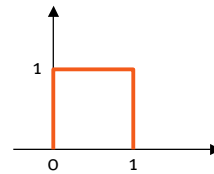

SUBDIVISION I: THE UNIVARIATE SETTING

Peter Schröder

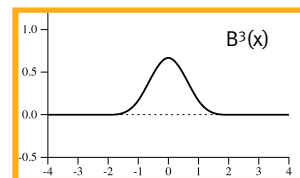
B - SPLINES (UNIFORM)

Through repeated integration

$$B^1(x) := \begin{cases} 1 & \text{if } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$



$$B^m(x) = \int_0^1 B^{m-1}(x-t) dt$$



B - SPLINES

Obvious properties

- piecewise polynomial:

$$\forall i \in \mathbb{N} : B^m(x) \Big|_{[i, i+1]} \in \Pi^{m-1}, B^m \in C^{m-2}(\mathbb{R})$$

- unit integral: $\int_{\mathbb{R}} B^m(x) dx = 1$

- non-negative: $B^m(x) \geq 0, x \in \mathbb{R}$

- partition of unity: $\sum_i B^m(x-i) = 1$

- support: $B^m(x) \neq 0, x \in [0, m]$

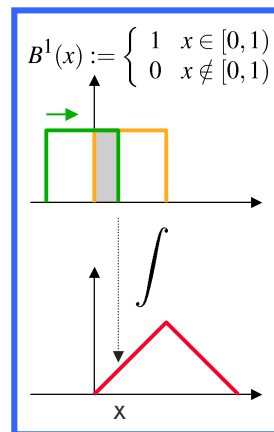
B - SPLINES

Repeated convolution

- box function

$$\begin{aligned} B^2(x) &= \int B^1(t)B^1(x-t) dt \\ &= B^1 \otimes B^1(x) \end{aligned}$$

$$B^m(x) = B^1 \otimes B^{m-1}(x)$$



CONVOLUTION

Reminder

- definition:

$$g \otimes h(x) = \int g(t)h(x-t) dt$$

- translation:

$$g(\cdot - i) \otimes h(\cdot - j)(x) = g \otimes h(x - i - j)$$

- dilation:

$$g(2\cdot) \otimes h(2\cdot)(x) = \frac{1}{2}g \otimes h(2x)$$

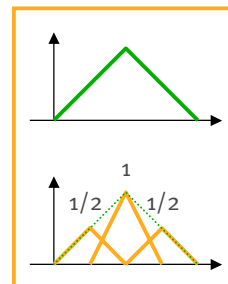
REFINABILITY I

B-Spline refinement equation

- a B-spline can be written as a linear combination of dilates and translates of itself

- example

- linear B-spline
- and all others...



REFINABILITY II

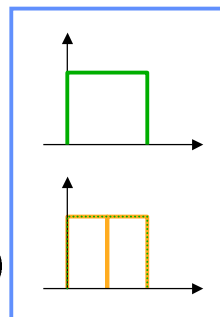
Refinement equation for B-splines

■ take advantage of box refinement

$$B^1(x) = B^1(2x) + B^1(2x - 1)$$

$$B^m(x) = \bigotimes_{i=1}^m B^1(x)$$

$$= \bigotimes_{i=1}^m (B^1(2x) + B^1(2x - 1))$$

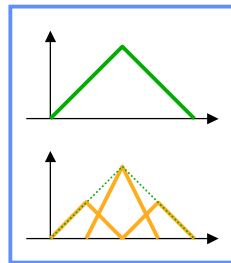


REFINABILITY

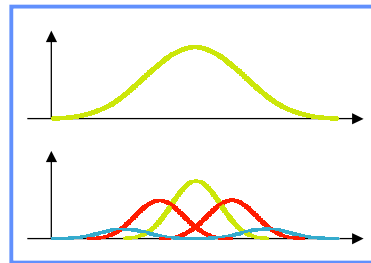
$$\begin{aligned} B^m(x) &= \bigotimes_{i=1}^m (B^1(2x) + B^1(2x - 1)) \\ &= \sum_{i=0}^m \binom{m}{i} (B^1(2\cdot))^{\otimes i} \otimes (B^1(2\cdot - 1))^{\otimes m-i}(x) \\ &= 2^{-m+1} \sum_{i=0}^m \binom{m}{i} B^m(2x - i) \\ &= \sum_{i=0}^m s_i B^m(2x - i) \quad S(z) = 2 \left(\frac{1+z}{2} \right)^m \end{aligned}$$

