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Multiscale Domain Decomposition Methods for Elliptic Problems with High Aspect Ratios

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Abstract In this paper we study some nonoverlapping domain decomposition methods for solving a class of elliptic problems arising from composite materials and flows in porous media which contain many spatial scales. Our preconditioner differs from traditional domain decomposition preconditioners by using a coarse solver which is adaptive to small scale heterogeneous features. While the convergence rate of traditional domain decomposition algorithms using coarse solvers based on linear or polynomial interpolations may deteriorate in the presence of rapid small scale oscillations or high aspect ratios, our preconditioner is applicable to multiple-scale problems without restrictive assumptions and seems to have a convergence rate nearly independent of the aspect ratio within the substructures. A rigorous convergence analysis based on the Schwarz framework is carried out, and we demonstrate the efficiency and robustness of the proposed preconditioner through numerical experiments which include problems with multiple-scale coefficients, as well problems with continuous scales.

Keywords Multiscale elliptic problems, Domain decomposition, Schwarz methods, Porous media
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1 Introduction

Many problems of fundamental and practical importance have multiple-scale solutions. Typical examples include transport of flows in strongly heterogeneous media and heat conduction in composite materials. When applying conventional domain decomposition methods to these problems using linear or polynomial interpolations, the convergence rate deteriorates because the coarse grid solver does not account for fine scale heterogeneous features. To attain a satisfactory convergence rate it is therefore important to construct a coarse grid solver which reflects the small scale structures. Such a solver has been developed by Hou et al.[4,5] who introduced the Multiscale Finite Element Method (MsFEM). The basic idea behind the MsFEM is to construct base functions which are adaptive to the local property of the differential operator and contain the important subgrid information.

An important property with the MsFEM solver is that the coarse space is "generalized" discrete harmonic with respect to the physical elliptic operator that contains small scale coefficients. From a theoretical point of view, this property implies that the MsFEM solver is in a sense optimal within a general class of coarse solvers. Furthermore it allows us to interpret the MsFEM solver as a natural extension of coarse solvers using discrete harmonic coarse spaces to

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problems which contain multiple-scale coefficients. In particular, if the coarse grid is a triangulation of the physical domain and the elliptic coefficients are quasi-homogeneous, i.e. constant on each coarse grid element, then the corresponding MsFEM using linear boundary conditions to construct the multiscale base functions simply reduces to standard linear finite elements.

Our main objective in this paper is to develop, analyze and test a class of nonoverlapping domain decomposition methods using multiscale coarse grid solvers. These methods fall into the category of Schwarz methods, and the main steps in our analysis is based on the general abstract framework for the analysis of Schwarz methods, see [3,8,11,15,16]. We first demonstrate that the MsFEM induces an ideal nonoverlapping domain decomposition preconditioner in 1D which converges in one iteration. In 2D and 3D the ability to select proper boundary conditions for the multiscale base functions will be important to achieve a fast convergence rate. We derive condition number estimates for the multiscale domain decomposition preconditioner by splitting the convergence rate into a homogenized part, which essentially depends on the selection of boundary conditions for the base functions, and a multiscale part which depends on the nature of the heterogeneous structures within the coarse grid elements. Our analysis shows that the multiscale preconditioner almost achieves the same rate of convergence for multiscale elliptic problems as traditional preconditioners with "discrete harmonic" coarse solvers achieve for elliptic problems with quasi-homogeneous coefficients. The extra multiscale factor has a relatively weak dependence on the elliptic coefficients, and is hard to observe in practice.

We perform a series of numerical experiments to test the performance of our preconditioner for elliptic partial differential equations in two dimensions. We choose the coarse grid and the boundary conditions for the multiscale base functions so that the MsFEM solver is the "multiscale extension" of bilinear finite elements. The elliptic coefficient function is chosen to be the product of a quasi-homogeneous coefficient function and a periodic oscillatory coefficient function. We demonstrate that the MsFEM induced preconditioner proposed in this paper shows a logarithmic dependence on the mesh ratio H/h and is almost insensitive to the local aspect ratios (for aspect ratios as high as 10^{10}). This confirms that the rate of convergence of our preconditioner for elliptic problems with high aspect ratios is essentially the same as standard nonoverlapping domain decomposition methods using conventional conforming finite element coarse solvers achieve for elliptic problems with quasi-homogeneous coefficients.

We compare our preconditioner with the preconditioners obtained by replacing the MsFEM solver with the linear and bilinear finite element solvers. The convergence behavior for these preconditioners may deteriorate rapidly if the aspect ratio within the coarse grid elements blows up. As we are not aware of any other coarse solvers which successfully handles high aspect ratios, the linear and bilinear finite element solvers serve the purpose of illustrating the need for coarse solvers which are adaptive to the small scale structures.

The paper is organized as follows. In Section 2 we define the model problem and the MsFEM, and outline the abstract Schwarz framework. In Section 3 we present the domain decomposition preconditioner and provide the convergence analysis. The numerical results are reported in Section 4 and we conclude with some remarks and directions for further work in Section 5.

2 Mathematical Formulations

In Section 2.1 we introduce the elliptic model problem and outline the multiscale finite element method in its original form which was defined and analyzed in [4,5]. The selection of boundary functions for the multiscale base functions is addressed in Section 2.2. In Section 2.3 we give some general remarks and show that the MsFEM solver is an ideal preconditioner for the Schur compliment in 1D. Finally, in Section 2.4 we outline the abstract framework for the analysis of Schwarz methods and give the abstract form of our preconditioner.

2.1 Governing Equations and the MsFEM

We consider solving the second-order elliptic equation

$$
\begin{aligned}
-\nabla \cdot \big(a(x)\nabla u\big) &= f \qquad \text{in } \Omega \subset \mathcal{R}^d, && (1)\\
u &= 0 \qquad \text{on } \partial\Omega, && (2)
\end{aligned}
$$

where $a(x) = (a_{ij}(x))$ is the conductivity tensor, assumed to be symmetric and positive definite with upper and lower bounds. We also assume that Ω is a polygon if $d = 2$ and a polyhedron if $d = 3$. Eq. (1) may represent single-phase porous media flow or steady state heat conduction through a composite material. These are typical examples of problems where $a(x)$ can be highly oscillatory and the solution of (1)–(2) displays a multiple-scale structure.

