A Nearly Optimal Existence Result for Slightly Perturbed 3-D Vortex Sheets

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Revised in May 8, 2002

Abstract

In this article, we study existence of analytic solution for slightly perturbed three-dimensional vortex sheets in the absence of surface tension. The key in our analysis is to derive a local leading order system, which captures the leading order behavior of the full three-dimensional vortex sheet equation. This is accomplished by a series of changes of variables, for both dependent and independent variables in order to single out the leading order contributions of the 3-D vortex sheet equations. The changes of variables are guided by properties of certain singular integral operators. Moreover, by using the extended abstract Cauchy-Kowalewski theorem, we can control the growth of the nonlinear nonlocal terms. For small initial analytic data, we show that the existence time can be sufficiently close to the time of singularity formation. Thus our existence result is nearly optimal.

1 Introduction

One of the classical examples of hydrodynamic instability occurs when two fluids are separated by a free surface across which the tangential velocity has a jump discontinuity. This is called Kelvin-Helmholtz instability. Kelvin-Helmholtz instability is a fundamental instability of incompressible fluid flow at high Reynolds number. The idealization of a shear layered flow as a vortex sheet separating two regions of potential flow has often been used as a model to study mixing properties, boundary layers and coherent structures of fluids (see, e.g. [21]).

It is well known that small initial perturbations on a vortex sheet may grow rapidly due to Kelvin-Helmholtz instability. The problem is ill-posed in the Hadamard sense [4]. Finite time existence and uniqueness have been obtained only for analytic initial data, see, e.g., [24]. Due to the rapid growth in high frequency modes and the nonlinear interaction among these high frequency modes, an initially analytic vortex sheet may develop finite time curvature singularities. Moore [19] was the first who studied finite time singularity formation of a 2D vortex sheet with a small sinusoidal initial disturbance. His analysis predicted that close to the singularity, the curvature of

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the sheet is proportional to $|\Gamma - \Gamma_s|^{-1/2}$, where Γ is the circulation in the sheet measured from a fixed reference particle and Γ_s is the position of the singularity. Moreover, his analysis gave an accurate prediction of the singularity time. Although Moore's result was based on asymptotic analysis, his result was subsequently verified numerically by a number of researchers, including Meiron, Baker & Orszag [18], Krasny [17], Shelley [22]. As a rigorous validation of Moore's analysis, Caflisch & Orellana [7] proved a nearly optimal existence result for a slightly perturbed two-dimensional vortex sheet using Moore's initial condition. Using a constructive proof, Duchon & Roberts [11] and Caflisch & Orellana [8] showed independently that an initial analytic vortex sheet can develop a finite time singularity. More recently, Cowley, Baker & Tanveer [10] provided further detailed study to singularity formation of the two-dimensional vortex sheet problem, revealing the the mechanism of generating the singularities of order $3/2$ for two-dimensional vortex sheets.

Three-dimensional vortex sheets are more difficult to analyze than the two-dimensional ones. This is because one can formulate and analyze the two-dimensional vortex sheet problem using complex variables. Sulem, Sulem, Bardos, & Frisch [24] were the first who provided a finite time existence proof of three-dimensional vortex sheets for general analytic initial data. Studies of singularity formation in 3D vortex sheets are limited. Among them, Ishihara & Kaneda [16] provided some evidence of the singularity formation in the three-dimensional problem by directly generalizing Moore's analysis to the three-dimensional problem. However, their result does not give a clear description of the singularity structure of the 3-D vortex sheet problem. Brady & Pullin [5] studied three-dimensional vortex sheets which have cylindrical shape and normal mode initial data. They showed that for this type of special initial data, the three-dimensional vortex sheet problem can be reduced exactly to a two-dimensional vortex sheet problem.

In this article, we obtain a nearly optimal existence result for three-dimensional vortex sheets with small analytic initial data. We do not consider the effect of surface tension in this study. The key in our analysis is to derive a linear leading order system, which captures the leading order behavior of the full three-dimensional vortex sheet equation. This is accomplished by a series of changes of variables, for both dependent and independent variables in order to single out the leading order contributions of the 3-D vortex sheet equations. This analysis is guided by a related stability analysis for 3-D fluid interfaces by Hou and Zhang in [14] which uses properties of certain pseudo-differential operators defined on moving interfaces. Although a nonlinear leading order system (similar to Moore's system) can also be derived, we find that the linear leading order system gives a better structure for our analysis. Based on the approximated system, we split the solution into a leading order part and a lower order part as in [7]. The existence of the leading order part of the solution can be obtained immediately from the linear theory. On the other hand, we apply the extended abstract Cauchy-Kowalewski theorem to estimate the nonlinear nonlocal lower order part. We show that the second part of the solution is indeed of lower order and smaller amplitude in a suitable norm. This proves the existence of the three-dimensional vortex sheet solution.

Specifically, we consider a three-dimensional vortex sheet with analytic initial data. More precisely, we assume that the initial data are analytic within the complex domain of strip width $\max(|Im(\alpha_1)|, |Im(\alpha_2)|) < \rho_0$, where (α_1, α_2) are the Lagrangian parameters. Under these assumptions, we show that the smooth three-dimensional vortex sheet exists up to time $T = 2\rho_0/(1+2\kappa)$, where $\kappa > 0$ is a parameter depending on the amplitude of the initial disturbance and can be made arbitrarily small for small initial disturbance. For the special initial data considered by Brady and Pullin [5] of wave length π and amplitude ϵ, Moore's analysis can be used to show that singularities of order 3/2 develop at $t_c = |\log|\epsilon| + O(\log|\log \epsilon|)$. Our existence results prove existence for $t < \mu|\log \epsilon|$, if ϵ is sufficiently small, with $\mu \to 1$ as $\epsilon \to 0$. Thus we show that the existence time can be sufficiently close to the singularity time of the leading order system. In this sense, our result is nearly optimal.

We remark that our existence result is different from that of Sulem, Sulem, Bardos, & Frisch [24]. The existence result of [24] is a short time existence result for general analytic data. Our analyses focus on slightly perturbed initial analytic data, and we obtain a nearly optimal long time existence result for small initial analytic data. Our aim is to establish existence of analytic solutions for 3D vortex sheets arbitrarily close to the time when the vortex sheet develops a curvature singularity of order 3/2 [5]. The derivation of the leading order system plays a crucial role in obtaining this nearly optimal existence result. In particular, the analysis on the leading order system shows that along the direction of the tangential velocity jump, the three-dimensional vortex sheet problem can be effectively reduced to a corresponding two-dimensional problem to the leading order approximation. In a separate paper [15], we have used this leading order analysis to study singularity formation of 3D vortex sheets. We show that the singularity type of the three-dimensional vortex sheet is essentially the same as that of the two-dimensional vortex sheet.

The organization of the rest of the paper is as follows. Section 2 provides a general introduction to the formulation of the 3-D vortex sheet problem, and states our main result. In Section 3, we derive a nonlinear system with linear leading order terms. This system is crucial in obtaining our nearly optimal existence result. We outline the proof without giving the detailed estimates of our approximated system in Section 4. In Section 5 and in the Appendix, we provide technical details omitted in Section 4.

2 Formulation and Main Result

2.1 General Formulation

We consider an interface Γ separating two infinite layers of incompressible, inviscid, irrotational and identical fluids in the absence of surface tension. Using the Lagrangian frame, the interface location at any instant t is given by:

$$\mathbf{z}(\alpha_1, \alpha_2, t) = (x(\alpha_1, \alpha_2, t)\ , y(\alpha_1, \alpha_2, t), z(\alpha_1, \alpha_2, t))^T\ , \tag{1}$$

where (α_1, α_2) is the Lagrangian surface parameter. Thus, the normalized tangential vectors to the surface, $\mathbf{T}_1$ and $\mathbf{T}_2$, are defined by

$$\mathbf{T}_1 = \frac{\mathbf{z}_{\alpha_1}}{|\mathbf{z}_{\alpha_1}|}\ , \qquad \mathbf{T}_2 = \frac{\mathbf{z}_{\alpha_2}}{|\mathbf{z}_{\alpha_2}|}\ , \tag{2}$$

and the unit normal vector to the surface $\mathbf{N}$ is defined by

$$\mathbf{N} = \frac{\mathbf{z}_{\alpha_1} \times \mathbf{z}_{\alpha_2}}{|\mathbf{z}_{\alpha_1} \times \mathbf{z}_{\alpha_2}|}\ . \tag{3}$$

We label the region below the interface as Region 1 and the region above the interface as Region 2. Therefore, the velocity field $\mathbf{u}_1$ ($\mathbf{u}_2$) is the velocity below (above) the interface. We define $\mathbf{u}_+$ to be the limit of $\mathbf{u}_2$ approaching the interface from Region 2 and $\mathbf{u}_-$ to be the limit of $\mathbf{u}_1$ approaching the interface from Region 1. Since the flow in each region is irrotational, we can introduce the velocity potentials ϕ_1 and ϕ_2 so that

$$\mathbf{u}_1 = \nabla\phi_1 \ , \qquad\qquad \mathbf{u}_2 = \nabla\phi_2 \ . \tag{4}$$

Furthermore, since the flows are incompressible, the velocity potentials satisfy the Laplace equation:

$$\nabla^2\phi_1 = 0 \quad \text{and} \quad \nabla^2\phi_2 = 0 \ . \tag{5}$$

Therefore, the potentials in the fluid domain can be written in the following dipole representation [2]:

$$\phi(\mathbf{z}) = \int \mu(\alpha')(\mathbf{z}_{\alpha_1} \times \mathbf{z}_{\alpha_2})(\alpha') \cdot \nabla_{z'} G(\mathbf{z} - \mathbf{z}(\alpha'))d\alpha' \ , \tag{6}$$

where

$$\begin{aligned} G(\mathbf{z} - \mathbf{z}') &= -\frac{1}{4\pi|\mathbf{z} - \mathbf{z}'|} \ , \\ \nabla_{z'} G(\mathbf{z} - \mathbf{z}') &= -\frac{\mathbf{z} - \mathbf{z}'}{4\pi|\mathbf{z} - \mathbf{z}'|^3} \ , \end{aligned}$$

and $\mu(\alpha) = \phi_- - \phi_+$. By differentiating equation (6) with respect to $\mathbf{z}$ and then integrating by parts, we obtain

$$\nabla\phi(\mathbf{z}) = \int |\nabla_\alpha \mu(\alpha')^T, \nabla_\alpha \mathbf{z}(\alpha')^T| \times \nabla_{\mathbf{z}'} G(\mathbf{z} - \mathbf{z}(\alpha'))\, d\alpha' \ , \tag{7}$$

where we have used the notation

$$|\nabla_\alpha \mu^T, \nabla_\alpha \mathbf{z}^T| = \frac{\partial \mu}{\partial \alpha_1}\mathbf{z}_{\alpha_2} - \frac{\partial \mu}{\partial \alpha_2}\mathbf{z}_{\alpha_1} \ .$$

In the Lagrangian formulation of the interface problem, the motion of the interface is governed by

$$\frac{\partial \mathbf{z}}{\partial t}(\alpha, t) = \mathbf{u}(\mathbf{z}(\alpha, t), t) \ , \tag{8}$$

where $\mathbf{u} = (u, v, w)$ is the velocity of fluid particles on the interface. The kinematic condition that ensures that the interface moves with the fluid requires that the normal component of the velocity be continuous at the interface. However, the tangential velocity at the interface is arbitrary and can be chosen at our convenience.

For the vortex sheet problem, we apply Bernoulli's equation to the upper and lower layer of fluid respectively. Based on the continuity of the normal stress, and combining with equation (6) and equation (7), we can show that by choosing the interface velocity in (8) to be the average of the interface velocity from above $\mathbf{u}_+$ and the interface velocity from below $\mathbf{u}_-$ from above and from below respectively, i.e. $\mathbf{u} = \frac{1}{2}(\mathbf{u}_+ + \mathbf{u}_-)$, then we have [3]

$$\frac{\partial \mu}{\partial t} = 0. \tag{9}$$

Equation (9) says that the circulation stays constant along the trajectories whose motions are determined by the average fluid velocity.

With this particular choice of tangential velocity, the velocity of the vortex sheet interface can be obtained by the average of the limiting velocities in equation (7) approaching from the upper and lower layer of fluid. The equation of the surface particle motion can be written as (see, e.g. [6, 12] for a derivation):

$$\begin{aligned} \frac{\partial \mathbf{z}}{\partial t}(\alpha, t) &= \int |\nabla_\alpha \mu(\alpha')^T, \nabla_\alpha \mathbf{z}(\alpha', t)^T| \times \nabla_{\mathbf{z}'} G(\mathbf{z}(\alpha, t) - \mathbf{z}(\alpha', t))\, d\alpha' \ , &(10)\\ \mathbf{z}(\alpha, 0) &= \mathbf{z}_0(\alpha) &(11) \end{aligned}$$

where $\mathbf{z} \in S$ and the integral takes the Cauchy principal value.

2.2 Main Result

Throughout the paper, we study the existence of a unique solution to the initial value problem (10). The main result is to prove the existence of such a solution given a slightly perturbed periodic initial condition from an equilibrium flat state. More precisely, we write the interface variable $\mathbf{z}$ as

$$\mathbf{z} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{pmatrix} + \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} , \tag{12}$$

where S_1, S_2 and S_3 are periodic functions with period of $(2\pi \times 2\pi)$ and small analytic initial values in an analytic norm (see (13) below). Furthermore, without loss of generality, we assume $\mu = \gamma_1 \alpha_1 + \gamma_2 \alpha_2$, with γ_1 and γ_2 being two constants. The existence of such a set of coordinates has been proved in [15] in which the singularity formation of three-dimensional vortex sheets is investigated.

It has been shown that the vortex sheet problem is not well-posed in any Sobolev norm [7]. Therefore, we establish the well-posedness in an analytic norm. Particularly, we define the Lipschitz norm within a certain complex strip as follows:

$$\|f\|_{\alpha,\rho} = \sup_{\substack{|Im(\kappa_1)|<\rho \\ |Im(\kappa_2)|<\rho}} |f(\kappa_1, \kappa_2)| + \sup_{\substack{|Im(\kappa_1)|<\rho,\ |Im(\kappa_2)|<\rho \\ |Im(\kappa_1')|<\rho,\ |Im(\kappa_2')|<\rho \\ (\kappa_1,\kappa_2)\neq(\kappa_1',\kappa_2')}} \frac{|f(\kappa_1, \kappa_2) - f(\kappa_1', \kappa_2')|}{|(\kappa_1, \kappa_2) - (\kappa_1', \kappa_2')|^\alpha} .$$

The following theorem is our main result.

Theorem 2.1 *(Existence of 3D Vortex Sheet Solutions) Let* $0 < \alpha < 1$, *and* $\rho_0 > 0$. *Assume that* $\mathbf{z}$ *has the form of (12) with* $S_i(\beta_1, \beta_2, 0)$ *satisfying*

$$\sup_{\substack{|Im(\beta_1)|<\rho_0 \\ |Im(\beta_2)|<\rho_0}} \Big(\ |S_i(\beta_1,\beta_2,0)| + |\nabla S_i(\beta_1,\beta_2,0)| + \sum_{k=1}^{2}\sum_{j=1}^{2} |(\partial_{\beta_k}\partial_{\beta_j} S_i(\beta_1,\beta_2,0))| \ \Big) < \varepsilon \ , \tag{13}$$

where $i = 1, 2, 3$ *and* ε *is sufficiently small. Then there exists a solution* $\mathbf{z} = (\beta_1, \beta_2, 0)^T + (S_1, S_2, S_3)^T$ *for a time* $0 \le t \le T$, *where* T *satisfies*

$$T < \frac{2\rho_0}{1+2\kappa} \ .$$

Here $\kappa > 0$ *is a parameter depending on* ε. *It satisfies the properties that* $\kappa \to 0$ *as* $\varepsilon \to 0$ *and* $\kappa^{-2}\varepsilon\rho_0 \to 0$ *as* $\varepsilon \to 0$. *Moreover,* $S_i(t)$ *satisfies:*

$$\sum_{i=1}^{3} \|S_i(t)\|_{\alpha,\rho} \ \le \ c\kappa^{-1}t\varepsilon \le c\kappa^{-1}T\varepsilon \ ,$$

for any ρ *and* t *such that*

$$0 < \rho < \rho_0 - \frac{t}{\frac{1}{2}+\kappa} \ ,$$

where c *is independent of* ε, κ *and* ρ_0.

