Subdivision I: The Univariate Setting

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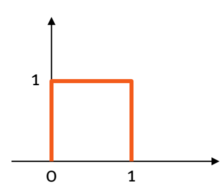
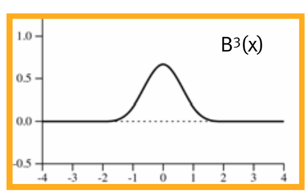
B-Splines (Uniform)

Through repeated integration

$$B^1(x) := \begin{cases} 1 & \text{if } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$



$$B^m(x) = \int_0^1 B^{m-1}(x-t)\,dt$$



B-Splines

Obvious properties

- piecewise polynomial:

$$\forall i \in \mathbb{N} : B^m(x)\Big|_{[i,i+1]} \in \Pi^{m-1}, B^m \in C^{m-2}(\mathbb{R})$$

- unit integral: $\int_{\mathbb{R}} B^m(x)\,dx = 1$
- non-negative: $B^m(x) \geq 0, x \in \mathbb{R}$
- partition of unity: $\sum_i B^m(x-i) = 1$
- support: $B^m(x) \neq 0, x \in [0,m]$

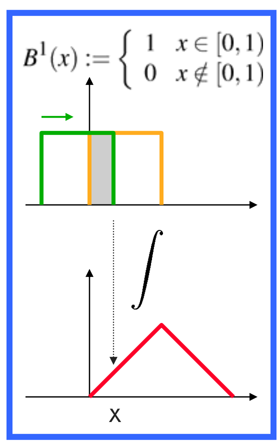
B-Splines

Repeated convolution

- box function

$$B^2(x) = \int B^1(t)B^1(x-t)\,dt = B^1 \otimes B^1(x)$$

$$B^m(x) = B^1 \otimes B^{m-1}(x)$$



Convolution

Reminder

- definition:

$$g \otimes h(x) = \int g(t)h(x-t)\,dt$$

- translation:

$$g(.-i) \otimes h(.-j)(x) = g \otimes h(x-i-j)$$

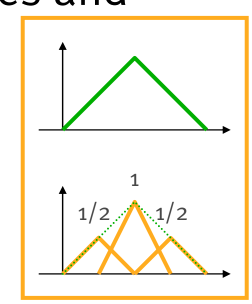
- dilation:

$$g(2\cdot) \otimes h(2\cdot)(x) = \frac{1}{2} g \otimes h(2x)$$

Refinability I

B-Spline refinement equation

- a B-spline can be written as a linear combination of dilates and translates of itself
- example
  - linear B-spline
  - and all others...



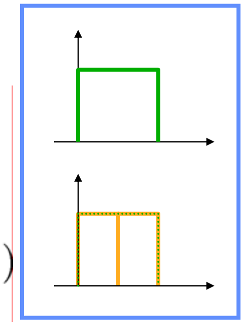
Refinability II

Refinement equation for B-splines

- take advantage of box refinement

$$B^1(x) = B^1(2x) + B^1(2x-1)$$

$$\begin{aligned} B^m(x) &= \bigotimes_{i=1}^{m} B^1(x) \\ &= \bigotimes_{i=1}^{m} B^1(2x) + B^1(2x-1) \end{aligned}$$

