in time and using (14), we establish a sharp nonlinear dynamic estimate for $(\int_\Omega \phi u^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}z)^{1/2}$, which enables us to prove finite time blow-up of the 3D model.

Another interesting result is that we prove the finite time blow-up of the 3D model with partial viscosity. Under similar assumptions on u_0, ψ_0 and ω_0 as in the inviscid case and by assuming that ω satisfies a boundary condition similar to ψ, we can prove that the 3D model with partial viscosity

$$u_t = 2u\psi_z, \tag{15}$$

$$\omega_t = \left(u^2\right)_z + \nu\Delta\omega, \tag{16}$$

$$-\Delta\psi = \omega, \tag{17}$$

develops a finite time singularity.

We also study singularity formation of the 3D model with $\beta = 0$ in (6). This case is interesting because the smooth solution of the corresponding 3D model satisfies an energy identity. In this case, we can establish a finite time blow-up under an additional condition:

$$\int_0^a\int_0^a (\psi - \psi_0)|_\Gamma \sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\mathrm{d}\mathbf{x} < c_0 \int_\Omega \psi_{0z}\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z,$$

as long as the solution remains regular, where $c_0 > 0$ depends only on the size of the domain.

We remark that although the 3D model using the mixed Dirichlet Robin boundary condition with $\beta \neq 0$ does not conserve energy exactly, we prove that the energy remains bounded as long as the solution is smooth and $\beta < c_0$ for some $c_0 > 0$. We also establish the local well-posedness of the initial boundary problem with the mixed Dirichlet Robin boundary condition. Our numerical study shows that the energy is still bounded up to the blow-up time even if $\beta > c_0$. Our numerical study also suggests that the nature of the singularity in the case of $\beta > c_0$ is qualitatively similar to that in the case of $\beta < c_0$.

Study of singularity formation for various model equations for the 3D Euler/Navier–Stokes equations or the surface quasi-geostrophic equation has been investigated by a number of people, including Constantin, Lax and Majda [6], Constantin [5], De Gregorio [8,9], Kerr [21], Caflisch and Siegel [3], Cordoba, Cordoba and Fontelos [7], Chae, Cordoba, Cordoba and Fontelos [4], Matsumotoa, Becb and Frisch [25], Hou and Li [17], Li and Sinai [23], Li and Rodrigo [22], and Hou, Li, Shi, Wang and Yu [19]. The effect of convection has also been studied by Hou and Li in a recent paper [18] via a new 1D model. They proved dynamic stability of this 1D model by exploiting the nonlinear cancellation between the convection and the vortex stretching term, and constructing a Lyapunov function which gives rise to a global pointwise estimate for the derivatives of the vorticity in their model.

We would like to point out that the study of [18,15] is based on a reduced model for some special flow geometry. One should not conclude that convection term could lead to depletion of singularity of the Navier–Stokes equations in general. It is possible that convection term may act as a destabilizing term for a different flow geometry. One of the main findings of [18,15] and the present paper is that convection term carries important physical information that should not be neglected in our analysis of the Navier–Stokes equations. Since the behavior of the 3D model is very different from that of the Navier–Stokes equations, it is important to develop a method of analysis that could take into account the physical significance of convection term in an essential way.

The rest of the paper is organized as follows. In Section 2, we study the local well-posedness of the 3D inviscid model and some properties of the model. In Section 3, we prove the finite time blow-up of the 3D inviscid model with mixed Dirichlet and Robin boundary conditions. In Section 4, we prove finite time blow-up of the 3D model with partial viscosity. Section 5 is devoted to analyzing the finite time blow-up of the 3D inviscid model with some conservative boundary conditions. In Section 6, we prove the global regularity of the 3D inviscid model for a class of small initial data with some appropriate boundary condition. A technical lemma is proved in Appendix A.

2. Properties of the 3D model

2.1. Local well-posedness in H^s

In this section, we will establish the local well-posedness of the initial boundary problem of the 3D model with the mixed Dirichlet Robin boundary condition. We will present our analysis for the semi-infinite domain using the Sobolev space H^s. The same result is also true in a bounded domain.

Consider the 3D model with the following mixed initial boundary condition:

$$\begin{cases} u_t = 2u\psi_z, \\ -\Delta\psi_t = (u^2)_z, \end{cases} \quad (\mathbf{x}, z) \in \Omega = \Omega_{\mathbf{x}} \times (0, \infty), \tag{18}$$

$$\psi|_{\partial\Omega\setminus\Gamma} = 0, \qquad (\psi_z + \beta\psi)|_\Gamma = 0, \tag{19}$$

$$\psi|_{t=0} = \psi_0(\mathbf{x}, z), \qquad u|_{t=0} = u_0(\mathbf{x}, z) \geqslant 0, \tag{20}$$

where $\mathbf{x} = (x_1, x_2)$, $\Omega_{\mathbf{x}} = (0, a) \times (0, a)$, $\Gamma = \{(\mathbf{x}, z) \mid \mathbf{x} \in \Omega_{\mathbf{x}},\ z = 0\}$ and $\Delta = \Delta_{\mathbf{x}} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial z^2}$.

The local well-posedness analysis depends on an important property of the elliptic operator with the mixed Dirichlet Robin boundary condition.

Lemma 2.1. *There exists a unique solution $v \in H^s(\Omega)$ to the boundary value problem:*

$$-\Delta v = f, \quad (\mathbf{x}, z) \in \Omega, \tag{21}$$

$$v|_{\partial\Omega\setminus\Gamma} = 0, \qquad (v_z + \beta v)|_\Gamma = 0, \tag{22}$$

if $\beta \in S_\infty \equiv \{\beta \mid \beta \neq \frac{\pi|k|}{a}$ for all $k \in \mathbb{Z}^2\}$, $f \in H^{s-2}(\Omega)$ with $s \geqslant 2$ and $f|_{\partial\Omega\setminus\Gamma} = 0$. Moreover we have

$$\|v\|_{H^s(\Omega)} \leqslant C_s \|f\|_{H^{s-2}(\Omega)}, \tag{23}$$

where C_s is a constant depending on s, $|k| = \sqrt{k_1^2 + k_2^2}$.

We defer the proof of Lemma 2.1 to Appendix A.

Remark 2.1. We remark that we can prove the same result as in Lemma 2.1 for a bounded domain $\Omega = \Omega_{\mathbf{x}} \times (0, b)$ with the same boundary condition by assuming that $\beta \in S_b$ where

$$S_b = \left\{\beta \;\middle|\; \beta \neq \frac{\pi|k|}{a} \text{ and } \beta \neq \frac{\pi|k|}{a}\left(\frac{1 + e^{-2|k|\pi b/a}}{1 - e^{-2|k|\pi b/a}}\right) \text{ for all } k \in \mathbb{Z}^2\right\}. \tag{24}$$

Remark 2.2. We would like to point out that regularity estimates for the second order elliptic problem with mixed Dirichlet and Robin boundary conditions have been studied by Temam and Ziane [29] in the context of geophysical flows. However, there is an important difference between the case investigated by Temam and Ziane and the case considered by us here. Although the problem is formulated slightly differently, the case considered by Temam and Ziane corresponds to the case of $\beta < 0$ on Γ, which gives rise to a dissipative boundary condition. The case of $\beta > 0$ is the main focus of our present study. This case is more difficult because the boundary contribution from the Robin boundary condition produces the wrong sign when we perform energy estimates. In our analysis, we need to study the spectral property of the differential operator and exclude an infinite number of discrete eigenvalues from β in order to obtain well-posedness of the elliptic problem with this mixed Dirichlet Robin boundary condition.

Definition 2.1. Let $\mathcal{K} : H^{s-2}(\Omega) \to H^s(\Omega)$ be a linear operator defined as following:

$$\text{for all } f \in H^{s-2}(\Omega), \quad \mathcal{K}(f) \text{ is the solution of the boundary value problem (21)–(22).}$$

It follows from Lemma 2.1 that for any $f \in H^{s-2}(\Omega)$, we have

$$\big\|\mathcal{K}(f)\big\|_{H^s(\Omega)} \leqslant C_s \|f\|_{H^{s-2}(\Omega)}. \tag{25}$$

We also need the following well-known Sobolev inequality [14].

Lemma 2.2. *Let $u, v \in H^s(\Omega)$ with $s > 3/2$. We have*

$$\|uv\|_{H^s(\Omega)} \leqslant c\|u\|_{H^s(\Omega)}\|v\|_{H^s(\Omega)}. \tag{26}$$

Now we can state the local well-posedness result for the 3D model with the mixed Dirichlet Robin boundary condition.

Theorem 2.1. *Assume that $u_0 \in H^s(\Omega)$, $\psi_0 \in H^{s+1}(\Omega)$ for some $s > 3/2$, $u_0|_{\partial\Omega} = 0$ and ψ_0 satisfies* (19). *Moreover, we assume that $\beta \in S_\infty$ (or S_b) as defined in Lemma* 2.1. *Then there exists a finite time $T = T(\|u_0\|_{H^s(\Omega)}, \|\psi_0\|_{H^{s+1}(\Omega)}) > 0$ such that the system* (18)–(20) *has a unique solution, $u \in C^1([0,T), H^s(\Omega))$ and $\psi \in C^1([0,T), H^{s+1}(\Omega))$.*

Proof. Let $v = u^2$, then we obtain an equivalent system for v and ψ as follows:

$$v_t = 4v\psi_z, \tag{27}$$

$$\psi_t = \mathcal{K}(v_z), \tag{28}$$

where $\mathcal{K}$ is defined in Definition 2.1. To prove the local well-posedness of system (27)–(28), we introduce the space

$$V^{s+1} = \big\{\psi \in H^{s+1}(\Omega)\colon \psi|_{\partial\Omega\setminus\Gamma} = 0,\ (\psi_z + \beta\psi)|_\Gamma = 0\big\}.$$

By the Trace Theorem [11], the trace of ψ and ψ_z on $\partial\Omega$ is well defined since we assume that $\psi \in H^{s+1}(\Omega)$ with $s > 3/2$. Then we can write the system (27)–(28) as an ODE in the Banach space $X := H^s(\Omega) \times V^{s+1}(\Omega)$:

$$U_t = F(U), \tag{29}$$

where $U = (U_1, U_2) = (v, \psi)$, $F(U) = (F_1(U), F_2(U)) = (4v\psi_z, \mathcal{K}(v_z))$ and the norm $\|\cdot\|_X$ of the space X is defined as follows:

$$\|U\|_X = \|U_1\|_{H^s(\Omega)} + \|U_2\|_{H^{s+1}(\Omega)}.$$

We will use the well-known Picard theorem on a Banach space (see e.g. Theorem 3.1 in [24]) to prove the local well-posedness of system (29). In order to apply the Picard theorem on a Banach space, we need to check the following two conditions:

1. F maps $O \subset X$ to X, where O is an open subset of X.
2. F is locally Lipschitz continuous, i.e. for any $U \in O$, there exist $L > 0$ and an open neighborhood of U, $B_U \subset O$, such that

$$\big\|F(\bar{U}) - F(\tilde{U})\big\|_X \leqslant L\|\bar{U} - \tilde{U}\|_X, \quad \text{for all } \bar{U}, \tilde{U} \in B_U.$$

First, we choose the open set O to be a bounded set defined as following:

$$O = \big\{U \in X\colon\ \|U\|_X < M\big\}, \tag{30}$$

where $M > 0$ is a constant.

To verify the first condition, we obtain by using estimate (25) and Lemma 2.2 that

$$\begin{aligned}
\big\|F(U)\big\|_X &= \big\|F_1(U)\big\|_{H^s} + \big\|F_2(U)\big\|_{H^{s+1}} \\
&= \|4U_1U_{2z}\|_{H^s} + \big\|\mathcal{K}(U_{1z})\big\|_{H^{s+1}} \\
&\leqslant 4C_s\|U_1\|_{H^s}\|U_{2z}\|_{H^s} + C_s\|U_{1z}\|_{H^{s-1}} \\
&\leqslant 4C_s\|U_1\|_{H^s}\|U_2\|_{H^{s+1}} + C_s\|U_1\|_{H^s} \\
&\leqslant 4C_s\|U\|_X\big(1 + \|U\|_X\big) < 4C_sM(1+M),
\end{aligned} \tag{31}$$

where $U_{iz} \equiv (U_i)_z$ $(i = 1, 2)$.

