ON THE FINITE-TIME BLOWUP OF A 1D MODEL FOR THE 3D INCOMPRESSIBLE EULER EQUATIONS

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ABSTRACT

We study a 1D model for the 3D incompressible Euler equations in axisymmetric geometries, which can be viewed as a local approximation to the Euler equations near the solid boundary of a cylindrical domain. We prove the local well-posedness of the model in spaces of zero-mean functions, and study the potential formation of a finite-time singularity under certain convexity conditions for the velocity field. It is hoped that the results obtained on the 1D model will be useful in the analysis of the full 3D problem, whose loss of regularity in finite time has been observed in a recent numerical study (Luo and Hou, 2013).

1. BACKGROUND

The purpose of this note is to summarize some of the results we obtained on a 1D model for the 3D incompressible Euler equations. In a recently completed computation (Luo and Hou, 2013), we have numerically studied the 3D Euler equations in axisymmetric geometries and identified a class of potentially singular solutions. The equations being solved take the form

$$u_{1,t} + u^r u_{1,r} + u^z u_{1,z} = 2u_1 \psi_{1,z}, \tag{1.1a}$$

$$\omega_{1,t} + u^r \omega_{1,r} + u^z \omega_{1,z} = (u_1^2)_z, \tag{1.1b}$$

$$-\big[\partial_r^2 + (3/r)\partial_r + \partial_z^2\big]\psi_1 = \omega_1, \tag{1.1c}$$

where

$$u_1 = u^\theta/r, \qquad \omega_1 = \omega^\theta/r, \qquad \psi_1 = \psi^\theta/r,$$

are transformed angular velocity, vorticity, and stream functions and

$$u^r = -r\psi_{1,z}, \qquad u^z = 2\psi_1 + r\psi_{1,r},$$

are radial and axial velocity components. The solutions of (1.1) were computed in the cylinder

$$D(1,L) = \Big\{(r,z)\colon 0 \le r \le 1,\ 0 \le z \le L\Big\},$$

with carefully chosen initial data and no-flow (in r) and periodic (in z) boundary conditions. It was observed that the vorticity function $|\omega|$ develops a point singularity in finite time at the corner $\tilde{q}_0 = (1,0)^T$, which corresponds to a "singularity ring" on the solid boundary of the cylinder. The numerical data has been carefully checked against all major blowup (non-blowup) criteria including Beale-Kato-Majda (Beale et al., 1984), Constantin-Fefferman-Majda (Constantin et al., 1996), and

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Deng-Hou-Yu (Deng et al., 2005), to confirm the validity of the singularity. A local analysis near the point of the singularity also suggests the existence of a self-similar blowup. The interested readers are referred to Luo and Hou (2013) for more details.

2. The 1D Model and Its Well-Posedness

For the particular solution studied in Luo and Hou (2013), it is observed that, near the point of the singularity $\tilde{q}_0 = (1,0)^T$, the axial velocity u^z is negative when $z > 0$ and positive when $z < 0$. This creates a compression mechanism along the wall which seems to be responsible for the generation of the finite-time singularity. Motivated by these observations, we consider in this note the following 1D model

$$u_t + vu_z = 0, \qquad z \in (0, L), \tag{2.1a}$$

$$\omega_t + v\omega_z = u_z, \tag{2.1b}$$

with the nonlocal, zero-mean velocity v determined by

$$v_z(z) = H\omega(z) := \frac{1}{L}\,\text{P.V.}\int_0^L \omega(y)\cot\big[\mu(z-y)\big]\,dy, \qquad \mu = \pi/L. \tag{2.1c}$$

The problem is complemented by periodic boundary conditions and zero-mean initial data.

This 1D model can be viewed as the "restriction" of the 3D axisymmetric Euler equations (1.1) to the wall $r = 1$, with the identification

$$u(z) \sim u_1^2(1,z), \qquad \omega(z) \sim \omega_1(1,z), \qquad v(z) \sim \psi_{1,r}(1,z).$$

Indeed, the no-flow boundary condition ($\psi_1(1,z) = 0$) implies that

$$u^r = -r\psi_{1,z} = 0 \qquad \text{on} \qquad r = 1,$$

hence the evolution equations (1.1a)–(1.1b) reduce to (2.1a)–(2.1b) on the wall. To define the velocity v, we observe that

$$\psi_{1,r}(r,z) \ll \psi_{1,rr}(r,z), \qquad \omega_1(r,z) \approx \omega_1(1,z),$$

near the point of the singularity (Luo and Hou, 2013). Hence the Poisson equation (1.1c) can be locally approximated by

$$-\big[\partial_r^2 + \partial_z^2\big]\psi_1 = \omega_1(1,z),$$

the solution of which on the stretched domain $r \in (-\infty, 1)$ satisfies

$$\psi_{1,rz}(1,z) = H\omega_1(1,z).$$

This is precisely equation (2.1c) which provides the key relation needed to close (2.1a)–(2.1b).

Problems similar to (2.1b) have been studied in the past as models for the 3D Euler equations. In Constantin et al. (1985), the equation

$$\omega_t - v_x\omega = 0, \qquad v_x = H\omega, \tag{2.2a}$$

was introduced as a model for the dynamics of vorticity in incompressible inviscid flows. The finite-time blowup of (2.2a) was established for a large class of initial data as a consequence of the explicit solution formula

$$\omega(x,t) = \frac{4\omega_0(x)}{[2 - tH\omega_0(x)]^2 + t^2\omega_0^2(x)}.$$

In De Gregorio (1990, 1996), the model (2.2a) was modified to include a convection term:

$$\omega_t + v\omega_x - v_x\omega = 0, \qquad v_x = H\omega, \tag{2.2b}$$

and the resulting problem was conjectured to admit globally regular solutions. In Córdoba et al. (2005), the equation

$$\theta_t + \theta_x H\theta = 0, \tag{2.2c}$$

was proposed as the simplest model for transport equations with a nonlocal velocity. The finite-time blowup of (2.2c) was rigorously proved for a large class of initial data as a consequence of the estimate

$$-\int_0^\infty \frac{f_x(x)Hf(x)}{x^{1+\delta}}\,dx \ge C_\delta \int_0^\infty \frac{f^2(x)}{x^{2+\delta}}\,dx, \qquad \forall\delta\in(0,1),$$

which holds true for any even function f decaying sufficiently fast at ∞ and vanishing at 0 (see also Córdoba et al., 2006). In Okamoto et al. (2008), a generalization of the models (2.2a)–(2.2c):

$$\omega_t + av\omega_x - v_x\omega = 0, \qquad v_x = H\omega, \tag{2.2d}$$

was studied. The model reduces to (2.2a) if $a = 0$, to (2.2b) if $a = 1$, and to (2.2c) if $a = -1$ and $\omega = -\theta_x$. The global regularity of (2.2d) was numerically demonstrated for the case $a = 1$ and was rigorously proved in the limit of $a \to \infty$, in which case (2.2d) reduces to

$$\omega_t + v\omega_x = 0, \qquad v_x = H\omega.$$

Other similar models were also proposed and analyzed in the literature. The interested readers are referred to Chae et al. (2005); Castro and Córdoba (2008, 2010); Castro et al. (2010) for further readings.

Compared with the existing models, the 1D model (2.1) is distinct in that it consists of a *system* of two equations while all other models considered so far are *scalar* equations. In addition, the 1D model (2.1) provides a natural approximation to the 3D axisymmetric Euler equations along the wall $r = 1$, while no such explicit connection exists in other models. The purpose of this note is to study the basic properties of (2.1) including its (local) well-posedness and potential finite-time blowup. It is hoped that the results will be useful in the analysis of the full problem (1.1).

To study the well-posedness of the 1D model (2.1), define

$$V^k(S) = \Big\{f\colon f\in H^k(S),\ \bar f = 0\Big\}, \qquad k\ge 0,$$

where S denotes the circle on the plane with circumference L, $H^k(S)$ the usual (real) Sobolev space on S, and

$$\bar f := \frac{1}{L}\int_0^L f(z)\,dz$$

the mean of f on S. In view of the zero-mean property of functions in $V^k(S)$ and Poincaré's inequality (see (4.6a)), a suitable norm on $V^k(S)$ can be chosen as

$$\|f\|_{V^k} = \left[\int_0^L |\partial_z^k f(z)|^2\,dz\right]^{1/2}, \qquad f\in V^k(S),$$

with associated inner product

$$(f,g)_{V^k} = \int_0^L \partial_z^k f(z)\cdot\partial_z^k g(z)\,dz, \qquad f,\, g\in V^k(S).$$

The (local) well-posedness of the 1D model (2.1) is contained in the following three theorems.

Theorem 2.1 (Local existence and uniqueness)**.** *Let $m \geq 1$ be any positive integer. For any initial data*

$$u_0 \in V^{m+1}(S), \qquad \omega_0 \in V^m(S), \tag{2.3a}$$

there exists $T > 0$ depending only on $\|u_0\|_{V^{m+1}}$ and $\|\omega_0\|_{V^m}$ such that the 1D model (2.1) has a unique solution

$$\begin{aligned} u &\in C([0,T]; V^{m+1}(S)) \cap C^1([0,T]; V^m(S)), \\ \omega &\in C([0,T]; V^m(S)) \cap C^1([0,T]; V^{m-1}(S)). \end{aligned} \tag{2.3b}$$

We say the solution (u, ω) belongs to class CV^m on $[0, T]$ if it satisfies (2.3b).

Theorem 2.2 (Regularity)**.** *Let $m \geq 1$ be any positive integer and let*

$$u \in C([0,T]; V^2(S)), \qquad \omega \in C([0,T]; V^1(S)),$$

be a solution of (2.1) with initial data $u_0 \in V^{m+1}(S)$, $\omega_0 \in V^m(S)$. Then

$$u \in C([0,T]; V^{m+1}(S)), \qquad \omega \in C([0,T]; V^m(S)).$$

In particular, $u(\cdot, t)$, $\omega(\cdot, t) \in C^\infty(S)$ for each $t \in [0, T]$ if u_0, $\omega_0 \in C^\infty(S)$.

In essence, the regularity theorem says that the existence interval $[0, T]$ of the solution depends only on the low-norm $\|u_0\|_{V^2}$, $\|\omega_0\|_{V^1}$ of the initial data.