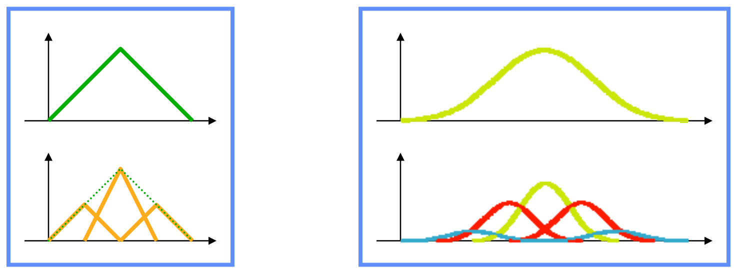


Refinability

$$\begin{aligned} B^m(x) &= \bigotimes_{i=1}^{m} (B^1(2x) + B^1(2x-1)) \\ &= \sum_{i=0}^{m} \binom{m}{i} (B^1(2\cdot))^{\otimes i} \otimes (B^1(2\cdot-1))^{\otimes m-i}(x) \\ &= 2^{-m+1} \sum_{i=0}^{m} \binom{m}{i} B^m(2x-i) \\ &= \sum_{i=0}^{m} s_i B^m(2x-i) \qquad S(z) = 2\left(\frac{1+z}{2}\right)^m \end{aligned}$$

B-Spline Refinement

Examples



1/2(1, 2, 1) 1/8(1, 4, 6, 4, 1)

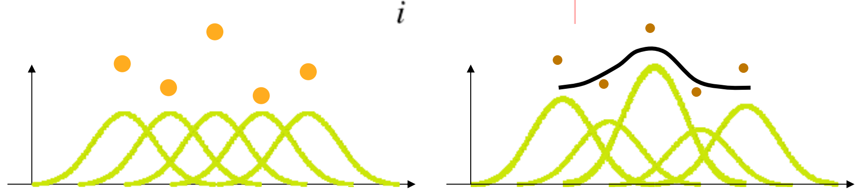
Spline Curves I

Sum of B-splines

- curve as linear combination

control points

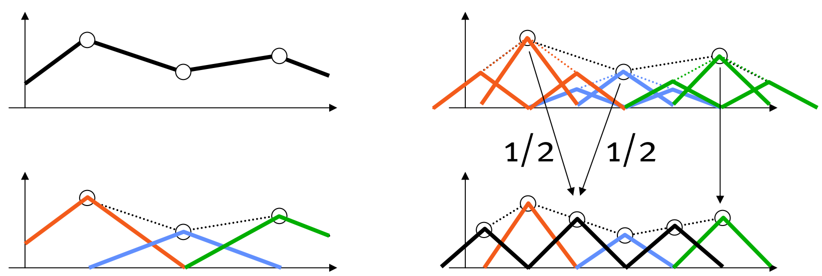
$$\gamma(t) = \sum_i p_i B(t-i)$$



Spline Curves II

Refine each B-spline in sum

- example: linear B-spline



Spline Curves III

Refinement for curves

- refine each B-spline in sum

$$
\begin{aligned}
\gamma(t) &= \sum_i p_i B(t-i) \\
&= \sum_i p_i \Big(\sum_k s_k B(2(t-i)-k)\Big) \\
&= \sum_i B(2t-i) \Big(\sum_k s_{i-2k} p_k\Big)
\end{aligned}
$$

refined bases

refinement of control points

Refinement of Curves

Linear operation on control points

- succinctly

$$\gamma(t) = \sum_i p_i B(t-i) \qquad \boxed{\gamma(t) = B(t)p}$$

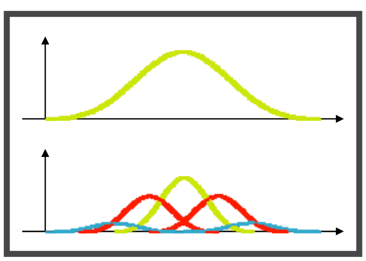
$$B(t) = \sum_k s_k B(2t-k) \qquad \boxed{B(t) = B(2t)S}$$