We remark that existence of 3D vortex sheets has been obtained by Sulem, Sulem, Bardos, & Frisch [24] for general analytic initial data. Our existence result is different from that of [24] in the sense that we are interested in establishing a nearly optimal existence result for slightly perturbed vortex sheets arbitrarily close to the time when the vortex sheet forms a curvature singularity [5, 15]. In order to achieve this objective, it is essential to explore the leading order structure of the 3D vortex sheet equations. The derivation of the leading order system plays a crucial role in obtaining this nearly optimal existence result for slightly perturbed 3D vortex sheets. For the special initial data considered by Brady and Pullin [5] of wave length π and amplitude ϵ, Moore's analysis can be used to show that singularities of order 3/2 develop at $t_c = |\log|\epsilon| + O(\log|\log\epsilon|)$. Our existence result proves existence for $t < \mu|\log\epsilon|$, if ϵ is sufficiently small, with $\mu \to 1$ as $\epsilon \to 0$. In comparison, applying the existence analysis of [24] to this initial data would give only a short time existence result, which is not optimal.

3 A Nonlinear System with Linear Leading Order Terms

In this section, we derive a nonlinear system with linear leading order terms which approximates the full vortex sheet equation (10). As in 2-D vortex sheets, the linear leading order system is of elliptic type, whose initial value problem leads to Kelvin-Helmholtz instability. We will show that the nonlinear terms are small in the Lipschitz norm for analytic solutions within a strip in the complex domain. The bounds of the nonlinear terms are proved rigorously in the next section.

To estimate the growth of Kelvin-Helmholtz instability, we extend the independent variables into the complex domain. With this complexification, the system can be considered as a hyperbolic system with complex characteristic speeds. With its characteristic lines propagating within the complex domain, the ill-posed problem in the physical domain becomes a well-posed problem in the extended complex domain with shrinking analyticity strip.

Before we start deriving the system, it is necessary to introduce the Riesz transforms, which will be used extensively throughout this paper.

Define:

$$H_1(f) = \frac{1}{2\pi} \iint \frac{(\alpha_1 - \alpha_1')f(\alpha')}{((\alpha_1-\alpha_1')^2 + (\alpha_2-\alpha_2')^2)^{\frac{3}{2}}} \, d\alpha' \ , \tag{14}$$

$$H_2(f) = \frac{1}{2\pi} \iint \frac{(\alpha_2 - \alpha_2')f(\alpha')}{((\alpha_1-\alpha_1')^2 + (\alpha_2-\alpha_2')^2)^{\frac{3}{2}}} \, d\alpha' \ , \tag{15}$$

$$\Lambda(f) = \frac{1}{2\pi} \iint \frac{f(\alpha) - f(\alpha')}{((\alpha_1-\alpha_1')^2 + (\alpha_2-\alpha_2')^2)^{\frac{3}{2}}} \, d\alpha' \ ,, \tag{16}$$

for $f \in L^p(R^2)$, where $1 < p < \infty$, $\alpha' = (\alpha_1', \alpha_2')$. The integrals take the Cauchy principal value.

In [23], Stein proved that the Riesz transformations have the following spectral representations:

$$\widehat{H_1 f} = \frac{-i\xi_1}{(\xi_1^2 + \xi_2^2)^{1/2}} \hat{f} \ , \tag{17}$$

$$\widehat{H_2 f} = \frac{-i\xi_2}{(\xi_1^2 + \xi_2^2)^{1/2}} \hat{f} \ , \tag{18}$$

$$\widehat{\Lambda f} = (\xi_1^2 + \xi_2^2)^{1/2} \hat{f} \ , \tag{19}$$

if $(\xi_1, \xi_2) \neq (0,0)$, and $\widehat{H_i f} = 0$ if $(\xi_1, \xi_2) = (0,0)$. Here $\hat{f}$ stands for the Fourier transformation of $f \in L^2(R^2)$. From (17)-(19), we can prove the following Lemma directly.

Lemma 3.1 *Assume that $f \in H^1(R^2)$ and $\hat{f}(0) = 0$, where $H^1(R^2)$ is the Sobolev H^1 space on R^2 and $\hat{f}$ is the Fourier transform of f. Then the following equalities hold:*

$$H_1 H_2(f) = H_2 H_1(f), \tag{20}$$

$$H_1 D_2(f) = H_2 D_1(f), \tag{21}$$

$$(H_1^2 + H_2^2)(f) = -f, \tag{22}$$

$$(H_1 D_1 + H_2 D_2)(f) = \Lambda(f) \tag{23}$$

where D_1 (D_2) stands for derivative operator with respect to α_1 (α_2).

Up to now, we have defined the Riesz transforms for L^2 functions in the infinite domain. We would like to extend the definition to periodic functions. This can be implemented in two ways. One is to use the Fourier representations, in which the Fourier transforms in (17) and (18) will be written in the form of Fourier coefficients for periodic functions. This can be done for Lipschitz continuous functions $Lip_\alpha([0,2\pi]\times[0,2\pi])$ because of the fact that $Lip_\alpha([0,2\pi]\times[0,2\pi]) \subset L^2([0,2\pi]\times[0,2\pi])$.

Another way to extend the definition is to modify the integral kernel. We denote by $K(\alpha_1,\alpha_2)$ the integral kernel of the Riesz transform, and assume that f is periodic with period of $2\pi \times 2\pi$ and

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\alpha_1,\alpha_2)d\alpha_1 d\alpha_2 = 0 \ . \tag{24}$$

The Riesz transform with kernel $K(\alpha_1,\alpha_2)$ can be written as:

$$\begin{aligned}
H_i(f)(\zeta) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\zeta-\alpha)K(\alpha)d\alpha_1 d\alpha_2 \\
&= \lim_{n_1,n_2\to\infty}\int_{-(2n_1+1)\pi}^{(2n_1+1)\pi}\int_{-(2n_2+1)\pi}^{(2n_2+1)\pi} f(\zeta-\alpha)K(\alpha)d\alpha_1 d\alpha_2 \\
&= \lim_{n_1,n_2\to\infty}\sum_{k_1=-n_1}^{n_1}\sum_{k_2=-n_2}^{n_2}\int_{(2k_1-1)\pi}^{(2k_1+1)\pi}\int_{(2k_2-1)\pi}^{(2k_2+1)\pi} f(\zeta-\alpha)K(\alpha)d\alpha_1 d\alpha_2 \\
&= \lim_{n_1,n_2\to\infty}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\zeta-\alpha)[K(\alpha) \\
&\qquad + \sum_{\substack{(k_1,k_2)=(-n_1,-n_2)\\(k_1,k_2)\neq(0,0)}}^{(n_1,n_2)} K(\alpha_1-2k_1\pi,\alpha_2-2k_2\pi)]d\alpha_1 d\alpha_2 \\
&= \lim_{n_1,n_2\to\infty}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\zeta-\alpha)[K(\alpha) \\
&\qquad + \sum_{\substack{(k_1,k_2)=(-n_1,-n_2)\\(k_1,k_2)\neq(0,0)}}^{(n_1,n_2)} (K(\alpha-2k\pi)-K(-2k\pi))]d\alpha_1 d\alpha_2 \ ,
\end{aligned} \tag{25}$$

where $\alpha=(\alpha_1,\alpha_2)$, $\zeta=(\zeta_1,\zeta_2)$, and $k=(k_1,k_2)$ with k_1,k_2 both being integers. We have used the fact that $\int_{[-\pi,\pi]^2} f(\alpha)\,d\alpha = 0$ in the last step.

If for the kernel $K(\alpha)$, the sum

$$\sum_{\substack{(k_1,K_2)=(-n_1,-n_2)\\(k_1,k_2)\neq(0,0)}}^{(n_1,n_2)} (K(\alpha-2k\pi)-K(-2k\pi))$$

converges absolutely and uniformly for $\alpha\in[0,2\pi]\times[0,2\pi]$, we can take the limit into the integral and define:

$$
\begin{aligned}
K^*(\alpha) &= K(\alpha) + \sum_{k\neq(0,0)} [K(\alpha + 2k\pi) - K(2k\pi)] \\
&\equiv K(\alpha) + \overline{K}(\alpha) \ ,
\end{aligned} \tag{26}
$$

for each $(\alpha_1, \alpha_2) \in [0, 2\pi] \times [0, 2\pi]$.

It is well-known [9] that the sum $\overline{K}(\alpha_1, \alpha_2)$ does converge absolutely and uniformly to a bounded function for each $(\alpha_1, \alpha_2) \in [0, 2\pi] \times [0, 2\pi]$. Particularly, K^* converges to $\frac{1}{2}\cot(\frac{1}{2}\alpha)$ if K is the kernel of the Hilbert transform in one dimension. This shows that the Riesz transform for periodic functions is well-defined. Thus, we can write the integral of the Riesz transform either over one period with the periodic kernel or over the infinite domain with its original kernel; both forms are equivalent.

We derive similarly the periodic kernel of the vortex sheet integral. The result is analogous to obtaining the kernel $\frac{1}{2}\cot(\frac{z-z'}{2})$ in one-dimensional space.

Denote:

$$
K_z^1(\alpha, \alpha - \zeta) = \frac{\zeta_1}{|\mathbf{z}(\alpha) - \mathbf{z}(\alpha - \zeta)|^3} \ , \tag{27}
$$

where $\alpha = (\alpha_1, \alpha_2)$ and $\zeta = (\zeta_1, \zeta_2)$. Since $\mathbf{z}$ has the form of (12), the denominator can be re-written as:

$$
|\mathbf{z}(\alpha) - \mathbf{z}(\alpha - \zeta)|^3 = \left| \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ 0 \end{pmatrix} + \begin{pmatrix} S_1(\alpha) - S_1(\alpha - \zeta) \\ S_2(\alpha) - S_2(\alpha - \zeta) \\ S_3(\alpha) - S_3(\alpha - \zeta) \end{pmatrix} \right|^3 . \tag{28}
$$

Using the periodicity of S_1, S_2 and S_3, one can show that

$$
\begin{aligned}
&|\mathbf{z}(\alpha) - \mathbf{z}(\alpha - (\zeta + 2k\pi))|^3 = \\
&\left| \begin{pmatrix} \zeta_1 + 2k_1\pi \\ \zeta_2 + 2k_2\pi \\ 0 \end{pmatrix} + \begin{pmatrix} S_1(\alpha) - S_1(\alpha - \zeta) \\ S_2(\alpha) - S_2(\alpha - \zeta) \\ S_3(\alpha) - S_3(\alpha - \zeta) \end{pmatrix} \right|^3 ,
\end{aligned} \tag{29}
$$

where $k = (k_1, k_2)$. Consequently, we can define:

$$
\begin{aligned}
&K_z^{1*}(\alpha, \alpha - \zeta) \\
&= K_z^1(\alpha, \alpha - \zeta) + \sum_{k\neq(0,0)} (K_z^1(\alpha, \alpha - (\zeta + 2k\pi)) - K^1(2k\pi)) \\
&\equiv K_z^1(\alpha, \alpha - \zeta) + \overline{K_z^1}(\alpha, \alpha - \zeta)
\end{aligned} \tag{30}
$$

where K^1 is the kernel of the Riesz transform in the α_1-direction,

$$
K^1(\zeta) = \frac{\zeta_1}{|\zeta|^3} \ .
$$

Similarly to equation (26), $\overline{K_z^1}$ converges absolutely and uniformly to a bounded function for every $(\alpha_1, \alpha_2) \in [0, 2\pi] \times [0, 2\pi]$ provided that the perturbation from $\mathbf{z}$ to a flat plane is sufficiently small.

We illustrate the above idea for the Hilbert transform, where

$$\begin{aligned} K_z(\alpha, \alpha-\zeta) &= \frac{1}{z(\alpha)-z(\alpha-\zeta)} \\ &= \frac{1}{\zeta+s(\alpha)-s(\alpha-\zeta)} \, , \end{aligned}$$

and

$$K(\alpha) = \frac{1}{\alpha} \, .$$

After some manipulations, we can obtain a closed form for the periodic kernel $K_z^*(\alpha, \alpha-\zeta)$:

$$\begin{aligned} & K_z^*(\alpha, \alpha-\zeta) \\ = \; & K_z(\alpha, \alpha-\zeta) + \sum_{k\neq 0} (K_z(\alpha, \alpha-(\zeta+2k\pi)) - K(2k\pi)) \\ = \; & \frac{1}{z(\alpha)-z(\alpha-\zeta)} + \sum_{k\neq 0} \left(\frac{1}{2k\pi + z(\alpha)-z(\alpha-\zeta)} - \frac{1}{2k\pi} \right) \\ = \; & \frac{1}{2} \cot\left(\frac{1}{2}(z(\alpha)-z(\alpha-\zeta)) \right) \, . \end{aligned} \tag{31}$$

Remarks: 1. By using the periodic kernel mentioned above, the vortex sheet integral can be rewritten as an integral over one period.

2. As we will show rigorously in the last section, if $\mathbf{z}$ is a small perturbation of a flat plane under the Lipschitz norm, i.e. ,

$$\mathbf{z} \sim (\alpha_1, \alpha_2, 0)^T + O(\varepsilon) \, ,$$

the vortex sheet kernel $K_z^{1*}(\alpha, \alpha-\zeta)$ defined in (30) is close to the Riesz transform kernel $K^{1*}(\alpha)$ defined in (26) under the same Lipschitz norm. This observation will be used extensively in the following derivation.

3. All the derivations in this section are formal. We write the remaining terms as $O(\varepsilon^2)$ since they are, as we will show later, of smaller amplitude. At the end of this section, we denote them as R_1, R_2 and R_3 respectively.

Next we derive our leading order system. Based on the assumptions stated in the last section, equation (10) can be re-written as:

$$\begin{aligned} \frac{\partial \mathbf{z}}{\partial t} &= -\frac{1}{4\pi} \iint \frac{(\gamma_1 \mathbf{z}'_{\alpha_2} - \gamma_2 \mathbf{z}'_{\alpha_1}) \times (\mathbf{z}-\mathbf{z}')}{|\mathbf{z}-\mathbf{z}'|^3} d\alpha' \\ &= -\frac{1}{4\pi} \iint \left[\gamma_1 \begin{pmatrix} S'_{1\alpha_2} \\ 1+S'_{2\alpha_2} \\ S'_{3\alpha_2} \end{pmatrix} - \gamma_2 \begin{pmatrix} 1+S'_{1\alpha_1} \\ S'_{2\alpha_1} \\ S'_{3\alpha_1} \end{pmatrix} \right] \\ & \quad \times \begin{pmatrix} \alpha_1-\alpha'_1+S_1-S'_1 \\ \alpha_2-\alpha'_2+S_2-S'_2 \\ S_3-S'_3 \end{pmatrix} \frac{d\alpha'}{|\mathbf{z}-\mathbf{z}'|^3} \, . \end{aligned}$$

Since $S_i \sim O(\varepsilon)$ (see Theorem 2.1), it is reasonable to consider the linear terms in the numerator of the integrand in the above interface equation as the leading order terms. By writing down every component separately and keeping only the linear terms, we obtain the equations for S_1, S_2 and S_3 as follows:

$$
\begin{aligned}
\frac{\partial S_1}{\partial t} &= -\frac{1}{4\pi}\iint \frac{(\gamma_1(1+S'_{2\alpha_2})-\gamma_2 S'_{2\alpha_1})(S_3-S'_3)}{|\mathrm{z}-\mathbf{z}'|^3} \\
&\quad - \frac{(\gamma_1 S'_{3\alpha_2}-\gamma_2 S'_{3\alpha_1})(\alpha_2-\alpha'_2+S_2-S'_2)}{|\mathbf{z}-\mathrm{z}'|^3}\, d\alpha' \\
&= -\frac{1}{4\pi}\iint \frac{\gamma_1(S_3-S'_3)-(\gamma_1 S'_{3\alpha 2}-\gamma_2 S'_{3\alpha_1})(\alpha_2-\alpha'_2)}{|\mathbf{z}-\mathbf{z}'|^3}\, d\alpha' \\
&\quad + O(\varepsilon^2)\ ,
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
\frac{\partial S_2}{\partial t} &= -\frac{1}{4\pi}\iint \frac{(\gamma_1 S'_{3\alpha_2}-\gamma_2 S'_{3\alpha_1})(\alpha_1-\alpha'_1+S_1-S'_1)}{|\mathbf{z}-\mathbf{z}'|^3} \\
&\quad - \frac{(\gamma_1 S'_{1\alpha_2}-\gamma_2(1+S'_{1\alpha_1}))(S_3-S'_3)}{|\mathbf{z}-\mathbf{z}'|^3}\, d\alpha' \\
&= -\frac{1}{4\pi}\iint \frac{\gamma_2(S_3-S'_3)+(\gamma_1 S'_{3\alpha 2}-\gamma_2 S'_{3\alpha_1})(\alpha_1-\alpha'_1)}{|\mathbf{z}-\mathbf{z}'|^3}\, d\alpha' \\
&\quad + O(\varepsilon^2)\ ,
\end{aligned}
\tag{33}
$$

and

$$
\begin{aligned}
\frac{\partial S_3}{\partial t} &= -\frac{1}{4\pi}\iint \frac{(\gamma_1 S'_{1\alpha_2}-\gamma_2(1+S'_{1\alpha_1}))(\alpha_2-\alpha'_2+S_2-S'_2)}{|\mathbf{z}-\mathbf{z}'|^3} \\
&\quad - \frac{(\gamma_1(1+S'_{2\alpha_2})-\gamma_2 S'_{2\alpha_1})(\alpha_1-\alpha'_1+S_1-S'_1)}{|\mathbf{z}-\mathbf{z}'|^3}\, d\alpha' \\
&= -\frac{1}{4\pi}\iint \frac{(\gamma_1 S'_{1\alpha_2}-\gamma_2 S'_{1\alpha_1})(\alpha_2-\alpha'_2)}{|\mathbf{z}-\mathbf{z}'|^3} \\
&\quad - \frac{(\gamma_1 S'_{2\alpha_2}-\gamma_2 S'_{2\alpha_1})(\alpha_1-\alpha'_1)}{|\mathbf{z}-\mathbf{z}'|^3} \\
&\quad - \frac{\gamma_1(\alpha_1-\alpha'_1+S_1-S'_1)+\gamma_2(\alpha_2-\alpha'_2+S_2-S'_2)}{|\mathbf{z}-\mathbf{z}'|^3}\, d\alpha' \\
&\quad + O(\varepsilon^2)\ .
\end{aligned}
\tag{34}
$$

Guided by the stability analysis by Hou and Zhang in [14], we introduce the following change of variables:

$$\psi_1 = H_2(S_1) - H_1(S_2) \ , \qquad (35)$$

$$\psi_2 = H_1(S_1) + H_2(S_2) \ . \qquad (36)$$

The ill-posedness or instability will become more apparent using these new variables. In particular, we will show that using this change of variables, we can remove the Kelvin-Helmholtz instability from the ψ_1 variable to the leading order. The Kelvin-Helmholtz instability is only present through the coupling between ψ_2 and S_3 along certain one-dimensional direction. This observation is essential for us to obtain a nearly optimal existence result.