Next, we show that F is locally Lipschitz continuous. For any $\bar{U}, \tilde{U} \in O$, we have by using (25) and Lemma 2.2 that

$$\begin{aligned}
\big\|F(\bar{U}) - F(\tilde{U})\big\|_X &= \big\|F_1(\bar{U}) - F_1(\tilde{U})\big\|_{H^s} + \big\|F_2(\bar{U}) - F_2(\tilde{U})\big\|_{H^{s+1}} \\
&= 4\|\bar{U}_1\bar{U}_{2z} - \tilde{U}_1\tilde{U}_{2z}\|_{H^s} + \big\|\mathcal{K}\big((\bar{U}_1 - \tilde{U}_1)_z\big)\big\|_{H^{s+1}} \\
&\leqslant 4C_s\|\bar{U}_1\|_{H^s}\big\|(\bar{U}_2 - \tilde{U}_2)_z\big\|_{H^s} \\
&\quad + 4C_s\|\tilde{U}_{2z}\|_{H^s}\|\bar{U}_1 - \tilde{U}_1\|_{H^s} + C_s\big\|(\bar{U}_1 - \tilde{U}_1)_z\big\|_{H^{s-1}}
\end{aligned}$$

$$\begin{aligned}
&\leqslant 4C_s\|\bar{U}_1\|_{H^s}\|\bar{U}_2-\tilde{U}_2\|_{H^{s+1}} \\
&\quad + 4C_s\|\tilde{U}_2\|_{H^{s+1}}\|\bar{U}_1-\tilde{U}_1\|_{H^s} + C_s\|\bar{U}_1-\tilde{U}_1\|_{H^s} \\
&\leqslant (4C_sM + C_s)\big(\|\bar{U}_1-\tilde{U}_1\|_{H^s} + \|\bar{U}_2-\tilde{U}_2\|_{H^{s+1}}\big) \\
&= C_s(4M+1)\|\bar{U}-\tilde{U}\|_X,
\end{aligned} \tag{32}$$

which proves that F is locally Lipschitz continuous.

Now we can apply the Picard theorem on a Banach space to conclude that there exists a time $T(\|u_0\|_{H^s(\Omega)}, \|\psi_0\|_{H^{s+1}(\Omega)}) > 0$ such that the system

$$U_t = F(U), \qquad U|_{t=0} = U_0 \in O,$$

has a unique solution $U = (v, \psi) \in C^1([0,T), H^s(\Omega) \times V^{s+1}(\Omega))$. □

2.2. Bounded energy for the 3D model with mixed boundary conditions

Proposition 2.1. *Let $\Omega = \Omega_{\mathbf{x}} \times (0,b)$ and $\Gamma = \{(\mathbf{x},z) \mid \mathbf{x} \in \Omega_{\mathbf{x}},\ z=0\}$. Assume $u_0|_{z=0} = u_0|_{z=b} = 0$, $u_0 \in H^2(\Omega)$, $\psi_0 \in H^3(\Omega)$ and satisfies* (33)*. Moreover, we assume that ψ satisfies the following mixed Dirichlet Robin boundary condition:*

$$\psi|_{\partial\Omega\setminus\Gamma} = 0, \qquad (\psi_z + \beta\psi)|_\Gamma = 0. \tag{33}$$

Let T be the largest time up to which the 3D *inviscid model* (4)–(5) *has a smooth solution with $u(t) \in H^2(\Omega)$ and $\psi(t) \in H^3(\Omega)$ for $0 \leqslant t < T$. Then the following identity holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \big(u^2 + 2|\nabla\psi|^2\big)\,\mathrm{d}\mathbf{x}\,\mathrm{d}z - 2\beta\int_0^a\int_0^a \psi^2\big|_{z=0}\,\mathrm{d}\mathbf{x}\right) = 0, \quad 0 \leqslant t < T. \tag{34}$$

Moreover, we have for $0 \leqslant t < T$ that

$$\int_\Omega \big(u^2 + 2(1-\beta b)|\nabla\psi|^2\big)\,\mathrm{d}\mathbf{x}\,\mathrm{d}z \leqslant \int_\Omega \big(u_0^2 + 2|\nabla\psi_0|^2\big)\,\mathrm{d}\mathbf{x}\,\mathrm{d}z - 2\beta\int_0^a\int_0^a \psi_0^2|_{z=0}\,\mathrm{d}\mathbf{x}. \tag{35}$$

Remark 2.3. One immediate consequence of the above proposition is that if $\beta < 1/b$, both $\int_\Omega u^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}z$ and $\int_\Omega |\nabla\psi|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}z$ are bounded.

Proof of Proposition 2.1. First of all, we know by the local existence result in Theorem 2.1 that there exists a T_0 such that the 3D inviscid model (4)–(5) has a unique smooth solution with $u(t) \in H^2(\Omega)$ and $\psi(t) \in H^3(\Omega)$ for $0 \leqslant t < T_0$. Let T be the largest time up to which the 3D inviscid model (4)–(5) has a smooth solution with $u(t) \in H^2(\Omega)$ and $\psi(t) \in H^3(\Omega)$ for $0 \leqslant t < T$. In the following, we will perform energy estimates for (4)–(5) for $0 \leqslant t < T$.

First, we multiply (4) by u and integrate over Ω. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}z = 4\int_\Omega u^2\psi_z\,\mathrm{d}\mathbf{x}\,\mathrm{d}z. \tag{36}$$

Next, we multiply (5) by ψ and integrate over Ω to obtain

$$-\int_{\Omega} \Delta\psi_t \psi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z = \int_{\Omega} \left(u^2\right)_z \psi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z. \tag{37}$$

Integrating by parts and using boundary condition (19), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} |\nabla\psi|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}z + 2\int_{\Omega_{\mathbf{x}}} \psi_{zt}\psi|_{z=0} \,\mathrm{d}\mathbf{x} = -2\int_{\Omega} u^2\psi_z \,\mathrm{d}\mathbf{x}\,\mathrm{d}z. \tag{38}$$

Multiplying (38) by 2 and adding the resulting equation to (36) gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} \left(u^2 + 2|\nabla\psi|^2\right) \mathrm{d}\mathbf{x}\,\mathrm{d}z &= -4\int_{\Omega_{\mathbf{x}}} \psi_{zt}\psi|_{z=0} \,\mathrm{d}\mathbf{x} \\ &= 4\beta\int_{\Omega_{\mathbf{x}}} \psi_t\psi|_{z=0} \,\mathrm{d}\mathbf{x} \\ &= 2\beta\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{\mathbf{x}}} \psi^2\big|_{z=0} \,\mathrm{d}\mathbf{x},\end{aligned} \tag{39}$$

which gives (34). On the other hand, we have the following estimate

$$\begin{aligned}\int_{\Omega_{\mathbf{x}}} \psi^2\big|_{z=0} \,\mathrm{d}\mathbf{x} &= \int_{\Omega_{\mathbf{x}}} \left(\int_0^b \psi_z \,\mathrm{d}z\right)^2 \mathrm{d}\mathbf{x} \\ &\leqslant b\int_{\Omega_{\mathbf{x}}} \int_0^b \psi_z^2 \,\mathrm{d}z\,\mathrm{d}\mathbf{x} \leqslant b\int_{\Omega} |\nabla\psi|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}z.\end{aligned} \tag{40}$$

This implies that

$$\int_{\Omega} \left(u^2 + 2|\nabla\psi|^2\right) \mathrm{d}\mathbf{x}\,\mathrm{d}z - 2\beta\int_{\Omega_{\mathbf{x}}} \psi^2\big|_{z=0} \,\mathrm{d}\mathbf{x} \geqslant \int_{\Omega} \left(u^2 + 2(1-\beta b)|\nabla\psi|^2\right) \mathrm{d}\mathbf{x}\,\mathrm{d}z. \tag{41}$$

Combining (34) with (41), we obtain

$$\int_{\Omega} \left(u^2 + 2(1-\beta b)|\nabla\psi|^2\right) \mathrm{d}\mathbf{x}\,\mathrm{d}z \leqslant \int_{\Omega} \left(u_0^2 + 2|\nabla\psi_0|^2\right) \mathrm{d}\mathbf{x}\,\mathrm{d}z - 2\beta\int_{\Omega_{\mathbf{x}}} \psi_0^2|_{z=0} \,\mathrm{d}\mathbf{x}. \tag{42}$$

This completes the proof of Proposition 2.1. □

3. Blow-up of the 3D inviscid model

In this section, we will prove that the 3D model (18)–(19) develops a finite time singularity for a class of smooth initial data with finite energy. The finite time blow-up is proved in a semi-infinite and a bounded domain with mixed Dirichlet–Robin boundary conditions.

3.1. Blow-up in a semi-infinite domain

First, we consider the initial boundary value problem (18)–(20) in a semi-infinite domain with $\Omega = \Omega_{\mathbf{x}} \times (0, \infty)$. The main result is stated in the theorem below:

Theorem 3.1. *Assume that $u_0 \in H^2(\Omega)$, $u_0|_{\partial\Omega} = 0$, $u_0|_{\Omega} > 0$, $\psi_0 \in H^3(\Omega)$ and satisfies* (19). *Further we assume that $\beta > \frac{\sqrt{2}\pi}{a}$ and $\beta \in S_\infty$ as defined in Lemma* 2.1. *Choose $\alpha = \frac{2\pi^2}{\beta a^2}$, and define*

$$\phi(\mathbf{x}, z) = e^{-\alpha z}\phi_1(\mathbf{x}), \qquad \phi_1(\mathbf{x}) = \sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a}, \quad (\mathbf{x}, z) \in \Omega,$$

$$A = \int_\Omega (\log u_0)\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z, \qquad B = 2\int_\Omega \psi_{0z}\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z,$$

$$D = \frac{\pi\alpha^{5/2}}{a(2(\frac{\pi}{a})^2 - \alpha^2)}, \qquad I_\infty = \int_0^\infty \frac{\mathrm{d}x}{\sqrt{x^3+1}}. \tag{43}$$

If $A > 0$ and $B > 0$, then the 3*D inviscid model* (18), *with the boundary condition* (19) *and the initial data* (20) *will develop a finite time singularity in the H^2-norm no later than*

$$T^* = \left(\frac{2DB}{3}\frac{\sqrt{\alpha}\,\pi}{2a}\right)^{-1/3} I_\infty.$$

Proof. By Theorem 2.1, we know that there exists a finite time $T > 0$ such that the system (18)–(20) has a unique smooth solution with $u \in C^1([0, T), H^2(\Omega))$ and $\psi \in C^1([0, T), H^3(\Omega))$. Let T_b be the largest time such that the system (18)–(19) with initial condition u_0, ψ_0 has a smooth solution with $u \in C^1([0, T_b); H^2(\Omega))$ and $\psi \in C^1([0, T_b); H^3(\Omega))$. We claim that $T_b < \infty$. We prove this by contradiction.