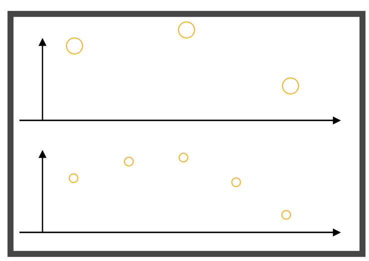
$$\boxed{\gamma(t) = B(t)p = B(2t)Sp}$$

Refinement of Curves

Bases and control points

$$\boxed{B(t) = B(2t)S} \qquad \boxed{p^1 = Sp^0}$$





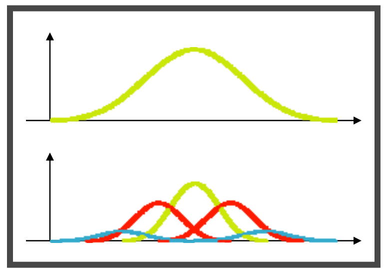
Subdivision Operator

Example

- cubic splines

$$S = \frac{1}{8}\begin{pmatrix} O & M & M & M & M & O \\ \Lambda & 1 & 0 & 0 & 0 & \Lambda \\ \Lambda & 4 & 0 & 0 & 0 & \Lambda \\ \Lambda & 6 & 1 & 0 & 0 & \Lambda \\ \Lambda & 4 & 4 & 0 & 0 & \Lambda \\ \Lambda & 1 & 6 & 1 & 0 & \Lambda \\ \Lambda & 0 & 4 & 4 & 0 & \Lambda \\ \Lambda & 0 & 1 & 6 & 1 & \Lambda \\ \Lambda & 0 & 0 & 4 & 4 & \Lambda \\ \Lambda & 0 & 0 & 1 & 6 & \Lambda \\ \Lambda & 0 & 0 & 0 & 4 & \Lambda \\ \Lambda & 0 & 0 & 0 & 1 & \Lambda \\ O & M & M & M & M & O \end{pmatrix}$$

$$B(t) = B(2t)S$$

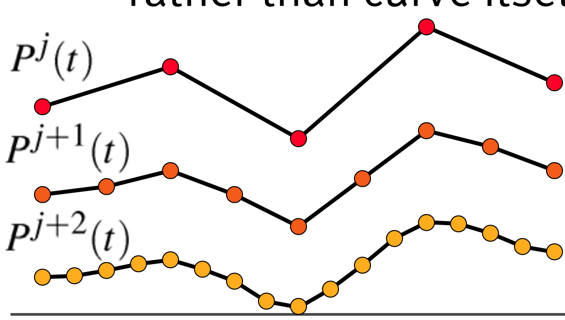


1/8(1, 4, 6, 4, 1)

Subdivision

Apply subdivision to control points

- draw successive control polygons rather than curve itself



$$p^{j+1} = Sp^j$$

Summary so far

Splines through refinement

- B-splines satisfy refinement eq.
- basis refinement corresponds to control point refinement
- instead of drawing curve, draw control polygon
- subdivision is refinement of control polygon

Analysis

Setup

- polygon mapped to polygon

  $$p^j = [p_i^j] \mapsto p^{j+1} = [p_i^{j+1}]$$

  - finite or (bi-)infinite, $p_i^j \in \mathbb{R}^d$
- subdivision operator (linear for now)

$$p^{j+1} = S p^j \qquad p_i^{j+1} = \sum_k s_{i,k} p_k^j$$

Subdivision Schemes

Some properties

- affine invariance
- compact support
- index invariance (topologic symmetry)
- local definition

Subdivision Operator

Properties

- compact support

  $\exists l_0, l_1 \in \mathbb{N} : \forall k : s_{i,k} = 0, i \notin [2k - l_0, 2k + l_1]$

- affine invariance

  $$ap_i^{j+1} + b = \sum_k s_{i,k}(ap_k^j + b)$$

- index invariance/symmetry

  $s_{i,k} = s_{i+2,k+1} \quad s_{2k-l,k} = s_{2k+l,k}$

Subdivision Operator

- local definition: weights depend only on local neighborhood

$$p_i^{j+1} = \sum_k s_{i-2k} p_k^j$$

Terms

- stationary: level independence
- uniform: location independence
  - no boundaries (for now)

Generating Functions

$$P^j(z) = \sum_i p_i^j z^i \qquad c_k = \sum_{i+j=k} a_i b_j \qquad C(z) = A(z)B(z)$$