It follows from Lemma 3.1 that (S_1, S_2) can be represented by (ψ_1, ψ_2) through the following equations:

$$S_1 = -H_2(\psi_1) - H_1(\psi_2) + < S_1 > \ , \qquad (37)$$

$$S_2 = H_1(\psi_1) - H_2(\psi_2) + < S_2 > \ , \qquad (38)$$

where $< S_i >= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} S_i(\alpha) d\alpha$. Therefore, it follows from differentiating equation (35) and (36) with respect to time t that

$$\frac{\partial \psi_1}{\partial t} = H_2(\frac{\partial S_1}{\partial t}) - H_1(\frac{\partial S_2}{\partial t}) \ , \qquad (39)$$

$$\frac{\partial \psi_2}{\partial t} = H_1(\frac{\partial S_1}{\partial t}) + H_2(\frac{\partial S_2}{\partial t}) \ . \qquad (40)$$

To derive the leading order terms of the evolution equation for ψ_1, we substitute (32) and (33) into (39) to obtain

$$\begin{aligned} \frac{\partial \psi_1}{\partial t} &= H_2(\frac{\partial S_1}{\partial t}) - H_1(\frac{\partial S_2}{\partial t}) \\ &= H_2(-\frac{1}{4\pi}\int \frac{\gamma_1(S_3 - S_3') - (\gamma_1 S'_{3\alpha_2} - \gamma_2 S'_{3\alpha_1})(\alpha_2 - \alpha_2')}{|\mathbf{z} - \mathbf{z}'|^3} d\alpha') \\ &\quad - H_1(-\frac{1}{4\pi}\int \frac{\gamma_2(S_3 - S_3') + (\gamma_1 S'_{3\alpha_2} - \gamma_2 S'_{3\alpha_1})(\alpha_1 - \alpha_1')}{|\mathbf{z} - \mathbf{z}'|^3} d\alpha') \\ &\quad + O(\varepsilon^2) \ . \end{aligned} \qquad (41)$$

Further, we observe that the vortex sheet kernel is close to the Riesz transform kernel under the Lipschitz norm. Thus one can show that

$$\begin{aligned} \frac{\partial \psi_1}{\partial t} &= -\frac{1}{2}H_2(\gamma_1 \Lambda S_3 - \gamma_1 H_2 D_2 S_3 + \gamma_2 H_2 D_1 S_3) \\ &\quad + \frac{1}{2}H_1(\gamma_2 \Lambda S_3 + \gamma_1 H_1 D_2 S_3 - \gamma_2 H_1 D_1 S_3) + O(\varepsilon^2) \\ &= -\frac{1}{2}H_2(\gamma_1 H_1 D_1 S_3 + \gamma_2 H_2 D_1 S_3) \\ &\quad + \frac{1}{2}H_1(\gamma_1 H_1 D_2 S_3 + \gamma_2 H_2 D_2 S_3) + O(\varepsilon^2) \\ &= O(\varepsilon^2) \ , \end{aligned} \qquad (42)$$

where we have applied Lemma 3.1 and performing integration by parts in the last step.

Similarly, to derive the leading order terms in the evolution equation of ψ_2, we substitute (32) and (33) into (40) and get

$$\begin{aligned}\frac{\partial \psi_2}{\partial t} &= H_1(\frac{\partial S_1}{\partial t}) + H_2(\frac{\partial S_2}{\partial t}) \\ &= H_1(-\frac{1}{4\pi}\int \frac{\gamma_1(S_3 - S_3') - (\gamma_1 S'_{3\alpha_2} - \gamma_2 S'_{3\alpha_1})(\alpha_2 - \alpha_2')}{|\mathbf{z} - \mathbf{z}'|^3}\, d\alpha') \\ &\quad + H_2(-\frac{1}{4\pi}\int \frac{\gamma_2(S_3 - S_3') + (\gamma_1 S'_{3\alpha_2} - \gamma_2 S'_{3\alpha_1})(\alpha_1 - \alpha_1')}{|\mathbf{z} - \mathbf{z}'|^3}\, d\alpha') \\ &\quad + O(\varepsilon^2)\ . \end{aligned} \tag{43}$$

Using that $|\mathbf{z} - \mathbf{z}'| \sim |\alpha - \alpha'|$ to the leading order, we can further simplify the above expression to

$$\begin{aligned}\frac{\partial \psi_2}{\partial t} &= -\frac{1}{2}H_1(\gamma_1 \Lambda S_3 - \gamma_1 H_2 D_2 S_3 + \gamma_2 H_2 D_1 S_3) \\ &\quad - \frac{1}{2}H_2(\gamma_2 \Lambda S_3 + \gamma_1 H_1 D_2 S_3 - \gamma_2 H_1 D_1 S_3) + O(\varepsilon^2) \\ &= -\frac{1}{2}H_1(\gamma_1 H_1 D_1 S_3 + \gamma_2 H_2 D_1 S_3) \\ &\quad - \frac{1}{2}H_2(\gamma_1 H_1 D_2 S_3 + \gamma_2 H_2 D_2 S_3) + O(\varepsilon^2)\ . \end{aligned} \tag{44}$$

By applying Lemma 3.1, we reduce the above equation to

$$\begin{aligned}\frac{\partial \psi_2}{\partial t} &= -\frac{1}{2}\gamma_1 D_1(H_1H_1 + H_2H_2)S_3 \\ &\quad - \frac{1}{2}\gamma_2 D_2(H_1H_1 + H_2H_2)S_3 + O(\varepsilon^2) \\ &= \frac{1}{2}(\gamma_1 D_1 + \gamma_2 D_2)S_3 + O(\varepsilon^2)\ . \end{aligned} \tag{45}$$

For the evolution equation of S_3, we substitute (37) and (38) into (34), and extract the leading order terms:

$$
\begin{aligned}
\frac{\partial S_3}{\partial t} \;=\; & -\frac{1}{4\pi}\iint \frac{\gamma_1(-H_2(\psi_1') - H_1(\psi_2'))_{\alpha_2}(\alpha_2-\alpha_2')}{|\mathbf{z}-\mathbf{z}'|^3} \\
& + \frac{\gamma_2(H_2(\psi_1') + H_1(\psi_2'))_{\alpha_1}(\alpha_2-\alpha_2')}{|\mathbf{z}-\mathbf{z}'|^3} \\
& + \frac{\gamma_2(H_1(\psi_1') - H_2(\psi_2'))_{\alpha_1}(\alpha_1-\alpha_1')}{|\mathbf{z}-\mathbf{z}'|^3} \\
& - \frac{\gamma_1(H_1(\psi_1') - H_2(\psi_2'))_{\alpha_2}(\alpha_1-\alpha_1')}{|\mathbf{z}-\mathbf{z}'|^3} \\
& - \frac{\gamma_2(H_1(\psi_1-\psi_1') - H_2(\psi_2-\psi_2'))}{|\mathbf{z}-\mathbf{z}'|^3} \\
& - \frac{\gamma_1(-H_2(\psi_1-\psi_1') - H_1(\psi_2-\psi_2'))}{|\mathbf{z}-\mathbf{z}'|^3} \\
& - \frac{\gamma_1(\alpha_1-\alpha_1') + \gamma_2(\alpha_2-\alpha_2')}{|\mathbf{z}-\mathbf{z}'|^3}\,d\alpha' + O(\varepsilon^2)\ . \qquad (46)
\end{aligned}
$$

By using Lemma 3.1, and the fact that $|\mathbf{z}-\mathbf{z}'| \sim |\alpha-\alpha'|$ to the leading order, we obtain

$$
\begin{aligned}
\frac{\partial S_3}{\partial t} \;=\; & \frac{1}{2}\gamma_1 H_2^2 D_2\psi_1 + \frac{1}{2}\gamma_1 H_2 H_1 D_2\psi_2 - \frac{1}{2}\gamma_2 H_2^2 D_1\psi_1 - \frac{1}{2}\gamma_2 H_2 H_1 D_1\psi_2 \\
& + \frac{1}{2}\gamma_1 H_1^2 D_2\psi_1 - \frac{1}{2}\gamma_1 H_1 H_2 D_2\psi_2 - \frac{1}{2}\gamma_2 H_1^2 D_1\psi_1 + \frac{1}{2}\gamma_2 H_1 H_2 D_1\psi_2 \\
& + \frac{1}{2}\gamma_2\Lambda H_1\psi_1 - \frac{1}{2}\gamma_2\Lambda H_2\psi_2 - \frac{1}{2}\gamma_1\Lambda H_2\psi_1 - \frac{1}{2}\gamma_1\Lambda H_1\psi_2 \\
& + \frac{1}{4\pi}\int \frac{\gamma_2(\alpha_2-\alpha_2') + \gamma_1(\alpha_1-\alpha_1')}{|\mathbf{z}-\mathbf{z}'|^3}\,d\alpha' + O(\varepsilon^2) \\
\;=\; & \frac{1}{2}(\gamma_1 D_1 + \gamma_2 D_2)\psi_2 \\
& + \frac{1}{4\pi}\int \frac{\gamma_2(\alpha_2-\alpha_2') + \gamma_1(\alpha_1-\alpha_1')}{|\mathbf{z}-\mathbf{z}'|^3}\,d\alpha' + O(\varepsilon^2)\ . \qquad (47)
\end{aligned}
$$

It is necessary to analyze the integral term of equation (47) and extract the leading order contributions. By further expanding the integral in terms of S_i's, we find that the leading order terms are:

$$
\begin{aligned}
& \frac{1}{4\pi}\int \frac{\gamma_2(\alpha_2-\alpha_2') + \gamma_1(\alpha_1-\alpha_1')}{|\mathbf{z}-\mathbf{z}'|^3}\,d\alpha' \\
= \; & -\frac{3}{4\pi}\int \frac{\gamma_2(\alpha_2-\alpha_2')[(S_1-S_1')(\alpha_1-\alpha_1') + (S_2-S_2')(\alpha_2-\alpha_2')]}{|\mathbf{z}-\mathbf{z}'|^5} \\
+ \; & O(\epsilon^2)\ , \qquad (48)
\end{aligned}
$$

where we have applied the matrix equality of

$$
\nabla_{z'}\nabla_{z'}G(\mathbf{z}-\mathbf{z}') = \frac{1}{4\pi}\left(\frac{I}{|\mathbf{z}-\mathbf{z}'|^3} - \frac{3(\mathbf{z}-\mathbf{z}')(\mathbf{z}-\mathbf{z}')^T}{|\mathbf{z}-\mathbf{z}'|^5}\right)
$$

from [13]. Therefore, we obtain using Lemma 3.1 and integration by parts that

$$\begin{aligned}
&\frac{1}{4\pi}\int \frac{\gamma_2(\alpha_2-\alpha_2')+\gamma_1(\alpha_1-\alpha_1')}{|\mathbf{z}-\mathbf{z}'|^3}\,d\alpha' \\
&\qquad + \frac{\gamma_1(\alpha_1-\alpha_1')[(S_1-S_1')(\alpha_1-\alpha_1')+(S_2-S_2')(\alpha_2-\alpha_2')]}{|\mathbf{z}-\mathbf{z}'|^5} + O(\varepsilon^2) \\
&= -\frac{1}{2}\gamma_2(H_1D_2(S_1)+(2H_2D_2+H_1D_1)(S_2)) \\
&\quad -\frac{1}{2}\gamma_1((2H_1D_1+H_2D_2)(S_1)+H_1D_2(S_2))+O(\varepsilon^2)\ . \qquad (49)
\end{aligned}$$

By substituting (37) and (38) into (49), we write the leading order terms in terms of ψ_1 and ψ_2:

$$\begin{aligned}
&\frac{1}{4\pi}\int \frac{\gamma_2(\alpha_2-\alpha_2')+\gamma_1(\alpha_1-\alpha_1')}{|\mathbf{z}-\mathbf{z}'|^3}\,d\alpha' \\
&= -\frac{1}{2}\gamma_2(H_1D_2)(-H_2\psi_1-H_1\psi_2) \\
&\quad -\frac{1}{2}\gamma_2(2H_2D_2+H_1D_1)(H_1\psi_1-H_2\psi_2) \\
&\quad -\frac{1}{2}\gamma_1(2H_1D_1+H_2D_2)(-H_2\psi_1-H_1\psi_2) \\
&\quad -\frac{1}{2}\gamma_1H_1D_2(H_1\psi_1-H_2\psi_2)+O(\varepsilon^2)\ , \qquad (50)
\end{aligned}$$

which can be further simplified to

$$\begin{aligned}
&\frac{1}{4\pi}\int \frac{\gamma_2(\alpha_2-\alpha_2')+\gamma_1(\alpha_1-\alpha_1')}{|\mathbf{z}-\mathbf{z}'|^3}\,d\alpha' \\
&= -\frac{1}{2}(H_1D_1+H_2D_2)(-\gamma_1H_2+\gamma_2H_1)\psi_1 \\
&\quad +(H_1D_1+H_2D_2)(\gamma_1H_1+\gamma_2H_2)\psi_2+O(\varepsilon^2)) \\
&= -(\gamma_1D_1+\gamma_2D_2)\psi_2-\frac{1}{2}(\gamma_1D_2-\gamma_2D_1)\psi_1+O(\varepsilon^2)\ . \qquad (51)
\end{aligned}$$

To unify our notations, we define

$$\psi_3 = S_3 \qquad (52)$$

Combining (32), (33), and (47) into a system we get

$$\frac{\partial\psi_1}{\partial t} = O(\varepsilon^2)\ , \qquad (53)$$

$$\frac{\partial\psi_2}{\partial t} = \frac{1}{2}(\gamma_1D_1+\gamma_2D_2)\psi_3+O(\varepsilon^2)\ , \qquad (54)$$

$$\frac{\partial\psi_3}{\partial t} = -\frac{1}{2}(\gamma_1D_1+\gamma_2D_2)\psi_2+\frac{1}{2}(\gamma_2D_1-\gamma_1D_2)\psi_1+O(\varepsilon^2)\ , \qquad (55)$$

where D_1 (D_2) stands for differentiation with respect to the α_1 (α_2) variable.

We compare our leading order terms to the linearized system derived in the article of Hou & Zhang [14]. Their linearized system is

$$\begin{aligned}
\frac{\partial \dot{\psi}_1}{\partial t} &= 0 \ , \\
\frac{\partial \dot{\psi}_2}{\partial t} &= \frac{1}{2}(\gamma_1 D_1 + \gamma_2 D_2)\dot{z} \ , \\
\frac{\partial \dot{z}}{\partial t} &= -\frac{1}{2}(\gamma_1 D_1 + \gamma_2 D_2)\dot{\psi}_2 + \frac{1}{2}(\gamma_2 D_1 - \gamma_1 D_2)(\dot{\psi}_1) \ ,
\end{aligned}$$

where $\dot{\psi}_1$,$\dot{\psi}_2$ and $\dot{z}$ are perturbations of ψ_1, ψ_2 and z respectively. This comparison confirms that our linear system does capture the leading order terms of the three-dimensional vortex sheet equation when perturbed around an equilibrium state.

