
INTRODUCTION TO POLYNOMIAL CURVES PART II

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POLAR FORMS

Blossom

- For every polynomial $F(u)$ of degree n there exists a unique symmetric multi-affine map $f(u_1, \dots, u_n)$ for which $F(u) = f(u, \dots, u)$. This map is called the polar form. $F(u)$ is called its diagonal and f is called the blossom of F .

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EXAMPLE

Cubic case

$$F(u) = a_0 + a_1u + a_2u^2 + a_3u^3$$

$$f(u_1, u_2, u_3) = a_0 + \frac{a_1}{3}(u_1 + u_2 + u_3) + \frac{a_2}{3}(u_1u_2 + u_2u_3 + u_3u_1) + a_3u_1u_2u_3$$

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POLAR FORM

Correspondence

$$F(u) = \sum_{i=0}^n a_i u^i \quad f(u_1, \dots, u_n) = \sum_{i=0}^n a_i \binom{n}{i}^{-1} \sum_{\substack{S \subseteq \{1, \dots, n\} \\ |S|=i}} \prod_{j \in S} u_j$$

Differentiation

- differencing

$$\Delta[\tau]f(u_1, \dots, u_{n-1}) := f(u_1, \dots, u_n + \tau) - f(u_1, \dots, u_n)$$

- well defined, i.e., independent of u_n

- linear in τ

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POLAR FORM

Differentiation

$$\begin{aligned} F^{(q)}(u) &= \frac{d^q}{du^q} f(\overbrace{u, \dots, u}^n) \\ &= \frac{d^{q-1}}{du^{q-1}} (n \Delta[1] f(\overbrace{u, \dots, u}^{n-1})) \\ &= n \Delta[1] \frac{d^{q-1}}{du^{q-1}} f(\overbrace{u, \dots, u}^{n-1}) \\ &\dots \\ &= \frac{n!}{(n-q)!} \Delta[1^q] f(\overbrace{u, \dots, u}^{n-q}) \end{aligned}$$

$$a_q = \frac{F^{(q)}(0)}{q!} = \binom{n}{q} \Delta[1^q] f(0^{n-q})$$

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POLAR FORM

Differentiation

- let f be a polar form and $0 \leq p \leq n$, and $u \in \mathbb{R}$ (fixed) then the following are equivalent

$$\forall q = 0, \dots, p : \Delta[1^q] f(u^{n-q}) = 0$$

$$\forall v_1, \dots, v_p \in \mathbb{R} : f(u^{n-p}, v_1, \dots, v_p) = 0$$

Proof $f(\dots, u) = f(\dots, v) - \Delta[v - u]f(\dots)$

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CONTINUITY

Two polynomials

- let F and G be two polynomials of degree n and let u be a point on the real line. Then F and G are C^q continuous at u iff

$$f(u^{n-q}, u_1, \dots, u_q) = g(u^{n-q}, u_1, \dots, u_q)$$

$$u_1, \dots, u_q \in \mathbb{R}$$

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BEZIER SEGMENTS

Continuity conditions

$$f(000) \quad f(001) \quad f(011) \quad f(111) \quad g(000) \quad g(001) \quad g(011) \quad g(111)$$

$$C^0 \rightsquigarrow f(111) = g(000)$$

$$\Delta[1]f(u) = f(uu1) - f(uu0) \rightsquigarrow f(111) - f(110)$$

$$\Delta[1]g(u) = g(uu1) - g(uu0) \rightsquigarrow g(001) - g(000)$$

$$C^1 \rightsquigarrow f(111) = g(000) = 1/2(g(001) + f(110))$$

$$\Delta[11]f(u) = f(u11) - 2f(u10) + f(u00)$$

$$\Delta[11]g(u) = g(u11) - 2g(u10) + g(u00)$$

$$C^2 \rightsquigarrow f(111) = g(000) = 1/4(g(011) - f(001))$$

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BEZIER POINTS

Finding control points

- the Bezier control points of F with respect to $[s,t]$ are

$$b_i = f(s^{n-i}t^i) \quad i = 0, \dots, n \quad u = \frac{t-u}{t-s}s + \frac{u-s}{t-s}t$$

Proof

$$\begin{aligned} F(u) = f(u^n) &= \left(\frac{t-u}{t-s}\right) f(su^{n-1}) + \left(\frac{u-s}{t-s}\right) f(u^{n-1}t) \\ &= \dots = \sum_{i=0}^n B_i^{\Delta, n}(u) f(s^{n-i}t^i) \end{aligned}$$

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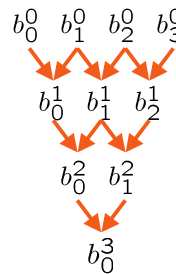
DE CASTELJAU ALG.