The variational formulation of (1)–(2) is to seek $u \in H_0^1(\Omega)$ such that

$$a(u, v) := \int_\Omega (\nabla u)^T a \nabla v \, dx = \int_\Omega f v \, dx := f(v), \qquad \forall\, v \in H_0^1(\Omega). \tag{3}$$

In the conforming finite element method the approximate solution u^h is sought in a finite dimensional subspace $V^h \subset H_0^1(\Omega)$, i.e. we seek $u^h \in V^h$ such that

$$a(u^h, v) = f(v), \qquad \forall\, v \in V^h. \tag{4}$$

We assume linear finite elements is applied for the fine mesh discretization. Thus, let $\mathcal{T}^h = \{\tau\}$ be a quasi-uniform triangulation of Ω with mesh parameter h and let V^h be the space of piecewise linear functions determined by its values at the triangle (tetrahedral) vertices $\mathcal{N}_\mathcal{T}$. Furthermore, let $\mathcal{K}^H = \{K\}$ be a quasi-uniform partitioning of Ω with mesh parameter H and a corresponding set of nodal points $\mathcal{N}_\mathcal{K} \subset \mathcal{N}_\mathcal{T} \cap \Gamma$, $\Gamma = \underset{K\in\mathcal{K}^H}{\cup} \partial K$, such that $\mathcal{T}^h$ also forms a triangulation of each $K \in \mathcal{K}^H$. Then, for each $K \in \mathcal{K}^H$ we define the multiscale base functions ϕ_K^i on K by

$$a(\phi_K^i, v) = 0, \qquad \forall\, v \in V^h \cap H_0^1(K), \quad i = 1, \cdots, n(K), \tag{5}$$

where $n(K)$ is the number of base functions with support in K, i.e. the number of nodal points $x_i \in \mathcal{N}_\mathcal{K} \cap \partial K$. To make (5) well posed, we need to specify the boundary condition for ϕ_K^i. For now, assume the base functions are continuous across the boundaries of the elements so that

$$V_0 = \operatorname{span}\big\{\phi_K^i : K \in \mathcal{K}^H,\ i = 1, \cdots, n(K)\big\} \subset V^h.$$

The MsFEM solution $u_0 \in V_0$ is thus defined by

$$a(u_0, v) = f(v), \qquad \forall\, v \in V_0. \tag{6}$$

For easy reference we write this equation in operator form, $A_0 u^h = u_0$, where A_0 is referred to as the MsFEM operator.

2.2 Boundary Conditions for the Base Functions

The selection of proper boundary conditions for the base functions is important to achieve good approximation properties. In fact, since the base functions satisfy the homogeneous equation (5), the boundary conditions determines how well the local property of the operator is sampled into the base functions. It was eg. observed in [4,5] that $u - u_0$ may display a boundary layer structure if improper boundary conditions are chosen.

From computational experience we found that boundary conditions which adapt to heterogeneous structures along the element boundaries in general lead to better accuracy than simple (smooth) boundary conditions. An appealing approach is to let the boundary condition μ_K^i for ϕ_K^i be the solution of the reduced elliptic problem obtained from (1) by deleting terms with partial derivatives in the direction normal to ∂K and having the coordinate normal to ∂K as a parameter. The boundary data for μ_K^i is such that $\mu_K^i(x_j) = \delta_{ij}$, $x_j \in \mathcal{N}_\mathcal{K}$. One may verify that this choice implies $V_0 \subset V^h$. Another possibility, which was proposed and analyzed in [4,5], is by oversampling, i.e. by constructing base-functions ψ_S^i from a sampling element $S \supset K$, and defining

$$\phi_K^i = \sum_{j=1}^{n(K)} c_{ij}\psi_S^i,$$

where the constants c_{ij} are determined by the condition $\phi_K^i(x_j) = \delta_{ij}$. Because the boundary conditions constructed in this way in general does not allow the inclusion $V_0 \subset V^h$, we see that this leads to a non-conforming MsFEM.

Though the MsFEM solvers induced by the boundary conditions above may have better approximation properties than eg. linear boundary conditions, they do not necessarily achieve faster convergence of the domain decomposition iteration consisting of additional local solves. In fact, preliminary tests showed that choosing the boundary conditions μ_K^i for ϕ_K^i to be the solution of the reduced elliptic problem described above did not give faster convergence than if linear boundary conditions for ϕ_K^i were used. To understand this we should keep in mind that the coarse solver usually act the role of removing low frequency errors, while the local solves remove high frequency errors. But most importantly, the local solves should be somewhat complimentary to the coarse solver. Therefore, using non-smooth boundary conditions may call for non-standard local solves.

2.3 General Remarks

Let V^h be the linear finite element space in one dimension and let $u^h \in V^h$. Then the multiscale finite element method inherit the special super convergence property $u_0 = u_I$, where u_I is the coarse scale interpolant of u^h in V_0. Indeed, since $u^h - u_I$ vanishes at the coarse grid nodal points $\mathcal{N}_K = \partial\mathcal{K}\backslash\partial\Omega$, we have

$$a(u_I, v) = a(u^h, v) = f(v), \qquad \forall\, v \in V_0.$$

Thus, in particular, by (6) and choosing $v = u_I - u_0$ we obtain

$$a(u_I - u_0,\, u_I - u_0) = 0,$$

which implies $u_0 = u_I$. It is interesting to observe that this super-convergence result remains valid, using the same reasoning, for general functions $u \in H_0^1(\Omega)$ if we replace the roles of $V^h \cap H_0^1(K)$ in (5) with $H_0^1(K)$. The solution to (3) can thus be decomposed as $u = u_0 + u_*$ where u_* is the sum of the local solutions $u_{*,K} \in H_0^1(K)$ with

$$a(u_{*,K}, v) = f(v), \qquad \forall\, v \in H_0^1(K).$$

This result implies that A_0 is an ideal preconditioner for nonoverlapping domain decomposition methods in 1D, i.e. $\kappa(A_0^{-1}S) = 1$ where S is the Schur compliment matrix.