Suppose that $T_b = \infty$, this means that for the given initial data u_0, ψ_0, the system (18)–(20) has a globally smooth solution $u \in C^1([0, \infty); H^2(\Omega))$ and $\psi \in C^1([0, \infty); H^3(\Omega))$. Multiplying ϕ_z to the both sides of (5) and integrating over Ω, we get

$$-\int_\Omega \Delta\psi_t\phi_z \,\mathrm{d}\mathbf{x}\,\mathrm{d}z = \int_\Omega \left(u^2\right)_z \phi_z \,\mathrm{d}\mathbf{x}\,\mathrm{d}z. \tag{44}$$

Note that $u|_{\partial\Omega} = 0$ as long as the solution remains smooth. By integrating by parts and using the boundary condition on ψ and the property of ϕ to eliminate the boundary terms, we have

$$-\int_{\Omega} \psi_{zt}\,\Delta\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z - \int_{\Omega_{\mathbf{x}}} \psi_{zt}\phi_z|_{z=0}\,\mathrm{d}\mathbf{x} - \int_{\Omega_{\mathbf{x}}} \psi_t\,\Delta_{\mathbf{x}}\phi|_{z=0}\,\mathrm{d}\mathbf{x} = \int_{\Omega} u^2\phi_{zz}\,\mathrm{d}\mathbf{x}\,\mathrm{d}z. \tag{45}$$

Substituting ϕ into the above equation, we obtain

$$\begin{aligned}\left(\frac{2\pi^2}{a^2}-\alpha^2\right)\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} \psi_z\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z &= \alpha^2\int_{\Omega} u^2\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z - \int_{\Omega_{\mathbf{x}}}\left(\alpha\psi_{zt}+\frac{2\pi^2}{a^2}\psi_t\right)\bigg|_{z=0}\phi_1(\mathbf{x})\,\mathrm{d}\mathbf{x}\\ &= \alpha^2\int_{\Omega} u^2\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z + \int_{\Omega_{\mathbf{x}}}\left(\alpha\beta-\frac{2\pi^2}{a^2}\right)\psi_t|_{z=0}\phi_1(\mathbf{x})\,\mathrm{d}\mathbf{x}. \end{aligned} \tag{46}$$

By the definition of α, we have

$$\alpha\beta-\frac{2\pi^2}{a^2}=0 \quad \text{and} \quad \alpha=\frac{2\pi^2}{\beta a^2}<\frac{\sqrt{2}\pi}{a}, \tag{47}$$

since $\beta > \frac{\sqrt{2}\pi}{a}$. Thus the boundary term on the right-hand side of (46) vanishes. We get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} \psi_z\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z = \frac{\alpha^2}{2(\frac{\pi}{a})^2-\alpha^2}\int_{\Omega} u^2\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z. \tag{48}$$

Next, we multiply ϕ to (8) and integrate over Ω. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} (\log u)\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z = 2\int_{\Omega} \psi_z\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z. \tag{49}$$

Combining (48) with (49), we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_{\Omega} (\log u)\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z = \frac{2\alpha^2}{2(\frac{\pi}{a})^2-\alpha^2}\int_{\Omega} u^2\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z. \tag{50}$$

Integrating the above equation twice in time, we get

$$\begin{aligned}\int_{\Omega} (\log u)\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z &= \frac{2\alpha^2}{2(\frac{\pi}{a})^2-\alpha^2}\int_0^t\int_0^s\left(\int_{\Omega} u^2\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z\right)\mathrm{d}\tau\,\mathrm{d}s + A + Bt\\ &\geqslant \frac{2\alpha^2}{2(\frac{\pi}{a})^2-\alpha^2}\int_0^t\int_0^s\left(\int_{\Omega} u^2\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z\right)\mathrm{d}\tau\,\mathrm{d}s + Bt.\end{aligned} \tag{51}$$

Note that $u > 0$ for $(\mathbf{x}, z) \in \Omega$ and $t < T_b$. It is easy to show that

$$\begin{aligned}\int_\Omega (\log u)\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z &\leqslant \int_\Omega (\log u)^+\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z \leqslant \int_\Omega u\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z \\ &\leqslant \left(\int_\Omega \phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z\right)^{1/2}\left(\int_\Omega u^2\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z\right)^{1/2} \\ &= \frac{2a}{\sqrt{\alpha}\,\pi}\left(\int_\Omega u^2\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z\right)^{1/2},\end{aligned} \tag{52}$$

where $(\log u)^+ = \max(\log u, 0)$. Combining (51) with (52) gives us the crucial nonlinear dynamic estimate:

$$\left(\int_\Omega u^2\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z\right)^{1/2} \geqslant \frac{2\alpha^2}{2(\frac{\pi}{a})^2 - \alpha^2}\frac{\sqrt{\alpha}\,\pi}{2a}\int_0^t\int_0^s\left(\int_\Omega u^2\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z\right)\mathrm{d}\tau\,\mathrm{d}s + \frac{\sqrt{\alpha}\,\pi}{2a}Bt. \tag{53}$$

Define

$$F(t) = \frac{\pi\alpha^{5/2}}{a(2(\frac{\pi}{a})^2 - \alpha^2)}\int_0^t\int_0^s\left(\int_\Omega u^2\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z\right)\mathrm{d}\tau\,\mathrm{d}s + \frac{\sqrt{\alpha}\,\pi}{2a}Bt. \tag{54}$$

Then we have $F(0) = 0$ and $F_t(0) = \frac{\sqrt{\alpha}\,\pi}{2a}B > 0$. By differentiating (54) twice in time and substituting the resulting equation into (53), we obtain

$$\frac{\mathrm{d}^2F}{\mathrm{d}t^2} = \frac{\pi\alpha^{5/2}}{a(2(\frac{\pi}{a})^2 - \alpha^2)}\int_\Omega u^2\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z \geqslant DF^2, \tag{55}$$

where $D = \frac{\pi\alpha^{5/2}}{a(2(\frac{\pi}{a})^2-\alpha^2)}$. Note that $F_t = D\int_0^t(\int_\Omega u^2\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z)\,\mathrm{d}s + \frac{\sqrt{\alpha}\,\pi}{2a}B > 0$. Multiplying F_t to (55) and integrating in time, we get

$$\frac{\mathrm{d}F}{\mathrm{d}t} \geqslant \sqrt{\frac{2D}{3}F^3 + C}, \tag{56}$$

where $C = (F_t(0))^2 = \frac{\alpha\pi^2}{4a^2}B^2$. Define

$$I(x) = \int_0^x \frac{\mathrm{d}y}{\sqrt{y^3+1}}, \qquad J = \left(\frac{3C}{2D}\right)^{1/3}.$$

Then, integrating (56) in time gives

$$I\left(\frac{F(t)}{J}\right) \geqslant \frac{\sqrt{C}t}{J}, \quad \forall t \in \left[0, T^*\right]. \tag{57}$$

Note that both I and F are strictly increasing functions, and $I(x)$ is uniformly bounded for all $x > 0$ while the right-hand side increases linearly in time. It follows from (57) that $F(t)$ must blow up no later than

$$\frac{J}{\sqrt{C}} I_\infty = T^*.$$

This contradicts with the assumption that the 3D model has a globally smooth solution. This contradiction implies that the solution of the system (18) must develop a finite time singularity no later than T^*. □

Remark 3.1. As we can see in the proof of Theorem 3.1, the same conclusion still holds if we replace the boundary condition

$$(\psi_z + \beta\psi)|_{z=0} = 0,$$

by the following integral constraint

$$\int_{\Omega_{\mathbf{x}}} (\psi_z + \beta\psi)|_{z=0} \sin\frac{\pi x_1}{a} \sin\frac{\pi x_2}{a} \, d\mathbf{x} = 0.$$

3.2. Blow-up in a bounded domain

In this subsection, we will prove finite time blow-up of the 3D model in a bounded domain. First, we formulate the initial boundary problem of the 3D model as follows:

$$\begin{cases} u_t = 2u\psi_z, \\ -\Delta\psi_t = \left(u^2\right)_z, \end{cases} \quad (\mathbf{x}, z) \in \Omega = \Omega_{\mathbf{x}} \times (0, b), \tag{58}$$

$$\psi|_{\partial\Omega\setminus\Gamma} = 0, \qquad (\psi_z + \beta\psi)|_\Gamma = 0,$$

$$\psi|_{t=0} = \psi_0(\mathbf{x}, z), \qquad u|_{t=0} = u_0(\mathbf{x}, z) \geqslant 0, \tag{59}$$

where $\mathbf{x} = (x_1, x_2)$, $\Omega_{\mathbf{x}} = (0, a) \times (0, a)$, $\Gamma = \{(\mathbf{x}, z) \in \Omega \mid \mathbf{x} \in \Omega_{\mathbf{x}},\ z = 0\}$. We can get a similar blow-up result which is summarized below:

Theorem 3.2. *Assume that $u_0 \in H^2(\Omega)$, $u_0|_{\partial\Omega} = 0$, $u_0|_\Omega > 0$, $\psi_0 \in H^3(\Omega)$ and satisfies* (59). *Further, we assume that $\beta \in S_b$ as defined in Lemma* 2.1 *and satisfies $\beta > \frac{\sqrt{2}\pi}{a}(\frac{e^{\sqrt{2}\pi b/a} + e^{-\sqrt{2}\pi b/a}}{e^{\sqrt{2}\pi b/a} - e^{-\sqrt{2}\pi b/a}})$. Define*

$$\phi(\mathbf{x}, z) = \frac{e^{-\alpha(z-b)} + e^{\alpha(z-b)}}{2} \sin\frac{\pi x_1}{a} \sin\frac{\pi x_2}{a}, \quad (\mathbf{x}, z) \in \Omega, \tag{60}$$

where α satisfies $0 < \alpha < \sqrt{2}\pi/a$ and $2(\frac{\pi}{a})^2 \frac{e^{\alpha b}-e^{-\alpha b}}{\alpha(e^{\alpha b}+e^{-\alpha b})} = \beta$. Let

$$A = \int_{\Omega} (\log u_0)\phi \, \mathrm{d}\mathbf{x}\, \mathrm{d}z, \qquad B = 2\int_{\Omega} \psi_{0z}\phi \, \mathrm{d}\mathbf{x}\, \mathrm{d}z,$$

$$D = \frac{\pi \alpha^{5/2}}{a(2(\frac{\pi}{a})^2 - \alpha^2)}, \qquad I_\infty = \int_0^\infty \frac{\mathrm{d}\mathbf{x}}{\sqrt{x^3+1}}.$$

If $A > 0$ and $B > 0$, then the solution of (58)–(59) *will blow up no later than*

$$T^* = \left(\frac{2DB}{3}\frac{\sqrt{\alpha}\pi}{2a}\right)^{-1/3} I_\infty.$$

Proof. We follow the same strategy as in the proof of Theorem 3.1. By Theorem 2.1, we know that there exists a finite time $T > 0$ such that the system (58) has a unique smooth solution with $u \in C^1([0,T), H^2(\Omega))$ and $\psi \in C^1([0,T), H^3(\Omega))$ for $0 \leqslant t < T$. Let T_b be the largest time such that the system (58)–(59) with initial condition u_0, ψ_0 has a smooth solution with $u \in C^1([0,T_b); H^2(\Omega))$ and $\psi \in C^1([0,T_b); H^3(\Omega))$. We claim that $T_b < \infty$. We prove this by contradiction.

Suppose that $T_b = \infty$, this means that for the given initial data u_0, ψ_0, the system (58) has a globally smooth solution with $u \in C^1([0,\infty); H^2(\Omega))$ and $\psi \in C^1([0,\infty); H^3(\Omega))$. Multiplying ϕ_z to the both sides of the ψ-equation and integrating over Ω, we get

$$-\int_{\Omega} \Delta\psi_t \phi_z \, \mathrm{d}\mathbf{x}\, \mathrm{d}z = \int_{\Omega} \left(u^2\right)_z \phi_z \, \mathrm{d}\mathbf{x}\, \mathrm{d}z. \tag{61}$$

Note that $u|_{\partial\Omega} = 0$ as long as the solution remains smooth. By integrating by parts and using the boundary condition on ψ and the property of ϕ to eliminate the boundary terms, we have

$$-\int_{\Omega} \psi_{zt}\Delta\phi \, \mathrm{d}\mathbf{x}\, \mathrm{d}z - \int_{\Omega_{\mathbf{x}}} \psi_{zt}\phi_z|_{z=0} \, \mathrm{d}\mathbf{x} - \int_{\Omega_{\mathbf{x}}} \psi_t \Delta_{\mathbf{x}}\phi|_{z=0} \, \mathrm{d}\mathbf{x} = \int_{\Omega} u^2\phi_{zz} \, \mathrm{d}\mathbf{x}\, \mathrm{d}z. \tag{62}$$

Substituting ϕ to (62) and using the boundary condition for ψ, we obtain

$$\begin{aligned}
&\left(2\frac{\pi^2}{a^2} - \alpha^2\right)\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} \psi_z\phi \, \mathrm{d}\mathbf{x}\, \mathrm{d}z \\
&= \alpha^2 \int_{\Omega} u^2\phi \, \mathrm{d}\mathbf{x}\, \mathrm{d}z - \int_{\Omega_{\mathbf{x}}} \left(\frac{\alpha}{2}\left(e^{\alpha b} - e^{-\alpha b}\right)\psi_{zt} + \frac{\pi^2}{a^2}\left(e^{\alpha b} + e^{-\alpha b}\right)\psi_t\right)\bigg|_{z=0} \phi_1(\mathbf{x}) \, \mathrm{d}\mathbf{x} \\
&= \alpha^2 \int_{\Omega} u^2\phi \, \mathrm{d}\mathbf{x}\, \mathrm{d}z + \int_{\Omega_{\mathbf{x}}} \left(\frac{\alpha}{2}\left(e^{\alpha b} - e^{-\alpha b}\right)\beta - \frac{\pi^2}{a^2}\left(e^{\alpha b} + e^{-\alpha b}\right)\right)\psi_t|_{z=0}\, \phi_1(\mathbf{x}) \, \mathrm{d}\mathbf{x}
\end{aligned}$$

$$= \alpha^2 \int\limits_{\Omega} u^2 \phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z$$
$$+ \frac{\alpha}{2}\big(e^{\alpha b} - e^{-\alpha b}\big) \int\limits_{\Omega_{\mathbf{x}}} \left(\beta - 2\left(\frac{\pi}{a}\right)^2 \frac{e^{\alpha b} + e^{-\alpha b}}{\alpha(e^{\alpha b} - e^{-\alpha b})}\right) \psi_t|_{z=0}\, \phi_1(\mathbf{x})\,\mathrm{d}\mathbf{x}, \tag{63}$$

where $\phi_1(\mathbf{x}) = \sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a}$. Let $h(\alpha) = 2(\frac{\pi}{a})^2 \frac{e^{\alpha b}+e^{-\alpha b}}{\alpha(e^{\alpha b}-e^{-\alpha b})}$. Direct computations show that $\frac{\mathrm{d}}{\mathrm{d}\alpha} h(\alpha) < 0$ for all $\alpha > 0$. Thus we have

$$\frac{\sqrt{2}\pi}{a}\left(\frac{e^{\sqrt{2}\pi b/a} + e^{-\sqrt{2}\pi b/a}}{e^{\sqrt{2}\pi b/a} - e^{-\sqrt{2}\pi b/a}}\right) = h\left(\frac{\sqrt{2}\pi}{a}\right) < h(\alpha) < h(0_+) = \infty, \quad 0 < \alpha < \frac{\sqrt{2}\pi}{a}. \tag{64}$$