B-SPLINE REFINEMENT

Examples



$1/2(1, 2, 1)$



$1/8(1, 4, 6, 4, 1)$

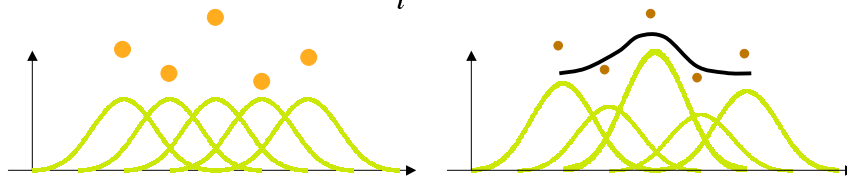
SPLINE CURVES I

Sum of B-splines

- curve as linear combination

$$\gamma(t) = \sum_i p_i B(t-i)$$

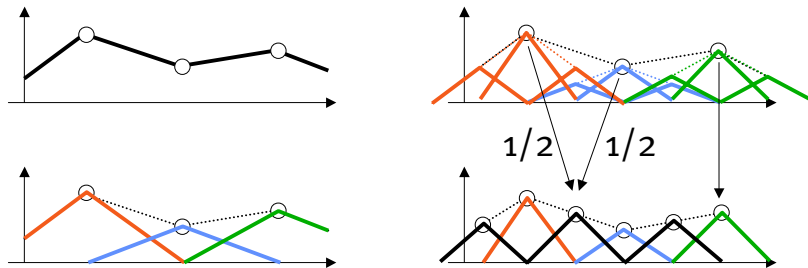
control points



SPLINE CURVES II

Refine each B-spline in sum

■ example: linear B-spline



SPLINE CURVES III

Refinement for curves

■ refine each B-spline in sum

$$\begin{aligned}
 \gamma(t) &= \sum_i p_i B(t-i) \\
 &= \sum_i p_i \left(\sum_k s_k B(2(t-i)-k) \right) \\
 &= \sum_t B(2t-i) \left(\sum_k s_{k-2i} p_i \right)
 \end{aligned}$$

refined bases

refinement of control points

REFINEMENT OF CURVES

Linear operation on control points

■ succinctly

$$\gamma(t) = \sum_i p_i B(t-i)$$

$$\gamma(t) = B(t)p$$

$$B(t) = \sum_k s_k B(2t-k)$$

$$B(t) = B(2t)S$$

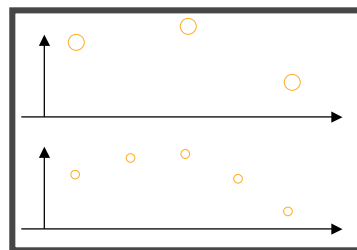
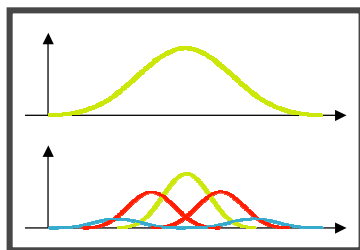
$$\gamma(t) = B(t)p = B(2t)Sp$$