There is, however, a fundamental difference between 1D and higher dimensional problems since the "resonance error" caused by non-matching boundary conditions only occurs in multi-D. This is clearly important since nonoverlapping domain decomposition methods act on the set of interface variables $M^h = V^h|_\Gamma$ and the convergence rate of the domain decomposition iteration is closely related to the approximation properties of A_0 on Γ, which, in turn, is determined by the space $M_0 = V_0\,|_\Gamma$ spanned by the boundary conditions for the multiscale space functions. To clarify the relation between the approximation properties of A_0 on Γ and the selected boundary conditions for the multiscale base functions, define the space of generalized discrete harmonic functions,

$$W^h = \{ w \in V^h : a(w,v) = 0, \ \forall v \in V^h \cap H_0^1(\mathcal{K}^H) \},$$

and the generalized discrete harmonic extension operator $H_a^h : M^h \to W^h$,

$$a(H_a^h \mu, v) = 0, \qquad \forall v \in V^h \cap H_0^1(\mathcal{K}^H).$$

If $a(x)$ is quasi-homogeneous, then H_a^h coincides with the ordinary discrete harmonic extension and we omit the subscript a and write H^h. In nonoverlapping domain decomposition methods we reformulate (4) as follows,

$$\text{Find } \mu^h \in M^h : a(H_a^h \mu^h, H_a^h \nu) = (f, H_a^h \nu), \qquad \forall \nu \in M^h, \tag{7}$$

$$\text{Find } u_{*,K} \in V^h \cap H_0^1(K) : a(u_{*,K}, v) = f(v), \qquad \forall v \in V^h \cap H_0^1(K), \tag{8}$$

and write $u^h = H_a^h \mu^h + \sum_{K \in \mathcal{K}^H} u_{*,K}$. It thus follows that the relevant bilinear form for the nonoverlapping domain decomposition formulation (7) is given by $(\mu,\nu)_M = a(H_a^h \mu, H_a^h \nu)$. Now, since $V_0 \subset W^h$ and u_0 is the orthogonal projection of u^h onto V_0 with respect to $a(\cdot,\cdot)^{1/2}$, it follows that $\mu_0 = u_0|_\Gamma$ is the orthogonal projection of μ^h onto M_0 with respect to $(\cdot,\cdot)_M^{1/2}$. This implies that A_0 is optimal, in a certain sense, for nonoverlapping domain decomposition algorithms among all coarse solvers $A^H : V^h \to V^H$ with $V^H|_\Gamma = M_0$. In the remainder of this paper we shall view A_0 as an operator acting on the set of interface variables so that $A_0 \mu^h = \mu_0$.

2.4 Framework for Analysis

Many nonoverlapping domain decomposition methods can be categorized as so called Schwarz methods for which a simple framework for the convergence analysis exists, see [3,8,11,15,16]. This framework was originally developed for the analysis of domain decomposition preconditioners for linear elliptic partial differential equations, but has later been extended to include non-linear elliptic partial differential equations, see [13,14]. The abstract Schwarz framework is based on a splitting of a finite dimensional Hilbert space V into subspaces with in general much smaller dimension.

Thus, let V_i be a sequence of finite dimensional Hilbert spaces and let $I_i : V_i \to V$ be a corresponding sequence of interpolationlike operators such that V allows the following decomposition,

$$V = \sum_{i=0}^{p} I_i V_i := \Big\{ v : v = \sum_i I_i v_i, \ v_i \in V_i \Big\},$$

The space V_0 represents a coarse global approximation space while V_i, $1 \le i \le p$, are subspaces corresponding to some localized region in space. Let V be supplied with a symmetric positive definite bilinear form $a(\cdot,\cdot)$ and assume that each V_i is supplied with an auxiliary symmetric positive definite bilinear form $(\cdot,\cdot)_i$ on V_i which approximates $a(\cdot,\cdot)$ on V_i in the following sense:

$$a(I_i v, I_i v) \le \omega (v,v)_i, \qquad \forall v \in V_i, \ \ \forall i.$$

The parameter ω is assumed to be bounded and plays a special role in the analysis of Schwarz methods. Now, define the projectionlike operators $T_i : V \to V_i$ by,

$$(T_i u, v)_i = a(u, I_i v), \qquad \forall u \in V, \ \ \forall v \in V_i.$$

Finally, let $\mathcal{P}$ be a polynomial with no zero order term and suppose we want to find $u^* \in V$ such that

$$a(u^*, v) = f(v), \qquad \forall v \in V, \ \ f \in V'. \tag{9}$$

The idea behind the general abstract Schwarz method is to replace (9) with a better conditioned operator equation

$$\mathcal{P}(T_0, \cdots, T_p)u = g^*,$$

where $g^* = \mathcal{P}(T_0, \cdots, T_p)u^*$. The Additive Schwarz method is eg. obtained by choosing $\mathcal{P}(\cdot) = \sum_{i=0}^{p} T_i$. The following result bounds the condition number for the preconditioned abstract additive Schwarz method (see [2,8,11,15]).