Subdivision operator as convolution

$$s_{i,k} = s_{i-2k} \iff s_{2i+k,i} = s_k$$

$$\hat{s}_{i+k,i} = s_k \qquad \uparrow P^j(z) = P^j(z^2)$$

$$P^{j+1}(z) = S(z)P^j(z^2)$$

Examples

Splines

- linear: $1/2 + z + 1/2z^2 = 1/2(1+z)^2$

$$P^{j+1}(z) = \left(\frac{1+z}{2}\right)((1+z)P^j(z^2))$$

- quadratic:

$$P^{j+1}(z) = \left(\frac{1+z}{2}\right)^2((1+z)P^j(z^2))$$

- higher order...

Examples

Quadratic splines

- Chaikin's algorithm computes new points with weights 1/4(1, 3) and 1/4(3, 1)
- what happens if we change the weights?

Convergence

How much leeway do we have?

- design of other subdivision rules
  - example: 4pt scheme
- establish convergence
- establish order of continuity

Analysis

Simple facts

- affine invariance necessary condition for uniform convergence

$$S(1) = 2 \qquad S(-1) = 0$$

$$p_0^{j+1} = \sum_k s_{0-2k} p_k^j$$

$$p_1^{j+1} = \sum_k s_{1-2k} p_k^j$$

$$\text{supp } s_k \subset [-M, M]$$

$$\lim_{j \to \infty} p_k^j = f(0)$$

$$k \in [-M, M]$$

Analysis

Convergence

- define linear interpolant over given control points and associated parametric values (knot vector)

$$L[t^j, p^j](t) \qquad L[t^j, p^j](t_i^j) = p_i^j$$

- typically $t^j = 2^{-j}\mathbb{Z}$
- define pointwise $\lim_{j\to\infty} L[t^j, p^j](t) = f(t)$

Analysis

Convergence

- in max/sup norm $\|v\|_\infty = \sup_i |v_i|$

$$\|A\|_\infty = \sup_{v\neq 0} \frac{\|Av\|_\infty}{\|v\|_\infty} = \sup_i \sum_j |a_{i,j}|$$

Theorem

- if $\exists \beta > 0 \wedge \alpha \in (0,1) \forall j > 0 : \|\Delta p^j\|_\infty < \beta\alpha^j$ then convergence of $\lim_{j\to\infty} L[t^j, p^j](t)$ is uniform

Uniform Convergence

Proof

linear spline subdivision operator

$$L[t^{j+1}, p^{j+1}](t) - L[t^j, p^j](t) = L[t^{j+1}, (S - S_1)p^j](t)$$

$$S - S_1 = A\Delta \qquad \Delta(z) = (1 - z)$$

$$\|L[t^{j+1}, A\Delta p^j](t)\|_\infty \leq \|A\|_\infty \, \|\Delta p^j\|_\infty \leq \beta\alpha^j$$

$$L[t^0, p^0](t) + \sum_{j>0} (L[t^{j+1}, p^{j+1}](t) - L[t^j, p^j](t))$$

$$\|f(t) - L[t^j, p^j](t)\|_\infty < \|A\| \frac{\beta}{1-\alpha} a^j$$

Difference Decay

Sufficient condition

- continuous limit if $\|\Delta p^j\|_\infty < \beta a^j$
- analysis by examining associated difference scheme

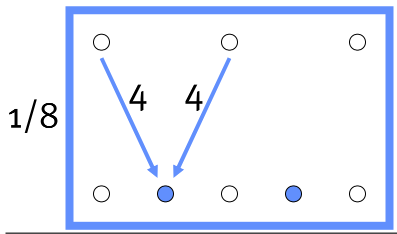
$$\Delta p^{j+1} = D\Delta p^j$$

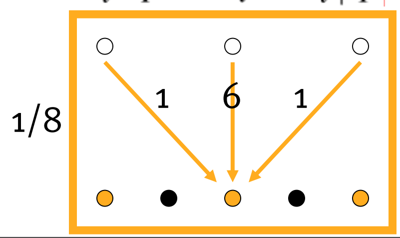
Example

Cubic B-splines

- stencils

$$p_{2i+1}^{j+1} = 1/8(4p_i^j + 4p_{i+1}^j)$$
$$p_{2i}^{j+1} = 1/8(p_{i-1}^j + 6p_i^j + p_{i+1}^j)$$