Since γ_1 and γ_2 are constants, we introduce a change of variables from (α_1, α_2) to (β_1, β_2) as follows:

$$\beta_1 = \frac{1}{\sqrt{\gamma_1^2 + \gamma_2^2}}(-\gamma_2 \alpha_1 + \gamma_1 \alpha_2) \ , \tag{56}$$

$$\beta_2 = \frac{1}{\sqrt{\gamma_1^2 + \gamma_2^2}}(\gamma_1 \alpha_1 + \gamma_2 \alpha_2) \ . \tag{57}$$

Upon substitution of this variable change into system (53) – (55), we obtain the leading order system as

$$\frac{\partial \psi_1}{\partial t} = O(\varepsilon^2) \ , \tag{58}$$

$$\frac{\partial \psi_2}{\partial t} = \frac{1}{2}\gamma D_{\beta_2}\psi_3 + O(\varepsilon^2) \ , \tag{59}$$

$$\frac{\partial \psi_3}{\partial t} = -\frac{1}{2}\gamma D_{\beta_2}\psi_2 - \frac{1}{2}\gamma D_{\beta_1}\psi_1 + O(\varepsilon^2) \ , \tag{60}$$

where $\gamma = \sqrt{\gamma_1^2 + \gamma_2^2}$. In the new coordinates, we can see that the system will suffer from the Kelvin-Helmholtz instability because of the coupling of (59) and (60). It also shows that the β_2 direction is the unstable direction responsible for generating Kelvin-Helmholtz instability. Moreover, since the β_2 direction is the tangential velocity jump direction between the upper and lower layers of fluid, the leading order terms confirm that the tangential velocity jump is the physical driving force of the instability of the three-dimensional vortex sheet.

During the above derivation, we write down only the leading order terms. As we will show later, the remaining terms on the right-hand side of the equations are of smaller magnitude. We denote them as R_1, R_2, and R_3 and define them as:

$$R_1(\psi_1, \psi_2, \psi_3) = \frac{\partial \psi_1}{\partial t} \ , \tag{61}$$

$$R_2(\psi_1, \psi_2, \psi_3) = \frac{\partial \psi_2}{\partial t} - \frac{1}{2}\gamma D_{\beta_2}\psi_3 \ , \tag{62}$$

$$R_3(\psi_1, \psi_2, \psi_3) = \frac{\partial \psi_3}{\partial t} + \frac{1}{2}\gamma D_{\beta_2}\psi_2 + \frac{1}{2}\gamma D_{\beta_1}\psi_1 \ . \tag{63}$$

Next, we extend the independent variables β_1 and β_2 into the complex domain. We remark that it is important that we complexify β_1 and β_2 variables instead of α_1 and α_2 variables. The leading order singular structure will become more apparent using the complexification of β_1 and β_2 variables. As a result of this complexification of the independent variables, we can analytically continue the system into the two-dimensional complex domain. We assume that S_1, S_2, and S_3 are initially small analytic functions within a strip of $\max(|Im(\alpha_1)|, |Im(\alpha_2)|) < \rho$, where the strip width ρ depends on their initial amplitude ε. Since ψ_1, ψ_2 are Riesz transforms of S_1 and S_2, it can be shown that ψ_1 and ψ_2 are also analytic functions within the same strip, since the Riesz transforms preserve analyticity. Therefore, from the fact that γ_1 and γ_2 are constants, we conclude that ψ_1, ψ_2, and ψ_3 are initially analytic functions in the strip of $\max(|Im(\beta_1)|, |Im(\beta_2)|) < \rho_0$, where ρ_0 and ρ are of the same order.

By analytically extending system (61) – (63) into the complex domain, the system can be considered as a hyperbolic system with complex characteristic speeds. Furthermore, simple calculations show that the characteristic speeds of the linear system are 0 and $\pm\frac{i\gamma}{2}$. Later, we will prove that these are the leading order terms of the characteristic speed of the nonlinear system.

Remark: Even though we will analyze a complex system instead of a real system, the domain of interest is the real (β_1, β_2) plane. This means that we have the flexibility to shrink the imaginary strip without affecting the physical solution.

As was done in [7], we differentiate the governing equation (61) to reduce a nonlinear system to a quasilinear system. The resulting system is larger, but more amenable for error control for the high order terms. Specifically, we differentiate (61)–(63) in space and derive a system of the space derivatives of ψ_1, ψ_2 and ψ_3. Define

$$\psi_{11} = \frac{\partial \psi_1}{\partial \beta_1} , \qquad \psi_{12} = \frac{\partial \psi_1}{\partial \beta_2} , \tag{64}$$

$$\psi_{21} = \frac{\partial \psi_2}{\partial \beta_1} , \qquad \psi_{22} = \frac{\partial \psi_2}{\partial \beta_2} , \tag{65}$$

$$\psi_{31} = \frac{\partial \psi_3}{\partial \beta_1} , \qquad \psi_{32} = \frac{\partial \psi_3}{\partial \beta_2} . \tag{66}$$

Then equations (61) – (63) become:

$$\frac{\partial \psi_{11}}{\partial t} = \frac{\partial}{\partial \beta_1} R_1(\psi_1, \psi_2, \psi_3) , \tag{67}$$

$$\frac{\partial \psi_{12}}{\partial t} = \frac{\partial}{\partial \beta_2} R_1(\psi_1, \psi_2, \psi_3) , \tag{68}$$

$$\frac{\partial \psi_{21}}{\partial t} = \frac{\gamma}{2} D_{\beta_1} \psi_{32} + \frac{\partial}{\partial \beta_1} R_2(\psi_1, \psi_2, \psi_3) , \tag{69}$$

$$\frac{\partial \psi_{22}}{\partial t} = \frac{\gamma}{2} D_{\beta_2} \psi_{32} + \frac{\partial}{\partial \beta_2} R_2(\psi_1, \psi_2, \psi_3) , \tag{70}$$

$$\frac{\partial \psi_{31}}{\partial t} = -\frac{\gamma}{2} D_{\beta_1} \psi_{22} - \frac{\gamma}{2} D_{\beta_1} \psi_{11} + \frac{\partial}{\partial \beta_1} R_3(\psi_1, \psi_2, \psi_3) , \tag{71}$$

$$\frac{\partial \psi_{32}}{\partial t} = -\frac{\gamma}{2} D_{\beta_2} \psi_{22} - \frac{\gamma}{2} D_{\beta_2} \psi_{11} + \frac{\partial}{\partial \beta_3} R_3(\psi_1, \psi_2, \psi_3) . \tag{72}$$

This is our target nonlinear system. Further, we define

$$E_{11} = \frac{\partial}{\partial \beta_1} R_1, \qquad E_{12} = \frac{\partial}{\partial \beta_2} R_1, \tag{73}$$

$$E_{21} = \frac{\partial}{\partial \beta_1} R_2, \qquad E_{22} = \frac{\partial}{\partial \beta_2} R_2, \tag{74}$$

$$E_{31} = \frac{\partial}{\partial \beta_1} R_3, \qquad E_{32} = \frac{\partial}{\partial \beta_2} R_3. \tag{75}$$

Assuming that $E'_{ij}s$ are given, to solve system (67) – (72), we can first solve system (67), (68), (70), (72) by integrating along the characteristic lines, and then substituting the calculated solutions into system (69), (71) and solve the resulting O.D.E.. The procedure will be used in the last section where we prove a lemma on energy estimates. Furthermore, without loss of generality, we assume that $\gamma = 1$ for the rest of the paper. In this case, the leading order term of the characteristic speed becomes $\frac{1}{2}$.

4 The Existence Proof

In this section, we prove the main result of this paper, the nearly optimal existence of the three-dimensional vortex sheet equation. The main idea is to separate the governing equation (67) – (72) into a leading order linear system and a smaller nonlinear system. The linear system can be analyzed easily. We will apply the extension of the abstract Cauchy-Kowalewski Theorem introduced by Caflisch & Orellana [7] to estimate the nonlinear system. This requires estimates of the nonlinear terms in system (67) – (72). Since the estimation itself is rather technical, to show a clearer outline of our main proof, we just state the results in this section and leave the detailed derivation to the next section.

The subsections in this section are arranged as follows. In the next subsection, we present two lemmas about error estimates and energy estimates respectively. The proofs of the lemmas are deferred to the next section. In subsection 2, we state the extended abstract Cauchy-Kowalewski theorem. Furthermore, we devote subsection 3 to solving the linear system with full initial condition as the first part of the solution to the full nonlinear system. The existence of the second part of the solution will be proved in the last subsection.

4.1 Results on Error Estimates and Linear Systems

In this subsection, we state two lemmas related to the error estimates and energy estimates respectively. First of all, it is necessary to define the following Lipschitz norms:

$$|f|_\rho = \sup_{\substack{|Im(\kappa_1)|<\rho \\ |Im(\kappa_2)|<\rho}} |f(\kappa_1, \kappa_2)| \,, \tag{76}$$

$$\|f\|_{\alpha,\rho} = |f|_\rho + \sup_{\substack{|Im(\kappa_1)|<\rho,\ |Im(\kappa_2)|<\rho \\ |Im(\kappa'_1)|<\rho,\ |Im(\kappa'_2)|<\rho \\ (\kappa_1,\kappa_2)\neq(\kappa'_1,\kappa'_2)}} \frac{|f(\kappa_1,\kappa_2) - f(\kappa'_1,\kappa'_2)|}{|(\kappa_1,\kappa_2) - (\kappa'_1,\kappa'_2)|^\alpha} \,, \tag{77}$$

$$\|f\|_{\alpha,\rho+} = |f|_\rho + \sup_{\substack{|Im(\kappa_1)|<\rho,\ |Im(\kappa_2)|<\rho \\ |Im(\kappa_1')|<\rho,\ |Im(\kappa_2')|<\rho \\ Im(\kappa_1)=Im(\kappa_1'),\ Im(\kappa_2)=Im(\kappa_2') \\ (\kappa_1,\kappa_2)\neq(\kappa_1',\kappa_2')}} \frac{|f(\kappa_1,\kappa_2) - f(\kappa_1',\kappa_2')|}{|(\kappa_1,\kappa_2) - (\kappa_1',\kappa_2')|^\alpha}\ , \tag{78}$$

where $0 < \alpha < 1$ and $|(\kappa_1, \kappa_2) - (\kappa_1', \kappa_2')| = \sqrt{(\kappa_1 - \kappa_1')^2 + (\kappa_2 - \kappa_2')^2}$. We also define $\|\cdot\|_{1,\alpha,\rho}$ and $\|\cdot\|_{2,\alpha,\rho}$ as the $\|\cdot\|_{\alpha,\rho}$ norm on the α_1 and α_2 directions respectively:

$$\|f\|_{1,\alpha,\rho} = |f|_\rho + \sup_{\substack{\max(|Im(\kappa_1)|,|Im(\kappa_1')|,|Im(\kappa_2)|)<\rho \\ \kappa_1\neq\kappa_1'}} \frac{|f(\kappa_1,\kappa_2) - f(\kappa_1',\kappa_2)|}{|(\kappa_1 - \kappa_1')|^\alpha}\ , \tag{79}$$

$$\|f\|_{2,\alpha,\rho} = |f|_\rho + \sup_{\substack{\max(|Im(\kappa_1)|,|Im(\kappa_2)|,|Im(\kappa_2')|)<\rho \\ \kappa_2\neq\kappa_2'}} \frac{|f(\kappa_1,\kappa_2) - f(\kappa_1,\kappa_2')|}{|(\kappa_2 - \kappa_2')|^\alpha}\ . \tag{80}$$

Similarly, we can define $\|\cdot\|_{1,\alpha,\rho+}$ and $\|\cdot\|_{2,\alpha,\rho+}$ for the $\|\cdot\|_{\alpha,\rho+}$ norm on the α_1 and α_2 directions respectively:

$$\|f\|_{1,\alpha,\rho+} = |f|_\rho + \sup_{\substack{\max(|Im(\kappa_1)|,|Im(\kappa_1')|,|Im(\kappa_2)|)<\rho \\ Im(\kappa_1)=Im(\kappa_1') \\ \kappa_1\neq\kappa_1'}} \frac{|f(\kappa_1,\kappa_2) - f(\kappa_1',\kappa_2)|}{|(\kappa_1 - \kappa_1')|^\alpha}\ , \tag{81}$$

$$\|f\|_{2,\alpha,\rho+} = |f|_\rho + \sup_{\substack{\max(|Im(\kappa_1)|,|Im(\kappa_2)|,|Im(\kappa_2')|)<\rho \\ Im(\kappa_2)=Im(\kappa_2') \\ \kappa_2\neq\kappa_2'}} \frac{|f(\kappa_1,\kappa_2) - f(\kappa_1,\kappa_2')|}{|(\kappa_2 - \kappa_2')|^\alpha}\ . \tag{82}$$

Note that for $\kappa_1 \neq \kappa_1'$ and $\kappa_2 \neq \kappa_2'$,

$$\begin{aligned} & \frac{|f(\kappa_1,\kappa_2) - f(\kappa_1',\kappa_2')|}{|(\kappa_1,\kappa_2) - (\kappa_1',\kappa_2')|^\alpha} \\ \leq\ & \frac{|f(\kappa_1,\kappa_2) - f(\kappa_1',\kappa_2)|}{|(\kappa_1,\kappa_2) - (\kappa_1',\kappa_2')|^\alpha} + \frac{|f(\kappa_1',\kappa_2) - f(\kappa_1',\kappa_2')|}{|(\kappa_1,\kappa_2) - (\kappa_1',\kappa_2')|^\alpha} \\ \leq\ & \frac{|f(\kappa_1,\kappa_2) - f(\kappa_1',\kappa_2)|}{|(\kappa_1,\kappa_2) - (\kappa_1',\kappa_2)|^\alpha} + \frac{|f(\kappa_1',\kappa_2) - f(\kappa_1',\kappa_2')|}{|(\kappa_1',\kappa_2) - (\kappa_1',\kappa_2')|^\alpha} \\ =\ & \frac{|f(\kappa_1,\kappa_2) - f(\kappa_1',\kappa_2)|}{|\kappa_1 - \kappa_1'|^\alpha} + \frac{|f(\kappa_1',\kappa_2) - f(\kappa_1',\kappa_2')|}{|\kappa_2' - \kappa_2'|^\alpha}\ . \end{aligned} \tag{83}$$

We conclude that $\|\cdot\|_{\alpha,\rho}$ is equivalent to $(\|\cdot\|_{1,\alpha,\rho} + \|\cdot\|_{2,\alpha,\rho})$. Similarly, we can show that $\|\cdot\|_{\alpha,\rho+}$ is equivalent to $(\|\cdot\|_{1,\alpha,\rho+} + \|\cdot\|_{2,\alpha,\rho+})$. This property will be used extensively in the later analysis.

Definitions (79), (80), (81), and (82) are natural extensions from the corresponding one-dimensional norms. In particular, Caflisch and Orellana [7] have proved that for one-dimensional analytic functions:

$$\|f\|_{\alpha,\rho} \leq c\|f\|_{\alpha,\rho+}\ ,$$
$$\|f\|_{\alpha,\rho} \leq c\|f\|_{\alpha,\rho+}\ .$$

Therefore, for two-dimensional analytic functions, the following inequality holds,

$$\|f\|_{\alpha,\rho} \leq (\|f\|_{1,\alpha,\rho} + \|f\|_{2,\alpha,\rho}) \leq c(\|f\|_{1,\alpha,\rho+} + \|f\|_{2,\alpha,\rho+}) \leq 2c\|f\|_{\alpha,\rho+} \ . \tag{84}$$

In the next section, we will prove that

Lemma 4.1 *Assume that x, y, z defined in (1) are small perturbations of a flat plane, and S_1, S_2, S_3 satisfy*

1. *S_1, S_2, S_3 are analytic functions within the strip*

$$\{(\beta_1, \beta_2) | \max(|Im(\beta_1)|, |Im(\beta_2)|) < \rho\} \ ;$$

2. *S_1, S_2, S_3 are periodic functions with period of $(2\pi, 2\pi)$;*

3.

$$\|S_j\|_\alpha(\cdot + i\mu_1, \cdot + i\mu_2) \ \ \leq \ \ \frac{1}{8} \ , \quad j = 1, 2, 3,$$

where

$$\begin{aligned} \|f\|_\alpha(\cdot + i\mu_1, \cdot + i\mu_2) \ \ &= \ \ \sup_{Im(\kappa_1)=\mu_1, Im(\kappa_2)=\mu_2} |f(\kappa_1, \kappa_2)| \\ &+ \ \ \sup_{\substack{Im(\kappa_1)=\mu_1,\ Im(\kappa_2)=\mu_2 \\ Im(\kappa_1')=\mu_1,\ Im(\kappa_2')=\mu_2 \\ (\kappa_1,\kappa_2)\neq(\kappa_1',\kappa_2')}} \frac{|f(\kappa_1, \kappa_2) - f(\kappa_1', \kappa_2')|}{|(\kappa_1, \kappa_2) - (\kappa_1', \kappa_2')|^\alpha} \ , \end{aligned}$$

and (μ_1, μ_2) satisfy $|\mu_1| < \rho$ and $|\mu_2| < \rho$.