Since $\beta > h(\frac{\sqrt{2}\pi}{a})$ by assumption, we can choose a unique α with $0 < \alpha < \frac{\sqrt{2}\pi}{a}$ such that

$$\frac{2\pi^2}{a^2} \frac{e^{\alpha b} + e^{-\alpha b}}{\alpha(e^{\alpha b} - e^{-\alpha b})} = \beta. \tag{65}$$

With this choice of α, the boundary term in (63) vanishes. Therefore we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\Omega} \psi_z \phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z = \frac{\alpha^2}{2(\frac{\pi}{a})^2 - \alpha^2} \int\limits_{\Omega} u^2 \phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z. \tag{66}$$

Next, we multiply ϕ to (8) and integrate over Ω. We get

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\Omega} (\log u)\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z = 2 \int\limits_{\Omega} \psi_z \phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z. \tag{67}$$

Combining (67) with (66), we get

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int\limits_{\Omega} (\log u)\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z = \frac{2\alpha^2}{2(\frac{\pi}{a})^2 - \alpha^2} \int\limits_{\Omega} u^2 \phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z. \tag{68}$$

Now we can follow the exactly same procedure as in the proof of Theorem 3.1 to prove that the 3D model must develop a finite time blow-up. □

Remark 3.2. We remark that the same conclusion is still true if we replace the Dirichlet boundary condition $\psi|_{z=b} = 0$ by the Neumann boundary condition $\psi_z|_{z=b} = 0$. The only difference is that the weight function ϕ is now changed to

$$\phi(\mathbf{x}, z) = \frac{e^{-\alpha(z-b)} - e^{\alpha(z-b)}}{2} \sin\frac{\pi x_1}{a} \sin\frac{\pi x_2}{a}, \quad (\mathbf{x}, z) \in \Omega, \tag{69}$$

where $0 < \alpha < \sqrt{2}\pi/a$, and β satisfies a variant of (24) in Lemma 2.1 and

$$\frac{\sqrt{2}\pi}{a}\left(\frac{e^{\sqrt{2}\pi b/a} - e^{-\sqrt{2}\pi b/a}}{e^{\sqrt{2}\pi b/a} + e^{-\sqrt{2}\pi b/a}}\right) < \beta < 2b\left(\frac{\pi}{a}\right)^2. \tag{70}$$

We omit the proof here.

3.3. Blow-up of a generalized 3D model

In this section, we study singularity formation of a generalized 3D model by changing the sign of the Laplace operator in the ψ-equation (5). Specifically, we consider the following generalized 3D model:

$$\begin{cases} u_t = 2u\psi_z, \\ \Delta\psi_t = (u^2)_z, \end{cases} \quad (\mathbf{x}, z) \in \Omega = \Omega_{\mathbf{x}} \times (0, a) = (0, a) \times (0, a) \times (0, a). \tag{71}$$

The boundary and initial conditions are below:

$$\begin{gathered} \psi|_{\partial\Omega\setminus\Gamma} = 0, \qquad \psi_z|_\Gamma = 0, \qquad \Gamma = \left\{(\mathbf{x}, z) \,\middle|\, \mathbf{x} \in \Omega_{\mathbf{x}},\ z = 0,\ \text{or } z = a\right\}, \\ \psi|_{t=0} = \psi_0(\mathbf{x}, z), \qquad u|_{t=0} = u_0(\mathbf{x}, z) \geqslant 0. \end{gathered} \tag{72}$$

In this subsection, we will generalize the singularity analysis presented in the previous subsection to prove that the solution of the generalized 3D model will develop a finite time singularity. The main result is summarized in the following theorem.

Theorem 3.3. *Assume that $u_0 \in H^2(\Omega)$, $u_0|_{\partial\Omega} = 0$, $u_0|_\Omega > 0$, $\psi_0 \in H^3(\Omega)$ and satisfies* (72). *Further, we define*

$$\phi(\mathbf{x}, z) = \sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a}\sin\frac{\pi z}{a}, \quad (\mathbf{x}, z) \in \Omega. \tag{73}$$

Let

$$A = \int_\Omega (\log u_0)\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z, \qquad B = 2\int_\Omega \psi_{0z}\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z, \qquad I_\infty = \int_0^\infty \frac{\mathrm{d}\mathbf{x}}{\sqrt{x^3 + 1}}.$$

If $A > 0$ and $B > 0$, then the solution of (71)–(72) *will blow up no later than $T^* = (\frac{B}{18})^{-1/3} I_\infty$.*

Proof. First, by using an argument similar to the local well-posedness result in Theorem 2.1, we can prove that the system (71)–(72) is locally well-posed. We prove the theorem by contradiction. Suppose that the system (71)–(72) has a globally smooth solution with $u \in C^1([0, \infty); H^2(\Omega))$ and $\psi \in C^1([0, \infty); H^3(\Omega))$. Multiplying ϕ_z to the both sides of the ψ-equation and integrating over Ω, we get

$$-\int_\Omega \Delta\psi_t\phi_z \,\mathrm{d}\mathbf{x}\,\mathrm{d}z = \int_\Omega (u^2)_z\phi_z \,\mathrm{d}\mathbf{x}\,\mathrm{d}z. \tag{74}$$

Note that $u|_{\partial\Omega} = 0$ as long as the solution remains smooth. By integrating by parts and using the boundary condition on ψ and the property of ϕ to eliminate the boundary terms, we have

$$-\int_{\Omega} \psi_{zt} \Delta\phi \, \mathrm{d}\mathbf{x} \, \mathrm{d}z = \int_{\Omega} u^2 \phi_{zz} \, \mathrm{d}\mathbf{x} \, \mathrm{d}z. \tag{75}$$

Substituting ϕ to (75) and using the boundary condition for ψ, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \psi_z \phi \, \mathrm{d}\mathbf{x} \, \mathrm{d}z = \frac{1}{3} \int_{\Omega} u^2 \phi \, \mathrm{d}\mathbf{x} \, \mathrm{d}z. \tag{76}$$

Next, we multiply ϕ to (8) and integrate over Ω. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} (\log u)\phi \, \mathrm{d}\mathbf{x} \, \mathrm{d}z = 2 \int_{\Omega} \psi_z \phi \, \mathrm{d}\mathbf{x} \, \mathrm{d}z. \tag{77}$$

Combining (76) with (77), we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_{\Omega} (\log u)\phi \, \mathrm{d}\mathbf{x} \, \mathrm{d}z = \frac{2}{3} \int_{\Omega} u^2 \phi \, \mathrm{d}\mathbf{x} \, \mathrm{d}z. \tag{78}$$

Integrating the above equation twice in time, we get

$$\int_{\Omega} (\log u)\phi \, \mathrm{d}\mathbf{x} \, \mathrm{d}z = \frac{2}{3} \int_0^t \int_0^s \left(\int_{\Omega} u^2 \phi \, \mathrm{d}\mathbf{x} \, \mathrm{d}z \right) \mathrm{d}\tau \, \mathrm{d}s + A + Bt. \tag{79}$$

Using (79), following the same argument as in Theorem 3.1, we can prove that the solution of the initial boundary value problem (71)–(72) blows up no later than T^*. □

4. Blow-up of the 3D model with partial viscosity

In this section, we prove finite blow-up of the 3D model with partial viscosity. Specifically, we consider the following initial boundary value problem in a semi-infinite domain:

$$\begin{cases} u_t = 2u\psi_z, \\ \omega_t = (u^2)_z + \nu \Delta\omega, \qquad (\mathbf{x}, z) \in \Omega = \Omega_{\mathbf{x}} \times (0, \infty). \\ -\Delta\psi = \omega, \end{cases} \tag{80}$$

The initial and boundary conditions are given as follows:

$$\psi|_{\partial\Omega \setminus \Gamma} = 0, \qquad (\psi_z + \beta\psi)|_{\Gamma} = 0, \tag{81}$$

$$\omega|_{\partial\Omega \setminus \Gamma} = 0, \qquad (\omega_z + \gamma\omega)|_{\Gamma} = 0, \tag{82}$$

$$\omega|_{t=0} = \omega_0(\mathbf{x}, z), \qquad u|_{t=0} = u_0(\mathbf{x}, z) \geqslant 0, \tag{83}$$

where $\Gamma = \{(\mathbf{x}, z) \in \Omega \mid \mathbf{x} \in \Omega_{\mathbf{x}},\ z = 0\}$.

Now we state the main result of this section.

Theorem 4.1. *Assume that* $u_0|_{\partial\Omega} = 0$, $u_{0z}|_{\partial\Omega} = 0$, $u_0|_\Omega > 0$, $u_0 \in H^2(\Omega)$, $\psi_0 \in H^3(\Omega)$, $\omega_0 \in H^1(\Omega)$, ψ_0 *satisfies* (81) *and* ω_0 *satisfies* (82). *Further, we assume that* $\beta \in S_\infty$ *as defined in Lemma* 2.1 *and* $\beta > \frac{\sqrt{2}\pi}{a}$, $\gamma = \frac{2\pi^2}{\beta a^2}$. *Let*

$$\phi(\mathbf{x}, z) = e^{-\alpha z} \sin\frac{\pi x_1}{a} \sin\frac{\pi x_2}{a}, \quad (\mathbf{x}, z) \in \Omega, \tag{84}$$

where $\alpha = \frac{2\pi^2}{\beta a^2}$ *satisfies* $0 < \alpha < \sqrt{2}\pi/a$. *Define*

$$A = \int_\Omega (\log u_0)\phi \, d\mathbf{x}\, dz, \qquad B = -\int_\Omega \omega_0 \phi_z \, d\mathbf{x}\, dz, \qquad D = \frac{2}{2(\frac{\pi}{a})^2 - \alpha^2}, \tag{85}$$

$$I_\infty = \int_0^\infty \frac{d\mathbf{x}}{\sqrt{x^3+1}}, \qquad T^* = \left(\frac{\pi\alpha^3 D^2 B}{12a}\right)^{-1/3} I_\infty. \tag{86}$$

If $A > 0$, $B > 0$, *and* $T^* < (\log 2)(\nu(\frac{2\pi^2}{a^2} - \alpha^2))^{-1}$, *then the solution of model* (80) *with initial and boundary conditions* (81)–(83) *will develop a finite time singularity before* T^*.

Proof. First of all, we can prove that the 3D model (80) with initial and boundary conditions given by (81)–(83) has a unique solution, $u \in C([0,T], H^2(\Omega))$, $\omega \in C([0,T], H^1(\Omega))$ and $\psi \in C([0,T], H^3(\Omega))$ for some $T > 0$ depending on initial data. There are two key ingredients in this analysis. The first one is to design a Picard iteration for the 3D model. The second one is to show that the mapping that generates the Picard iteration is a contraction mapping and the Picard iteration converges to a fixed point of the Picard mapping by using the Contraction Mapping Theorem. To establish the contraction property of the Picard mapping, we need to use the well-posedness property of the heat equation with the same Dirichlet Robin boundary condition as ω. The well-posedness analysis of the heat equation with a mixed Dirichlet Robin boundary has been studied in the literature. The case of $\gamma > 0$ is more subtle because there is a growing eigenmode. Since the complete analysis of the local well-posedness of 3D model with partial viscosity is quite technical, we will not present the analysis here and refer the reader to [20] for the details of the analysis.