Theorem 2.3 (Continuous dependence)**.** *Let $m \geq 3$ be a positive integer and let*

$$u \in C([0,T]; V^{m+1}(S)), \qquad \omega \in C([0,T]; V^m(S)),$$

be a solution of (2.1) with initial data $u_0 \in V^{m+1}(S)$, $\omega_0 \in V^m(S)$. Let

$$u_{0,j} \in V^{m+1}(S), \quad \omega_{0,j} \in V^m(S), \qquad j = 1, 2, \ldots,$$

be a sequence of functions such that $u_{0,j} \to u_0$ in V^{m+1} and $\omega_{0,j} \to \omega_0$ in V^m. Then there exists $T' \in (0, T]$ and solutions

$$u_j \in C([0,T']; V^{m+1}(S)), \qquad \omega_j \in C([0,T']; V^m(S)),$$

of (2.1) with initial data $(u_{0,j}, \omega_{0,j})$ for sufficiently large j, such that

$$u_j \to u \quad in \quad C([0,T']; V^{m+1}(S)), \qquad \omega_j \to \omega \quad in \quad C([0,T']; V^m(S)).$$

The local existence theorem (Theorem 2.1) is a direct consequence of an abstract existence theorem of Kato and Lai (1984) and various calculus inequalities. To prove Theorem 2.2, we need the following energy estimate (see Proposition 4.7)

$$\max_{t\in[0,T]} \Big\{ \|u(\cdot,t)\|_{V^{m+1}}^2 + \|\omega(\cdot,t)\|_{V^m}^2 \Big\} \leq M_m(T) \Big\{ \|u_0\|_{V^{m+1}}^2 + \|\omega_0\|_{V^m}^2 \Big\},$$

where $M_m(T)$ is a constant depending on $\|u_0\|_{V^{\min(m,2)}}$, $\|\omega_0\|_{V^{\min(m,2)}}$, and

$$M_0(T) := \exp\bigg\{ \int_0^T \|H\omega(\cdot,t)\|_{L^\infty}\, dt \bigg\}.$$

Finally, Theorem 2.3 can be proved using a standard regularization technique. The details of these proofs are given in Section 4.

3. The Finite-Time Blowup of the 1D Model

To study the finite-time blowup of the 1D model (2.1), it is convenient to establish the following

Theorem 3.1 (Beale-Kato-Majda type criterion)**.** *Suppose that*

$$u_0 \in V^{m+1}(S), \qquad \omega_0 \in V^m(S),$$

and that the solution of (2.1) *in class* CV^m *exists on* $[0,T)$. *Then the solution cannot be continued in class* CV^m *up to and beyond* T *if and only if*

$$\int_0^T \|H\omega(\cdot,t)\|_{L^\infty}\, dt = \infty. \tag{3.1}$$

This criterion is similar to Theorem 3.2 proved in Okamoto et al. (2008) and is an analogue of the celebrated Beale-Kato-Majda theorem (Beale et al., 1984). Its proof is given in Section 4.6.

We shall now argue that the 1D model (2.1) develops a singularity in finite time, for the particular initial data

$$u_0(z) = a\sin^2(\mu z), \quad a > 0, \qquad \omega_0(z) = 0. \tag{3.2}$$

More specifically, we shall show that the velocity gradient

$$v_z(0) = H\omega(0) = -\frac{1}{L}\int_0^L \omega(z)\cot(\mu z)\, dz$$

at $z = 0$ satisfies a lower bound estimate

$$-v_z(0,t) = |v_z(0,t)| \geq 2c_0\tan(\tfrac{1}{2}c_0 t), \qquad c_0 = \left[\frac{\mu}{L}\int_0^L u_0(z)\cot^2(\mu z)\, dz\right]^{1/2} = (\tfrac{1}{2}a\mu)^{1/2}. \tag{3.3}$$

The finite-time blowup of (2.1) is a consequence of (3.3) in view of Theorem 3.1. Note that, for the given initial data, the solution has the property that u is even and has a *double zero* at $z = 0,\ \frac{1}{2}L$, and $\omega,\ v$ are odd at $z = 0,\ \frac{1}{2}L$. In addition, u, u_z, $\omega > 0$ and $v < 0$ on $(0, \frac{1}{2}L)$ for all $t > 0$ (for the proof of the last assertion, see (3.5)). These symmetry and sign-preserving properties mimic the behavior of the solutions of the 3D Euler equations (1.1) on the wall $r = 1$. In particular, they create a compression flow near $z = 0$ with $v < 0$ for $z > 0$ and $v > 0$ for $z < 0$, completely similar to the scenario observed in 3D (Luo and Hou, 2013). This provides an intuitive explanation for the finite-time blowup of the 1D model.

The proof of (3.3) proceeds as follows. First, we multiply the ω-equation (2.1b) by $\cot(\mu z)/L$ and integrate the resulting equation on $[0, L]$; this yields

$$-v_{zt}(0,t) + I = \frac{\mu}{L}\int_0^L u(z)\csc^2(\mu z)\, dz, \tag{3.4a}$$

where

$$\begin{aligned} I &= \frac{1}{L}\int_0^L v(z)\omega_z(z)\cot(\mu z)\,dz \\ &= -\frac{1}{L}\int_0^L \omega(z)\big[v_z(z)\cot(\mu z) - \mu v(z)\csc^2(\mu z)\big]\,dz \\ &= H(\omega v_z)(0) + \frac{\mu}{L}\int_0^L \omega(z)v(z)\csc^2(\mu z)\,dz =: \frac{1}{2}\,(v_z)^2(0) - I_1. \end{aligned} \tag{3.4b}$$

A direct computation using the definition of v shows

$$\begin{aligned} v(z) &= \frac{1}{\pi}\left[\int_0^{L/2} + \int_{L/2}^L\right]\omega(y)\log\big|\sin[\mu(z-y)]\big|\,dy \\ &= \frac{1}{\pi}\int_0^{L/2}\omega(y)\Big\{\log\big|\sin[\mu(z-y)]\big| - \log\big|\sin[\mu(z+y)]\big|\Big\}\,dy \\ &= \frac{1}{\pi}\int_0^{L/2}\omega(y)\log\left|\frac{\tan(\mu z)-\tan(\mu y)}{\tan(\mu z)+\tan(\mu y)}\right|dy < 0, \qquad \forall z \in (0, \tfrac{1}{2}L). \end{aligned} \tag{3.5}$$

Substituting (3.5) into the definition of I_1 (see (3.4b)), we deduce

$$\begin{aligned} I_1 &= -\frac{\mu}{L}\int_0^L \omega(z)v(z)\csc^2(\mu z)\,dz \ge -\frac{2\mu}{L}\int_0^{L/2}\omega(z)v(z)\cot^2(\mu z)\,dz \\ &= -\frac{2\mu}{\pi L}\int_0^{L/2} F(z)\int_0^{L/2} F(y)K(y,z)\,dy\,dz, \end{aligned}$$

where $F(z) = \omega(z)\cot(\mu z)$ and

$$K(y,z) = -w\log\left|\frac{w+1}{w-1}\right|, \qquad w = \frac{\tan(\mu y)}{\tan(\mu z)}.$$

By introducing the decomposition

$$\begin{aligned} I_{11} &= \frac{2\mu}{\pi L}\int_0^{L/2} F(z)\int_0^{L/2} F(y)\,dy\,dz, \\ I_{12} &= -\frac{2\mu}{\pi L}\int_0^{L/2} F(z)\int_0^{L/2} F(y)\big[K(y,z)+1\big]\,dy\,dz, \end{aligned}$$

we write $I_1 \ge I_{11} + I_{12}$ and compute

$$I_{11} = \frac{1}{2}\left[\frac{2}{L}\int_0^{L/2}\omega(z)\cot(\mu z)\,dz\right]^2 = \frac{1}{2}\,(v_z)^2(0).$$

To estimate I_{12}, we introduce another decomposition

$$I_{12} = -\frac{2\mu}{\pi L}\int_0^{L/2} F(z)\left[\int_0^z + \int_z^{L/2}\right](\cdots)\,dy\,dz =: J_{11} + J_{12},$$

and rearrange J_{12} using Fubini's theorem:

$$\begin{aligned} J_{12} &= -\frac{2\mu}{\pi L}\int_0^{L/2} F(y)\int_0^y F(z)\big[K(y,z)+1\big]\,dz\,dy \\ &= -\frac{2\mu}{\pi L}\int_0^{L/2} F(z)\int_0^z F(y)\big[K(z,y)+1\big]\,dy\,dz. \end{aligned}$$

This yields

$$I_{12} = -\frac{2\mu}{\pi L}\int_0^{L/2} F(z)\int_0^z F(y)\big[K(y,z)+K(z,y)+2\big]\,dy\,dz.$$

Since

$$K(y,z)+K(z,y)+2 = -w\log\left|\frac{w+1}{w-1}\right| - \frac{1}{w}\log\left|\frac{w+1}{w-1}\right| + 2 \le 0, \qquad \forall w\ge 0,$$

and $F\ge 0$ on $[0,\frac{1}{2}L]$, we conclude that $I_{12}\ge 0$ and hence

$$I_1 \ge I_{11} = \frac{1}{2}\,(v_z)^2(0).$$

This, combined with (3.4), leads to the estimate:

$$-v_{zt}(0,t) \ge \frac{\mu}{L}\int_0^L u(z)\cot^2(\mu z)\,dz. \tag{3.6}$$

Next, we multiply the u-equation (2.1a) by $\mu\cot^2(\mu z)/L$ and integrate the resulting equation on $[0,L]$; this yields

$$\frac{d}{dt}\left[\frac{\mu}{L}\int_0^L u(z,t)\cot^2(\mu z)\,dz\right] - I_2 = 0, \tag{3.7a}$$

where

$$\begin{aligned} I_2 &= -\frac{\mu}{L}\int_0^L v(z)u_z(z)\cot^2(\mu z)\,dz \\ &= -\frac{2\mu}{\pi L}\int_0^{L/2} G(z)\int_0^{L/2} F(y)K(y,z)\,dy\,dz, \end{aligned} \tag{3.7b}$$

where $G(z)=u_z(z)\cot(\mu z)$ and $F(y),\ K(y,z)$ are given as before. By introducing the decomposition

$$\begin{aligned} I_{21} &= \frac{2\mu}{\pi L}\int_0^{L/2} G(z)\int_0^{L/2} F(y)\,dy\,dz, \\ I_{22} &= -\frac{2\mu}{\pi L}\int_0^{L/2} G(z)\int_0^{L/2} F(y)\big[K(y,z)+1\big]\,dy\,dz, \end{aligned}$$

we write $I_2 = I_{21}+I_{22}$ and compute

$$I_{21} \ge -\frac{1}{2}\,v_z(0)\left[\frac{\mu}{L}\int_0^L u(z)\cot^2(\mu z)\,dz\right].$$

To estimate I_{22}, we introduce another decomposition

$$I_{22} = -\frac{2\mu}{\pi L}\int_0^{L/2} G(z)\left[\int_0^z + \int_z^{L/2}\right](\cdots)\,dy\,dz =: J_{21}+J_{22},$$

where

$$\begin{aligned} J_{21} &= -\frac{2\mu}{\pi L}\int_0^{L/2} G(z)\int_0^z F(y)\big[K(y,z)+1\big]\,dy\,dz, \\ J_{22} &= -\frac{2\mu}{\pi L}\int_0^{L/2} F(z)\int_0^z G(y)\big[K(z,y)+1\big]\,dy\,dz. \end{aligned}$$

