Computational Complexity and Truth in Auctions

Dave Buchfuhrer Chris Umans

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- You assign items to the bidders
- You charge the bidders for their winnings

Example: Video Game Auction

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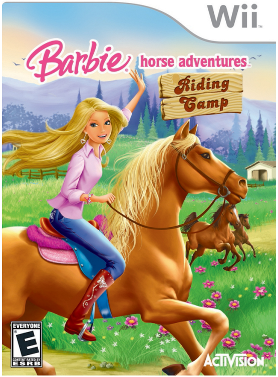

Value: 60

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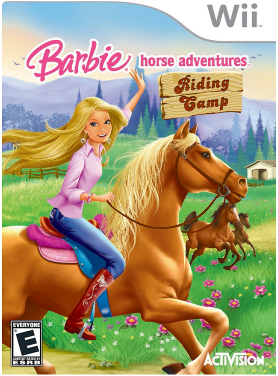

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- Each bidder receives some value from the set received
- The sum of the values for each player is the social welfare
- The social welfare does not depend on charges to bidders

The VCG Mechanism

- By participating in the auction, each bidder harms the others

80 80 80

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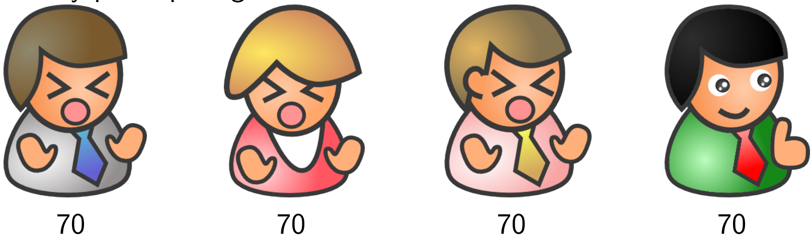

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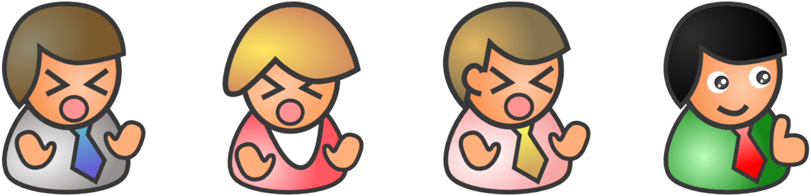

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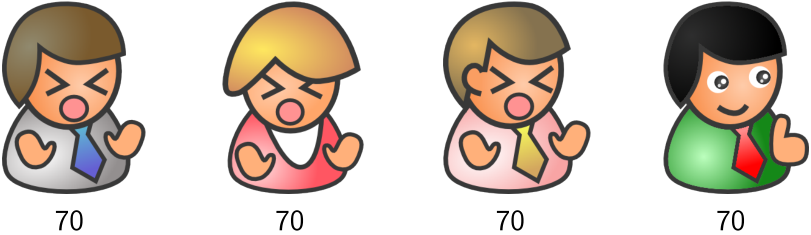

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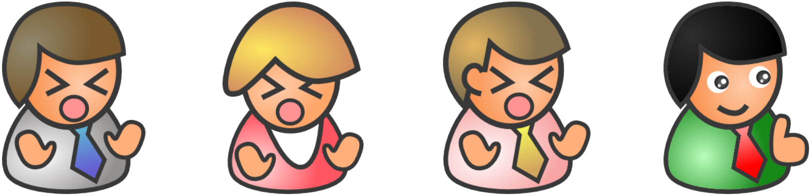

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- Intuitively, the player wants the social welfare maximized
- This all depends on being maximal-in-range

Maximal-In-Range

- An allocation function maps bids to distributions of items
- Each allocation function f has a range R
- f is Maximal-In-Range if it maximizes over R

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Example

Grouping all items into one lot, we can maximize over a range of size n. This yields a $1/n$ approximation.

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  - but using it encourages bidders to game the system
- It is difficult to have both computability and truthfulness

The Model

- Each bidder has a valuation function v_i
- For each item j, bidder i has a value $v_{i,j}$
- Each bidder i has a budget b_i
- For each subset $S \subseteq [m]$ of the items,

$$v_i(S) = \min\left(\sum_{j \in S} v_{i,j}, b_i\right)$$

Previous Work

- Inapproximability for Combinatorial Public Projects (Schapira, Singer, 2008)
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- The key to both of of these was VC dimension
- We show that n-bidder actions can't do better than $1/n$

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- 1221 means bidder 1 values 1 and 4, bidder 2 values 2 and 3
- All values are 1 or 0, budgets are infinite
- Social welfare is just how well the vectors match

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- Fix an allocation r in the range
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- So there is a subset of δm items on which we can solve exactly
- Using this subset as advice, we can solve welfare maximization
- So approximating to $1/2 + \epsilon$ is impossible unless $NP \subseteq P/poly$

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- So we focus in on some items where it's close to true
- VC dimension doesn't generalize well to more than 2 bidders
- So we form a meta-bidder out of all but one of the bidders

Coverings

- Suppose we have an approximation ratio of $1/n + \epsilon$
- For every $v \in [n]^m$, some $r \in R$ matches $(1/n + \epsilon)m$ indices

$$\begin{aligned} v &= 122221112212 \\ r &= 111221012210 \end{aligned}$$

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- For each S, T_S projects R to S
- T_S filters out $r \in R$ such that any $s \in S$ is unassigned
- $t \in T_S$ covers v if it is the projection of v to S

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- c is constant when $|S| = \alpha m$, for $\alpha < \epsilon$

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$$1231 \rightarrow 100\ 010\ 001\ 100$$

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- For each bidder other than i, $b = \infty$, $v_j = a_j$
- For bidder i, $b = 2\tau$, $v_j = 2a_j$
- A subset sums to τ iff we get welfare $\sum_j a_j + \tau$

Done

So if a maximal-in-range mechanism approximates the social welfare to $1/n + \epsilon$, subset sum is in $P/poly$

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- Non-constant number of bidders remains an open problem
- The more general question of how well truthful mechanisms can perform is left open