We are now ready to prove the finite time singularity of the 3D model with partial viscosity with the given initial boundary data. We will prove the theorem by contradiction. Assume that the 3D model (80) with initial and boundary conditions (81)–(83) has a globally smooth solution, $u \in C^1([0,\infty); H^2(\Omega))$, $\psi \in C^1([0,\infty); H^3(\Omega))$, and $\omega \in C^1([0,\infty); H^1(\Omega))$. Multiplying ϕ to the both sides of the ψ-equation and integrating over Ω, we get

$$-\int_\Omega \Delta\psi \phi_z \, d\mathbf{x}\, dz = \int_\Omega \omega\phi_z \, d\mathbf{x}\, dz. \tag{87}$$

By integrating by parts and using boundary conditions (81)–(82) and the property of ϕ, we obtain

$$\int_{\Omega} \psi_z \Delta\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z - \int_{\Omega_{\mathbf{x}}} \psi_z \phi_z|_{z=0} \,\mathrm{d}\mathbf{x}\,\mathrm{d}z - \int_{\Omega_{\mathbf{x}}} \psi\,\Delta_{\mathbf{x}}\phi|_{z=0} \,\mathrm{d}\mathbf{x}\,\mathrm{d}z = \int_{\Omega} \omega\phi_z \,\mathrm{d}\mathbf{x}\,\mathrm{d}z. \tag{88}$$

Substituting ϕ defined in (84) into the above equation, we have

$$\begin{aligned} -\left(\frac{2\pi^2}{a^2} - \alpha^2\right) \int_{\Omega} \psi_z \phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z &= \int_{\Omega} \omega\phi_z \,\mathrm{d}\mathbf{x}\,\mathrm{d}z - \int_{\Omega_{\mathbf{x}}} \left(\alpha\psi_z + \frac{2\pi^2}{a^2}\psi\right)\Big|_{z=0} \phi_1(\mathbf{x}) \,\mathrm{d}\mathbf{x} \\ &= \int_{\Omega} \omega\phi_z \,\mathrm{d}\mathbf{x}\,\mathrm{d}z + \int_{\Omega_{\mathbf{x}}} \left(\alpha\beta - \frac{2\pi^2}{a^2}\right)\psi|_{z=0}\,\phi_1(\mathbf{x}) \,\mathrm{d}\mathbf{x}, \end{aligned} \tag{89}$$

where $\phi_1(\mathbf{x}) = \sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a}$. Since $\beta > \frac{\sqrt{2}\pi}{a}$, we can choose

$$\alpha = \frac{2\pi^2}{\beta a^2} < \frac{\sqrt{2}\pi}{a}, \tag{90}$$

to eliminate the boundary term in (89). This gives rise to the following identity:

$$\int_{\Omega} \psi_z \phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z = -\frac{1}{2(\frac{\pi}{a})^2 - \alpha^2} \int_{\Omega} \omega\phi_z \,\mathrm{d}\mathbf{x}\,\mathrm{d}z. \tag{91}$$

Next, we multiply ϕ_z to the both sides of the ω-equation and integrate over Ω

$$\int_{\Omega} \omega_t \phi_z \,\mathrm{d}\mathbf{x}\,\mathrm{d}z = \int_{\Omega} \left(u^2\right)_z \phi_z \,\mathrm{d}\mathbf{x}\,\mathrm{d}z + \nu \int_{\Omega} \Delta\omega\phi_z \,\mathrm{d}\mathbf{x}\,\mathrm{d}z. \tag{92}$$

Integrating by parts and using $u|_{\partial\Omega} = 0$, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} \omega\phi_z \,\mathrm{d}\mathbf{x}\,\mathrm{d}z = {} & -\int_{\Omega} u^2 \phi_{zz} \,\mathrm{d}\mathbf{x}\,\mathrm{d}z \\ & + \nu\left(-\int_{\Omega_{\mathbf{x}}} \omega_z\phi_z|_{z=0} \,\mathrm{d}\mathbf{x} + \int_{\Omega_{\mathbf{x}}} \omega\phi_{zz}|_{z=0} \,\mathrm{d}\mathbf{x} + \int_{\Omega} \omega\Delta\phi_z \,\mathrm{d}\mathbf{x}\,\mathrm{d}z\right) \\ = {} & -\alpha^2 \int_{\Omega} u^2\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z - \nu\left(\frac{2\pi^2}{a^2} - \alpha^2\right)\int_{\Omega} \omega\phi_z \,\mathrm{d}\mathbf{x}\,\mathrm{d}z \\ & + \nu \int_{\Omega_{\mathbf{x}}} \left(\alpha\omega_z + \alpha^2\omega\right)\big|_{z=0} \phi_1(\mathbf{x}) \,\mathrm{d}\mathbf{x} \\ = {} & -\alpha^2 \int_{\Omega} u^2\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z - \nu\left(\frac{2\pi^2}{a^2} - \alpha^2\right)\int_{\Omega} \omega\phi_z \,\mathrm{d}\mathbf{x}\,\mathrm{d}z \end{aligned}$$

$$+\nu\int_{\Omega_{\mathbf{x}}}\alpha(\alpha-\gamma)\omega|_{z=0}\,\phi_1(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

$$=-\alpha^2\int_\Omega u^2\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z-\nu\left(\frac{2\pi^2}{a^2}-\alpha^2\right)\int_\Omega \omega\phi_z\,\mathrm{d}\mathbf{x}\,\mathrm{d}z, \tag{93}$$

where we have used $\alpha=\gamma$ to eliminate the boundary term in the above estimates. Solving the above ordinary equation for $\int_\Omega \omega\phi_z\,\mathrm{d}\mathbf{x}\,\mathrm{d}z$ gives

$$\int_\Omega \omega\phi_z\,\mathrm{d}\mathbf{x}\,\mathrm{d}z=e^{-\lambda t}\int_\Omega \omega_0\phi_z\,\mathrm{d}\mathbf{x}\,\mathrm{d}z-\alpha^2\int_0^t e^{-\lambda(t-s)}\left(\int_\Omega u^2\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z\right)\mathrm{d}s, \tag{94}$$

where $\lambda=\nu(\frac{2\pi^2}{a^2}-\alpha^2)$. Using the reformulated u-equation (8), (91) and (94), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(\log u)\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z=2\int_\Omega \psi_z\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z$$

$$=\frac{2}{2(\frac{\pi}{a})^2-\alpha^2}\left(-e^{-\lambda t}\int_\Omega \omega_0\phi_z\,\mathrm{d}\mathbf{x}\,\mathrm{d}z+\alpha^2\int_0^t e^{-\lambda(t-s)}\left(\int_\Omega u^2\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z\right)\mathrm{d}s\right). \tag{95}$$

Integrating the above equation in time, we get

$$\int_\Omega(\log u)\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z=\int_\Omega(\log u_0)\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z-\frac{2}{2(\frac{\pi}{a})^2-\alpha^2}\left(\frac{1-e^{-\lambda t}}{\lambda}\right)\left(-\int_\Omega \omega_0\phi_z\,\mathrm{d}\mathbf{x}\,\mathrm{d}z\right)$$

$$+\frac{2\alpha^2}{2(\frac{\pi}{a})^2-\alpha^2}\int_0^t\int_0^s e^{-\lambda(s-\tau)}\left(\int_\Omega u^2\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z\right)\mathrm{d}\tau\,\mathrm{d}s. \tag{96}$$

Let $T_0=\frac{\log 2}{\lambda}$, then $e^{-\lambda t}\geqslant\frac{1}{2}$ over the interval $[0,T_0]$. Note that $\frac{\mathrm{d}}{\mathrm{d}t}(\frac{1-e^{-\lambda t}}{\lambda})=e^{-\lambda t}\geqslant\frac{1}{2}$ for $0\leqslant t\leqslant T_0$. This implies that $\frac{1-e^{-\lambda t}}{\lambda}\geqslant\frac{t}{2}$ for $0\leqslant t\leqslant T_0$. Thus we have from (96) that

$$\int_\Omega(\log u)\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z\geqslant A+\frac{1}{2}DBt+\frac{1}{2}D\alpha^2\int_0^t\int_0^s\left(\int_\Omega u^2\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z\right)\mathrm{d}\tau\,\mathrm{d}s, \tag{97}$$

for all $t\in[0,T_0]$. Now we can follow exactly the same procedure as in the proof of Theorem 3.1 to prove that the 3D model must develop a finite time blow-up before

$$T^*=\left(\frac{\alpha^3\pi D^2B}{12a}\right)^{-1/3}I_\infty. \tag{98}$$

Since $T^*<T_0$, we conclude that the solution must blow up before T^*. □

Remark 4.1. We can also prove the finite time blow-up of the 3D model with partial viscosity in a bounded domain following a similar argument. We omit the analysis here.

5. Blow-up of the 3D model with conservative boundary conditions

In this section, we will consider boundary conditions for ψ that will conserve energy. Under some additional condition, we can prove that the solution of the 3D model with conservative boundary conditions will also develop a finite time singularity.

5.1. Blow-up in a semi-infinite domain

Consider the following initial boundary value problem:

$$\begin{cases} u_t = 2u\psi_z, \\ -\Delta\psi_t = (u^2)_z, \end{cases} \quad (x,z) \in \Omega = \Omega_{\mathbf{x}} \times (0,\infty), \tag{99}$$

$$\psi|_{\partial\Omega\setminus\Gamma} = 0, \qquad \psi_z|_{\Gamma} = 0,$$

$$\psi|_{t=0} = \psi_0(\mathbf{x},z), \qquad u|_{t=0} = u_0(\mathbf{x},z) \geqslant 0, \tag{100}$$

where $\Omega_{\mathbf{x}} = (0,a)\times(0,a)$, and $\Gamma = \{(\mathbf{x},z)\in\Omega \mid \mathbf{x}\in\Omega_{\mathbf{x}},\ z=0\}$.

Theorem 5.1. *Assume that $u_0 \in H^2(\Omega)$, $u_0|_{\partial\Omega} = 0$, $u_0|_{\Omega} > 0$, $\psi_0 \in H^3(\Omega)$ and satisfies* (100). *Let*

$$\phi(\mathbf{x},z) = e^{-\alpha z}\sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a}, \quad (x,z)\in\Omega, \tag{101}$$

with $\alpha = \frac{\pi}{a}$, and

$$A = \int_{\Omega} (\log u_0)\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z, \qquad B = 2\int_{\Omega} \psi_{0z}\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z,$$

$$r(t) = \frac{4(\frac{\pi}{a})^2}{2(\frac{\pi}{a})^2 - \alpha^2}\int_{\Omega_{\mathbf{x}}} (\psi - \psi_0)|_{z=0}\sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a}\,\mathrm{d}\mathbf{x}.$$

If $A > 0$, $B > 0$ and $r(t) \leqslant \frac{B}{2}$ as long as u, ψ remain regular, then the solution of (99)–(100) *will develop a finite time singularity in the H^2 norm.*

Proof. First, by using an argument similar to the local well-posedness result in Theorem 2.1, we can prove that the system (99)–(100) is locally well-posed. We prove the theorem by contradiction. Assume that the initial boundary value problem has a globally smooth solution with $u \in C^1([0,\infty); H^2(\Omega))$ and $\psi \in C^1([0,\infty); H^3(\Omega))$. Multiplying ϕ_z to the both sides of the ψ-equation and integrating over Ω, we get

$$-\int_{\Omega} \Delta\psi_t\phi_z \,\mathrm{d}\mathbf{x}\,\mathrm{d}z = \int_{\Omega} (u^2)_z\phi_z \,\mathrm{d}\mathbf{x}\,\mathrm{d}z. \tag{102}$$

Note that $u|_{z=0} = 0$ since $u_0|_{z=0} = 0$. By integrating by parts and using the boundary condition of ψ and the property of ϕ, we have

$$-\int_{\Omega} \psi_{zt} \Delta\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z - \int_{\Omega_{\mathbf{x}}} \psi_{zt}\phi_z|_{z=0} \,\mathrm{d}\mathbf{x}\,\mathrm{d}z - \int_{\Omega_{\mathbf{x}}} \psi_t \Delta_{\mathbf{x}}\phi|_{z=0} \,\mathrm{d}\mathbf{x}\,\mathrm{d}z = \int_{\Omega} u^2\phi_{zz} \,\mathrm{d}\mathbf{x}\,\mathrm{d}z. \tag{103}$$