REFINEMENT OF CURVES

Bases and control points

$$B(t) = B(2t)S$$

$$p^1 = Sp^0$$

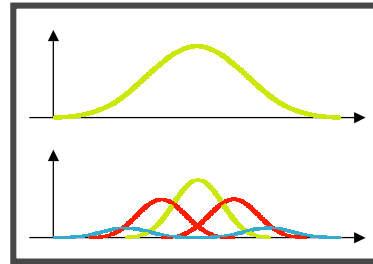


SUBDIVISION OPERATOR

Example

■ cubic splines

$$B(t) = B(2t)S$$

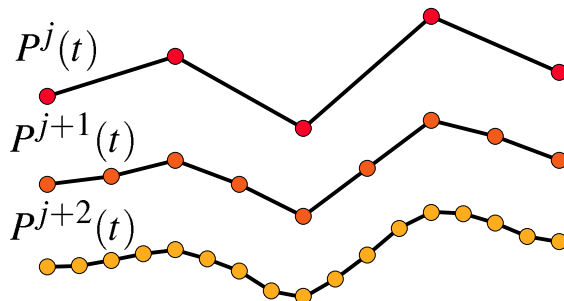
$$S = \frac{1}{8} \begin{pmatrix} \dots & 1 & 0 & 0 & 0 & \dots \\ \dots & 4 & 0 & 0 & 0 & \dots \\ \dots & 6 & 1 & 0 & 0 & \dots \\ \dots & 4 & 4 & 0 & 0 & \dots \\ \dots & 1 & 6 & 1 & 0 & \dots \\ \dots & 0 & 4 & 4 & 0 & \dots \\ \dots & 0 & 1 & 6 & 1 & \dots \\ \dots & 0 & 0 & 4 & 4 & \dots \\ \dots & 0 & 0 & 1 & 6 & \dots \\ \dots & 0 & 0 & 0 & 4 & \dots \\ \dots & 0 & 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$


$$\frac{1}{8}(1, 4, 6, 4, 1)$$

SUBDIVISION

Apply subdivision to control points

■ draw successive control polygons rather than curve itself



$$P^{j+1} = SP^j$$

SUMMARY SO FAR

Splines through refinement

- B-splines satisfy refinement eq.
- basis refinement corresponds to control point refinement
- instead of drawing curve, draw control polygon
- subdivision is refinement of control polygon

ANALYSIS

Setup

- polygon mapped to polygon
 $p^j = [p_i^j] \mapsto p^{j+1} = [p_i^{j+1}]$
- finite or (bi-)infinite, $p_i^j \in \mathbb{R}^d$
- subdivision operator (linear for now)

$$p^{j+1} = Sp^j \quad p_i^{j+1} = \sum_k s_{i,k} p_k^j$$

SUBDIVISION SCHEMES

Some properties

- affine invariance
- compact support
- index invariance (topologic symmetry)
- local definition

SUBDIVISION OPERATOR

Properties

- compact support

$$\exists l_0, l_1 \in \mathbb{N} : \forall k : s_{i,k} = 0, i \notin [2k - l_0, 2k + l_1]$$

- affine invariance

$$ap_i^{j+1} + b = \sum_k s_{i,k} (ap_k^j + b)$$

- index invariance/symmetry

$$s_{i,k} = s_{i+2,k+1} \quad s_{2k-l,k} = s_{2k+l,k}$$

SUBDIVISION OPERATOR

- local definition: weights depend only on local neighborhood

$$p_i^{j+1} = \sum_k s_{i-2k} p_k^j$$

Terms

- stationary: level independence
- uniform: location independence
- no boundaries (for now)

GENERATING FUNCTIONS

$$P^j(z) = \sum_i p_i^j z^i \quad c_k = \sum_{i+j=k} a_i b_j \quad C(z) = A(z)B(z)$$

Subdivision operator as convolution

$$s_{i,k} = s_{i-2k} \iff s_{2i+k,i} = s_k$$

$$\hat{s}_{i+k,i} = s_k \quad \uparrow P^j(z) = P^j(z^2)$$

$$P^{j+1}(z) = S(z)P^j(z^2)$$

EXAMPLES

Splines

- linear: $1/2 + z + 1/2z^2 = 1/2(1+z)^2$

$$P^{j+1}(z) = \left(\frac{1+z}{2}\right) ((1+z)P^j(z^2))$$

- quadratic:

$$P^{j+1}(z) = \left(\frac{1+z}{2}\right)^2 ((1+z)P^j(z^2))$$

- higher order...

EXAMPLES

Quadratic splines

- Chaikin's algorithm computes new points with weights $1/4(1, 3)$ and $1/4(3, 1)$
- what happens if we change the weights?

CONVERGENCE

How much leeway do we have?