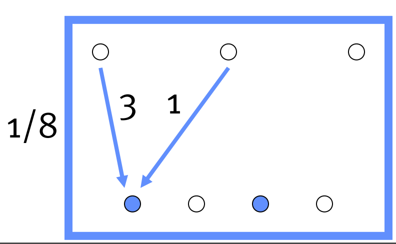


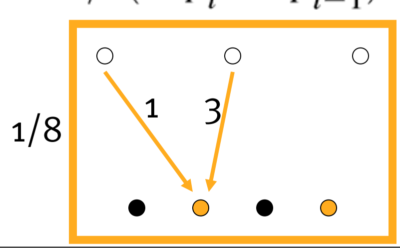
Example

Cubic B-splines

- differences

$$\Delta p_{2i}^{j+1} = p_{2i+1}^{j+1} - p_{2i}^{j+1}$$
$$= 1/8(3\Delta p_i^j + \Delta p_{i-1}^j)$$





Difference Decay

Analysis of difference scheme

- construction from the subdivision scheme itself

$$\Delta S p^j = D \Delta p^j$$

$$\rightsquigarrow \quad (1-z)P^{j+1}(z) = D(z)(1-z^2)P^j(z^2)$$

$$\rightsquigarrow \quad (1-z)S(z)P^j(z^2) = D(z)(1-z^2)P^j(z^2)$$

$$\rightsquigarrow \quad S(z) = D(z)(1+z)$$

Higher Orders

Smoothness

- how to show C^1?
- divided differences must converge
  - check difference of divided differences
- example
  - 4pt scheme

Smoothness

Consequences

$$S(z) \in C^0 \Longrightarrow 2^{-k}(1+z)^k S(z) \in C^k$$

- 4pt scheme: decay estimate

$$\begin{aligned} S(z) &= 1/16(-z^{-3}+4z^{-2}-z^{-1})(1+z)^4 \\ &= 1/16(-z^{-3}+9z^{-1}+16+9z-z^3) \\ DD(z) &= 1/8(-z^{-3}+4z^{-2}-z^{-1})(1+z)^2 \end{aligned}$$

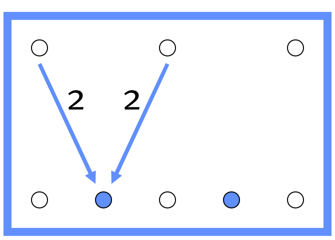
$$\|DD\|_\infty = 1 \qquad \boxed{\|DD^2\|_\infty = 3/4}$$

Example

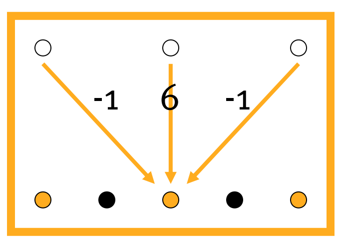
4pt scheme

- differences of divided differences

1/8



1/8



$$\|DD\|_\infty = 1$$

Analysis

Fundamental solution

- gives basis functions

$$L[t^j, p^j](t) = \sum_i p_i^j L[t^j, e_i^j]$$

$$\lim_{j\to\infty} L[t^j, S^j e_i^0](t) = \phi(t-i)$$

$$V^j = \text{span}\{\phi_i^j(t) := \phi(2^j t - i) | i \in \mathbb{Z}\}$$

Fundamental Solution

Properties

- refinement relation (why?)

$$\phi(t) = \Sigma_{i=-l_0}^{l_1} s_i\, \phi(2x-i)$$

- support? non-zero coefficients:

$$p^1_{-l_0} \cdots p^1_{l_1} \rightsquigarrow p^k_{-l_0(2^k-1)} \cdots p^k_{l_1(2^k-1)}$$

$$\lim_{k\to\infty} t^k_{-l_0(2^k-1)} = t^0_{-l_0} \cdots \qquad \text{supp}\,\phi \subset [t^0_{-l_0}, t^0_{l_1}]$$

So Far, So Good I

What do we know now?