Since

$$K(y,z) = -w \log\left|\frac{w+1}{w-1}\right| \le 0, \qquad w \in [0,1),$$
$$K(z,y) = -\frac{1}{w} \log\left|\frac{w+1}{w-1}\right| \le -2, \qquad w \in [0,1),$$

and F, $G \ge 0$ on $[0, \frac{1}{2}L]$, we conclude that

$$I_{22} \ge \frac{2\mu}{\pi L} \int_0^{L/2} \cot(\mu z) \int_0^z \cot(\mu y) D(y,z)\, dy\, dz,$$

where $D(y,z) = \omega(z)u_y(y) - u_z(z)\omega(y)$. *Assuming* for now that

$$D(y,z) \ge 0, \qquad \text{for all } 0 \le y \le z \le \tfrac{1}{2}L. \tag{3.8}$$

Then

$$I_2 \ge -\frac{1}{2}\, v_z(0) \left[\frac{\mu}{L} \int_0^L u(z) \cot^2(\mu z)\, dz\right],$$

and estimates (3.6)–(3.7) reduce to

$$h_1'(t) \ge h_2(t), \tag{3.9a}$$

$$h_2'(t) \ge \frac{1}{2}\, h_1(t) h_2(t), \tag{3.9b}$$

where

$$h_1(t) := -v_z(0,t) \ge 0, \qquad h_2(t) := \frac{\mu}{L} \int_0^L u(z,t) \cot^2(\mu z)\, dz \ge 0.$$

The lower bound estimate (3.3) can now be easily derived from (3.9). Indeed, integrating (3.9a) from 0 to t and using the initial condition $h_1(0) = 0$, we see

$$h_1(t) \ge H(t) := \int_0^t h_2(s)\, ds.$$

Substituting this estimate into (3.9b) and rearranging, we then deduce

$$H''(t) \ge \frac{1}{2}\, H(t) H'(t),$$

the repeated integration of which yields

$$H(t) \ge 2c_0 \tan(\tfrac{1}{2} c_0 t), \qquad c_0 = h_2^{1/2}(0).$$

It follows that $H(t)$, hence $h_1(t) = |v_z(0,t)|$, blows up no later than $T^* = \pi/c_0$. For the initial data considered in (3.2), we have

$$h_2(0) = \frac{\mu}{L} \int_0^L a \sin^2(\mu z) \cot^2(\mu z)\, dz = \frac{1}{2}\, a\mu,$$

so the solution blows up no later than $T^* = \sqrt{2\pi L/a}$.

To complete the proof of (3.3) and hence the finite-time blowup of the 1D model (2.1), it remains to prove (3.8). Since $u_z > 0$ on $(0, \frac{1}{2}L)$ and

$$\frac{D(y,z)}{u_z(z)u_y(y)} = \frac{\omega(z)}{u_z(z)} - \frac{\omega(y)}{u_y(y)} =: Q(z) - Q(y), \qquad Q(z) = \frac{\omega(z)}{u_z(z)},$$

we see that $D(y,z) \ge 0$ for $0 \le y \le z \le \frac{1}{2}L$ if $Q_z \ge 0$ on $(0, \frac{1}{2}L)$. Since Q_z satisfies the evolution equation

$$Q_{zt} + vQ_{zz} = v_{zz}Q, \qquad Q_z(z,0) = 0,$$

we see that $Q_z \ge 0$ on $(0, \frac{1}{2}L)$ if (recall $Q > 0$ on $(0, \frac{1}{2}L)$)

$$v_{zz}(z) \ge 0, \qquad \forall z \in (0, \tfrac{1}{2}L). \tag{3.10}$$

We have not been able to prove (3.10) rigorously but have verified this condition *numerically* for solutions generated from (3.2). This strongly indicates the existence of a finite-time singularity for the 1D model (2.1).

It is interesting to note that condition (3.10) can be interpreted from a geometric point of view: if the flow field v is odd at $z = 0$, $\frac{1}{2}L$ and is convex on $(0, \frac{1}{2}L)$, it will necessarily be negative for $z > 0$ and positive for $z < 0$, creating a compression flow near $z = 0$. If the convexity of the velocity field is preserved by the flow, then the compression mechanism near $z = 0$ will be sustained and reinforced, eventually leading to the formation of a finite-time singularity.

4. Proof of the Well-Posedness

In this section we give the proofs of Theorems 2.1–2.3 and 3.1.

4.1. **An Abstract Existence Theorem.**

The local existence of solutions of (2.1) can be proved using various techniques such as successive approximation, fixed-point theorem, or Galerkin approximation. In what follows, we shall make use of an abstract existence theorem of Kato and Lai (1984), which is based on a variant of Galerkin approximation. To state the theorem, we consider abstract nonlinear evolution equations of the form

$$u_t + A(t,u) = 0, \qquad t \ge 0, \; u(0) = u_0, \tag{4.1}$$

where A is a nonlinear operator. To define A precisely, we introduce the notion of *admissible triplet*, which consists of three real separable Banach spaces $\{Y, H, X\}$ with the properties:

(a) $Y \subset H \subset X$, with the inclusions continuous and dense;
(b) H is a Hilbert space, with inner product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H = (\cdot,\cdot)_H^{1/2}$;
(c) there is a continuous, nondegenerate bilinear form on $Y \times X$, denoted by $\langle \cdot,\cdot \rangle$, such that

$$\langle v, u\rangle = (v,u)_H, \qquad \forall v \in Y, \; u \in H.$$

With these notions, the existence theorem of Kato and Lai (1984) reads

Theorem 4.1 (Abstract existence theorem)**.** *Let $\{Y, H, X\}$ be an admissible triplet. Let A be a sequentially weakly continuous map on $[0,T_0] \times H$ into X such that*

$$\langle v, A(t,v)\rangle \ge -\beta(\|v\|_H^2), \qquad \forall t \in [0,T_0], \; v \in Y, \tag{4.2}$$

where $\beta(r) \ge 0$ is a monotone increasing function of $r \ge 0$. Then for any $u_0 \in H$, there exists $T \in (0, T_0]$ and a solution u of (4.1) in the class

$$u \in C_w([0,T]; H) \cap C_w^1([0,T]; X),$$

where the subscript w in C_w and C_w^1 indicates weak continuity. Moreover, one has

$$\|u(t)\|_H^2 \le \rho(t), \qquad t \in [0,T],$$

where ρ solves the scalar differential equation

$$\rho'(t) = 2\beta(\rho), \qquad \rho(0) = \|u_0\|_H^2. \tag{4.3}$$

If the solution to (4.3) *is not unique, ρ is understood as the maximal solution.*

Theorem 4.1 is not concerned with the uniqueness of the solution, neither is it concerned with the existence of strongly continuous solutions. However, both issues can be settled in a straightforward manner in our case.

With the aid of Theorem 4.1, we shall prove the local existence part of Theorem 2.1 by introducing, for any $k \geq 0$, the tensor product space

$$W^k(S) = V^{k+1}(S) \times V^k(S).$$

We equip $W^k(S)$ with the (natural) inner product

$$(f, g)_{W^k} = (f_1, g_1)_{V^{k+1}} + (f_2, g_2)_{V^k}, \qquad f,\, g \in W^k(S),$$

and the norm

$$\|f\|_{W^k} = (f, f)_{W^k}^{1/2}, \qquad f \in W^k(S).$$

In addition, we define for any $m \geq 1$ the triplet

$$Y = W^{2m}(S), \qquad H = W^m(S), \qquad X = W^0(S),$$

and the continuous bilinear form $\langle \cdot, \cdot \rangle \colon Y \times X \to \mathbb{R}$:

$$\langle f, g \rangle = (-1)^m \int_0^L \partial_z^{2m+1} f_1 \cdot \partial_z g_1 \, dz + (-1)^m \int_0^L \partial_z^{2m} f_2 \cdot g_2 \, dz, \qquad f \in Y,\ g \in X.$$

Integration by parts shows that

$$\langle f, g \rangle = (f_1, g_1)_{V^{m+1}} + (f_2, g_2)_{V^m} = (f, g)_H, \qquad f \in Y,\ g \in H,$$

so $\{Y, H, X\}$ forms an admissible triplet. Finally, we introduce the nonlinear mapping

$$A(h) = (vu_z, v\omega_z - u_z), \qquad h = (u, \omega) \in W^m(S),\ v \in V^{m+1}(S) \text{ with } v_z = H\omega. \tag{4.4}$$

To apply Theorem 4.1, we need to show that A defines a map from H into X, that A is sequentially weakly continuous, and that $\langle h, A(h) \rangle$ satisfies the estimate (4.2). The proof of these facts relies on two basic estimates of the operator A, which we shall derive in the next section.

4.2. **Basic Estimates.**

The basic tool that we shall need is the following

Proposition 4.2 (Calculus inequalities)**.** *Let $k \geq 0$ be any nonnegative integer and let $f,\, g \in L^\infty(S) \cap V^k(S)$. Then*

$$\|fg\|_{V^k} \leq C\Big\{\|f\|_{L^\infty}\|g\|_{V^k} + (1 - \delta_{k,0})\|f\|_{V^k}\|g\|_{L^\infty}\Big\}, \tag{4.5a}$$

where C is an absolute constant depending only on k and L and $\delta_{k,0}$ is the usual Kronecker delta symbol, with value 1 at $k = 0$ and 0 otherwise. If $k \geq 1$, then there also holds

$$\|\partial_z^k(fg) - f\partial_z^k g\|_{V^0} \leq C\Big\{\|f_z\|_{L^\infty}\|g\|_{V^{k-1}} + (1 - \delta_{k,1})\|f\|_{V^k}\|g\|_{L^\infty}\Big\}. \tag{4.5b}$$

Inequalities (4.5) are well known and hold true more generally for functions $f,\, g \in L^\infty \cap H^k$. For a proof of these results, see for example Majda and Bertozzi (2002). It is also worth noting that (4.5b) holds true for $g \in L^\infty \cap H^{k-1}$ by the usual density argument, even if the individual terms on the left-hand side may not belong to H^0.

Besides Proposition 4.2, the following well-known facts will also be used in the sequel.

(a) The Hilbert transform H is an isometry on $V^k(S)$ for all $k \geq 0$, with $\|Hf\|_{V^k} = \|f\|_{V^k}$. In addition, H commutes with ∂_z, i.e. $H(f_z) = (Hf)_z$.

(b) The Poincaré inequality asserts that $V^k(S) \subset V^j(S)$ for all $k > j \geq 0$, with

$$\|f\|_{V^j} \leq c_0^{k-j} \|f\|_{V^k}, \qquad f \in V^k(S),\ k > j \geq 0. \tag{4.6a}$$

The constant c_0 in the above inequality can be computed explicitly, with $c_0 = L/(2\pi)$.

(c) The Sobolev imbedding theorem asserts that $V^k(S) \subset L^\infty(S)$ for all $k \geq 1$, with

$$\|f\|_{L^\infty} \leq \tilde{c}_0 \|f_z\|_{V^0} \leq \tilde{c}_0 c_0^{k-1} \|f\|_{V^k}, \qquad f \in V^k(S),\ k \geq 1. \tag{4.6b}$$

The constant $\tilde{c}_0$ in the above inequality can be computed explicitly, with $\tilde{c}_0 = L/(2\sqrt{3})$.