- design of other subdivision rules
 - example: 4pt scheme
- establish convergence
- establish order of continuity

ANALYSIS

Simple facts

- affine invariance necessary
condition for uniform convergence

$$S(1) = 2 \quad S(-1) = 0$$

$$p_0^{j+1} = \sum_k s_{0-2k} p_k^j$$

$$p_1^{j+1} = \sum_k s_{1-2k} p_k^j$$

$$\text{supp } s_k \subset [-M, M]$$

$$\lim_{j \rightarrow \infty} p_k^j = f(0)$$

$$k \in [-M, M]$$

ANALYSIS

Convergence

- define linear interpolant over given control points and associated parametric values (knot vector)

$$L[t^j, p^j](t) \quad L[t^j, p^j](t_i^j) = p_i^j$$

- typically $t^j = 2^{-j}\mathbb{Z}$
- define pointwise $\lim_{j \rightarrow \infty} L[t^j, p^j](t) = f(t)$

ANALYSIS

Convergence

- in max/sup norm $\|v\|_\infty = \sup_i |v_i|$

$$\|A\|_\infty = \sup_{v \neq 0} \frac{\|Av\|_\infty}{\|v\|_\infty} = \sup_i \sum_j |a_{i,j}|$$

Theorem

- if $\exists \beta > 0 \wedge \alpha \in (0, 1) \forall j > 0 : \|\Delta p^j\|_\infty < \beta \alpha^j$
then convergence of $\lim_{j \rightarrow \infty} L[t^j, p^j](t)$ is uniform

UNIFORM CONVERGENCE

Proof

linear spline subdivision operator

$$L[t^{j+1}, p^{j+1}](t) - L[t^j, p^j](t) = L[t^{j+1}, (S - S_1)p^j](t)$$

$$S - S_1 = A\Delta \quad \Delta(z) = (1 - z)$$

$$\|L[t^{j+1}, A\Delta p^j](t)\|_\infty \leq \|A\|_\infty \|\Delta p^j\|_\infty \leq \beta \alpha^j$$

$$L[t^0, p^0](t) + \sum_{j>0} (L[t^{j+1}, p^{j+1}](t) - L[t^j, p^j](t))$$

$$\|f(t) - L[t^j, p^j](t)\|_\infty < \|A\| \frac{\beta}{1 - \alpha} \alpha^j$$

DIFFERENCE DECAY

Sufficient condition

- continuous limit if $\|\Delta p^j\|_\infty < \beta \alpha^j$
- analysis by examining associated difference scheme

$$\Delta p^{j+1} = D\Delta p^j$$

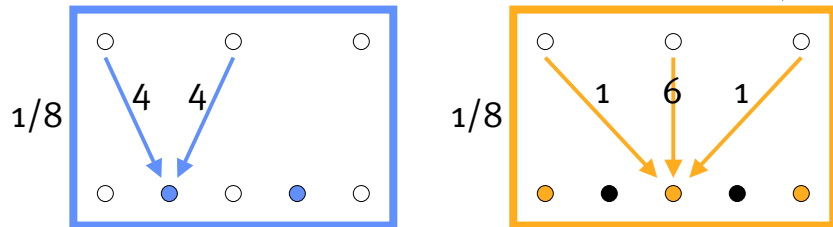
EXAMPLE

Cubic B-splines

■ stencils

$$p_{2i+1}^{j+1} = 1/8(4p_i^j + 4p_{i+1}^j)$$

$$p_{2i}^{j+1} = 1/8(p_{i-1}^j + 6p_i^j + p_{i+1}^j)$$



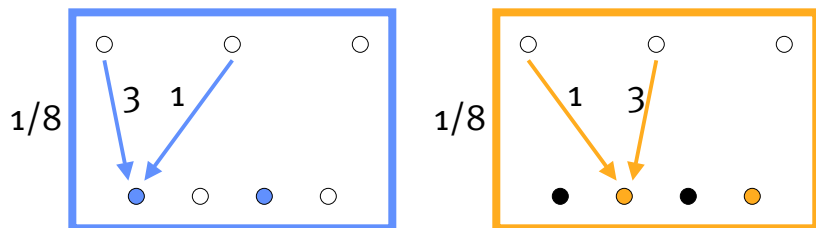
EXAMPLE

Cubic B-splines

■ differences

$$\Delta p_{2i}^{j+1} = p_{2i+1}^{j+1} - p_{2i}^{j+1}$$

$$= 1/8(3\Delta p_i^j + \Delta p_{i-1}^j)$$



DIFFERENCE DECAY

Analysis of difference scheme

- construction from the subdivision scheme itself

$$\Delta S p^j = D \Delta p^j$$

$$\rightsquigarrow (1-z)P^{j+1}(z) = D(z)(1-z^2)P^j(z^2)$$

$$\rightsquigarrow (1-z)S(z)P^j(z^2) = D(z)(1-z^2)P^j(z^2)$$

$$\rightsquigarrow S(z) = D(z)(1+z)$$

HIGHER ORDERS

Smoothness

- how to show C^1 ?
- divided differences must converge
 - check difference of divided differences
- example
 - 4pt scheme