- regular setting
- approximating
  - B-splines
- interpolating
  - 4pt scheme (Deslaurier-Dubuc)

So Far, So Good II

What do we know now?

- differences
  - continuity
  - differentiability
- not quite general enough
  - current setting assumes a particular parameterization

More General Settings

Non-uniform

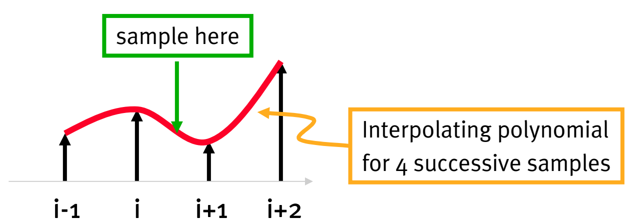
- in spline case better curves
- subdivision weights will vary
  - knot insertion
  - interpolation

$$S = \begin{pmatrix} a & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ c & f & 0 & 0 \\ d & g & 0 & 0 \\ e & h & k & 0 \\ 0 & i & l & 0 \\ 0 & j & m & p \\ 0 & 0 & n & q \\ 0 & 0 & o & r \\ 0 & 0 & 0 & s \\ 0 & 0 & 0 & t \end{pmatrix}$$

4pt Scheme I

Where do the weights come from?

- example of Deslaurier-Dubuc
- given set of samples use interpolating polynomial to refine



4pt Scheme II

Weight computation

- grind out interpolating polynomial
  - resulting weights: 1/16(-1, 9, 9, -1)
- Deslaurier-Dubuc
  - generalization of same idea
  - higher orders yield higher continuity
  - tends to exhibit “ringing” (as is to be expected...)

Deslaurier-Dubuc

Local polynomial reproduction

- choose s_k accordingly (d =1 for 4pt)

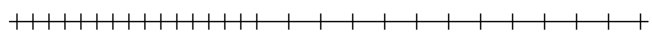
$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_{-d} & t_{-d+1} & \cdots & t_{d+1} \\ \vdots & \vdots & \ddots & \vdots \\ t_{-d}^{2d+1} & t_{-d+1}^{2d+1} & \cdots & t_{d+1}^{2d+1} \end{pmatrix} \begin{pmatrix} s_{-d} \\ s_{-d+1} \\ \vdots \\ s_{d+1} \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \\ \vdots \\ (1/2)^{2d+1} \end{pmatrix}$$

- non-uniform possible, increasing smoothness, approximation power, limit for increasing d is sync fn.

Irregular Analysis

New tools

- generating functions not applicable
- instead: spectral analysis (why?)
- for irregular spacing only one parameter: ratio of spacing



On to spectral analysis

Analysis

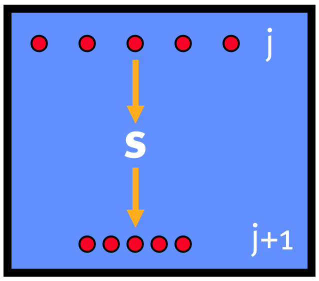
We need a different approach

- the subdivision matrix
  - a finite submatrix representative of overall subdivision operation
- based on invariant neighborhoods
- structure of this matrix key to understanding subdivision