Theorem 1. *Let C_0 be a positive constant such that for any $v \in V$ there exists a decomposition $v = \sum_{i=0}^{p} I_i v_i$, $v_i \in V_i$ with*

$$\sum_{i=0}^{p} (v_i, v_i)_i \leq C_0 a(v, v),$$

and let

$$C_1 = \max_{1 \leq j \leq p} \sum_{i=1}^{p} \varepsilon_{ij},$$

where $\varepsilon_{ij} = 0$ if $T_i T_j = 0$, and 1 otherwise. Then the abstract additive Schwarz method admits the following estimate

$$\kappa\Big(\sum_{i=0}^{p} T_i\Big) \leq \omega\, C_0 (1 + C_1). \tag{10}$$

We assume $T_0 = A_0$ and study the preconditioner

$$\mathcal{P}(\,\cdot\,) = T_0 + \Big(I - \sum_{i=1}^{p} T_i\Big) T_0, \tag{11}$$

for which it is known that $\kappa(\mathcal{P}) \leq \kappa\big(\sum_{i=0}^{p} T_i\big)$, see [7,11]. Note that (11) can be viewed as a multiplicative Schwarz preconditioner on the splitting $V = V_0 + V_*$, $V_* = V$, where the bilinear form on V_* is approximated with an additive Schwarz splitting into V_1 to V_p. Hence, in each iteration we first update the solution for the coarse subspace correction and then perform an additive block Jakobi sweep to correct for the local "high frequency" error. This splitting isolates the coarse subspace correction in each iteration, and it is therefore easier to single out the effect of replacing conventional finite element solvers with the MsFEM operator A_0. Moreover, since $V_0 \subset V_* = V$, this multiplicative coarse-local splitting of V ensures that we do not correct for the same error twice, regardless of the selected boundary conditions for the multiscale base functions. Finally, it should be noted that the operator (11) can be symmetrized by including a coarse subspace correction before and after the local additive solves. However, this will not affect the result since A_0 is an orthogonal projection and we do not gain anything by applying A_0 more than once in consecutive order.

3 The Domain Decomposition Preconditioner

We now study a class of domain decomposition methods arising from (11) with $T_0 = A_0$. We shall leave the boundary conditions for the multiscale base functions undetermined, but we assume that they induce a well defined problem with $V_0 \subset V^h$. We wish to point out, however, that allowing non-conforming boundary conditions can improve the convergence rate of the domain decomposition iteration in the same way the non-conforming finite element methods may give superior performance to the analogous conforming finite element methods.

For instance, it is well known that if we allow the subdomains to have sufficient overlap, then the non-conforming linear finite element coarse solver allows an optimal rate of convergence for the preconditioning of elliptic problems with quasi-homogeneous coefficients, see [10]. On the other hand, we know that the conforming linear finite element method coarse solver induce a suboptimal rate of convergence, see [1].

The local solves can be viewed as an overlapping additive Schwarz method where we use a partition of unity to assemble the local solves. The idea of using an overlapping domain decomposition strategy for nonoverlapping domain decomposition algorithms has been studied before in various forms, see eg. [16] Section 7.2 and [12]. The main difference between the local solves proposed below and those proposed in [16] is that we use "vertex based" subdomains similar to the vertex domains in [12], while Xu et al. used "domain based" subdomains in [16]. The overlapping subdomains for our local solves are chosen to coincide with the support of the multiscale base functions. This implies that the construction of A_0 is comparable to one iteration of the preconditioner, and it is therefore easy to measure the computational savings we achieve with the MsFEM induced preconditioner.

We now proceed to develop the components the Schwarz preconditioner (11) for the preconditioning of (7). First, let the bilinear form on $V = M^h$ be given by $(\cdot\,,\cdot)_M$ and note that $(\cdot\,,\cdot)_0 = (\cdot\,,\cdot)_M$ by the definition of A_0. To define the local components of (11), let Ω be decomposed into the overlapping subdomains

$$\Omega_i = \cup\{K \in \mathcal{K}^H : \partial K \cap x_i \neq \emptyset,\ x_i \in \mathcal{N}_{\mathcal{K}}\},$$

where we assume $\Omega_i \cap \mathcal{N}_{\mathcal{K}} = x_i$. Let $\Gamma_i = \Omega_i \cap \Gamma$ and let $\Theta = \{\theta_i : \Gamma_i \to (0,1]\}$ be a partition of unity on Γ with $\theta_i(x_j) = \delta_{ij}$ for $x_j \in \mathcal{N}_{\mathcal{K}}$ where δ_{ij} is the Kronecker delta function. In 2D it is natural to let θ_i be linear on each edge $E \subset \Gamma_i$. Similarly, in 3D one can let θ_i be eg. linear or bilinear on each face $F \subset \Gamma$, depending on the number of vertices for F. Now, let $M_i = M^h|_{\Gamma_i}$ and define the local Schwarz operators on M_i according to,

$$I_i = I^h(1/\theta_i) \qquad \text{and} \qquad T_i = I^h(\theta_i P_i),$$

where P_i is the orthogonal projection onto M_i with respect to $(\cdot\,,\cdot)_M^{1/2}$ and $I^h : H^{1/2}(\partial\mathcal{K}) \to M^h$ is the nodal operator mapping $\mu \in H^{1/2}(\partial\mathcal{K})$ onto $\mu^h \in M^h$ with $\mu(x) = \mu^h(x)$ for every $x \in \mathcal{N}_{\mathcal{T}} \cap \Gamma$. The appropriate bilinear form on M_i is now defined by

$$\big(I^h(\theta_i\mu),\, I^h(\theta_i\nu)\big)_i = (\mu,\nu)_M, \qquad \forall\, \mu,\, \nu \in M_i.$$

The choice of T_i, and thus of I_i and $(\cdot\,,\cdot)_i$, is unconventional, but has a very intuitive explanation. Observe that each P_i has best approximation properties away from $\partial\Omega_i$ and not so good close to $\partial\Omega_i$ since P_i correspond to solving (4) in Ω_i with homogeneous Dirichlet boundary conditions on $\partial\Omega_i$. Hence, each θ_i is in correspondence with P_i in such a way that θ_i is close to one where P_i performs well, and close to zero where P_i performs poorly. Thus, by defining T_i like above we gather the local updates such that the most reliable solution at any point is weighted the most.