(d) As a result of the Sobolev imbedding theorem and Proposition 4.2, $V^k(S)$ is a Banach algebra for all $k \geq 1$, with

$$\|fg\|_{V^k} \leq C \|f\|_{V^k} \|g\|_{V^k}, \qquad f,\, g \in V^k(S),\ k \geq 1, \tag{4.6c}$$

where C is an absolute constant depending only on k and L.

With the aid of these tools, we shall prove two basic estimates for the nonlinear operator A defined in (4.4). The first estimate concerns the boundedness and (strong) continuity of A.

Lemma 4.3. *Let $k \geq 1$ be any positive integer. Let $u \in V^{k+1}(S)$, $\omega, \tilde{\omega} \in V^k(S)$ and $\tilde{v} \in V^{k+1}(S)$ be such that $\tilde{v}_z = H\tilde{\omega}$. Then $\tilde{v}u_z \in V^k(S)$, $\tilde{v}\omega_z \in V^{k-1}(S)$, and*

$$\|\tilde{v}u_z\|_{V^k} \leq C\Big\{ \|\tilde{\omega}\|_{V^0} \|u\|_{V^{k+1}} + \|u\|_{V^2} \|\tilde{\omega}\|_{V^{k-1}} \Big\} \leq C \|\tilde{\omega}\|_{V^{k-1}} \|u\|_{V^{k+1}}, \tag{4.7a}$$

$$\|\tilde{v}\omega_z\|_{V^{k-1}} \leq C\Big\{ \|\tilde{\omega}\|_{V^0} \|\omega\|_{V^k} + (1-\delta_{k,1}) \|\omega\|_{V^2} \|\tilde{\omega}\|_{V^{k-2}} \Big\} \leq C \|\tilde{\omega}\|_{V^{k-1}} \|\omega\|_{V^k}, \tag{4.7b}$$

where C is an absolute constant depending only on k and L.

Proof. (4.7a) is a direct consequence of the calculus inequality (4.5a):

$$\|\tilde{v}u_z\|_{V^k} \leq C\Big\{ \|\tilde{v}\|_{L^\infty} \|u_z\|_{V^k} + \|u_z\|_{L^\infty} \|\tilde{v}\|_{V^k} \Big\},$$

and the isometry property of the Hilbert transform:

$$\|\tilde{v}\|_{V^k} = \|\tilde{v}_z\|_{V^{k-1}} = \|H\tilde{\omega}\|_{V^{k-1}} = \|\tilde{\omega}\|_{V^{k-1}}.$$

Combining these estimates, using Sobolev's imbedding theorem and noting that $k \geq 1$, we obtain

$$\begin{aligned} \|\tilde{v}u_z\|_{V^k} &\leq C\Big\{ \|\tilde{v}_z\|_{V^0} \|u\|_{V^{k+1}} + \|u_{zz}\|_{V^0} \|\tilde{\omega}\|_{V^{k-1}} \Big\} \\ &\leq C\Big\{ \|\tilde{\omega}\|_{V^0} \|u\|_{V^{k+1}} + \|u\|_{V^2} \|\tilde{\omega}\|_{V^{k-1}} \Big\} \leq C \|\tilde{\omega}\|_{V^{k-1}} \|u\|_{V^{k+1}}, \end{aligned}$$

which is (4.7a). To prove (4.7b), we follow the same steps, utilizing the calculus inequality (4.5a):

$$\|\tilde{v}\omega_z\|_{V^{k-1}} \leq C\Big\{ \|\tilde{v}\|_{L^\infty} \|\omega_z\|_{V^{k-1}} + (1-\delta_{k,1}) \|\omega_z\|_{L^\infty} \|\tilde{v}\|_{V^{k-1}} \Big\},$$

and the isometry property of the Hilbert transform (for $k \geq 2$):

$$\|\tilde{v}\|_{V^{k-1}} = \|\tilde{v}_z\|_{V^{k-2}} = \|H\tilde{\omega}\|_{V^{k-2}} = \|\tilde{\omega}\|_{V^{k-2}}.$$

Combining these estimates and using Sobolev's imbedding theorem then yields

$$\begin{aligned}\|\tilde{v}\omega_z\|_{V^{k-1}} &\leq C\Big\{\|\tilde{v}_z\|_{V^0}\|\omega\|_{V^k} + (1-\delta_{k,1})\|\omega_{zz}\|_{V^0}\|\tilde{\omega}\|_{V^{k-2}}\Big\} \\ &\leq C\Big\{\|\tilde{\omega}\|_{V^0}\|\omega\|_{V^k} + (1-\delta_{k,1})\|\omega\|_{V^2}\|\tilde{\omega}\|_{V^{k-2}}\Big\} \leq C\|\tilde{\omega}\|_{V^{k-1}}\|\omega\|_{V^k},\end{aligned}$$

which is (4.7b). □

As an immediate consequence of Lemma 4.3, we have the following

Proposition 4.4. *Let $m \geq 1$ be any positive integer. The nonlinear operator A defined by*

$$A(h) = (vu_z, v\omega_z - u_z), \qquad h = (u,\omega) \in W^m(S),\ v \in V^{m+1}(S) \text{ with } v_z = H\omega,$$

maps $W^m(S)$ continuously (in the strong topology) into $W^k(S)$ for all $0 \leq k \leq m-1$. In particular, we have

$$\begin{aligned}\|A(h)\|_{W^k} &\leq C\Big\{\|h\|_{W^m} + 1\Big\}\|h\|_{W^m}, && \forall h \in W^m(S), \\ \|A(h_1) - A(h_2)\|_{W^k} &\leq C\Big\{\|h_1\|_{W^m} + \|h_2\|_{W^m} + 1\Big\}\|h_1 - h_2\|_{W^m}, && \forall h_1,\, h_2 \in W^m(S),\end{aligned}$$

where C is an absolute constant depending only on m and L.

Proof. Using Poincaré's inequality and Lemma 4.3, we have, for each $0 \leq k \leq m-1$,

$$\begin{aligned}\|vu_z\|_{V^{k+1}} &\leq C\|vu_z\|_{V^m} \leq C\|\omega\|_{V^m}\|u\|_{V^{m+1}}, \\ \|v\omega_z - u_z\|_{V^k} &\leq C\|v\omega_z - u_z\|_{V^{m-1}} \leq C\Big\{\|\omega\|_{V^m}^2 + \|u\|_{V^{m+1}}\Big\}.\end{aligned}$$

Hence

$$\|A(h)\|_{W^k} \leq C\Big\{\|h\|_{W^m} + 1\Big\}\|h\|_{W^m}, \qquad \forall h \in W^m(S),$$

which shows that A maps $W^m(S)$ into $W^k(S)$. In addition, for any $h_1 = (u_1,\omega_1),\ h_2 = (u_2,\omega_2) \in W^m(S)$, we have

$$\begin{aligned}\|v_1u_{1,z} - v_2u_{2,z}\|_{V^m} &\leq \|\tilde{v}u_{1,z}\|_{V^m} + \|v_2\tilde{u}_z\|_{V^m}, \\ \|v_1\omega_{1,z} - v_2\omega_{2,z}\|_{V^{m-1}} &\leq \|\tilde{v}\omega_{1,z}\|_{V^{m-1}} + \|v_2\tilde{\omega}_z\|_{V^{m-1}},\end{aligned}$$

where $(\tilde{u},\tilde{\omega}) = (u_1 - u_2, \omega_1 - \omega_2)$ and $\tilde{v} = v_1 - v_2$. Hence another application of Lemma 4.3 yields

$$\begin{aligned}\|v_1u_{1,z} - v_2u_{2,z}\|_{V^{k+1}} &\leq C\Big\{\|u_1\|_{V^{m+1}}\|\tilde{\omega}\|_{V^m} + \|\omega_2\|_{V^m}\|\tilde{u}\|_{V^{m+1}}\Big\}, \\ \|v_1\omega_{1,z} - v_2\omega_{2,z}\|_{V^k} &\leq C\Big\{\|\omega_1\|_{V^m} + \|\omega_2\|_{V^m}\Big\}\|\tilde{\omega}\|_{V^m},\end{aligned}$$

from which we deduce that

$$\|A(h_1) - A(h_2)\|_{W^k} \leq C\Big\{\|h_1\|_{W^m} + \|h_2\|_{W^m} + 1\Big\}\|h_1 - h_2\|_{W^m}, \qquad \forall h_1,\, h_2 \in W^m(S).$$

This shows that A is strongly continuous from $W^m(S)$ to $W^k(S)$. □

In particular, Proposition 4.4 shows that A maps $H = W^m(S)$ continuously into $X = W^0(S)$.

The second estimate we shall prove for the operator A concerns the semi-boundedness of the nonlinear pairing $\langle h, A(h)\rangle$. Note that by Proposition 4.4 and Poincaré's inequality,

$$A(h) \in W^{2m-1}(S) \subset H, \qquad \forall h \in Y = W^{2m}(S),\ m \geq 1,$$

so to study $\langle h, A(h)\rangle$ it suffices to consider $(h, A(h))_H$.

Lemma 4.5. *Let $k \geq 1$ be any positive integer. Let $u, \tilde{u} \in V^{k+1}(S)$, $\omega, \tilde{\omega} \in V^k(S)$ and $v, \tilde{v} \in V^{k+1}(S)$ be such that $v_z = H\omega$, $\tilde{v}_z = H\tilde{\omega}$. Then*

$$|(\tilde{u}, \tilde{v}u_z)_{V^k}| \leq C\|\tilde{u}\|_{V^k}\Big\{\|\tilde{\omega}\|_{V^0}\|u\|_{V^{k+1}} + \|u\|_{V^2}\|\tilde{\omega}\|_{V^{k-1}}\Big\}, \tag{4.8a}$$

$$|(\tilde{\omega}, \tilde{v}\omega_z)_{V^{k-1}}| \leq C\|\tilde{\omega}\|_{V^{k-1}}\Big\{\|\tilde{\omega}\|_{V^0}\|\omega\|_{V^k} + (1-\delta_{k,1})\|\omega\|_{V^2}\|\tilde{\omega}\|_{V^{k-2}}\Big\}, \tag{4.8b}$$

$$|(\tilde{u}, v\tilde{u}_z)_{V^k}| \leq C\|\tilde{u}\|_{V^k}\Big\{\|H\omega\|_{L^\infty}\|\tilde{u}\|_{V^k} + (1-\delta_{k,1})\|\tilde{u}_z\|_{L^\infty}\|\omega\|_{V^{k-1}}\Big\}, \tag{4.8c}$$

$$|(\tilde{\omega}, v\tilde{\omega}_z)_{V^{k-1}}| \leq C\|\tilde{\omega}\|_{V^{k-1}}\Big\{\|H\omega\|_{L^\infty}\|\tilde{\omega}\|_{V^{k-1}} + (1-\delta_{k,1})(1-\delta_{k,2})\|\tilde{\omega}_z\|_{L^\infty}\|\omega\|_{V^{k-2}}\Big\}, \tag{4.8d}$$

where C is an absolute constant depending only on k and L.