Example

Cubic B-spline

- 5 control points for 1 segment on either side of the origin

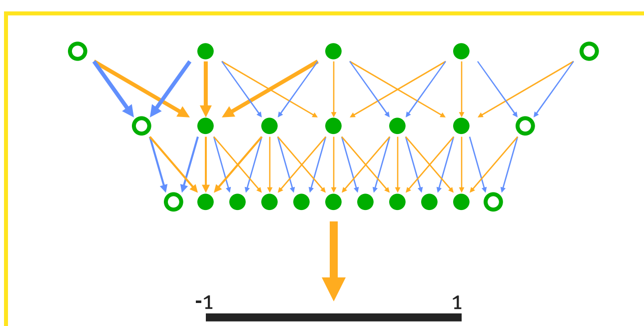


$$\begin{pmatrix} p^{j+1}_{-2} \\ p^{j+1}_{-1} \\ p^{j+1}_{0} \\ p^{j+1}_{1} \\ p^{j+1}_{2} \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 1 & 6 & 1 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 \\ 0 & 1 & 6 & 0 & 0 \\ 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 1 & 6 & 1 \end{pmatrix} \begin{pmatrix} p^{j}_{-2} \\ p^{j}_{-1} \\ p^{j}_{0} \\ p^{j}_{1} \\ p^{j}_{2} \end{pmatrix}$$

Neighborhoods

Which points influence a region?

- for analysis around a point



Subdivision Matrix

Invariant neighborhood

- which $\phi(i\text{-}t)$ overlap the origin?
  - tells smoothness story

$$\bar{S} = \frac{1}{16}\begin{pmatrix} -1 & 9 & 9 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 9 & 9 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 9 & 9 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 9 & 9 & -1 \end{pmatrix}$$

Eigen Analysis

What happens in the limit?

- behavior in neighborhood of point
  - apply S infinitely many times...
- suppose S has complete set of EVs

control points in invariant neighborhood

eigen vectors

$$p = \sum_{i=0}^{n-1} a_i x_i$$

$$Sp = \sum_{i=0}^{n-1} \lambda_i a_i x_i$$

Subdivision Matrix

Properties

- eigen vectors of non-zero eigen values identical

$$\lambda \neq 0, \bar{S}y = \lambda y \;\Rightarrow\; \exists x : Sx = \lambda x \wedge \bar{x} = y$$

  - proof by extension of y
- if defective, use generalized eigen vectors and values

Subdivision Matrix

Eigen vectors and eigen functions

$$\bar{S}\bar{x}_i = \lambda_i \bar{x}_i$$

$$|\lambda_0| > |\lambda_1| \geq |\lambda_2| \geq \ldots |\lambda_l|$$

$$l = -l_0 + l_1, \; i = 0, \ldots, l$$

$$\bar{p} = \sum_{i=0}^{l} c_i \bar{x}_i \rightsquigarrow F[t^0, p^0](t) = \sum_{i=0}^{l} c_i F[t^0, x_i](t)$$

no overbar

$$t \in (t^0_{-1}, t^0_1)$$

Scaling Relation

Eigen functions scale

$$Sx = \lambda x \quad \Longrightarrow \quad \lambda F[t^0,x](t) = F[t^0,x](t/2)$$

$$\begin{aligned}
\lambda F[t^0,x](t) &= F[t^0,\lambda x](t) = F[t^0,Sx](t) \\
&= \lim_{j\to\infty} L[t^j, S^j(Sx)](t) \\
&= \lim_{j\to\infty} L[2\,t^{j+1}, S^{j+1}x](t) \\
&= F[t^0,x](t/2)
\end{aligned}$$

in neighborhood of the origin

Smoothness (at Origin)

Lemma for functions which scale

$$\lambda g(t) = g(t/2)$$

I $\max_{t\in(1/2,1]} |g(t)| < \infty \wedge |\lambda| < 2^{-k} \Rightarrow \lim_{t\to 0} \frac{g(t)}{t^k} = 0$

