A complexity measure on matrices

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... because all good math begins with a vague idea.
Complexity is defined on a set. Simple elements are used to construct other elements, and as the number of simple elements used or the number of operations in this construction increases, so does the complexity.

Examples

- (Linear algebra) The family is the set of linear operators, the simple elements are rank one operators, and complexity is the rank.
- (Computational complexity) The family is the set of boolean functions on 2^n, simple elements are constant functions and the coordinate projections. AND, OR, and NOT are the composition operators, and complexity of an element is the number of composition steps required to construct it from the simple elements.

What I'm interested in:
The family is the set of square matrices, the simple elements are the rank one sign matrices, and matrices are built up using addition. Complexity is the smallest ℓ_1 norm of the coefficients in a sum which expresses a matrix as the sum of such basic matrices.

$$\begin{pmatrix} 1 & 1 & 3 & 3 \\ 3 & 3 & 1 & -1 \\ 6 & -3 & 2 & 7 \\ 5 & 2 & 2 & 1 \end{pmatrix}$$

has what complexity? Seems like a hard question. Now consider non-integer entries.

This complexity measure is given by

$$
\begin{aligned}
|||A||| &= \inf\left\{\|d\|_1 \;:\; A = U\mathrm{diag}(d)V^t,\ \|U\|_{1\to\infty} \le 1,\ \|V\|_{1\to\infty} \le 1\right\} \\
&= \inf\left\{\sum_i d_i \;:\; A = \sum_i d_i u_i v_i^t,\ u_i \text{ and } v_i \text{ are sign vectors}\right\}
\end{aligned}
$$

- Why the equivalence?
- Note that the number of terms in the sum is not specified (but it is finite)

Recall that if A, B are conformal matrices, then

$$\langle A, B\rangle = \mathrm{trace}(AB^t)$$

is an inner product. The trace pairing can be used to define dual norms.
Conveniently $|||\cdot|||$ is the trace dual of the $\|\cdot\|_{\infty\to 1}$ norm:

$$|||A||| = \max_{\|B\|_{\infty\to 1}\leq 1} \langle A, B\rangle \equiv \|A\|^{\star}_{\infty\to 1}$$

Why conveniently? Because it allows us to estimate $|||A|||$ easily.

Theorem (Grothendieck's Theorem)

There is a universal constant K_G such that if $A \in \mathrm{R}^{n \times n}$ is such that for any vectors u and v such that $\|u\|_\infty \leq 1$ and $\|v\|_\infty \leq 1$, $\left|\sum_{ij} Au_i v_j\right| \leq 1$, then

$$\sup_{\substack{\|u_i\|_H = 1 \\ \|v_j\|_H = 1}} \left| \sum_{ij} A_{ij} \langle u_i, v_j \rangle \right| \leq K_G$$

where the supremum is taken over all Hilbert spaces H.

Another way to state Grothendieck's theorem is

Theorem

There is a universal constant K_G so that is A is a real, square matrix, then

$$\gamma_2^\star(A) \leq K_G \|A\|_{\infty\to 1}$$

where $\gamma_2^\star(A)$ is given by

$$\sup_{\substack{\|u_i\|_H=1\\ \|v_j\|_H=1}} \left| \sum_{ij} A_{ij} \langle u_i, v_j \rangle \right|$$

and turns out to be a norm!

It isn't hard to see that we also have $\|A\|_{\infty\to 1} \leq \gamma_2^\star(A)$ (just take $H = \mathrm{R}$), which gives the final version of Grothendieck's theorem:

Theorem (Grothendieck's theorem)

There is a universal constant K_G such that if A is a real, square matrix, then

$$\|A\|_{\infty\to 1} \leq \gamma_2^\star(A) \leq K_G\|A\|_{\infty\to 1}$$

and by duality,

$$\gamma_2(A) \leq \|A\|_{\infty\to 1}^\star \leq K_G\gamma_2(A)$$

... duality is a beautiful thing.
Goemans-Williamson and Alon-Naor exploit the first duality to (under-)estimate $\|\cdot\|_{\infty\to 1}$. We exploit the second duality to (over-)estimate $\|\cdot\|^{\star}_{\infty\to 1}$.

First, note

$$\gamma_2(A) = \inf\{\|U\|_{2\to\infty}\|V\|_{2\to\infty} \ : A = UV^t\}$$

To interpret, consider $A : \ell_1 \to \ell_\infty$ to be factorized through ℓ_2 as $A = UV^t$ where $V^t : \ell_1 \to \ell_2$ and $U : \ell_2 \to \ell_\infty$, then

$$\|A\|_{\ell_1\to\ell_\infty} \leq \|U\|_{\ell_2\to\ell_\infty}\|V\|_{\ell_2\to\ell_\infty},$$

so $\|A\|_{\ell_1\to\ell_\infty} \leq \gamma_2(A)$ is the tightest possible bound one can get from such a factorization.

The dimension of the ℓ_2 space we factorize through is not predetermined, but we can show that we can take it to be $2n$. Then we can write $\gamma_2(A)$ as the solution to the SDP

$$
\begin{aligned}
&\text{minimize } t \\
&\text{subject to} \\
&\qquad X \in \mathrm{R}^{4n \times 4n} \succeq 0 \\
&\qquad X(1{:}2n, 2n+1{:}end) = A \\
&\qquad \mathrm{diag}(X) \leq t
\end{aligned}
$$

Why?

Ok, so now we can estimate $|||A|||$, but how can we find a corresponding rank one sign matrix decomposition? NO IDEA.

- there's a Grothendieck factorization for $C^\star$ algebras; maybe can do a similar procedure of SDP relaxation and rounding to computationally realize this
- greedy algorithms (pretty horrible)