Substituting ϕ defined in (101) into the above equation, we have

$$\left(2\left(\frac{\pi}{a}\right)^2 - \alpha^2\right)\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} \psi_z\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z = \alpha^2\int_{\Omega} u^2\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z - 2\left(\frac{\pi}{a}\right)^2\int_{\Omega_{\mathbf{x}}} \psi_t|_{z=0}\,\phi_1(\mathbf{x})\,\mathrm{d}\mathbf{x}, \tag{104}$$

where $\phi_1(\mathbf{x}) = \sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a}$. Finally we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} \psi_z\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z = \frac{\alpha^2}{2(\frac{\pi}{a})^2 - \alpha^2}\int_{\Omega} u^2\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z - \frac{2(\frac{\pi}{a})^2}{2(\frac{\pi}{a})^2 - \alpha^2}\int_{\Omega_{\mathbf{x}}} \psi_t|_{z=0}\,\phi_1(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{105}$$

Next, we multiply ϕ to (8) and integrate over Ω. We get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} (\log u)\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z = 2\int_{\Omega} \psi_z\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z. \tag{106}$$

Combining (105) with (106), we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_{\Omega} (\log u)\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z = \frac{2\alpha^2}{2(\frac{\pi}{a})^2 - \alpha^2}\int_{\Omega} u^2\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z - \frac{4(\frac{\pi}{a})^2}{2(\frac{\pi}{a})^2 - \alpha^2}\int_{\Omega_{\mathbf{x}}} \psi_t|_{z=0}\,\phi_1(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{107}$$

Integrating the above equation in time and using the assumption that $r(t) \leqslant \frac{B}{2}$, we get

$$\begin{aligned}
\int_{\Omega} (\log u)\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z &= \frac{2\alpha^2}{2(\frac{\pi}{a})^2 - \alpha^2}\int_0^t\int_0^s\left(\int_{\Omega} u^2\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z\right)\mathrm{d}\tau\,\mathrm{d}s + A + Bt \\
&\quad - \frac{4(\frac{\pi}{a})^2}{2(\frac{\pi}{a})^2 - \alpha^2}\int_0^t\left(\int_{\Omega_{\mathbf{x}}} (\psi - \psi_0)|_{z=0}\,\phi_1(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)\mathrm{d}s \\
&\geqslant \frac{2\alpha^2}{2(\frac{\pi}{a})^2 - \alpha^2}\int_0^t\int_0^s\left(\int_{\Omega} u^2\phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}z\right)\mathrm{d}\tau\,\mathrm{d}s + A + \frac{1}{2}Bt.
\end{aligned} \tag{108}$$

Using (108), following the same argument as in the proof of Theorem 3.1, we can prove that the solution of the initial boundary value problem of the 3D model blows up in a finite time. □

5.2. Blow-up in a bounded domain

In this subsection, we will prove the finite time blow-up of the 3D model with a conservative boundary condition in a bounded domain. Specifically, we consider the following initial boundary value problem:

$$\begin{cases} u_t = 2u\psi_z, \\ -\Delta\psi_t = (u^2)_z, \end{cases} \quad (\mathbf{x}, z) \in \Omega = \Omega_{\mathbf{x}} \times (0, b), \tag{109}$$

$$\psi|_{\partial\Omega\setminus\Gamma} = 0, \qquad \psi_z|_\Gamma = 0,$$

$$\psi|_{t=0} = \psi_0(\mathbf{x}, z), \qquad u|_{t=0} = u_0(\mathbf{x}, z) \geqslant 0, \tag{110}$$

where $\mathbf{x} = (x_1, x_2)$, $\Omega_{\mathbf{x}} = (0, a) \times (0, a)$, $\Gamma = \{(\mathbf{x}, z) \in \Omega \mid \mathbf{x} \in \Omega_{\mathbf{x}},\ z = 0 \text{ or } z = b\}$.

The main result is stated in the following theorem.

Theorem 5.2. *Assume that $u_0 \in H^2(\Omega)$, $u_0|_{\partial\Omega} = 0$, $u_0|_\Omega > 0$, $\psi_0 \in H^3(\Omega)$ and satisfies* (110). *Let*

$$\phi(\mathbf{x}, z) = \frac{e^{-\alpha(z-b)} - e^{\alpha(z-b)}}{2} \sin\frac{\pi x_1}{a} \sin\frac{\pi x_2}{a}, \quad (\mathbf{x}, z) \in \Omega, \tag{111}$$

with $\alpha = \frac{\pi}{a}$, and

$$A = \int_\Omega (\log u_0)\phi \, \mathrm{d}\mathbf{x}\, \mathrm{d}z, \qquad B = 2\int_\Omega \psi_{0z}\phi \, \mathrm{d}\mathbf{x}\, \mathrm{d}z,$$

$$r(t) = \frac{2(\frac{\pi}{a})^2(e^{\alpha b} - e^{-\alpha b})}{2(\frac{\pi}{a})^2 - \alpha^2} \int_{\Omega_{\mathbf{x}}} (\psi - \psi_0)|_{z=0} \sin\frac{\pi x_1}{a} \sin\frac{\pi x_2}{a} \, \mathrm{d}\mathbf{x} \leqslant \frac{B}{2}.$$

If $A > 0$, $B > 0$ and $r(t) \leqslant \frac{B}{2}$ as long as u, ψ remain regular, then the solution of (109)–(110) *will develop a finite time singularity in the H^2 norm.*

Proof. Again, the local well-posedness of (109)–(110) can be established by using an argument similar to the proof of Theorem 2.1. We prove the theorem by contradiction. Assume that the initial boundary value problem has a globally smooth solution with $u \in C^1([0, \infty); H^2(\Omega))$ and $\psi \in C^1([0, \infty); H^3(\Omega))$. Multiplying ϕ_z to the both sides of the ψ-equation and integrating over Ω, we have

$$-\int_\Omega \Delta\psi_t \phi_z \, \mathrm{d}\mathbf{x}\, \mathrm{d}z = \int_\Omega (u^2)_z \phi_z \, \mathrm{d}\mathbf{x}\, \mathrm{d}z. \tag{112}$$

Note that $u|_\Gamma = 0$ as long as the solution remains regular. By integrating by parts and using the boundary condition of ψ and the property of ϕ, we get

$$-\int_\Omega \psi_{zt}\Delta\phi \, \mathrm{d}\mathbf{x}\, \mathrm{d}z + \int_{\Omega_{\mathbf{x}}} \psi_t \Delta_{\mathbf{x}}\phi|_{z=0}^{z=b} \, \mathrm{d}\mathbf{x}\, \mathrm{d}z = \int_\Omega u^2\phi_{zz} \, \mathrm{d}\mathbf{x}\, \mathrm{d}z. \tag{113}$$

Substituting ϕ to the above equation, we obtain

$$\left(2\left(\frac{\pi}{a}\right)^2-\alpha^2\right)\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \psi_z\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z = \alpha^2\int_\Omega u^2\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z-\left(\frac{\pi}{a}\right)^2\left(e^{\alpha b}-e^{-\alpha b}\right)\int_{\Omega_\mathbf{x}}\psi_t|_{z=0}\,\phi_1(\mathbf{x})\,\mathrm{d}\mathbf{x}, \tag{114}$$

where $\phi_1(\mathbf{x})=\sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a}$. Thus we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \psi_z\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z=\frac{\alpha^2}{2(\frac{\pi}{a})^2-\alpha^2}\int_\Omega u^2\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z-\frac{(\frac{\pi}{a})^2(e^{\alpha b}-e^{-\alpha b})}{2(\frac{\pi}{a})^2-\alpha^2}\int_{\Omega_\mathbf{x}}\psi_t|_{z=0}\,\phi_1(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{115}$$

Next, we multiply ϕ to (8) and integrate over Ω. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(\log u)\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z=2\int_\Omega\psi_z\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z. \tag{116}$$

Combining (115) with (116), we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_\Omega(\log u)\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z=\frac{2\alpha^2}{2(\frac{\pi}{a})^2-\alpha^2}\int_\Omega u^2\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z - \frac{2(\frac{\pi}{a})^2(e^{\alpha b}-e^{-\alpha b})}{2(\frac{\pi}{a})^2-\alpha^2}\int_{\Omega_\mathbf{x}}\psi_t|_{z=0}\,\phi_1(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{117}$$

Integrating the above equation in time and using the assumption that $r(t)\leqslant\frac{B}{2}$, we get

$$\begin{aligned}\int_\Omega(\log u)\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z&=\frac{2\alpha^2}{2(\frac{\pi}{a})^2-\alpha^2}\int_0^t\int_0^s\left(\int_\Omega u^2\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z\right)\mathrm{d}\tau\,\mathrm{d}s+A+Bt\\&\quad-\frac{2(\frac{\pi}{a})^2(e^{\alpha b}-e^{-\alpha b})}{2(\frac{\pi}{a})^2-\alpha^2}\int_0^t\left(\int_{\Omega_\mathbf{x}}(\psi-\psi_0)|_{z=0}\,\phi_1(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)\mathrm{d}s\\&\geqslant\frac{2\alpha^2}{2(\frac{\pi}{a})^2-\alpha^2}\int_0^t\int_0^s\left(\int_\Omega u^2\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}z\right)\mathrm{d}\tau\,\mathrm{d}s+A+\frac{1}{2}Bt.\end{aligned} \tag{118}$$

Using (118) and following the same argument as in the proof of Theorem 3.1, we can prove that the solution of the 3D model will develop a finite time singularity in the H^2 norm. □

5.3. Blow-up of the 3D model with other conservative boundary conditions

The singularity analysis we present in the previous subsection can be generalized to study the finite time blow-up of the 3D model with the same boundary condition along the x_1 and x_2 directions as in Section 5.2, but changing the Neumann boundary condition along the z-direction to a periodic boundary condition. The assumption on u_0 and ψ_0 remains the same as in Section 5.2. In this case, we can prove the finite time blow-up of the corresponding initial boundary value problem with two minor modifications in the statement of the blow-up theorem. The first change is to replace ϕ by the following definition:

$$\phi(\mathbf{x}, z) = \frac{e^{-\alpha z} + e^{-\alpha(z-b)}}{2} \sin\frac{\pi x_1}{a} \sin\frac{\pi x_2}{a}, \quad (\mathbf{x}, z) \in \Omega = (0, a) \times (0, a) \times (0, b), \tag{119}$$

with $\alpha = \frac{\pi}{a}$. The second change is to modify the definition of $r(t)$ as follows:

$$r(t) = \frac{2\alpha(1 - e^{-\alpha b})}{2(\frac{\pi}{a})^2 - \alpha^2} \int\limits_{\Omega_{\mathbf{x}}} (\psi_z - \psi_{0z})|_{z=0} \sin\frac{\pi x_1}{a} \sin\frac{\pi x_2}{a}\, d\mathbf{x} \leqslant \frac{B}{2},$$

where A and B are the same as in Theorem 5.2. If $A > 0$, $B > 0$ and $r(t) \leqslant \frac{B}{2}$ as long as u, ψ remain regular, then we can prove that the solution of the corresponding initial boundary value problem will develop a finite time singularity in the H^2 norm.

The same singularity analysis can be applied to study the finite time blow-up of the 3D model with the same boundary condition along the x_1 and x_2 directions as in Section 5.2, but changing the Neumann boundary condition along the z-direction to the Dirichlet boundary condition. The assumption on u_0 and ψ_0 remains the same as in Section 5.2. In this case, we can prove the finite time blow-up of the corresponding initial boundary value problem with two minor modifications in the statement of the blow-up theorem. The first change is to replace ϕ by the following definition:

$$\phi(\mathbf{x}, z) = \frac{e^{\alpha(z-b)} + e^{-\alpha(z-b)}}{2} \sin\frac{\pi x_1}{a} \sin\frac{\pi x_2}{a}, \quad (\mathbf{x}, z) \in \Omega = (0, a) \times (0, a) \times (0, b), \tag{120}$$

with $\alpha = \frac{\pi}{a}$. The second change is to modify the definition of $r(t)$ as follows:

$$r(t) = \frac{\alpha(e^{\alpha b} - e^{-\alpha b})}{2(\frac{\pi}{a})^2 - \alpha^2} \int\limits_{\Omega_{\mathbf{x}}} (\psi_z - \psi_{0z})|_{z=0} \sin\frac{\pi x_1}{a} \sin\frac{\pi x_2}{a}\, d\mathbf{x} \leqslant \frac{B}{2},$$

where A and B are the same as in Theorem 5.2. If $A > 0$, $B > 0$ and $r(t) \leqslant \frac{B}{2}$ as long as u, ψ remain regular, then we can prove that the solution of the corresponding initial boundary value problem will develop a finite time singularity in the H^2 norm.