II $\lambda = 1 \wedge g(t) \in C^0([0,1]) \Rightarrow g(t) = c$

III $g(d) \neq 0,\ d \neq 0 \wedge |\lambda| > 1 \Rightarrow \lim_{t\to 0} g(t) = \infty$

Necessary Conditions

Continuity at eigen functions

$$Sx = \lambda x \wedge |\lambda| \geq 2^{-k}$$

$$F[t,x](t) \in C^k \Rightarrow \exists 0 \leq i \leq k :$$

$$\lambda = 2^{-i} \wedge F[t,x](t) = c_i t^i \; c_i \neq 0$$

Necessary Conditions

Spectrum

- must be 2^{-i} for $0 \leq i \leq k$ and corresponding eigen functions must be monomials
- generalized eigen vectors?
  - λ_0 must be simple

$$Sx_0 = x_0$$
$$Sx_1 = \lambda_1 x_1 + x_0$$

$\lambda_1 = 1$?

$$\lambda_1 F[t,x_1](t) + F[t,x_0](t) = F[t,x_1](t/2)$$

Sufficient Conditions

Check at origin

- eigen functions for $|\lambda| < 2^{-k}$ must be checked

$$Sx = \lambda x \wedge |\lambda| < 2^{-k}$$

$$F[t,x](t) \in C^k((0,1]) \Rightarrow F[t,x](t) \in C^k([0,1])$$

Subdivision Operator

Spectrum

- necessary conditions
  - for C^k must have $\lambda_i=2^{-i}$ for $i \leq k$
    - eigen functions are polynomials
  - generally not enough
    - 4pt scheme has 1,1/2,1/4,1/4,1/8,-1/16, -1/16
  - approximation properties

Convergence

Limit position

- let j go to infinity

$$p^j = \lambda_0^j a_0 x_0 + \lambda_1^j a_1 x_1 + \ldots$$

- if λ_0=1 and $|\lambda_i|$<1, i=1,…,n-1

$$p_0^\infty = a_0$$

- example: cubic B-spline

$$p_i^\infty = a_0 = 1/6(p_{i-1}^j + 4p_i^j + p_{i+1}^j)$$

Geometric Behavior

Move limit point to origin

- look at higher order behavior

$$p^j = \lambda_0^j a_0 x_0 + \lambda_1^j a_1 x_1 + \ldots$$

$$|\lambda_2|, \ldots, |\lambda_{n-1}| < |\lambda_1|$$

$$p^j / \lambda_1^j = a_1 x_1 + (\lambda_2/\lambda_1)^j a_2 x_2 + \ldots$$

tangent vector

More General Settings

Subtleties

- generalized eigen values
- more subtle smoothness analysis
- non-uniform subdivision
- completely irregular subdivision
- boundaries
- formule de commutation

A Note

Size of subdivision matrix

- for *analysis* need enough support to parameterize a finite neighborhood of the origin
- for *evaluation* need only enough support to zoom in on origin
- e.g., cubic spline needs 5 respectively 3 control points

Affine Invariance

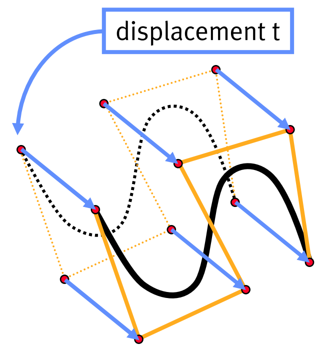
Sanity condition

- also necessary for convergence

$$p_i^{j+1} = \sum_k s_{k-2i} p_k$$

$$p_i^{j+1} + t = \sum_k s_{k-2i}(p_k + t)$$

$$= p_i^{j+1} + \Big(\underbrace{\sum_k s_{k-2i}}_{=1}\Big)t$$



Eigen Analysis

Summary

- invariant neighborhood to understand behavior around point
- Eigen decomposition of subdivision matrix helpful
  - limit point: a_o, tangent: a_1

General setting more complicated...