SMOOTHNESS

Consequences

$$S(z) \in C^0 \implies 2^{-k}(1+z)^k S(z) \in C^k$$

4pt scheme: decay estimate

$$\begin{aligned} S(z) &= 1/16(-z^{-3} + 4z^{-2} - z^{-1})(1+z)^4 \\ &= 1/16(-z^{-3} + 9z^{-1} + 16 + 9z - z^3) \end{aligned}$$

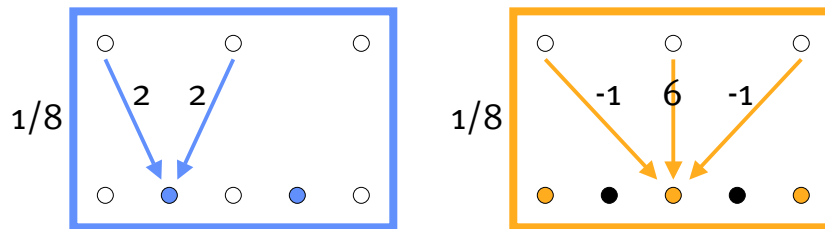
$$DD(z) = 1/8(-z^{-3} + 4z^{-2} - z^{-1})(1+z)^2$$

$$\|DD\|_\infty = 1 \quad \|DD^2\|_\infty = 3/4$$

EXAMPLE

4pt scheme

differences of divided differences



$$\|DD\|_\infty = 1$$

ANALYSIS

Fundamental solution

- gives basis functions

$$L[t^j, p^j](t) = \sum_i p_i^j L[t^j, e_i^j]$$
$$\lim_{j \rightarrow \infty} L[t^j, S^j e_i^0](t) = \phi(t - i)$$

$$V^j = \text{span}\{\phi_i^j(t) := \phi(2^j t - i) \mid i \in \mathbb{Z}\}$$

FUNDAMENTAL SOLUTION

Properties

- refinement relation (why?)

$$\phi(t) = \sum_{i=-l_0}^{l_1} s_i \phi(2x - i)$$

- support? non-zero coefficients:

$$p_{-l_0}^1 \cdots p_{l_1}^1 \rightsquigarrow p_{-l_0(2^k-1)}^k \cdots p_{l_1(2^k-1)}^k$$

$$\lim_{k \rightarrow \infty} t_{-l_0(2^k-1)}^k = t_{-l_0}^0 \cdots \text{supp } \phi \subset [t_{-l_0}^0, t_{l_1}^0]$$

SO FAR, SO GOOD I

What do we know now?

- regular setting
- approximating
 - B-splines
- interpolating
 - 4pt scheme (Deslaurier-Dubuc)

SO FAR, SO GOOD II

What do we know now?

- differences
 - continuity
 - differentiability
- not quite general enough
 - current setting assumes a particular parameterization

MORE GENERAL SETTINGS

Non-uniform

- in spline case better curves
- subdivision weights will vary
 - knot insertion
 - interpolation

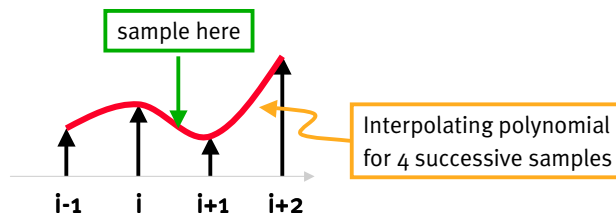
$S =$

a	0	0	0
b	0	0	0
c	f	0	0
d	g	0	0
e	h	k	0
0	i	l	0
0	j	m	p
0	0	n	q
0	0	o	r
0	0	0	s
0	0	0	t

4PT SCHEME I

Where do the weights come from?