Furthermore, we assume that ψ_{ij} and $\tilde{\psi}_{ij}$ are two sets of functions defined in (35), (36), and (64) – (66). Then, for $0 < \rho' < \rho$ and $0 < \alpha < 1$, the following inequalities hold:

$$\|E_{ij}\|_{\alpha,\rho'} \leq c(\rho - \rho')^{-1} \left(\sum_{k_1=1}^{3} \sum_{k_2=1}^{2} \|\psi_{k_1 k_2}\|_{\alpha,\rho} \right)^2 \ , \tag{85}$$

and

$$\begin{aligned} \|E_{ij} - \tilde{E}_{ij}\|_{\alpha,\rho'} \ \ &\leq \ \ c(\rho - \rho')^{-1} \left(\sum_{k_1=1}^{3} \sum_{k_2=1}^{2} (\|\psi_{k_1 k_2}\|_{\alpha,\rho} + \|\tilde{\psi}_{k_1 k_2}\|_{\alpha,\rho} \right) \\ &\cdot \left(\sum_{k_1=1}^{3} \sum_{k_2=1}^{2} \|\psi_{k_1 k_2} - \tilde{\psi}_{k_1, k_2}\|_{\alpha,\rho} \right) \ , \end{aligned} \tag{86}$$

where $i, j = 1, 2, 3$, and E_{ij} and $\tilde{E}_{ij}$ are defined in (73) – (75).

Lemma 4.2 *Consider the following system of* $\mathbf{u} = (u_{11}, u_{12}, u_{21}, u_{22}, u_{31}, u_{32})^T$ *with analytic forcing terms* $\mathbf{g} = (g_{11}, g_{12}, g_{21}, g_{22}, g_{31}, g_{32})^T$,

$$\frac{\partial u_{11}}{\partial t} = g_{11} \ , \tag{87}$$

$$\frac{\partial u_{12}}{\partial t} = g_{12} \ , \tag{88}$$

$$\frac{\partial u_{21}}{\partial t} = \frac{1}{2} D_{\beta_1} u_{32} + g_{21} \ , \tag{89}$$

$$\frac{\partial u_{22}}{\partial t} = \frac{1}{2} D_{\beta_2} u_{32} + g_{22} \ , \tag{90}$$

$$\frac{\partial u_{31}}{\partial t} = -\frac{1}{2} D_{\beta_1} u_{22} - \frac{1}{2} D_{\beta_1} u_{11} + g_{31} \ , \tag{91}$$

$$\frac{\partial u_{32}}{\partial t} = -\frac{1}{2} D_{\beta_2} u_{22} - \frac{1}{2} D_{\beta_2} u_{11} + g_{32} \ , \tag{92}$$

with zero initial condition

$$\mathbf{u}|_{t=0} = 0 \ . \tag{93}$$

Then, the following inequality holds,

$$\|\mathbf{u}\|_{\alpha,\rho} \le c\kappa^{-1} \int_0^t \|\mathbf{g}(\cdot,\cdot,\tau)\|_{\alpha,(\rho+(\frac{1}{2}+\kappa)(t-\tau))} d\tau \ , \tag{94}$$

for any $0 < \kappa < 1$, $0 < \alpha < 1$, *with* c *being a constant.*

4.2 Extended Abstract Cauchy-Kowalewski Theorem

In this subsection, we state the result of the extended abstract Cauchy-Kowalewski theorem for future use.

Consider the equation

$$\frac{\partial}{\partial t} u + L[u,t] = G[u,t] \ , \tag{95}$$

with

$$u|_{t=0} = 0 \ , \tag{96}$$

in which L is a linear operator on u, and G may be nonlinear. Assume that there are positive functions $\rho_0(t)$, $\rho_1(\tau, t, \rho)$, $d_1(\tau, t)$, $d_2(t)$ and positive scalars c_1, c_2, c_3, R and K that satisfy the following conditions:

1. If u solves $(\partial/\partial t)u + L[u,t] = g(t)$, with $u(t=0) = 0$ for some g, then for any $\rho < \rho_0(t)$,

$$\|u(t)\|_\rho \le c_1 \int_0^t d_1(\tau, t) \|g(\cdot, \tau)\|_{\rho_1(\tau,t,\rho)} d\tau \ ; \tag{97}$$

2. If $\|u(t)\|_\rho \le R$, $\|\tilde{u}(t)\|_\rho \le R$, and $0 < \rho' < \rho < \rho_0(t)$, then

$$\|G[u,t] - G[\tilde{u},t]\|_{\rho'} \le c_2(\rho-\rho')^{-1}(d_2(t) + \|u(t)\|_\rho + \|\tilde{u}(t)\|_\rho)\|(u-\tilde{u})(t)\|_\rho \ , \tag{98}$$

   in which d_2 is an increasing function of t;

3. $\|G[u=0,t]\|_\rho \le K d_2(t)(\rho_0(t)-\rho)^{-1}$ if $\rho < \rho_0(t)$;

4. ρ is positive and decreasing for $0 < t < T_0$; $\rho_1(\tau,t,\rho)$ is decreasing in τ and increasing in ρ. Moreover, if $0 < \tau < t$ and $0 < \rho < \rho_0(t)$

$$\rho < \rho_0(\tau,t,\rho) \le \rho_0(\tau) - (\rho_0(t) - \rho) \ ; \tag{99}$$

5. If $0 \le \tau \le t < T_0$, then

$$d_1(\tau,t)d_2(\tau) < d_2(t) < c_3 \ . \tag{100}$$

Theorem 4.1 *(Extended Abstract Cauchy-Kowalewski Theorem) Under assumption (1)-(5) above, equation (95) with initial condition (96) has a unique solution u for the time interval $0 \le t \le T$. The solution satisfies*

$$\|u(t)\|_\rho < \beta d_2(t) < R \tag{101}$$

for all ρ and $t < T$ for which $0 < t < a(\rho_0(t) - \rho)$, where a, β, and T are any numbers satisfying

$$\gamma(1+2\beta)a < 1, \tag{102}$$

$$2ac_1K(3-\gamma(1+2\beta)a)(1-\gamma(1+2\beta)a)^{-3} < \beta, \tag{103}$$

$$T = \min(T_0, \max(t : 2\beta d_2(t) < R)), \tag{104}$$

with $\gamma = 8c_1c_2c_3$ and $R, d_2, K, c_1, c_2, c_3, T_0$ defined in (1)-(5).

We refer to [20] or the Appendix C of [7] for the proof of the above extended abstract Cauchy-Kowalewski theorem.

Remark: 1. In the proof of [7], Condition (2) was replaced by:

$$\|G[u,t] - G[\tilde{u},t]\|_{\rho'} \le c_2(\rho-\rho')^{-1}(d_2(t) + \|u(t)\|_{\rho'} + \|\tilde{u}(t)\|_{\rho'})\|(u-\tilde{u})(t)\|_\rho \ . \tag{105}$$

The same proof can be carried out using the Condition (2) in our statement, and this does not effect the result.

2. The proof uses the iteration method since the system is basically a linear system with weak nonlinear terms. Among all the constraints, Condition (1) provides the energy estimates in every step of the iteration. Condition (2) controls the nonlinear terms during the iteration. Condition (3) describes the nonlinear terms at the initial moment.

3. In our case, $\rho_0(t)$ is to describe the outer boundary where ψ_{ij} are still small and analytic at time t for our nonlinear problem. The function $\rho_1(\tau, t, \rho)$ corresponds to the downwards moving characteristic for the linear problem in Condition (1). Condition (4) says that the linear characteristic stays within the domain of dependence for the nonlinear problem.

4. The inequalities (102), (103), and (104) are due to some technical estimates in the proof. These inequalities set a bound on a, which is the speed in which the complex domain shrinks in addition to $\rho_0(t)$.

4.3 Linear System with Full Initial Condition

In this subsection, we study the linear system with full initial condition. Consider the following linear system:

$$\begin{aligned}
\frac{\partial \tilde{\psi}_{11}}{\partial t} &= 0 \,, && (106)\\
\frac{\partial \tilde{\psi}_{12}}{\partial t} &= 0 \,, && (107)\\
\frac{\partial \tilde{\psi}_{21}}{\partial t} &= \frac{1}{2}\gamma D_{\beta_1}\tilde{\psi}_{32} \,, && (108)\\
\frac{\partial \tilde{\psi}_{22}}{\partial t} &= \frac{1}{2}\gamma D_{\beta_2}\tilde{\psi}_{32} \,, && (109)\\
\frac{\partial \tilde{\psi}_{31}}{\partial t} &= -\frac{1}{2}\gamma D_{\beta_1}\tilde{\psi}_{22} - \frac{1}{2}\gamma D_{\beta_1}\tilde{\psi}_{11} \,, && (110)\\
\frac{\partial \tilde{\psi}_{32}}{\partial t} &= -\frac{1}{2}\gamma D_{\beta_2}\tilde{\psi}_{22} - \frac{1}{2}\gamma D_{\beta_2}\tilde{\psi}_{11} \,, && (111)
\end{aligned}$$

with

$$\tilde{\psi}_{ij}|_{t=0} = \psi_{ij}|_{t=0}, \quad i = 1,2,3, \quad j = 1,2.$$

If we make a change of variables, $\psi'_{ij} = \tilde{\psi}_{ij} - \tilde{\psi}_{ij}|_{t=0}$ and still write them as $\tilde{\psi}_{ij}$, we get a system as (87) – (92) in Lemma 4.2. Therefore, we can apply Lemma 4.2 to prove the existence and estimate the boundedness of $\tilde{\psi}_{ij}$.

Assume that $\psi_{ij}|_{t=0}$'s are analytic within the strip $\max(|Im(\beta_1)|, |Im(\beta_2)|) < \rho_0$ with their Lipschitz norms satisfying:

$$\|\nabla \psi_{ij}|_{t=0}\|_{\alpha,\rho_0} < \varepsilon \,, \tag{112}$$

where $i = 1,2,3$; and $j = 1,2$. From Lemma 4.2, we know that $\tilde{\psi}_{ij}$ is analytic within a shrinking domain of width

$$\rho_0(t) = \rho_0 - (\frac{1}{2} + \kappa)t \,, \tag{113}$$

where $0 < \kappa < 1$. As we will see later, κ could be taken as small as pleased provided that ε is sufficiently small. Furthermore, the estimates in Lemma 4.2 show that

$$\begin{aligned}
\|\tilde{\psi}_{ij}(t)\|_{\alpha,\rho_0(t)} &\leq c\kappa^{-1}\int_0^t \|\nabla(\psi_{ij}|_{t=0})\|_{\alpha,\rho_0} d\tau \\
&\leq c\varepsilon(\kappa^{-1}t) \,. && (114)
\end{aligned}$$

We remark that $\rho_0(t)$ in the three-dimensional problem has an expression similar to the corresponding two-dimensional problem in [7]. In addition, similar linear growth rate with respect to time is observed in the two-dimensional vortex sheet problem.

4.4 Existence Theorem

In this subsection, we prove the existence theorem using the lemmas and the Cauchy-Kowalewski theorem stated in previous subsections. First, we will split the solution into two parts. We define:

$$\psi'_{ij} = \psi_{ij} - \tilde{\psi}_{ij} \ ,$$

where $i = 1, 2, 3,\ j = 1, 2$, ψ_{ij}'s are the solution of the full nonlinear system while $\tilde{\psi}_{ij}$ are solution of the linear leading order system defined in the previous subsection. By substituting it into the nonlinear system, we can derive equations for ψ'_{ij} as follows:

$$\frac{\partial \psi'_{11}}{\partial t} = \frac{\partial}{\partial \beta_1} R_1(\psi_1, \psi_2, \psi_3) \ , \tag{115}$$

$$\frac{\partial \psi'_{12}}{\partial t} = \frac{\partial}{\partial \beta_2} R_1(\psi_1, \psi_2, \psi_3) \ , \tag{116}$$

$$\frac{\partial \psi'_{21}}{\partial t} = \frac{1}{2} D_{\beta_1} \psi'_{32} + \frac{\partial}{\partial \beta_1} R_2(\psi_1, \psi_2, \psi_3) \ , \tag{117}$$

$$\frac{\partial \psi'_{22}}{\partial t} = \frac{1}{2} D_{\beta_2} \psi'_{32} + \frac{\partial}{\partial \beta_2} R_2(\psi_1, \psi_2, \psi_3) \ , \tag{118}$$

$$\frac{\partial \psi'_{31}}{\partial t} = -\frac{1}{2} D_{\beta_1} \psi'_{22} - \frac{1}{2} D_{\beta_1} \psi'_{11} + \frac{\partial}{\partial \beta_1} R_3(\psi_1, \psi_2, \psi_3) \ , \tag{119}$$

$$\frac{\partial \psi'_{32}}{\partial t} = -\frac{1}{2} D_{\beta_2} \psi'_{22} - \frac{1}{2} D_{\beta_2} \psi'_{11} + \frac{\partial}{\partial \beta_3} R_3(\psi_1, \psi_2, \psi_3) \ , \tag{120}$$

with

$$\psi'_{ij}(t=0) = 0 \quad i = 1, 2, 3 \quad j = 1, 2.$$

The existence of ψ'_{ij}'s will imply the existence of S_i's. Moreover, the following theorem implies Theorem 2.1.

Theorem 4.2 *(Existence Theorem) Let* $0 < \alpha < 1$ *and* $\rho_0 > 0$*. Assume that* $S_i(\beta_1, \beta_2, 0)$ *satisfy*

$$\sup_{\substack{|Im(\alpha_1)|<\rho_0 \\ |Im(\alpha_2)|<\rho_0}} \left(|S_i(\alpha_1, \alpha_2, 0)| + |\nabla(S_i(\alpha_1, \alpha_2, 0))| + |\sum_{k=1}^{2} \sum_{j=1}^{2} (\partial_{\alpha_k} \partial_{\alpha_j} S_i(\alpha_1, \alpha_2, 0))|\right) < \varepsilon \ , \tag{121}$$

where $i = 1, 2, 3,$ *and* ε *is sufficiently small. Then system (67) – (72) with* $\mathbf{z} = (\alpha_1, \alpha_2, 0) + (s_1, s_2, s_3)$ *has a solution for a time* $0 \le t \le T$ *where* T *satisfies*

$$T \le \frac{\rho_0}{\frac{1}{2} + \kappa} \ .$$

Here $\kappa > 0$ is a parameter depending on ε. It satisfies the properties that $\kappa \to 0$ as $\varepsilon \to 0$ and $\kappa^{-2}\varepsilon 2\rho_0 \to 0$ as $\varepsilon \to 0$. Moreover, the corresponding functions ψ_{ij} satisfy

$$\sum_{i=1,2,3;j=1,2} \|\psi_{ij}(t) - \tilde{\psi}_{ij}(t)\|_{\alpha,\rho} \leq c\varepsilon\kappa^{-1}t < \frac{1}{8} \ , \tag{122}$$

for any ρ, t satisfying

$$0 < \rho < \rho_0 - \frac{t}{\frac{1}{2} + \kappa} \ ,$$

where the functions $\tilde{\psi}_{ij}$ are solutions of the linear system (106) – (111) with initial data corresponding to $S_i(t = 0)$, and c is independent of ε, κ and ρ_0.