Proof. (4.8a) is a direct consequence of the Cauchy-Schwarz inequality and estimate (4.7a):

$$|(\tilde{u}, \tilde{v}u_z)_{V^k}| \leq \|\tilde{u}\|_{V^k}\|\tilde{v}u_z\|_{V^k} \leq C\|\tilde{u}\|_{V^k}\Big\{\|\tilde{\omega}\|_{V^0}\|u\|_{V^{k+1}} + \|u\|_{V^2}\|\tilde{\omega}\|_{V^{k-1}}\Big\}.$$

Likewise, (4.8b) follows from the Cauchy-Schwarz inequality and estimate (4.7b):

$$|(\tilde{\omega}, \tilde{v}\omega_z)_{V^{k-1}}| \leq \|\tilde{\omega}\|_{V^{k-1}}\|\tilde{v}\omega_z\|_{V^{k-1}} \leq C\|\tilde{\omega}\|_{V^{k-1}}\Big\{\|\tilde{\omega}\|_{V^0}\|\omega\|_{V^k} + (1-\delta_{k,1})\|\omega\|_{V^2}\|\tilde{\omega}\|_{V^{k-2}}\Big\}.$$

To prove (4.8c), we introduce the decomposition

$$(\tilde{u}, v\tilde{u}_z)_{V^k} = (\partial_z^k\tilde{u}, \partial_z^k(v\tilde{u}_z) - v\partial_z^{k+1}\tilde{u})_{V^0} + (\partial_z^k\tilde{u}, v\partial_z^{k+1}\tilde{u})_{V^0} =: I_1 + I_2.$$

The first term I_1 on the right-hand side can be bounded using the calculus inequality (4.5b):

$$\begin{aligned}|I_1| &\leq \|\partial_z^k\tilde{u}\|_{V^0}\|\partial_z^k(v\tilde{u}_z) - v\partial_z^{k+1}\tilde{u}\|_{V^0}\\ &\leq C\|\tilde{u}\|_{V^k}\Big\{\|v_z\|_{L^\infty}\|\tilde{u}\|_{V^k} + (1-\delta_{k,1})\|v\|_{V^k}\|\tilde{u}_z\|_{L^\infty}\Big\}.\end{aligned}$$

As for I_2, integration by parts yields

$$I_2 = \int_0^L \partial_z^k\tilde{u}\cdot v\partial_z^{k+1}\tilde{u}\,dz = -\frac{1}{2}\int_0^L v_z(\partial_z^k\tilde{u})^2\,dz,$$

hence

$$|I_2| \leq C\|v_z\|_{L^\infty}\|\tilde{u}\|_{V^k}^2.$$

In summary,

$$|(\tilde{u}, v\tilde{u}_z)_{V^k}| \leq C\|\tilde{u}\|_{V^k}\Big\{\|H\omega\|_{L^\infty}\|\tilde{u}\|_{V^k} + (1-\delta_{k,1})\|\tilde{u}_z\|_{L^\infty}\|\omega\|_{V^{k-1}}\Big\},$$

as is to be shown.

Finally, to prove (4.8d) we write

$$(\tilde{\omega}, v\tilde{\omega}_z)_{V^{k-1}} = (\partial_z^{k-1}\tilde{\omega}, \partial_z^{k-1}(v\tilde{\omega}_z) - v\partial_z^k\tilde{\omega})_{V^0} + (\partial_z^{k-1}\tilde{\omega}, v\partial_z^k\tilde{\omega})_{V^0} =: I_3 + I_4,$$

where

$$\begin{aligned} k = 1: &\qquad I_3 = 0, \\ k \geq 2: &\qquad |I_3| \leq \|\partial_z^{k-1}\tilde{\omega}\|_{V^0}\|\partial_z^{k-1}(v\tilde{\omega}_z) - v\partial_z^k\tilde{\omega}\|_{V^0} \\ &\qquad\quad \leq C\|\tilde{\omega}\|_{V^{k-1}}\Big\{\|v_z\|_{L^\infty}\|\tilde{\omega}\|_{V^{k-1}} + (1-\delta_{k,2})\|v\|_{V^{k-1}}\|\tilde{\omega}_z\|_{L^\infty}\Big\}, \end{aligned}$$

and

$$I_4 = \int_0^L \partial_z^{k-1}\tilde{\omega}\cdot v\partial_z^k\tilde{\omega}\,dz = -\frac{1}{2}\int_0^L v_z(\partial_z^{k-1}\tilde{\omega})^2\,dz.$$

Therefore,

$$|(\tilde{\omega}, v\tilde{\omega}_z)_{V^{k-1}}| \leq C\|\tilde{\omega}\|_{V^{k-1}}\Big\{\|H\omega\|_{L^\infty}\|\tilde{\omega}\|_{V^{k-1}} + (1-\delta_{k,1})(1-\delta_{k,2})\|\tilde{\omega}_z\|_{L^\infty}\|\omega\|_{V^{k-2}}\Big\},$$

as is to be shown. □

As an immediate consequence of Lemma 4.5, we have the following

Proposition 4.6. *Let $m \geq 1$ be any positive integer. The nonlinear operator A defined by*

$$A(h) = (vu_z, v\omega_z - u_z), \qquad h = (u,\omega) \in H = W^m(S),\ v \in V^{m+1}(S) \text{ with } v_z = H\omega,$$

satisfies the estimate

$$\langle h, A(h)\rangle \geq -\beta(\|h\|_H^2), \qquad \forall h \in Y = W^{2m}(S),$$

where

$$\beta(r) = Cr(1 + r^{1/2}),$$

with C being an absolute constant depending only on m and L.

Proof. For each $h = (u,\omega) \in Y \subset W^{m+1}(S)$, we have, by Proposition 4.4 and Poincaré's inequality, $A(h) \in W^{2m-1}(S) \subset H$. Hence

$$\langle h, A(h)\rangle = (h, A(h))_H = (u, vu_z)_{V^{m+1}} + (\omega, v\omega_z - u_z)_{V^m}.$$

Using estimates (4.8c)–(4.8d) from Lemma 4.5 with $k = m+1$, $\tilde{u} = u$ and $\tilde{\omega} = \omega$, we have, for $m \geq 1$,

$$\begin{aligned} |(u, vu_z)_{V^{m+1}}| &\leq C\|u\|_{V^{m+1}}\Big\{\|H\omega\|_{L^\infty}\|u\|_{V^{m+1}} + \|u_z\|_{L^\infty}\|\omega\|_{V^m}\Big\} \\ &\leq C\|u\|_{V^{m+1}}\Big\{\|\omega\|_{V^m}\|u\|_{V^{m+1}} + \|u\|_{V^{m+1}}\|\omega\|_{V^m}\Big\} \leq C\|\omega\|_{V^m}\|u\|_{V^{m+1}}^2, \\ |(\omega, v\omega_z)_{V^m}| &\leq C\|\omega\|_{V^m}\Big\{\|H\omega\|_{L^\infty}\|\omega\|_{V^m} + (1-\delta_{m,1})\|\omega_z\|_{L^\infty}\|\omega\|_{V^{m-1}}\Big\} \\ &\leq C\|\omega\|_{V^m}\Big\{\|\omega\|_{V^m}^2 + (1-\delta_{m,1})\|\omega\|_{V^m}^2\Big\} \leq C\|\omega\|_{V^m}^3. \end{aligned}$$

This shows that

$$|\langle h, A(h)\rangle| \leq C\Big\{\|h\|_H + 1\Big\}\|h\|_H^2, \qquad \forall h \in Y,$$

and hence the proposition follows. □

4.3. **Proof of Theorem 2.1.** Now we are ready to prove Theorem 2.1. By Proposition 4.4 and Proposition 4.6, the conditions of Theorem 4.1 are satisfied, hence for any given initial data $h_0 = (u_0, \omega_0) \in H = W^m(S)$, there exists $T > 0$ depending only on $\|h_0\|_H^2 = \|u_0\|_{V^{m+1}}^2 + \|\omega_0\|_{V^m}^2$ such that the 1D model (2.1) has a solution

$$h = (u, \omega) \in C_w([0,T]; H) \cap C_w^1([0,T]; X).$$

To prove uniqueness, assume $h_1 = (u_1, \omega_1),\ h_2 = (u_2, \omega_2)$ are two solutions to (2.1) with the same initial data h_0. Subtracting the two equations satisfied by $h_1,\ h_2$, taking the X-inner product of the resulting equation with $\tilde{h} = (\tilde{u}, \tilde{\omega}) = (u_1 - u_2, \omega_1 - \omega_2)$, and observing that $\tilde{h} \in \mathrm{Lip}([0,T]; X)$ (which implies $\partial_t \|\tilde{h}\|_X^2 = 2(\tilde{h}, \tilde{h}_t)_X$ a.e. $t \in [0,T]$), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\tilde{h}\|_X^2 = -(\tilde{h}, A(h_1) - A(h_2))_X = I_1 + I_{21} + I_{22},$$

where

$$\begin{aligned} I_1 &= -(\tilde{u}, v_1 u_{1,z} - v_2 u_{2,z})_{V^1} = -(\tilde{u}, \tilde{v} u_{1,z})_{V^1} - (\tilde{u}, v_2 \tilde{u}_z)_{V^1}, && \tilde{v} = v_1 - v_2, \\ I_{21} &= -(\tilde{\omega}, v_1 \omega_{1,z} - v_2 \omega_{2,z})_{V^0} = -(\tilde{\omega}, \tilde{v}\omega_{1,z})_{V^0} - (\tilde{\omega}, v_2 \tilde{\omega}_z)_{V^0}, && I_{22} = (\tilde{\omega}, \tilde{u}_z)_{V^0}. \end{aligned}$$

Since $h_1,\ h_2 \in H \subset W^1(S)$, Lemma 4.5 applies with $k = 1$, yielding

$$\begin{aligned} |I_1| &\le C\Big\{\|\tilde{\omega}\|_{V^0}\|\tilde{u}\|_{V^1}\|u_1\|_{V^2} + \|\omega_2\|_{V^1}\|\tilde{u}\|_{V^1}^2\Big\}, \\ |I_{21}| &\le C\Big\{\|\tilde{\omega}\|_{V^0}^2\|\omega_1\|_{V^1} + \|\omega_2\|_{V^1}\|\tilde{\omega}\|_{V^0}^2\Big\}. \end{aligned}$$

On the other hand, the Cauchy-Schwarz inequality implies that

$$|I_{22}| \le \|\tilde{\omega}\|_{V^0}\|\tilde{u}\|_{V^1}.$$

Hence $\|\tilde{h}\|_X^2$ satisfies

$$\frac{d}{dt}\|\tilde{h}\|_X^2 \le C\Big\{\|h_1\|_{W^1} + \|h_2\|_{W^1} + 1\Big\}\|\tilde{h}\|_X^2.$$

Now by Gronwall's lemma,

$$\|\tilde{h}(t)\|_X^2 \le C\|\tilde{h}(0)\|_X^2 \exp\Big\{\int_0^t \Big[\|h_1(s)\|_{W^1} + \|h_2(s)\|_{W^1} + 1\Big]\, ds\Big\}.$$

Since $\tilde{h}(0) = 0$ and

$$h_1,\ h_2 \in C_w([0,T]; H) \subset L^\infty([0,T]; H) \subset L^1([0,T]; W^1(S)),$$

we conclude that $\tilde{h} \equiv 0$ on $[0,T]$, which proves the uniqueness of the solution.