Evaluate Bezier form

$$b_i^0(u) = b_i \quad i = 0, \dots, n$$

$$b_i^l(u) = \left(\frac{t-u}{t-s}\right) b_i^{l-1}(u) + \left(\frac{u-s}{t-s}\right) b_{i+1}^{l-1}(u)$$

- very stable
- convex combinations only



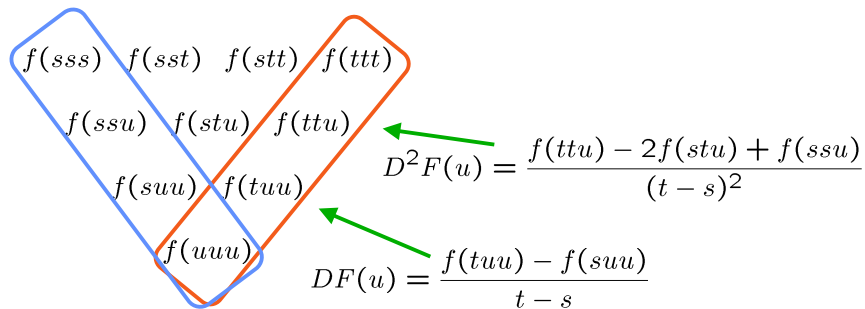
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DE CASTELJAU

Properties

- segment splitting, derivatives



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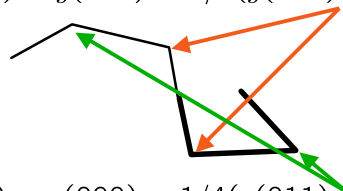
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MULTIPLE SEGMENTS

How to stick them together?

- can match first derivative easily
- second derivative?

$$f(111) = g(000) = 1/2(g(001) - f(110))$$



$$f(111) = g(000) = 1/4(g(011) - f(001))$$

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MODELING

Bezier curves

- requires to keep track of continuity conditions explicitly... Possible, but very messy
- better: use basis which has continuity conditions built in!

B-Splines

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B - S P L I N E S

Basic idea

- share arguments of control points

$$u \in [0, 1]$$

$$f(-2 - 10) \quad f(-101) \quad f(012) \quad f(123)$$

$$f(-10u) \quad f(01u) \quad f(12u)$$

$$f(0uu) \quad f(1uu)$$

$$f(uuu)$$

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ARGUMENT “BAGS”

Knotvector

- definition

$$T = (t_i)_{i \in \mathbb{Z}} \quad t_i < t_{i+n+1}$$

- non-decreasing sequence, can have multiple entries (multiplicity)

- curve:

$$c(u) = \sum N_i^n(u) d_i$$

KNOT VECTOR

Example

- Bezier segments

$$\begin{array}{ccccccc} 0 & 0 & 0 & 1 & 1 & 1 & \\ (000) & (001) & (011) & (111) & & & \end{array}$$

- B-splines segments

$$\begin{array}{ccccccc} -2 & -1 & 0 & 1 & 2 & 3 & \\ (-2-10) & (-101) & (012) & (123) & & & \end{array}$$

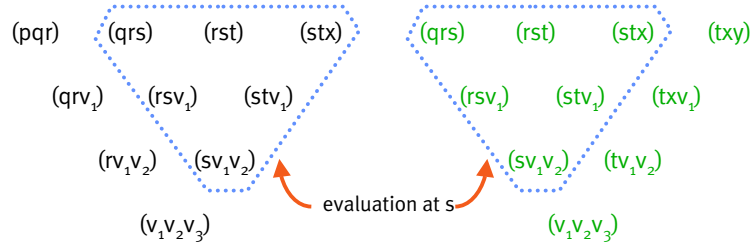
- $2n$ knots yield $n+1$ control points

- moving window of size n (why?)