Proof: From equation (113) in the last section, we have that

$$\rho_0(t) = \rho_0 - (\frac{1}{2} + \kappa)t \ .$$

From Lemma 4.2, choosing any fixed $\kappa' > \kappa$, we can always derive inequality (97) such that

$$\rho_1(\rho, t, \tau) = \rho + (\frac{1}{2} + \kappa')(t - \tau) \ .$$

From Lemma 4.1, the following inequalities hold:

$$\begin{aligned} \|E_{ij}(0,t)\|_{\alpha,\rho'} &\leq c(\rho - \rho')^{-1} \left(\sum_{k_1=1}^{3} \sum_{k_2=1}^{2} \|\tilde{\psi}_{k_1k_2}\|_{\alpha,\rho} \right)^2 \\ &\leq c(\rho - \rho')^{-1}(\varepsilon\kappa^{-1}t)^2 \ , \end{aligned} \tag{123}$$

and

$$\begin{aligned} &\|E_{ij}(\psi'_{ij}, t) - E_{ij}(\tilde{\psi}'_{ij}, t)\|_{\alpha,\rho'} \\ &\leq c(\rho - \rho')^{-1} \left[c\varepsilon\kappa^{-1}t + \left(\sum_{k_1=1}^{3} \sum_{k_2=1}^{2} (\|\psi_{k_1k_2}\|_{\alpha,\rho} + \|\tilde{\psi}_{k_1k_2}\|_{\alpha,\rho} \right) \right] \\ &\qquad \cdot \left[\sum_{k_1=1}^{3} \sum_{k_2=1}^{2} \|\psi'_{k_1k_2} - \tilde{\psi}'_{k_1,k_2}\|_{\alpha,\rho} \right] \ , \end{aligned} \tag{124}$$

for any $0 < \rho' < \rho < \rho_0(t)$. Thus, the assumptions (1) to (5) in the statement of Theorem (4.1) are satisfied with our choice of $\rho_0(t)$, $\rho_1(\tau, t, \rho)$ as above and with

$$d_1(\tau, t) = 1; \quad c_1 = c\kappa^{-1}; \quad c_2 = c;$$

$$d_2(t) = c\varepsilon t\kappa^{-1}; \quad K = \varepsilon T\kappa^{-1}; \quad c_3 = c\varepsilon T\kappa^{-1}.$$

We can simply take $\beta = 1$ and any constant a for ε sufficiently small. In particular, we take $a = \kappa' - \kappa$ to fulfill the conditions on $\rho_0(t)$ and $\rho_1(\rho, t, \tau)$. Therefore, it is straight-forward to apply the Cauchy-Kowalewski theorem to our system. This guarantees the existence of the solution to (10) throughout the time interval. This also proves that the magnitude of the solution remains small since the ψ_{ij}'s are small up to T for sufficiently small ε.

5 Estimate on the Error Terms

Our goal in this section is to provide proofs for Lemma 4.1 and 4.2. Since the proof for lemma 4.2 is quite straightforward, we present it in the first subsection. The proof for Lemma 4.1 is much more complicated and we divide it into three subsubsections.

5.1 Bounds on a Linear System

Consider the linear, spatially inhomogeneous complex $N \times N$ system

$$\frac{\partial}{\partial t}\mathbf{u} + F\frac{\partial}{\partial y}\mathbf{u} = \mathbf{g}(x, y, t) \ , \tag{127}$$

with

$$\mathbf{u}(t = 0) = 0 \ ,$$

in which the complex N-vector $\mathbf{g}$ and the complex $N \times N$ matrix F are given. Further assume that F is constant matrix and can be diagonalized as:

$$F = P^{-1}\Lambda P \ ,$$

$$\Lambda = diag(\lambda_1, \cdots, \lambda_n) \ .$$

Define the backward characteristics by

$$\frac{\partial}{\partial \tau} Y_i(\tau, t, y) = \lambda_i \ , \tag{128}$$

with

$$Y_i(t, t, y) = y \ , \tag{129}$$

and for $\tau < t$, define the dependence set $\Omega(\tau, t, y)$ as those y' which can be reached at time τ going backwards along characteristics starting from y at time t, i.e. ,

$$\Omega(t, t, y) = \{y\} \ , \tag{130}$$

$$\begin{aligned} \Omega(\tau, t, y) \quad = \quad & \{y' : y' = Y_i(\tau, t, y'') \text{ for some } i \text{ and} \\ & \text{for some } y'' \in \Omega(t_1, t, y), \tau < t_1 < t\} \ . \end{aligned} \tag{131}$$

Lemma 5.1 *Suppose that* $\mathbf{g}$ *is analytic in* x *and* y*. Moreover, we assume that*

$$|P| + |P^{-1}| < p \ , \tag{132}$$

where $|P|$ *is the maximal norm of matrix* P*. Then the solution* $\mathbf{u}$ *of (127) is analytic and satisfies*

$$|u_i(x, y, t)| \le c \int_0^t \sup_{\bar{y} \in \Omega(\tau, t, y)} |g(x, \bar{y}, \tau)| d\tau \ , \tag{133}$$

$$\begin{aligned}\frac{|u_i(x,y,t)-u_i(x,y',t)|}{|y-y'|^\alpha} &\leq c\int_0^t \sup_{\bar{y}\in\Omega(\tau,t,y)\cup\Omega(\tau,t,y')} |g(x,\bar{y},\tau)|d\tau \\ &\quad + c\int_0^t \sup_{\substack{\bar{y}\in\Omega(\tau,t,x,y)\\ \bar{y}'\in\Omega(\tau,t,x,y')\\ \bar{y}\neq\bar{y}'}} \left(\frac{|g(x,\bar{y},\tau)-g(x,\bar{y}',\tau)|}{|\bar{y}-\bar{y}'|}\right) d\tau \ ,\end{aligned} \tag{134}$$

$$\frac{|u_i(x,y,t)-u_i(x',y,t)|}{|x-x'|^\alpha} \leq c\int_0^t \sup_{\bar{y}\in\Omega(\tau,t,y)} \frac{|g(x,\bar{y},\tau)-g(x',\bar{y},\tau)|}{|x-x'|^\alpha} d\tau \ , \tag{135}$$

where c only depends on p.

Proof: The proof of inequality (133) and (134) can be obtained by directly applying Proposition B.1 in [7]. Therefore, we only need to prove (135). Note that in equation (127), x is only a parameter. Therefore, if $u(x,y,t)$ is a solution of (127),

$$\frac{u(x,y,t)-u(x',y,t)}{|x-x'|^\alpha}$$

is a solution of

$$\frac{\partial}{\partial t}\mathbf{u} + F\frac{\partial}{\partial y}\mathbf{u} = \frac{\mathbf{g}(x,y,t)-\mathbf{g}(x',y,t)}{|x-x'|^\alpha} \ .$$

And thus, by applying inequality (133), we get (135).

As we mentioned at the end of last section, if we treat the error terms in system (67) – (72) as given functions, we can solve the equation by first solving system (67), (68), (70), (72) by integrating along the characteristic lines, and then substituting the solution into system (69), (71) to solve the resulting O.D.E. This was the procedure used in the proof of the extended abstract Cauchy-Kowalewski theorem in which the following results are used to carry over the iteration [7].

Using Lemma 5.1 as a tool, we can prove Lemma 4.2.

Proof of Lemma 4.2: Considering the following matrix

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \end{pmatrix},$$

Straightforward calculation shows that its eigenvalues are $0, 0, \frac{i}{2}$, and $-\frac{i}{2}$. By applying Lemma 5.1, we can show that

$$\begin{aligned}\|u_{11}(t),u_{12}(t),u_{22}(t),u_{32}(t)\|_{\alpha,\rho} &\leq c\int_0^t \|\mathbf{g}(\cdot,\cdot,\tau)\|_{\alpha,(\rho+\frac{1}{2}(t-\tau))}d\tau \\ &\leq c\int_0^t \|\mathbf{g}(\cdot,\cdot,\tau)\|_{\alpha,(\rho+(\frac{1}{2}+\kappa)(t-\tau))}d\tau \ .\end{aligned} \tag{136}$$

Furthermore, we can solve the remaining two equations as O.D.E.'s. Taking u_{21} as example, we can show that

$$\begin{aligned}\|u_{21}\|_{\alpha,\rho} &\leq \int_0^t \frac{1}{2}\|D_{\beta_1}u_{32}(\cdot,\cdot,\tau)\|_{\alpha,\rho} + \|g_{21}(\cdot,\cdot,\tau)\|_{\alpha,\rho}d\tau \\ &\leq \int_0^t \frac{1}{2}\kappa^{-1}t^{-1}\|u_{32}(\cdot,\cdot,\tau)\|_{\alpha,(\rho+\kappa t)}d\tau + \int_0^t \|g_{21}(\cdot,\cdot,\tau)\|_{\alpha,\rho}d\tau,\end{aligned} \tag{137}$$

by applying the Cauchy Inequality. Then, it follows from the estimates on the first integral that

$$\begin{aligned}\|u_{21}\|_{\alpha,\rho} &\leq \frac{1}{2}\kappa^{-1}\max_{0<t'<t}\|u_{32}(\cdot,\cdot,t')\|_{\alpha,(\rho+\kappa t)} + \int_0^t \|g_{21}(\cdot,\cdot,\tau)\|_{\alpha,\rho}d\tau \\ &\leq \frac{1}{2}\kappa^{-1}\max_{0<t'<t}\int_0^{t'} c\|\mathbf{g}(\cdot,\cdot,\tau)\|_{\alpha,(\rho+\kappa t+\frac{1}{2}(t'-\tau))}d\tau \\ &\quad + \int_0^t \|g_{21}(\cdot,\cdot,\tau)\|_{\alpha,\rho}d\tau \ ,\end{aligned} \tag{138}$$

where we used inequality (136) in the last step. Moreover, using the monotonicity of the Lipschitz norm, we have

$$\begin{aligned}\|u_{21}\|_{\alpha,\rho} &\leq \frac{c}{2}\kappa^{-1}\int_0^t \|\mathbf{g}(\cdot,\cdot,\tau)\|_{\alpha,(\rho+\kappa t+\frac{1}{2}(t-\tau))}d\tau + \int_0^t \|g_{21}(\cdot,\cdot,\tau)\|_{\alpha,\rho}d\tau \\ &\leq c\kappa^{-1}\int_0^t \|\mathbf{g}(\cdot,\cdot,\tau)\|_{\alpha,(\rho+\kappa t+\frac{1}{2}(t-\tau))}d\tau \\ &\leq c\kappa^{-1}\int_0^t \|\mathbf{g}(\cdot,\cdot,\tau)\|_{\alpha,(\rho+(\frac{1}{2}+\kappa)(t-\tau))}d\tau \ .\end{aligned} \tag{139}$$

The estimate on u_{32} can be obtained similarly.

5.2 Bound on E_{ij}

The error terms E_{ij} are defined in system (67) – (72), which are space derivatives of the R_k's. Since we can apply the Cauchy inequality for analytic functions in the complex plane [1], it is sufficient to obtain the bounds for R_k. Moreover, since the R_k's are combinations of the Riesz transforms of the error terms in the equations for $\partial S_i/\partial t$'s, from the boundedness of the Riesz transforms which we show later, we are able to derive the bounds of E_{ij} from the bounds in the $\partial S_i/\partial t$ equations.

Along this line, we perform the error estimates in three steps: the error estimates on the Riesz transforms, the error estimates on the R_i terms, and the error estimates on the E_{ij} terms.

First of all, we define a special Lipschitz norm, which will be used only in this section.

$$\|f(\cdot + i\mu_1, \cdot + i\mu_2)\|_0 = \sup_{\kappa_1,\kappa_2 \in R} |f(\kappa_1 + i\mu_1, \kappa_2 + i\mu_2)| \ , \tag{140}$$

$$\begin{aligned} \|f(\cdot + i\mu_1, \cdot + i\mu_2)\|_\alpha &= \|f(\cdot + i\mu_1, \cdot + i\mu_2)\|_0 \\ &+ \sup_{\substack{(\kappa_1,\kappa_2),(\kappa_1',\kappa_2') \in R\times R \\ (\kappa_1,\kappa_2)\neq(\kappa_1',\kappa_2')}} \frac{|f(\kappa_1 + i\mu_1, \kappa_2 + i\mu_2) - f(\kappa_1' + i\mu_1, \kappa_2' + i\mu_2)|}{|(\kappa_1,\kappa_2) - (\kappa_1',\kappa_2')|^\alpha} \ . \end{aligned} \tag{141}$$

5.2.1 Bounds on the Hilbert Transform

The following lemma has been proved by Calderon & Zygmund [9] and Taibleson [25]:

Lemma 5.2 *If f has period of $2\pi \times 2\pi$, and satisfies*

$$\int_{-\pi}^{\pi}\int_{\pi}^{\pi} f(\beta_1 + i\mu_1, \beta_2 + i\mu_2) d\beta_1 d\beta_2 = 0 \ ,$$

for any μ_1 and μ_2, then

$$\|H_k f(\cdot + i\mu_1, \cdot + i\mu_2)\|_\alpha \leq c\|f(\cdot + i\mu_1, \cdot + i\mu_2)\|_\alpha \tag{142}$$

where H_k is the Riesz transform in the k-th variable, $k = 1, 2$, $0 < \alpha < 1$, and c depends only on α.

5.2.2 Bounds on the R_i Terms in System (61) – (63)

The R_i terms are defined in (61) – (63). Taking equation (61) as an example, we see that R_1 is the sum of the Riesz transforms of the residue terms in the $\frac{\partial S_1}{\partial t}$ and the $\frac{\partial S_2}{\partial t}$ equations. The same is true for equation (62). Therefore, by the boundedness of the Riesz transform, we claim that the boundedness of the residue terms in equations (32), (33), (34) are equivalent to the boundedness of the R_i terms. To obtain estimates for R_i, we need to use the following lemmas.

Lemma 5.3 *Let x, y, z be small perturbations of a flat plane. Assume that (μ_1, μ_2) satisfy $|\mu_1| < \rho$ and $|\mu_2| < \rho$, and f, S_1, S_2, S_3 satisfy:*

1. *f, S_1, S_2, S_3 are analytic functions within the strip*

$$\{(\beta_1, \beta_2)|\max(|Im(\beta_1)|, |Im(\beta_2)|) < \rho\} \ ;$$

2. f*,* S_1*,* S_2*,* S_3 *are periodic functions with period of* $(2\pi, 2\pi)$*;*

3.

$$\begin{aligned} \|f\|_\alpha(\cdot + i\mu_1, \cdot + i\mu_2) &\leq \frac{1}{8} \ , \\ \|S_j\|_\alpha(\cdot + i\mu_1, \cdot + i\mu_2) &\leq \frac{1}{8} \ , \end{aligned}$$

where $j = 1, 2, 3$ *and* $\|f\|_\alpha(\cdot + i\mu_1, \cdot + i\mu_2)$ *is defined as in Lemma 4.1. Further, we define:*

$$\begin{aligned} &Diff^k[f, S_1, S_2, S_3](\beta_1 + i\mu_1, \beta_2 + i\mu_2) = \\ &\quad H_k f(\beta_1 + i\mu_1, \beta_2 + i\mu_2) - \frac{1}{2\pi} \int \frac{(\beta_k - \beta_k')f(\beta_1' + i\mu_1, \beta_2' + i\mu_2)}{|\mathbf{z} - \mathbf{z}'|^3} \, d\beta' \ , \end{aligned} \tag{143}$$

where

$$\begin{aligned} \mathbf{z} &= (\beta_1 + S_1(\beta + i\mu), \beta_2 + S_2(\beta + i\mu), S_3(\beta + i\mu))^T \ , \\ \mathbf{z}' &= (\beta_1' + S_1(\beta' + i\mu), \beta_2' + S_2(\beta' + i\mu), S_3(\beta' + i\mu))^T \ , \end{aligned}$$

and

$$\begin{aligned} \beta + i\mu &= (\beta_1 + i\mu_1, \beta_2 + i\mu_2) \ , \\ \beta' + i\mu &= (\beta_1' + i\mu_1, \beta_2' + i\mu_2) \ . \end{aligned}$$

Then the following inequalities hold

$$\begin{aligned} &\|Diff^k[f, S_1, S_2, S_3](\cdot + i\mu_1, \cdot + i\mu_2)\|_\alpha \leq \\ &\quad c(\|\nabla S_1\|_\alpha + \|\nabla S_2\|_\alpha + \|\nabla S_3\|_\alpha)\|f\|_\alpha(\cdot + i\mu_1, \cdot + i\mu_2) \ , \end{aligned} \tag{144}$$

and

$$\|(Diff^k[f, S_1, S_2, S_3] - Diff^k[\tilde{f}, \tilde{S}_1, \tilde{S}_2, \tilde{S}_3])(\cdot + i\mu_1, \cdot + i\mu_2)\|_\alpha \leq cN(\alpha, \mu_1, \mu_2) \ , \tag{145}$$

for $0 < \alpha < 1$*, where*

$$\begin{aligned} N(\alpha, \mu_1, \mu_2) &= (\|f\|_\alpha + \|\tilde{f}\|_\alpha + \sum_{j=1}^{3}(\|\nabla S_j\|_\alpha + \|\nabla \tilde{S}_j\|_\alpha)) \cdot \\ &\qquad (\|f - \tilde{f}\|_\alpha + \sum_{j=1}^{3}(\|\nabla(S_j - \tilde{S}_j)\|_\alpha))(\cdot + i\mu_1, \cdot + i\mu_2) \ , \end{aligned} \tag{146}$$

and $k = 1, 2$.