It is easy to see that the auxiliary bilinear forms $(\cdot\,,\cdot)_i$ are symmetric positive definite. Moreover, for any $\mu \in M^h$, we have

$$(T_i\mu,\, \nu)_i = (P_i\mu,\, I_i\nu)_M = (\mu,\, I_i\nu)_M, \qquad \forall\, \nu \in M_i,$$

and, by definition,

$$(I_i\mu,\, I_i\mu)_M = (\mu,\mu)_i, \qquad \forall\, \mu \in M_i,$$

which implies $\omega = 1$. It is therefore clear that M_i, $(\cdot\,,\cdot)_i$, T_i and I_i fulfill the prerequisites for Schwarz analysis. Hence, to obtain a bound on the condition number of our preconditioner we only need to bound the parameters C_0 and C_1. The parameter C_1 is bounded independent of the mesh parameters by a standard coloring argument, see eg. [11,16]. To bound C_0 we first observe that, since $(\cdot,\cdot)_0 = (\cdot,\cdot)_M$, estimating C_0 amounts to showing, for each $\mu \in M^h$, the existence of a representation $\mu = \sum_{i=0}^{p} \mu_i$, $\mu_i \in M_i$, with

$$\sum_{i=0}^{p} (\mu_i,\, \mu_i)_M \leq C_0(\mu, \mu)_M. \tag{12}$$

Note that this is the usual estimate for C_0 corresponding to $T_i = P_i$, $1 \leq i \leq p$.

Before we continue with the convergence analysis, let us outline the main steps in the proposed domain decomposition algorithm. One loop of the domain decomposition iteration consists of the following steps:

If not converged,
(a) $u_0^{k+1} = u^k + A_0(u^h - u^k)$,
(b) $u_i^{k+1} = P_i(u^h - u_0^{k+1})$, $i = 1, \cdots, p$,
(c) $u^{k+1} = u_0^{k+1} + \sum_{i=1}^{p} I^h(\theta_i u_i^{k+1})$,

where u^h is the solution to (7) and μ^k is the current approximation to μ^h after k iterations.

3.1 Analysis and Error Estimates

The purpose of the following is to estimate C_0 and thereby obtain an estimate for the condition number of our preconditioner (11). Our primary objective is to clarify the advantages of using the MsFEM solver as opposed to conventional coarse finite element solvers. In particular we want to establish that the MsFEM induced preconditioner is insensitive to the local aspect ratios and thus show the same performance for elliptic problems with high aspect ratios as traditional preconditioners with conventional coarse solvers show for elliptic problems with quasi-homogeneous coefficients. The idea is to split the analysis into a homogenized part which depends on the selected boundary conditions for the multiscale base functions, and a multiscale part which only depends on the heterogeneous structures within the coarse grid elements.

To analyze the homogenized part of the multiscale algorithm we assume that there exists an operator $J : V^h \to H^h M_0$ such that the following local stability estimates hold for each $v \in V^h$ and $K \in \mathcal{K}^H$,

$$|Jv|^2_{H^1(K)} \lesssim \beta\, |v|^2_{H^1(K)}, \tag{13}$$

$$\|v - Jv\|^2_{L^2(K)} \lesssim \beta H^2 |v|^2_{H^1(K)}. \tag{14}$$

Estimates of this kind is the main ingredient in the analysis of traditional Schwarz algorithms and has been established for a great variety of coarse spaces. For instance if $\mathcal{K}^H$ is a triangulation of Ω and M_0 is the space of piecewise linear functions on Γ determined by its values at the nodal points $\mathcal{N}_{\mathcal{K}}$, then it is well known that $\beta \sim \log(H/h)$ in 2D and $\beta \sim H/h$ in 3D. Similar estimates holds if K is a polygon and linear on each edge or face $F \subset \partial K$, see [16] Section 5.2. To eliminate this mesh dependence one need to allow non-conforming boundary conditions so that $M_0 \not\subset M^h$. In particular, the non-conforming linear finite element method allows β to be independent of the mesh parameters, see [10]. We now state the following lemma.

Lemma 2. *If the stability estimates (13)–(14) hold, then, for each $v \in V^h$ we have a decomposition $v = \sum_{i=0}^{p} v_i$, $v_0 \in H^h M_0$, $v_i \in V^h \cap H_0^1(\Omega_i)$, with*

$$\sum_{i=0}^{p} |v_i|^2_{H^1(K)} \leq \beta\, |v|^2_{H^1(K)}, \qquad \forall\, K \in \mathcal{K}^H. \tag{15}$$

Proof. Since the subdomains have generous overlap, there exists a partition of unity $\Psi = \{\psi_i : \Omega_i \to (0,1],\ 1 \le i \le p\}$ on Ω such that $\|\nabla \psi_i\|_{L^\infty(\Omega)} \lesssim H^{-1}$. Thus, letting $v_0 = Jv$, $v_* = v - v_0$ and $v_i = I^h \psi_i v_*$, we get (15) by following the proof in eg. [11], pp. 166–167. This proves (15).

As a direct consequence of (15) we have that for each $\mu \in M^h$, there exists a decomposition $\mu = \sum_{i=0}^{p} \mu_i$, $\mu_i \in M_i$, such that

$$\sum_{i=0}^{p} |H^h \mu_i|^2_{H^1(K)} \le \beta\, |H^h \mu|^2_{H^1(K)}, \forall\, K \in \mathcal{K}. \tag{16}$$

Indeed, since $H^h M^h \subset V^h$, the estimate (15) holds for each $v = H^h \mu$, $\mu \in M^h$. Now, if $v = \sum_{i=0}^{p} v_i$, then we also have $v = \sum_{i=0}^{p} H^h \mu_i$, $\mu_i = v_i|_\Gamma$. Thus, by the minimal energy property of discrete harmonic functions we have $|H^h \mu_i|_{H^1(K)} \le |v_i|_{H^1(K)}$ and (16) follows.

