CS 175 HOMEWORK 7: DUE 5-27-2003 IN CLASS

5-20-2003

1. Barycentric Coordinates (5 pts)

Consider a closed convex polygon Ω with vertices $v_1, \ldots, v_n$, $v_i \in \mathbb{R}^2$, ordered counterclockwise. Then a set of functions $w_i : \text{Interior}(\Omega) \to \mathbb{R}$, $i = 1, \ldots, n$ are called *affine coordinates* if

$$\sum_{i=1}^{n} w_i(v)(v_i - v) = 0. \tag{1}$$

For any point $v \in \text{Interior}(\Omega)$ let $\alpha_i(v)$, $\beta_i(v)$, and $\gamma_i(v)$ be the angles of the triangle $[v, v_i, v_{i+1}]$ at v, v_i, and v_{i+1}, respectively. Then the *discrete conformal coordinates* are defined as

$$w_i^c = \cot \beta_{i-1} + \cot \gamma_i \tag{2}$$

and the *mean value coordinates* are defined as

$$w_i^m = \frac{\tan(\alpha_{i-1}/2) + \tan(\alpha_i/2)}{r_i} \tag{3}$$

with $r_i(v) = \|v - v_i\|$. Show that both are affine coordinates.

Remark: The mean value coordinates are also positive, $w_i^m(v) > 0$, and define the barycentric coordinates

$$\lambda_i^m = \frac{w_i^m}{\sum_{j=1}^{n} w_j^m}. \tag{4}$$

In general, any set of functions $\lambda_i : \Omega \to \mathbb{R}$ are called *barycentric coordinates* if they satisfy the three properties

$$\lambda_i(v) \geq 0, \qquad i = 1, 2, \ldots, n, \tag{5}$$

$$\sum_{i=1}^{n} \lambda_i(v) = 1 \tag{6}$$

and

$$\sum_{i=1}^{n} \lambda_i(v) v_i = v. \tag{7}$$

2. Parameterizations (5 pts)

Both coordinates in the previous exercise have been used for parameterizing triangulations in the following way. For each edge $e = [v, w]$ that connects two vertices v and w, a weight d_{vw} is specified as

$$d^c_{vw} = \cot\beta + \cot\gamma \tag{8}$$

and

$$d^m_{vw} = \frac{\tan(\alpha/2) + \tan(\delta/2)}{\|v - w\|} \tag{9}$$

where β and γ are the angles opposite the edge e in the adjacent triangles and α and δ are the angles at v in both triangles. Note that $d^c_{vw} = d^c_{wv}$, but in general $d^m_{vw} \neq d^m_{wv}$. After specifying the parameter values $\psi(v)$ for the boundary vertices $v \in V_B$ in some way, the parameter values for the interior vertices are then found by solving the linear system

$$\sum_{w \in N_v} d_{vw}(\psi(v) - \psi(w)) = 0, \qquad v \in V_I. \tag{10}$$

If the parameter values for the boundary vertices form a convex polygon and if all the weights are positive, then the resulting parameterization ψ is guaranteed to be a bijection. While the mean value weights d^m_{vw} are always positive, the discrete conformal weight d^c_{vw} can be negative.

Find a simple example for which the parameterization with conformal weights and a convex boundary becomes non-bijective, i.e., parameter triangles overlap. List the coordinates of the vertices and the triangles of your example as well as the parameter values for both conformal and mean value parameterization.