Lemma 5.4 *Let f, x, y, z be defined as in Lemma 5.3, then the same inequalities hold if we re-define Diff as,*

$$\mathit{Diff}^k[f,S_1,S_2,S_3](\beta_1+i\mu_1,\beta_2+i\mu_2) =$$

$$\frac{1}{2\pi}\int \frac{(S_k-S_k')f'}{|\mathbf{z}-\mathbf{z}'|^3}\,d\beta' - \frac{1}{2\pi}\int \frac{[(\beta_1-\beta_1')S_{k_{\beta_1}}+(\beta_2-\beta_2')S_{k_{\beta_2}}]f'}{|(\beta_1,\beta_2)-(\beta_1',\beta_2')|^3}\,d\beta' \ , \tag{147}$$

where $k=1,2,3$.

Lemma 5.5 *Let x, y, z be given as in Lemma 5.3. Define*

$$\begin{aligned}
&\mathit{Diff}[S_1,S_2,S_3](\beta_1+i\mu_1,\beta_2+i\mu_2) = \\
&\frac{1}{2\pi}\int \frac{\gamma_1(\beta_1-\beta_1')+\gamma_2(\beta_2-\beta_2')}{|\mathbf{z}(\beta+i\mu)-\mathbf{z}(\beta'+i\mu')|^3}\,d\beta' \\
&+\gamma_2(H_1D_2(S_1)+(2H_2D_2+H_1D_1)(S_2))(\beta+i\mu) \\
&+\gamma_1((2H_1D_1+H_2D_2)(S_1)+H_1D_2(S_2))(\beta+i\mu) \ ,
\end{aligned} \tag{148}$$

where D_1 (D_2) stands for the space derivative on the β_1 (β_2) direction, and

$$\begin{aligned}
\mathbf{z} &= (\beta_1+S_1(\beta+i\mu),\beta_2+S_2(\beta+i\mu),S_3(\beta+i\mu))^T \ , \\
\mathbf{z}' &= (\beta_1'+S_1(\beta'+i\mu),\beta_2'+S_2(\beta'+i\mu),S_3(\beta'+i\mu))^T \ ,
\end{aligned}$$

and

$$\begin{aligned}
\beta+i\mu &= (\beta_1+i\mu_1,\beta_2+i\mu_2) \ , \\
\beta'+i\mu &= (\beta_1'+i\mu_1,\beta_2'+i\mu_2) \ .
\end{aligned}$$

Then the following inequalities hold:

$$\begin{aligned}
&\|\mathit{Diff}[S_1,S_2,S_3](\cdot+i\mu_1,\cdot+i\mu_2)\|_\alpha \le \\
&c(\|\nabla S_1\|_\alpha+\|\nabla S_2\|_\alpha+\|\nabla S_3\|_\alpha)^2(\cdot+i\mu_1,\cdot+i\mu_2) \ ,
\end{aligned} \tag{149}$$

and

$$\|(\mathit{Diff}[S_1,S_2,S_3]-\mathit{Diff}[\tilde{S}_1,\tilde{S}_2,\tilde{S}_3])(\cdot+i\mu_1,\cdot+i\mu_2)\|_\alpha \le cN(\alpha,\mu_1,\mu_2) \ , \tag{150}$$

for $0<\alpha<1$, where $N(\alpha,\mu_1,\mu_2)$ is defined in (146).

Remarks: The above lemmas confirm our observation in the previous section where we derived our leading order system. We have proved that, if the interface is close to a flat plane, the vortex sheet kernel is close to the Riesz transform kernel. The difference is a smaller term in the Lipschitz norm. The proofs of the above lemmas are quite technical. Thus, we will defer them to the Appendix.

Combining Lemma 5.3, 5.4, and Lemma 5.5, we get the following bound.

Lemma 5.6 *Let x, y, z be given as that in Lemma 5.3, then the following inequalities hold:*

$$\|R_k[S_1,S_2,S_3]\|_\alpha(\cdot+i\mu_1,\cdot+i\mu_2) \le c\left(\sum_{j=1}^{3}\|\nabla S_j\|_\alpha\right)^2(\cdot+i\mu_1,\cdot+i\mu_2) \ , \tag{151}$$

and

$$\begin{aligned}&\|R_k[S_1,S_2,S_3]-R_k[\tilde{S}_1,\tilde{S}_2,\tilde{S}_3\|_\alpha(\cdot+i\mu_1,\cdot+i\mu_2) \le \\ &\quad c\left(\sum_{j=1}^{3}(\|\nabla S_j\|_\alpha+\|\nabla\tilde{S}_j\|_\alpha)\right)\left(\sum_{j=1}^{3}\|\nabla(S_j-\tilde{S}_j)\|_\alpha\right)(\cdot+i\mu_1,\cdot+i\mu_2) \ ,\end{aligned} \tag{152}$$

where $k=1,2,3$.

Now that we have obtained the estimates of the R_i terms, the bounds of E_{ij} terms can be derived using the Cauchy inequality. This will be presented in the next subsection.

5.2.3 Bounds on E_{ij}

The error terms E_{ij} are defined in system (67) – (72), which are space derivatives of the R_k's.

By the definition of the E_{ij}'s and using the Cauchy inequality for analytic functions, we get:

$$\begin{aligned}\|E_{ij}\|_{\alpha,\rho'} &\le c\|E_{ij}\|_{\alpha,\rho'+} \\ &\le c\sup_{\max(|\mu_1|,|\mu_2|)<\rho'}\|\nabla R_k[S_1,S_2,S_3]\|_\alpha(\cdot+i\mu_1,\cdot+i\mu_2) \\ &\le c(\rho-\rho')^{-1}\sup_{\max(|\mu_1|,|\mu_2|)<\rho}\|R_k[S_1,S_2,S_3]\|_\alpha(\cdot+i\mu_1,\cdot+i\mu_2) \ .\end{aligned} \tag{153}$$

It further follows from Lemma 5.6 that

$$\begin{aligned}\|E_{ij}\|_{\alpha,\rho'} &\leq c(\rho-\rho')^{-1}\sup_{\max(|\mu_1|,|\mu_2|)<\rho}\left(\sum_{k=1}^{3}\|\nabla S_k\|_\alpha\right)^2(\cdot+i\mu_1,\cdot+i\mu_2)\\ &\leq c(\rho-\rho')^{-1}\left(\sum_{k=1}^{3}\|\nabla S_k\|_{\alpha,\rho}\right)^2\\ &\leq c(\rho-\rho')^{-1}\left(\sum_{k_1=1}^{3}\sum_{k_2=1}^{2}\|\psi_{k_1k_2}\|_{\alpha,\rho}\right), \end{aligned} \tag{154}$$

where in the last step we have used the definition of ψ_{ij}.

Similarly, we can get the bounds of $(E_{ij}-\tilde{E}_{ij})$. We conclude this part of the estimate with a final lemma which is identical to Lemma 4.1.

Lemma 5.7 *Let x, y, z be given as in Lemma 5.3. Suppose ϕ_{ij} and $\tilde{\phi}_{ij}$ are analytic in* $\max(|Im(\kappa_1)|,|Im(\kappa_2)|)$ ρ. *Then for $0<\rho'<\rho$, and $0<\alpha<1$, the following inequalities hold:*

$$\|E_{ij}\|_{\alpha,\rho'}\leq c(\rho-\rho')^{-1}\left(\sum_{k_1=1}^{3}\sum_{k_2=1}^{2}\|\psi_{k_1k_2}\|_{\alpha,\rho}\right)^2, \tag{155}$$

and

$$\begin{aligned}\|E_{ij}-\tilde{E}_{ij}\|_{\alpha,\rho'} &\leq c(\rho-\rho')^{-1}\left(\sum_{k_1=1}^{3}\sum_{k_2=1}^{2}(\|\psi_{k_1k_2}\|_{\alpha,\rho}+\|\tilde{\psi}_{k_1k_2}\|_{\alpha,\rho}\right)\\ &\quad\cdot\left(\sum_{k_1=1}^{3}\sum_{k_2=1}^{2}\|\psi_{k_1k_2}-\tilde{\psi}_{k_1,k_2}\|_{\alpha\rho}\right), \end{aligned} \tag{156}$$

where $i,j=1,2,3$.

Appendix. Detailed Proof of the Error Estimates

In the Appendix, we will prove Lemma 5.3. Lemma 5.4 and Lemma 5.5 can be proved similarly.

Proof of Lemma 5.3: We only prove the inequalities for Diff^1. The inequalities of Diff^2 can be proved similarly. Furthermore, we suppress μ_1 and μ_2 and just keep the real part β_1 and β_2 throughout the proof.

Note that inequality (144) can be derived from inequality (145) by taking $\tilde{f} = 0$, and $\tilde{S}_i = 0$ for $i = 1, 2, 3$. Therefore, it is sufficient to prove (145).

To perform our analysis, we first write the integrals in the periodic form as

$$\begin{aligned}
\mathit{Diff}^1[f, S_1, S_2, S_3] &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(K^{1*}(\zeta) - K_z^{1*}(\beta, \beta - \zeta))f(\beta - \zeta)d\zeta_1 d\zeta_2 \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(K^{1}(\zeta) - K_z^{1}(\beta, \beta - \zeta))f(\beta - \zeta)d\zeta_1 d\zeta_2 \\
&\quad + \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(\overline{K^{1}}(\zeta) - \overline{K_z^{1}}(\beta, \beta - \zeta))f(\beta - \zeta)d\zeta_1 d\zeta_2 \\
&\triangleq I_1[f, S_1, S_2, S_3](\beta) + I_2[f, S_1, S_2, S_3](\beta) \ .
\end{aligned} \tag{157}$$

We need to show that both I_1 and I_2 satisfy (145). Since I_2 does not contain any singularity, it is just a regular integral on a bounded domain. Therefore, the maximum value of I_2 is bounded by the maximum value of f, and S_i's. In addition, the Hölder norm of I_2 is bounded by the Hölder norm of f, and ∇S_i's. Along this line, we can prove that I_2 satisfy (145).

We focus on the first integral I_1. What we need to prove is

$$|(I_1 - \tilde{I}_1)(\beta)| \leq cN(\alpha) \tag{158}$$

and

$$|(I_1 - \tilde{I}_1)(\beta) - (I_1 - \tilde{I}_1)(\beta')| \leq c|\beta - \beta'|^{\alpha} N(\alpha) \ , \tag{159}$$

where $N(\alpha)$ is defined in (146) with $i\mu$ being suppressed.

We split the rest of our proof into three parts: the preparation, the proof of (158) and the proof of (159).

Preparation: Before we go on to prove (158) and (159), it is necessary to analyze further the integrand of I_1 and derive several inequalities for later use.

From the definition of I_1, we get:

$$\begin{aligned}
I_1 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(K^{1}(\zeta) - K_z^{1}(\beta, \beta - \zeta))f(\beta - \zeta)d\zeta_1 d\zeta_2 \\
&= -\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\beta + \zeta)\left(\frac{\zeta_1}{|\zeta|^3} - \frac{\zeta_1}{|\mathbf{z}(\beta + \zeta) - \mathbf{z}(\beta)|^3}\right)d\zeta_1 d\zeta_2 \\
&= -\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\zeta_1 f(\beta + \zeta)}{|\zeta|^3}\left(\frac{|\mathbf{z}(\beta + \zeta) - \mathbf{z}(\beta)|^3 - |\zeta|^3}{|\mathbf{z}(\beta + \zeta) - \mathbf{z}(\beta)|^3}\right)d\zeta_1 d\zeta_2 \ ,
\end{aligned} \tag{160}$$

where $\beta = (\beta_1, \beta_2)$ and $\zeta = (\zeta_1, \zeta_2)$.

Define

$$G(\beta, \beta+\zeta) = f(\beta+\zeta) G_1(\beta, \beta+\zeta) \ , \tag{161}$$

where

$$G_1(\beta, \beta+\zeta) = \frac{|\mathbf{z}(\beta+\zeta) - \mathbf{z}(\beta)|^3 - |\zeta|^3}{|\mathbf{z}(\beta+\zeta) - \mathbf{z}(\beta)|^3} \ . \tag{162}$$

Similarly, we can also write down:

$$\tilde{G}(\beta, \beta+\zeta) = \tilde{f}(\beta+\zeta) \tilde{G}_1(\beta, \beta+\zeta) \ , \tag{163}$$

where

$$\tilde{G}_1(\beta, \beta+\zeta) = \frac{|\tilde{\mathbf{z}}(\beta+\zeta) - \tilde{\mathbf{z}}(\beta)|^3 - |\zeta|^3}{|\tilde{\mathbf{z}}(\beta+\zeta) - \tilde{\mathbf{z}}(\beta)|^3} \ . \tag{164}$$

Under the assumptions of Lemma 5.3, we can prove the following bounds:

$$|\mathbf{z}(\beta+\zeta) - \mathbf{z}(\beta)| \geq c|\zeta| \ , \tag{165}$$

$$|G_1(\beta, \beta+\zeta)| \leq c \sum_{i=1}^{3} \|\nabla S_i\|_0 \ , \tag{166}$$

$$|(G_1 - \tilde{G}_1)(\beta, \beta+\zeta)| \leq c \sum_{i=1}^{3} \|\nabla (S_i - \tilde{S}_i)\|_0 \ , \tag{167}$$

$$\begin{aligned} & |(G_1 - \tilde{G}_1)(\beta, \beta+\zeta) - (G_1 - \tilde{G}_1)(\beta, \beta-\zeta)| \\ \leq \ & c|\zeta|^\alpha \left(\sum_{i=1}^{3} \|\nabla (S_i - \tilde{S}_i)\|_\alpha \right) \ , \end{aligned} \tag{168}$$

$$|\frac{\partial}{\partial S_i} G_1| \leq c|\zeta|^{-1} \ . \tag{169}$$

However, to focus on the main idea of the proof of Lemma 5.3, we will defer the verification of the above inequalities to the end.

From (166), it is straightforward to derive the following bound:

$$\begin{aligned} |G(\beta, \beta+\zeta)| & \leq |f(\beta+\zeta)||G_1(\beta, \beta+\zeta)| \\ & \leq c\|f\|_0 \sum_{i=1}^{3} \|\nabla S_i\|_0 \ . \end{aligned} \tag{170}$$

Similarly, from (167), it can be shown that

$$\begin{aligned} & |(G - \tilde{G})(\beta, \beta+\zeta)| \\ \leq \ & |(f - \tilde{f})(\beta+\zeta)||G_1(\beta, \beta+\zeta)| + |\tilde{f}(\beta+\zeta)||(G_1 - \tilde{G}_1)(\beta, \beta+\zeta)| \\ \leq \ & c\|f - \tilde{f}\|_0 \sum_{i=1}^{3} \|\nabla S_i\|_0 + c\|f\|_0 \sum_{i=1}^{3} \|\nabla (S_i - \tilde{S}_i)\|_0 \\ \leq \ & cN(\alpha) \ , \end{aligned} \tag{171}$$

where $N(\alpha)$ again is defined in the statement of the lemma as $N(\alpha, \mu_1, \mu_2)$.

Using an argument similar to (171), we can prove the following inequality from (168):

$$|(G - \tilde{G})(\beta, \beta + \zeta) - (G - \tilde{G})(\beta, \beta - \zeta)| \leq c|\zeta|^\alpha N(\alpha) \ , \tag{172}$$

From (169), we can derive

$$|\nabla_\beta (G_1 - \tilde{G}_1)(\beta, \beta + \zeta)| \leq c|\zeta|^{-1+\alpha} \sum_{i=1}^{3} \|\nabla(S_i - \tilde{S}_i)\|_\alpha \ , \tag{173}$$

which implies

$$\begin{aligned} &|(G_1 - \tilde{G}_1)(\beta, \beta + \zeta) - (G_1 - \tilde{G}_1)(\beta', \beta' + \zeta)| \\ \leq \ & c|\beta - \beta'||\zeta|^{-1+\alpha} \sum_{i=1}^{3} \|\nabla(S_i - \tilde{S}_i)\|_\alpha \ . \end{aligned} \tag{174}$$

Proof of (158): We are now ready to derive (158). We split the integration domain in two regions as:

$$\begin{aligned} |I_1 - \tilde{I}_1| &= \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\zeta_1}{|\zeta|^3} (G(\beta, \beta + \zeta) - \tilde{G}(\beta, \beta + \zeta)) d\zeta \right| \\ &= \frac{1}{2\pi} \left| \left(\int_{-\pi}^{\pi} \int_{-\pi}^{0} + \int_{-\pi}^{\pi} \int_{0}^{\pi} \right) \right. \\ &\qquad \left. \frac{\zeta_1}{|\zeta|^3} (G(\beta, \beta + \zeta) - \tilde{G}(\beta, \beta + \zeta)) d\zeta \right| \ . \end{aligned} \tag{175}$$

$$\tag{176}$$

From the oddness of the kernel $\frac{\zeta_1}{|\zeta|^3}$, we change the variable $\zeta' = -\zeta$ in the second integral and get

$$\begin{aligned} |I_1 - \tilde{I}_1| &= \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} \int_{0}^{\pi} \frac{\zeta_1}{|\zeta|^3} [(G - \tilde{G})(\beta, \beta + \zeta) \right. \\ &\qquad \left. - (G - \tilde{G})(\beta, \beta - \zeta)] d\zeta_1 d\zeta_2 \right| \\ &\leq \frac{c}{2\pi} \left| \int_{-\pi}^{\pi} \int_{0}^{\pi} \frac{\zeta_1}{|\zeta|^3} |\zeta|^\alpha d\zeta_1 d\zeta_2 \right| N(\alpha) \\ &\leq cN(\alpha) \ , \end{aligned} \tag{177}$$

where we have applied inequality (172) in the second step.