We now turn to the general multiscale case. We thus want to replace H^h in (16) with H^h_a and the H^1 seminorm $|\cdot|_{H^1(K)}$ with the following local weighted norm on $H^1(K)$,

$$|u|^2_{a,K} = \int_K (\nabla u)^T a(x) \nabla u.$$

To this end, we introduce positive constants $\gamma_1(K)$ and $\gamma_2(K)$ such that

$$\gamma_1(K)\, |H^h \mu|^2_{H^1(K)} \le |H^h_a \mu|^2_{a,K} \le \gamma_2(K)\, |H^h \mu|^2_{H^1(K)}, \qquad \forall\, \mu \in M^h. \tag{17}$$

Now, let $\gamma_1(K)$ and $\gamma_2(K)$ be the sharpest possible bounds in (17), and define

$$\gamma = \max_{K \in \mathcal{K}^{\mathcal{H}}} \frac{\gamma_2(K)}{\gamma_1(K)}. \tag{18}$$

We have the following estimate for the condition number of (11).

Theorem 3. *Let β and γ be as defined by (13)–(14) and (18). An upper bound for the condition number of the multiscale domain decomposition preconditioner (11) is then given by*

$$\kappa\Big(T_0 + \Big(I - \sum_{i=1}^{p} T_i\Big) T_0\Big) \lesssim \gamma\, \beta. \tag{19}$$

Proof. Let $\mu \in M^h$ have the decomposition $\mu = \sum_{i=0}^{p} \mu_i$ used in (16). Then, as a direct consequence of (16) and (17) we obtain,

$$\sum_{i=0}^{p} |H^h_a \mu_i|^2_{a,K} \le \sum_{i=0}^{p} \gamma_2(K)\, |H^h \mu_i|^2_{H^1(K)} \le \gamma_2(K)\, \beta\, |H^h \mu|^2_{H^1(K)} \le \frac{\gamma_2(K)}{\gamma_1(K)} \beta\, |H^h_a \mu|^2_{a,K}.$$

Thus, by (18) and summing over $\forall\, K \in \mathcal{K}^H$ we have,

$$\sum_{i=0}^{p} (\mu_i, \mu_i)_M = \sum_{i=0}^{p} a(H^h_a \mu_i, H^h_a \mu_i) \lesssim \gamma\beta a(H^h_a \mu, H^h_a \mu) = \gamma\beta(\mu, \mu)_M. \tag{20}$$

This bounds the parameter C_0 in (12) and the desired result follows from (10). This completes the proof of Theorem 3.

It follows from Voigt-Reiss Inequality in the homogenization theory that $\gamma_1(K)$ is bounded below by the "harmonic mean" of $a(x)$ over K and $\gamma_2(K)$ is bounded above by the "arithmetic mean" of $a(x)$ over K. Indeed, if $a_\varepsilon(x)$ is a symmetric periodic matrix in $\mathcal{R}^{d\times d}$, then the homogenized matrix a_0 satisfies the convergence of energies, see [6] Section 1.3,

$$\lim_{\varepsilon\to 0} |H^h_{a_\varepsilon}\mu|^2_{a_\varepsilon,K} = |H^h_{a_0}\mu|^2_{a_0,K}, \qquad \forall\, \mu\in M^h(\partial K).$$

Moreover, by the Voigt-Reiss's inequality (see [6] Section 1.6), we have

$$\Big(\frac{1}{K}\int_K a^{-1}(x)\,dx\Big)^{-1} \le a_0 \le \Big(\frac{1}{K}\int_K a(x)\,dx\Big),$$

where the inequalities are to be interpreted in a spectral sense. Though $a(x)$ may not be periodic, the multiscale analysis only concerns the local nature of $a(x)$ and we may think of K as a periodic cell in an infinite periodic media. The bounds for $\gamma_1(K)$ and $\gamma_2(K)$ follows and we obtain,

$$\gamma \le \max_{K\in\mathcal{K}^H} \frac{\kappa\left(\int_K a(x)\,dx\,\int_K a^{-1}(x)\,dx\right)}{|K|^2}.$$

The harmonic and arithmetic means are sharp lower and upper bounds for the homogenized matrix in the sense that they are attained for perfectly stratified media with flow either perpendicular or parallel to the layers. But, they are also very crude in the sense that they do not account for the heterogeneous structures within K. Many other and better bounds can be found in the literature on upscaling for porous media flow, and some of these bounds can be found in [9].

Since we always interpret coarse spaces for nonoverlapping domain decomposition in terms of values on Γ, it seems appropriate to emphasize what makes our algorithm special. Philosophically we see that the MsFEM coarse space V_0 for general polygonal partitionings is a generalization of the coarse spaces discussed in [16], Section 5.2. However, they have not discussed the technicalities which need to be considered if we have strongly varying coefficients within the substructures. The important thing to keep in mind is that we have to select the coarse grid operator so that it is close to the orthogonal projection onto V_0 with respect to $a(\cdot,\cdot)^{1/2}$. For this purpose it is no longer sufficient to apply "discrete harmonic" coarse solvers.

Again we use the one dimensional case to illustrate. We now assume that $a(\tau) = c_\tau > 0$ for each $\tau\in\mathcal{T}^h$ since this is the resolution of the fine mesh discretization. First note that $\gamma_1(K)$ and $\gamma_2(K)$ coincide with the harmonic mean of $a(x)$ over K. To see this, let $\mu\in M^h$ and define $u = H^h_a\mu$ and $v = H^h\mu$. Then, since v is linear on each K and $a(x)\partial_x u$ is constant on each K, we have

$$\partial_x v = \frac{1}{|K|}\int_K \partial_x v = \frac{1}{|K|}\int_K \partial_x u = \frac{a(x)\partial_x u}{|K|}\int_K a(x)^{-1}dx.$$

Thus we have,

$$|u|^2_{a,K} = a(x)\partial_x u\int_K \partial_x u\,dx = \frac{|K|^2(\partial_x v)^2}{\int_K a^{-1}dx} = \frac{|K|}{\int_K a^{-1}dx}|v|^2_{H^1(K)}.$$

It follows that $\gamma = 1$ and we confirm the optimal convergence rate since $\beta\sim 1$ in one dimension. In contrast, if we replace A_0 with the coarse grid linear finite element solver, then we need to replace $H^h_a\mu_0$ in (20) with $v_0 = H^h\mu_0$. Thus, if v_0 is the interpolant of $u\in W^h$ in H^hM_0 so that $\mu_0 = \mu$, then

$$|v_0|^2_{a,K} = \|a\|_{L^1(K)}\|1/a\|_{L^1(K)}|u|^2_{a,K}.$$

The convergence of the corresponding linear finite element induced preconditioner thus depend on the ratio of the arithmetic mean over the harmonic mean.

