Introduction to Polynomial Curves Part II

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Polar Forms

Blossom

- For every polynomial F(u) of degree n there exists a unique symmetric multiaffine map $f(u_1,\ldots,u_n)$ for which $F(u)=f(u,\ldots,u)$. This map is called the polar form. F(u) is called its diagonal and f is called the blossom of F.

Example

Cubic case

$$F(u) = a_0 + a_1 u + a_2 u^2 + a_3 u^3$$

$$\begin{aligned} f(u_1, u_2, u_3) \;=\; & a_0 + \\ & \frac{a_1}{3}(u_1 + u_2 + u_3) + \\ & \frac{a_2}{3}(u_1 u_2 + u_2 u_3 + u_3 u_1) + \\ & a_3 u_1 u_2 u_3 \end{aligned}$$

Polar Form

Correspondence

$$F(u) = \sum_{i=0}^{n} a_i u^i \quad f(u_1, \ldots, u_n) = \sum_{i=0}^{n} a_i \binom{n}{i}^{-1} \sum_{\substack{S \subseteq \{1,\ldots,n\} \\ |S|=i}} \prod_{j \in S} u_j$$

Differentiation

- differencing

$$\Delta[\tau] f(u_1, \ldots, u_{n-1}) := f(u_1, \ldots, u_n + \tau) - f(u_1, \ldots, u_n)$$

- well defined, i.e., independent of u_n
- linear in τ

Polar Form

Differentiation

$$\begin{aligned} F^{(q)}(u) &= \frac{d^q}{du^q} f(\overbrace{u,\ldots,u}^{n}) \\ &= \frac{d^{q-1}}{du^{q-1}}(n\Delta[1]f(\overbrace{u,\ldots,u}^{n-1})) \\ &= n\Delta[1]\frac{d^{q-1}}{du^{q-1}} f(\overbrace{u,\ldots,u}^{n-1}) \\ &\ldots \\ &= \frac{n!}{(n-q)!}\Delta[1^q]f(\overbrace{u,\ldots,u}^{n-q}) \end{aligned}$$

$$a_q = \frac{F^{(q)}(0)}{q!} = \binom{n}{q}\Delta[1^q]f(0^{n-q})$$

Polar Form

Differentiation

- let f be a polar form and 0≤ p≤ n, and u $\in \mathbb{R}$ (fixed) then the following are equivalent

$$\forall q = 0,\ldots,p : \Delta[1^q]f(u^{n-q}) = 0$$

$$\forall v_1,\ldots,v_p \in \mathbb{R} : f(u^{n-p}, v_1,\ldots,v_p) = 0$$

Proof $f(\ldots,u) = f(\ldots,v) - \Delta[v-u]f(\ldots)$

Continuity

Two polynomials

- let F and G be two polynomials of degree n and let u be a point on the real line. Then F and G are C^q continuous at u iff

$$f(u^{n-q}, u_1, \ldots, u_q) = g(u^{n-q}, u_1, \ldots, u_q)$$

$$u_1, \ldots, u_q \in \mathbb{R}$$

Bezier Segments

Continuity conditions

$f(000)\quad f(001)\quad f(011)\quad f(111)\quad g(000)\quad g(001)\quad g(011)\quad g(111)$

$$C^0 \rightsquigarrow f(111) = g(000)$$

$$\Delta[1]f(uu) = f(uu1) - f(uu0) \rightsquigarrow f(111) - f(110)$$

$$\Delta[1]g(uu) = g(uu1) - g(uu0) \rightsquigarrow g(001) - g(000)$$

$$C^1 \rightsquigarrow f(111) = g(000) = 1/2(g(001)+f(110))$$

$$\Delta[11]f(u) = f(u11) - 2f(u10) + f(u00)$$

$$\Delta[11]g(u) = g(u11) - 2g(u10) + g(u00)$$

$$C^2 \rightsquigarrow f(111) = g(000) = 1/4(g(011)-f(001))$$

Bezier Points

Finding control points

- the Bezier control points of F with respect to [s,t] are

$$b_i = f(s^{n-i}t^i) \quad i = 0, \dots, n \quad u = \frac{t-u}{t-s}s + \frac{u-s}{t-s}t$$

Proof

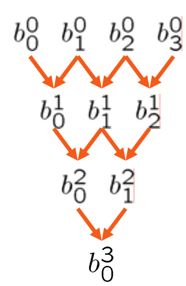
$$\begin{aligned} F(u) = f(u^n) &= \left(\frac{t-u}{t-s}\right) f(su^{n-1}) + \left(\frac{u-s}{t-s}\right) f(u^{n-1}t) \\ &= \dots = \sum_{i=0}^{n} B_i^{\Delta,n}(u) f(s^{n-i}t^i) \end{aligned}$$

de Casteljau Alg.

