
INTRODUCTION TO POLYNOMIAL CURVES PART I

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CURVES AND SURFACE

Representations

- parameterized
- implicit
- piecewise linear
 - lines and polygons
- higher order (not good... Why?)
 - typically polynomials

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PARAMETRIC DESC.

Curves (piecewise)

- linear
 - quadratic? (why is this not enough?)
- cubic
 - Hermite, Bezier, B-Spline

Surfaces (patches)

- tensor products (typically...)

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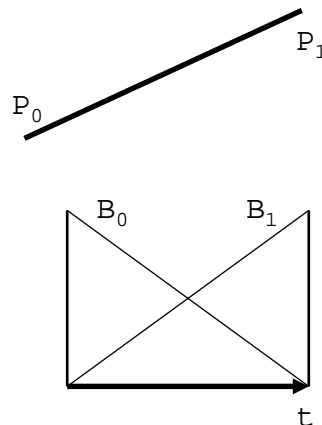
PARAMETRIC CURVES

Linear interpolation

$$C(t) = P_0 + t(P_1 - P_0)$$

$$C(t) = \begin{pmatrix} C_x(t) \\ C_y(t) \\ C_z(t) \end{pmatrix} = (P_0 \ P_1) \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t \\ 1 \end{pmatrix}$$

$$C(t) = G^T B T(t) \quad C'(t) = G^T B T'(t)$$



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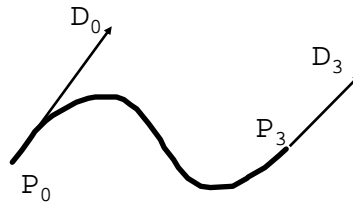
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HERMITE CURVES

Specify position and derivative at start and end point

■ 4DOF, need cubic polynomial

$$G^T = (P_0 \ P_3 \ D_0 \ D_3)$$
$$T^T(t) = \begin{pmatrix} t^3 & t^2 & t & 1 \end{pmatrix}$$
$$T'^T(t) = \begin{pmatrix} 3t^2 & 2t & 1 & 0 \end{pmatrix}$$



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HERMITE CURVES

Substitute constraints to find basis

$$G^T B \begin{pmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} = G^T \quad B = \begin{pmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$$

$$C(t) = B_0(t)P_0 + B_1(t)P_3 + B_2(t)D_0 + B_3(t)D_3$$

■ how to transform?

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MULTIPLE SEGMENTS

What are the basis functions?

- why is it useful to think in these terms?
- what do they look like?
- does the number of DOFs seem to be right?

Need something better!

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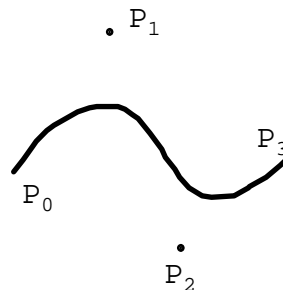
BEZIER CURVES

Bernstein blending polynomials

$$B_i^k(t) = \binom{k}{i} t^i (1-t)^{k-i}$$

■ cubic case

$$\begin{aligned} B_0^3(t) &= (1-t)^3 \\ B_1^3(t) &= 3t(1-t)^2 \\ B_2^3(t) &= 3t^2(1-t) \\ B_3^3(t) &= t^3 \end{aligned}$$



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BEZIER CURVES

Definition

$$c(t) = p_0B_0^3(t) + p_1B_1^3(t) + p_2B_2^3(t) + p_3B_3^3(t)$$

$$c'(0) = 3(p_1 - p_0) = d_0$$

$$c'(1) = 3(p_3 - p_2) = d_3$$

Fun facts

- partition of unity $\sum_{i=0}^k B_i^k(t) = 1$
- affine combinations...
- why do they matter?

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BEZIER CURVES

Properties

- curve is affine combination of control points
- endpoint interpolating
- convex hull property
- shape is invariant under affine transforms

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CHANGE OF BASIS

Example

■ Hermite to Bezier

$$B(t) = \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} t^3 \\ t^2 \\ t \\ 1 \end{pmatrix} \quad H(t) = \begin{pmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} t^3 \\ t^2 \\ t \\ 1 \end{pmatrix}$$

$$(p_0 \ p_1 \ p_2 \ p_3) = (p_0 \ p_3 \ d_0 \ d_3)HB^{-1}$$

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EVALUATION

Direct

$$c(t_0) = G^T B \begin{pmatrix} t_0^3 \\ t_0^2 \\ t_0 \\ 1 \end{pmatrix}$$

- not very efficient
- Horner's rule
- can be unstable for high orders

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FORWARD DIFFERENCING

Step along curve

$$\begin{aligned}C(t+\delta) &= C(t) + \Delta C(t) & C_{n+1} &= C_n + \Delta C_n \\ \Delta C(t+\delta) &= \Delta C(t) + \Delta^2 C(t) & \Delta C_{n+1} &= \Delta C_n + \Delta^2 C_n \\ \Delta^2 C(t+\delta) &= \Delta^2 C(t) + \Delta^3 C(t) & \Delta^2 C_{n+1} &= \Delta^2 C_n + \Delta^3 C_n\end{aligned}$$

■ choose step size

■ how many flops?