- example of Deslaurier-Dubuc
- given set of samples use interpolating polynomial to refine



4PT SCHEME II

Weight computation

- grind out interpolating polynomial
 - resulting weights: $1/16(-1, 9, 9, -1)$
- Deslaurier-Dubuc
 - generalization of same idea
 - higher orders yield higher continuity
 - tends to exhibit “ringing” (as is to be expected...)

DESLAURIER - DUBUC

Local polynomial reproduction

- choose s_k accordingly ($d=1$ for 4pt)

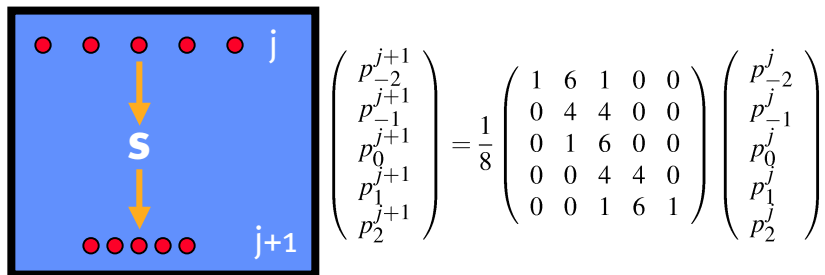
$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ t_{-d} & t_{-d+1} & \dots & t_{d+1} \\ \vdots & \vdots & \ddots & \vdots \\ t_{-d}^{2d+1} & t_{-d+1}^{2d+1} & \dots & t_{d+1}^{2d+1} \end{pmatrix} \begin{pmatrix} s_{-d} \\ s_{-d+1} \\ \vdots \\ s_{d+1} \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \\ \vdots \\ (1/2)^{2d+1} \end{pmatrix}$$

- non-uniform possible, increasing smoothness, approximation power, limit for increasing d is sinc fn.

EXAMPLE

Cubic B-spline

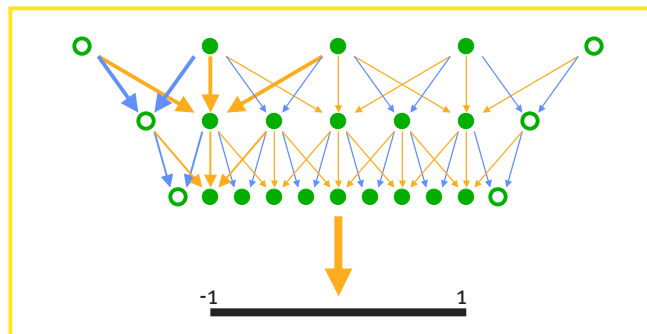
- 5 control points for 1 segment on either side of the origin



NEIGHBORHOODS

Which points influence a region?

- for analysis around a point



SUBDIVISION MATRIX

Invariant neighborhood

- which $\phi(i-t)$ overlap the origin?
- tells smoothness story

$$\bar{S} = \frac{1}{16} \begin{pmatrix} -1 & 9 & 9 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 9 & 9 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 9 & 9 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 9 & 9 & -1 \end{pmatrix}$$

EIGEN ANALYSIS

What happens in the limit?

- behavior in neighborhood of point
 - apply S infinitely many times...
- suppose S has complete set of EVs

control points in invariant neighborhood $\rightarrow p = \sum_{i=0}^{n-1} a_i x_i$ \leftarrow eigen vectors

$$Sp = \sum_{i=0}^{n-1} \lambda_i a_i x_i$$

SUBDIVISION MATRIX

Properties

- eigen vectors of non-zero eigen values identical

$$\lambda \neq 0, \bar{S}y = \lambda y \Rightarrow \exists x : Sx = \lambda x \wedge \bar{x} = y$$

- proof by extension of y
- if defective, use generalized eigen vectors and values

SUBDIVISION MATRIX

Eigen vectors and eigen functions

$$\bar{S}\bar{x}_i = \lambda_i \bar{x}_i$$

$$|\lambda_0| > |\lambda_1| \geq |\lambda_2| \geq \dots |\lambda_l|$$

$$l = -l_0 + l_1, i = 0, \dots, l$$

$$\bar{p} = \sum_{i=0}^l c_i \bar{x}_i \rightsquigarrow F[t^0, p^0](t) = \sum_{i=0}^l c_i F[t^0, x_i](t)$$

$$t \in (t_{-1}^0, t_1^0)$$

no overbar

SCALING RELATION

Eigen functions scale

$$Sx = \lambda x \implies \lambda F[t^0, x](t) = F[t^0, x](t/2)$$

$$\lambda F[t^0, x](t) = F[t^0, \lambda x](t) = F[t^0, Sx](t)$$

$$= \lim_{j \rightarrow \infty} L[t^j, S^j(Sx)](t)$$

in neighborhood
of the origin

$$= \lim_{j \rightarrow \infty} L[2^j t^{j+1}, S^{j+1}x](t)$$

$$= F[t^0, x](t/2)$$

SMOOTHNESS (AT ORIGIN)