Proof of (159): Our next step is to prove

$$|(I_1 - \tilde{I}_1)(\beta) - (I_1 - \tilde{I}_1)(\beta')| \leq c|\beta - \beta'|^\alpha N(\alpha) \ . \tag{178}$$

For a complicated inequality like (178), we would like to break it down into several integrals and estimate them one by one. It can be shown that

$$\begin{aligned}
&|(I_1-\tilde{I}_1)(\beta)-(I_1-\tilde{I}_1)(\beta')| \\
= &\left|\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left(f(\beta+\zeta)G_1(\beta,\beta+\zeta)-\tilde{f}(\beta+\zeta)\tilde{G}_1(\beta,\beta+\zeta)\right)\frac{\zeta_1}{|\zeta|^3}d\zeta\right. \\
&\left.-\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left(f(\beta'+\zeta)G_1(\beta',\beta'+\zeta)-\tilde{f}(\beta'+\zeta)\tilde{G}_1(\beta',\beta'+\zeta)\right)\frac{\zeta_1}{|\zeta|^3}d\zeta\right| \\
\le &\left|\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left((f-\tilde{f})(\beta+\zeta)G_1(\beta,\beta+\zeta)-(f-\tilde{f})(\beta'+\zeta)G_1(\beta',\beta'+\zeta)\right)\frac{\zeta_1}{|\zeta|^3}d\zeta\right| \\
&+\left|\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\tilde{f}(\beta'+\zeta)\left((G_1-\tilde{G}_1)(\beta,\beta+\zeta)-(G_1-\tilde{G}_1)(\beta',\beta'+\zeta)\right)\frac{\zeta_1}{|\zeta|^3}d\zeta\right| \\
&+\left|\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(\tilde{f}(\beta+\zeta)-\tilde{f}(\beta'+\zeta))(G_1-\tilde{G}_1)(\beta,\beta+\zeta)\frac{\zeta_1}{|\zeta|^3}d\zeta\right| \\
\triangleq &\ I_{11}+I_{12}+I_{13}\ .
\end{aligned} \tag{179}$$

We prove the inequalities of I_{11} and I_{12} in detail. The estimates of I_{13} can be obtained similar to that of I_{11}.

Bounds on I_{12}: To estimate I_{12}, we first break the integration domain into two regions, $|\zeta|<|\beta-\beta'|$ and $|\zeta|\ge|\beta-\beta'|$,

$$\begin{aligned}
&\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\zeta_1}{|\zeta|^3}\tilde{f}(\beta'+\zeta)[(G_1-\tilde{G}_1)(\beta,\beta+\zeta)-(G_1-\tilde{G}_1)(\beta',\beta'+\zeta)]d\zeta\right| \\
= &\left|\frac{1}{2\pi}\left(\int_{|\zeta|<|\beta-\beta'|}+\int_{|\zeta|\ge|\beta-\beta'|}\right)\right. \\
&\left.\frac{\zeta_1}{|\zeta|^3}\tilde{f}(\beta'+\zeta)[(G_1-\tilde{G}_1)(\beta,\beta+\zeta)-(G_1-\tilde{G}_1)(\beta',\beta'+\zeta)]d\zeta\right|\ .
\end{aligned} \tag{180}$$

For the first integral we use the oddness of the kernel in a way similar to the proof of (177), while for the second integral, we apply inequality (174). Therefore, it follows that

$$\begin{aligned}
&\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\zeta_1}{|\zeta|^3}\tilde f(\beta'+\zeta)[(G_1-\tilde G_1)(\beta,\beta+\zeta)-(G_1-\tilde G_1)(\beta',\beta'+\zeta)]d\zeta\right| \\
\leq\ & \frac{1}{2\pi}\left|\int_{\substack{|\zeta|<|\beta-\beta'| \\ \zeta_1>0}}\frac{\zeta_1}{|\zeta|^3}\cdot\right. \\
&\quad [\tilde f(\beta'+\zeta)((G_1-\tilde G_1)(\beta,\beta+\zeta)-(G_1-\tilde G_1)(\beta',\beta'+\zeta)) \\
&\quad \left. -\tilde f(\beta'-\zeta)((G_1-\tilde G_1)(\beta,\beta-\zeta)-(G_1-\tilde G_1)(\beta',\beta'-\zeta))]d\zeta\right| \\
&+\frac{1}{2\pi}\left|\int_{|\zeta|\geq|\beta-\beta'|}\frac{\zeta_1}{|\zeta|^3}\tilde f(\beta'+\zeta)[(G_1-\tilde G_1)(\beta,\beta+\zeta)\right. \\
&\qquad \left. -(G_1-\tilde G_1)(\beta',\beta'+\zeta)]d\zeta\right| \\
\leq\ & \frac{c}{2\pi}\int_{|\zeta|<|\beta-\beta'|}\frac{\zeta_1}{|\zeta|^3}|\zeta|^\alpha d\zeta N(\alpha) \\
&+\frac{c}{2\pi}\int_{|\zeta|\geq|\beta-\beta'|}\frac{\zeta_1}{|\zeta|^3}|\zeta|^{-1+\alpha}d\zeta|\beta-\beta'|N(\alpha) \\
\leq\ & c|\beta-\beta'|^\alpha N(\alpha)\ ,
\end{aligned} \tag{181}$$

where we have applied (168) and (174) in the above proof.

Therefore, we get

$$I_{12}\leq c|\beta-\beta'|^\alpha N(\alpha)\ . \tag{182}$$

Bounds on I_{11}: It is sufficient to prove

$$\begin{aligned}
&\left|\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(\beta+\zeta)G_1(\beta,\beta+\zeta)\frac{\zeta_1}{|\zeta|^3}d\zeta-\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(\beta'+\zeta)G_1(\beta',\beta'+\zeta)\frac{\zeta_1}{|\zeta|^3}d\zeta\right| \\
&\qquad\leq c|\beta-\beta'|^\alpha\|f\|_\alpha\sum_{i=1}^{3}\|\nabla S_i\|_\alpha,
\end{aligned} \tag{183}$$

because by taking $f'=f-\tilde f$ and still writing as f in I_{11}, the bound satisfies

$$c|\beta-\beta'|^\alpha\|f-\tilde f\|_\alpha\sum_{i=1}^{3}\|\nabla S_i\|_\alpha\leq c|\beta-\beta'|^\alpha N(\alpha)\ .$$

From (181), by taking $\tilde G_1=0$ and $\tilde f=1$, we get:

$$\begin{aligned}
&\left|f(\beta)\left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}G_1(\beta,\beta+\zeta)\frac{\zeta_1}{|\zeta|^3}d\zeta-\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}G_1(\beta',\beta'+\zeta)\frac{\zeta_1}{|\zeta|^3}d\zeta\right)\right| \\
&\qquad\leq c|\beta-\beta'|^\alpha\|f\|_0\sum_{i=1}^{3}\|\nabla S_i\|_\alpha\ .
\end{aligned} \tag{184}$$

Therefore, to prove (183), we only need to prove that

$$\Big|\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(f(\beta+\zeta)-f(\beta))G_1(\beta,\beta+\zeta)\frac{\zeta_1}{|\zeta|^3}d\zeta- \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(f(\beta'+\zeta)-f(\beta))G_1(\beta',\beta'+\zeta)\frac{\zeta_1}{|\zeta|^3}d\zeta\Big| \le c|\beta-\beta'|^{\alpha}\|f\|_0\sum_{i=1}^{3}\|\nabla S_i\|_{\alpha} \ . \tag{185}$$

For simplicity, we denote $h=\beta'-\beta$ and denote its component by $h=(h_1,h_2)$. By changing variable from ζ to ζ' as

$$\zeta'=\zeta+h \ ,$$

and still writing it as ζ, we re-write the second integral above as

$$\int_{-\pi+h_1}^{\pi+h_1}\int_{-\pi+h_2}^{\pi+h_2}(f(\beta+\zeta)-f(\beta))G_1(\beta',\beta+\zeta)\frac{\zeta_1-h_1}{|\zeta-h|^3}d\zeta \ .$$

For h sufficiently small, the integral

$$\left(\int_{-\pi+h_1}^{\pi+h_1}\int_{-\pi+h_2}^{\pi+h_2}-\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\right)(f(\beta+\zeta)-f(\beta))G_1(\beta',\beta+\zeta)\frac{\zeta_1-h_1}{|\zeta-h|^3}d\zeta$$

does not contain any singular points for sufficiently small h. Because the integration area is of order $O(|h|)$, it can be shown that

$$\left(\int_{-\pi+h_1}^{\pi+h_1}\int_{-\pi+h_2}^{\pi+h_2}-\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\right)(f(\beta+\zeta)-f(\beta))G_1(\beta',\beta+\zeta)\frac{\zeta_1-h_1}{|\zeta-h|^3}d\zeta \le c|h|\|f\|_0\sum_{i=1}^{3}\|\nabla S_i\|_0 \le c|\beta-\beta'|^{\alpha}N(\alpha) \ .$$

Therefore, to prove (185), we only need to prove

$$\Big|\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(f(\beta+\zeta)-f(\beta))\left[G_1(\beta,\beta+\zeta)\frac{\zeta_1}{|\zeta|^3}-G_1(\beta',\beta+\zeta)\frac{\zeta_1-h_1}{|\zeta-h|^3}\right]d\zeta\Big| \le c|\beta-\beta'|^{\alpha}\|f\|_0\sum_{i=1}^{3}\|\nabla S_i\|_{\alpha} \quad . \tag{186}$$

We denote the above integral as I_3 and split it into two parts:

$$\begin{aligned} I_3 \ \le\ & \Big|\int_{|\zeta|<3|h|}(f(\beta+\zeta)-f(\beta))\left[G_1(\beta,\beta+\zeta)\frac{\zeta_1}{|\zeta|^3}-G_1(\beta',\beta+\zeta)\frac{\zeta_1-h_1}{|\zeta-h|^3}\right]d\zeta\Big| \\ & +\Big|\int_{|\zeta|\ge 3|h|}(f(\beta+\zeta)-f(\beta))\left[G_1(\beta,\beta+\zeta)\frac{\zeta_1}{|\zeta|^3}-G_1(\beta',\beta+\zeta)\frac{\zeta_1-h_1}{|\zeta-h|^3}\right]d\zeta\Big| \ . \\ \triangleq\ & I_{31}+I_{32} \ . \end{aligned} \tag{187}$$

Because the integration area is of order $O(h^2)$, we obtain

$$I_{31} \leq c|\beta - \beta'|^{\alpha} \|f\|_{\alpha} \sum_{i=1}^{3} \|\nabla S_i\|_{\alpha} \ , \tag{188}$$

where we have used (166).

To estimate I_{32}, we further split I_{32} into two parts:

$$\begin{aligned} I_{32} \quad &\leq \quad \int_{|\zeta| \geq 3|h|} |\zeta|^{\alpha} \|f\|_{\alpha} \left| [G_1(\beta, \beta+\zeta) - G_1(\beta', \beta+\zeta)] \frac{\zeta_1}{|\zeta|^3} \right| d\zeta \\ &\quad + \int_{|\zeta| \geq 3|h|} |\zeta|^{\alpha} \|f\|_{\alpha} \left| G_1(\beta', \beta+\zeta) \left(\frac{\zeta_1}{|\zeta|^3} - \frac{\zeta_1 - h_1}{|\zeta - h|^3} \right) \right| d\zeta \\ &\triangleq \quad I_{321} + I_{322} \ , \end{aligned} \tag{189}$$

which were denoted as I_{321} and I_{322}.

We estimate them separately. First, it can be shown that

$$I_{322} \leq c|\beta - \beta'|^{\alpha} \|f\|_{\alpha} \sum_{i=1}^{3} \|\nabla S_i\|_{\alpha} \ , \tag{190}$$

since

$$\left| \frac{\zeta_1}{|\zeta|^3} - \frac{\zeta_1 - h_1}{|\zeta - h|^3} \right| \leq c \frac{|h|}{|\zeta|^3} \ . \tag{191}$$

Furthermore, the following bound can be proved.

$$I_{321} \leq c|\beta - \beta'|^{\alpha} \|f\|_{\alpha} \sum_{i=1}^{3} \|\nabla S_i\|_{\alpha} \ , \tag{192}$$

since

$$|G_1(\beta, \beta+\zeta) - G_1(\beta', \beta+\zeta)| \leq c \frac{|h|}{|\zeta|} \sum_{i=1}^{3} \|\nabla S_i\|_0 \ , \tag{193}$$

where we have applied (166), (168), and (174). The derivation of (193) is similar to that of (191).

Moreover, as we mentioned earlier, by replacing f with $f - \tilde{f}$, we can prove

$$I_{11} \leq c|\beta - \beta'|^{\alpha} \|f - \tilde{f}\|_{\alpha} \sum_{i=1}^{3} \|\nabla S_i\|_{\alpha} \ ,$$

which further implies that

$$I_{11} \leq c|\beta - \beta'|^{\alpha} N(\alpha) \ .$$

In summary, we obtained inequality (145). Inequality (144) will follow if we take $\tilde{f} = \tilde{S_k} = 0$.

Up to now, the only thing left is to prove (165) – (169). Since the proofs of (165) – (169) are all very similar, we just prove the first two estimates.

1. The proof of (165)

Basically, we need to prove

$$\frac{|\mathbf{z}(\beta+\zeta)-\mathbf{z}(\beta)|}{|\zeta|} \geq c\ ,$$

which is equivalent to

$$\frac{|\mathbf{z}(\beta+\zeta)-\mathbf{z}(\beta)|^2}{|\zeta|^2} \geq c^2\ .$$

Substituting the formulation of x, y, and z into the left-hand side of the inequality, we get

$$\left(\frac{\zeta_1+S_1(\beta+\zeta)-S_1(\beta)}{|\zeta|}\right)^2+\left(\frac{\zeta_2+S_2(\beta+\zeta)-S_2(\beta)}{|\zeta|}\right)^2+\left(\frac{S_3(\beta+\zeta)-S_3(\beta)}{|\zeta|}\right)^2,$$

which is greater than

$$1-2(\|\nabla S_1\|_0+|\nabla S_2\|_0)-\sum_{i=1}^{3}\|\nabla S_i\|_0^2\ .$$

The above quantity has a lower bound of $\frac{29}{64}$ if

$$\|\nabla S_i\|_0 \leq \frac{1}{8}\ .$$

2. The proof of (166)

Similarly to the proof of (165), we can show that

$$\frac{|\mathbf{z}(\beta+\zeta)-\mathbf{z}(\beta)|}{|\zeta|} \leq c\ .$$

Therefore, to prove (166), we only need to show that

$$\frac{|\mathbf{z}(\beta+\zeta)-\mathbf{z}(\beta)|^3-|\zeta|^3}{|\zeta|^3} \leq c\left(\sum_{i=1}^{3}\|\nabla S_i\|_0\right)\ .$$

This is true if

$$\frac{|\mathbf{z}(\beta+\zeta)-\mathbf{z}(\beta)|^2-|\zeta|^2}{|\zeta|^2} \leq c\left(\sum_{i=1}^{3}\|\nabla S_i\|_0\right)\ ,$$

because by denoting $Df=\mathbf{z}(\beta+\zeta)-\mathbf{z}(\beta)$, we have

$$\frac{|Df|^3-|\zeta|^3}{|\zeta|^3}=\frac{|Df|^2-|\zeta|^2}{|\zeta|^2}\cdot\frac{|Df|^2+|Df||\zeta|+|\zeta|^2}{|\zeta|(|Df|+|\zeta|)}\ .$$

It comes down to show that for each S_i

$$|S_i(\beta+\zeta)-S_i(\beta)| \leq c\left(\sum_{i=1}^{3}\|\nabla S_i\|_0\right)|\zeta|$$

holds, which is obviously true. This concludes our proof.

Acknowledgments. We would like to thank Drs. Stephen Cowley, Dale Pullin, Patrick Guidotti, and Pingwen Zhang for many helpful discussions during the preparation of this work. We also would like to thank Dr. Hector Ceniceros for proof reading the original manuscript and making a number of useful suggestions which improve the presentation of the paper. The research was in part supported by a grant from National Science Foundation under the contract DMS-0073916, and by a grant from Army Research Office under the contract DAAD19-99-1-0141.

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