Remark 5.1. All the results in this section can be generalized to a cylindrical domain Ω in high dimension space $\mathbb{R}^N$, with $\Omega = \{(\mathbf{x}, z) \mid \mathbf{x} \in \Omega_{\mathbf{x}} \subset \mathbb{R}^{N-1},\ z \in [a, b] \subset \mathbb{R}\}$. In this case, the weight function $\phi(\mathbf{x}, z)$ is chosen to be the product of two functions:

$$\phi(\mathbf{x}, z) = \phi_1(\mathbf{x})\eta(z). \tag{121}$$

Here the eigen-function, $\eta(z)$, is the same in the Eulerian coordinate in the previous sections. The eigen-function, $\phi_1(\mathbf{x})$, defined in the $\mathbf{x}$ space, is chosen to be the first eigen-function of the following eigenvalue problem:

$$-\Delta_{\mathbf{x}}\phi_1 = \lambda\phi_1, \tag{122}$$

$$\phi_1|_{\partial\Omega_{\mathbf{x}}} = 0, \tag{123}$$

with $\lambda > 0$, where $\Delta_{\mathbf{x}}$ is the $(N-1)$-dimensional Laplace operator, $\Delta_{\mathbf{x}} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{N-1}^2}$.

6. Global regularity of the 3D inviscid model with small data

In this section, we will prove the global regularity of the 3D inviscid model for a class of small initial data with some appropriate boundary condition. We remark that since we consider the inviscid version of the 3D model, there is no viscosity in the model equation. Although we impose some smallness condition on the initial data, such result is still very interesting since there is currently no global regularity result for the 3D incompressible Euler equations even for small initial data.

To simplify the presentation of our analysis, we use u^2 and ψ_z as our new variables. We will define $v = \psi_z$ and still use u to stand for u^2. Then the 3D model now has the form:

$$\begin{cases} u_t = 4uv, \\ -\Delta v_t = u_{zz}, \end{cases} \quad (\mathbf{x}, z) \in \Omega = (0,\delta) \times (0,\delta) \times (0,\delta). \tag{124}$$

We choose the following boundary condition for v:

$$v|_{\partial\Omega} = -4, \tag{125}$$

and denote $v|_{t=0} = v_0(\mathbf{x}, z)$ and $u|_{t=0} = u_0(\mathbf{x}, z) \geqslant 0$.

In our regularity analysis, we need to use the following Sobolev inequality [14]:

Lemma 6.1. *For all $s \in \mathbb{Z}^+$, there exists $C_s > 0$, such that, for all $u, v \in L^\infty \cap H^s(\mathbb{R}^N)$,*

$$\left(\sum_{0\leqslant|\alpha|\leqslant s} \left\|\partial^\alpha(uv) - \partial^\alpha u \cdot v\right\|_{L^2}^2\right)^{1/2} \leqslant C_s\big(\|u\|_{L^\infty}\|v\|_{H^s} + \|\nabla v\|_{L^\infty}\|u\|_{H^{s-1}}\big). \tag{126}$$

Now we state the main result of this section.

Theorem 6.1. *Assume that $u_0, v_0 \in H^s(\Omega)$ with $s \geqslant 4$, $u_0|_{\partial\Omega} = 0$, $v_0|_{\partial\Omega} = -4$ and $v_0 \leqslant -4$ over Ω, then the solution of* (124)–(125) *remains regular in $H^s(\Omega)$ for all time as long as the following holds*

$$\delta(4C_s + 1)\big(\|v_0\|_{H^s} + C_s\|u_0\|_{H^s}\big) < 1, \tag{127}$$

where C_s is an interpolation constant. Moreover, we have $\|u\|_{L^\infty} \leqslant \|u_0\|_{L^\infty}e^{-7t}$, $\|u\|_{H^s(\Omega)} \leqslant \|u_0\|_{H^s(\Omega)}e^{-7t}$ and $\|v\|_{H^s(\Omega)} \leqslant C$ for some constant C which depends on u_0, v_0 and s only.

Proof. First of all, we note that v_t satisfies the homogeneous boundary condition on $\partial\Omega$ since $v = -4$ on $\partial\Omega$. Let $K = (-\Delta)^{-1}$ be the inverse Laplacian operator with homogeneous Dirichlet boundary condition. Then, we can rewrite (124) as follows:

$$\begin{cases} u_t = 4uv, \\ v_t = K(u_{zz}), \end{cases} \qquad (\mathbf{x}, z) \in \Omega = (0,\delta) \times (0,\delta) \times (0,\delta). \tag{128}$$

Standard elliptic theory implies that K is a linear bounded operator from $H^{s-2}(\Omega)$ to $H^s(\Omega)$, that is, for any $f \in H^{s-2}(\Omega)$, we have

$$\big\|K(f)\big\|_{H^s(\Omega)} \leqslant C_s\|f\|_{H^{s-2}(\Omega)}, \tag{129}$$

for $s \geqslant 2$. Such estimate can be also obtained directly by using an argument similar to the proof of Lemma 2.1.

Next, we define $V^s = \{v \in H^s(\Omega)\colon v|_{\partial\Omega} = -4\}$. Since $s \geqslant 4$, the trace of v on $\partial\Omega$ is well defined. Let $X := H^s(\Omega) \times V^s(\Omega)$ be a Banach space with the norm $\|\cdot\|_X$ of the space X defined as follows:

$$\|U\|_X = \|U_1\|_{H^s(\Omega)} + \|U_2\|_{H^s(\Omega)}.$$

Further we express the system (128) as an ODE in the Banach space X:

$$U_t = F(U), \tag{130}$$

where $U = (U_1, U_2) = (u, v)$ and $F(U) = (F_1(U), F_2(U)) = (4uv, K(u_{zz}))$.

We note that $K\partial_{zz}$ is a bounded linear operator from $H^s(\Omega)$ to $H^s(\Omega)$. By using an argument similar to the local well-posedness analysis presented in Section 2.1, we can show that the system (128) is locally well-posed and there exists $T_0 > 0$ such that $\|u\|_{H^s}$ and $\|v\|_{H^s}$ are bounded for $0 \leqslant t \leqslant T_0$. Furthermore, by using Lemma 2.2 and the fact that $K\partial_{zz}$ is a bounded operator from H^s to H^s, we can easily obtain the following *a priori* estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|U\|_X \leqslant C_s\|U\|_X^2,$$

for $0 \leqslant t \leqslant T_0$, which implies that $\|U\|_X$ is bounded by a constant M that depends on $\|U_0\|_X$ only for $0 \leqslant t \leqslant \overline{T}_0 < \min(T_0, 1/(C_s\|U_0\|_X))$.

On the other hand, since $K\partial_{zz}$ is a bounded operator from H^s to H^s, we obtain by standard energy estimates that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{H^s(\Omega)} \leqslant C_s\|u\|_{H^s(\Omega)} \leqslant C_s M\big(\|u_0\|_{H^s(\Omega)}, \|v_0\|_{H^s(\Omega)}\big),$$

from which we conclude that $\|v(t)\|_{H^s(\Omega)}$ can be made as close to $\|v_0\|_{H^s(\Omega)}$ as we wish for $0 \leqslant t \leqslant \overline{T}_0$ by making $\overline{T}_0$ small enough. Similarly, since $s \geqslant 4$, we have by using the Sobolev embedding theorem and the *a priori* estimate that

$$\|v_t\|_{L^\infty(\Omega)} \leqslant C_0\|v_t\|_{H^s(\Omega)} \leqslant C_0\big\|K(u_{zz})\big\|_{H^s(\Omega)} \leqslant C_s M\big(\|u_0\|_{H^s(\Omega)}, \|v_0\|_{H^s(\Omega)}\big).$$

Thus we can also make $\|v(t) - v_0\|_{L^\infty(\Omega)}$ as small as we wish for $0 \leqslant t \leqslant \overline{T}_0$ by making $\overline{T}_0$ small enough.

Note that (127) implies that $2C_s\delta\|v_0\|_{H^s} < \frac{1}{2}$. By our assumption, we also have $v_0 \leqslant -4$ in Ω. Based on the above argument, we can choose $\overline{T}_0$ small enough so that we have $v(t) < -2$ on Ω, and $2C_s\delta\|v(t)\|_{H^s} < 1$ for $0 \leqslant t < \overline{T}_0$.

Let $[0, T)$ be the largest time interval on which $\|u\|_{H^s}$ and $\|v\|_{H^s}$ are bounded, and both of the following inequalities hold:

$$v \leqslant -2 \quad \text{over } \Omega, \qquad 2C_s\delta\|v\|_{H^s} \leqslant 1.$$

We will show that $T = \infty$.

For $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_j \geqslant 0$ $(j = 1, 2, 3)$ and $|\alpha| \leqslant s$, we have for $0 \leqslant t < T$ that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle \partial^\alpha u, \partial^\alpha u\rangle &= 8\langle \partial^\alpha (uv), \partial^\alpha u\rangle \\
&= 8\langle \partial^\alpha u \cdot v, \partial^\alpha u\rangle + 8\langle \partial^\alpha (uv) - \partial^\alpha u \cdot v, \partial^\alpha u\rangle \\
&= 8\int_\Omega |\partial^\alpha u|^2 v \,\mathrm{d}\mathbf{x}\,\mathrm{d}z + 8\langle \partial^\alpha (uv) - \partial^\alpha u \cdot v, \partial^\alpha u\rangle \\
&\leqslant -16\int_\Omega |\partial^\alpha u|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}z + 8\|\partial^\alpha (uv) - \partial^\alpha u \cdot v\|_{L^2}\|\partial^\alpha u\|_{L^2}.
\end{aligned} \tag{131}$$

Using Lemma 6.1, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^s} \leqslant -8\|u\|_{H^s} + C_s\big(\|u\|_{L^\infty}\|v\|_{H^s} + \|\nabla v\|_{L^\infty}\|u\|_{H^{s-1}}\big). \tag{132}$$

Since $u|_{\partial\Omega} = u_0|_{\partial\Omega} = 0$, we obtain

$$\begin{aligned}
u(\mathbf{x}, z, t) &= \int_0^z \partial_{z'} u(\mathbf{x}, z', t)\,\mathrm{d}z' \\
&= \int_0^z \int_0^{x_1} \int_0^{x_2} \partial_{x_1'}\partial_{x_2'}\partial_{z'} u(x_1', x_2', z', t)\,\mathrm{d}x_1'\,\mathrm{d}x_2'\,\mathrm{d}z' \\
&\leqslant \delta^{3/2}\|\partial_{x_1}\partial_{x_2}\partial_z u\|_{L^2} \leqslant \delta\|u\|_{H^s},
\end{aligned} \tag{133}$$

since $s \geqslant 4$. Notice that $v_{x_i}|_{z=0} = 0$, so we have

$$v_{x_i} = \int_0^z v_{x_i z'}\,\mathrm{d}z' \leqslant \int_0^\delta |v_{x_i z'}|\,\mathrm{d}z' \leqslant \delta\|v_{x_i z}\|_{L^\infty}. \tag{134}$$

Similarly, since $v_z|_{x_1=0} = 0$, we have

$$v_z = \int_0^{x_1} v_{x_1' z}\,\mathrm{d}x_1' \leqslant \int_0^{\delta} |v_{x_1' z}|\,\mathrm{d}x_1' \leqslant \delta \|v_{x_1 z}\|_{L^\infty}. \tag{135}$$

Combining (134) with (135), we get

$$\|\nabla v\|_{L^\infty} \leqslant \delta \max_{i=1,2}\big(\|v_{x_i z}\|_{L^\infty}\big). \tag{136}$$

Since $s \geqslant 4 > 2 + 3/2$ by our assumption, we obtain by using the Sobolev embedding theorem [14] that

$$\|v_{x_i z}\|_{L^\infty} \leqslant C_s \|v_{x_i z}\|_{H^{s-2}} \leqslant C_s \|v\|_{H^s}. \tag{137}$$

It follows from (136) and (137) that

$$\|\nabla v\|_{L^\infty} \leqslant C_s \delta \|v\|_{H^s}. \tag{138}$$

Combining (132)–(133) with (138), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^s} \leqslant \big(-8 + 2C_s\delta\|v\|_{H^s}\big)\|u\|_{H^s}. \tag{139}$$