4 Numerical Results

Let $\mathcal{T}^h$ be a uniform triangulation of $\Omega = [0,1]^2$ with mesh parameter h and let $\mathcal{K}^H$ be a uniform partitioning of Ω into squares with mesh parameter H. We assume the coarse to fine grid mesh ratio is determined by the relation $H/h = H^{-1}$. The right hand side in (1) is chosen to be $f \equiv 1$. This choice of f implies that the solution u^h will have a multiple scale structure if the elliptic coefficients are oscillatory. The stopping criteria is set to be when the relative size of the current residual to the initial residual drops below 10^{-5}, i.e. when $\|r^k\|_2/\|b\|_2 \leq 10^{-5}$ where $\|\cdot\|_2$ is the Euclidean norm.

The scalar coefficient function is assumed to have the form $a(x) = a_H(x)a_h(x)$ where a_h is an H-periodic function and a_H is quasi-homogeneous. It is clear that a_h is determined by its values in the unit cell $Y = [0,H]^d$. We report results for three different choices of a_h, a uniformly oscillatory function $a_{h,1}$, and two rather special functions with high aspect ratios, $a_{h,2}$ and $a_{h,3}$, used to identify situations where multiscale domain decomposition methods may give a substantial improvement of the convergence rate. Hence, let $k(\tau)$, $\tau \in \mathcal{T}^h(Y)$, be randomly sampled from a uniform probability distribution on $(0,1]$ and define $a_{h,1}|_Y(x) = k(x)^p$,

$$a_{h,2}|_Y(x) = \begin{cases} 10^p k(x), & \text{if } \operatorname{dist}(x,\partial Y) \leq h, \\ k(x), & \text{if } \operatorname{dist}(x,\partial Y) > h, \end{cases}$$

$$a_{h,3}|_Y(x) = \begin{cases} k(x), & \text{if } \operatorname{dist}(x,\partial Y) \leq h, \\ 10^p k(x), & \text{if } \operatorname{dist}(x,\partial Y) > h, \end{cases}$$

where p is some specified power which has the effect of scaling the local aspect ratios. The quasi-homogeneous coefficient function a_H is chosen to be either $a_H \equiv 1$, the periodic media case, or $a_H(K) = k(K)^2$ where $k(K)$ is randomly sampled from a uniform probability distribution on $(0,100]$.

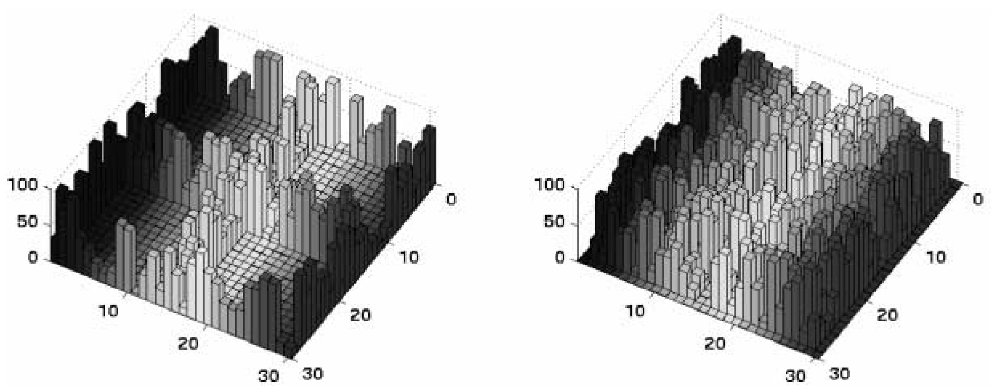

Fig. 1 Plots of $a_{h,2}$ and $a_{h,3}$ on a 2-by-2 coarse grid with H/h=15 and p=2.

The boundary conditions for the multiscale base functions are chosen to be linear and determined by the requirement $\phi^i_K(x_j) = \delta_{ij}$ where x_j range over the set of vertices for K. Hence, if $a_h \equiv 1$, then A_0 coincides with the coarse bilinear finite element solver. It is therefore natural to use this solver for comparison below. We also compare with the standard linear finite elements solver as this solver is perhaps the most popular coarse solver for domain decomposition algorithms in 2D. We thus denote by A_L and A_{BL} the coarse grid finite element operators constructed from linear and bilinear base functions respectively with coarse grid nodal points $\mathcal{N}_\mathcal{K}$. We denote by $\mathcal{P}_0$, the Schwarz preconditioner (11) induced by A_0. Similarly we denote by $\mathcal{P}_L$ and $\mathcal{P}_{BL}$ the Schwarz operators induced by replacing A_0 with $H^h_a A_L$ and $H^h_a A_{BL}$ respectively.

Table 1. The table shows iteration counts for each of the Schwarz operators $\mathcal{P}_0$, $\mathcal{P}_L$ and $\mathcal{P}_{BL}$ for a periodic media in $\mathcal{R}^2$ with the prescribed choices of a_h.

p	H^{-1}	$a_h = a_{h,1}$			$a_h = a_{h,2}$			$a_h = a_{h,3}$		
		$\mathcal{P}_0$	$\mathcal{P}_{BL}$	$\mathcal{P}_L$	$\mathcal{P}_0$	$\mathcal{P}_{BL}$	$\mathcal{P}_L$	$\mathcal{P}_0$	$\mathcal{P}_{BL}$	$\mathcal{P}_L$
1	4	5	6	6	5	7	8	7	16	17
1	8	5	6	6	6	11	11	7	18	19
1	16	6	7	7	7	15	15	7	18	18
1	32	6	7	7	6	14	15	7	14	14
2	4	6	7	9	7	10	11	7	30	30
2	8	7	11	11	8	18	19	9	93	94
2	16	6	11	11	7	19	19	10	>100	>100
2	32	7	11	11	8	23	23	9	>100	>100
3	4	7	8	9	8	11	12	6	34	34
3	8	6	13	14	6	17	17	9	>100	>100
3	16	7	19	19	8	24	24	12	>100	>100
3	32	8	21	22	9	28	28	12	>100	>100