CONTINUITY

Simple knots example

- $n=3$ just drawing arguments



- C^2 at s : cubic B-splines are C^2

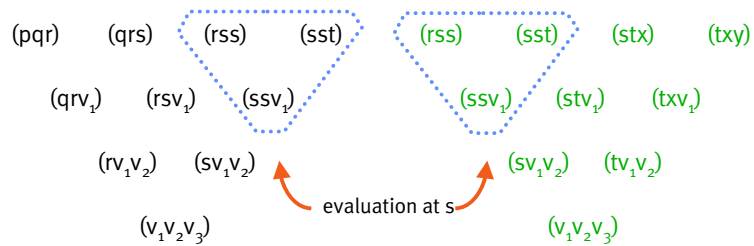
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CONTINUITY

Doubling a knot

- lose one order of continuity



- now only C^1

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B - S P L I N E S

Terms

- degree: highest exponent in polynomial
- order: 1+degree
- multiplicity of a knot: how many times repeated
- continuity: C^{n-m} with multiplicity m

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B - S P L I N E S

Definition

- recursive

$$N_i^0(u) = \begin{cases} 1 & t_i \leq u < t_{i+1} \\ 0 & \text{else} \end{cases}$$

$$N_i^l(u) = \frac{u - t_i}{t_{i+l} - t_i} N_i^{l-1}(u) + \frac{t_{i+l+1} - u}{t_{i+l+1} - t_{i+1}} N_{i+1}^{l-1}(u)$$

- support

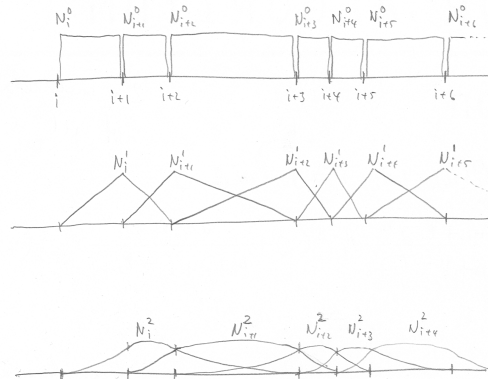
$$N_i^l(u) = 0 \quad u \notin [t_i, t_{i+l+1})$$

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B-SPLINE DEFINITION

Basis functions



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DE BOOR ALGORITHM

Recursion

- recursive definition of the basis is reflected in recursive structure on control points

$$d_i^0(u) = d_i \quad i = j - n, \dots, j$$
$$d_i^l(u) = \frac{t_{i+n+1} - u}{t_{i+n+1} - t_{i+l}} d_i^{l-1}(u) + \frac{u - t_{i+l}}{t_{i+n+1} - t_{i+l}} d_{i+1}^{l-1}(u)$$
$$l = 1, \dots, n \quad i = j - n, \dots, j - l$$

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EVALUATION

$$t_{j-2} \quad t_{j-1} \quad t_j \quad t_{j+1} \quad t_{j+2} \quad t_{j+3}$$

$$d_{j-3}^0 = (t_{j-2}t_{j-1}t_j) \quad d_{j-2}^0 = (t_{j-1}t_jt_{j+1}) \quad cc$$

$$d_{j-1}^0 = (t_jt_{j+1}t_{j+2}) \quad d_j^0 = (t_{j+1}t_{j+2}t_{j+3})$$

$$d_{j-3}^1 = \frac{t_{j+1} - u}{t_{j+1} - t_{j-2}} d_{j-3}^0 + \frac{u - t_{j-2}}{t_{j+1} - t_{j-2}} d_{j-2}^0$$

$$d_{j-2}^1 = \frac{t_{j+2} - u}{t_{j+2} - t_{j-1}} d_{j-2}^0 + \frac{u - t_{j-1}}{t_{j+2} - t_{j-1}} d_{j-1}^0$$

$$d_{j-1}^1 = \frac{t_{j+3} - u}{t_{j+3} - t_j} d_{j-1}^0 + \frac{u - t_j}{t_{j+3} - t_j} d_j^0$$

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EVALUATION

Cubic spline

■ two successive segments

(pqr) (qrs) (rst) (stx) (qrs) (rst) (stx) (txy)

(qrv₁) (rsv₁) (stv₁) (rsv₁) (stv₁) (txv₁)

(rv₁v₂) (sv₁v₂) (sv₁v₂) (tv₁v₂)

(v₁v₂v₃) (v₁v₂v₃)

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B - SPLINES

de Boor Algorithm

$$r_n \leq \dots \leq r_1 < s_1 \leq \dots \leq s_n$$

$$\begin{aligned} d_j^l(u) &= f(r_1, \dots, r_{n-l-j}, u^l, s_1, \dots, s_j) \\ &= \frac{s_{j+1} - u}{s_{j+1} - r_{n-l-j+1}} f(r_1, \dots, r_{n-l-j+1}, u^{l-1}, s_1, \dots, s_j) + \\ &\quad \frac{u - r_{n-l-j+1}}{s_{j+1} - r_{n-l-j+1}} f(r_1, \dots, r_{n-l-j}, u^{l-1}, s_1, \dots, s_{j+1}) \\ &= \frac{s_{j+1} - u}{s_{j+1} - r_{n-l-j+1}} d_j^{l-1}(u) + \frac{u - r_{n-l-j+1}}{s_{j+1} - r_{n-l-j+1}} d_{j+1}^{l-1}(u) \end{aligned}$$