Since $2C_s\delta\|v\|_{H^s} \leqslant 1$ for $t < T$ by the assumption of T, we have for $t < T$ that

$$\|u\|_{H^s} \leqslant \|u_0\|_{H^s} e^{-7t}. \tag{140}$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\langle \partial^\alpha v, \partial^\alpha v\big\rangle = 2\big\langle \partial^\alpha v_t, \partial^\alpha v\big\rangle \leqslant 2\big\|\partial^\alpha v_t\big\|_{L^2}\big\|\partial^\alpha v\big\|_{L^2}. \tag{141}$$

Recall that $\Delta v_t = u_{zz}$. We can easily generalize the proof of Lemma 2.1 to show that

$$\|v_t\|_{H^s(\Omega)} \leqslant C_s \|u_{zz}\|_{H^{s-2}(\Omega)} \leqslant C_s \|u\|_{H^s(\Omega)}. \tag{142}$$

Using (142), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{H^s}^2 \leqslant 2\|v_t\|_{H^s}\|v\|_{H^s} \leqslant 2C_s\|u\|_{H^s}\|v\|_{H^s}. \tag{143}$$

Substituting (140) to the above equations, we get for $t < T$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{H^s} \leqslant C_s\|u\|_{H^s} \leqslant C_s\|u_0\|_{H^s} e^{-7t}. \tag{144}$$

Integrating the above inequality in time, we obtain the estimate of $\|v\|_{H^s}$ over $[0, T)$:

$$\|v\|_{H^s} \leqslant \|v_0\|_{H^s} + C_s\|u_0\|_{H^s} \int_0^t e^{-7s}\,\mathrm{d}s$$
$$\leqslant \|v_0\|_{H^s} + \frac{C_s}{7}\|u_0\|_{H^s} \leqslant \|v_0\|_{H^s} + C_s\|u_0\|_{H^s}. \tag{145}$$

Since $v|_{\partial\Omega} = -4$, we can use the same argument as in the proof of (133) to show that

$$|v+4| \leqslant \delta\|v\|_{H^s} \leqslant \delta\big(\|v_0\|_{H^s} + C_s\|u_0\|_{H^s}\big), \tag{146}$$

where we have used (145). Now we have for $t < T$ that

$$v \leqslant -4 + \delta\big(\|v_0\|_{H^s} + C_s\|u_0\|_{H^s}\big),$$
$$2C_s\delta\|v\|_{H^s} \leqslant 2C_s\delta\big(\|v_0\|_{H^s} + C_s\|u_0\|_{H^s}\big).$$

By our assumption on the initial data, we have

$$\delta(4C_s+1)\big(\|v_0\|_{H^s} + C_s\|u_0\|_{H^s}\big) < 1. \tag{147}$$

Therefore, we have proved that if

$$v \leqslant -2 \quad \text{on } \Omega \quad \text{and} \quad 2C_s\delta\|v\|_{H^s} \leqslant 1, \quad 0 \leqslant t < T, \tag{148}$$

then we actually have

$$v \leqslant -3 \quad \text{on } \Omega \quad \text{and} \quad 2C_s\delta\|v\|_{H^s} \leqslant \frac{1}{2}, \quad 0 \leqslant t < T. \tag{149}$$

This implies that we can extend the time interval beyond T so that (148) is still valid. This contradicts the assumption that $[0, T)$ is the largest time interval on which (148) is valid. This contradiction shows that T cannot be a finite number, i.e. (148) is true for all time. This implies that $\|u\|_{H^s(\Omega)}$ and $\|v\|_{H^s(\Omega)}$ are bounded for all time. Moreover, we have shown that $\|u\|_{L^\infty} \leqslant \|u_0\|_{L^\infty}e^{-7t}$, $\|u\|_{H^s(\Omega)} \leqslant \|u_0\|_{H^s(\Omega)}e^{-7t}$ and $\|v\|_{H^s(\Omega)} \leqslant \|v_0\|_{H^s} + C_s\|u_0\|_{H^s}$. □

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Appendix A

Proof of Lemma 2.1. We present the proof for the case of $a = \pi$. The case of $a \neq \pi$ can be proved similarly. First, we perform the sine transform along x_1 and x_2 directions to both sides of (21). We have

$$|k|^2 \hat{v}(k,z) - \hat{v}_{zz}(k,z) = \hat{f}(k,z), \tag{A.1}$$

where $k = (k_1, k_2)$, $|k| = \sqrt{k_1^2 + k_2^2}$, and the sine transform of v is defined as follows:

$$\hat{v}(k,z) = \left(\frac{2}{\pi}\right)^2 \int_0^\pi \int_0^\pi v(x_1, x_2, z) \sin(k_1 x_1) \sin(k_2 x_2) \, \mathrm{d}x_1 \, \mathrm{d}x_2. \tag{A.2}$$

Applying the sine transform to the boundary condition gives

$$\big(\hat{v}_z(k,z) + \beta \hat{v}(k,z)\big)\big|_{z=0} = 0. \tag{A.3}$$

The second order ODE (A.1) can be solved analytically. The general solution is given by

$$\hat{v}(k,z) = \frac{e^{|k|z}}{|k|}\left(-\frac{1}{2}\int_0^z \hat{f} e^{-|k|z'} \, \mathrm{d}z' + C_1(k)\right) + \frac{e^{-|k|z}}{|k|}\left(\frac{1}{2}\int_0^z \hat{f} e^{|k|z'} \, \mathrm{d}z' + C_2(k)\right). \tag{A.4}$$

The boundary condition (A.3) and the constraint that $v \in L^2(\Omega)$ determine the constants C_1 and C_2 uniquely as follows:

$$C_1(k) = \frac{1}{2}\int_0^\infty \hat{f}(k,z) e^{-|k|z'} \, \mathrm{d}z', \qquad C_2(k) = \frac{|k| + \beta}{|k| - \beta} C_1(k). \tag{A.5}$$

Let $\chi(x)$ be the characteristic function

$$\chi(x) = \begin{cases} 0, & x \leqslant 0, \\ 1, & x > 0. \end{cases} \tag{A.6}$$

Then $\hat{v}(k,z)$ has the following integral representation (note that $\beta \neq |k|$ by our assumption):

$$\hat{v}(k,z) = -\frac{1}{2|k|}\int_0^\infty \hat{f}(k,z) e^{-|k|(z'-z)} \chi\big(z'-z\big) \, \mathrm{d}z' + \frac{1}{2|k|}\int_0^\infty \hat{f}(k,z) e^{-|k|(z-z')} \chi\big(z-z'\big) \, \mathrm{d}z'$$
$$+ \frac{|k| + \beta}{|k|(|k| - \beta)} \int_0^\infty \hat{f}(k,z) e^{-|k|(z+z')} \, \mathrm{d}z'$$

$$= -\frac{1}{2|k|}\int_0^\infty \hat{f}(k,z)K_1(z'-z)\,\mathrm{d}z' + \frac{1}{2|k|}\int_0^\infty \hat{f}(k,z)K_1(z-z')\,\mathrm{d}z' + \frac{|k|+\beta}{|k|(|k|-\beta)}\int_0^\infty \hat{f}(k,z)K_2(z+z')\,\mathrm{d}z', \tag{A.7}$$

where $K_1(z) = e^{-|k|z}\chi(z)$, $K_2(z) = e^{-|k|z}$. Using Young's inequality (see e.g. page 232 of [13]), we obtain:

$$\begin{aligned}\|\hat{v}(k,\cdot)\|_{L^2[0,\infty)} &\leqslant \frac{1}{2|k|}\left(2\|K_1\|_{L^1[0,\infty)} + 2\left|\frac{|k|+\beta}{|k|-\beta}\right|\|K_2\|_{L^1[0,\infty)}\right)\|\hat{f}(k,\cdot)\|_{L^2[0,\infty)} \\ &\leqslant \frac{1}{|k|^2}\left(1+\left|\frac{|k|+\beta}{|k|-\beta}\right|\right)\|\hat{f}(k,\cdot)\|_{L^2[0,\infty)} \leqslant \frac{M}{|k|^2}\|\hat{f}(k,\cdot)\|_{L^2[0,\infty)},\end{aligned} \tag{A.8}$$

where $M = \max_{k_1,k_2>0}(1+|\frac{|k|+\beta}{|k|-\beta}|) < \infty$ since $\beta \neq |k|$ for any $k \in \mathbb{Z}^2$ by our assumption.

Next, we estimate $\hat{v}_z(k,z)$. Differentiating (A.4) with respect to z, we get

$$\hat{v}_z(k,z) = -\frac{1}{2}\int_z^\infty \hat{f}(k,z)e^{-|k|(z'-z)}\,\mathrm{d}z' - \frac{1}{2}\int_0^z \hat{f}(k,z)e^{-|k|(z-z')}\,\mathrm{d}z' - \frac{|k|+\beta}{|k|-\beta}\int_0^\infty \hat{f}(k,z)e^{-|k|(z+z')}\,\mathrm{d}z'. \tag{A.9}$$

Following the same procedure as in our estimate for $\hat{v}(k,z)$, we obtain a similar estimate for $\hat{v}_z(k,z)$:

$$\|\hat{v}_z(k,\cdot)\|_{L^2[0,\infty)} \leqslant \frac{1}{|k|}\left(1+\left|\frac{|k|+\beta}{|k|-\beta}\right|\right)\|\hat{f}(k,\cdot)\|_{L^2[0,\infty)} \leqslant \frac{M}{|k|}\|\hat{f}(k,\cdot)\|_{L^2[0,\infty)}. \tag{A.10}$$

Let $\alpha = (\alpha_1,\alpha_2,\alpha_3)$ with $\alpha_j \geqslant 0$ $(j=1,2,3)$. We will prove $\|\partial^\alpha v\|_{L^2}^2 \leqslant M^2\|f\|_{H^{|\alpha|-2}}^2$ for all $|\alpha| \geqslant 2$. We will prove this using an induction argument on α_3. First, we establish this estimate for $\alpha_3 = 0$ and $\alpha_1+\alpha_2 \geqslant 2$. Below we use the case of $\alpha_1 \geqslant 1$ and $\alpha_2 \geqslant 1$ as an example to illustrate the main idea. By using the Parseval equality and (A.8), we obtain

$$\begin{aligned}\|\partial^\alpha v\|_{L^2(\Omega)}^2 &= \sum_{k_1=1}^\infty\sum_{k_2=1}^\infty k_1^{2\alpha_1}k_2^{2\alpha_2}\int_0^\infty |\hat{v}(k,z)|^2\,\mathrm{d}z \\ &= \sum_{k_1=1}^\infty\sum_{k_2=1}^\infty k_1^{2\alpha_1}k_2^{2\alpha_2}\|\hat{v}(k,\cdot)\|_{L^2[0,\infty)}^2 \\ &\leqslant \sum_{k_1=1}^\infty\sum_{k_2=1}^\infty M^2k_1^{2\alpha_1}k_2^{2\alpha_2}|k|^{-4}\|\hat{f}(k,\cdot)\|_{L^2[0,\infty)}^2\end{aligned}$$

$$\leqslant M^2 \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \left\| k_1^{\alpha_1-1} k_2^{\alpha_2-1} \hat{f}(k,\cdot) \right\|_{L^2[0,\infty)}^2$$

$$= M^2 \left\| \partial_x^{\alpha_1-1} \partial_y^{\alpha_2-1} f \right\|_{L^2(\Omega)}^2 \leqslant M^2 \| f \|_{H^{|\alpha|-2}(\Omega)}^2. \tag{A.11}$$

Similarly, we can prove (A.11) for $\alpha_3 = 0$ and $\alpha_1 + \alpha_2 \geqslant 2$ by distributing the appropriate order of derivatives to x_1 and/or x_2 direction.

Using (A.10) and following the same procedure as in the proof of (A.11), we can prove (A.11) for the case of $\alpha_3 = 1$ and $\alpha_1 + \alpha_2 \geqslant 1$. Finally, using (A.1) and differentiating (A.1) with respect to z as many times as needed, we can prove

$$\left\| \partial^\alpha v \right\|_{L^2(\Omega)}^2 \leqslant C_\alpha \| f \|_{H^{|\alpha|-2}(\Omega)}^2, \tag{A.12}$$

for all $\alpha_3 \geqslant 2$ and $\alpha_1 + \alpha_2 \geqslant 0$ by using an induction argument and (A.11) for $\alpha_3 = 0$ and $\alpha_3 = 1$. Using (A.12) and (A.7), we obtain

$$\| v \|_{H^s(\Omega)} \leqslant C_s \| f \|_{H^{s-2}(\Omega)}, \tag{A.13}$$

for all $s \geqslant 2$, where C_s is a constant depending only on s. The uniqueness of the solution follows from the solution formula (A.4) and (A.5). This completes the proof of Lemma 2.1. □

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