Table 2. The table shows iteration counts for each of the Schwarz operators $\mathcal{P}_0$, $\mathcal{P}_L$ and $\mathcal{P}_{BL}$ in $\mathcal{R}^2$ with the prescribed choices of a_H and a_h.

p	H^{-1}	$a_h = a_{h,1}$			$a_h = a_{h,2}$			$a_h = a_{h,3}$		
		$\mathcal{P}_0$	$\mathcal{P}_{BL}$	$\mathcal{P}_L$	$\mathcal{P}_0$	$\mathcal{P}_{BL}$	$\mathcal{P}_L$	$\mathcal{P}_0$	$\mathcal{P}_{BL}$	$\mathcal{P}_L$
1	4	5	6	8	5	7	8	6	14	15
1	8	7	8	11	7	13	15	12	27	33
1	16	12	13	17	9	16	18	23	40	47
1	32	16	18	23	13	19	23	26	39	50
2	4	6	6	8	6	9	11	7	58	62
2	8	8	10	14	10	20	21	14	>100	>100
2	16	14	19	23	13	24	26	25	>100	>100
2	32	18	24	30	13	27	28	43	>100	>100
3	4	5	6	9	6	8	10	9	56	57
3	8	8	14	17	12	23	24	14	>100	>100
3	16	18	27	32	17	32	34	30	>100	>100
3	32	21	35	42	22	46	48	49	>100	>100

We first consider the periodic media case $a_H \equiv 1$. Table 1 shows that the condition number of $\mathcal{P}_0$ seems to be bounded independent of the mesh parameters H and h, and, in particular, that the convergence rate seems to be nearly independent of a_h. We wish to emphasize that the scaling of the local aspect ratio through the parameter p has very little effect on the convergence of the MsFEM induced preconditioner. We observe that the iteration count for $\mathcal{P}_L$ and $\mathcal{P}_{BL}$ are comparable for all choices of a_h. They perform reasonably well for moderate local aspect ratios, i.e. for $p = 1$, but both solvers are clearly very sensitive to a scaling of the local aspect ratios, especially for $a_{h,3}$. This reflect that neither the linear finite element or bilinear finite element coarse grid solver accounts for small scale features in the elliptic coefficients.

We now study the general non-periodic media problem where a is the product of a quasi-homogeneous coefficient function a_H and a periodic oscillatory coefficient function a_h. Note that multiplying a_h with a_H does not affect the local aspect ratios since a_H only scales the "mean" of a over the coarse grid elements. The results depicted in table 3 demonstrate that the iteration count for $\mathcal{P}_0$ still seems to be almost insensitive to the parameter p which scales the local aspect ratios, even for $p = 3$ where the elliptic coefficients can vary more than 10 orders of magnitude within each coarse grid element. The iteration counts for $\mathcal{P}_0$ are consistent with a logarithmic growth in $H^{-1} = H/h$. These results thus illustrate that the coarse MsFEM solver A_0 reveals only a weak dependence on the heterogeneous structures within the coarse grid blocks.

We again observe that $\mathcal{P}_L$ and $\mathcal{P}_{BL}$ perform reasonably well for $p = 1$, but are clearly very sensitive to a scaling of p, in particular for $a_{h,3}$ where we again see that the coarse subspace correction does not seem to have an important effect on the convergence rate for $p = 2$ and $p = 3$. Hence, we conclude that the linear bilinear finite element coarse solvers perform very poorly for high aspect ratios and are sensitive to the heterogeneous formation within the coarse grid blocks.

5. Concluding Remarks

We have demonstrated that the proposed multiscale domain decomposition preconditioner is applicable to problems with high aspect ratios as well as problems with continuous scales. The numerical tests indicate a convergence rate nearly independent of the local aspect ratios and a logarithmic dependence on the mesh ratio H/h. This latter dependence is of the same order as conventional overlapping Schwarz methods using conforming finite element coarse solvers achieve for quasi-homogeneous coefficients. It is possible to eliminate this mesh dependence by considering non-conforming finite element spaces such as the oversampling strategy or selecting boundary conditions for the multiscale base functions which correspond to eg. non-conforming linear finite elements. These solvers will be considered in future work.

We have shown that the proposed preconditioner can lead to a significant gain in iterations compared with the linear and bilinear finite element induced preconditioners discussed in this paper. Since the construction of the multiscale finite element solver is comparable with one loop of the domain decomposition iteration, we also see a substantial improvement in computational cost. The linear and bilinear finite element induced preconditioners show good performance for problems with small aspect ratios, and therefore serve the purpose of providing a valid measure of the performance of the MsFEM induced preconditioner. We are not aware of any coarse solvers in the literature which are able to handle high aspect ratios within the coarse grid elements. The blow up of the iteration count for the linear and bilinear finite element induced preconditioners therefore display a typical situation for conventional domain decomposition preconditioners.

The key to the success of the proposed domain decomposition algorithm is that the coarse subspace correction accounts for small scale heterogeneous features. In porous media flow problems, the permeability tensor can often vary several orders of magnitude in the microscale level. Thus, to obtain effective preconditioners for these problems it is very important that we have a coarse solver which is adaptive to the small scale heterogeneities. We have shown that the proposed preconditioner can be a very effective tool for these problems. In multi-phase porous media flows, the savings can be even greater since the elliptic problem is only a part of a set of coupled equations and involves solving the elliptic equation repeatedly.

We have also carried out preliminary tests for periodic oscillatory coefficients in 3D and these show that the MsFEM solver retains its insensitivity to the local aspect ratios. The performance of the MsFEM induced preconditioner to general three dimensional problems with oscillatory coefficients will be studied in more detail in further work.

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