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B - SPLINES

Knot insertion

- create new degree of freedom along curve

$$t_j \leq t \leq t_{j+1} \quad d_i^* = \alpha_i d_i + (1 - \alpha_i) d_{i-1}$$

$$\alpha_i = \begin{cases} 1 & i \leq j - n \\ \frac{t - t_i}{t_{i+n} - t_i} & j - n + 1 \leq i \leq j \\ 0 & j + 1 \leq i \end{cases}$$

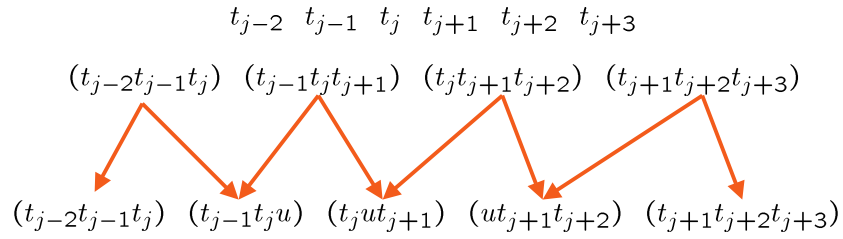
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KNOT INSERTION

Example

- cubic spline: new knot $u \in [t_j, t_{j+1}]$



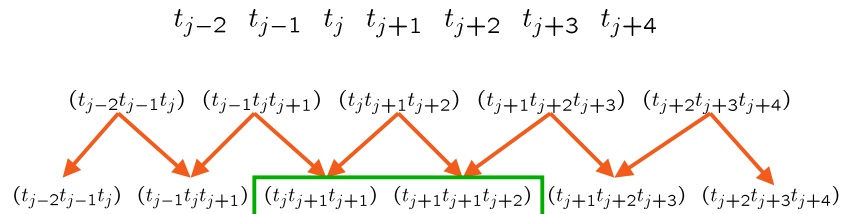
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KNOT DOUBLING

Example

- increasing multiplicity: $u = t_{j+1}$



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OTHER PROPERTIES

Integrals and derivatives

- derivatives as differences

- B-spline: $DB_j^l(t) = \frac{lB_j^{l-1}(t)}{t_{l+j-1} - t_{j-1}} - \frac{lB_{j+1}^{l-1}(t)}{t_{j+l} - t_j}$

- integrals

- Bernstein basis:

$$\int_0^x B_i^l(t) dt = \frac{1}{l+1} \sum_{j=i+1}^{l+1} B_j^{l+1}(x)$$

$$\int_0^1 \sum_{i=0}^n b_i B_i^n(x) dx = \frac{1}{n+1} \sum_{j=0}^n b_j$$

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DEGREE ELEVATION

Re-express as higher order curve

- by uniqueness

$$f^{n+1}(u_1, \dots, u_{n+1}) = \frac{1}{n+1} \sum_{i=0}^{n+1} f^n(\{u_j\}_{j \neq i})$$

- example: quadratic to cubic

$$f^2(u_1 u_2) = a_0 + a_1/2(u_1 + u_2) + a_2 u_1 u_2$$

$$f^3(u_1 u_2 u_3) = a_0 + a_1/3(u_1 + u_2 + u_3) + a_2/3(u_1 u_2 + u_2 u_3 + u_3 u_1) + a_3 u_1 u_2 u_3$$

- Bezier control points?

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POLAR FORMS

Curve manipulations

- splitting
- differentiation
- knot insertion
- increasing knot multiplicity
- degree elevation

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WHAT ELSE?

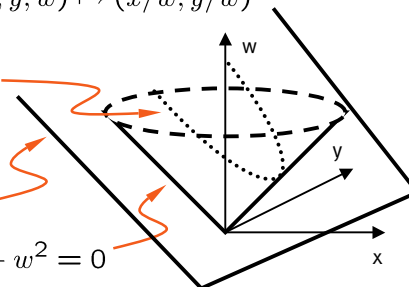
Rational curves

- conic sections in homogeneous coordinates $(x, y, w) \mapsto (x/w, y/w)$

$$\begin{pmatrix} x(t) \\ y(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} 1/2(1-t^2) \\ t \\ 1 - 1/2(1-t^2) \end{pmatrix}$$

$$x + w - 1 = 0$$

$$x^2 + y^2 - w^2 = 0$$



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