Evaluate Bezier form

$$b_i^0(u) = b_i \quad i = 0, \dots, n$$

$$b_i^l(u) = \left(\frac{t-u}{t-s}\right) b_i^{l-1}(u) + \left(\frac{u-s}{t-s}\right) b_{i+1}^{l-1}(u)$$

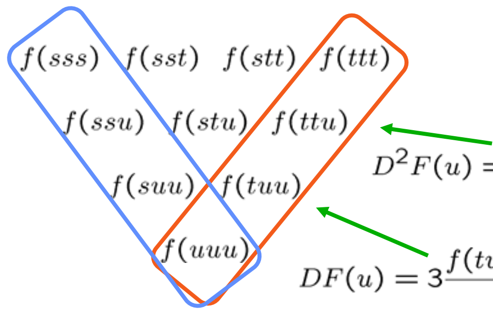


- very stable
- convex combinations only

de Casteljau

Properties

- segment splitting, derivatives



$$D^2F(u) = 6\frac{f(ttu) - 2f(stu) + f(ssu)}{(t-s)^2}$$

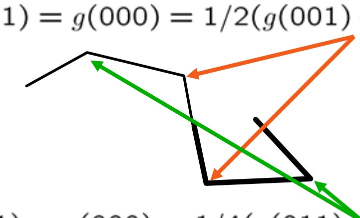
$$DF(u) = 3\frac{f(tuu) - f(suu)}{t-s}$$

Multiple Segments

How to stick them together?

- can match first derivative easily
- second derivative?

$$f(111) = g(000) = 1/2(g(001) - f(110))$$



$$f(111) = g(000) = 1/4(g(011) - f(001))$$

Modeling

Bezier curves

- requires to keep track of continuity conditions explicitly... Possible, but very messy
- better: use basis which has continuity conditions built in!

B-Splines

B-Splines

Basic idea

- share arguments of control points

$$u \in [0,1]$$

$$f(-2-10) \quad f(-101) \quad f(012) \quad f(123)$$

$$f(-10u) \quad f(01u) \quad f(12u)$$

$$f(0uu) \quad f(1uu)$$

$$f(uuu)$$

Argument "Bags"

Knotvector

- definition

$$T = (t_i)_{i \in \mathbb{Z}} \quad t_i < t_{i+n+1}$$

- non-decreasing sequence, can have multiple entries (multiplicity)
- curve:

$$c(u) = \sum N_i^n(u) d_i$$

Knot Vector

Example

- Bezier segments

$$0 \; 0 \; 0 \; 1 \; 1 \; 1$$
$$(000) \; (001) \; (011) \; (111)$$

- B-splines segments

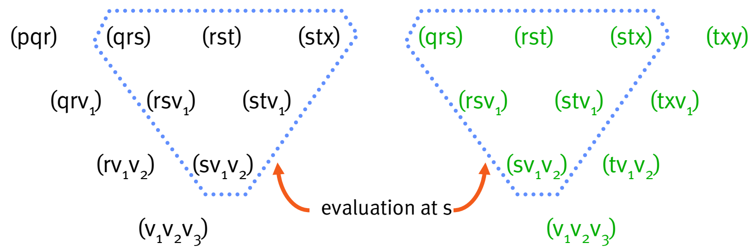
$$-2 \; -1 \; 0 \; 1 \; 2 \; 3$$
$$(-2-10) \; (-101) \; (012) \; (123)$$

- 2n knots yield n+1 control points
  - moving window of size n (why?)