Lemma for functions which scale

$$\lambda g(t) = g(t/2)$$

I $\max_{t \in (1/2, 1]} |g(t)| < \infty \wedge |\lambda| < 2^{-k} \implies \lim_{t \rightarrow 0} \frac{g(t)}{t^k} = 0$

II $\lambda = 1 \wedge g(t) \in C^0([0, 1]) \implies g(t) = c$

III $g(d) \neq 0, d \neq 0 \wedge |\lambda| > 1 \implies \lim_{t \rightarrow 0} g(t) = \infty$

NECESSARY CONDITIONS

Continuity at eigen functions

$$Sx = \lambda x \wedge |\lambda| \geq 2^{-k}$$

$$F[t,x](t) \in C^k \Rightarrow \exists 0 \leq i \leq k :$$

$$\lambda = 2^{-i} \wedge F[t,x](t) = c_i t^i \quad c_i \neq 0$$

NECESSARY CONDITIONS

Spectrum

- must be 2^{-i} for $0 \leq i \leq k$ and corresponding eigen functions must be monomials
- generalized eigen vectors?

$$\lambda_1 = 1$$

- λ_0 must be simple

$$\begin{aligned} Sx_0 &= x_0 \\ Sx_1 &= \lambda_1 x_1 + x_0 \end{aligned}$$

? $\rightarrow \lambda_1 F[t,x_1](t) + F[t,x_0](t) = F[t,x_1](t/2)$

SUFFICIENT CONDITIONS

Check at origin

- eigen functions for $|\lambda| < 2^{-k}$ must be checked

$$Sx = \lambda x \wedge |\lambda| < 2^{-k}$$

$$F[t,x](t) \in C^k((0,1]) \Rightarrow F[t,x](t) \in C^k([0,1])$$

SUBDIVISION OPERATOR

Spectrum

- necessary conditions
 - for C^k must have $\lambda_i = 2^{-i}$ for $i \leq k$
 - eigen functions are polynomials
 - generally not enough
 - 4pt scheme has $1, 1/2, 1/4, 1/4, 1/8, -1/16, -1/16$
 - approximation properties

CONVERGENCE

Limit position

- let j go to infinity

$$p^j = \lambda_0^j a_0 x_0 + \lambda_1^j a_1 x_1 + \dots$$

- if $\lambda_0 = 1$ and $|\lambda_i| < 1$, $i = 1, \dots, n-1$

$$p_0^\infty = a_0$$

- example: cubic B-spline

$$p_i^\infty = a_0 = 1/6(p_{i-1}^j + 4p_i^j + p_{i+1}^j)$$

GEOMETRIC BEHAVIOR

Move limit point to origin

- look at higher order behavior

$$p^j = \lambda_0^j a_0 x_0 + \lambda_1^j a_1 x_1 + \dots$$

$$|\lambda_2|, \dots, |\lambda_{n-1}| < |\lambda_1|$$

$$p^j / \lambda_1^j = a_1 x_1 + (\lambda_2 / \lambda_1)^j a_2 x_2 + \dots$$

tangent vector

MORE GENERAL SETTINGS

Subtleties

- generalized eigen values
- more subtle smoothness analysis
- non-uniform subdivision
- completely irregular subdivision
- boundaries
- formule de commutation

A NOTE

Size of subdivision matrix

- for *analysis* need enough support to parameterize a finite neighborhood of the origin
- for *evaluation* need only enough support to zoom in on origin
- e.g., cubic spline needs **5** respectively **3** control points

EIGEN ANALYSIS

Summary

- invariant neighborhood to understand behavior around point
- Eigen decomposition of subdivision matrix helpful
 - limit point: a_0 , tangent: a_1

General setting more complicated...