To establish the strong continuity of the solution, we follow a standard argument which starts by showing that the solution determined by Theorem 4.1 is strongly right continuous at $t = 0$. Indeed, Theorem 4.1 implies that $\|u(t)\|_H^2 \le \rho(t)$ and, in particular, (see (4.3))

$$\limsup_{t\to 0^+}\|h(t)\|_H^2 \le \limsup_{t\to 0^+}\rho(t) = \|h_0\|_H^2.$$

On the other hand, the weak continuity of $h(t)$ at $t = 0$ implies that

$$\|h_0\|_H^2 \le \liminf_{t\to 0^+}\|h(t)\|_H^2.$$

Hence $\|h(t)\|_H \to \|h(0)\|_H$ as $t \to 0^+$, which establishes the strong right continuity of $h(t)$ at $t = 0$. To prove the right continuity of $h(t)$ at any $t_0 \in [0,T]$, let $\tilde{h}(t)$ be the solution of the 1D model (2.1) for $t \ge t_0$ with initial data $\tilde{h}(t_0) = h(t_0)$. Then $\tilde{h}(t)$ is strongly right continuous at $t = t_0$.

But the two solutions $h(t)$ and $\tilde{h}(t)$ must coincide for $t \geq t_0$ by uniqueness, so $h(t)$ is strongly right continuous at $t = t_0$. This shows that $h(t)$ is strongly right continuous on $[0, T]$. Since the 1D model (2.1) is time-reversible, which is apparent from the two-sided estimate of the nonlinear pairing $\langle h, A(h) \rangle$, it follows that $h(t)$ is strongly continuous on $[0, T]$. Finally, the higher regularity and strong continuity of $h_t(t)$ follows directly from Proposition 4.4, which asserts that

$$h_t(t) = -A(h(t)) \in W^{m-1}(S), \qquad \forall t \in [0, T],$$

and

$$\begin{aligned}\|h_t(t_1) - h_t(t_2)\|_{W^{m-1}} &= \|A(h(t_1)) - A(h(t_2))\|_{W^{m-1}} \\ &\leq C\Big\{\|h(t_1)\|_{W^m} + \|h(t_2)\|_{W^m} + 1\Big\}\|h(t_1) - h(t_2)\|_{W^m}, \qquad \forall t_1,\, t_2 \in [0, T].\end{aligned}$$

The proof of Theorem 2.1 is complete.

4.4. **Proof of Theorem 2.2.** We next prove Theorem 2.2. The key of the proof is to find an estimate of the existence time T that depends only on the low-norm $\|u_0\|_{V^2}$, $\|\omega_0\|_{V^1}$ of the initial data. In Kato and Lai (1984), this is accomplished using the technique of norm compression. Here we give a different proof where the idea is to directly bound the high-norms of the solution in terms of its low-norms. In particular, we have

Proposition 4.7. *Let $m \geq 1$ be any positive integer and let*

$$h = (u, \omega) \in C([0, T]; W^m(S)) \cap C^1([0, T]; W^{m-1}(S))$$

be a solution of (2.1) *in class CV^m on $[0, T]$, with the initial data $h_0 \in W^m(S)$. Then*

$$\max_{t \in [0,T]} \|h(t)\|_{W^m} \leq M_m(T)\|h_0\|_{W^m}, \tag{4.9a}$$

where $M_m(T)$ is a constant depending on m, $\|h_0\|_{W^{\min(m,2)}}$, and

$$M_0(T) = \exp\Big\{\int_0^T \|H\omega(t)\|_{L^\infty}\, dt\Big\}.$$

In addition, there exists an absolute constant C_0 depending only on L such that

$$\|h(t)\|_{W^1} \leq \frac{e^{C_0 t}\|h_0\|_{W^1}}{1 - (e^{C_0 t} - 1)\|h_0\|_{W^1}}, \qquad \forall t \in [0, T^*), \tag{4.9b}$$

where T^* is the first time at which the right-hand side of (4.9b) *becomes unbounded.*

Proof. We first derive an upper bound for $\|u_z\|_{L^\infty}$ in terms of $\|H\omega\|_{L^\infty}$. To begin with, we differentiate (2.1a) with respect to z (note that $u_t \in C([0, T]; V^1(S))$):

$$u_{zt} + v u_{zz} = -v_z u_z,$$

and introduce the characteristic variable

$$\frac{d}{dt} z(t) = v[z(t), t], \qquad z(0) = \xi \in [0, L].$$

This leads to the ODE

$$\frac{d}{dt} u_z[z(t), t] = -H\omega[z(t), t] \cdot u_z[z(t), t].$$

Its solution is easily found to be

$$u_z[z(t), t] = u_{0z}(\xi) \exp\Big\{-\int_0^t H\omega[z(s), s]\, ds\Big\},$$

hence

$$\|u_z(t)\|_{L^\infty} \le \|u_{0z}\|_{L^\infty} \exp\bigg\{ \int_0^t \|H\omega(s)\|_{L^\infty}\, ds \bigg\} =: M_0(t)\|u_{0z}\|_{L^\infty},$$

where

$$M_0(t) = \exp\bigg\{ \int_0^t \|H\omega(s)\|_{L^\infty}\, ds \bigg\},$$

which is the desired estimate.

Next, we derive an upper bound for the high-norm $\|h\|_{W^m}$ in terms of $\|H\omega\|_{L^\infty}$. To simplify the argument, we first assume h is sufficiently smooth, e.g. h belongs to class CV^{m+1}; then for each $1 \le k \le m$, taking the W^k-inner product of equations (2.1a)–(2.1b) with h yields

$$\frac{1}{2}\frac{d}{dt}\|h\|_{W^k}^2 = -(h, A(h))_{W^k} = -(u, vu_z)_{V^{k+1}} - (\omega, v\omega_z - u_z)_{V^k}. \tag{4.10a}$$

Applying estimates (4.8c)–(4.8d) from Lemma 4.5 with $k \leftarrow k+1$, $\tilde{u} = u$ and $\tilde{\omega} = \omega$ (again assuming a sufficiently smooth h), we obtain

$$|(u, vu_z)_{V^{k+1}}| \le C\|u\|_{V^{k+1}} \Big\{ \|H\omega\|_{L^\infty}\|u\|_{V^{k+1}} + \|u_z\|_{L^\infty}\|\omega\|_{V^k} \Big\}, \tag{4.10b}$$

$$|(\omega, v\omega_z)_{V^k}| \le C\|\omega\|_{V^k} \Big\{ \|H\omega\|_{L^\infty}\|\omega\|_{V^k} + (1-\delta_{k,1})\|\omega_z\|_{L^\infty}\|\omega\|_{V^{k-1}} \Big\}. \tag{4.10c}$$

For $k = 1$, (4.10) implies

$$\frac{d}{dt}\|h\|_{W^1}^2 \le C\Big\{ \|H\omega\|_{L^\infty} + \|u_z\|_{L^\infty} + 1 \Big\}\|h\|_{W^1}^2 \le K_0\Big\{ \|H\omega\|_{L^\infty} + M_0 \Big\}\|h\|_{W^1}^2,$$

where K_0 is a constant depending on $\|h_0\|_{W^1}$ (without loss of generality we assume $K_0 \ge 2$). Then by Gronwall's lemma,

$$\|h(t)\|_{W^1} \le M_1(t)\|h_0\|_{W^1}, \qquad \forall t \in [0, T],$$

where

$$M_1(t) = \exp\bigg\{ \frac{1}{2}K_0 \int_0^t \Big[\|H\omega(s)\|_{L^\infty} + M_0(s) \Big]\, ds \bigg\} \ge M_0(t).$$

For $k = 2$, (4.10) implies

$$\frac{d}{dt}\|h\|_{W^2}^2 \le C\Big\{ \|H\omega\|_{L^\infty} + \|u_z\|_{L^\infty} + \|\omega\|_{V^1} + 1 \Big\}\|h\|_{W^2}^2 \le K_1\Big\{ \|H\omega\|_{L^\infty} + M_1 \Big\}\|h\|_{W^2}^2,$$

where K_1 is a constant depending on $\|h_0\|_{W^1}$ (without loss of generality we assume $K_1 \ge K_0$). Then by Gronwall's lemma,

$$\|h(t)\|_{W^2} \le M_2(t)\|h_0\|_{W^2}, \qquad \forall t \in [0, T],$$

where

$$M_2(t) = \exp\bigg\{ \frac{1}{2}K_1 \int_0^t \Big[\|H\omega(s)\|_{L^\infty} + M_1(s) \Big]\, ds \bigg\} \ge M_1(t).$$

Finally, for $2 < k \le m$, (4.10) becomes

$$\frac{d}{dt}\|h\|_{W^k}^2 \le C\Big\{ \|H\omega\|_{L^\infty} + \|u_z\|_{L^\infty} + \|\omega\|_{V^2} + 1 \Big\}\|h\|_{W^k}^2 \le K_{k-1}\Big\{ \|H\omega\|_{L^\infty} + M_2 \Big\}\|h\|_{W^k}^2,$$

where K_{k-1} is a constant depending on k and $\|h_0\|_{W^2}$. Gronwall's lemma then implies

$$\|h(t)\|_{W^k} \le M_k(t)\|h_0\|_{W^k}, \qquad \forall t \in [0, T],$$

where

$$M_k(t) = \exp\left\{\frac{1}{2}\,K_{k-1}\int_0^t \Big[\|H\omega(s)\|_{L^\infty} + M_2(s)\Big]\,ds\right\}.$$

This establishes the high-norm estimate (4.9a) for $\|h\|_{W^m}$ under the assumption of sufficiently smooth h. To prove the low-norm estimate (4.9b), it suffices to note that $\|h\|_{W^1}$ can alternatively be estimated by

$$\frac{d}{dt}\|h\|_{W^1}^2 \le C\Big\{\|H\omega\|_{L^\infty} + \|u_z\|_{L^\infty} + 1\Big\}\|h\|_{W^1}^2 \le 2C_0\Big\{\|h\|_{W^1} + 1\Big\}\|h\|_{W^1}^2,$$

where C_0 is an absolute constant depending only on L. The desired estimate then follows from Gronwall's lemma.