Continuity

Simple knots example

- n=3 just drawing arguments

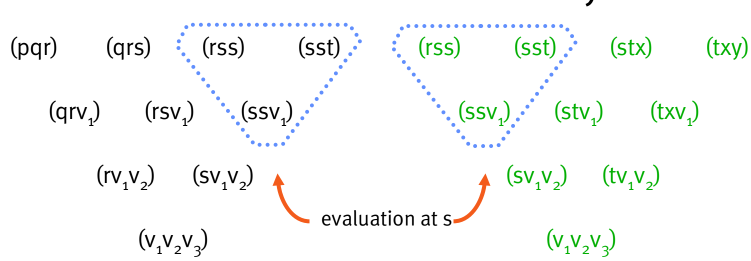


- C^2 at s: cubic B-splines are C^2

Continuity

Doubling a knot

- lose one order of continuity



- now only C^1

B-Splines

Terms

- degree: highest exponent in polynomial
- order: 1+degree
- multiplicity of a knot: how many times repeated
- continuity: C^{n-m} with multiplicty m

B-Splines

Definition

- recursive

$$N_i^0(u) = \begin{cases} 1 & t_i \le u < t_{i+1} \\ 0 & \text{else} \end{cases}$$

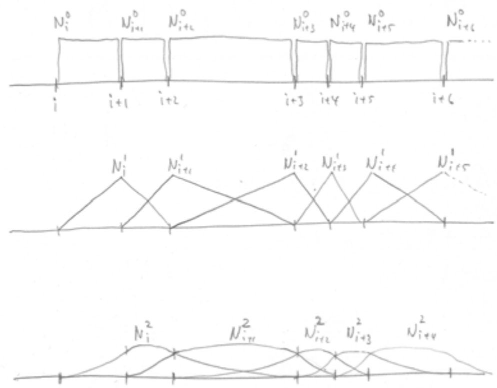
$$N_i^l(u) = \frac{u - t_i}{t_{i+l} - t_i} N_i^{l-1}(u) + \frac{t_{i+l+1} - u}{t_{i+l+1} - t_{i+1}} N_{i+1}^{l-1}(u)$$

- support

$$N_i^l(u) = 0 \quad u \notin [t_i, t_{i+l+1})$$

B-Spline Definition

Basis functions



de Boor Algorithm

Recursion

- recursive definition of the basis is reflected in recursive structure on control points

$$d_i^0(u) = d_i \quad i = j-n, \ldots, j$$

$$d_i^l(u) = \frac{t_{i+n+1} - u}{t_{i+n+1} - t_{i+l}} d_i^{l-1}(u) + \frac{u - t_{i+l}}{t_{i+n+1} - t_{i+l}} d_{i+1}^{l-1}(u)$$

$$l = 1, \ldots, n \quad i = j-n, \ldots, j-l$$

Evaluation

$$t_{j-2} \quad t_{j-1} \quad t_j \quad t_{j+1} \quad t_{j+2} \quad t_{j+3}$$

$$d^0_{j-3} = (t_{j-2}t_{j-1}t_j) \quad d^0_{j-2} = (t_{j-1}t_jt_{j+1})$$

$$d^0_{j-1} = (t_jt_{j+1}t_{j+2}) \quad d^0_j = (t_{j+1}t_{j+2}t_{j+3})$$

$$d^1_{j-3} = \frac{t_{j+1}-u}{t_{j+1}-t_{j-2}} d^0_{j-3} + \frac{u-t_{j-2}}{t_{j+1}-t_{j-2}} d^0_{j-2}$$

$$d^1_{j-2} = \frac{t_{j+2}-u}{t_{j+2}-t_{j-1}} d^0_{j-2} + \frac{u-t_{j-1}}{t_{j+2}-t_{j-1}} d^0_{j-1}$$

$$d^1_{j-1} = \frac{t_{j+3}-u}{t_{j+3}-t_j} d^0_{j-1} + \frac{u-t_j}{t_{j+3}-t_j} d^0_j$$

Evaluation

Cubic spline

- two successive segments

(pqr) (qrs) (rst) (stx) (qrs) (rst) (stx) (txy)

(qrv_1) (rsv_1) (stv_1) (rsv_1) (stv_1) (txv_1)

(rv_1v_2) (sv_1v_2) (sv_1v_2) (tv_1v_2)

($v_1v_2v_3$) ($v_1v_2v_3$)