■ accumulation of error

$$D = \begin{pmatrix} 0 & 0 & 0 & 1 \\ \delta^3 & \delta^2 & \delta & 0 \\ 6\delta^3 & 2\delta^2 & 0 & 0 \\ 6\delta^3 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

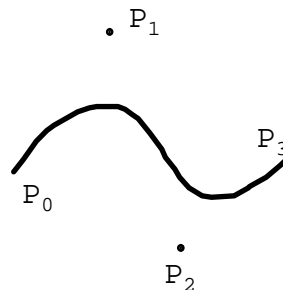
BEZIER CURVES

Bernstein blending polynomials

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■ cubic case

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BERNSTEIN POLYNOMIALS

Evaluation

■ recurrence

$$\begin{aligned} B_i^k(t) &= \binom{k}{i} t^i (1-t)^{k-i} \\ &= t \binom{k-1}{i-1} t^{i-1} (1-t)^{k-1-(i-1)} + \\ &\quad (1-t) \binom{k-1}{i} t^i (1-t)^{k-1-i} \\ &= t B_{i-1}^{k-1}(t) + (1-t) B_i^{k-1}(t) \end{aligned}$$

■ affine (convex!) combination

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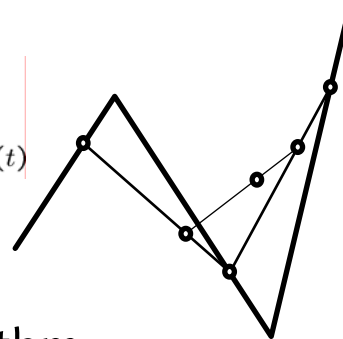
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BERNSTEIN POLYNOMIALS

Evaluation

$$b_i^0(t) = p_i \quad i = 0 \dots 3$$

$$b_i^l(t) = (1-t)b_i^{l-1}(t) + t b_{i+1}^{l-1}(t)$$



de Casteljau Algorithm

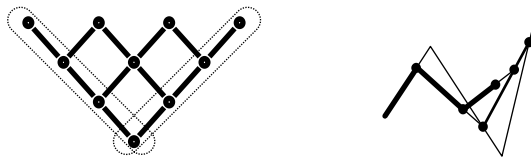
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DE CASTELJAU ALG.

Properties

- very stable, uses only affine combinations
- use to split curve into pieces
- recursive, adaptive subdivision



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AFFINE SPACE

Affine Combination

$$u_k \in \mathbb{R}^n \quad \boxed{\sum \alpha_k u_k} \quad \sum \alpha_k = 1$$

Affine Map

$$f : \mathbb{R}^n \mapsto \mathbb{R}^m \quad f(\sum \alpha_k u_k) = \sum \alpha_k f(u_k) \quad \sum \alpha_k = 1$$

- preserves affine combinations
- exactly the linear maps and translation

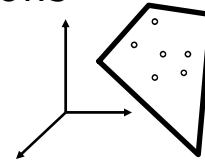
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AFFINE SPACE

Definition

- subset of vector space closed under affine combinations



- exactly the linear subspaces plus translation (points, lines, planes...)

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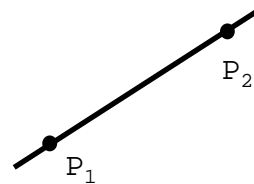
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AFFINE FORMS

Evaluation

$$\begin{aligned} f(p) &= f((1-t)p_1 + tp_2) \\ &= (1-t)f(p_1) + tf(p_2) \end{aligned}$$

$$f(t) = (1-t)f(p_1) + tf(p_2)$$



Differentiation

$$\begin{aligned} Df(t) &= \lim_{\tau \rightarrow 0} \frac{f(t+\tau) - f(t)}{\tau} \\ &= f'(1) - f'(0) \end{aligned}$$

$$\begin{aligned} f(t) &= at + b \\ Df(t) &= f'(1) - f'(0) = a \end{aligned}$$

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