To complete the proof of the proposition, we need to rigorously justify the above formal manipulations, in particular (4.10) for $k = m$. We achieve this by regarding $h = (u, \omega)$ as solutions of the *linear* hyperbolic equation

(4.11a) $$h_t + vh_z = f, \qquad f = (0, u_z),$$

with initial data

(4.11b) $$h_0 = (u_0, \omega_0) \in W^m(S),$$

and velocity

(4.11c) $$v \in C([0,T]; V^{m+1}(S)) \cap C^1([0,T]; V^m(S)).$$

It is a standard result from the theory of linear hyperbolic equations that problem (4.11) has a unique solution in class $C([0,T]; W^m(S)) \cap C^1([0,T]; W^{m-1}(S))$, and by approximating the data h_0 using smooth functions, (4.10) and the resulting high-norm estimates can be readily established as shown above. The same estimate for h then follows from a density argument. □

Now we are ready to prove Theorem 2.2. Suppose

$$h = (u, \omega) \in C([0,T]; W^1(S)) \cap C^1([0,T]; W^0(S))$$

is a solution of the 1D model (2.1) in class CV^1 on $[0,T]$, with the initial data $h_0 = (u_0, \omega_0) \in W^m(S)$. By Theorem 2.1, there exists another $T_m > 0$ that may depend on $\|h_0\|_{W^m}^2 = \|u_0\|_{V^{m+1}}^2 + \|\omega_0\|_{V^m}^2$ such that (2.1) has a solution in class CV^m on $[0, T_m]$:

$$\tilde{h} = (\tilde{u}, \tilde{\omega}) \in C([0,T_m]; W^m(S)) \cap C^1([0,T_m]; W^{m-1}(S)).$$

By uniqueness $\tilde{h} = h$ on $[0, T_m]$, so h belongs to class CV^m on $[0, T_m]$. Now let $T_m^* \ge T_m > 0$ denote the first time at which h ceases to be a solution in CV^m. We shall show that $T_m^* \ge T$. Suppose this is not the case, i.e. $T_m^* < T$. Then by Proposition 4.7,

$$\|h(t)\|_{W^m} \le M_m(T)\|h_0\|_{W^m}, \qquad \forall t \in [0, T_m^*),$$

where $M_m(T)$ is a constant depending only on

$$M_0(T) = \exp\left\{\int_0^T \|H\omega(t)\|_{L^\infty}\,dt\right\} \le \exp\left\{CT \max_{t\in[0,T]}\|h(t)\|_{W^1}\right\}$$

and $\|h_0\|_{W^{\min(m,2)}}$. Consequently, $\|h(t)\|_{W^m}$ is uniformly bounded on $[0, T_m^*)$ with a bound depending only on $\|h_0\|_{W^m}$ and $\max_{t\in[0,T]}\|h(t)\|_{W^1}$, and by Theorem 2.1, there exists, for each $t_0 < T_m^*$, a $\delta > 0$ independent of t_0 such that (2.1) has a solution $\tilde{h}$ in CV^m on $[t_0, t_0 + \delta]$ with initial data $\tilde{h}(t_0) = h(t_0)$. By uniqueness $\tilde{h}$ and h must coincide on $[t_0, t_0 + \delta]$, which then contradicts the assumption that h cannot be continued in class CV^m beyond T_m^*. This completes the proof of Theorem 2.2.

4.5. **Proof of Theorem 2.3.** We next prove Theorem 2.3. The key of the proof is to find, for any pair of solutions $h_1 = (u_1, \omega_1)$, $h_2 = (u_2, \omega_2)$ to (2.1), an appropriate bound for the difference $\|A(h_1) - A(h_2)\|_{W^m}$ where A is the nonlinear operator defined in (4.4). Since this bound generally involves higher norms $\|h_1\|_{W^{m+1}}$, $\|h_2\|_{W^{m+1}}$ of the solution, it is necessary to introduce some form of regularizations so that the desired estimates carry through.

To this end, we follow the idea of Kato and Lai (1984) and introduce the family of smoothing operators

$$J_\epsilon f = \eta_\epsilon * f, \qquad f \in V^0(S),\ \epsilon \in (0,1], \tag{4.12}$$

where η_ϵ is a (smooth) approximation to identity on $V^0(S)$ (e.g. the Poisson kernel or, for $\epsilon = 1/n$, $n \in \mathbb{N}$, the Fejér kernel), and

$$f * g(z) = \frac{1}{L}\int_0^L f(z-y)g(y)\,dy = \frac{1}{L}\int_0^L g(z-y)f(y)\,dy$$

denotes the convolution of two L-periodic functions f and g. The following properties of J_ϵ are well known (see, for example, Majda and Bertozzi, 2002):

Proposition 4.8. *Let J_ϵ be the smoothing operator (mollifier) defined in* (4.12)*. Then for any m, $k \geq 0$ and $f \in V^m(S)$:*

(a) $J_\epsilon f \in V^{m+k}(S)$;
(b) *there exists constant C depending only on m, k and L such that* $\|J_\epsilon f\|_{V^{m+k}} \leq C\epsilon^{-k}\|f\|_{V^m}$;
(c) $\|J_\epsilon f - f\|_{V^m} \to 0$ *and, if $m \geq 1$,* $\epsilon^{-1}\|J_\epsilon f - f\|_{V^{m-1}} \to 0$ *as* $\epsilon \to 0^+$.

Returning to problem (2.1), for any given data $h_0 = (u_0, \omega_0) \in W^m(S)$ with $m \geq 3$, we consider the smoothed problem with h_0 replaced by $h_0^\epsilon = J_\epsilon h_0$ and denote the corresponding solutions by $h^\epsilon = (u^\epsilon, \omega^\epsilon)$. Similarly, for any given data $h_{0,j} = (u_{0,j}, \omega_{0,j}) \in W^m(S)$, we consider the smoothed problem with $h_{0,j}$ replaced by $h_{0,j}^\epsilon = J_\epsilon h_{0,j}$ and denote the corresponding solutions by $h_j^\epsilon = (u_j^\epsilon, \omega_j^\epsilon)$. Since, by Proposition 4.8, the set of data $\{h_0^\epsilon, h_{0,j}^\epsilon\}$ is uniformly bounded in $W^m(S)$ for all $\epsilon \in (0,1]$ and $j \in \mathbb{N}$, it follows from Theorem 2.1 that there exists a $T' > 0$ independent of ϵ and j such that the solutions $\{h^\epsilon, h_j^\epsilon\}$ exist in class CV^m on $[0, T']$. In view of Proposition 4.7, we may also choose T' sufficiently small so that $\{h^\epsilon, h_j^\epsilon\}$ are uniformly bounded in $W^m(S)$ on $[0, T']$. Since each h_0^ϵ and $h_{0,j}^\epsilon$ belongs to $W^{m+1}(S)$, Theorem 2.2 shows that h^ϵ and h_j^ϵ indeed belongs to class CV^{m+1}.

The key step in the proof of Theorem 2.3 is the following proposition, which establishes the uniform convergence of the smoothed solutions h^ϵ to h, and similarly the uniform convergence of h_j^ϵ to h_j, in $W^m(S)$ as $\epsilon \to 0^+$.

Proposition 4.9. *Let $m \geq 3$ be a positive integer and let*

$$h = (u, \omega) \in C([0,T]; W^m(S)) \cap C^1([0,T]; W^{m-1}(S))$$

be a solution of (2.1) *in class CV^m on $[0,T]$, with the initial data $h_0 \in W^m(S)$. Let h^ϵ be the solution of the smoothed problem with initial data $h_0^\epsilon = J_\epsilon h_0$ where J_ϵ is the smoothing operator* *defined in* (4.12)*, and let* $T' > 0$ *be the common existence time of the solutions* $\{h, h^\epsilon\}$ *chosen as above. Then*

$$\max_{t\in[0,T']} \|h^\epsilon(t) - h(t)\|_{W^m} \to 0 \qquad \text{as} \qquad \epsilon \to 0^+,$$

and the convergence is uniform when h_0 *varies over compact subsets of* $W^m(S)$*.*

Proof. For any $0 < \delta < \epsilon \le 1$ and $2 \le k \le m$, we subtract the two equations satisfied by h^ϵ, h^δ, take the W^k-inner product of the resulting equation with $\tilde{h} = (\tilde{u}, \tilde{\omega}) = (u^\epsilon - u^\delta, \omega^\epsilon - \omega^\delta)$, and use the observation that $\tilde{h} \in C^1([0, T']; W^k)$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|\tilde{h}\|_{W^k}^2 = -(\tilde{h}, A(h^\epsilon) - A(h^\delta))_{W^k} = I_{11} + I_{12} + I_{21} + I_{22} + I_{23},$$

where

$$I_{11} + I_{12} = -(\tilde{u}, v^\epsilon u_z^\epsilon - v^\delta u_z^\delta)_{V^{k+1}} = -(\tilde{u}, \tilde{v} u_z^\epsilon)_{V^{k+1}} - (\tilde{u}, v^\delta \tilde{u}_z)_{V^{k+1}}, \qquad \tilde{v} = v^\epsilon - v^\delta,$$
$$I_{21} + I_{22} = -(\tilde{\omega}, v^\epsilon \omega_z^\epsilon - v^\delta \omega_z^\delta)_{V^k} = -(\tilde{\omega}, \tilde{v}\omega_z^\epsilon)_{V^k} - (\tilde{\omega}, v^\delta \tilde{\omega}_z)_{V^k}, \qquad I_{23} = (\tilde{\omega}, \tilde{u}_z)_{V^k}.$$

Since h^ϵ, $h^\delta \in W^{m+1}(S) \subset W^{k+1}(S)$, Lemma 4.5 applies with $k \leftarrow k + 1$, yielding

$$|I_{11}| \le C\|\tilde{u}\|_{V^{k+1}}\Big\{\|\tilde{\omega}\|_{V^0}\|u^\epsilon\|_{V^{k+2}} + \|u^\epsilon\|_{V^2}\|\tilde{\omega}\|_{V^k}\Big\},$$
$$|I_{12}| \le C\|\tilde{u}\|_{V^{k+1}}\Big\{\|\omega^\delta\|_{V^1}\|\tilde{u}\|_{V^{k+1}} + \|\tilde{u}\|_{V^2}\|\omega^\delta\|_{V^k}\Big\},$$
$$|I_{21}| \le C\|\tilde{\omega}\|_{V^k}\Big\{\|\tilde{\omega}\|_{V^0}\|\omega^\epsilon\|_{V^{k+1}} + \|\omega^\epsilon\|_{V^2}\|\tilde{\omega}\|_{V^{k-1}}\Big\},$$
$$|I_{22}| \le C\|\tilde{\omega}\|_{V^k}\Big\{\|\omega^\delta\|_{V^1}\|\tilde{\omega}\|_{V^k} + \|\tilde{\omega}\|_{V^2}\|\omega^\delta\|_{V^{k-1}}\Big\}.$$