B-Splines

de Boor Algorithm

$$r_n \le \ldots \le r_1 < s_1 \le \ldots \le s_n$$

$$\begin{aligned} d_j^l(u) &= f(r_1, \ldots, r_{n-l-j}, u^l, s_1, \ldots, s_j) \\ &= \frac{s_{j+1} - u}{s_{j+1} - r_{n-l-j+1}} f(r_1, \ldots, r_{n-l-j+1}, u^{l-1}, s_1, \ldots, s_j) + \\ &\quad \frac{u - r_{n-l-j+1}}{s_{j+1} - r_{n-l-j+1}} f(r_1, \ldots, r_{n-l-j}, u^{l-1}, s_1, \ldots, s_{j+1}) \\ &= \frac{s_{j+1} - u}{s_{j+1} - r_{n-l-j+1}} d_j^{l-1}(u) + \frac{u - r_{n-l-j+1}}{s_{j+1} - r_{n-l-j+1}} d_{j+1}^{l-1}(u) \end{aligned}$$

B-Splines

Knot insertion

- create new degree of freedom along curve

$$t_j \le t \le t_{j+1} \quad d_i^\star = \alpha_i d_i + (1 - \alpha_i) d_{i-1}$$

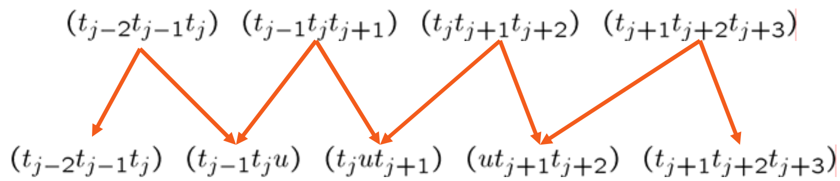
$$\alpha_i = \begin{cases} 1 & i \le j - n \\ \frac{t - t_i}{t_{i+n} - t_i} & j - n + 1 \le i \le j \\ 0 & j + 1 \le i \end{cases}$$

Knot Insertion

Example

- cubic spline: new knot $u \in [t_j, t_{j+1}]$

$t_{j-2} \quad t_{j-1} \quad t_j \quad t_{j+1} \quad t_{j+2} \quad t_{j+3}$

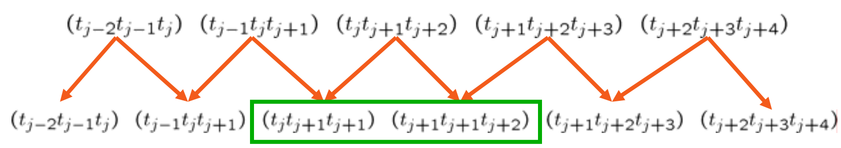


Knot Doubling

Example

- increasing multiplicity: $u = t_{j+1}$

$t_{j-2} \quad t_{j-1} \quad t_j \quad t_{j+1} \quad t_{j+2} \quad t_{j+3} \quad t_{j+4}$



Other Properties

Integrals and derivatives

- derivatives as differences
  - B-spline: $DB_j^l(t) = \frac{lB_j^{l-1}(t)}{t_{l+j-1} - t_{j-1}} - \frac{lB_{j+1}^{l-1}(t)}{t_{j+l} - t_j}$
- integrals
  - Bernstein basis:

$$\int_0^x B_i^l(t)dt = \frac{1}{l+1} \sum_{j=i+1}^{l+1} B_j^{l+1}(x)$$

$$\int_0^1 \sum_{i=0}^{n} b_i B_i^n(x)dx = \frac{1}{n+1} \sum_{j=0}^{n} b_j$$

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Degree Elevation

Re-express as higher order curve

- by uniqueness

$$f^{n+1}(u_1, \ldots, u_{n+1}) = \frac{1}{n+1} \sum_{i=0}^{n+1} f^n(\{u_j\}_{j \neq i})$$

- example: quadratic to cubic

$$f^2(u_1 u_2) = a_0 + a_1/2(u_1 + u_2) + a_2 u_1 u_2$$

$$f^3(u_1 u_2 u_3) = a_0 + a_1/3(u_1 + u_2 + u_3) + a_2/3(u_1 u_2 + u_2 u_3 + u_3 u_1) + a_3 u_1 u_2 u_3$$

- Bezier control points?

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Polar Forms

Curve manipulations

- splitting
- differentiation
- knot insertion
- increasing knot multiplicity
- degree elevation

What Else?

Rational curves

- conic sections in homogeneous coordinates $(x, y, w) \mapsto (x/w, y/w)$

$$\begin{pmatrix} x(t) \\ y(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} 1/2(1-t^2) \\ t \\ 1-1/2(1-t^2) \end{pmatrix}$$

$x + w - 1 = 0$

$x^2 + y^2 - w^2 = 0$