Summing up these estimates and invoking the Cauchy-Schwarz inequality

$$|I_{23}| \le \|\tilde{\omega}\|_{V^k}\|\tilde{u}\|_{V^{k+1}},$$

we deduce

$$\frac{d}{dt}\|\tilde{h}\|_{W^k}^2 \le C\Big\{\|h^\epsilon\|_{W^2} + \|h^\delta\|_{W^1} + 1\Big\}\|\tilde{h}\|_{W^k}^2 + C\Big\{\|h^\epsilon\|_{W^{k+1}} + \|h^\delta\|_{W^k}\Big\}\|\tilde{h}\|_{W^2}\|\tilde{h}\|_{W^k}. \tag{4.13}$$

We shall now derive an estimate for $\|\tilde{h}\|_{W^m}$ and use the result to show that $\{h^\epsilon\}$ is Cauchy in

$$X := C([0, T']; W^m(S)) \cap C^1([0, T']; W^{m-1}(S)).$$

To begin with, we set $k = 2$ in (4.13) to obtain

$$\frac{d}{dt}\|\tilde{h}\|_{W^2}^2 \le C\Big\{\|h^\epsilon\|_{W^3} + \|h^\delta\|_{W^2} + 1\Big\}\|\tilde{h}\|_{W^2}^2.$$

Since $\{h_0^\epsilon\}$ is uniformly bounded in $W^m(S)$ for all $\epsilon \in (0, 1]$:

$$\sup_{\epsilon\in(0,1]} \|h_0^\epsilon\|_{W^m} \le K,$$

and T' is chosen sufficiently small, there exists, by Proposition 4.7, a constant K_1 depending on m, K and T' such that

$$\sup_{\epsilon\in(0,1]} \max_{t\in[0,T']} \|h^\epsilon(t)\|_{W^m} \le C \sup_{\epsilon\in(0,1]} \|h_0^\epsilon\|_{W^m} \le K_1.$$

Using Gronwall's lemma and noting that $m \geq 3$, we then deduce

$$\max_{t\in[0,T']} \|\tilde{h}(t)\|_{W^2} \leq \mathrm{e}^{CK_1T'} \|\tilde{h}(0)\|_{W^2}.$$

Now setting $k = m$ in (4.13) we obtain

$$\frac{d}{dt}\|\tilde{h}\|_{W^m} \leq C\Big\{\|h^\epsilon\|_{W^2} + \|h^\delta\|_{W^1} + 1\Big\}\|\tilde{h}\|_{W^m} + C\Big\{\|h^\epsilon\|_{W^{m+1}} + \|h^\delta\|_{W^m}\Big\}\|\tilde{h}\|_{W^2}.$$

By Proposition 4.7 and Proposition 4.8, there holds

$$\sup_{\epsilon\in(0,1]} \max_{t\in[0,T']} \epsilon\|h^\epsilon(t)\|_{W^{m+1}} \leq C \sup_{\epsilon\in(0,1]} \epsilon\|h_0^\epsilon\|_{W^{m+1}} \leq K_2,$$

where K_2 is another constant depending on $m,\ K$ and T', and (recall that $\delta < \epsilon$)

$$\begin{aligned}\max_{t\in[0,T']} \epsilon^{-1}\|\tilde{h}(t)\|_{W^2} &\leq \mathrm{e}^{CK_1T'}\epsilon^{-1}\|\tilde{h}(0)\|_{W^2} \\ &\leq \mathrm{e}^{CK_1T'}\Big\{\epsilon^{-1}\|h_0^\epsilon - h_0\|_{W^2} + \delta^{-1}\|h_0^\delta - h_0\|_{W^2}\Big\} \to 0 \qquad \text{as} \qquad \epsilon \to 0^+.\end{aligned}$$

Hence Gronwall's lemma implies that

$$\max_{t\in[0,T']} \|\tilde{h}(t)\|_{W^m} \leq \mathrm{e}^{CK_1T'}\Big\{\|\tilde{h}(0)\|_{W^m} + CK_2\epsilon^{-1}\int_0^{T'} \|\tilde{h}(s)\|_{W^2}\,ds\Big\} \to 0 \qquad \text{as} \qquad \epsilon \to 0^+,$$

which shows that $\{h^\epsilon\}$ is uniformly Cauchy in $W^m(S)$. To see $\{h_t^\epsilon\}$ is also uniformly Cauchy in $W^{m-1}(S)$, it suffices to recall from Proposition 4.4 that $A(h)$ is strongly continuous from $W^m(S)$ to $W^{m-1}(S)$, i.e.

$$\begin{aligned}\|\tilde{h}_t\|_{W^{m-1}} &= \|A(h^\epsilon) - A(h^\delta)\|_{W^{m-1}} \\ &\leq C\Big\{\|h^\epsilon\|_{W^m} + \|h^\delta\|_{W^m} + 1\Big\}\|\tilde{h}\|_{W^m} \leq CK_1\|\tilde{h}\|_{W^m}.\end{aligned}$$

Hence $\{h^\epsilon\}$ is Cauchy in

$$X = C([0,T']; W^m(S)) \cap C^1([0,T']; W^{m-1}(S)),$$

as claimed.

Since X is complete (with the obvious choice of the norm), there exists a unique $\hat{h} \in X$ such that

$$\max_{t\in[0,T']}\Big\{\|h^\epsilon(t) - \hat{h}(t)\|_{W^m} + \|h_t^\epsilon(t) - \hat{h}_t(t)\|_{W^{m-1}}\Big\} \to 0 \qquad \text{as} \qquad \epsilon \to 0^+.$$

Since

$$\begin{aligned}\|\hat{h}_t + A(\hat{h})\|_{W^{m-1}} &= \limsup_{\epsilon\to 0^+}\|\hat{h}_t + A(\hat{h}) - h_t^\epsilon - A(h^\epsilon)\|_{W^{m-1}} \\ &\leq \limsup_{\epsilon\to 0^+}\Big\{\|\hat{h}_t - h_t^\epsilon\|_{W^{m-1}} + CK_1\|\hat{h} - h^\epsilon\|_{W^m}\Big\} = 0,\end{aligned}$$

$\hat{h}$ is also a solution of (2.1) in class CV^m. But by uniqueness, $\hat{h}$ must coincide with h on $[0,T']$, so

$$\max_{t\in[0,T']}\Big\{\|h^\epsilon(t) - h(t)\|_{W^m} + \|h_t^\epsilon(t) - h_t(t)\|_{W^{m-1}}\Big\} \to 0 \qquad \text{as} \qquad \epsilon \to 0^+.$$

In addition, this convergence is uniform when h_0 varies over compact subsets of $W^m(S)$, since the convergence of $h_0^\epsilon = J_\epsilon h_0$ to h_0 is uniform over compact subsets of $W^m(S)$. Hence the proposition follows. □

Now we are ready to prove Theorem 2.3. Since $\{h_0^\epsilon, h_{0,j}^\epsilon\}$ is compact in $W^m(S)$, for any given $\delta > 0$ there exists, by Proposition 4.9, an $\epsilon \in (0,1]$ such that

$$\max_{t\in[0,T']}\|h^\epsilon(t) - h(t)\|_{W^m} < \frac{\delta}{3}, \qquad \sup_{j\in\mathbb{N}}\max_{t\in[0,T']}\|h_j^\epsilon(t) - h_j(t)\|_{W^m} < \frac{\delta}{3}.$$

For this fixed ϵ, a computation similar to the one leading to (4.13) shows that

$$\frac{d}{dt}\|\tilde{h}\|_{W^m}^2 \le C\Big\{\|h_j^\epsilon\|_{W^{m+1}} + \|h^\epsilon\|_{W^m} + 1\Big\}\|\tilde{h}\|_{W^m}^2 \le CK_2\epsilon^{-1}\|\tilde{h}\|_{W^m}^2,$$

where $\tilde{h} = h_j^\epsilon - h^\epsilon \in W^{m+1}(S)$. Gronwall's lemma then implies that

$$\max_{t\in[0,T']}\|\tilde{h}(t)\|_{W^m} \le \mathrm{e}^{CK_2\epsilon^{-1}T'}\|\tilde{h}(0)\|_{W^m} < \frac{\delta}{3},$$

provided that $j > j_0$ is sufficiently large. Hence

$$\begin{aligned}\max_{t\in[0,T']}\|h_j(t) - h(t)\|_{W^m} \le \max_{t\in[0,T']}\Big\{&\|h_j(t) - h_j^\epsilon(t)\|_{W^m} \\ &+ \|h_j^\epsilon(t) - h^\epsilon(t)\|_{W^m} + \|h^\epsilon(t) - h(t)\|_{W^m}\Big\} < \delta, \qquad \forall j > j_0,\end{aligned}$$

which shows that

$$\max_{t\in[0,T']}\|h_j(t) - h(t)\|_{W^m} \to 0 \qquad \text{as} \qquad j \to \infty.$$

This completes the proof of Theorem 2.3.

4.6. **Proof of Theorem 3.1.** Finally, we give the proof of Theorem 3.1. Suppose first that (3.1) holds, i.e.

$$\int_0^T \|H\omega(t)\|_{L^\infty}\,dt = \infty,$$

then necessarily

$$\limsup_{t\to T^-}\|H\omega(t)\|_{L^\infty} = \infty.$$

But by Sobolev's imbedding theorem and Poincaré's inequality,

$$\|H\omega(t)\|_{L^\infty} \le C\|\omega(t)\|_{V^1} \le C\|\omega(t)\|_{V^m},$$

so

$$\limsup_{t\to T^-}\|\omega(t)\|_{V^m} = \infty.$$

This shows that the solution cannot be continued in class CV^m up to $t = T$.

Next, suppose that (3.1) does not hold, i.e.

$$\int_0^T \|H\omega(t)\|_{L^\infty}\,dt < \infty. \tag{4.14}$$

Then by Proposition 4.7,

$$\|h(t)\|_{W^m} \le M_m(T)\|h_0\|_{W^m}, \qquad \forall t \in [0,T),\ h = (u,\omega),$$

where $M_m(T)$ is a constant depending only on

$$M_0(T) = \exp\Big\{\int_0^T \|H\omega(t)\|_{L^\infty}\,dt\Big\} < \infty$$

and $\|h_0\|_{W^{\min(m,2)}}$. Consequently, $\|h(t)\|_{W^m}$ is uniformly bounded on $[0,T)$ with a bound depending only on $\|h_0\|_{W^m}$ and $M_0(T)$, and by Theorem 2.1, there exists, for each $t_0 < T$, a $\delta > 0$ independent of t_0 such that (2.1) has a solution $\tilde{h}$ in CV^m on $[t_0, t_0+\delta]$ with initial data $\tilde{h}(t_0) = h(t_0)$. By uniqueness $\tilde{h}$ and h must coincide on $[t_0, t_0+\delta]$, which then shows that h can be continued in class CV^m to $t = T + \frac{1}{2}\delta$. This completes the proof of Theorem 3.1.

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